See HW4.pdf for rendered LaTeX equations. Github does not render them in markdown files.

Q1

a)

Symmetric

$$K(x_i, x_j) = \sum_{a=1}^{m} w_a K_a(x_i, x_j)$$

$$\text{Since } K_1, ..., K_a \text{ are valid kernel functions:}$$

$$K(x_i, x_j) = \sum_{a=1}^{m} w_a K_a(x_j, x_i) \triangleq K(x_j, x_i)$$

$$\therefore \text{K is symmetric}$$

Positive Semi-Definite

$$\text{Using Mercer's Theorem, a Kernel is positive semi-definite if:}$$

$$\underline{c}^T \underline{K} \underline{c} \geq 0$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j K(x_i, x_j) \geq 0$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \sum_{a=1}^{m} w_a K_a(x_i, x_j) \geq 0$$

$$\sum_{a=1}^{m} w_a \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \langle \underline{\phi}(x_i), \underline{\phi}(x_j) \rangle \geq 0$$

$$\sum_{a=1}^{m} w_a \sum_{i=1}^{n} \sum_{j=1}^{n} c_i c_j \sum_{k=1}^{p} \phi_k(x_i) \phi_k(x_j) \geq 0$$

$$\sum_{a=1}^{m} w_a \sum_{k=1}^{p} \sum_{i=1}^{n} \sum_{j=1}^{n} c_i \phi_k(x_i) c_j \phi_k(x_j) \geq 0$$

$$\sum_{a=1}^{m} w_a \sum_{k=1}^{p} \left(\sum_{i=1}^{n} c_i \phi_k(x_i) \right)^2 \geq 0$$

$$\text{Since } w_i \geq 0, \forall w_i, \text{ The inequality holds, } \therefore \text{ K is positive semi-definite.}$$

The function is symmetric and positive semi-definite, $\therefore K$ is a Kernel.

b)

Symmetric

$$K(x_i, x_j) = K_1(x_i, x_j) K_2(x_i, x_j)$$

Since K_1 and K_2 are valid kernel functions:

$$K(x_i, x_j) = K_1(x_j, x_i)K_2(x_j, x_i) \triangleq K(x_j, x_i)$$

$\therefore$ K is symmetric

Positive Semi-Definite

Using Mercer's Theorem, a Kernel is positive semi-definite if:

$$\underline{c}^T \underline{Kc} \geq 0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K(x_i, x_j) \geq 0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K_1(x_i, x_j)K_2(x_i, x_j) \geq 0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \langle \underline{\phi_1}(x_i), \underline{\phi_1}(x_j)\rangle\langle \underline{\phi_2}(x_i), \underline{\phi_2}(x_j)\rangle \geq 0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{p} c_i c_j \phi_{(1,k)}(x_i)\phi_{(1,k)}(x_j)\phi_{(2,k)}(x_i)\phi_{(2,k)}(x_j) \geq 0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{p} c_i \phi_{(1,k)}(x_i)\phi_{(2,k)}(x_i) c_j \phi_{(1,k)}(x_j)\phi_{(2,k)}(x_j) \geq 0$$

$$\sum_{k=1}^{p}\left(\sum_{i=1}^{n} c_i \phi_{(1,k)}(x_i)\phi_{(2,k)}(x_i)\right)^2 \geq 0$$

Inequality holds, $\therefore$ K is positive semi-definite.

The function is symmetric and positive semi-definite, $\therefore K$ is a Kernel.

c)

Symmetric

$$K(x, x') = (xx' + 1)^{2015}$$

$$K(x, x') = (x'x + 1)^{2015} \triangleq K(x', x)$$

$\therefore$ K is symmetric

Positive Semi-Definite

$$K(x, x') = (xx' + 1)^{2015}$$

$$K(x, x') = \sum_{i=1}^{2015}\left(\frac{2015!}{i!(2015-i)!}(x)^i(x')^i\right)$$

Using Mercer's Theorem, a Kernel is positive semi-definite if:

$$\underline{c}^T \underline{Kc} \geq 0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K(x_i, x_j) \geq 0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \sum_{k=1}^{2015}\left(\frac{2015!}{k!(2015-k)!}(x_i)^k (x_j)^k\right) \geq 0$$

$$\sum_{k=1}^{2015}\frac{2015!}{k!(2015-k)!}\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j (x_i)^k (x_j)^k \geq 0$$

$$\sum_{k=1}^{2015}\frac{2015!}{k!(2015-k)!}\left(\sum_{i=1}^{n} c_i (x_i)^k\right)^2 \geq 0$$

$$\text{Inequality holds, } \therefore \text{ K is positive semi-definite.}$$

The function is symmetric and positive semi-definite, $\therefore K$ is a Kernel.

d)

Symmetric

$$K(x, x') = \exp\left(-\frac{(x-x')^2}{2}\right)$$

$$\text{Since } x - x' \text{ is being squared, it is also equal to:}$$

$$K(x, x') = \exp\left(-\frac{(x'-x)^2}{2}\right) \triangleq K(x', x)$$

$$\therefore \text{K is symmetric}$$

Positive Semi-Definite

$$K(x, x') = \exp\left(-\frac{(x-x')^2}{2}\right)$$

$$K(x, x') = \exp\left(-\frac{x^2 - 2xx' + (x')^2}{2}\right)$$

$$K(x, x') = \exp\left(-\frac{x^2}{2} + xx' - \frac{(x')^2}{2}\right)$$

$$K(x, x') = \frac{\exp(xx')}{\exp\left(\frac{x^2}{2}\right)\exp\left(\frac{(x')^2}{2}\right)}$$

$$K(x, x') = \frac{\sum_{k=0}^{\infty}\frac{(xx')^k}{k!}}{\exp\left(\frac{x^2}{2}\right)\exp\left(\frac{(x')^2}{2}\right)}$$

$$K(x, x') = \sum_{k=0}^{\infty}\frac{(x)^k (x')^k}{k!\exp\left(\frac{x^2}{2}\right)\exp\left(\frac{(x')^2}{2}\right)}$$

$$\text{Using Mercer's Theorem, a Kernel is positive semi-definite if:}$$

$$\underline{c}^T \underline{K}\,\underline{c} \geq 0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j K(x_i, x_j) \geq 0$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \sum_{k=0}^{\infty} \frac{(x_i)^k (x_j)^k}{k! \exp\left(\frac{x_i^2}{2}\right) \exp\left(\frac{x_j^2}{2}\right)} \geq 0$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \frac{(x_i)^k}{\exp\left(\frac{x_i^2}{2}\right)} \frac{(x_j)^k}{\exp\left(\frac{x_j^2}{2}\right)} \geq 0$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{i=1}^{n} c_i \frac{(x_i)^k}{\exp\left(\frac{x_i^2}{2}\right)} \right)^2 \geq 0$$

$$\text{Inequality holds, } \therefore \text{ K is positive semi-definite.}$$

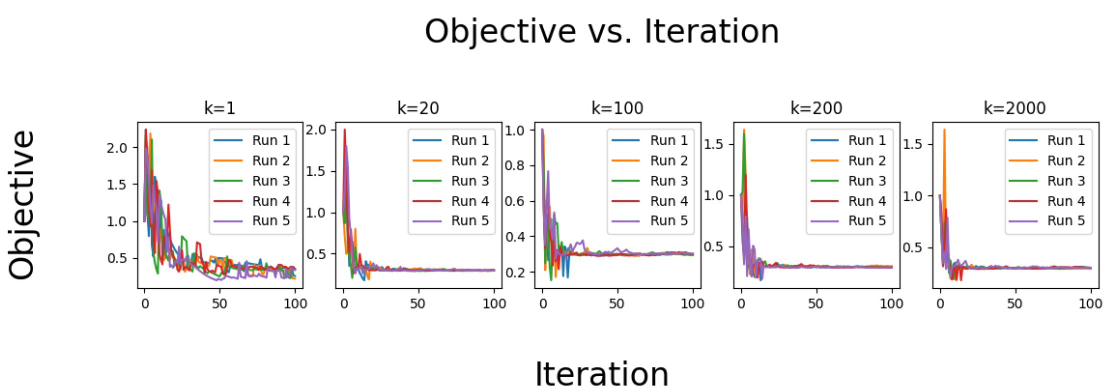
The function is symmetric and positive semi-definite, $\therefore K$ is a Kernel.

Q2

Summary and Results

max_iter = 100

lambda = 1e5 (A high regularization coefficient was used to stabilize the plots quickly)




```
Avg runtime for 5 runs with minibatch size of 1: 1.39 sec
Std runtime for 5 runs with minibatch size of 1: 0.03 sec
Avg runtime for 5 runs with minibatch size of 20: 1.48 sec
Std runtime for 5 runs with minibatch size of 20: 0.03 sec
Avg runtime for 5 runs with minibatch size of 100: 1.58 sec
Std runtime for 5 runs with minibatch size of 100: 0.04 sec
Avg runtime for 5 runs with minibatch size of 200: 1.75 sec
Std runtime for 5 runs with minibatch size of 200: 0.03 sec
Avg runtime for 5 runs with minibatch size of 2000: 3.97 sec
Std runtime for 5 runs with minibatch size of 2000: 0.03 sec
```

Code

Code can be found on Github.

Pegasos algorithm:

https://github.com/DeBestTrap/Intro-to-Machine-Learning/blob/main/HW4/pegasos.py

Code to run and plot pegasos algorithm (for instructions to run code see README.md):

https://github.com/DeBestTrap/Intro-to-Machine-Learning/blob/main/HW4/mysgdsvm.py