Problem 1: Basicc probability Joseph is waiting for a friend at a bus stop located on a highway and, in order to kill time, observes passing cars and buses. He notices that the buses are scheduled - each bus passes the stop exactly five minutes after the previous one - exactly 12 per hour. The cars, on the other hand, seem to arrive at random times. However, by observing lots of cars he comes to a conclusion that the probability for a car to pass by in a short interval of time $[t, t+dt]$ is independent of time t and is equal to dt/τ with $\tau = 5$ minutes.

1.1 What is the average number of cars passing in one hour?

1.2 Write the mathematical expression for the probability distribution $P(x)$ that the time elapsed between two subsequent cars is x? For buses?

1.3 If another observer arrives at the road at a randomly chosen time, what is the probability distribution for the time he/she has to wait for the 1^{st} bus to arrive? What is the probability distribution for the time she has to wait for the 1rst car to pass by?

1.4 What is the probability P_n that n buses will pass during a randomly chosen 20 min interval (trivial)? What is this probability for cars? Hint: recall the Poisson distribution.

1.5 It has been determined in field observations that the probability of an individual of certain animal species to survive till the age a is $S(a) = (1 + bt)e^{-bt}$. What is the probability of an animal to actually die in a small interval da around age a? What is the *conditional* probability to die in the same interval, conditioned on the fact that the animal is currently of age a?

Problem 2: Simple random walks. Using a programming language of your own choice, simulate a random walk in one dimension. At each timestep, the walker jumps a fixed distance a to the right or to the left with probability $1/2$. Time between subsequent jumps, τ, is fixed. Generate a sufficient number of trajectories (choose any numerical values for a and τ).

2.1. a) Estimate the $\langle r \rangle$ as a function the number of steps or the time t. b) Then estimate what is the probability of a single trajectory to actually return to the origin. (*Hint:* count the number of such trajectories compared with the total number of trajectories you have generated). Is there a contradiction between a) and b)?

2.2. Estimate $\langle r^2 \rangle$ as a function the number of steps or the time t. Plot $\langle r^2 \rangle$ vs. t. Estimate the diffusion coefficient. Does it agree with the expression derived in class?

2.3. Repeat **2.**1-2 under the condition that the time between jumps is fixed, but the jump distance is not a constant but a random, exponentially distributed number, so that the probability density of a jump of a length l is $p(l) = \frac{1}{\langle l \rangle} \exp(-l/\langle l \rangle)$. Is the dependence of the $\langle r^2 \rangle$ on t different between this and the previous case? Why? Hint: if x is a random number uniformly distributed between 0 and 1, then the random number $y = -k \ln(x)$ is exponentially distributed with $\langle y \rangle = k$.

Assuming reflective boundaries on both sides, derive the equilibrium steady state probability distribution $P(x)$.

Problem 3: Master equation. A molecule can transition between three conformations $1, 2, 3$, with the corresponding rates r_{ij} from state j to i.

3.1 Write the Master equation for the system. For all equal rates, $r_{ij} = r$, solve it analytically to find $P_i(t)$ assuming the molecule starts in a state 1. What is the probability to be at the same starting state at time t?

3.2 Find the steady state of the system and calculate the fluxes between the conformations in this steady state. Is this an equilibrium steady state? Confirm if the Kolmogorov condition is satisfied.

3.3 Repeat 3.2 for a case when all the rates are equal to $r = 1$ except for $r_{12} = 0.5r$. Comment on the differences with 5.2

3.4 For all rates equal r, use Gillespie algorithm to simulate the system and verify that the simulations agree with the analytical solution of 5.1. Perform the simulation for the rates set in 5.2 and discuss the result.

Problem 4. Simple random walk with bias At each timestep, a walker jumps a fixed distance a to the right with probability p or to the left with probability q. Time between subsequent jumps, τ, is fixed.

4.1 What is the probability distribution $P_n(t)$ to find the walker at site n at time t (starting from $n = 0$)? Calculate its mean and variance.

4.2 Bonus (10 pt) Following derivation in class, using Sterling approximation, derive Gaussian approximation for the distribution in 4.1. Calculate its mean and variance. Does it agree with 3.1? Comment.

4.3. Starting with the master equation, derive the Fokker-Planck equation for a random walk in Problem 4.1. What is the solution for this equation for a walker starting at $x = 0$? (Hint: try a Gaussian around the mean). Does it agree with the exact solution and/or the Gaussian approximation derived in Problem 4.2? Comment.