`conexp-clj`
Exploratory Programming for FCA
Exercise Sheet

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The mathematical background of these exercises is described in [1].

Exercise 1 Given $n \in \mathbb{N}$, compute a contextual representation of the closure system of all closure systems on $\{1, \ldots, n\}$. For $n \in \{1, 2, 3, 4\}$, compute the number of formal concepts of these formal contexts.

Hint:

- A contextual representation of the lattice of closure systems on $M = \{1, \ldots, n\}$ is given by
$$(\mathfrak{P}(M), \{ A \to \{x\} \mid A \subseteq M, x \in M \setminus A \}, \models),$$
where $N \models (A \to \{x\})$ if and only if N respects $A \to \{x\}$.
- `(set-of-range 1 n)` yields the set `#{1, ..., n}`
- `(subsets M)` yields a lazy sequence of all subsets of the set M.
- `(respects? M implication)` returns true if and only if `M` respects `implication`.

Exercise 2 Given $n \in \mathbb{N}$, compute a contextual representation of the Tamari lattice on n symbols. Draw the concept lattice of these formal contexts for suitable (i. e. not too large) values of n.

Hint: Such a contextual representation can be obtained as

$$(P, P, \{ ((i,j),(p,q)) \mid (i,j),(p,q) \in P, i = q \text{ or not } (p \leqslant i \leqslant q \leqslant j) \})$$

where $P = \{ (a,b) \mid a, b \in \{1, \ldots, n\}, a < b \}$.

Exercise 3 Given $n \in \mathbb{N}$, compute a contextual representation of the lattice of all permutations on $\{ 1, \ldots, n \}$.

Hint: If $\mathbb{K}_n$ is such a contextual representation, then it can be obtained by

$$
\begin{aligned}
\mathbb{K}_0 := \mathbb{L}_0 &:= \boxed{\times} \\
\mathbb{L}_{n+1} &:= \begin{array}{c|c} \varnothing & \mathbb{L}_n \\ \hline \mathbb{L}_n & \mathbb{L}_n \end{array} \\
\mathbb{K}_{n+1} &:= \begin{array}{c|c} \mathbb{K}_n & \mathbb{K}_n \\ \hline \mathbb{K}_n & \mathbb{L}_n \end{array}
\end{aligned}
$$

Contextual apposition and subposition are implemented in `context-apposition` and `context-subposition`, respectively.

Exercise 4 Compute the number of linear extensions of the free distributive lattice on 3 generators.

Hint:

- A contextual representation of the free distributive lattice on 3 generators can be obtained by

  $$
  \begin{array}{|c|c|} \hline & \times \\ \hline & \\ \hline \end{array}
  \times
  \begin{array}{|c|c|} \hline & \times \\ \hline & \\ \hline \end{array}
  \times
  \begin{array}{|c|c|} \hline & \times \\ \hline & \\ \hline \end{array}
  $$

  where $\times$ denotes the contextual product, which is implemented in `conexp-clj` as `context-product`.

- Let $\mathbb{K}$ be the contraordinal scale of an ordered set $(S, \leqslant)$, i. e. $\mathbb{K} = (S, S, \not\leqslant)$. The number of linear extensions of $(S, \leqslant)$ is then the value $\mu_{\mathfrak{B}(\mathbb{K})}(\varnothing, \varnothing'')$, where

  $$
  \mu_{\mathfrak{B}(\mathbb{K})}(A, B) := \begin{cases} 1 & \text{if } A = S, \\ \sum_{(C,D) \succ (A,B)} \mu_{\mathfrak{B}(\mathbb{K})}(C, D) & \text{otherwise} \end{cases}
  $$

  where $\succ$ denotes the neighborhood relation in $(\mathfrak{B}(\mathbb{K}), \leq)$. Direct upper neighbors can be obtained by means of the function `direct-upper-concepts`.

References

[1] Bernhard Ganter and Rudolf Wille. *Formal Concept Analysis: Mathematical Foundations*. Berlin-Heidelberg: Springer, 1999.