On equivalence relations of Boolean functions

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Notations and definitions

PN and APN functions:

Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$ be a Vectorial Boolean function. We define

$$\delta_F(a, b) = |\{x \in \mathbb{F}_2^n \, : \, F(x + a) - F(x) = b\}.$$

The **differential uniformity** of F is

$$\delta(F) = \max_{a \in \mathbb{F}_2^n \setminus \{0\}, \, b \in \mathbb{F}_2^m} \delta_F(a, b).$$

If $\delta(F) = 2^{n-m}$ then F is said **Perfect Nonlinear** (PN) or **Bent**.
Best resistance to differential attack.

K. Nyberg: Bent functions exist only when n is even and $m \leq n/2$.

If $m = n$, then $\delta(F) \geq 2$.

If $\delta(F) = 2$, then F is called **almost perfect nonlinear** (APN).

AB functions:

The **nonlinearity** of a vectorial Boolean function F is the minimum Hamming distance between

- all component functions $v \cdot F(x)$, $v \neq 0$ and
- all affine functions $u \cdot x + \varepsilon$, $u \in \mathbb{F}_2^n$ $\varepsilon \in \mathbb{F}_2$.

The nonlinearity can be given in terms of the **Walsh transform** of F

$$\mathcal{W}_F(a,b) = \sum_{x \in \mathbb{F}_2^n} (-1)^{a \cdot x + b \cdot F(x)}.$$

The nonlinearity equals:

$$\mathcal{N}\ell(F) = 2^{n-1} - \frac{1}{2} \max_{\substack{a \in \mathbb{F}_2^n, \\ b \in \mathbb{F}_2^m \setminus \{0\}}} |\mathcal{W}_F(a,b)|.$$

Bounds on nonlinearity

$$\mathcal{N}\ell(F) \leq 2^{n-1} - 2^{n/2-1}.$$

The equality holds iff F is bent (best resistance to linear attack).
If $n = m$ the Sidelnikov-Chabaud-Vaudenay bound states

$$\mathcal{N}\ell(F) \leq 2^{n-1} - 2^{\frac{n-1}{2}}.$$

In case of equality (n necessarily odd) F is called **almost bent** (AB).

AB $\Rightarrow$ APN

From now on, we assume that $m = n$. In this case we can identify $\mathbb{F}_2^n$ with $\mathbb{F}_{2^n}$ and then we can take $x \cdot y = tr(xy)$.

Table: Known APN power functions x^d over $\mathbb{F}_{2^n}$

Functions	Exponents d	Conditions	Degree
Gold	2^i+1	$\gcd(i,n)=1$	2
Kasami	$2^{2i}-2^i+1$	$\gcd(i,n)=1$	$i+1$
Welch	2^t+3	$n=2t+1$	3
Niho	$2^t+2^{\frac{t}{2}}-1$, t even $2^t+2^{\frac{3t+1}{2}}-1$, t odd	$n=2t+1$	$\frac{t+2}{2}$ $t+1$
Inverse	$2^{2t}-1$	$n=2t+1$	$n-1$
Dobbertin	$2^{4i}+2^{3i}+2^{2i}+2^i-1$	$n=5i$	$i+3$

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Gold, Kasami, Welch and Niho functions are AB for n odd

Equivalence relations

Two functions $F, G : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ are **EA-equivalent** iff

$$G = A_2 \circ F \circ A_1(x) + A(x),$$

with A, A_1 and A_2 affine maps and A_1 and A_2 permutations.

Let $\Gamma_f = \{(x, f(x)) \mid x \in \mathbb{F}_{2^n}\}$.
Two functions $F, G : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ are **CCZ-equivalent** if and only if Γ_F and Γ_G are affine-equivalent, i.e. let $\mathcal{L}$ an affine permutation on $(\mathbb{F}_{2^n})^2$, $\mathcal{L}(\Gamma_F) = \Gamma_G$.
EA and CCZ-equivalence preserve the nonlinearity and the differential uniformity.

CCZ-equivalence

Let $\mathcal{L}$ be a linear permutation of $(\mathbb{F}_{2^n})^2$ such that $\mathcal{L}(\Gamma_F) = \Gamma_G$.
$\mathcal{L} = (L_1, L_2)$ for some linear $L_1, L_2 : (\mathbb{F}_{2^n})^2 \to \mathbb{F}_{2^n}$. Then

$$\mathcal{L}(x, F(x)) = (F_1(x), F_2(x)),$$

where $F_1(x) = L_1(x, F(x))$ and $F_2(x) = L_2(x, F(x))$.

$$\mathcal{L}(\Gamma_F) = \{(F_1(x), F_2(x)) \, : \, x \in \mathbb{F}_{2^n}\}.$$

$\mathcal{L}(\Gamma_F)$ is the graph of G iff the function F_1 is a permutation and $G = F_2 \circ F_1^{-1}$

EA-equivalence $\subset$ CCZ-equivalence

EA$\Rightarrow$ CCZ:

- If $(L_1(x,y), L_2(x,y)) = (x, A(x)+y)$ then $\mathcal{L}(x, F(x)) = (x, F(x)+A(x))$ and $G(x) = F(x)+A(x)$.
- If $(L_1(x,y), L_2(x,y)) = (A(x), y)$ then $\mathcal{L}(x, F(x)) = (A(x), F(x))$ and $G(x) = F \circ A^{-1}(x)$.
- If $(L_1(x,y), L_2(x,y)) = (x, A(y))$ then $\mathcal{L}(x, F(x)) = (x, A \circ F(x))$ and $G(x) = A \circ F(x)$.

inversion is a particular case of CCZ:

- $(L_1(x,y), L_2(x,y)) = (y, x)$ then $\mathcal{L}(x, F(x)) = (F(x), x)$ and $G(x) = F^{-1}(x)$.

Relation between CCZ- and EA-equivalences

Cases when CCZ-equivalence coincides with EA-equivalence:

- Boolean functions, $m = 1$. (Budaghyan and Carlet)
- Bent functions. (Budaghyan and Carlet)
- Two quadratic APN functions. (Yoshiara)
- A power function F is CCZ-equivalent to a power function F' iff F is EA-equivalent to F' or F'^{-1}. (for APN and $p = 2$ Yoshiara, any p and any power Dempwolff)
- A quadratic APN function is CCZ-equivalent to a power function iff it is EA-equivalent to one of the Gold functions. (Yoshiara)
- If n is even, a plateaued APN function is CCZ-equivalent to a plateaued power function iff it is EA-equivalent to it. (Yoshiara)

Cases when CCZ-equivalence differs from EA-equivalence:

- For functions from $\mathbb{F}_2^n$ to $\mathbb{F}_2^m$ with $m \geq 2$.

EA-equivalence preserves algebraic degree while inverse and CCZ-equivalence do not.

Relation between CCZ and EA-equivalence + Inverse

Proposition (L. Budaghyan, C. Carlet, A. Pott)

G is EA-equivalent to the function F or to F^{-1} (if it exists) iff there exists a linear permutation $\mathcal{L} = (L_1, L_2)$ on $(\mathbb{F}_{2^n})^2$ such that $\mathcal{L}(\Gamma_F) = \Gamma_G$ and $L_1(x, y) = L(x)$ or $L_1(x, y) = L(y)$.

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If we want to construct G which cannot be constructed from F via EA-equivalence and inverse transformation:

- To find a permutation $L_1(x, F(x)) = L(x) + R \circ F(x)$ where $L, R \neq 0$ are linear.
- Then find linear function $L_2(x, y) = L'(x) + R'(y)$such that $\mathcal{L}$ is a permutation. (Found L_1 then there always exists suitable L_2)

Fixed L_1 , different L' and R' produce EA-equivalent functions.

The condition that L_1 depends on both variables is necessary but not sufficient.

Example: Let $n = 2m + 1$ and $s = m \mod 2$. Then

$$\mathcal{L}(x, y) = (x + tr(x) + \sum_{i=0}^{m-s} y^{2^{2j+s}}, y + tr(x))$$

is a linear permutation and maps the graph of $F(x) = x^3$ to the graph of G which is EA-equivalent to F^{-1}.

CCZ-equivalence more general than EA-equivalence with inverse transformation

APN functions CCZ-equivalent to Gold functions and EA-inequivalent to power functions on $\mathbb{F}_{2^n}$

Function	conditions
$x^{2^i+1} + (x^{2^i} + x + tr(1) + 1)tr(x^{2^i} + 1 + xtr(1))$	$n \geq 4$, $\gcd(n, i) = 1$
$[x + tr_3^n(x^{2(2^i+1)} + x^{4(2^i+1)}) + tr(x)tr_3^n(x^{2^i+1} + x^{2^i(2^i+1)})]^{2^i+1}$	$6 \mid n$, $\gcd(n, i) = 1$
$x^{2^i+1} + tr_m^n(x^{2^i+1}) + x^{2^i}tr_m^n(x) + xtr_m^n(x)^{2^i}$ $+[tr_m^n(x)^{2^i+1} + tr_m^n(x^{2^i+1}) + tr_m^n(x)]^{1/(2^i+1)}](x^{2^i} + tr_m^n(x^{2^i}) + 1)$ $+[tr_m^n(x)^{2^i+1} + tr_m^n(x^{2^i+1}) + tr_m^n(x)]^{2^i/(2^i+1)}](x + tr_m^n(x)+)$	n odd, $m \mid n$ $\gcd(n, i) = 1$

Only for Gold functions it is known that CCZ>EA+inverse. For the rest of power functions it is an open problem.

A procedure for investigating if $CCZ \stackrel{?}{=} EA + Inv$

Let $L_1(x, y) = L(x) + R(y)$. $F_1(x) = L(x) + R(F(x))$ is a permutation iff any of its component is balanced.
In terms of Walsh coefficients

$$\mathcal{W}_{F_1}(0, \lambda) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{tr(\lambda L(x) + \lambda R \circ F(x))} = 0, \quad \text{for all } \lambda \in \mathbb{F}_{2^n}^*.$$

$$\Downarrow$$

$$\mathcal{W}_{F_1}(0, \lambda) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{tr(L^*(\lambda)x + R^*(\lambda)F(x))} = \mathcal{W}_F(L^*(\lambda), R^*(\lambda)).$$

(L^* is the adjoint operator)

We want to construct L^* and R^* so that F_1 is a permutation.
Let $\mathcal{ZW}(b) = \{a \mid \mathcal{W}_F(a, b) = 0\}$ for any $b \in \mathbb{F}_{2^n}$ and consider

$$S_F = \{b : \mathcal{ZW}(b) \neq \emptyset\}.$$

Note: if F_1 is a permutation then $Im(R^*) \subseteq S_F$.
For constructing F_1 we need to consider the possible vector subspaces contained in S_F.

Construction of R^*

Let $U \subseteq S_F$ be a vector subspace. Fixed any basis $\{u_1, \ldots, u_k\}$ of U, we can suppose that $R^*(e_i) = u_i$ for $i = 1, ..., k$ and $\mathrm{Ker}(R^*) = \mathrm{Span}(e_{k+1}, ..., e_n)$.
(e_i is the canonical vector.)

Fixed any basis $\{u_1, \ldots, u_k\}$ of U we can suppose that

$$R^* = \begin{bmatrix} u_1 \\ \vdots \\ u_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Construction of L^*

For any $a_1, ..., a_k$ with $a_1 \in \mathcal{ZW}(u_1), ..., a_k \in \mathcal{ZW}(u_k)$ we need to check if

(P1) $\sum_{i=1}^{k} \lambda_i a_i \in \mathcal{ZW}(\sum_{i=1}^{k} \lambda_i u_i)$ with $\lambda_i \in \mathbb{F}_2$ not all zero.

and if there exist $a_{k+1}, ..., a_n$ satisfying

(P2) $a_{k+1}, ..., a_n$ are linear independent;

(P3) for any $a \in Span(a_{k+1}, ..., a_n)$, $a + \sum_{i=1}^{k} \lambda_i a_i \in \mathcal{ZW}(\sum_{i=1}^{k} \lambda_i u_i)$, for any $\lambda_1, \ldots, \lambda_k \in \mathbb{F}_2$.

Then,

$$L^* = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Proposition

Let U be a subspace contained in S_F. Then, there exists a permutation of $\mathbb{F}_{2^n}$ $F_1(x) = L(x) + R \circ F(x)$, with L and R linear and $\mathrm{Im}(R^*) = U$ iff the procedure above is successful.

Budaghyan, Carlet, Pott

$$F' \sim_{EA} F \Leftrightarrow \exists \mathcal{L} = \begin{bmatrix} A_1 & 0 \\ A_3 & A_4 \end{bmatrix},$$

with A_1 and A_4 permutations and $\mathcal{L}(\Gamma_F) = \Gamma_{F'}$.

$$F' \sim_{EA} F^{-1} \overset{\text{inv}}{\rightarrow} F \Leftrightarrow \exists \mathcal{L} = \begin{bmatrix} 0 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

A_2 and A_3 permutations and $\mathcal{L}(\Gamma_F) = \Gamma_{F'}$.

Proposition

$$F' \sim_{EA} G \overset{inv}{\to} G^{-1} \sim_{EA} F \Leftrightarrow \exists\, \mathcal{L} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

A_2 a permutation and $A_1 \neq 0$ and $\mathcal{L}(\Gamma_F) = \Gamma_{F'}$.

$$F' \sim_{EA} G \overset{inv}{\to} G^{-1} \sim_{EA} F^{-1} \overset{inv}{\to} F, \Leftrightarrow \exists\, \mathcal{L} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

A_1 a permutation and $A_2 \neq 0$ and $\mathcal{L}(\Gamma_F) = \Gamma_{F'}$.

Note: G is a permutation.

Theorem

Let F be a function from $\mathbb{F}_{2^n}$ to itself. If for any nonzero vector subspace U in S_F different from $\mathbb{F}_{2^n}$ it is not possible to construct any matrix $A_1^* \neq 0$ with the procedure above, then any function F' CCZ-equivalent to F can be obtained from F applying only EA-equivalence and inverse transformation (when applicable) iteratively.

Theorem

Let F be a permutation over $\mathbb{F}_{2^n}$. If for any nonzero vector subspace U in S_F different from $\mathbb{F}_{2^n}$ it is not possible to construct a matrix $A_1^* \neq 0$ of $\mathrm{rank}(A_1^*) < n$ with the procedure above, then any function F' CCZ-equivalent to F can be obtained from F applying only EA-equivalence and inverse (when applicable) transformation iteratively.

Application to non-quadratic functions

Let $n = 6$, and $F : \mathbb{F}_{2^n} \to \mathbb{F}_{2^n}$ be

$$\begin{aligned} F(x) =& x^3 + u^{17}(x^{17} + x^{18} + x^{20} + x^{24}) \\ &+ u^{14}((u^{52}x^3 + u^6x^5 + u^{19}x^7 + u^{28}x^{11} + u^2x^{13}) + \\ &(u^{52}x^3 + u^6x^5 + u^{19}x^7 + u^{28}x^11 + u^2x^{13})^2 \\ &+ (u^{52}x^3 + u^6x^5 + u^{19}x^7 + u^{28}x^11 + u^2x^{13})^4 + \\ &(u^{52}x^3 + u^6x^5 + u^{19}x^7 + u^{28}x^11 + u^2x^{13})^8 \\ &+ (u^{52}x^3 + u^6x^5 + u^{19}x^7 + u^{28}x^11 + u^2x^{13})^{16} + \\ &(u^{52}x^3 + u^6x^5 + u^{19}x^7 + u^{28}x^11 + u^2x^{13})^{32} \\ &+ (u^2x)^9 + (u^2x)^{18} + (u^2x)^{36} + x^{21} + x^{42}), \end{aligned}$$

where u is a primitive element of $\mathbb{F}_{2^n}$.
F is the first example of APN function CCZ-inequivalent to a quadratic function.

Using the procedure it is possible to construct the functions L and R given by

$$L(x) = u^{50}x^{32} + u^{51}x^{16} + u^{43}x^8 + ux^4 + u^{26}x^2 + u^{26}x$$

and

$$R(x) = u^{26}x^{32} + u^{17}x^{16} + u^{56}x^8 + u^9x^4 + u^{54}x^2 + u^{46}x,$$

Considering the function $F_2(x) = L_2(x, F(x)) = F(x)$
we have

$$\begin{aligned}
F'(x) =& u^{41}x^{60} + u^{29}x^{58} + u^{46}x^{57} + u^3x^{56} + u^{39}x^{54} + u^{47}x^{53} \\
&+ u^3x^{52} + u^{62}x^{51} + u^{54}x^{50} + u^{62}x^{49} + u^{53}x^{48} + u^{14}x^{46} \\
&+ u^{39}x^{45} + u^{20}x^{44} + u^{26}x^{43} + u^{11}x^{42} + u^{31}x^{41} + u^{53}x^{40} \\
&+ u^{59}x^{39} + u^{53}x^{38} + u^{41}x^{37} + u^{19}x^{36} + u^{58}x^{35} + u^2x^{34} + \\
&u^7x^{33} + u^{39}x^{32} + u^{15}x^{30} + u^{17}x^{29} + u^{45}x^{28} + u^{39}x^{27} \\
&+ u^{57}x^{26} + u^{33}x^{25} + u^{61}x^{24} + u^{41}x^{23} + u^{50}x^{22} + u^{58}x^{21} \\
&+ u^{55}x^{20} + u^{26}x^{19} + u^{17}x^{18} + u^{37}x^{17} + u^{30}x^{16} + ux^{15} \\
&+ u^{46}x^{14} + u^{21}x^{13} + u^{13}x^{12} + u^{61}x^{11} + u^{20}x^{10} + x^9 + u^{61}x^8
\end{aligned}$$

The function F' cannot be constructed from F via EA-equivalence and inverse transformation.
F has algebraic degree equals to 3 and F' equals to 4.
Moreover to apply the inverse transformation at least once we need $F \sim_{EA} G$ with G permutation, but since F has quadratic components this cannot be possible.

Then we have that CCZ>EA+inversion also for APN functions inequivalent to quadratic functions

Note: F has quadratics components, that may be useful to crate the function F_1.

Why quadratic components could help

P. Charpin and G.M. Kyureghyan (2008) characterized permutation polynomial of type

$$F'(x) = G(x) + b \cdot tr(H(x))$$

with G a permutation.
In particular, consider a function F and let b a 0-linear structure of $tr(\lambda F)$ (i.e. $tr(\lambda F(x) + \lambda F(x+b)) \equiv 0$) we obtain that

$$F_1(x) = x + b \cdot tr(\lambda F(x))$$

is a permutation.
Moreover, if b a 1-linear structure of $tr(\lambda F)$ and $tr(b) = 1$ then

$$F_1'(x) = x + b \cdot tr(\lambda F(x) + x)$$

is a permutation.

$F_1(x)$ and $F_1'(x)$ are involution.

$$\begin{aligned} F_1 \circ F_1(x) =& x + b\, tr(\lambda F(x)) + b\, tr(\lambda F(x + b\, tr(\lambda F(x)))) \\ =& \begin{cases} x & \text{if } tr(\lambda F(x)) = 0 \\ x + b\, tr(\lambda F(x) + \lambda F(x + b)) = x & \text{if } tr(\lambda F(x)) = 1 \end{cases} \end{aligned}$$

Considering $F_2(x) = F(x)$ we obtain

$$F_2 \circ F_1(x) = F(x) + tr(\lambda F(x))(F(x) + F(x + b)),$$

or

$$F_2 \circ F_1'(x) = F(x) + tr(\lambda F(x) + x)(F(x) + F(x + b)).$$

APN power functions

Power functions

Let $n = 7$ and $F(x) = x^d$ with d not a Gold exponent, i,e, $d = 11, 13, 39, 57, 126$. Then, in these cases the CCZ-equivalence coincide with the EA-equivalence and the inverse transformation.

Let $n = 8$ and $F(x) = x^{57}$ (Kasami). Then in this case the CCZ-equivalence coincide with the EA-equivalence and the inverse transformation.

Classification of APN functions

APN polynomial families CCZ-inequivalent to power functions

N°	Functions	Conditions
C1-C2	$x^{2^s+1} + u^{2^k-1}x^{2^{ik}+2^{mk+s}}$	$n = pk$, $\gcd(k,p)=\gcd(s,pk)=1$, $p \in \{3,4\}$, $i = sk \bmod p$, $m = p-i$, $n \geq 12$, u primitive in $\mathbb{F}_{2^n}^*$
C3	$x^{2^{2i}+2^i} + cx^{q+1} + dx^{q(2^{2i}+2^i)}$	$q = 2^m$, $n = 2m$, $\gcd(i,m)=1$, $\gcd(2^i+1, q+1) \neq 1$, $dc^q + c \neq 0$, $d \notin \{\lambda^{(2^i+1)(q-1)}, \lambda \in \mathbb{F}_{2^n}\}$, $d^{q+1} = 1$
C4	$x(x^{2^i} + x^q + cx^{2^i q}) + x^{2^i}(c^q x^q + sx^{2^i q}) + x^{(2^i+1)q}$	$q = 2^m$, $n = 2m$, $\gcd(i,m)=1$, $c \in \mathbb{F}_{2^n}$, $s \in \mathbb{F}_{2^n} \setminus \mathbb{F}_q$, $X^{2^i+1} + cX^{2^i} + c^q X + 1$ has no solutions s.t. $x^{q+1} = 1$
C5	$x^3 + a^{-1}Tr(a^3x^9)$	$a \neq 0$
C6	$x^3 + a^{-1}Tr_n^3(a^3x^9 + a^6x^{18})$	$3 \mid n$, $a \neq 0$
C7	$x^3 + a^{-1}Tr_n^3(a^6x^{18} + a^{12}x^{36})$	$3 \mid n$, $a \neq 0$

Classification of APN functions

C8-C10	$ux^{2^s+1} + u^{2^k}x^{2^{-k}+2^{k+s}} + vx^{2^{-k}+1} + wu^{2^k+1}x^{2^s+2^{k+s}}$	$n = 3k$, $\gcd(k,3)=\gcd(s,3k)=1$, $v, w \in \mathbb{F}_{2^k}$, $vw \neq 1$, $3\|(k+s)$ u primitive in $\mathbb{F}_{2^n}^*$
C11	$dx^{2^s+1} + d^{2^k}x^{2^{k+s}+2^k} + cx^{2^k+1} + \sum_{i+1}^{k-1}\gamma_i x^{2^{k+i}+2^i}$	$n = 2k$, $\gcd(s,k)=1$, s,k odd, $c \notin \mathbb{F}_{2^k}$, $\gamma_i \in \mathbb{F}_{2^k}$, d not a cube
C12	$(x+x^{2^m})^{2^k+1} + u'(ux+u^{2^m}x^{2^m})^{(2^k+1)2^i} + u(x+x^{2^m})(ux+u^{2^m}x^{2^m})$	$n = 2m$, $m \geq 2$ even, $\gcd(k,m) = 1$ and i even u primitive in $\mathbb{F}_{2^n}^*$, $u' \in \mathbb{F}_{2^m}$ not cube
C13	$x^{2^k+1} + tr_m^n(x)^{2^k+1}$	$n = 2m = 4t$, $\gcd(k,n) = 1$
C14	$a^2x^{2^{2m+1}+1} + b^2x^{2^{m+1}+1} + ax^{2^{2m}+2} + bx^{2^m+2} + (c^2+c)x^3$	$n = 3m$, m odd Irene Villa's talk

Classification of APN functions

C8-C10	$ux^{2^s+1} + u^{2^k}x^{2^{-k}+2^{k+s}} +$ $vx^{2^{-k}+1} + wu^{2^k+1}x^{2^s+2^{k+s}}$	$n = 3k$, $\gcd(k,3)$=$\gcd(s,3k)$=1, $v, w \in \mathbb{F}_{2^k}$, $vw \neq 1$, $3\|(k+s)$ u primitive in $\mathbb{F}_{2^n}^*$
C11	$dx^{2^s+1} + d^{2^k}x^{2^{k+s}+2^k} +$ $cx^{2^k+1} + \sum_{i+1}^{k-1} \gamma_i x^{2^{k+i}+2^i}$	$n = 2k$, $\gcd(s,k)$=1, s, k odd, $c \notin \mathbb{F}_{2^k}$, $\gamma_i \in \mathbb{F}_{2^k}$, d not a cube
C12	$(x + x^{2^m})^{2^k+1} +$ $u'(ux + u^{2^m}x^{2^m})^{(2^k+1)2^i} +$ $u(x + x^{2^m})(ux + u^{2^m}x^{2^m})$	$n = 2m$, $m \geq 2$ even, $\gcd(k,m) = 1$ and i even u primitive in $\mathbb{F}_{2^n}^*$, $u' \in \mathbb{F}_{2^m}$ not cube
C13	$x^{2^k+1} + tr_m^n(x)^{2^k+1}$	$n = 2m = 4t$, $\gcd(k,n) = 1$
C14	$a^2x^{2^{2m+1}+1} + b^2x^{2^{m+1}+1} + ax^{2^{2m}+2}$ $+bx^{2^m+2} + (c^2+c)x^3$	$n = 3m$, m odd Irene Villa's talk

C13 is equivalent to $x^{2^{m-k}+1}$ (L. Budaghyan, T. Helleseth, N. Li, B. Sun)

C3, C4 and C11

C3	$x^{2^{2i}+2^i} + cx^{q+1} + dx^{q(2^{2i}+2^i)}$	$q = 2^m$, $n = 2m$, $\gcd(i, m)$=1, $\gcd(2^i + 1, q + 1) \neq 1$, $dc^q + c \neq 0$, $d \notin \{\lambda^{(2^i+1)(q-1)}, \lambda \in \mathbb{F}_{2^n}\}$, $d^{q+1} = 1$
C4	$x(x^{2^i} + x^q + cx^{2^i q}) + x^{2^i}(c^q x^q + sx^{2^i q}) + x^{(2^i+1)q}$	$q = 2^m$, $n = 2m$, $\gcd(i, m)$=1, $c \in \mathbb{F}_{2^n}$, $s \in \mathbb{F}_{2^n} \setminus \mathbb{F}_q$, $X^{2^i+1} + cX^{2^i} + c^q X + 1$ has no solutions s.t. $x^{q+1} = 1$
C11	$dx^{2^s+1} + d^q x^{q2^s+q} + cx^{q+1} + \sum_{i+1}^{m-1} \gamma_i x^{q2^i+2^i}$	$q = 2^m$, $n = 2m$, $\gcd(s, m)$=1, s, m odd, $c \notin \mathbb{F}_q$, $\gamma_i \in \mathbb{F}_q$, d not a cube

C3$\subseteq$C11

$n = 2m,\ q = 2^m$

$$F(x) = cx^{q+1} + x^{2^{2i}+2^i} + dx^{q(2^{2i}+2^i)}$$

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$d^{q+1} = 1 \Rightarrow \exists d'$ s.t. $d = d'^{q-1}$

$$F'(x) = d'F(x) = d'cx^{q+1} + d'x^{2^{2i}+2^i} + d'^q x^{q(2^{2i}+2^i)}$$

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$dc^q + c \neq 0 \Rightarrow d'c \notin \mathbb{F}_q$, so $\mathbb{F}_{2^n} = d'c\mathbb{F}_q \oplus \mathbb{F}_q$. We can apply a linear permutation which is the identity on $d'c\mathbb{F}_q$ and $x^{1/2^i}$ on $\mathbb{F}_q$.

$$L \circ F'(x) = d'cx^{q+1} + d''x^{2^i+1} + d''^q x^{q(2^i+1)} \in C11$$

$d'' = d'^{1/2^i}$

It is possible to prove also that C11$\subseteq$C3

Lemma

C3=C11

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C3=C11

Proposition

Let $n = 2m$ with m odd and let i be such that $\gcd(n, i) = 1$. Let

$$F(x) = cx^{2^m+1} + dx^{2^i+1} + d^{2^m}x^{2^m(2^i+1)}$$

and

$$F'(x) = c'x^{2^m+1} + d'x^{2^i+1} + d'^{2^m}x^{2^m(2^i+1)}.$$

be two APN functions of family C11. Then F and F' are affine equivalent.

C11⊆C4

$$F(x) = dx^{2^i+1} + d^q x^{q(2^i+1)} + cx^{q+1} + \sum_{s=1}^{k-1} \gamma_s x^{(q+1)2^s}$$

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Let $L(x) = (x + x^q)^{2^t} + w(x + x^q) + (c + c^q)^{2^t} x$

C11⊆C4

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$w \in \mathbb{F}_q \Rightarrow L(x)$ permutation

$$\frac{L \circ F(x)}{(c + c^q)^{2^t}} = dx^{2^i+1} + d^q x^{q(2^i+1)} + c' x^{q+1} + \sum_{\substack{s=1 \\ s \neq t}}^{k-1} \gamma_s x^{(q+1)2^s}$$

$c' = w(c + c^q)^{1-2^t} + c.$

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Wlog

$$F(x) = dx^{2^i+1} + d^q x^{q(2^i+1)} + cx^{q+1} + x^{(q+1)2^i}$$

Similarly

$$H(x) = \bar{d}x^{2^i(q+1)} + x^{(q+1)} + (x^{2^i+1} + x^{q(2^i+1)} + \bar{c}x^{q2^i+1} + \bar{c}^q x^{2^i+q})$$

is equivalent to

$$H'(x) = \bar{d}'x^{(q+1)} + (x^{2^i+1} + x^{q(2^i+1)} + \bar{c}x^{q2^i+1} + \bar{c}^q x^{2^i+q})$$

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We want to prove that $F(x) = dx^{2^i+1} + d^q x^{q(2^i+1)} + cx^{q+1} + x^{(q+1)2^i}$ is equivalent to $H'(x)$

Consider a permutation $x + \gamma x^q$ with $\gamma^{q+1} \neq 1$,

$$\begin{aligned} F(x+\gamma x^q) =& (c + c\gamma^{q+1})x^{q+1} + (1 + \gamma^{2^i(q+1)})x^{2^i(q+1)} \\ &+ (d + d^{2^m}\gamma^{q(2^i+1)})x^{2^i+1} + (d^{2^m} + d\gamma^{2^i+1})x^{q(2^i+1)} \\ &+ (d\gamma^{2^i} + d^{2^m}\gamma^q)x^{q2^i+1} + (d^{2^m}\gamma^{q2^i} + d\gamma)x^{2^i+q} \\ &+ \text{terms of } \deg \leq 1 \end{aligned}$$

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Which is EA-equivalent to

$$F'(x) = c'x^{q+1} + (ax^{2^i+1} + a^q x^{q(2^i+1)} + bx^{q2^i+1} + b^q x^{2^i+q}).$$

$a = (d + d^q\gamma^{q(2^i+1)})$ and $b = (d\gamma^{2^i} + d^q\gamma^q)$

Note: $q = 2^m$ with m odd and $gcd(i, n) = 1$. Thus x^{q2^i+1} is a permutation

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$$\Downarrow$$

there exists some λ s.t. $\lambda^{q2^i+1} = b$, substituting $x \mapsto \lambda^{-1}x$ we obtain

$$\bar{F}(x) = c''x^{q+1} + a''x^{2^i+1} + a''^q x^{q(2^i+1)} + x^{q2^i+1} + x^{2^i+q}.$$

Denoting by $j = m - i$ we have

$$\bar{F}(x) = c''x^{q+1} + x^{2^j+1} + x^{q(2^j+1)} + a''x^{q2^j+1} + a''^q x^{2^j+q}.$$

Now, $c'' \notin \mathbb{F}_q$ and $\bar{F}$ APN imply that

$$x^{2^j+1} + a''x^{2^j} + a''^q x + 1 = 0$$

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Theorem

$C3 = C11 \subseteq C4$.
Moreover we can rewrite the family of the hexanomials as:

$$H(x) = dx^{(q+1)} + (x^{2^i+1} + x^{q(2^i+1)} + cx^{q2^i+1} + c^q x^{2^i+q}).$$

Note that for m odd we can reduce the case i odd to the even case $j = m - i$.

Particular case: C12 with $k = 0$

When $k = 0$ for the family C12 we have that

$$F(x) = (x + x^q)^{2^i+1} + u'(ux + u^q x^q)^{2^i+1} + u(x + x^q)(ux + u^q x^q)$$
$$\sim_{EA} dx^{q+1} + ax^{2^i+1} + a^q x^{q(2^i+1)} + bx^{q2^i+1} + b^q x^{2^i+q},$$

with $d = u^2 + u^{q+1}$, $a = 1 + u'\, u^{2^i+1}$ and $b = 1 + u'\, u^{q2^i+1}$.

m even implies $\gcd(3, q+1) = 1$ and

$$\mathbb{F}_{2^n}^* = U \cup \rho U \cup \rho^2 U$$

with

$$U := \{x^{2^i+1} : x \in \mathbb{F}_{2^n}^*\} = \{x^3 : x \in \mathbb{F}_{2^n}^*\}$$

and $\rho = u^{q+1} \in \mathbb{F}_{2^m}$.

Now, we can have three cases

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- $a \in U$: $\Rightarrow \exists \lambda$ s.t. $\lambda^{2^i+1} = a$,

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- $a \in \rho^2 U$: $\Rightarrow \exists \lambda$ s.t. $\lambda^{2^i+1} = \rho a$.

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Now, we can have three cases

- $a \in U$: $\Rightarrow \exists \lambda$ s.t. $\lambda^{2^i+1} = a$,
- $a \in \rho U$: $\Rightarrow \exists \lambda$ s.t. $\lambda^{2^i+1} = \rho^2 a$,
- $a \in \rho^2 U$: $\Rightarrow \exists \lambda$ s.t. $\lambda^{2^i+1} = \rho a$.

Up to multiply by ρ we can perform the substitution $x \mapsto \lambda^{-1}x$ obtaining

$$F'(x) = d'x^{q+1} + x^{2^i+1} + x^{q(2^i+1)} + b'x^{q2^i+1} + b'^q x^{2^i+q}$$

Thanks for your attention!