Analysis, classification and construction of APN Boolean functions

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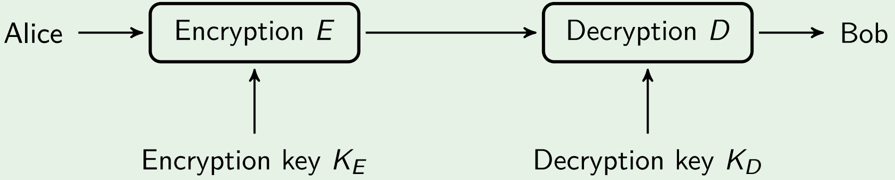
Cryptography



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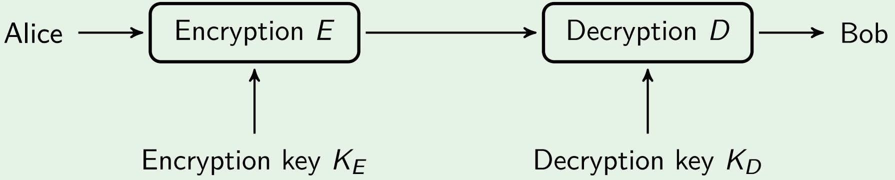

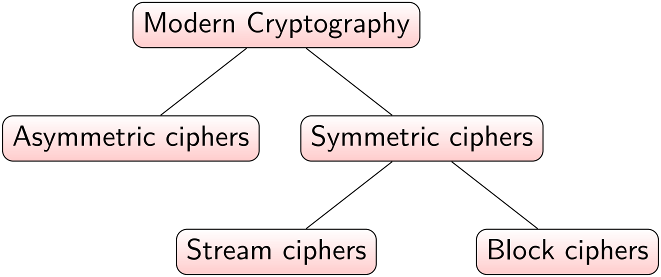

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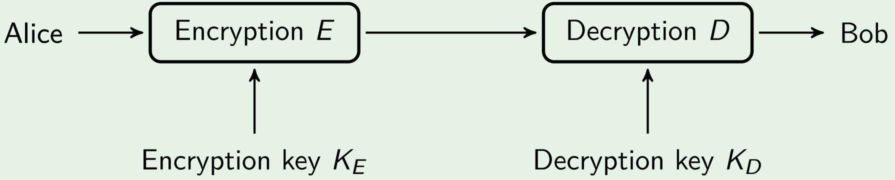

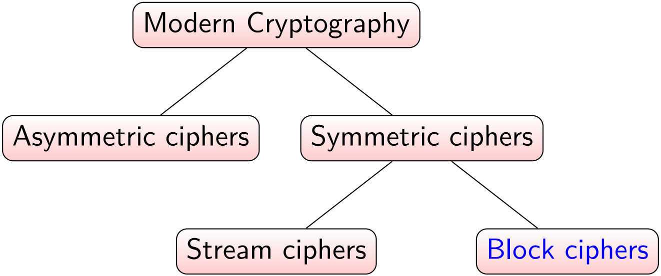

Block ciphers

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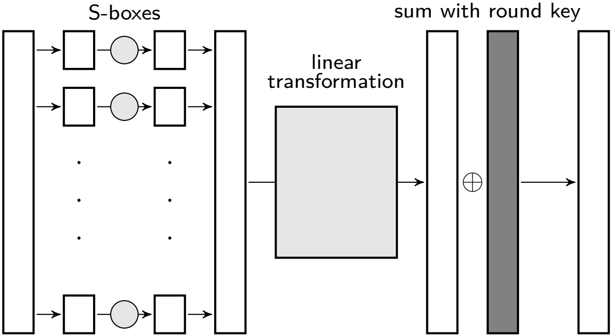

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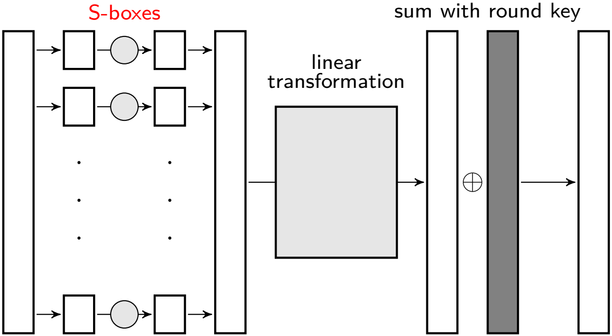

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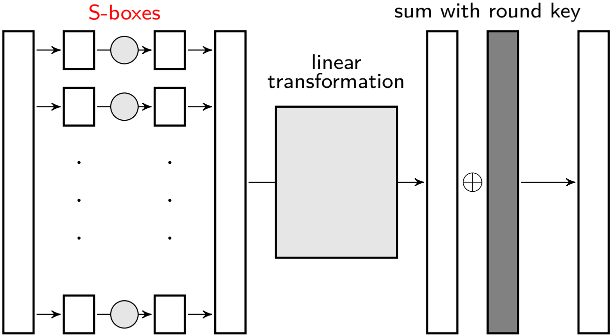

Substitution box (S-box)

- vectorial Boolean function
- nonlinear
- often invertible
- cryptographic properties

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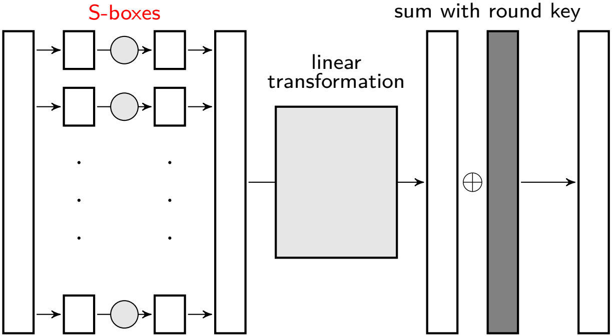

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Vectorial Boolean functions

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  - F is a permutation if $Im(F) = \mathbb{F}_{2^n}$

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Univariate representation of F

$$F(x) = \sum_{i=0}^{2^n-1} b_i x^i, \qquad b_i \in \mathbb{F}_{2^n}$$

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- F affine if $F =$ linear $+$ constant
- F DO polynomial if $F(x) = \sum_{i,j=0}^{n-1} b_{ij} x^{2^i+2^j}$, $i < j$
- F quadratic if $F =$ DO $+$ affine

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Example of linear function

The trace map: $\mathrm{Tr}(x) = x + x^2 + x^{2^2} + \ldots + x^{2^{n-1}}$

Differential uniformity

Important cryptographic property related to the differential attack

Differentially δ-uniform

For $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$

$$\delta = \max_{a,b \in \mathbb{F}_{2^n} a \neq 0} |\{x \in \mathbb{F}_{2^n} : F(x+a) + F(x) = b\}|$$

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F is called almost perfect nonlinear (APN) if $\delta = 2$

Differential uniformity

Equivalence relations

Differential uniformity is invariant under the following equivalence relations

- **linear equivalence** $F \overset{lin}{\sim} A_1 \circ F \circ A_2$ (A_1, A_2 linear permutations)
- **affine equivalence** $F \overset{aff}{\sim} A_1 \circ F \circ A_2$ (A_1, A_2 affine permutations)
- **EA-equivalence** (extended affine) $F \overset{EA}{\sim} F' + A$ ($F' \overset{aff}{\sim} F$, A affine)
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- construction methods of optimal functions
- classification of optimal functions
- invariants

On APN functions over $\mathbb{F}_{2^n}$

1994

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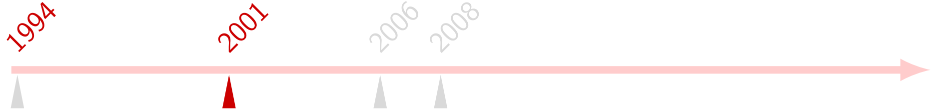
1994

- 6 families of power APN functions

Known APN power functions $F(x) = x^d$ over $\mathbb{F}_{2^n}$

Name	Exponent d	Conditions	Degree
Gold	$2^i + 1$	$\gcd(i, n)$=1	2
Kasami	$2^{2i} - 2^i + 1$	$\gcd(i, n)$=1	i+1
Welch	$2^t + 3$	$n = 2t + 1$	3
Niho	$2^t + 2^{\frac{t}{2}} - 1$, t even $2^t + 2^{\frac{3t+1}{2}} - 1$, t odd	$n = 2t + 1$	$\frac{t+2}{2}$ t+1
Inverse	$2^{2t} - 1$	$n = 2t + 1$	$n - 1$
Dobbertin	$2^{4i} + 2^{3i} + 2^{2i} + 2^i - 1$	$n = 5i$	$i + 3$

On APN functions over $\mathbb{F}_{2^n}$



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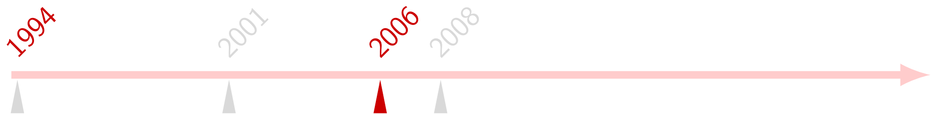

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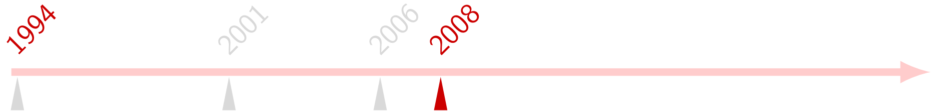

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On APN functions over $\mathbb{F}_{2^n}$

1994 2001 2006 2008

- 6 families of power APN function
- 16 families of quadratic APN functions

Known classes of quadratic APN polynomial over $\mathbb{F}_{2^n}$ CCZ-inequivalent fo power functions

N°	APN function over $\mathbb{F}_{2^n}$	Conditions
C1-C2	$x^{2^s+1}+\alpha^{2^k-1}x^{2^{ik}+2^{mk+s}}$	$n=pk$, $\gcd(k,3)=\gcd(s,3k)=1$, $p\in\{3,4\}$, $i=sk \mod p$, $m=p-i$, $n\geq 12$, α primitive in $\mathbb{F}_{2^n}^*$
C3	$x^{2^{2i}+2^i}+bx^{q+1}+cx^{q(2^{2i}+2^i)}$	$q=2^m$, $n=2m$, $\gcd(i,m)=1$, $gcd(2^i+1,q+1)\neq 1$, $cb^q+b\neq 0$, $c\notin\{\lambda^{(2^i+1)(q-1)},\lambda\in\mathbb{F}_{2^n}\}$, $c^{q+1}=1$
C4	$x(x^{2^i}+x^q+cx^{2^iq})+$ $x^{2^i}(c^qx^q+sx^{2^iq})+x^{(2^i+1)q}$	$q=2^m$, $n=2m$, $\gcd(i,m)=1$, $c\in\mathbb{F}_{2^n}$, $s\in\mathbb{F}_{2^n}\setminus\mathbb{F}_q$, $X^{2^i+1}+cX^{2^i}+c^qX+1$ is irreducible over $\mathbb{F}_{2^n}$
C5	$x^3+a^{-1}\mathrm{Tr}_n(a^3x^9)$	$a\neq 0$
C6	$x^3+a^{-1}\mathrm{Tr}_n^3(a^3x^9+a^6x^{18})$	$3\mid n$, $a\neq 0$
C7	$x^3+a^{-1}\mathrm{Tr}_n^3(a^6x^{18}+a^{12}x^{36})$	$3\mid n$, $a\neq 0$
C8-C10	$ux^{2^s+1}+u^{2^k}x^{2^{-k}+2^{k+s}}+$ $vx^{2^{-k}+1}+wu^{2^k+1}x^{2^s+2^{k+s}}$	$n=3k$, $\gcd(k,3)=\gcd(s,3k)=1$, $v,w\in\mathbb{F}_{2^k}$, $vw\neq 1$, $3\mid(k+s)$, $\mathbb{F}_{2^n}^\star=\langle u\rangle$

N°	APN function over $\mathbb{F}_{2^n}$	Conditions
C11	$dx^{2^i+1} + d^q x^{q(2^i+1)} +$ $cx^{q+1} + \sum_{s=1}^{m-1} \gamma_s x^{2^s(q+1)}$	$q = 2^m,\ n = 2m,\ \gcd(i, m) = 1,\ i, m$ odd, $c \notin \mathbb{F}_{2^m},\ \gamma_s \in \mathbb{F}_{2^m},\ d$ not a cube
C12	$(x + x^q)^{2^i+1} +$ $u'(ux + u^q x^q)^{(2^i+1)2^j} +$ $u(x + x^q)(ux + u^q x^q)$	$q = 2^m,\ n = 2m,\ m \geq 2$ even, $\gcd(i, m) = 1$ and j even $\mathbb{F}_{2^n}^\star = \langle u \rangle,\ u' \in \mathbb{F}_{2^m}$ not a cube
C13	$ut(x)(x^q + x) + t(x)^{2^{2i}+2^{3i}} +$ $at(x)^{2^{2i}}(x^q + x)^{2^i} + b(x^q + x)^{2^i+1}$	$q = 2^m,\ n = 2m,\ \gcd(i, m) = 1$ $a, b \in \mathbb{F}_{2^m}, u \notin \mathbb{F}_{2^m},\ t(x) = u^q x + x^q u$ and $X^{2^i+1} + aX + b$ has no solution over $\mathbb{F}_{2^m}$
C14	$x^3 + ax^{2^k(2^i+1)}$ $+bx^{3\cdot 2^m} + cx^{2^{n+k}(2^i+1)}$	$n = 2m = 10, (a, b, c) = (\beta, 1, 0), i = 3, k = 2, \mathbb{F}_4^* = \langle \beta \rangle$ $n = 2m, m$ odd, $3 \nmid m, (a, b, c) = (\beta, \beta^2, 1)$, $\mathbb{F}_4^* = \langle \beta \rangle,\ i \in \{m-2, m, 2m-1, (m-2)^{-1} \mod n\}$
C15	$u[t(x)^{2^i+1} + t(x)(x^q + x)^{2^i}$ $+(x^q + x)^{2^i+1}] + t(x)^{2^{2i}+1}$ $+t(x)^{2^{2i}}(x^q + x) + (x^q + x)^{2^{2i}+1}$	$q = 2^m,\ n = 2m,\ \gcd(3i, m) = 1$, $\mathbb{F}_{2^n}^\star = \langle u \rangle,\ t(x) = u^q x + ux^q$
C16	$u[t(x)^{2^i+1} + t(x)(x^q + x)^{2^i}$ $+(x^q + x)^{2^i+1}] + t(x)^{2^{3i}}(x^q + x)$ $+t(x)(x^q + x)^{2^{3i}}$	m odd, $q = 2^m,\ n = 2m,\ \gcd(3i, m) = 1$, $\mathbb{F}_{2^n}^\star = \langle u \rangle,\ t(x) = u^q x + ux^q$

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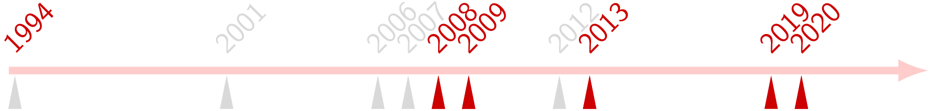

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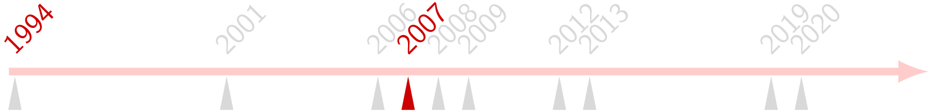

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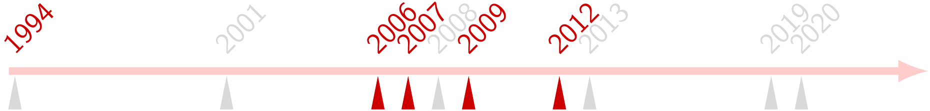

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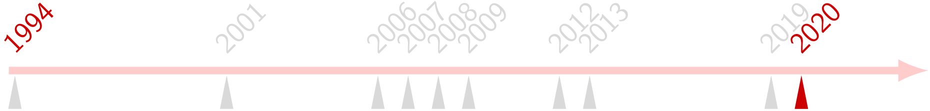

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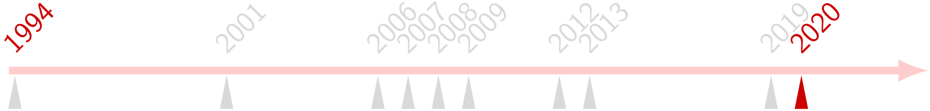

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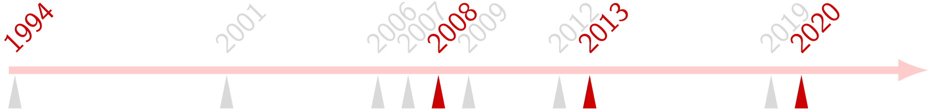

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- complete classification of quadratic APN functions with coefficients in $\mathbb{F}_2$ for $n = 8, 9$
- (not exhaustive) lists of CCZ-inequivalent quadratic APN functions for $n = 8, 9, 10$

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The big APN problem

- Construct other APN permutations in even dimension
- Construct an infinite family of APN permutations for even dimensions

Open problems and research interests

- The big APN problem
- Construct other infinite families of APN functions
- Increase the lists of known CCZ-inequivalent APN functions
- Find an equivalence relation more general than CCZ-equivalence
- Provide new CCZ/EA-invariants to help determining the equivalence between functions

Combination of theoretical analysis with computational results

List of papers

1. **Constructing APN functions through isotopic shifts**
   Lilya Budaghyan, Marco Calderini, Claude Carlet, Robert Coulter and I.V., *IEEE Transaction on Information Theory*, vol. 66, no. 8, pp. 5299-5309, 2020.
2. **Generalised isotopic shift construction for APN functions**
   Lilya Budaghyan, Marco Calderini, Claude Carlet, Robert Coulter and I.V., *Designs, Codes and Cryptography*, 2020.
3. **On equivalence between known families of quadratic APN functions**
   Lilya Budaghyan, Marco Calderini and I.V., *Finite Fields and their Applications*, vol. 66, 2020.
4. **Isotopic shift construction for planar functions**
   Lilya Budaghyan, Marco Calderini, Claude Carlet, Robert Coulter and I.V., *2019 IEEE International Symposium on Information Theory (ISIT)*, pp. 2962-2966, 2019.

Planar functions over $\mathbb{F}_{p^n}$

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F planar (or differentially 1-uniform)

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- linear, affine, EA- and CCZ-equivalence

Isotopic equivalence

- defined for quadratic planar functions $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$
- more general than CCZ-equivalence

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Theorem

For two isotopic equivalent quadratic planar functions F and F' there exists a linear permutation L such that $F' \overset{EA}{\sim} F_L$.

Isotopic equivalence

- defined for quadratic planar functions $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$
- more general than CCZ-equivalence

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The isotopic shift of F by L

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In the past years, known families of APN functions have been used to construct planar functions, and vice versa.

Isotopic shift applied to APN maps

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Some properties

- For $F \in \mathbb{F}_{2^n}[x]$ quadratic function and L linear, F_L is APN only if L is a permutation or 2-to-1.
- The isotopic shift does not preserve the differential uniformity.

Isotopic shift of Gold functions

For F a Gold function $F(x) = x^{2^i+1}$,

$$\boxed{F_L(x) = x^{2^i} L(x) + xL(x)^{2^i}}$$

Isotopic shift of Gold functions

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When L is a monomial

All such APN maps F_L over $\mathbb{F}_{2^n}$ for $3 \leq n \leq 12$ are affine equivalent to some Gold maps.

$$F_L(x) = x^{2^i} L(x) + x L(x)^{2^i}$$

For $n = km$, $L(x) = \sum_{i=0}^{k-1} b_i x^{2^{im}}$ is a 2^m-polynomial.

Theorem

Characterisation of APN maps F_L over $\mathbb{F}_{2^n}$ with $n = km$, $\gcd(n, i) = 1$ and L a 2^m-polynomial.

$F_L(x) = x^{2^i} L(x) + xL(x)^{2^i}$

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Constructed APN functions over $\mathbb{F}_{2^n}$ for $n = 6, 8, 9, 12, 18$.

- For $n = 9$ $(k = m = 3)$ and $i = 1$, constructed F_L APN and not CCZ-equivalent to any known map (**new**).
- For $n = 8$ $(k = 4,\ m = 2)$ and $i = 3$ constructed $F_L \overset{EA}{\sim} x^9 + \mathrm{Tr}(x^3)$, APN map known since 2006 but not part of any known family (**unclassified**).

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Conjecture

The Theorem covers APN functions for an infinite number of dimensions n.

The case $n = 6$

The isotopic shift connects any two quadratic APN maps over $\mathbb{F}_{2^6}$

For any $F, G \in \mathbb{F}_{2^6}[x]$ quadratic APN functions there exist L, L' linear maps with L permutation and L' 2-to-1 such that F_L and $F_{L'}$ are EA-equivalent to G.

Isotopic shift with L linear

$$F_L(x) = F(x + L(x)) + F(x) + F(L(x)).$$

When F is a Gold map,

$$F_L(x) = xL(x)^{2^i} + x^{2^i} L(x).$$

Isotopic shift with L linear

$$F_L(x) = F(x + L(x)) + F(x) + F(L(x)).$$

When F is a Gold map,

$$F_L(x) = x L_1(x)^{2^i} + x^{2^i} L_2(x).$$

Isotopic shift with L ~~linear~~ not linear

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Theorem

Characterisation of APN maps $xL_1(x)^{2^i} + x^{2^i}L_2(x)$ over $\mathbb{F}_{2^n}$ with $n = km$, $\gcd(i, m) = 1$ and L_1, L_2 2^m-polynomials.

$xL_1(x)^{2^i} + x^{2^i}L_2(x)$ with L_1, L_2 linear

Theorem

Characterisation of APN maps $xL_1(x)^{2^i} + x^{2^i}L_2(x)$ over $\mathbb{F}_{2^n}$ with $n = km$, $\gcd(i, m) = 1$ and L_1, L_2 2^m-polynomials.

- For $n = 8$ ($k = 4$, $m = 2$) with L_1, L_2 with coefficients over $\mathbb{F}_{2^4}$ many APN functions constructed (**several unclassified**).
- For $n = 9$ ($k = m = 3$) with L_1, L_2 with coefficients over $\mathbb{F}_{2^3}$ many APN functions constructed (**15 new** and **one unclassified**).

F_L with L not necessarily linear

F_L with L not necessarily linear

Theorem

Over $\mathbb{F}_{2^n}$ with n odd, F any known APN power function (except Dobbertin) there exists a monomial $L(x) = ax^d$ and a Gold map $G(x) = x^{2^i+1}$ such that $G_L \overset{EA}{\sim} F$.

New EA-invariant

New EA-invariant

$F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$

$$S(F) = \{b \in \mathbb{F}_{2^n} : \exists a \in \mathbb{F}_{2^n} \ s.t. \ \mathcal{W}_F(a, b) = 0\}$$

Proposition

Let N_i be the number of $\mathbb{F}_2$-vector subspaces of $\mathbb{F}_{2^n}$ contained in $S(F)$ of dimension i. The values N_i for $i = 0, \ldots, n$ are EA-invariant.

$\mathcal{W}_F(a, b) = \Sigma_{x \in \mathbb{F}_{2^n}} (-1)^{\mathrm{Tr}(ax+bF(x))}$ is the Walsh transform of F in (a, b)

Known classes of quadratic APN polynomial over $\mathbb{F}_{2^n}$ CCZ-inequivalent to power functions

N°	Functions
C1-C2	$x^{2^s+1}+u^{2^k-1}x^{2^{ik}+2^{mk+s}}$
C3	$x^{2^{2i}+2^i}+cx^{q+1}+dx^{q(2^{2i}+2^i)}$
C4	$x(x^{2^i}+x^q+cx^{2^iq})+$ $x^{2^i}(c^qx^q+sx^{2^iq})+x^{(2^i+1)q}$
C5	$x^3+a^{-1}Tr(a^3x^9)$
C6	$x^3+a^{-1}Tr_n^3(a^3x^9+a^6x^{18})$
C7	$x^3+a^{-1}Tr_n^3(a^6x^{18}+a^{12}x^{36})$
C8-C10	$ux^{2^s+1}+u^{2^k}x^{2^{-k}+2^{k+s}}+$ $vx^{2^{-k}+1}+wu^{2^k+1}x^{2^s+2^{k+s}}$
C11	$dx^{2^i+1}+d^qx^{q(2^i+1)}+$ $cx^{q+1}+\sum_{s=1}^{m-1}\gamma_sx^{2^s(q+1)}$
C17	$L_1(x)^{2^i}x+L_2(x)x^{2^i}$

N°	Functions
C12	$(x+x^q)^{2^i+1}+$ $u'(ux+u^qx^q)^{(2^i+1)2^j}+$ $u(x+x^q)(ux+u^qx^q)$
C13	$(u^{q+1}x+u^2x^q)(x^q+x)+$ $(u^qx+ux^q)^{2^{2i}+2^{3i}}+b(x^q+x)^{2^i+1}+$ $a(u^qx+ux^q)^{2^{2i}}(x^q+x)^{2^i}$
C14	$x^3+ax^{2^k(2^i+1)}+$ $bx^{3\cdot 2^m}+cx^{2^{n+k}(2^i+1)}$
C15	$u[t(x)^{2^i+1}+t(x)(x^q+x)^{2^i}+(x^q+x)^{2^i+1}]$ $+t(x)^{2^{2i}+1}+t(x)^{2^{2i}}(x^q+x)+(x^q+x)^{2^{2i}+1}$
C16	$u[t(x)^{2^i+1}+t(x)(x^q+x)^{2^i}+(x^q+x)^{2^i+1}]$ $+t(x)^{2^{3i}}(x^q+x)+t(x)(x^q+x)^{2^{3i}}$

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L. Budaghyan, T. Helleseth, N. Li, B. Sun *Some results on the known classes of quadratic APN functions.* Codes, Cryptology and Information Security - C2SI 2017 (2017).

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APN functions over $\mathbb{F}_{2^n}$:

- (C3)[1] $cx^{2^m+1} + x^{2^{2i}+2^i} + dx^{2^m(2^{2i}+2^i)}$

  with $n = 2m$, $\gcd(i, m) = 1$, $d^{2^m+1} = 1$, $d \notin \{\lambda^{(2^i+1)(2^m+1)} : \lambda \in \mathbb{F}_{2^n}\}$, $dc^{2^m} + c \neq 0$.

- (C11)[2] $cx^{2^m+1} + \sum_{s=1}^{m-1} \gamma_s x^{2^s(2^m+1)} + dx^{2^i+1} + d^{2^m} x^{2^m(2^i+1)}$

  with $n = 2m$, i odd, $\gcd(m, i) = 1$, $c \notin \mathbb{F}_{2^m}$, d not a cube, $\gamma_s \in \mathbb{F}_{2^m}$.

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[1] L. Budaghyan, C. Carlet *Classes of quadratic APN trinomials and hexanomials and related structures*. IEEE Transaction on Information Theory 54, 5 (2008).

[2] C. Bracken, E. Byrne, N. Markin, G. McGuire *New families of quadratic almost perfect nonlinear trinomials and multinomials*. Finite Fields and their Applications 14, 3 (2008).

[3] X.Y. Duan, Y.L. Deng *Two classes of quadratic crooked functions*. Applied Mechanics and Materials 513 (2014).

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- (C3*)[3] $cx^{2^m+1} + \sum_{l=1}^{m-1} \gamma_l x^{2^l(2^m+1)} + x^{2^i+2^j} + dx^{2^m(2^i+2^j)}$

  with $n = 2m$, m odd, $i > j$, $\gcd(i-j, m) = 1$, $d^{2^m+1} = 1$, $d \notin \{\lambda^{(2^i+2^j)(2^m-1)} : \lambda \in \mathbb{F}_{2^n}\}$, $dc^{2^m} + c \neq 0$, $d = \gamma_l^{1-2^m}$.

- (C11)[2] $cx^{2^m+1} + \sum_{s=1}^{m-1} \gamma_s x^{2^s(2^m+1)} + dx^{2^i+1} + d^{2^m} x^{2^m(2^i+1)}$

  with $n = 2m$, i odd, $\gcd(m, i) = 1$, $c \notin \mathbb{F}_{2^m}$, d not a cube, $\gamma_s \in \mathbb{F}_{2^m}$.

- (C11*)[3] $cx^{2^m+1} + \sum_{l=1}^{m-1} \gamma_l x^{2^l(2^m+1)} + L(dx^{2^i+2^j} + d^{2^m} x^{2^m(2^i+2^j)})$

  with $n = 2m$, m odd, $i > j$, $\gcd(i-j, m) = 1$, $c \notin \mathbb{F}_{2^m}$, $d \notin \{x^{2^i+2^j} : x \in \mathbb{F}_{2^n}\}$, $\gamma_l \in \mathbb{F}_{2^m}$, L linear permutation with coefficients in $\mathbb{F}_{2^m}$.

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Theorem

The families C3, C3, C11, C11* coincide and they are included in C4. In particular, the hexanomials (C4) admit a representation as pentanomials*

$$H(x) = dx^{2^m+1} + x^{2^i+1} + x^{2^m(2^i+1)} + cx^{2^{m+i}+1} + c^{2^m}x^{2^m+2^i},$$

$n = 2m$, $\gcd(m,i) = 1$, $d \notin \mathbb{F}_{2^m}$ *and* $x^{2^i+1} + cx^{2^i} + c^{2^m}x + 1 = 0$ *has no solution* x *such that* $x^{2^m+1} = 1$.

The updated list

Known classes of quadratic APN polynomial over $\mathbb{F}_{2^n}$ CCZ-inequivalent to power functions

N°	Functions	N°	Functions
F1-F2	$x^{2^s+1}+u^{2^k-1}x^{2^{ik}+2^{mk+s}}$	F11	$L_1(x)^{2^i}x+L_2(x)x^{2^i}$
F3	$sx^{q+1}+x^{2^i+1}+x^{q(2^i+1)}+$ $cx^{2^iq+1}+c^qx^{2^i+q}$	F12	$(u^{q+1}x+u^2x^q)(x^q+x)+$ $(u^qx+ux^q)^{2^{2i}+2^{3i}}+b(x^q+x)^{2^i+1}+$ $a(u^qx+ux^q)^{2^{2i}}(x^q+x)^{2^i}$
F4	$x^3+a^{-1}Tr(a^3x^9)$		
F5	$x^3+a^{-1}Tr_n^3(a^3x^9+a^6x^{18})$	F13	$x^3+ax^{2^k(2^i+1)}+$ $bx^{3\cdot 2^m}+cx^{2^{n+k}(2^i+1)}$
F6	$x^3+a^{-1}Tr_n^3(a^6x^{18}+a^{12}x^{36})$		
F7-F9	$ux^{2^s+1}+u^{2^k}x^{2^{-k}+2^{k+s}}+$ $vx^{2^{-k}+1}+wu^{2^k+1}x^{2^s+2^{k+s}}$	F14	$u[t(x)^{2^i+1}+t(x)(x^q+x)^{2^i}+(x^q+x)^{2^i+1}]$ $+t(x)^{2^{2i}+1}+t(x)^{2^{2i}}(x^q+x)+(x^q+x)^{2^{2i}+1}$
F10	$(x+x^q)^{2^i+1}+$ $u'(ux+u^qx^q)^{(2^i+1)2^j}+$ $u(x+x^q)(ux+u^qx^q)$	F15	$u[t(x)^{2^i+1}+t(x)(x^q+x)^{2^i}+(x^q+x)^{2^i+1}]$ $+t(x)^{2^{3i}}(x^q+x)+t(x)(x^q+x)^{2^{3i}}$

Finite presemifield $\mathcal{S} = (\mathbb{F}_{p^n}, +, \star)$

$\mathcal{S}$ is a ring with left and right distributivity and no zero divisor (not necessarily associative).

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Isotopic equivalence

$\mathcal{S}_1 = (\mathbb{F}_{p^n}, +, \star)$ and $\mathcal{S}_2 = (\mathbb{F}_{p^n}, +, *)$ are isotopic equivalent if there exist M, N, L linear permutations such that for any $x, y \in \mathbb{F}_{p^n}$

$$T(x \star y) = M(x) * N(y).$$

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$$T(x \star y) = M(x) * N(y).$$

For p odd:

- Given $\mathcal{S}$ commutative, then $F_{\mathcal{S}}(x) = \frac{1}{2} x \star x \in \mathbb{F}_{p^n}[x]$ is planar DO.
- Given $F \in \mathbb{F}_{p^n}[x]$ planar DO polynomial, then $\mathcal{S}_F = (\mathbb{F}_{p^n}, +, \star)$ with $x \star y = F(x+y) - F(x) - F(y)$ is a commutative presemifield.

Quadratic planar functions $F : \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{S}_1 = (\mathbb{F}_{p^n}, +, \star) \arrow[r, dash, "\text{isotopic equivalent}", "T(x \star y) = M(x) * N(y)"'] \arrow[d, leftrightarrow] & \mathcal{S}_2 = (\mathbb{F}_{p^n}, +, *) \arrow[d, leftrightarrow] \\
F_{\mathcal{S}_1} & F_{\mathcal{S}_2}
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- Isotopic equivalence can be extended to planar DO polynomials;
  [F, G isotopic equivalent if $\mathcal{S}_F, \mathcal{S}_G$ isotopic equivalent]
- Isotopic equivalence is more general than CCZ-equivalence;
  [CCZ-equivalence corresponds to $M = N$]
- two planar functions are CCZ-equivalent if and only if they are EA-equivalent.

Isotopic shift of quadratic planar maps over $\mathbb{F}_{p^n}$

$$\boxed{F_L(x) = F(x + L(x)) - F(x) - F(L(x))}$$

- It describes completely (up to EA-equivalence) the isotopic equivalence ($F \overset{isot}{\sim} G \to G \overset{EA}{\sim} F_L$ for some L linear permutation).

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- If F_L is planar then L must be a permutation.

Given $F, G \in \mathbb{F}_{p^n}[x]$ quadratic maps, does the existence of a linear isotopic shift connecting F to G ($F_L \overset{EA}{\sim} G$) imply the existence of a linear isotopic shift connecting G to F?

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YES over $\mathbb{F}_{3^3}, \mathbb{F}_{5^3}, \mathbb{F}_{7^3}$, with $j = 1$

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The isotopic shift (together with EA-transformation), even when applied to quadratic planar functions, is not an equivalence relation.

Isotopic shift over $\mathbb{F}_{p^n}$

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Proposition ($F(x) = x^2$)

For $n = 3m$ and L a p^m-polynomial with coefficients in $\mathbb{F}_{p^m}$, if $F_L(x) = 2xL(x)$ is planar then it is affine equivalent to x^2 or to x^{p^m+1}.

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Theorem

Characterisation of planar maps $xL_1(x)^{p^i} + x^{p^i}L_2(x)$ over $\mathbb{F}_{p^n}$ with $n = km$, $m/\gcd(m,i)$ odd and L_1, L_2 p^m-polynomials.

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Possible future works

- study further the isotopic shift construction over $\mathbb{F}_{2^n}$
  - construct new families of differentially 4-uniform permutations
- study further the isotopic shift construction over $\mathbb{F}_{p^n}$
  - construct new planar families

Thank you for your attention