Cryptanalysis of AES

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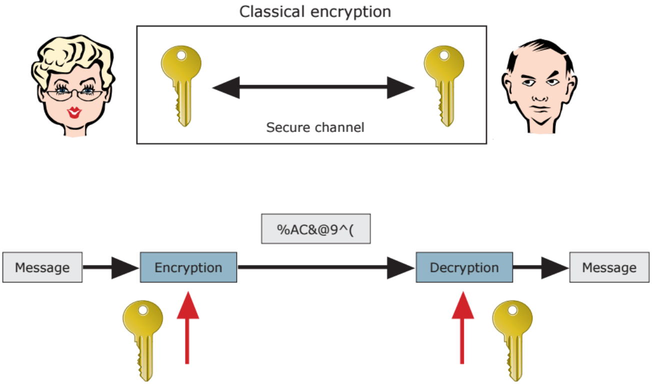
Part I

Background: Symmetric Cryptography

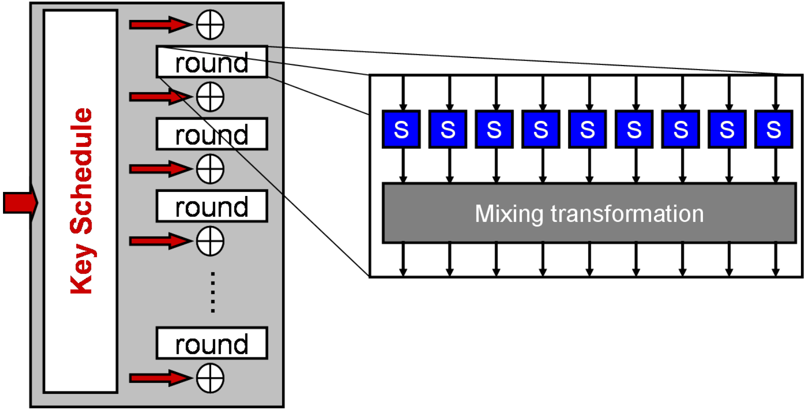
Symmetric Cryptography

Cryptography is communication in the presence of an adversary (Ron Rivest)



Reprinted from `https://www.cosic.esat.kuleuven.be/summer_school_sardinia_2015/slides/LRKnudsen.pdf` by *Lars R. Knudsen*

Design of SPN Round Function



Secure Ciphers - Symmetric Encryption

How can you tell if a cipher is secure?

Definition (Kerckhoffs' Principle)

The security of a cryptosystem must lie in the choice of its keys only. *Everything else (including the algorithm itself) should be considered public knowledge*.

A cipher is secure if there is no attack better than brute force: a solid cipher must resist all known attacks!

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A cipher is secure if there is no attack better than brute force: a solid cipher must resist all known attacks!

Key-Recovery Attack

Any attempt of the adversary to find the secret key.

A possible (but not only) way to set up a key-recovery attack is to exploit *secret-key distinguishers* - which are independent of the secret key - as starting points.

Given a set of chosen/known plaintexts, assume a *non-random property which is independent of the secret key* is known after s-round encryption:

$$\text{plaintexts} \xrightarrow[\text{distinguisher}]{R^s(\cdot)} \text{“}property \xleftarrow[\text{key guessing}]{R^{-r}(\cdot)} \text{ciphertexts}$$

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$$\text{plaintexts} \xrightarrow[\text{distinguisher}]{R^s(\cdot)} \text{“}property\text{”} \xleftarrow[\text{key guessing}]{R^{-r}(\cdot)} \text{ciphertexts}$$

Secret-Key Distinguisher

Setting: *Two Oracles*:

- one simulates the block cipher for which the cryptography key has been chosen at random;
- the other simulates a truly random permutation.

Goal: distinguish the two oracles, i.e. decide which oracle is the cipher.

Part II

AES

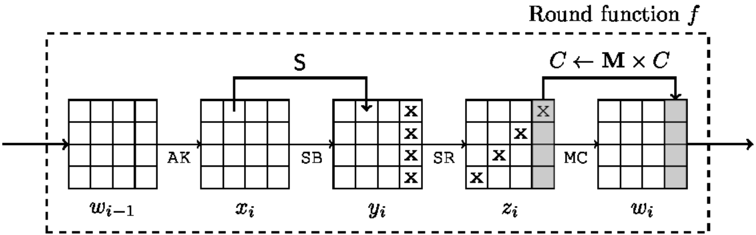
AES

High-level description of AES:

- block cipher based on a design principle known as *substitution-permutation network*;
- block size of 128 bits = 16 bytes, organized in a 4×4 matrix;
- key size of 128/192/256 bits;
- 10/12/14 rounds:

$$R^i(x) = k^i \oplus MC \circ SR \circ \text{S-Box}(x).$$

AES Round



Source-code of the Figure — by Jérémy Jean -– copied from `https://www.iacr.org/authors/tikz/`

Distinguishers for AES

(State of the Art) Distinguishers for up to 4-round of AES which are *independent of the secret key*:

Rounds	**Data** (CP/CC)	**Complexity**	**Property**
1 - 2	2	2 XOR	Truncated Diff.
3	$20 \simeq 2^{4.3}$	$2^{7.6}$ M	Truncated Diff.
3	2^8	2^8 XOR	Integral
4	$2^{16.25}$	$2^{31.5}$ M	Impossible Diff.
4	2^{32}	2^{32} XOR	Integral

Differential Cryptanalysis

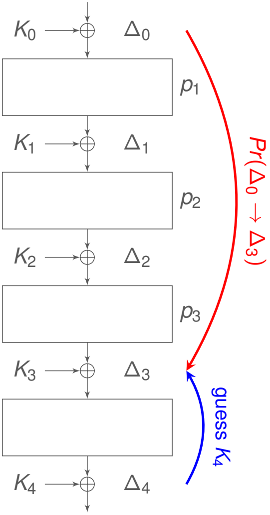
- One of the best attack methods in cryptanalysis: Introduced by Biham and Shamir to attack DES (1993)
- *Deduce information about the secret key by tracing differences between pairs of plaintexts during the encryption (and decryption)*
- R-rounds characteristic:

$$\Delta_0 \to \Delta_1 \to \Delta_2 \to \ldots \to \Delta_R$$

- R-rounds differential:

$$\Delta_0 \to ? \to ? \to \ldots \to \Delta_R$$

Basic Approach of a Differential Attack



1. Find “good” differential characteristic

$$\Delta_0 \to \Delta_1 \to \Delta_2 \to \Delta_3$$

2. Guess final key K_4' and compute backward through the S-Boxes to determine Δ_3'

3. The right key satisfies $\Delta_3' = \Delta_3$ with prob. $Pr(\Delta_0 \to \Delta_3)$, while a wrong key satisfies $\Delta_3' = \Delta_3$ with prob. $1/|\mathcal{P}|$.

4. *Necessary condition* for the attack: $Pr(\Delta_0 \to \Delta_3) \gg 1/|\mathcal{P}|$.

Truncated Differential Cryptanalysis

- First published by Knudsen in 1994
- Generalization of differential cryptanalysis
  - the main idea is to leave parts of the difference unspecified
  - by ignoring some bits we allow more differences which increases the probability
  - example truncated differential: ?0??0000 $\rightarrow$?0??0000
- Powerful against word/byte oriented ciphers

Diagonal of a Matrix - Definition

Diagonals

- 1-st (first) diagonal
- 2-nd (second) diagonal
- 3-rd (third) diagonal
- 4-th (fourth) diagonal

of a 4×4 matrix are

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

Anti-Diagonal of a Matrix - Definition

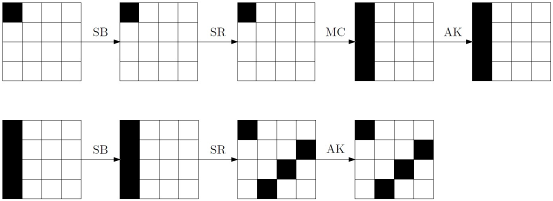
Anti-Diagonals

- 1-st (first) anti-diagonal
- 2-nd (second) anti-diagonal
- 3-rd (third) anti-diagonal
- 4-th (fourth) anti-diagonal

of a 4×4 matrix are

$$\begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

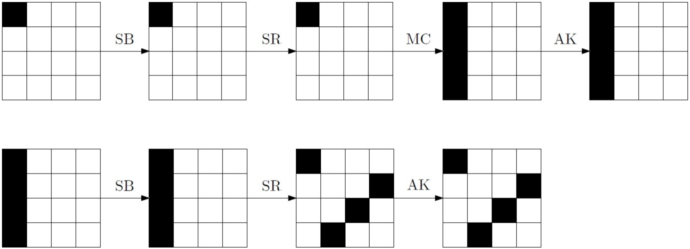
A 2-round AES Truncated Differential (1/2)



where

- *final MixColumns is omitted for simplicity*;
- *white box* □ denotes a byte for which the difference of the two texts is zero;
- *black box* ■ denotes a byte – active byte – for which the difference of the two texts is non-zero (note: ■ can take 255 possible values);
- S-Box is bijective & Branch number of *MC* matrix is 5.

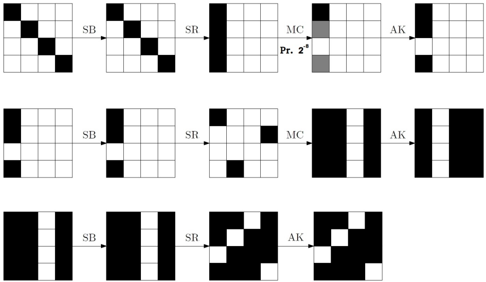
A 2-round AES Truncated Differential (2/2)



$$Prob[R^2(p^1) \oplus R^2(p^2) \in \mathcal{ID}_0 \,|\, p^1 \oplus p^2 \in \mathcal{D}_0] = 1$$

$$Prob[\Pi(p^1) \oplus \Pi(p^2) \in \mathcal{ID}_0 \,|\, p^1 \oplus p^2 \in \mathcal{D}_0] = 2^{-96}$$

A 3-round AES Truncated Differential

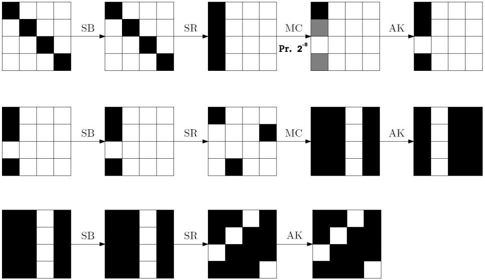


A *gray box* ■ denotes a byte for which the difference of the two texts is unknown.

$$Prob[R^3(p^1) \oplus R^3(p^2) \in \mathcal{ID}_{0,1,2} \,|\, p^1 \oplus p^2 \in \mathcal{D}_0] = 2^{-8}$$

$$Prob[\Pi(p^1) \oplus \Pi(p^2) \in \mathcal{ID}_{0,1,2} \,|\, p^1 \oplus p^2 \in \mathcal{D}_0] = 2^{-32}$$

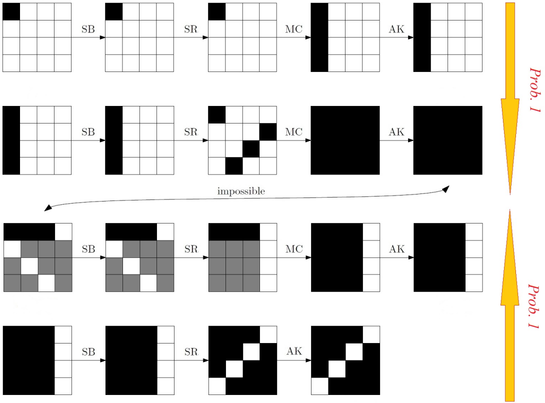
A 3-round AES Truncated Differential



A *gray box* ■ denotes a byte for which the difference of the two texts is unknown.

$$Prob[R^3(p^1) \oplus R^3(p^2) \in \mathcal{ID}_{0,1,2} \,|\, p^1 \oplus p^2 \in \mathcal{D}_0] = 2^{-8}$$
$$Prob[\Pi(p^1) \oplus \Pi(p^2) \in \mathcal{ID}_{0,1,2} \,|\, p^1 \oplus p^2 \in \mathcal{D}_0] = 2^{-32}$$

Impossible Differential on 4-round AES



$$Prob[R^4(p^1) \oplus R^4(p^2) \in \mathcal{ID}_{0,1,2} \mid p^1 \oplus p^2 \in \mathcal{D}_0] = 0$$

$$Prob[\Pi(p^1) \oplus \Pi(p^2) \in \mathcal{ID}_{0,1,2} \mid p^1 \oplus p^2 \in \mathcal{D}_0] = 2^{-32}$$

Our New Distinguishers for AES

In bold, our new distinguishers for up to 5-round AES: they are all independent of the secret key!

Rounds	Data (CP/CC)	Complexity	Property
4	$2^{16.25}$	$2^{31.5}$ M	Impossible Diff.
4	$\mathbf{2^{17}}$	$\mathbf{2^{23.1}}$ **M** $\approx$ $\mathbf{2^{16.75}}$ **E**	**Mixture Diff.**◊
4	2^{32}	2^{32} XOR	Integral [**DLR97**]
5	$2^{25.8}$ ACC	$2^{24.8}$ XOR	*Yoyo* [**RBH17**]
5	$\mathbf{2^{32}}$	$\mathbf{2^{35.6}}$ **M** $\approx$ $\mathbf{2^{29}}$ **E**	**Multiple-of-8**⋆
5	$\mathbf{2^{38}}$	$\mathbf{2^{41.6}}$ **M** $\approx$ $\mathbf{2^{35}}$ **E**	**Variance - Trunc. Diff.**
5	$\mathbf{2^{47.4}}$	$\mathbf{2^{51}}$ **M** $\approx$ $\mathbf{2^{44.3}}$ **E**	**Mean - Trunc. Diff.**

◊ ToSC/FSE 2019
⋆ Eurocrypt 2017

Part III

New Distinguishers for 5-round AES:
Multiple-of-8 Property
Mixture Differential Cryptanalysis

Multiple-of-8 Property for 5-round AES (EC'17)

Assume 5-round AES without the final MixColumns operation. Consider a set of 2^{32} chosen plaintexts with one active diagonal

$$\begin{bmatrix} A & C & C & C \\ C & A & C & C \\ C & C & A & C \\ C & C & C & A \end{bmatrix}$$

The number of *different* pairs of ciphertexts which are equal in one (fixed) anti-diagonal

$$\begin{bmatrix} 0 & ? & ? & ? \\ ? & ? & ? & 0 \\ ? & ? & 0 & ? \\ ? & 0 & ? & ? \end{bmatrix}$$

is a multiple of 8 with probability 1 **independent of the secret key, of the details of S-Box and of MixColumns matrix.**

From Multiple-of-8 to Mixture Diff. Cryptanalysis

Remember:

$$R(\underbrace{\begin{bmatrix} x_0 & 0 & 0 & 0 \\ 0 & x_1 & 0 & 0 \\ 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \end{bmatrix}}_{\equiv \mathcal{D}_I} \oplus a) = \underbrace{\begin{bmatrix} y_0 & 0 & 0 & 0 \\ y_1 & 0 & 0 & 0 \\ y_2 & 0 & 0 & 0 \\ y_3 & 0 & 0 & 0 \end{bmatrix}}_{\equiv \mathcal{C}_I} \oplus b$$

and

$$R(\underbrace{\begin{bmatrix} y_0 & 0 & 0 & 0 \\ y_1 & 0 & 0 & 0 \\ y_2 & 0 & 0 & 0 \\ y_3 & 0 & 0 & 0 \end{bmatrix}}_{\equiv \mathcal{C}_I} \oplus b) = \underbrace{MC \times \begin{bmatrix} z_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_1 \\ 0 & 0 & z_2 & 0 \\ 0 & z_3 & 0 & 0 \end{bmatrix}}_{\equiv MC \times \mathcal{ID}_I = \mathcal{M}_I} \oplus c$$

From Multiple-of-8 to Mixture Diff. Cryptanalysis

Since

$$\mathcal{D}_I \oplus a \xrightarrow[\text{prob. } 1]{R(\cdot)} \mathcal{C}_I \oplus b \xrightarrow{R^2(\cdot)} \mathcal{D}_J \oplus a' \xrightarrow[\text{prob. } 1]{R^2(\cdot)} \mathcal{M}_J \oplus b',$$

we can focus only on the middle round, and prove an equivalent result!

In the following, we prove a stronger result that holds on the last four rounds, called

Mixture Differential Cryptanalysis (ToSC/FSE 2019)

From Multiple-of-8 to Mixture Diff. Cryptanalysis

Since

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In the following, we prove a stronger result that holds on the last four rounds, called

Mixture Differential Cryptanalysis (ToSC/FSE 2019)

Mixture Diff. Cryptanalysis – 1*st* Case (1/2)

Consider $p^1, p^2 \in \mathcal{C}_0 \oplus a$:

$$p^1 = a \oplus \begin{bmatrix} x^1 & 0 & 0 & 0 \\ y^1 & 0 & 0 & 0 \\ z^1 & 0 & 0 & 0 \\ w^1 & 0 & 0 & 0 \end{bmatrix}, \qquad p^2 = a \oplus \begin{bmatrix} x^2 & 0 & 0 & 0 \\ y^2 & 0 & 0 & 0 \\ z^2 & 0 & 0 & 0 \\ w^2 & 0 & 0 & 0 \end{bmatrix}$$

where $x^1 \neq x^2$, $y^1 \neq y^2$, $z^1 \neq z^2$ and $w^1 \neq w^2$.

For the follow-up:

$$p^1 \equiv (x^1, y^1, z^1, w^1) \quad \text{and} \quad p^2 \equiv (x^2, y^2, z^2, w^2).$$

Mixture Diff. Cryptanalysis – 1*st* Case (2/2)

Given $p^1, p^2 \in \mathcal{C}_0 \oplus a$ as before:

$$p^1 \equiv (x^1, y^1, z^1, w^1) \quad \text{and} \quad p^2 \equiv (x^2, y^2, z^2, w^2)$$

it follows that

$$R^4(p^1) \oplus R^4(p^2) \in \mathcal{M}_J \quad \text{if and only if} \quad R^4(\hat{p}^1) \oplus R^4(\hat{p}^2) \in \mathcal{M}_J$$

where

$$\begin{aligned}
\hat{p}^1 &\equiv (x^2, y^1, z^1, w^1), & \hat{p}^2 &\equiv (x^1, y^2, z^2, w^2);\\
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\hat{p}^1 &\equiv (x^1, y^2, z^2, w^1), & \hat{p}^2 &\equiv (x^2, y^1, z^1, w^2).
\end{aligned}$$

Mixture Diff. Cryptanalysis – 2*nd* Case

Given $p^1, p^2 \in \mathcal{C}_0 \oplus a$ as before:

$$p^1 \equiv (x^1, y^1, z^1, w) \quad \text{and} \quad p^2 \equiv (x^2, y^2, z^2, w)$$

it follows that

$$R^4(p^1) \oplus R^4(p^2) \in \mathcal{M}_J \quad \text{if and only if} \quad R^4(\hat{p}^1) \oplus R^4(\hat{p}^2) \in \mathcal{M}_J$$

where

$$\begin{aligned}
\hat{p}^1 &\equiv (x^1, y^1, z^2, \Omega), & \hat{p}^2 &\equiv (x^2, y^2, z^2, \Omega);\\
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\hat{p}^1 &\equiv (x^1, y^1, z^2, \Omega), & \hat{p}^2 &\equiv (x^2, y^2, z^1, \Omega);
\end{aligned}$$

where Ω can take any value in $\mathbb{F}_{2^8}$.

Mixture Diff. Cryptanalysis – 3*rd* Case

Given $p^1, p^2 \in \mathcal{C}_0 \oplus a$ as before:

$$p^1 \equiv (x^1, y^1, z, w) \quad \text{and} \quad p^2 \equiv (x^2, y^2, z, w)$$

it follows that

$$R^4(p^1) \oplus R^4(p^2) \in \mathcal{M}_J \quad \text{if and only if} \quad R^4(\hat{p}^1) \oplus R^4(\hat{p}^2) \in \mathcal{M}_J$$

where

$$\hat{p}^1 \equiv (x^1, y^1, \mathcal{Z}, \Omega), \qquad \hat{p}^2 \equiv (x^2, y^2, \mathcal{Z}, \Omega);$$
$$\hat{p}^1 \equiv (x^2, y^1, \mathcal{Z}, \Omega), \qquad \hat{p}^2 \equiv (x^1, y^2, \mathcal{Z}, \Omega);$$

where $\mathcal{Z}$ and Ω can take any value in $\mathbb{F}_{2^8}$.

Reduction to 2 Rounds AES

Since

$$Prob\big(R^2(x) \oplus R^2(y) \in \mathcal{M}_J \,\big|\, x \oplus y \in \mathcal{D}_J\big) = 1$$

we can focus only on the two initial rounds:

$$\mathcal{C}_I \oplus b \xrightarrow{R^2(\cdot)} \mathcal{D}_J \oplus a' \xrightarrow[\text{prob. } 1]{R^2(\cdot)} \mathcal{M}_J \oplus b'$$

Consider $p^1, p^2 \in \mathcal{C}_I \oplus a$. We are going to prove that

$$R^2(p^1) \oplus R^2(p^2) \in \mathcal{D}_J$$

if and only if

$$R^2(\hat{p}^1) \oplus R^2(\hat{p}^2) \in \mathcal{D}_J,$$

where $\hat{p}^1, \hat{p}^2 \in \mathcal{C}_I \oplus a$ are defined as before.

Reduction to 2 Rounds AES

Since

$$Prob\left(R^2(x) \oplus R^2(y) \in \mathcal{M}_J \,\middle|\, x \oplus y \in \mathcal{D}_J\right) = 1$$

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if and only if

$$R^2(\hat{p}^1) \oplus R^2(\hat{p}^2) \in \mathcal{D}_J,$$

where $\hat{p}^1, \hat{p}^2 \in \mathcal{C}_I \oplus a$ are defined as before.

Idea of the Proof

Given p^1, p^2 and $\hat{p}^1, \hat{p}^2$ in $\mathcal{C}_0 \oplus a$ as before, if

$$R^2(p^1) \oplus R^2(p^2) = R^2(\hat{p}^1) \oplus R^2(\hat{p}^2)$$

then the previous result

$$R^2(p^1) \oplus R^2(p^2) \in \mathcal{D}_J \quad \textit{iff} \quad R^2(\hat{p}^1) \oplus R^2(\hat{p}^2) \in \mathcal{D}_J$$

follows immediately!

Super-Box Notation (1/2)

Let $\textit{super-SB}(\cdot)$ be defined as

$$\textit{super-SB}(\cdot) = \text{S-Box} \circ ARK \circ MC \circ \text{S-Box}(\cdot).$$

2-round AES can be rewritten as

$$R^2(\cdot) = ARK \circ MC \circ SR \circ \textit{super-SB} \circ SR(\cdot)$$

Super-Box Notation (2/2)

By simple computation,

$$R^2(p^1) \oplus R^2(p^2) = R^2(\hat{p}^1) \oplus R^2(\hat{p}^2)$$

is equivalent to

$$\textit{super-SB}(P^1) \oplus \textit{super-SB}(P^2) = \textit{super-SB}(\hat{P}^1) \oplus \textit{super-SB}(\hat{P}^2),$$

where

$$P^i \equiv SR(p^i), \hat{P}^i \equiv SR(\hat{p}^i) \in SR(\mathcal{C}_I) \oplus a' \equiv \mathcal{ID}_I \oplus a'$$

for $i = 1, 2$.

Sketch of the Proof (1/2)

Given $P^1 = SR(p^1), P^2 = SR(p^2) \in \mathcal{ID}_0 \oplus a'$, note that

$$P^1 = a' \oplus \begin{bmatrix} x^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & y^1 \\ 0 & 0 & z^1 & 0 \\ 0 & w^1 & 0 & 0 \end{bmatrix}, \qquad P^2 = a' \oplus \begin{bmatrix} x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & y^2 \\ 0 & 0 & z^2 & 0 \\ 0 & w^2 & 0 & 0 \end{bmatrix}$$

Sketch of the Proof

Since

- each column depends on different and independent variables;
- the super-SB works independently on each column;
- the XOR-sum is commutative;

then

$$\textit{super-SB}(P^1) \oplus \textit{super-SB}(P^2) = \textit{super-SB}(\hat{P}^1) \oplus \textit{super-SB}(\hat{P}^2)$$

for each $\hat{P}^1$ and $\hat{P}^2$ obtained by mixing/swapping the columns of P^1 and P^2, e.g.

$$\hat{P}^1 = a' \oplus \begin{bmatrix} x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & y^1 \\ 0 & 0 & z^1 & 0 \\ 0 & w^1 & 0 & 0 \end{bmatrix}, \qquad \hat{P}^2 = a' \oplus \begin{bmatrix} x^1 & 0 & 0 & 0 \\ 0 & 0 & 0 & y^2 \\ 0 & 0 & z^2 & 0 \\ 0 & w^2 & 0 & 0 \end{bmatrix}$$

Mixture Diff. Distinguisher on 4-round AES

Consider $p^1 \equiv (x^1, y^1, z^1, w^1)$, $p^2 \equiv (x^2, y^2, z^2, w^2) \in \mathcal{C}_0 \oplus a$ s.t.

$$c^1 \oplus c^2 \equiv R^4(p^1) \oplus R^4(p^2) \in \mathcal{M}_J,$$

i.e. c^1 and c^2 are equal in $4 - J$ anti-diagonals.

Given $\hat{p}^1, \hat{p}^2 \in \mathcal{C}_0 \oplus a$ obtained my mixing/swapping the generating variables of p^1, p^2, then:

- 4-round AES: the event $R^4(\hat{p}^1) \oplus R^4(\hat{p}^2) \in \mathcal{M}_J$ occurs with prob. 1;
- Random Perm.: the event $\Pi(\hat{p}^1) \oplus \Pi(\hat{p}^2) \in \mathcal{M}_J$ occurs with prob. $2^{-32\cdot(4-|J|)}$;

independently of the secret-key.

Mixture Diff. Distinguisher + Key-Recovery Attack

Since

$$a \oplus \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{bmatrix} \xrightarrow{R(\cdot)} b \oplus MC \times \begin{bmatrix} \text{S-Box}(x \oplus k_{0,0}) & 0 & 0 & 0 \\ \text{S-Box}(y \oplus k_{1,1}) & 0 & 0 & 0 \\ \text{S-Box}(z \oplus k_{2,2}) & 0 & 0 & 0 \\ \text{S-Box}(w \oplus k_{3,3}) & 0 & 0 & 0 \end{bmatrix},$$

the relations among the generating variables of $R(p^1), R(p^2)$ and of $R(\hat{p}^1), R(\hat{p}^2)$ depend on the key.

Idea of the attack:

$$\mathcal{D}_0 \oplus a \xrightarrow[\text{key guessing}]{R(\cdot)} \mathcal{C}_0 \oplus b \xrightarrow[\text{distinguisher}]{R^4(\cdot)} \textit{Mixture Diff. Property}$$

where *the mixture differential property holds only for the secret-key!*

Mixture Diff. Distinguisher + Key-Recovery Attack

Since

$$a \oplus \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & w \end{bmatrix} \xrightarrow{R(\cdot)} b \oplus MC \times \begin{bmatrix} \text{S-Box}(x \oplus k_{0,0}) & 0 & 0 & 0 \\ \text{S-Box}(y \oplus k_{1,1}) & 0 & 0 & 0 \\ \text{S-Box}(z \oplus k_{2,2}) & 0 & 0 & 0 \\ \text{S-Box}(w \oplus k_{3,3}) & 0 & 0 & 0 \end{bmatrix},$$

the relations among the generating variables of $R(p^1)$, $R(p^2)$ and of $R(\hat{p}^1)$, $R(\hat{p}^2)$ depend on the key.

Idea of the attack:

$$\mathcal{D}_0 \oplus a \xrightarrow[\text{key guessing}]{R(\cdot)} \mathcal{C}_0 \oplus b \xrightarrow[\text{distinguisher}]{R^4(\cdot)} \textit{Mixture Diff. Property}$$

where *the mixture differential property holds only for the secret-key!*

Mixture Diff. Key-Recovery Attack (1/2)

Consider 2^{32} chosen plaintexts with one active diagonal, that is $p^i \in \mathcal{D}_0 \oplus a$ for $i = 1, ..., 2^{32}$.

Find a pair of plaintexts (p, p') s.t. the corresponding ciphertexts after 5-round ($c = R^5(p), c' = R^5(p')$) satisfy the property

$$c \oplus c' = R^5(p) \oplus R^5(p') \in \mathcal{M}_J$$

for a certain J, i.e. c and c' are equal in $4 - |J|$ anti-diagonal(s).

Mixture Diff. Key-Recovery Attack (2/2)

For each guessed value of $(k_{0,0}, k_{1,1}, k_{2,2}, k_{3,3})$:

- *partially* compute 1-round encryption of $R(p), R(p')$ w.r.t. the **guessed-key**;
- let q, q' be two texts obtained by swapping the generating variables of $R(p), R(p')$;
- *partially* compute 1-round decryption of $\hat{q} \equiv R^{-1}(q), \hat{q}' \equiv R^{-1}(q')$ w.r.t. the *guessed-key*;
- if
$$R^5(\hat{q}) \oplus R^5(\hat{q}') \notin \mathcal{M}_J,$$
  then the guessed key is wrong (where $R^5(\cdot)$ is computed under the **secret-key**).

Key-Recovery Attacks on 5-round AES-128

Property	**Data** (CP/CC)	**Cost** (E)	**Memory**
MitM [**Der13**]	8	2^{64}	2^{56}
Imp. Polytopic [**Tie16**]	15	2^{70}	2^{41}
Partial Sum [**Tun12**]	2^{8}	2^{38}	small
Integral (EE) [**DR02**]	2^{11}	$2^{45.7}$	small
Mixture Diff.* [**BDK+18**]	$\mathbf{2^{22.25}}$	$\mathbf{2^{22.25}}$	$\mathbf{2^{20}}$
Imp. Differential [**BK01**]	$2^{31.5}$	2^{33} $(+\,2^{38})$	2^{38}
Integral (EB) [**DR02**]	2^{33}	$2^{37.7}$	2^{32}
Mixture Diff.	$\mathbf{2^{33.6}}$	$\mathbf{2^{33.3}}$	$\mathbf{2^{34}}$

* $\equiv$ follow-up work

At Crypto 2018, Bar-On et al. [**BDK+18**] *present the best (mixture-differential) attacks on 7-round AES-192 which use practical amounts of data and memory.*

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Part IV

New Distinguishers for 5-round AES: Truncated Differential Distinguishers

Truncated Differential - 5-round AES

Consider all the 2^{32} plaintexts with one active diagonal (i.e. a coset of a diagonal space $\mathcal{D}_I$) and the corresponding ciphertexts after 5 rounds, i.e. $(p^i, c^i \equiv R^5(p^i))$.

The average number of different pairs of ciphertexts (c^i, c^j) with $i < j$ that are equal in one fixed anti-diagonal (assuming the final MC is omitted) is approximately equal to

$$2\,147\,484\,685.6 \simeq 2^{31} + 2^{10.1}$$

while for a random permutation it is approximately equal to

$$2\,147\,483\,647.5 \simeq 2^{31} - 2^{-1}$$

(difference of ≈ 1038.1 collisions).

Truncated Differential – Assumption on the S-Box

The previous result is **independent of the secret key**, but **it depends on the details of S-Box.**

In more detail, consider the following equation:

$$\text{S-Box}(x \oplus \Delta_{IN}) \oplus \text{S-Box}(x) = \Delta_{OUT}.$$

*The previous result holds if the solutions (*in particular, *the number of solutions) of the previous equation are ("almost") uniformly distributed for each* $(\Delta_{IN}, \Delta_{OUT}) \neq (0, 0)$.

This is close to be satisfied if the S-Box is APN, or if the SBox is "close" to be APN (like the AES S-Box).

Variance distinguisher - 5-round AES

Consider all the 2^{32} plaintexts with one active diagonal (i.e. a coset of a diagonal space $\mathcal{D}_I$) and the corresponding ciphertexts after 5 rounds, i.e. $(p^i, c^i \equiv R^5(p^i))$.

Consider the variance of the distribution of the number of different pairs of ciphertexts (c^i, c^j) with $i < j$ that are equal in one fixed anti-diagonal (assuming the final MC is omitted). For 5-round AES, it is approximately equal to

$$2^{36.154}$$

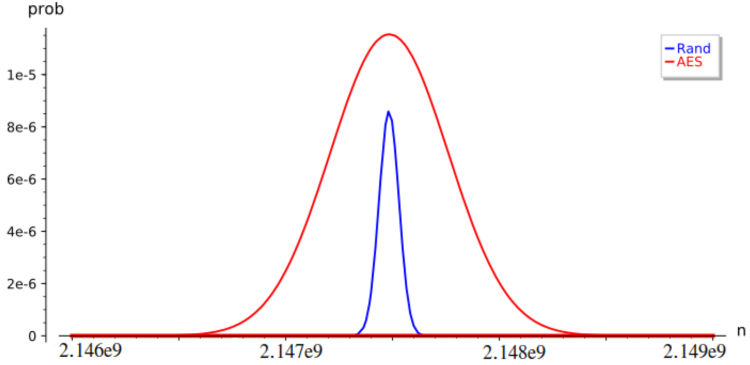
independent of the secret key, of the details of S-Box and of MixColumns matrix, while for a random permutation it is approximately equal to

$$2^{31}$$

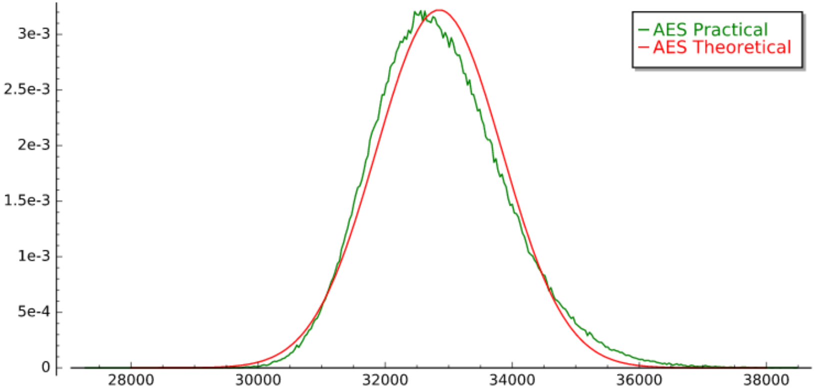
(difference of *factor* ≈ 35.6).

Probabilistic Distribution - AES vs Random

Probabilistic Distribution - 5-round AES vs Random:



Skew Distinguisher - 5-round (small-scale) AES



Note: do not confuse the mean and the mode!

Open Problem: theoretically compute the skew

Part V

Cryptanalysis of AES with a single Secret S-Box

AES with a single Secret S-Box

Consider AES with a single secret S-Box: the size of the secret information increases from 128-256 bits to

$$128 + \log_2 2^8! = 1812$$
$$256 + \log_2 2^8! = 1940$$

How does the security of the AES change when the S-Box is replaced by a secret S-Box, about which the adversary has no knowledge?

For all the attacks in literature:

1. determine the secret S-Box up to additive constants, i.e. S-Box$(a \oplus x) \oplus b$;
2. exploit this knowledge to find the key.

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2. exploit this knowledge to find the key.

AES with a single Secret S-Box

Is it possible to find directly the key, i.e. without finding or exploiting any information of S-Box?

Yes: exploit the fact that each row of the MixColumns matrix

$$MC \equiv \begin{bmatrix} 0x02 & 0x03 & 0x01 & 0x01 \\ 0x01 & 0x02 & 0x03 & 0x01 \\ 0x01 & 0x01 & 0x02 & 0x03 \\ 0x03 & 0x01 & 0x01 & 0x02 \end{bmatrix}$$

has two identical elements or that *the XOR-sum of "some" elements is equal to zero*.

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has two identical elements or that *the XOR-sum of “some” elements is equal to zero*.

Multiple-of-n Property - 5-round AES

Guess one byte of the key δ and consider the set of 2^{40} plaintexts V_δ

$$V_\delta \equiv \left\{ a \oplus \begin{bmatrix} x_0 & y & 0 & 0 \\ 0 & x_1 & y \oplus \delta & 0 \\ 0 & 0 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \end{bmatrix} \middle| \forall x_0, \ldots, x_3, y \in \mathbb{F}_{2^8} \right\}$$

Let N the number of different pairs of ciphertexts (c^1, c^2) that are equal in one fixed anti-diagonal, e.g.

$$c^1 \oplus c^2 = \begin{bmatrix} ? & ? & ? & 0 \\ ? & ? & 0 & ? \\ ? & 0 & ? & ? \\ 0 & ? & ? & ? \end{bmatrix}$$

(final MC omitted for simplicity)

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(final MC omitted for simplicity)

Multiple-of-n Property - 5-round AES

Let N the number of different pairs of ciphertexts (c^1, c^2) that are equal in one fixed anti-diagonal (final MC omitted for simplicity).

Since $MC_{3,0} = MC_{3,1}$:

- If $\delta = k_{0,1} \oplus k_{1,2}$, N is a multiple of 2 - i.e. $N = 2 \cdot N'$ - with prob. 1;
- If $\delta \neq k_{0,1} \oplus k_{1,2}$, N is a multiple of 2 with prob. 50% (same probability to be even or odd).

Part VI

Open Problems for Future Works

Recap and Future Works

- Open Problem (for the last 20 years) Solved:
  we have found new properties for 5-round AES which are independent of the secret key

- As a main result, cryptanalysis of AES is not “finished”:
  we have proposed new directions of research for AES-like ciphers that can lead to new distinguishers/attacks (e.g. new truncated differentials for 6-round AES have been proposed recently)

Recap and Future Works

Open Problems:

- how the details of the S-Box influence the truncated differentials for 5-/6-round AES?
- what about other distinguishers based on the variance/skewness?
- what about a truncated differential for 7-round AES?
- what about new key-recovery attacks?
- what about new boomerang distinguisher/attack based on multiple-of-8 property?
- is it possible to improve the attacks in the case of a secret S-Box(es)?
- ...

Thanks for your attention!

Questions?

Comments?

Sketch of the Proof - Reduction to a Single Round

Since

$$\mathcal{D}_I \oplus a \xrightarrow[\text{prob. } 1]{R^2(\cdot)} \mathcal{M}_I \oplus b \xrightarrow{R(\cdot)} \mathcal{D}_J \oplus a' \xrightarrow[\text{prob. } 1]{R^2(\cdot)} \mathcal{M}_J \oplus b',$$

we can focus only on the middle round!

For simplicity, we limit to consider

$$(\mathcal{C}_{0,1} \cap \mathcal{M}_0) \oplus b \xrightarrow{R(\cdot)} \mathcal{D}_{1,2,3} \oplus a'$$

where

$$\mathcal{C}_{0,1} \cap \mathcal{M}_0 \equiv \begin{bmatrix} 2 \cdot x & y & 0 & 0 \\ 3 \cdot x & 2 \cdot y & 0 & 0 \\ x & 3 \cdot y & 0 & 0 \\ x & y & 0 & 0 \end{bmatrix}$$

Sketch of the Proof - Reduction to a Single Round

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where

$$\mathcal{C}_{0,1} \cap \mathcal{M}_0 \equiv \begin{bmatrix} 2 \cdot x & y & 0 & 0 \\ 3 \cdot x & 2 \cdot y & 0 & 0 \\ x & 3 \cdot y & 0 & 0 \\ x & y & 0 & 0 \end{bmatrix}$$

Idea of the Proof

Given $p^1, p^2 \in (\mathcal{C}_{0,1} \cap \mathcal{M}_0) \oplus b$ where $p^1 \equiv (x^1, y^1)$, $p^2 \equiv (x^2, y^2)$ (where $x^1 \neq x^2$ and $y^1 \neq y^2$), they satisfy

$$R(p^1) \oplus R(p^2) \in \mathcal{D}_{1,2,3}$$

if and only if

$$(R(p^1) \oplus R(p^2))_{0,0} = 2 \cdot (\text{S-Box}(2 \cdot x^1 \oplus a_{0,0}) \oplus \text{S-Box}(2 \cdot x^2 \oplus a_{0,0})) \oplus$$
$$\oplus\, 3 \cdot (\text{S-Box}(y^1 \oplus a_{1,1}) \oplus \text{S-Box}(y^2 \oplus a_{1,1})) = 0,$$
$$(R(p^1) \oplus R(p^2))_{1,1} = \text{S-Box}(3 \cdot x^1 \oplus a_{3,0}) \oplus \text{S-Box}(3 \cdot x^2 \oplus a_{3,0}) \oplus$$
$$\oplus\, \text{S-Box}(y^1 \oplus a_{0,1}) \oplus \text{S-Box}(y^2 \oplus a_{0,1}) = 0,$$
$$(R(p^1) \oplus R(p^2))_{2,2} = 2 \cdot (\text{S-Box}(x^1 \oplus a_{2,0}) \oplus \text{S-Box}(x^2 \oplus a_{2,0})) \oplus$$
$$\oplus\, 3 \cdot (\text{S-Box}(2 \cdot y^1 \oplus a_{3,1}) \oplus \text{S-Box}(2 \cdot y^2 \oplus a_{3,1})) = 0,$$
$$(R(p^1) \oplus R(p^2))_{3,3} = \text{S-Box}(x^1 \oplus a_{1,0}) \oplus \text{S-Box}(x^2 \oplus a_{1,0}) \oplus$$
$$\oplus\, \text{S-Box}(3 \cdot y^1 \oplus a_{2,1}) \oplus \text{S-Box}(3 \cdot y^2 \oplus a_{2,1}) = 0.$$

Working on a single Equation

This means that four equations of the form

$$A \cdot \Big[\text{S-Box}(B \cdot x^1 \oplus a) \oplus \text{S-Box}(B \cdot x^2 \oplus a)\Big] \oplus$$
$$\oplus\, C \cdot \Big[\text{S-Box}(D \cdot y^1 \oplus c) \oplus \text{S-Box}(D \cdot y^2 \oplus c)\Big] = 0$$

must be satisfied, where A, B, C, D depend only on the MixColumns matrix, while a, c depend on the secret key and on the initial constant that defines the coset.

Equivalently:

$$\text{S-Box}(\hat{x} \oplus \Delta_I) \oplus \text{S-Box}(\hat{x}) = \Delta_O$$
$$\text{S-Box}(\hat{y} \oplus \Delta_I') \oplus \text{S-Box}(\hat{y}) = \Delta_O'$$
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Working on a single Equation

Note that for each $\Delta_O \neq 0$, the equation

$$\text{S-Box}(x \oplus \Delta_I) \oplus \text{S-Box}(x) = \Delta_O$$

admits 256 different solutions (x, Δ_I), where $\Delta_I \neq 0$.

As a result, there are

$$\underbrace{255}_{\text{values of } \Delta_O \neq 0} \times \underbrace{\frac{1}{2} \cdot 256^2}_{\text{different solutions} (\hat{x}, \Delta_I), (\hat{y}, \Delta_I')} = 255 \cdot 2^{15}$$

different solutions $(x^1, y^1), (x^2, y^2)$ of

$$A \cdot \Big[\text{S-Box}(B \cdot x^1 \oplus a) \oplus \text{S-Box}(B \cdot x^2 \oplus a)\Big] \oplus$$
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A system of four Equations

What is the probability that the two equations of the system admit a common solution $(x^1, y^1), (x^2, y^2)$*?*

Since *(1st)* $x^1 \neq x^2$ by assumption and since *(2nd)* $[(x^1, y^1), (x^2, y^2)]$ and $[(x^2, y^2), (x^1, y^1)]$ are equivalent solutions (e.g. a solution is "valid" if $y^2 < y^1$), this probability is equal to

$$\underbrace{(256 \cdot 255)^{-1}}_{\text{condition on } x^1, x^2} \times \underbrace{(255 \cdot 128)^{-1}}_{\text{condition on } y^1, y^2} = 2^{-15} \times 255^{-2}$$

Assumption: the solutions x of S-Box$(x \oplus \Delta_I) \oplus$ S-Box$(x) = \Delta_O$ are **uniform distributed** for each $(\Delta_I, \Delta_O) \neq (0, 0)$! Otherwise the previous probability is in general **not** correct!

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Conclusion

The number of texts $p^1, p^2 \in (\mathcal{C}_{0,1} \cap \mathcal{M}_0) \oplus b$ that satisfy $R(p^1) \oplus R(p^2) \in \mathcal{D}_{1,2,3}$ is

$$\left(255 \cdot 2^{15}\right)^4 \cdot \left(2^{-15} \cdot 255^{-2}\right)^3 = \frac{2^{15}}{255^2} = \frac{1}{2} + \underbrace{\frac{511}{2 \cdot 255^2}}_{\approx 2^{-8}}.$$

For a random permutation, the number of collisions is given by

$$\binom{2^{16}}{2} \cdot 2^{-32} = \frac{2^{16}-1}{2^{17}} = \frac{1}{2} - \frac{1}{2^{17}}.$$

Using the same strategy, it is possible to prove the results on 5-round AES!

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Using the same strategy, it is possible to prove the results on 5-round AES!

Variance - Idea of the Proof

The previous result is (almost) **independent of the secret key, of the details of S-Box and of MixColumns matrix.**

To theoretically derive the previous result:

- use the fact that the number of collisions is a multiple of 8;
- given a random variable X, remember that

$$Var(A \cdot X) = A^2 \cdot Var(X)$$

for any scalar A.

Variance - Sketch of the Proof (1/2)

Given 2^{32} texts in $\mathcal{D}_I \oplus a$, the corresponding pairs of texts are **not** independent! It is possible to divide such pairs in

- sets of cardinality 8 (different generating variables);
- sets of cardinality 2^{10} (one equal generating variable);
- sets of cardinality 2^{17} (two equal generating variables);

such that

1. *pairs of texts of different sets are independent*;
2. *pairs of texts in the same set have the same property*.

Variance - Sketch of the Proof (2/2)

If Y is the probabilistic distribution of the number of collisions, then

$$Y = 2^3 \times X_3 + 2^{10} \times X_{10} + 2^{17} \times X_{17}$$

where X_3, X_{10}, X_{17} is the probabilistic distribution of *independent/unrelated* pairs of texts.

The result follows from

$$\begin{aligned} Var(Y) &= Var(2^3 \times X_3 + 2^{10} \times X_{10} + 2^{17} \times X_{17}) = \\ &= Var(2^3 \times X_3) + Var(2^{10} \times X_{10}) + Var(2^{17} \times X_{17}) = \\ &= 2^6 \times Var(X_3) + 2^{20} \times Var(X_{10}) + 2^{34} \times Var(X_{17}) \end{aligned}$$

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