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1.1 Cars passing by is a counting process, which makes me want to think about Poisson distributions.

If, on average, cars pass at a rate of 1 every 5 minutes, then in 1 hour, one would expect

$$\frac{1}{5\text{min}} \times \frac{60\text{min}}{\text{hour}} = \boxed{12 \text{ cars}} \text{ in } 1 \text{ hour}$$

Which agrees with a Poisson dist. which has a mean of $\lambda = t/\tau$.

1.2 For time between subsequent cars, lets imagine starting our watch as soon as the first car goes by. The probability that no new car has gone by in some time interval is

$$P(t < s < t+h).$$

Similar to building master equations, this should be the same as the probability that there's been no new car up until t, <u>and</u> there is a car in the time interval $[t, t+h]$.

$$P(t < s < t+h) = \underbrace{P(N(t)=0)}_{\text{no cars up to } t} \underbrace{\times}_{\text{and}} \underbrace{P(N(t+h) - N(t) = 1)}_{\text{one car in } (t,\, t+h]}$$

For a Poisson distribution, $P(k \text{ events in } t) = \frac{(t/\tau)^k}{k!} e^{-t/\tau}$

Probability density becomes:

$$\rho(t)dt = P(t < s < t+h) = \frac{(t/\tau)^0}{0!} e^{-t/\tau} \times \underbrace{\frac{1}{\tau} h}_{\text{rate} \times \text{small time}} + O(h^2)$$

1.2 (cont.)

as $h \to dt$

$$\rho(t)\,dt = \frac{dt}{\tau} e^{-t/\tau}$$

$$\boxed{\rho_{car}(t) = \frac{1}{\tau} e^{-t/\tau}}$$

So if, for example we want to know the probability that the next car comes in the time interval $1\,min \leq t < 2\,min$ after the previous car, we have

$$\rho(1 \leq t\, 2) = \int_1^2 \frac{1}{\tau} e^{-t/\tau} = -e^{-t/\tau}\Big|_1^2 = 0.15$$

For the buses, unlike in real life, they run exactly on schedule, 5 minutes apart. If we just miss our bus and see it drive away, we must wait exactly 5 minutes

$$P(t) = \begin{cases} 1, & \text{for } t = 5\,min \\ 0, & \text{otherwise} \end{cases}$$

1.3 Buses:

Buses arrive at 5 minute intervals. We may arrive at $t = 0^+$ minutes all the way up to $t = 5^-$ minutes after the first bus left. We have an equal probability of arriving any time between these two, so

$$\rho(t) = \frac{dt'}{5\text{min}}$$

The probability is uniform.

Cars:

Exponential random variables are memoryless. If we don't know when the first car passed by we aren't any better of worse at knowing when the next will go by. The probability shouldn't change.

$$\rho(t) = \frac{1}{\tau} e^{-\lambda t}\, dt$$

1.4

If we start our timer just before the first bus arrives our arrivals would occur at

$$T_1 = 0^+\text{min},\ T_2 = 5^+\text{min},\ T_3 = 10^+\text{min},$$

$$T_4 = 15^+\text{min},\ \cancel{T_5 = 20^+\text{min}},\ \ldots$$

If we start our timer just after the bus leaves we would have arrivals at

$$\cancel{T_0 = 0\text{min}},\ T_1 = 5^-\text{min},\ T_2 = 10^-\text{min},\ T_3 = 15^-\text{min},\ T_4\ 20^-\text{min}$$

So again we get $\boxed{4}$ buses as one would expect.

1.4 (cont.) $P_{bus}^{20min}(N) = \begin{cases} 1, & \text{if } N=4 \\ 0, & \text{otherwise} \end{cases}$

Cars: For a Poisson process, the probability of some number of events happening in some time t is

$$P(n;t) = \frac{e^{-t/\tau}\left(\frac{t}{\tau}\right)^n}{n!}$$

$$\boxed{P\left(n, \underset{\tau=5min}{t=20min}\right) = \frac{e^{-4}\,4^n}{n!}}$$

$\underline{1.5}$ $S(a) = (1+bt)e^{-bt}$

a) For an animal to die in a time da, they must have survived until time a, then died in the next da. Another way to phrase this is that they survived until a, but not until a+da.

$$Pr(\text{Die } a \leq t < a+da) = S(a) - S(a+da)$$

Subtract probability that they survive through da

This looks like a derivative.

$$\frac{S(a+da) - S(a)}{da} = \left.\frac{dS}{da}\right|_a$$

$$S(a) - S(a+da) = -\frac{dS}{da}\,da$$

$$-\frac{dS}{da}\,da = -da\,\frac{d}{da}\left[(1+ba)e^{-ba}\right]$$

$$= -da\left[b\,e^{-ba} + (1+ba)(-b)e^{-ba}\right]$$

$$= -da\left(\cancel{b} + -\cancel{b} - b^2 a\right)e^{-ba}$$

$$\boxed{= b^2 a\,e^{-ba}\,da} = Pr(\text{die in } a < t < a+da)$$

b) Probability to die in next da, given we survived until a.

$$P(\text{die}(a+da) \mid \text{survive}(a)) = \frac{\cancel{P(\text{survive}(a) \mid \text{die}(a+da))}\,P(\text{die}(a))}{P(\text{survive}(a))}$$

1

Given we die at a+da, we must have survived up to a.

1.5 (cont.)

$$P(D(a+da) \mid S(a)) = \frac{P(D(a))}{P(S(a))} = \frac{b^2 a \cancel{e^{-ba}}}{(1+ba)\cancel{e^{-ba}}}$$

$$\Pr\left(D(a+da) \mid S(a)\right) = \boxed{\frac{b^2 a}{1+ba}}$$

Which makes some sense. When a is small, we'll probably survive, as a becomes large, it becomes certain that we'll die.

Problem 2: Simpler Random Walks

2.1 $\langle r \rangle$ does not seem to depend on t (number of steps). For $a=1$, 10^3 steps, averaged over 10^6 trajectories $\langle r \rangle$ just fluctuates close to 0. $\boxed{\langle r \rangle = 0}$ Larger fluctuations are on the order of ~ 0.02, which is small compared to the step size, $a=1$. For all these trajectories $\sim 97\%$ of them returned to their starting point at least once. I do not see a contradiction. It seems that the walks, on average, stay close to their starting point. If they stay close to their starting point, they could easily cross it.

2.2 $\langle r^2 \rangle$ linearly increases with steps. In class, we had

$$\langle r^2 \rangle = 2Dt, \quad D = a^2/2\tau \quad \rightarrow \quad \boxed{\langle r^2 \rangle = a^2 t/\tau}$$

Using the same parameters as above, performing a linear regression gave $2D = a^2/\tau = 1.001$. The powers of a and τ should be set by dimensionality, but to check, I used $a=3$, $\tau=2$ and found a slope of $2D = \frac{a^2}{\tau} = 4.504 \pm 0.006$.

2.2 (cont.) The diffusion constant has dimensions of Length^2/Time, so we must have a^2/τ.

As in class $\boxed{\langle r^2 \rangle = \frac{a^2}{\tau} t}$ and $\boxed{D = \frac{a^2}{2\tau}}$

2.3 For an exponential step (left or right), we expect $\langle r(t)\rangle = \langle N r_1\rangle = \frac{t}{\tau}\langle r_1\rangle$, where r_1 is the average displacement of 1 step.

$$\langle r_1\rangle = \frac{1}{2}\left[\int_0^\infty dl\, l(-\text{exponential} \langle l\rangle) + \int_0^\infty dl\, l\,\text{exponential}(\langle l\rangle)\right]$$

We define the exponential with its average,

$$\langle r_1\rangle = \frac{1}{2}\left[(-\langle l\rangle + \langle l\rangle\right] = 0$$

$$\boxed{\langle r(t)\rangle = 0}$$

For $\langle r^2(t)\rangle$, we have

$$\langle r_1^2\rangle = \frac{1}{2}\int_0^\infty dl\, \frac{l^2}{\langle l\rangle} e^{-l/\langle l\rangle} + \frac{1}{2}\int_0 dl\, \frac{l^2}{\langle l\rangle} e^{-l/\langle l\rangle}$$

$$= \frac{1}{\langle l\rangle}\int_0^\infty dl\, l^2 e^{-l/\langle l\rangle}$$

let $u = -\frac{l}{\langle l\rangle}$ $\quad \frac{du}{dl} = -\frac{1}{\langle l\rangle}$

$$= \frac{1}{\langle l\rangle}\int_0^{-\infty} -\langle l\rangle du\, \langle l\rangle^2 u^2 e^u = -\langle l^2\rangle\int_0^{-\infty} du\, u^2 e^u$$

$$\int f\,dg = fg - \int g\,df$$

let $f = u^2$ $\quad df = 2u\,du$
$dg = e^u$ $\quad g = e^u$

$$\langle r_1^2\rangle = \langle l^2\rangle\left[\cancel{u^2 e^u\Big|_0^{-\infty}} - \int_0^{-\infty} du\, e^u\, 2u\,du\right]$$

$f = u$ $\quad df = du$
$dg = e^u$ $\quad g = e^u$

$$= \langle l^2\rangle\left[-2\left[\cancel{u\, e^u\Big|_0^{-\infty}} - \int_0^{-\infty} du\, e^u\right]\right]$$

$$\boxed{= 2\langle l^2\rangle}$$

2.3 (cont.)

So we have $\boxed{\langle r^2 \rangle = \frac{t}{\tau} 2 \langle \ell^2 \rangle}$

recalling $\langle r^2 \rangle = 2Dt$ → $\boxed{D = \frac{\ell^2}{\tau}}$ lost a factor of 2.

98% of trajectories crossed zero. However, none in 10^5 runs actually landed on zero. I think this makes sense, as to land exactly on zero, we'd need to roll some random #s that match the previous random #, but in the opposite direction. Very low chance. $\langle r \rangle = 0$.

2.4 Reflective Boundaries

Simple:

All the middle spots from 1 to $N-1$ should be identical, the boundaries are not.

Ex. $N=1$ $\quad \vec{P}_n = \left(\frac{1}{2}, \frac{1}{2}\right)$

$N=2$ $\quad \vec{P}_n = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right)$

$N=3$ $\quad \vec{P}_n = \left(\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6}\right)$

$N=4$ $\quad \vec{P}_n = \left(\frac{1}{8}, \frac{2}{8}, \frac{2}{8}, \frac{2}{8}, \frac{1}{8}\right)$

Just count. $\frac{1}{2}$ left, $\frac{1}{2}$ right except boundaries, 1 inwards.

etc.

$$\boxed{\vec{P}_n = \begin{cases} \frac{1}{2N} & \text{for } n = 0, N \\ \frac{1}{N} & \text{for } 0 < n < k \end{cases}}$$

Exponential: For a steady state distribution, we require $\frac{\partial P}{\partial t} = \frac{-\partial J}{\partial x} = 0$, where

$$J = \underbrace{\cancel{v(x)}}_{\text{No drift}} P(x,t) - D\frac{\partial P}{\partial x} = -D\frac{\partial P}{\partial x}$$

$\frac{\partial^2 P}{\partial x^2} = 0$ at the boundaries, we also have

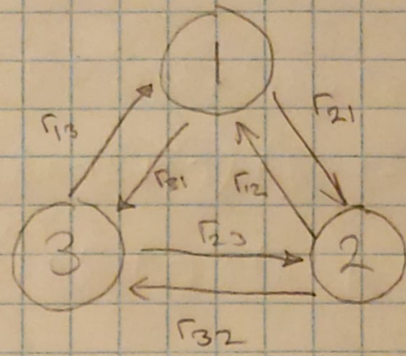
$\frac{\partial P}{\partial x} = 0$ (No probability leaves the box)

2.4 Exponentials (cont.)

Then, at steady state, $P(x) = $ constant.

$\boxed{\rho(x) = \frac{1}{a}\,dx}$, where we have boundaries at $x = a$, and $x = 0$.

3.1



$P_1(t+\Delta t) = P_2(t)\, r_{12}\Delta t + P_3(t)\, r_{13}\Delta t$

$P(t+\Delta t) \approx P(t) + \frac{dP}{dt}\Delta t$ all r equal

$\dot{P}_1 \Delta t = P_2 r \Delta t + P_3 r \Delta t - P$

$\dot{P}_1 = P_2 r + P_3 r - P/\Delta t$

we'll have similar equations for the other P_is

$\dot{\vec{P}} = \hat{M}\vec{P}$ $\hat{M} = \frac{1}{\Delta t}\begin{pmatrix} -1 & r & r \\ r & -1 & r \\ r & r & -1 \end{pmatrix}$ Columns must add to zero. $r = 1/2$

Try to find eigenvalues. Hoping for solutions of the form $\vec{P}(t) = \vec{v}e^{\lambda t}$

$$\text{Det}(M - \lambda \mathbb{1}) = (-1-\lambda)(1-r^2) - r(r(-1-\lambda) - r^2) + r(r^2 - r(-1-\lambda))$$

$$= (-1-\lambda)(1-r^2) + r(+r + r\lambda + r^2) + r(r^2 + r + r\lambda)$$

$$= -1 + r^2 - \lambda + \lambda r^2 + r^2 + \lambda r^2 + r^3 + r^3 + r^2 + \lambda r^2$$

$$= 2r^3 + 3r^2 + 3\lambda r^2 - \lambda - 1$$

We expect $\lambda = 0 = 2r - 1$ to be a solution

3.1 (cont.)

$$= 2r^3 + 3r^2 + 3\lambda r^2 - \lambda - 1$$

$$= (2r - 1 - \lambda)(r^2 + 2r + 2\lambda r + 1)$$

$$= (2r - 1 - \lambda)(r + \lambda + 1)(r + \lambda + 1)$$

$$\lambda_1 = 2r - 1 = 0$$

$$\lambda_2 = \lambda_3 = -1 - r = -\frac{3}{2}$$

Find eigenvectors: $(\hat{M} - \lambda)\vec{v} = 0$

λ_1

$$\begin{pmatrix} -1 & r & r \\ r & -1 & r \\ r & r & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0 \qquad \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$-v_1 + rv_2 + rv_3 = 0$ try $v_1 = 1$

$rv_1 - v_2 + rv_3 = 0$ $v_2 = v_3 = 1$

$rv_1 + rv_2 - 1v_3 = 0$

λ_2/λ_3

$$\begin{pmatrix} r & r & r \\ r & r & r \\ r & r & r \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0$$

$v_1 = 1$, $v_2 = 0$, $v_3 = -1$ $\qquad$ $v_1 = 1$, $v_2 = -1$, $v_3 = 0$

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

I forgot to carry $\Delta t \rightarrow \tau$ around the whole time. I think I put it back correctly.

$$\vec{P}(t) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-3t/2\tau} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-3/2\, t/\tau}$$

for molecule starting in site 1

$$\vec{P}(t) = \left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right) e^{-3t/2\tau} \right] \frac{1}{3}$$

$$\vec{P}_1(1, t | 1, t_0) = \frac{1 + 2e^{-3t/2\tau}}{3}$$

3.2 If $\vec{P}(t) = \left[\begin{pmatrix}1\\1\\1\end{pmatrix} + \left(\begin{pmatrix}1\\0\\-1\end{pmatrix} + \begin{pmatrix}-1\\1\\0\end{pmatrix}\right) e^{-3t/2\tau}\right]\frac{1}{3}$

then as $t \to \infty$ the second and third term go to zero, and we are left with a $1/3$ chance for the particle to be in each state.

In this steady state, the net fluxes are all zero.

$$\dot{P}_1 = rP_2 + rP_3 - P/\Delta t \qquad \frac{1}{\Delta t} = r_{31} + r_{21} = 2r$$

$$\frac{\dot{P}_1}{r} = \frac{1}{3} + \frac{1}{3} - \frac{2}{3} = 0 \quad ✓$$

$$\dot{P}_2 = rP_3 + rP_1 - 2rP_2 = 0 \quad ✓$$

Fluxes all 0.

$$\dot{P}_3 = rP_1 + rP_2 - 2rP_3 = 0 \quad ✓$$

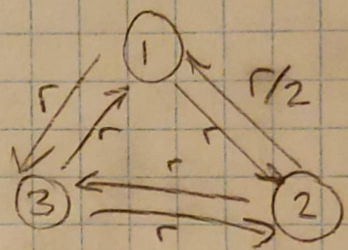
All fluxes are zero, so we must be at equilibrium steady state.

Kolmogorov condition says that the product of the rates around a closed loop must all be equal.

All of our rates are equal, so their products clockwise or counterclockwise will be equal. The condition must be satisfied.

3.3

$$\dot{\vec{P}} = \hat{M} \cdot \vec{P} = \frac{1}{\tau}\begin{pmatrix} -2 & r/2 & r \\ r & -3/2 & r \\ r & r & -2 \end{pmatrix} \vec{P}$$



Note that M_{22} has to be $-3/2$, not -2 in order for the column to sum to 0, as it should.

We are asked for the steady state distribution. This corresponds with $\lambda = 0$. Then we just need to find the corresponding eigenvector.

$$\hat{M}\vec{v} = \lambda \vec{v} = \vec{0}$$

$$\left(\begin{array}{ccc|c} -2 & r/2 & r & 0 \\ r & -3/2 & r & 0 \\ r & r & -2 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} -2 & r/2 & r & 0 \\ 0 & -3/2-r & r+2 & 0 \\ r & r & -2 & 0 \end{array}\right)$$

We know that we have $\sum_i v_i = 1$, so lets set $v_2 = 1$, then renormalize.

$$\left(-\frac{3}{2} - r\right)v_2 + (r+2)v_3 = 0 \qquad r v_1 + r v_2 - 2 v_3 = 0$$

$$-\frac{5}{2} + 3v_3 = 0 \qquad v_1 + 1 - \frac{5}{3} = 0$$

$$v_3 = 5/6 \qquad v_1 = \frac{2}{3}$$

$$\left(\frac{2}{3} + 1 + \frac{5}{6}\right)a = 1$$

$$\left(\frac{15}{6}\right)a = 1$$

$$\boxed{\vec{P}_{ss} = \frac{6}{15}\begin{pmatrix} 2/3 \\ 1 \\ 5/6 \end{pmatrix} = \frac{1}{15}\begin{pmatrix} 4 \\ 6 \\ 5 \end{pmatrix}}$$

3.3 (cont.)

Flux is not all equal, unlike in 3.2.

$J_{ij} = P_j r_{ij} - P_i r_{ji}$

$J_{21} = \frac{4}{15} r - \frac{6}{15} \frac{r}{2} = \frac{1}{15} r$

$J_{32} = \frac{6}{15} r - \frac{5}{15} r = \frac{1}{15} r$

$J_{13} = \frac{5}{15} r - \frac{4}{15} r = \frac{1}{15} r$

We are in a steady state, but the Kolmogorov condition is not satisfied, as all but one rate are equal. We are not at equilibrium because we have a nonzero flux between the states.

In 3.2, there was no such flux, and the Kolmogorov condition was satisfied, as all rates were equal.

3.4 For all rates equal the simulation agrees with theory. After 10^5 steps, the particle spent 33% of its time in each site. The system very close to being equilibrated within 1000 time steps.

With $r_{12} = 0.5r$ the simulation agrees with the steady state distribution. $P_1 = 0.27$, $P_2 = 0.40$, $P_3 = 0.33$ after 10^5 steps. Again, within 1000 time steps, the system is in steady state, or at least very close.

Plots attached.

4.1 A random walk with bias is a binomial distribution.

$$Pr(n,k) = \frac{n!}{k!(n-k)!} p^k q^{n-k}$$

With p chance to go right, and q chance to go left. Let's say k is steps to the right, and n is the total # of steps.

$$Pr(N, n_r, n_\ell) = \frac{N!}{n_r! \, n_\ell!} p^{n_r} q^{n_\ell}$$

We want $Pr = P_n(t)$

$N = n_r + n_\ell$ $\qquad$ $n = n_r - n_\ell$

Then $n_r = \frac{n+N}{2}$ $\qquad$ $n_\ell = \frac{N-n}{2}$

$$P(N,n) = \frac{N!}{\frac{N+n}{2}! \, \frac{N-n}{2}!} p^{\frac{N+n}{2}} q^{\frac{N-n}{2}}$$

We also have $N = t/\tau$,

$$P_n(t) = \frac{(t/\tau)!}{\left(\frac{t/\tau + n}{2}\right)! \left(\frac{t/\tau - n}{2}\right)!} p^{\frac{t/\tau + n}{2}} q^{\frac{t/\tau - n}{2}}$$

4.1 (cont.)

Mean: For N steps, I expect Np steps to the right, and Nq steps to the left.

$$\langle n(t)\rangle = N(p-q) = \boxed{\frac{t}{\tau}(p-q)}$$

Var: $Var(t) = \langle n(t)^2\rangle - \langle n(t)\rangle^2$

If we can find the variance of 1 step, each step is independent, so we can find $Var(t)$.

$$Var(t) = N\left(\langle n_1^2\rangle - \langle n_1\rangle^2\right)$$

$$\langle n_1^2\rangle = \sum_i (\Delta n_i)^2 P_i = 1^2 p + (-1)^2 q = p+q$$

$$\langle n_1\rangle = \sum_i \Delta n\, P_i = 1p + (-1)q = p-q$$

$$Var(t) = N\left[(p+q) - (p-q)^2\right] = N\left(p+q-p^2-q^2+2pq\right)$$

$$= N\left[p+(1-p)-p^2-(1-p)^2+2p(1-p)\right]$$

$$= N\left[\cancel{1}-p^2-\cancel{1}-p^2+2p+2p-2p^2\right]$$

$$= N(4p-4p^2) = 4N(1-p)p = 4Npq = \frac{4pq\,t}{\tau}$$

We can check with the $p=q$ case.

$$\langle n(t)\rangle = 0 \checkmark$$

$$2D\cancel{t} = \frac{\cancel{4(\tfrac{1}{2})^2 t}}{\tau} \rightarrow D = \frac{1}{2\tau}$$

I've been neglecting a this whole time.

$$\langle x(t)\rangle = \frac{at}{\tau}(p-q)$$

$$Var(t) = \frac{4pq\,a^2}{\tau}t = 8Dpq\,t$$

4.2 $P_r(n,N) = \dfrac{N!}{\left(\frac{n+N}{2}\right)!\left(\frac{N-n}{2}\right)!} p^{\frac{N+n}{2}} q^{\frac{N-n}{2}}$

Stirling's approximation says $N! \approx N^N e^{-N} \sqrt{2\pi N}$

$$P_r(n,N) \approx \frac{N^N \cancel{e^{-N}} \sqrt{2\pi N}\, p^{\frac{N+n}{2}} q^{\frac{N-n}{2}}}{\left(\frac{n+N}{2}\right)^{\frac{n+N}{2}} \cancel{e^{-\left(\frac{n+N}{2}\right)}} \sqrt{2\pi (N+n)/2} \left(\frac{N-n}{2}\right)^{\frac{N-n}{2}} \cancel{e^{-\left(\frac{N-n}{2}\right)}} \sqrt{2\pi (N-n)/2}}$$

$$= \left(\frac{2Np}{N+n}\right)^{\frac{N+n}{2}} \left(\frac{2Nq}{N-n}\right)^{\frac{N-n}{2}} \sqrt{\frac{N}{2\pi (N+n)/2\ (N-n)/2}}$$

Expand around the average. Let $\delta = n - N(p-q)$

$$\ln\left(\frac{2Np}{N+n}\right) = -\ln \frac{N+\delta+Np-Nq}{2Np} = -\ln\left(\frac{1}{2p} + \frac{\delta}{2Np} + \frac{1}{2} - \frac{q}{2p}\right) = -\ln\left(\frac{\delta}{2Np}+1\right)$$

$$\approx -\frac{\delta}{2Np} - \frac{1}{2}\left(\frac{\delta}{2Np}\right)^2$$

$$\ln\left(\frac{2Nq}{N-n}\right) = -\ln\left(\frac{N-\delta-Np+Nq}{2Nq}\right) = -\ln\left(\frac{1}{2q} - \frac{\delta}{2Nq} - \frac{p}{2q} + \frac{1}{2}\right) = -\ln\left(1 - \frac{\delta}{2Nq}\right)$$

$$\approx \frac{\delta}{2Nq} - \frac{1}{2}\left(\frac{\delta}{2Nq}\right)^2$$

Take ln and ignore stuff under sqrt for a bit

$$\frac{N+\delta+N(p-q)}{2}\left[-\frac{\delta}{2Np} - \frac{1}{2}\left(\frac{\delta}{2Np}\right)^2\right] + \frac{N-\delta-N(p-q)}{2}\left[\frac{\delta}{2Nq} - \frac{1}{2}\left(\frac{\delta}{2Nq}\right)^2\right]$$

$$-\left(Np+\frac{\delta}{2}\right)\left[\frac{\delta}{2Np} - \frac{1}{2}\left(\frac{\delta}{2Np}\right)^2\right] - \left(Nq - \frac{\delta}{2}\right)\left[\frac{-\delta}{2Nq} - \frac{1}{2}\left(\frac{\delta}{2Nq}\right)^2\right]$$

4.2 (cont.)

$$= \frac{-\delta}{2} + \frac{1}{2}\frac{\delta^2}{4Np} - \frac{\delta^2}{4Np} + \cancel{\frac{1}{4}\frac{\delta^3}{(2Np)^3}}^{\,\sim O[\delta^3]} + \frac{\delta}{2} + \frac{1}{2}\frac{\delta^2}{4Nq} - \frac{\delta^2}{4Nq} - \cancel{\frac{1}{4}\frac{\delta^3}{(2Nq)^3}}^{\,\sim O[\delta^3]}$$

$$= -\frac{\delta^2}{8Np} - \frac{\delta^2}{8Nq} = \frac{-\delta^2 q - \delta^2 p}{8Npq} = -\frac{\delta^2}{8Npq}$$

$$\boxed{\delta^2 = (n - N(p-q))^2}$$

$$\ln P_r = \frac{-\delta^2}{8Npq} + \ln\frac{\sqrt{N}}{\sqrt{2\pi (N+n)/2\,(N-n)/2}}$$

$$(N+\delta+N(p-q))(N-\delta-N(p-q))$$

$$N^2 - N\delta - N^2(p-q) + N\delta - \cancel{\delta^2} - \delta N(p-q) + N^2(p-q) - \delta N(p-q) - N^2(p-q)^2$$

$$N^2 - \cancel{N^2(p-q)} + \cancel{N^2(p-q)} - N^2(p-q)^2$$

$$N^2 \quad 1-(1-q-q)^2$$

$$(1-2q)^2$$

$$1-4q+4q^2$$

$$N^2\, 4pq$$

$$\ln P_r = \frac{-\delta^2}{8Npq} + \ln\frac{1}{\sqrt{2\pi Npq}}$$

$$\boxed{P_r = \frac{1}{\sqrt{2\pi 4Npq}}\, e^{\frac{-(n-N(p-q))^2}{8Npq}}}$$

Somehow I lost a factor of 4, and I can't find it. If you check normalization, it must be there.
Other than that, this is exactly what I expect.
This is just a Gaussian with mean $= N(p-q)$ and variance $4Npq$, which is what we found in part 1.
To convert to continuous, set $N \to t/\tau$, $n \to x$. a [illegible]

$$\frac{1}{\sqrt{2\pi\, 4a^2 pq\, t/\tau}} \exp\frac{-(x - a\frac{t}{\tau}(p-q))^2}{8a^2 pq\, t/\tau}$$

4.3 $$P(n, t+\Delta t) = qP(n+1, t) + pP(n-1, t)$$

$$\frac{\partial P_n}{\partial t}\Delta t \simeq q P_{n+1} + p P_{n-1} - P_n$$

$$\frac{\partial P_n}{\partial t} = \frac{q}{\Delta t} P_{n+1} + \frac{p}{\Delta t} P_{n-1} - \frac{1}{\Delta t} P_n$$

Change to continuous $x \pm a$

$$\frac{\partial P}{\partial t} = \frac{q}{\tau} P(x+a) + \frac{p}{\tau} P(x-a) - \frac{1}{\tau} P(x)$$

Expand in a, keep second order terms.

$$\frac{\partial P}{\partial t} = \frac{q}{\tau}\left(P(x) + \frac{\partial}{\partial x} a P(x) + \frac{1}{2}\frac{\partial^2}{\partial x^2} P(x) a^2\right) + \frac{p}{\tau}\left(P(x) - \frac{\partial}{\partial x} a P(x) + \frac{1}{2}\frac{\partial^2}{\partial x^2} P(x) a^2\right)$$
$$- \frac{1}{\tau} P(x)$$

$$= \cancel{\frac{q+p}{\tau} P(x)} + \frac{q-p}{\tau} a\frac{\partial}{\partial x}P(x) + \frac{q+p}{\tau}\frac{a^2}{2}\frac{\partial^2}{\partial x^2}P(x) - \cancel{\frac{1}{\tau} P(x)}$$

$q+p=1$ $\qquad$ $q+p=1$

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left[(p-q)\frac{a}{\tau} P(x) - \frac{a^2}{2\tau}\frac{\partial}{\partial x} P(x)\right]$$

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left[v(p-q) P(x) - D\frac{\partial}{\partial x} P(x)\right] \qquad v \equiv a/\tau \quad D \equiv a^2/2\tau$$

Which agrees with what we derived in class.

$$\boxed{P(x,t) = \frac{1}{\sqrt{16\pi pqDt}} \exp\left\{-\frac{(x-(p-q)vt)^2}{16pqDt}\right\}}$$

4.3 (cont.)

This should agree with 4.1 as we're basically just using central limit theorem to go from a binomial distribution to a normal distribution. It also agrees with 4.2, possibly within a factor of 2. (I lost a factor of 2 in 4.2 calculation, but it must be there for normalization.)