Ilaria Zappatore

An Algorithm-based Fault Tolerant Technique for solving Polynomial Linear Systems

a joint work with

Eleonora GUERRINI, Romain LEBRETON LIRMM, Université de Montpellier CNRS

Seminario UMI - Crittografia e Codici

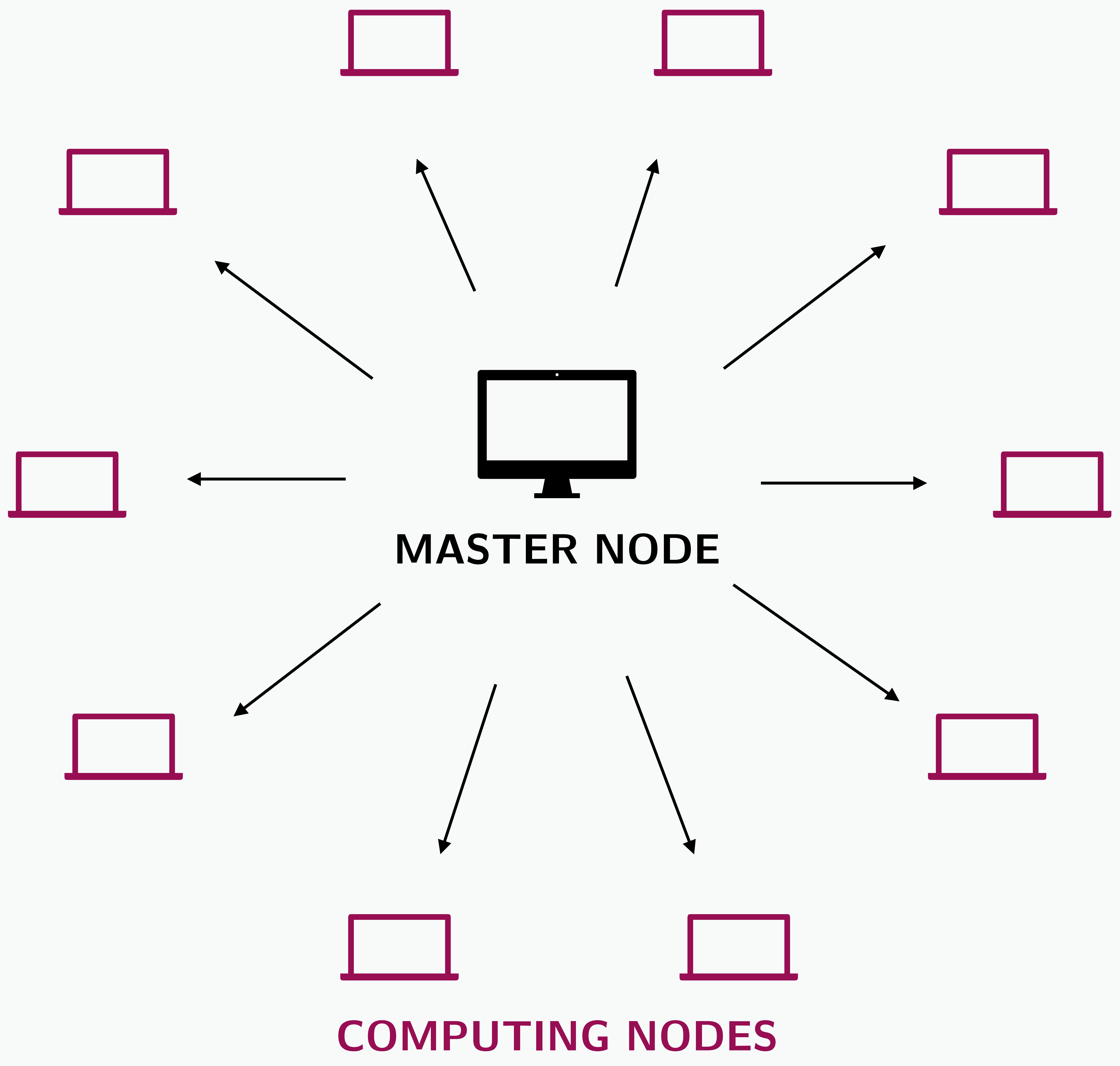
Starting Point

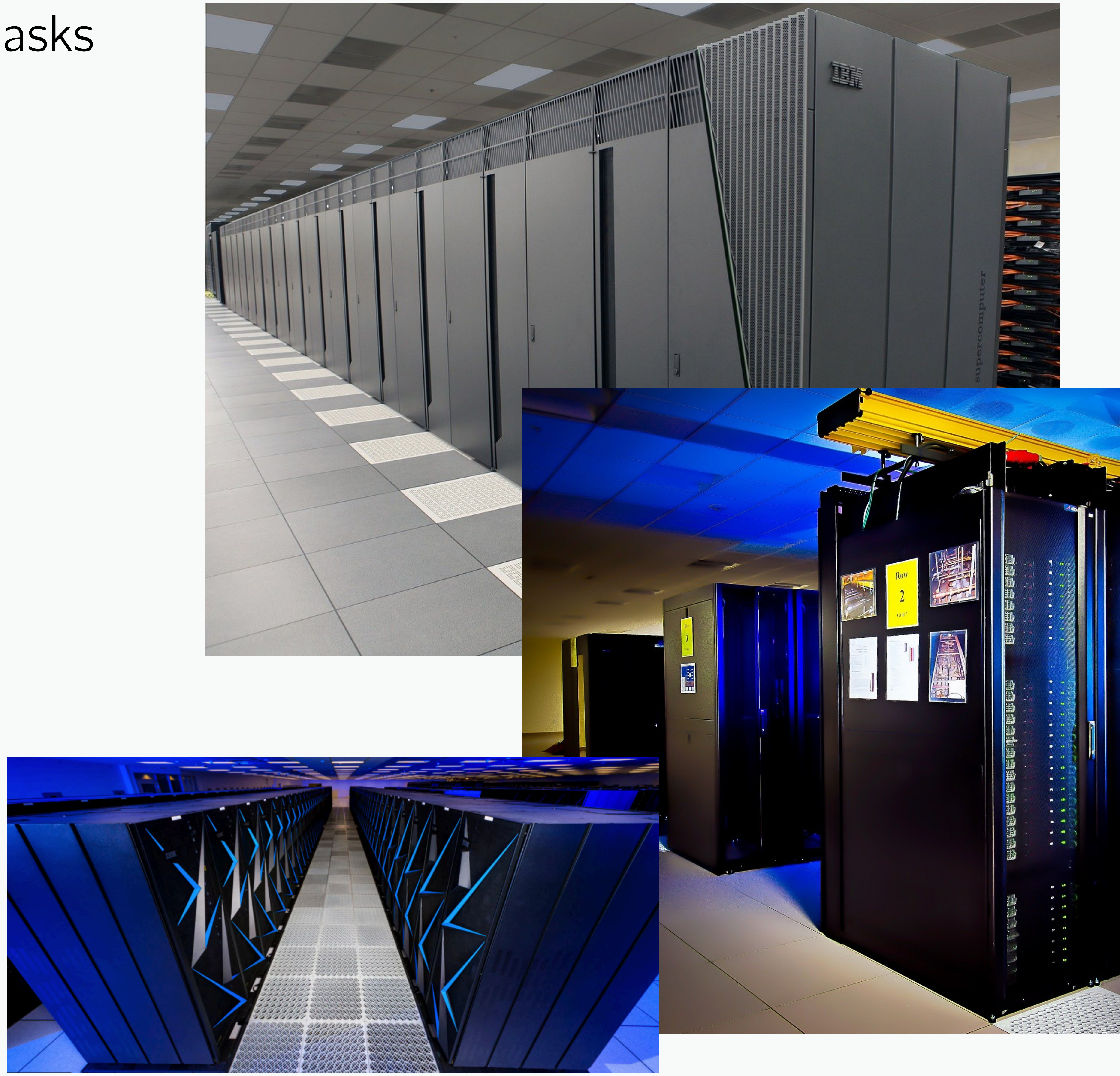
Overview & Motivations

Seminario UMI - Crittografia e Codici

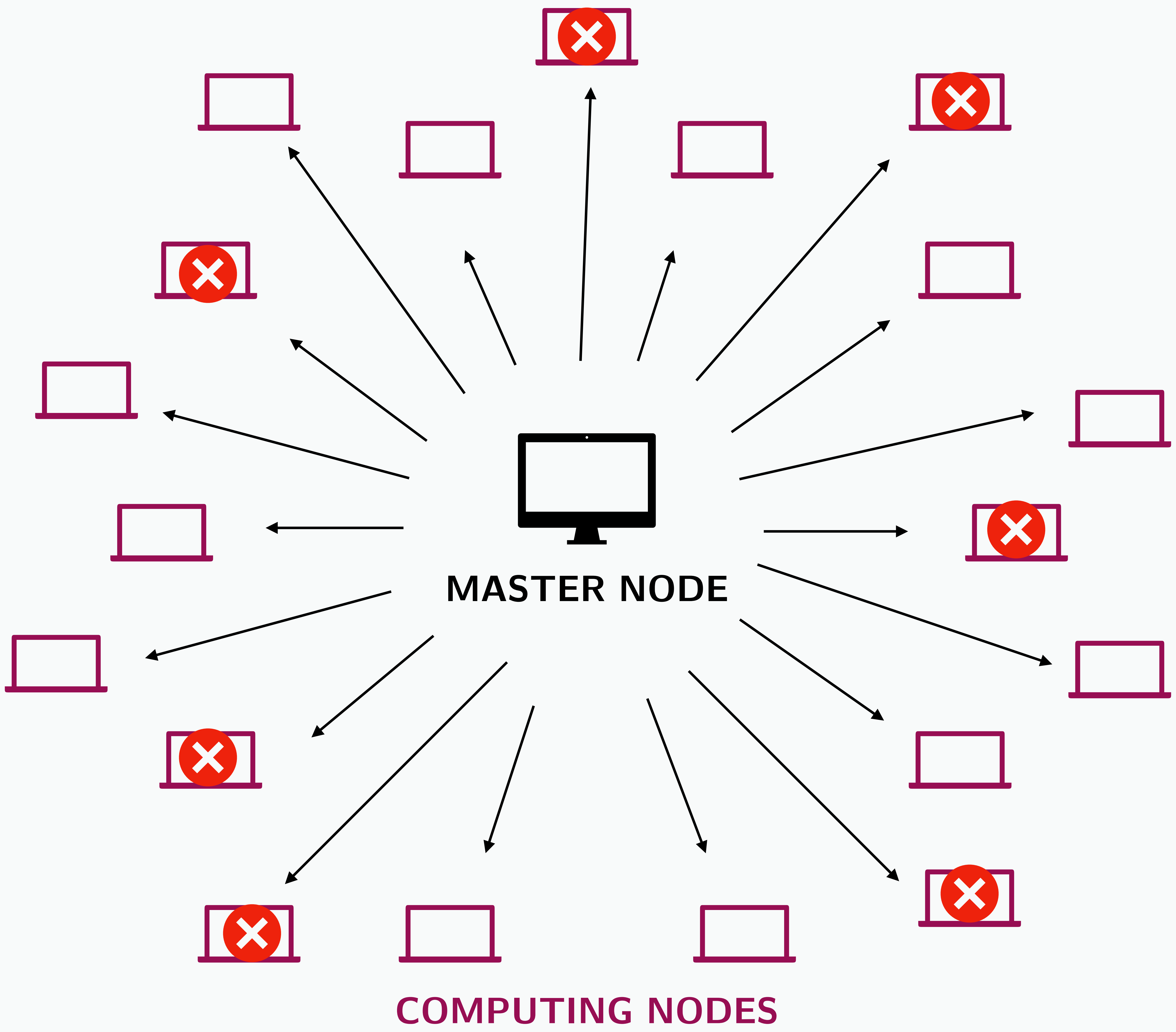
High Performance Computing Technologies

Goal: provide high performances and complete heavy tasks





Fault tolerant algorithms



The **more the number of system components** grows
the **more the failures** of computing nodes becomes **relevant**
(3,5 faults per day)
[DI, GUO, PERSHEY, SNIR, CAPPELLO, 2019], [LIU, CHEN, 2018]

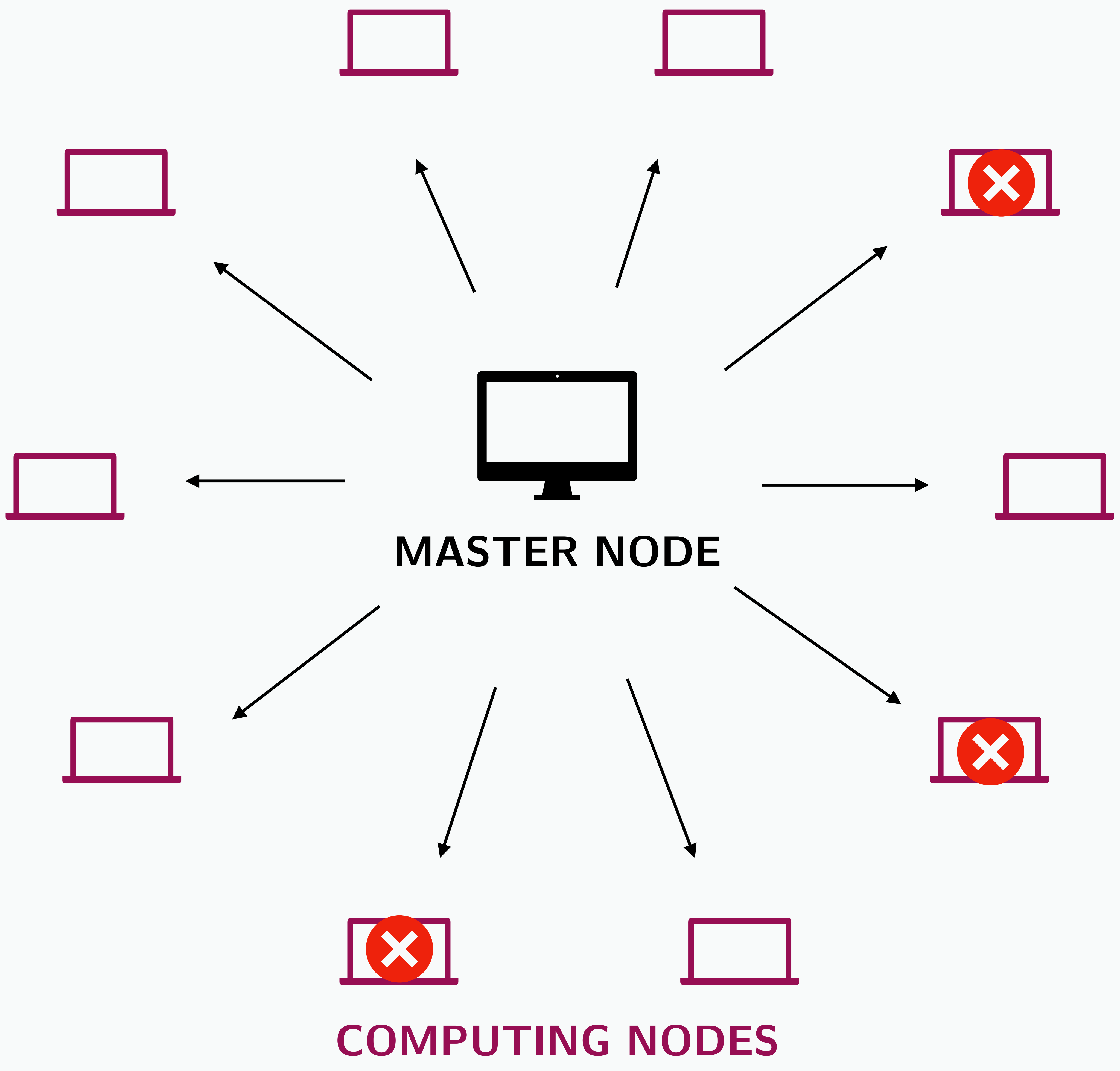
↓

construct **fault tolerant** algorithms

↓

detect/correct faults

Algorithm-based fault tolerant techniques (ABFT)



Algorithm-based fault tolerant techniques (ABFT)
[HUANG, ABRAHAM, 1984]

↓

exploits the **algorithm's characteristics**
to design a **fault tolerant algorithm**

↓

Goal: detect/correct computational errors (faults)

High Performance Computing Technologies

Algorithm-based fault tolerant techniques (ABFT)

[HUANG, ABRAHAM, 1984]

Goal: detect/correct computational errors (faults)

ALGORITHM

Algorithm-based fault tolerant techniques (ABFT)

[HUANG, ABRAHAM, 1984]

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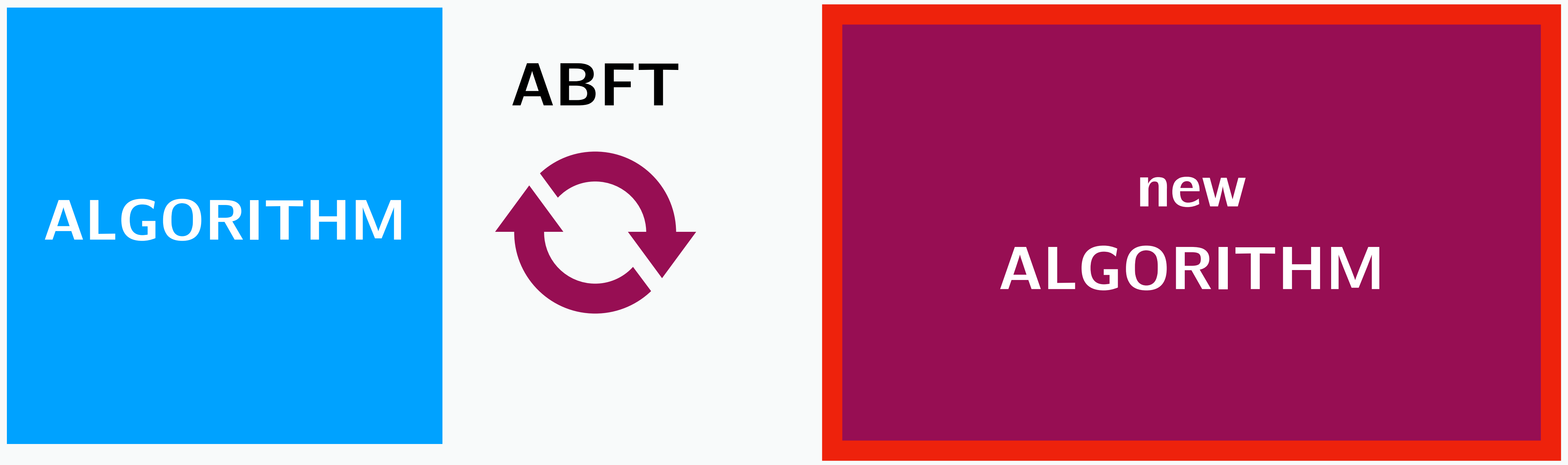
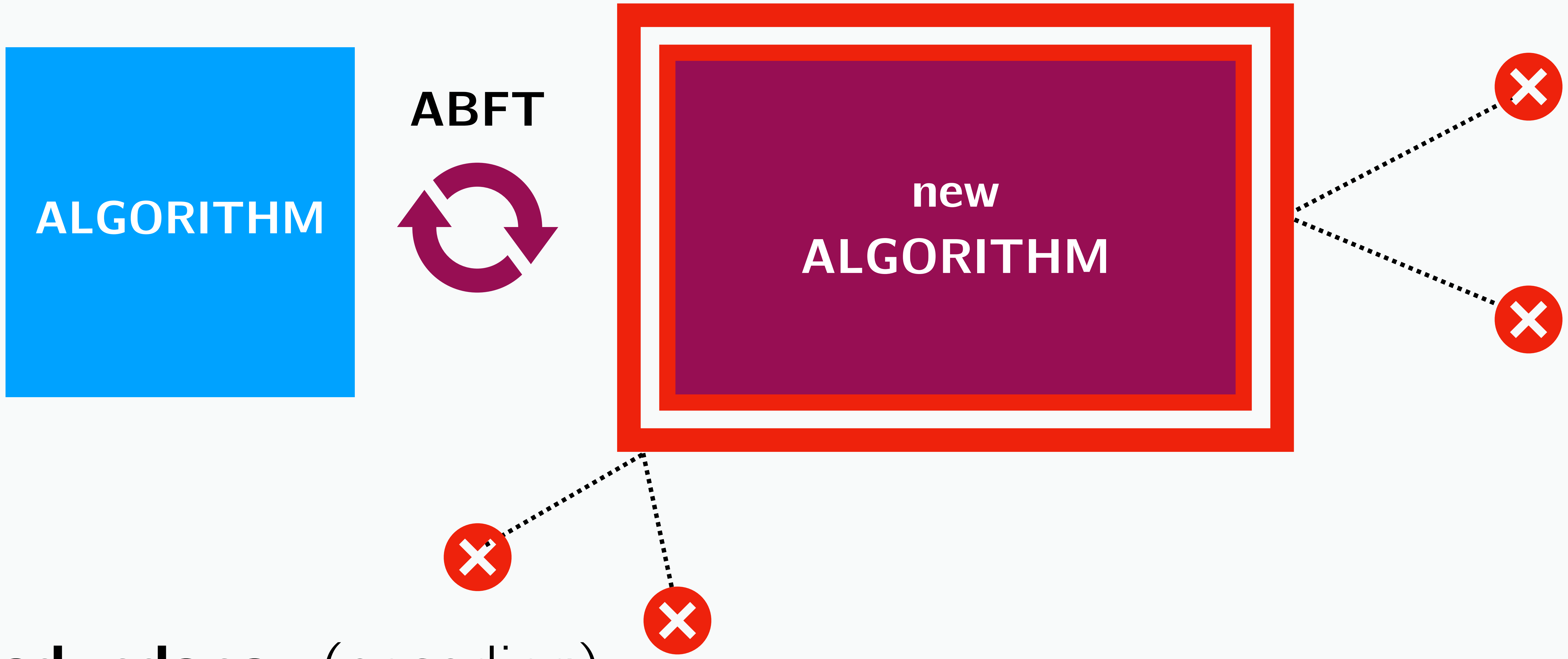


- **adding redundancy** (encoding)

Algorithm-based fault tolerant techniques (ABFT)

[HUANG, ABRAHAM, 1984]

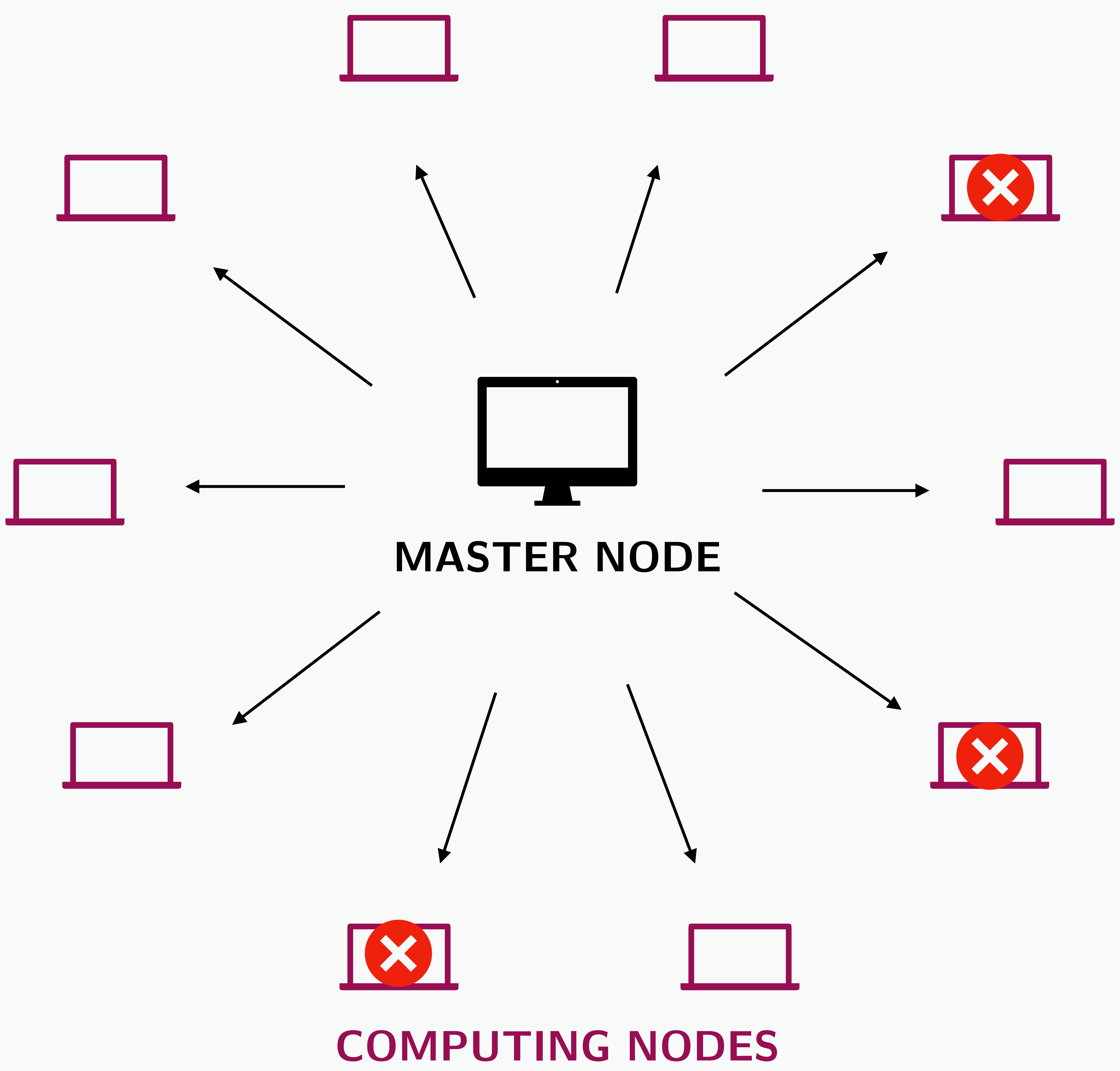
Goal: detect/correct computational errors (faults)



- **adding redundancy** (encoding)
- modify the algorithm, work on encoding data ⟵ robust to errors (faults)

error correcting codes

High Performance Computing Technologies



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ABFT for Polynomial Linear System Solving by Evaluation-Interpolation

[BOYER, KALTOFEN, 2014]

[KALTOFEN, PERNET, STORJOHANN, WADDEL, 2017]

[GUERRINI, LEBRETON, Z., 2019]

[GUERRINI, LEBRETON, Z., 2021]

: publication

Take a look inside

Seminario UMI - Crittografia e Codici

Polynomial Linear System Solving by Evaluation-Interpolation

Polynomial Linear System Solving

$$\underbrace{\begin{pmatrix} a_{1,1}(x) & a_{1,2}(x) & \dots & a_{1,n}(x) \\ a_{2,1}(x) & a_{2,2}(x) & \dots & a_{2,n}(x) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1}(x) & a_{n,2}(x) & \dots & a_{n,n}(x) \end{pmatrix}}_{A(x)} \underbrace{\begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix}}_{\boldsymbol{y}(\boldsymbol{x})} = \underbrace{\begin{pmatrix} b_1(x) \\ b_2(x) \\ \vdots \\ b_n(x) \end{pmatrix}}_{\boldsymbol{b}(x)}$$

Polynomial Linear System Solving by Evaluation-Interpolation

Polynomial Linear System Solving

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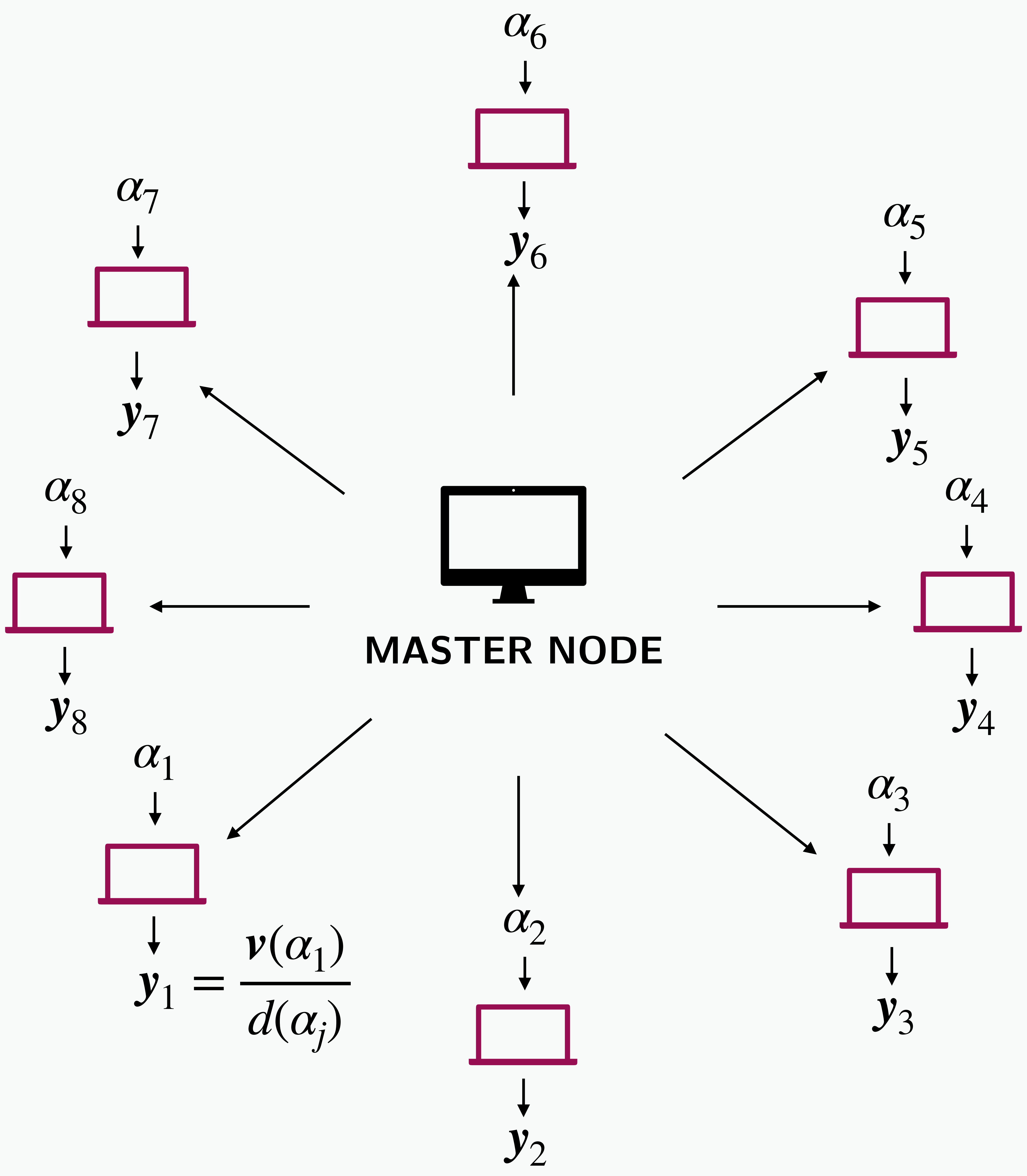
Evaluation-Interpolation Algorithm

1. **Evaluate** $A(x)$, $\boldsymbol{b}(x)$ in $\{\alpha_1, \dots, \alpha_L\}$, ($A(\alpha_j)$ full rank)
2. **Compute** $y_j = A(\alpha_j)^{-1}\boldsymbol{b}(\alpha_j) = \dfrac{\boldsymbol{v}(\alpha_j)}{d(\alpha_j)}$,



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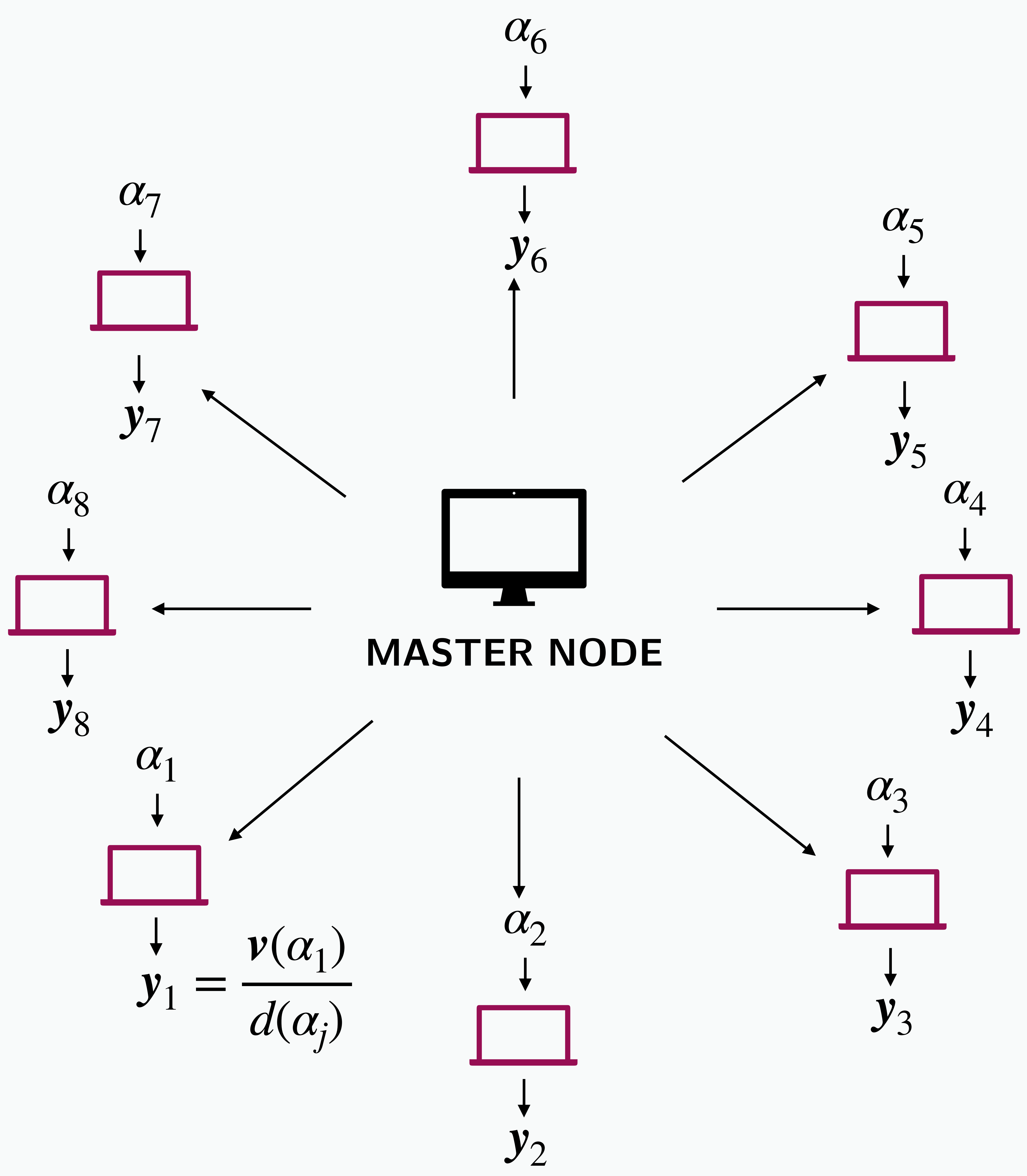

Polynomial Linear System Solving by Evaluation-Interpolation

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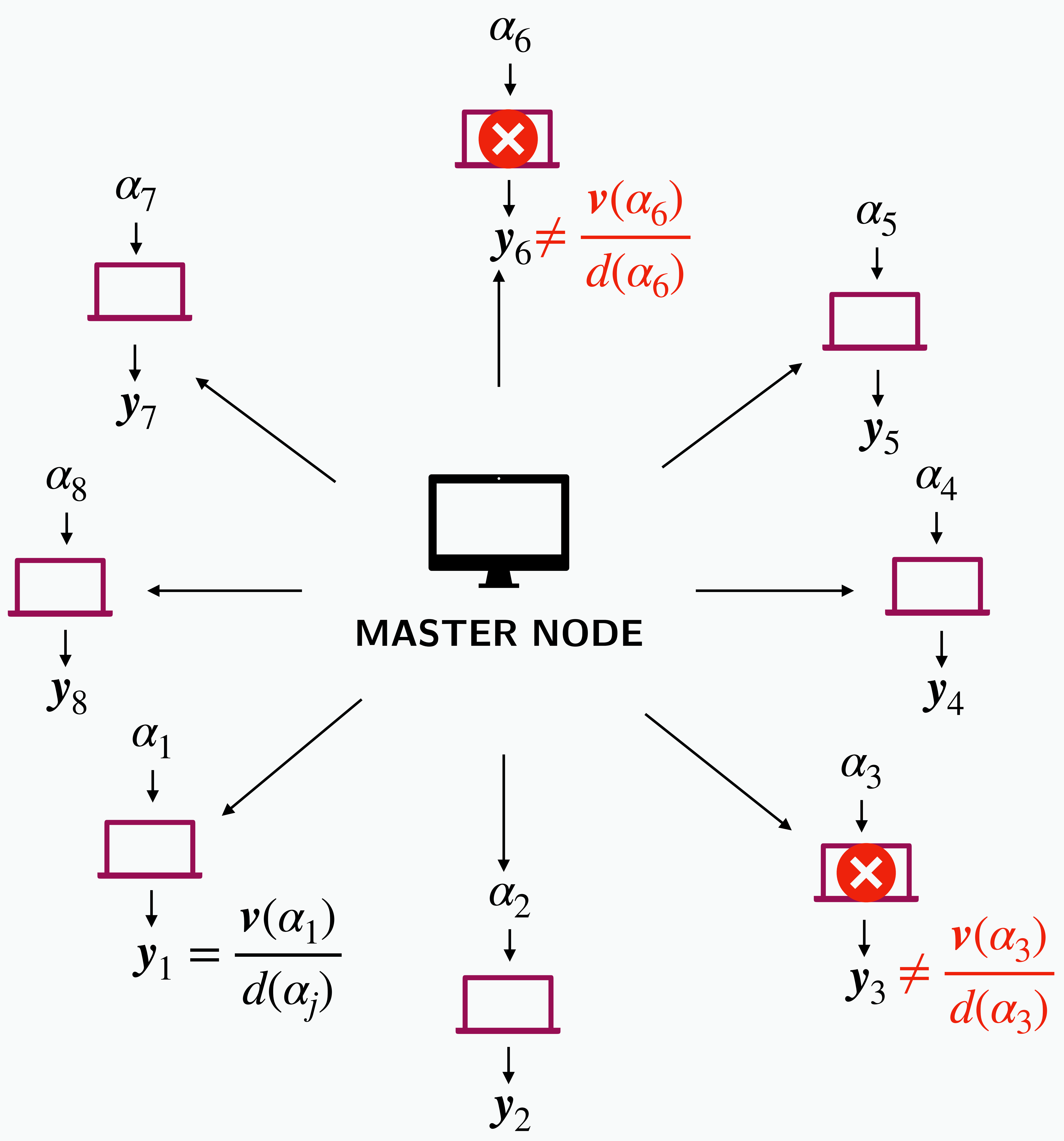
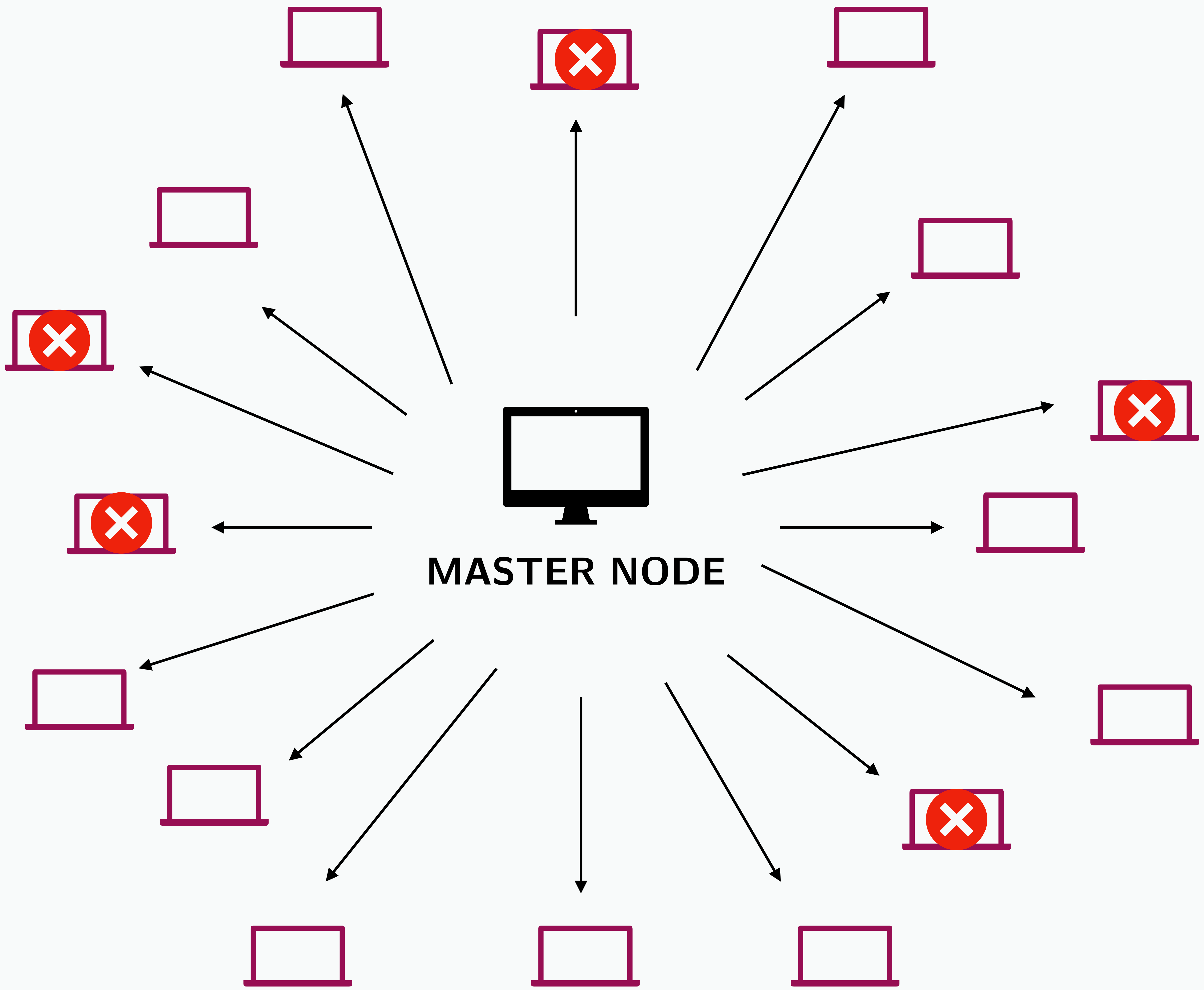


ABFT for PLS solving by Evaluation-Interpolation

- **adding** redundancy (encoding), consider many nodes

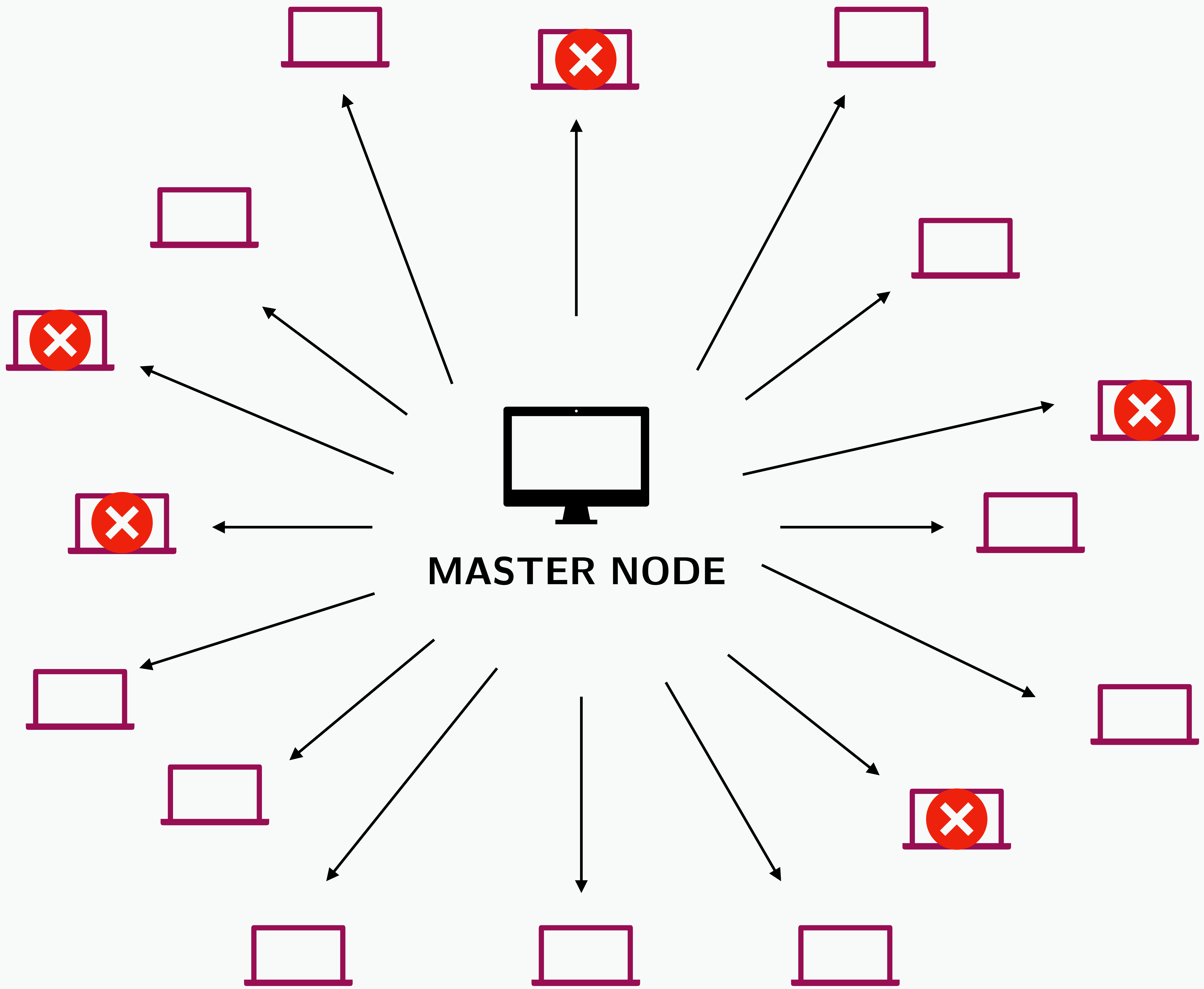


ABFT for Evaluation-Interpolation Algorithm

1. **Evaluate** $A(x)$, $\boldsymbol{b}(x)$ in many evaluation points
2. **Compute** $\boldsymbol{y}_j = A(\alpha_j)^{-1}\boldsymbol{b}(\alpha_j)$

ABFT for PLS solving by Evaluation-Interpolation

- **adding** redundancy (encoding), consider many nodes
- decoding, correcting errors



ABFT for Evaluation-Interpolation Algorithm

1. **Evaluate** $A(x)$, $\boldsymbol{b}(x)$ in many evaluation points
2. **Compute** $\boldsymbol{y}_j = A(\alpha_j)^{-1}\boldsymbol{b}(\alpha_j)$
3. Interpolate $\boldsymbol{y}(x)$ from $\boldsymbol{y}_1, \dots, \boldsymbol{y}_L$ where some errors occur

Simultaneous Cauchy Interpolation with Errors

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Simultaneous Cauchy Interpolation with Errors

Given $\boldsymbol{y}_1, \dots, \boldsymbol{y}_L$

- $\boldsymbol{y}_j = \dfrac{\boldsymbol{v}(\alpha_j)}{d(\alpha_j)}$ correct evaluations
- $\boldsymbol{y}_j \neq \dfrac{\boldsymbol{v}(\alpha_j)}{d(\alpha_j)}$ erroneous evaluations

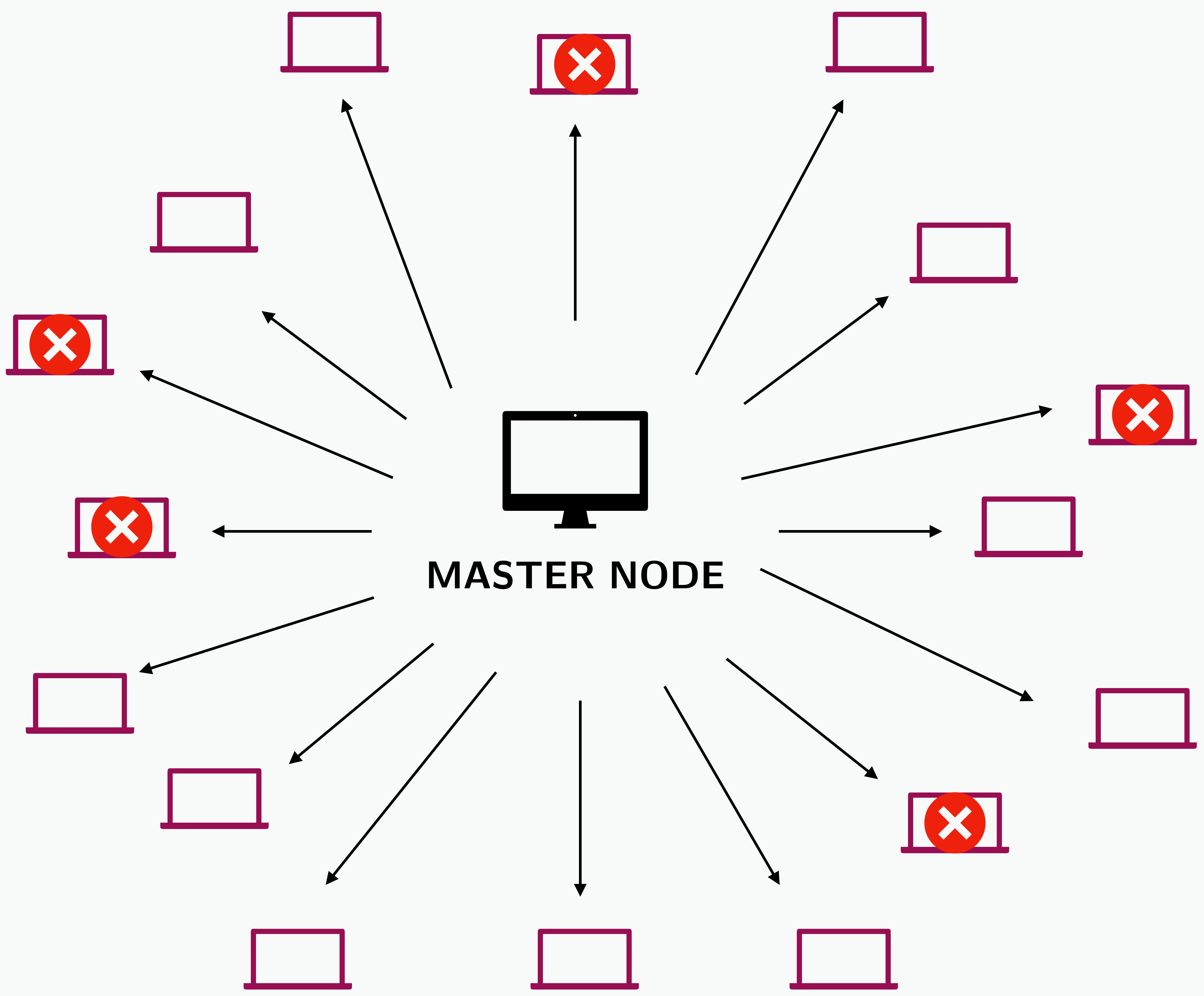
the *degree bounds* $N > \deg(\boldsymbol{v})$, $D > \deg(d)$

and *an upper bound* τ *on the number of errors*.

GOAL: **reconstruct** $(\boldsymbol{v}(x), d(x)) \longrightarrow \boldsymbol{y}(x)$

Simultaneous Cauchy Interpolation

Simultaneous Rational Function Reconstruction



Two Main Questions

1. How many evaluations (nodes) do we need to uniquely recover $(v(x), d(x))$?

fewer evaluations $\longrightarrow$ fewer computations

2. Can we reduce this number?

Number of Evaluations - Outline of this work

			uniqueness	uniqueness *almost always*
no-errors	$(\boldsymbol{v}(x), d(x))$		Cauchy Interpolation	**[GUERRINI, LEBRETON, Z., 2020]**
no-errors	$A(x)\frac{\boldsymbol{v}(x)}{d(x)} = \boldsymbol{b}(x)$		[CABAY, 1971]	?
with errors	$(\boldsymbol{v}(x), d(x))$	d constant $(D = 0)$	Unique Decoding Capability Theorem (IRS codes)	IRS codes [BLEICHENBACHER, KIAYIAS, YUNG, 2003]
with errors	$(\boldsymbol{v}(x), d(x))$	$D > 0$	[BOYER, KALTOFEN, 2014]	**[GUERRINI, LEBRETON, Z., 2019]**
with errors	$A(x)\frac{\boldsymbol{v}(x)}{d(x)} = \boldsymbol{b}(x)$	$D > 0$	[KALTOFEN, PERNET, STORJOHANN, WADDEL, 2017]	**[GUERRINI, LEBRETON, Z., 2021]**

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	$A(x)\frac{\boldsymbol{v}(x)}{d(x)} = \boldsymbol{b}(x)$		[KALTOFEN, PERNET, STORJOHANN, WADDEL, 2017]	[GUERRINI, LEBRETON, Z., 2021]

No-error case

Simultaneous Cauchy Interpolation

Given the vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_L$

and the *degree bounds* N, D

GOAL: **find** $\underbrace{(v_1(x), \ldots, v_n(x)}_{\boldsymbol{v}(x)}, d(x))$ s.t.

- $v_i(\alpha_j) = y_{i,j} d(\alpha_j)$
- $\deg(v_i) < N$
- $\deg(d) < D$

$\downarrow$

Simultaneous Rational Function Reconstruction

$\longrightarrow$

Vector Generalization

same denominator

Cauchy Interpolation

Given the evaluations $y_1, \ldots, y_L$

and the *degree bounds* N, D

GOAL: **find** $(v(x), d(x))$ s.t.

- $\dfrac{v(\alpha_j)}{d(\alpha_j)} = y_j, d(\alpha_j) \neq 0$
- $\deg(v) < N$
- $\deg(d) < D$

$\downarrow$

Rational Function Reconstruction

If the number of evaluations $L \geq N + D - 1$

$\downarrow$

Unique solution

$(v_1, d_1), (v_2, d_2)$ **solutions** $\longrightarrow \dfrac{v_1}{d_1} = \dfrac{v_2}{d_2}$

No-error case

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1. Apply the **Cauchy Interpolation** component-wise ($L \geq N + D - 1 \longrightarrow$ **uniqueness**)

No-error case

Simultaneous Cauchy Interpolation

Given the vectors $\boldsymbol{y}_1, \dots, \boldsymbol{y}_L$

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1. Apply the **Cauchy Interpolation** component-wise ($L \geq N + D - 1 \longrightarrow$ **uniqueness**)
2. Use the **common denominator feature** to reduce the number of equations of the related **homogeneous linear system**

$\underset{nL}{\textbf{\#equations}} = \underset{nN+D}{\textbf{\#unknowns}}$ **-1** $\longrightarrow L = N + (D-1)/n$ **uniqueness**?

No-error case

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If we want to recover a solution of a PLS:

- with $L \geq \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\}$ $\longrightarrow$ **uniqueness** [CABAY, 1971]
- for **specific degree constraints** N, D $\max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\} = N + (D-1)/n$ $\longrightarrow$ **uniqueness** [OLESH, STORJOHANN, 2007]

No-error case

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Theorem [GUERRINI, LEBRETON, Z., 2020]

If $L = N + (D-1)/n$, **for almost all instances** $\Longrightarrow$ **uniqueness**.

If $\mathbb{K} = \mathbb{F}_q$, the **proportion of instances** leading to **non-uniqueness** is $\leq (D-1)/q$

Just a hint of the technique...

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No-error case

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GOAL: **find** $\underbrace{(v_1(x), \ldots, v_n(x), d(x))}_{\boldsymbol{v}(x)}$ s.t.

- $v_i(x) = u_i(x)d(x) \bmod \prod (x - \alpha_j)$
- $\deg(v_i) < N$
- $\deg(d) < D$

$\boldsymbol{u}(x)$ vector of Lagrange interpolators

$\longrightarrow$ Specific case of

Simultaneous Rational Function Reconstruction

Given two vector of polynomials $\boldsymbol{u}(x), \boldsymbol{a}(x)$

and the *degree bounds* N, D

GOAL: **find** $\underbrace{(v_1(x), \ldots, v_n(x), d(x))}_{\boldsymbol{v}(x)}$ s.t.

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$$(\boldsymbol{v}, d)\underbrace{\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \\ -u_1 & -u_2 & \ldots & -u_n \end{pmatrix}}_{R_{\boldsymbol{u}}} = 0 \bmod \underbrace{\langle (0, \ldots, a_i, \ldots 0) \rangle}_{\mathscr{M}} \longleftrightarrow (\boldsymbol{v}, d) \in \mathscr{A}_{R_{\boldsymbol{u}}}$$

$$\mathscr{A}_{R_{\boldsymbol{u}}} = \{\boldsymbol{p} = (p_1(x), \ldots, p_n(x)) \mid \boldsymbol{p}R_{\boldsymbol{u}} = \boldsymbol{0} \bmod \mathscr{M}\}$$

Relation module

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No-error case

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$$\underbrace{(\deg(v_1) - N}_{<0}, \ldots, \underbrace{\deg(v_n) - N}_{<0}, \underbrace{\deg(d) - D}_{<0}) \xrightarrow{\text{take the max}} rdeg_{(-N,\ldots,-N,-D)}(\boldsymbol{v}, d) < 0$$

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Given two vector of polynomials $\boldsymbol{u}(x), \boldsymbol{a}(x)$ and the *degree bounds* N, D

GOAL: **find** $\underbrace{(v_1(x), \dots, v_n(x)}_{\boldsymbol{v}(x)}, d(x))$ s.t.

- $(\boldsymbol{v}, d) \in \mathscr{A}_{R_u}$
- $rdeg_{(-N,\dots,-N,-D)}(\boldsymbol{v}, d) < 0$

How to prove uniqueness?

- ***Minimal basis*** $\mathscr{B}$ *of* $\mathscr{A}_{R_u}$, for which the $s-$**row degrees are uniquely defined** *(Ordered Weak Popov)*
- **Solution space generated** by **elements of** $\mathscr{B}$ the with **negative** $s-$**row degrees**

$\downarrow$

$(-N, \dots, -N, -D)$-**row degrees** of $\mathscr{B}$ of the form $(0,0,\dots,-1)$

$\downarrow$

Solution space uniquely generated $\Rightarrow$ **uniqueness**

Theorem [GUERRINI, LEBRETON, Z., 2020]

If $L = \deg(\boldsymbol{a}) = N + (D-1)/n$, **for almost all instances,** $rdeg_{(-N,\dots,-N,-D)}(\mathscr{B}) = (0,\dots,0,-1)$

If $\mathbb{K} = \mathbb{F}_q$, the **proportion of instances** leading to **non-uniqueness** is $\leq (D-1)/q$

Number of Evaluations - Outline of this work

			uniqueness	**uniqueness** *almost always*
no-errors	$(\boldsymbol{v}(x), d(x))$		$L = N + D - 1$ Cauchy Interpolation	$L = N + \left\lceil \frac{(D-1)}{n} \right\rceil$
	$A(x)\frac{\boldsymbol{v}(x)}{d(x)} = \boldsymbol{b}(x)$		$L = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\}$ [CABAY, 1971]	
with errors	$(\boldsymbol{v}(x), d(x))$	d constant $(D = 0)$	Unique Decoding Capability Theorem (IRS codes)	IRS codes [BLEICHENBACHER, KIAYIAS, YUNG, 2003]
		$D > 0$	[BOYER, KALTOFEN, 2014]	**[GUERRINI, LEBRETON, Z., 2019]**
	$A(x)\frac{\boldsymbol{v}(x)}{d(x)} = \boldsymbol{b}(x)$		[KALTOFEN, PERNET, STORJOHANN, WADDEL, 2017]	**[GUERRINI, LEBRETON, Z., 2021]**

Number of Evaluations - Outline of this work

			uniqueness	uniqueness *almost always*
no-errors	$(\boldsymbol{v}(x), d(x))$		$L = N + D - 1$ Cauchy Interpolation	$L = N + \left\lceil \frac{(D-1)}{n} \right\rceil$
	$A(x)\frac{\boldsymbol{v}(x)}{d(x)} = \boldsymbol{b}(x)$		$L = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\}$ [CABAY, 1971]	
with errors	$(\boldsymbol{v}(x), d(x))$	d constant $(D = 0)$	Unique Decoding Capability Theorem (IRS codes)	IRS codes [BLEICHENBACHER, KIAYIAS, YUNG, 2003]
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	$A(x)\frac{\boldsymbol{v}(x)}{d(x)} = \boldsymbol{b}(x)$	$D > 0$	[KALTOFEN, PERNET, STORJOHANN, WADDEL, 2017]	**[GUERRINI, LEBRETON, Z., 2021]**

Simultaneous Cauchy Interpolation with Errors

Simultaneous Cauchy Interpolation with Errors

Given $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_L$

- $\boldsymbol{y}_j = \dfrac{\boldsymbol{v}(\alpha_j)}{d(\alpha_j)}$ correct evaluations
- $\boldsymbol{y}_j \neq \dfrac{\boldsymbol{v}(\alpha_j)}{d(\alpha_j)}$ erroneous evaluations

the *degree bounds* $N > \deg(\boldsymbol{v})$, $D > \deg(d)$

and *an upper bound* τ *on the number of errors*.

GOAL: **reconstruct** $(\boldsymbol{v}(x), d(x)) \longrightarrow \boldsymbol{y}(x)$

Simultaneous Interpolation with Errors (D=0)

Simultaneous Interpolation with Errors

Given $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_L$

- $\boldsymbol{y}_j = \boldsymbol{v}(\alpha_j)$ correct evaluations
- $\boldsymbol{y}_j \neq \boldsymbol{v}(\alpha_j)$ erroneous evaluations

the *degree bounds* $N > \deg(\boldsymbol{v})$

and *an upper bound* τ *on the number of errors*.

GOAL: **reconstruct** $\boldsymbol{v}(x)$

What happens if the denominator is constant?

GOAL: reconstruct a **vector** of **polynomials** given its **evaluations**, **some** of which are **erroneous**

$\downarrow$

decoding **Interleaved Reed-Solomon** codes

Algebraic coding theory

Interleaved Reed-Solomon codes

Interleaved Reed-Solomon Codes

Let $k \leq n \leq q$ and $\{\alpha_1, \ldots, \alpha_n\}$ distinct *evaluation points*,

$$\mathscr{C}_{IRS}(n,k) := \{(\boldsymbol{f}(\alpha_1), \ldots, \boldsymbol{f}(\alpha_n)) \mid \boldsymbol{f} \in \mathbb{F}_q[x]^{l \times 1}, \deg(\boldsymbol{f}) < k\}$$

The IRS code is an **MDS code**

the **minimum distance** is $d = n - k + 1$

$(m_{1,0}, \ldots, m_{1,k-1})$ $(m_{2,0}, \ldots, m_{2,k-1})$ $\cdots$ $(m_{l,0}, \ldots, m_{l,k-1})$

$$f_1(x) = \sum_{i=0}^{k-1} m_{1,i}x^i \quad f_2(x) = \sum_{i=0}^{k-1} m_{2,i}x^i \quad \cdots \quad f_l(x) = \sum_{i=0}^{k-1} m_{l,i}x^i \rightarrow \boldsymbol{f}(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_l(x) \end{pmatrix}$$

Encoding

$$C = \begin{pmatrix} f_1(\alpha_1) & f_1(\alpha_2) & f_1(\alpha_3) & \ldots & f_1(\alpha_n) \\ f_2(\alpha_1) & f_2(\alpha_2) & f_2(\alpha_3) & \ldots & f_2(\alpha_n) \\ f_3(\alpha_1) & f_3(\alpha_2) & f_3(\alpha_3) & \ldots & f_3(\alpha_n) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ f_l(\alpha_1) & f_l(\alpha_2) & f_l(\alpha_3) & \ldots & f_l(\alpha_n) \end{pmatrix} \begin{array}{l} \text{RS codeword} \\ \text{RS codeword} \\ \\ \\ \\ \end{array}$$

Interleaved Reed-Solomon codeword

Interleaved Reed-Solomon codes

Interleaved Reed-Solomon Codes

Let $k \leq n \leq q$ and $\{\alpha_1, \ldots, \alpha_n\}$ distinct *evaluation points,*

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Encoding

Channel

$$C = \begin{pmatrix} f_1(\alpha_1) & f_1(\alpha_2) & f_1(\alpha_3) & \ldots & f_1(\alpha_n) \\ f_2(\alpha_1) & f_2(\alpha_2) & f_2(\alpha_3) & \ldots & f_2(\alpha_n) \\ f_3(\alpha_1) & f_3(\alpha_2) & f_3(\alpha_3) & \ldots & f_3(\alpha_n) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ f_l(\alpha_1) & f_l(\alpha_2) & f_l(\alpha_3) & \ldots & f_l(\alpha_n) \end{pmatrix}$$

$$\begin{pmatrix} f_1(\alpha_1) & & & & \\ f_2(\alpha_1) & & & & \\ f_3(\alpha_1) & & & & \\ \vdots & & & & \\ f_l(\alpha_1) & & & & \end{pmatrix}$$

Interleaved Reed-Solomon codes

Interleaved Reed-Solomon Codes

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Channel

$$\begin{pmatrix} f_1(\alpha_1) & f_1(\alpha_2) & & \\ f_2(\alpha_1) & f_2(\alpha_2) & & \\ f_3(\alpha_1) & f_3(\alpha_2) & & \\ \vdots & \vdots & & \\ f_l(\alpha_1) & f_l(\alpha_2) & & \end{pmatrix}$$

burst error

Interleaved Reed-Solomon codes

Interleaved Reed-Solomon Codes

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Channel

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Interleaved Reed-Solomon codes

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burst error

Interleaved Reed-Solomon codes

Interleaved Reed-Solomon Codes

Let $k \leq n \leq q$ and $\{\alpha_1, \ldots, \alpha_n\}$ distinct *evaluation points*,

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Decoding

$$Y = \begin{pmatrix} f_1(\alpha_1) & f_1(\alpha_2) & f_1(\alpha_3) & \ldots & f_1(\alpha_n) \\ f_2(\alpha_1) & f_2(\alpha_2) & f_2(\alpha_3) & \ldots & f_2(\alpha_n) \\ f_3(\alpha_1) & f_3(\alpha_2) & f_3(\alpha_3) & \ldots & f_3(\alpha_n) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ f_l(\alpha_1) & f_l(\alpha_2) & f_l(\alpha_3) & \ldots & f_l(\alpha_n) \end{pmatrix}$$

burst error

The IRS code is an **MDS code**

the **minimum distance** is $d = n - k + 1$

Decoding IRS codes

Given the received matrix with columns $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$

- $\boldsymbol{y}_j = \boldsymbol{f}(\alpha_j)$ correct evaluations
- $\boldsymbol{y}_j \neq \boldsymbol{f}(\alpha_j)$ erroneous evaluations

the *degree bound* $k > \deg(\boldsymbol{f})$

and *an upper bound τ on the number of errors*.

GOAL: **reconstruct** $\boldsymbol{f}(x)$

Simultaneous Cauchy Interpolation

Interleaved Reed-Solomon codes

Interleaved Reed-Solomon Codes

Let $k \leq n \leq q$ and $\{\alpha_1, \ldots, \alpha_n\}$ distinct *evaluation points*,

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$$Y = \begin{pmatrix} f_1(\alpha_1) & f_1(\alpha_2) & f_1(\alpha_3) & \ldots & f_1(\alpha_n) \\ f_2(\alpha_1) & f_2(\alpha_2) & f_2(\alpha_3) & \ldots & f_2(\alpha_n) \\ f_3(\alpha_1) & f_3(\alpha_2) & f_3(\alpha_3) & \ldots & f_3(\alpha_n) \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ f_l(\alpha_1) & f_l(\alpha_2) & f_l(\alpha_3) & \ldots & f_l(\alpha_n) \end{pmatrix}$$

burst error

this is exactly our starting point!

Simultaneous Interpolation with Errors

The IRS code is an **MDS code**
the **minimum distance** is $d = n - k + 1$

Simultaneous Interpolation with Errors

Given the received matrix with columns $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$

- $\boldsymbol{y}_j = \boldsymbol{f}(\alpha_j)$ correct evaluations
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the *degree bound* $k > \deg(\boldsymbol{f})$
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Simultaneous Cauchy Interpolation

Decoding Interleaved Reed-Solomon codes

Simultaneous Interpolation with Errors

Given the received matrix with columns $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$

- $\boldsymbol{y}_j = \boldsymbol{f}(\alpha_j)$ correct evaluations
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the *degree bound* $k > \deg(\boldsymbol{f})$

and *an upper bound* τ *on the number of errors*.

GOAL: **reconstruct** $\boldsymbol{f}(x)$

Simultaneous Cauchy Interpolation

Given the vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$

and the *degree bounds* $\tau + k$, $\tau + 1$

GOAL: **find** $(\underbrace{\varphi_1(x), \ldots, \varphi_l(x)}_{\boldsymbol{\varphi}(x)}, \psi(x))$ s.t.

- $\varphi_i(\alpha_j) = y_{i,j}\psi(\alpha_j)$
- $\deg(\varphi_i) < \tau + k$
- $\deg(\psi) < \tau + 1$

$\downarrow$

Simultaneous Rational Function Reconstruction

Decoding Interleaved Reed-Solomon codes

Simultaneous Interpolation with Errors

Given the received matrix with columns $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$

- $\boldsymbol{y}_j = \boldsymbol{f}(\alpha_j)$ correct evaluations
- $\boldsymbol{y}_j \neq \boldsymbol{f}(\alpha_j)$ erroneous evaluations

the *degree bound* $k > \deg(\boldsymbol{f})$

and *an upper bound* τ *on the number of errors*.

GOAL: **reconstruct** $\boldsymbol{f}(x)$

Simultaneous Cauchy Interpolation

Given the vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$

and the *degree bounds* $\tau + k$, $\tau + 1$

GOAL: **find** $\underbrace{(\varphi_1(x), \ldots, \varphi_l(x)}_{\boldsymbol{\varphi}(x)}, \psi(x))$ s.t.

- $\Lambda(\alpha_j) f_i(\alpha_j) = y_{i,j} \Lambda(\alpha_j)$
- $\deg(\Lambda \boldsymbol{f}) < \tau + k$
- $\deg(\Lambda) < \tau + 1$

$(\Lambda(x)\boldsymbol{f}(x), \Lambda(x))$

solution

$$\Lambda(x) = \prod_{\alpha_j \textbf{ erroneous}} (x - \alpha_j)$$

Error Locator Polynomial

roots = erroneous evaluation points

$\deg(\Lambda)$ = nb errors

Decoding Interleaved Reed-Solomon codes

Simultaneous Interpolation with Errors

Given the received matrix with columns $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$

- $\boldsymbol{y}_j = \boldsymbol{f}(\alpha_j)$ correct evaluations
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the *degree bound* $k > \deg(\boldsymbol{f})$

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GOAL: **reconstruct** $\boldsymbol{f}(x)$

Simultaneous Cauchy Interpolation

Given the vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$

and the *degree bounds* $\tau + k$, $\tau + 1$

GOAL: **find** $(\underbrace{\varphi_1(x), \ldots, \varphi_l(x)}_{\boldsymbol{\varphi}(x)}, \psi(x))$ s.t.

- $\varphi_i(\alpha_j) = y_{i,j}\psi(\alpha_j)$
- $\deg(\varphi_i) < \tau + k$
- $\deg(\psi) < \tau + 1$

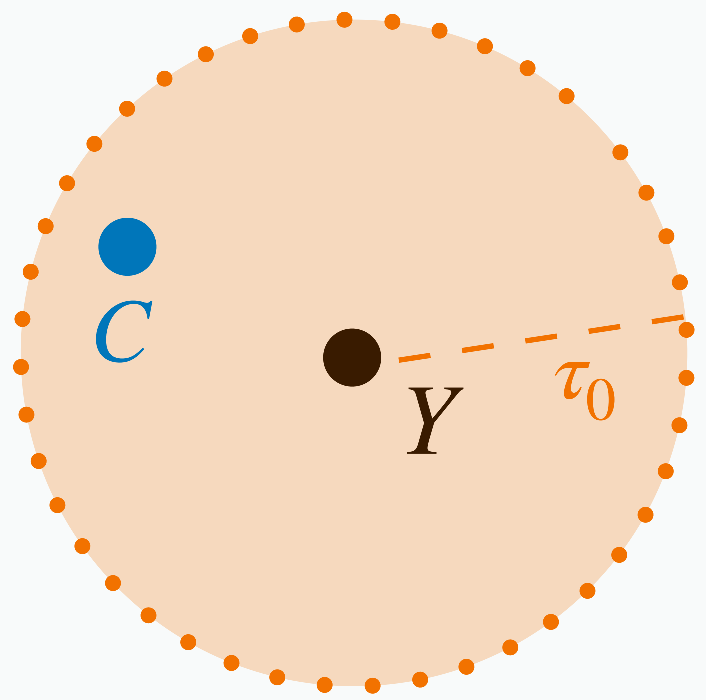
$\downarrow$

uniqueness

$(\Lambda(x)\boldsymbol{f}(x), \Lambda(x))$

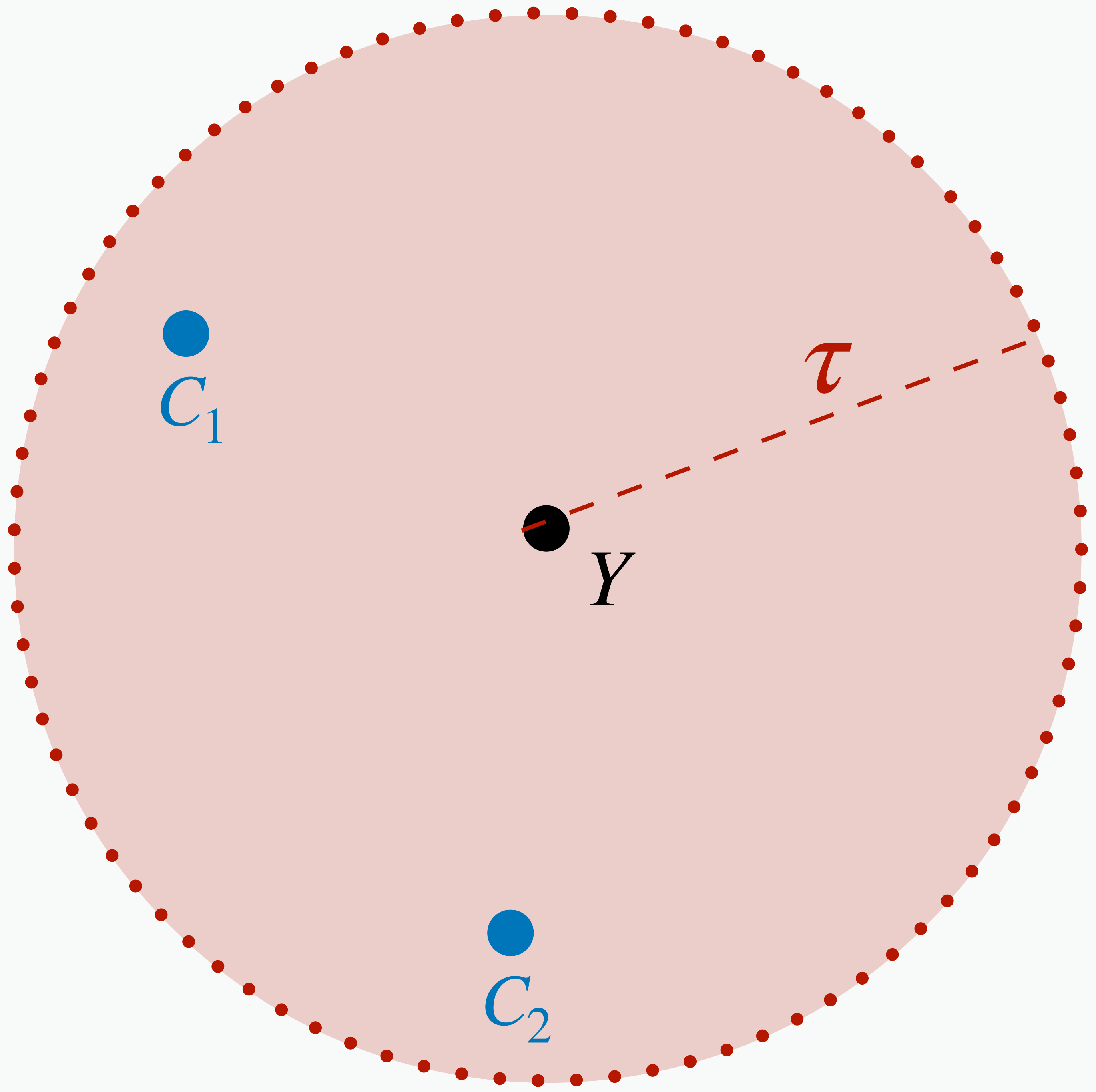
$\downarrow$

unique decoding



1. Unique decoding Capability

nb errors $\leq \dfrac{n-k}{2} = \dfrac{d-1}{2} := \tau_0 \longrightarrow$ unique decoding



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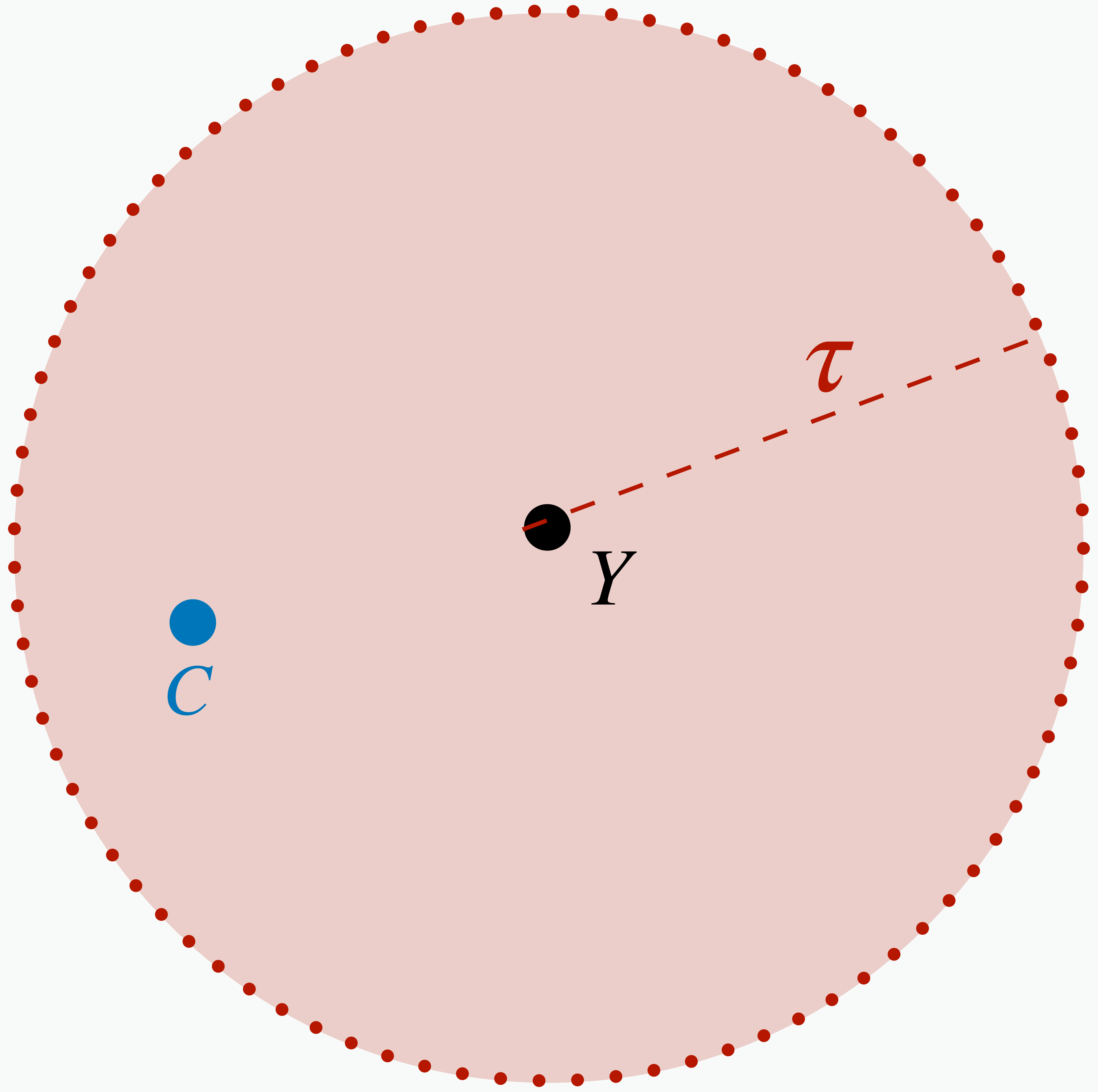


1. Unique decoding Capability

nb errors $\leq \dfrac{n-k}{2} = \dfrac{d-1}{2} := \tau_0 \longrightarrow$ unique decoding

2. [BLEICHENBACHER, KIAYIAS, YUNG, 2003]

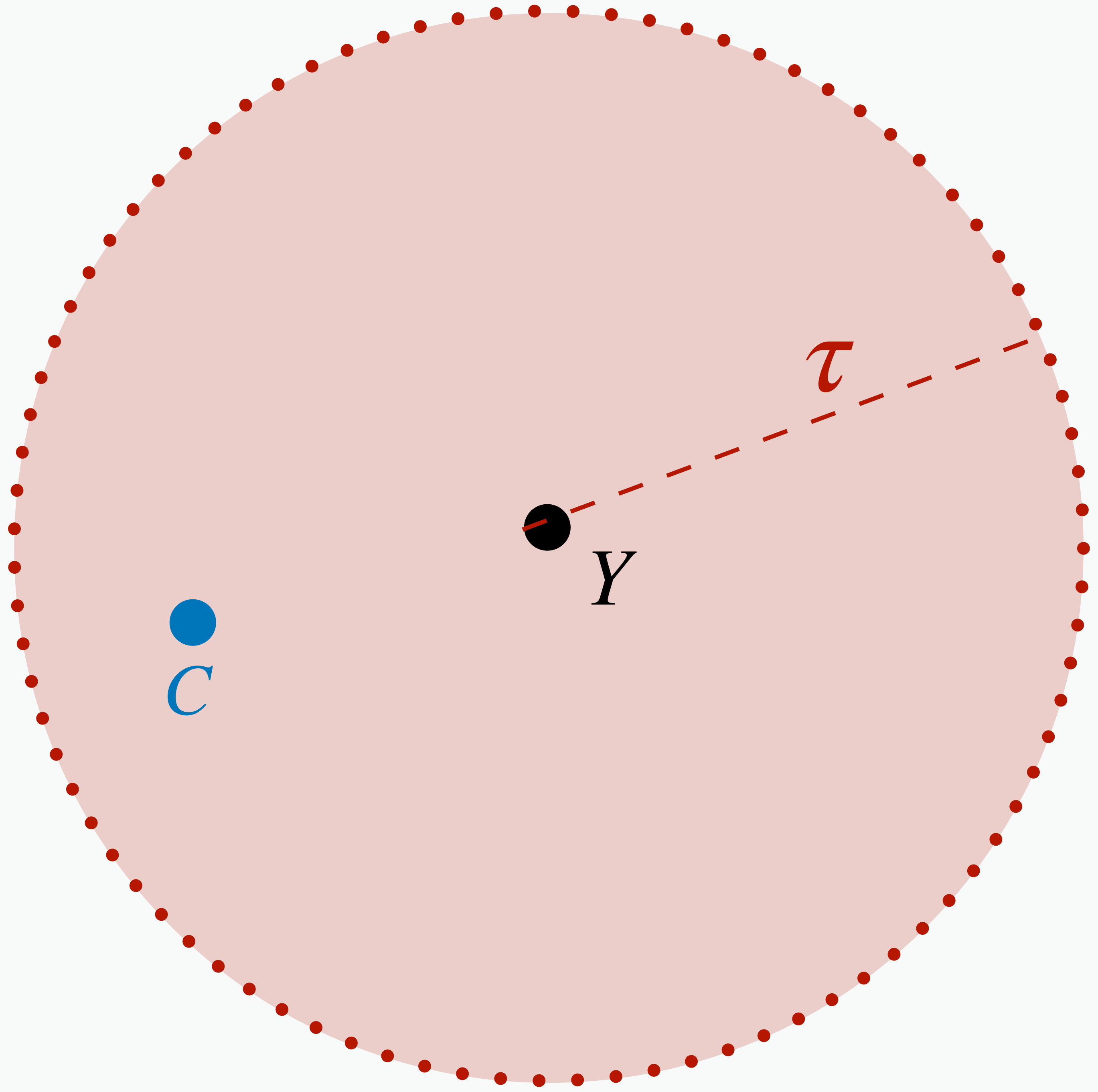
nb errors $\leq \dfrac{l(n-k)}{l+1} =: \tau_{IRS} \longrightarrow$ uniqueness for almost all error patterns

The proportion of error patterns leading to non-uniqueness $\leq$ nb errors$/q$

does not depend on errors, $\mathcal{O}(1/q)$

[BROWN, MINDER, SHOKROLLAHI, 2004]
[SCHIMDT, SIDORENKO, BOSSERT, 2009]
[PUCHINGER, ROSENKILDE, 2017]

Coding Theory & Computer Algebra



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Simultaneous Cauchy Interpolation

Given the vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_n$

and the *degree bounds* $\tau + k$, $\tau + 1$

GOAL: **find** $\underbrace{(\varphi_1(x), \ldots, \varphi_l(x), \psi(x))}_{\boldsymbol{\varphi}(x)}$ s.t.

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- $\deg(\varphi_i) < \tau + k$
- $\deg(\psi) < \tau + 1$

1. Cauchy Interpolation component-wise (RFR)

$n = \tau + k + \tau \iff$ nb errors $\leq \dfrac{n-k}{2} =: \tau_0 \longrightarrow$ uniqueness

2. Common denominator feature

$n = k + \left\lceil \dfrac{\tau}{l} \right\rceil + \tau \iff$ nb errors $\leq \dfrac{l(n-k)}{l+1} =: \tau_{IRS} \longrightarrow$ uniqueness for almost all error patterns

Simultaneous Interpolation with Errors

Reconstruct a **vector of polynomials** by its evaluations, some erroneous

$\updownarrow$

decoding IRS codes

Simultaneous Cauchy Interpolation with Errors

Reconstruct a **vector of rational functions** by its evaluations, some erroneous

$\downarrow$

🛠 **Simultaneous Cauchy Interpolation**

Number of Evaluations - Outline of this work

			uniqueness	**uniqueness** ***almost always***
no-errors	$(\boldsymbol{v}(x), d(x))$		$L = N + D - 1$ Cauchy Interpolation	$L = N + \left\lceil \frac{(D-1)}{n} \right\rceil$
	$A(x)\frac{\boldsymbol{v}(\boldsymbol{x})}{d(x)} = \boldsymbol{b}(x)$		$L = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\}$ [CABAY, 1971]	
with errors	$(\boldsymbol{v}(x), d(x))$	d constant $(D = 0)$	$L = N - 1 + 2\tau$ Unique Decoding Capability Theorem (IRS codes)	$L = N - 1 + \left\lceil \frac{\tau}{n} \right\rceil + \tau$ [BLEICHENBACHER, KIAYIAS, YUNG, 2003]
		$D > 0$	[BOYER, KALTOFEN, 2014]	**[GUERRINI, LEBRETON, Z., 2019]**
	$A(x)\frac{\boldsymbol{v}(\boldsymbol{x})}{d(x)} = \boldsymbol{b}(x)$		[KALTOFEN, PERNET, STORJOHANN, WADDEL, 2017]	**[GUERRINI, LEBRETON, Z., 2021]**

Number of Evaluations - Outline of this work

			uniqueness	uniqueness *almost always*
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Simultaneous Cauchy Interpolation with Errors

Simultaneous Cauchy Interpolation with Errors

Given $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_L$

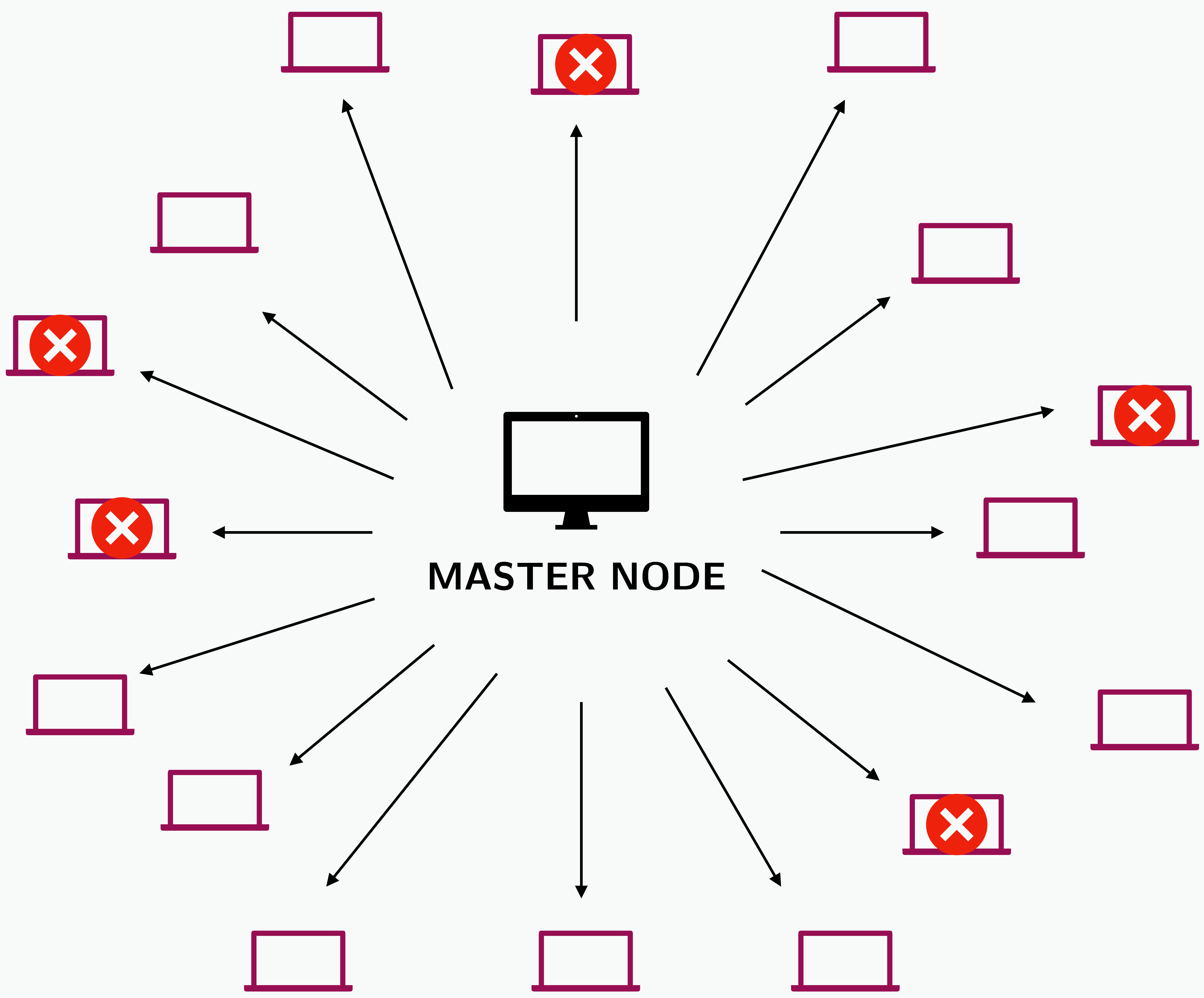
- $\boldsymbol{y}_j = \dfrac{\boldsymbol{v}(\alpha_j)}{d(\alpha_j)}$ correct evaluations
- $\boldsymbol{y}_j \neq \dfrac{\boldsymbol{v}(\alpha_j)}{d(\alpha_j)}$ erroneous evaluations

the *degree bounds* $N > \deg(\boldsymbol{v})$, $D > \deg(d)$ and *an upper bound* τ *on the number of errors*.

GOAL: **reconstruct** $(\boldsymbol{v}(x), d(x)) \longrightarrow \boldsymbol{y}(x)$

Simultaneous Cauchy Interpolation

Simultaneous Rational Function Reconstruction



Simultaneous Cauchy Interpolation with Errors

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GOAL: **reconstruct** $(\boldsymbol{v}(x), d(x)) \longrightarrow \boldsymbol{y}(x)$

Simultaneous Cauchy Interpolation

Given the vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_L$

and the *degree bounds* $N + \tau$, $D + \tau$

GOAL: **find** $\underbrace{(\varphi_1(x), \ldots, \varphi_n(x), \psi(x))}_{\boldsymbol{\varphi}(x)}$ s.t.

- $\varphi_i(\alpha_j) = y_{i,j}\psi(\alpha_j)$
- $\deg(\varphi_i) < N + \tau$
- $\deg(\psi) < D + \tau$

$(\Lambda(x)\boldsymbol{v}(x), \Lambda(x)d(x))$

solution

$$\Lambda(x) = \prod_{\alpha_j \textbf{ erroneous}} (x - \alpha_j)$$

Error Locator Polynomial

roots = erroneous evaluation points

$\deg(\Lambda)$ = nb errors

Simultaneous Cauchy Interpolation with Errors

Simultaneous Cauchy Interpolation with Errors

Given $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_L$

- $\boldsymbol{y}_j = \dfrac{\boldsymbol{v}(\alpha_j)}{d(\alpha_j)}$ correct evaluations
- $\boldsymbol{y}_j \neq \dfrac{\boldsymbol{v}(\alpha_j)}{d(\alpha_j)}$ erroneous evaluations

the *degree bounds* $N > \deg(\boldsymbol{v})$, $D > \deg(d)$

and *an upper bound* τ *on the number of errors*.

GOAL: **reconstruct** $(\boldsymbol{v}(x), d(x)) \longrightarrow \boldsymbol{y}(x)$

Simultaneous Cauchy Interpolation

Given the vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_L$

and the *degree bounds* $N + \tau$, $D + \tau$

GOAL: **find** $\underbrace{(\varphi_1(x), \ldots, \varphi_n(x), \psi(x))}_{\boldsymbol{\varphi}(x)}$ s.t.

- $\Lambda(\alpha_j) v_i(\alpha_j) = y_{i,j} \Lambda(\alpha_j)$
- $\deg(\Lambda v_i) < N + \tau$
- $\deg(\Lambda d) < D + \tau$

$(\Lambda(x)\boldsymbol{v}(x), \Lambda(x)d(x))$

solution

$$\Lambda(x) = \prod_{\alpha_j \text{ erroneous}} (x - \alpha_j)$$

Error Locator Polynomial

roots = erroneous evaluation points

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Simultaneous Cauchy Interpolation with Errors

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Simultaneous Cauchy Interpolation

Given the vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_L$

and the *degree bounds* $N + \tau$, $D + \tau$

GOAL: **find** $(\underbrace{\varphi_1(x), \ldots, \varphi_n(x)}_{\boldsymbol{\varphi}(x)}, \psi(x))$ s.t.

- $\varphi_i(\alpha_j) = y_{i,j}\psi(\alpha_j)$
- $\deg(\varphi_i) < N + \tau$
- $\deg(\psi) < D + \tau$

$\downarrow$

uniqueness

$(\Lambda(x)\boldsymbol{v}(x), \Lambda(x)d(x))$

Simultaneous Cauchy Interpolation with Errors

Simultaneous Cauchy Interpolation

Given the vectors $\boldsymbol{y}_1, \dots, \boldsymbol{y}_L$

and the *degree bounds* $N+\tau$, $D+\tau$

GOAL: **find** $\underbrace{(\varphi_1(x), \dots, \varphi_n(x)}_{\boldsymbol{\varphi}(x)}, \psi(x))$ s.t.

- $\varphi_i(\alpha_j) = y_{i,j}\psi(\alpha_j)$
- $\deg(\varphi_i) < N + \tau$
- $\deg(\psi) < D + \tau$

1. Cauchy Interpolation component-wise

   $L = N + D + 2\tau - 1 \longrightarrow$ uniqueness

   [BOYER, KALTOFEN, 2014]

2. If we want to recover a solution of a PLS

   $L = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\} + 2\tau \longrightarrow$ uniqueness

   [CABAY, 1971] $\longrightarrow$ [KALTOFEN, PERNET, STORJOHANN, WADDEL, 2017]

$L = N + D - 1 + \left\lceil \frac{\tau}{n} \right\rceil + \tau \longrightarrow$ uniqueness for almost all error patterns

If we want to recover a solution of a PLS

$L = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\} + \left\lceil \frac{\tau}{n} \right\rceil + \tau \longrightarrow$ uniqueness for almost all error patterns

generalizing and re-elaborating the result of [BLEICHENBACHER, KIAYIAS, YUNG, 2003] for IRS codes

The proportion of error patterns leading to non-uniqueness $\leq (D+\tau)/q$

Number of Evaluations - Outline of this work

			uniqueness	uniqueness *almost always*
no-errors	$(\boldsymbol{v}(x), d(x))$		$L = N + D - 1$ Cauchy Interpolation	$L = N + \left\lceil \frac{(D-1)}{n} \right\rceil$
no-errors	$A(x)\frac{\boldsymbol{v}(x)}{d(x)} = \boldsymbol{b}(x)$		$L = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\}$ [CABAY, 1971]	
with errors	$(\boldsymbol{v}(x), d(x))$	d constant $(D = 0)$	$L = N - 1 + 2\tau$ Unique Decoding Capability Theorem (IRS codes)	$L = N - 1 + \left\lceil \frac{\tau}{n} \right\rceil + \tau$ [BLEICHENBACHER, KIAYIAS, YUNG, 2003]
with errors	$(\boldsymbol{v}(x), d(x))$	$D > 0$	$L = N + D - 1 + 2\tau$ [BOYER, KALTOFEN, 2014]	$L = N + D - 1 + \left\lceil \frac{\tau}{n} \right\rceil + \tau$
with errors	$A(x)\frac{\boldsymbol{v}(x)}{d(x)} = \boldsymbol{b}(x)$	$D > 0$	$L = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\} + 2\tau$ [KALTOFEN, PERNET, STORJOHANN, WADDEL, 2017]	$L = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\} + \left\lceil \frac{\tau}{n} \right\rceil + \tau$

Early Termination

$$\overline{L} = \min\{L_{BK}, L_{KPS}\}$$

- $L_{BK} = N + D + 2\tau - 1,$
- $L_{KPSW} = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\} + 2\tau$

$\geq$

$$L_{new} = \min\{L_{GLZ19}, L_{GLZ20}\}$$

- $L_{GLZ19} = N + D - 1 + \tau + \left\lceil \frac{\tau}{n} \right\rceil,$
- $L_{GLZ20} = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\} + \tau + \left\lceil \frac{\tau}{n} \right\rceil$

All these bounds depend on

- $N > \deg(\boldsymbol{v})$
- $D > \deg(d)$
- $\tau \geq$ nb errors $|E|$

we don't know these quantities

If N, D, τ are too big compared to $\deg(\boldsymbol{v})$, $\deg(d)$, $|E|$ $\longrightarrow$ $\overline{L}$, L_{new} too big compared to the number we really need

EARLY TERMINATION TECHNIQUE [KALTOFEN, PERNET, STORJOHANN, WADDELL, 2017]

GOAL: decrease the number of evaluation points without knowing the real degrees

Early termination strategy

Simultaneous Cauchy Interpolation

Given the vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_L$

and the *degree bounds* $N + \tau$, $D + \tau$

GOAL: **find** $\underbrace{(\varphi_1(x), \ldots, \varphi_n(x)}_{\boldsymbol{\varphi}(x)}, \psi(x))$ s.t.

- $\varphi_i(\alpha_j) = y_{i,j}\psi(\alpha_j)$
- $\deg(\varphi_i) < N + \tau$
- $\deg(\psi) < D + \tau$

$\longrightarrow$ $(\Lambda \boldsymbol{v}, \Lambda d)$ is a solution

Early termination strategy

Simultaneous Cauchy Interpolation

Given the vectors $\boldsymbol{y}_1, \dots, \boldsymbol{y}_L$

and the *degree bounds* $N+\tau$, $D+\tau$

GOAL: **find** $\underbrace{(\varphi_1(x), \dots, \varphi_n(x)}_{\boldsymbol{\varphi}(x)}, \psi(x))$ s.t.

- $\varphi_i(\alpha_j) = y_{i,j}\psi(\alpha_j)$
- $\deg(\varphi_i) < \nu$
- $\deg(\psi) < \vartheta$

$\longrightarrow$ $(\Lambda \boldsymbol{v}, \Lambda d)$ is a solution

IDEA

Replace $\begin{matrix} N+\tau \\ D+\tau \end{matrix}$ with some parameters $\begin{matrix} \nu \\ \vartheta \end{matrix}$

$\mathscr{L}$ which depends on ν, ϑ

$\begin{matrix} \nu \\ \vartheta \end{matrix}$ $\longrightarrow$ **TEST**

$\nearrow$ uniqueness $(\Lambda \boldsymbol{v}, \Lambda d)$

$\searrow$ ν, ϑ too small

Early termination strategy

Simultaneous Cauchy Interpolation

Given the vectors $\boldsymbol{y}_1, \dots, \boldsymbol{y}_L$

and the *degree bounds* $N + \tau$, $D + \tau$

GOAL: **find** $\underbrace{(\varphi_1(x), \dots, \varphi_n(x)}_{\boldsymbol{\varphi}(x)}, \psi(x))$ s.t.

- $\varphi_i(\alpha_j) = y_{i,j}\psi(\alpha_j)$
- $\deg(\varphi_i) < \nu$
- $\deg(\psi) < \vartheta$

$\longrightarrow$ $(\Lambda \boldsymbol{v}, \Lambda d)$ is a solution

IDEA

Replace $\begin{matrix} N+\tau \\ D+\tau \end{matrix}$ with some parameters $\begin{matrix} \nu \\ \vartheta \end{matrix}$

$\mathscr{L}_{NEW}$ which depends on ν, ϑ

ν, ϑ $\longrightarrow$ **TEST**

- uniqueness $(\Lambda \boldsymbol{v}, \Lambda d)$
- ν, ϑ too small

Conclusions & Open Problems

Number of Evaluations - Outline of this work

			uniqueness	uniqueness *almost always*
no-errors	$(\boldsymbol{v}(x), d(x))$		$L = N + D - 1$ Cauchy Interpolation	$L = N + \left\lceil \frac{(D-1)}{n} \right\rceil$
	$A(x)\frac{\boldsymbol{v}(x)}{d(x)} = \boldsymbol{b}(x)$		$L = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\}$ [CABAY, 1971]	
with errors	$(\boldsymbol{v}(x), d(x))$	d constant $(D = 0)$	$L = N - 1 + 2\tau$ Unique Decoding Capability Theorem (IRS codes)	$L = N - 1 + \left\lceil \frac{\tau}{n} \right\rceil + \tau$ [BLEICHENBACHER, KIAYIAS, YUNG, 2003]
		$D > 0$	$L = N + D - 1 + 2\tau$ [BOYER, KALTOFEN, 2014]	$L = N + D - 1 + \left\lceil \frac{\tau}{n} \right\rceil + \tau$
	$A(x)\frac{\boldsymbol{v}(x)}{d(x)} = \boldsymbol{b}(x)$		$L = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\} + 2\tau$ [KALTOFEN, PERNET, STORJOHANN, WADDEL, 2017]	$L = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\} + \left\lceil \frac{\tau}{n} \right\rceil + \tau$

Early termination technique

Publications

- ***Polynomial Linear System Solving with Errors by Simultaneous Polynomial Reconstruction of Interleaved Reed-Solomon codes.*** E.Guerrini, R.Lebreton, I.Zappatore. *In Proceedings of ISIT'19*, IEEE, 2019
- ***On the Uniqueness of Rational Function Reconstruction.*** E.Guerrini, R.Lebreton, I.Zappatore. *In Proceedings of ISSAC'20.* ACM, 2020
- ***Polynomial Linear Systems Solving with Random Errors: New Bound and Early Termination Technique.*** E.Guerrini, R.Lebreton, I.Zappatore. *In Proceedings of ISSAC'21.* ACM, 2021

Perspectives

Improving previous results

- On the uniqueness of the simultaneous rational function reconstruction
- Failure probability of Simultaneous Cauchy Interpolation with Errors

Extending previous results

- Rational Function Codes
- Algorithm-based fault tolerant technique based on Hermite interpolation

On the uniqueness of SRFR

Simultaneous Cauchy Interpolation

Given two vector of polynomials $\boldsymbol{u}(x), \boldsymbol{a}(x)$ and the *degree bounds* N, D

GOAL: **find** $\underbrace{(v_1(x), \dots, v_n(x)}_{\boldsymbol{v}(x)}, d(x))$ s.t.

- $v_i(\alpha_j) = y_{ij} d(\alpha_j)$
- $\deg(v_i) < N$
- $\deg(d) < D$

$$\frac{v_i(\alpha_j)}{d(\alpha_j)} = y_{ij}, d(\alpha_j) \neq 0$$

If $L = N + (D-1)/n$, **for almost all instances** $y_i = v_i/d \implies$ **uniqueness?**

Theorem [GUERRINI, LEBRETON, Z., 2020]

If $L = N + (D-1)/n$, **for almost all instances** $\implies$ **uniqueness**.

If $\mathbb{K} = \mathbb{F}_q$, the **proportion of instances** leading to **non-uniqueness** is $\leq (D-1)/q$

Number of Evaluations - Outline of this work

			uniqueness	**uniqueness** *almost always*
no-errors	$\frac{v(x)}{d(x)}$		$L = N + D - 1$ Cauchy Interpolation	**?** $L = N + \left\lceil \frac{(D-1)}{n} \right\rceil$
	$A(x)\frac{\boldsymbol{v}(\boldsymbol{x})}{d(x)} = \boldsymbol{b}(x)$		$L = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\}$ [CABAY, 1971]	**?** $L = N + \left\lceil \frac{(D-1)}{n} \right\rceil$
with errors	$(\boldsymbol{v}(x), d(x))$	d constant $(D = 0)$	$L = N - 1 + 2\tau$ Unique Decoding Capability Theorem (IRS codes)	$L = N - 1 + \left\lceil \frac{\tau}{n} \right\rceil + \tau$ [BLEICHENBACHER, KIAYIAS, YUNG, 2003]
		$D > 0$	$L = N + D - 1 + 2\tau$ [BOYER, KALTOFEN, 2014]	$L = N + D - 1 + \left\lceil \frac{\tau}{n} \right\rceil + \tau$
	$A(x)\frac{\boldsymbol{v}(x)}{d(x)} = \boldsymbol{b}(x)$		$L = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\} + 2\tau$ [KALTOFEN, PERNET, STORJOHANN, WADDEL, 2017]	$L = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\} + \left\lceil \frac{\tau}{n} \right\rceil + \tau$

Simultaneous Cauchy Interpolation with Errors

Simultaneous Cauchy Interpolation

Given the vectors $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_L$

and the *degree bounds* $N+\tau$, $D+\tau$

GOAL: **find** $\underbrace{(\varphi_1(x), \ldots, \varphi_n(x), \psi(x))}_{\boldsymbol{\varphi}(x)}$ s.t.

- $\varphi_i(\alpha_j) = y_{i,j}\psi(\alpha_j)$
- $\deg(\varphi_i) < N+\tau$
- $\deg(\psi) < D+\tau$

$\downarrow$

uniqueness

$(\Lambda(x)\boldsymbol{v}(x), \Lambda(x)d(x))$

1. Cauchy Interpolation component-wise (RFR)

   $L = N + D + 2\tau - 1 \longrightarrow$ uniqueness

   [BOYER, KALTOFEN, 2014]

2. If we want to recover a solution of a PLS

   $L = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\} + 2\tau \longrightarrow$ uniqueness

   [CABAY, 1971] $\longrightarrow$ [KALTOFEN, PERNET, STORJOHANN, WADDEL, 2014]

Common denominator constraint

$$L = N + \left\lceil \frac{D-1+\tau}{n} \right\rceil + \tau \longrightarrow$$

uniqueness

for almost all error patterns

for almost all $\boldsymbol{v}/d$ **?**

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Number of Evaluations - Outline of this work

			uniqueness	uniqueness *almost always*
no-errors	$(\mathbf{v}(x), d(x))$		$L = N + D - 1$ Cauchy Interpolation	$L = N + \left\lceil \frac{(D-1)}{n} \right\rceil$
	$A(x)\frac{\mathbf{v}(x)}{d(x)} = \mathbf{b}(x)$		$L = \max\{\deg(A) + N, \deg(\mathbf{b}) + D\}$ [CABAY, 1971]	?
with errors	$(\mathbf{v}(x), d(x))$	d constant $(D = 0)$	$L = N - 1 + 2\tau$ Unique Decoding Capability Theorem (IRS codes)	$L = N - 1 + \left\lceil \frac{\tau}{n} \right\rceil + \tau$ [BLEICHENBACHER, KIAYIAS, YUNG, 2003]
		$D > 0$	$L = N + D - 1 + 2\tau$ [BOYER, KALTOFEN, 2014]	$L = N + D - 1 + \left\lceil \frac{\tau}{n} \right\rceil + \tau$
	$A(x)\frac{\mathbf{v}(x)}{d(x)} = \mathbf{b}(x)$	$D > 0$	$L = \max\{\deg(A) + N, \deg(\mathbf{b}) + D\} + 2\tau$ [KALTOFEN, PERNET, STORJOHANN, WADDEL, 2017]	$L = \max\{\deg(A) + N, \deg(\mathbf{b}) + D\} + \left\lceil \frac{\tau}{n} \right\rceil + \tau$

Simultaneous Cauchy Interpolation with Errors

Simultaneous Cauchy Interpolation

Given the vectors $\boldsymbol{y}_1, \dots, \boldsymbol{y}_L$

and the *degree bounds* $N + \tau$, $D + \tau$

GOAL: **find** $\underbrace{(\varphi_1(x), \dots, \varphi_n(x), \psi(x))}_{\boldsymbol{\varphi}(x)}$ s.t.

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- $\deg(\varphi_i) < N + \tau$
- $\deg(\psi) < D + \tau$

generalizing and re-elaborating the result of [BLEICHENBACHER, KIAYIAS, YUNG, 2003] for IRS codes

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   $L = N + D + 2\tau - 1 \longrightarrow$ uniqueness

   [BOYER, KALTOFEN, 2014]

2. If we want to recover a solution of a PLS

   $L = \max\{\deg(A) + N, \deg(\boldsymbol{b}) + D\} + 2\tau \longrightarrow$ uniqueness

   [CABAY, 1971] $\longrightarrow$ [KALTOFEN, PERNET, STORJOHANN, WADDEL, 2017]

$L = N +$ ... $\left\lceil \tau \right\rceil$ ... uniqueness

...atterns

If we want t...

$L = \max\{\deg$ ... eness

...error patterns

can we improve this bound?
prove that it does not depend on errors as IRS?

[BROWN, MINDER, SHOKROLLAHI, 2004]
[SCHMIDT, SIDORENKO, BOSSERT, 2009]
[PUCHINGER, ROSENKILDE, 2017]

The proportion of error patterns leading to non-uniqueness $\leq (D + \tau)/q$

Conclusions & Open Problems

Improving previous results

- On the uniqueness of the simultaneous rational function reconstruction
- Failure probability of Simultaneous Cauchy Interpolation with Errors

Extending previous results

- Rational Function Codes
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Simultaneous Cauchy Interpolation with Errors

Simultaneous Interpolation with Errors

Reconstruct a **vector of polynomials**
by its evaluations, some erroneous

decoding IRS codes

Simultaneous Cauchy Interpolation with Errors

Reconstruct a **vector of rational functions**
by its evaluations, some erroneous

decoding specific
Interleaved Rational Function codes
[PERNET, 2017]

Simultaneous Cauchy Interpolation

Conclusions & Open Problems

Rational Function Code [PERNET, 2017]

generalization of Reed-Solomon codes, **non linear**

Let $N, D \leq L \leq q$ and $\{\alpha_1, \ldots, \alpha_L\}$ distinct *evaluation points*,

$$\mathscr{C}_{RF}(n,k) := \left\{ \left(\frac{v(\alpha_1)}{d(\alpha_1)}, \ldots, \frac{v(\alpha_L)}{d(\alpha_L)} \right) | \frac{v}{d} \in \mathbb{F}_q(x), \deg(v) < N, \deg(d) < D, d(\alpha_j) \neq 0 \right\}$$

Interleaved Rational Function Code [PERNET, 2017]

the minimum distance $\geq L - (N + D + 2)$

Let $N, D \leq L \leq q$ and $\{\alpha_1, \ldots, \alpha_L\}$ distinct *evaluation points*,

$$\mathscr{C}_{RF}(n,k) := \left\{ \left(\frac{\boldsymbol{v}(\alpha_1)}{\boldsymbol{d}(\alpha_1)}, \ldots, \frac{\boldsymbol{v}(\alpha_L)}{\boldsymbol{d}(\alpha_L)} \right) | \frac{\boldsymbol{v}}{\boldsymbol{d}} \in \mathbb{F}_q(x)^{n\times 1}, \deg(v) < N, \deg(d) < D, d_i(\alpha_j) \neq 0 \right\}$$

Perspectives

Improving previous results

- On the uniqueness of the simultaneous rational function reconstruction
- Failure probability of Simultaneous Cauchy Interpolation with Errors

Extending previous results

- Rational Function Codes $\longrightarrow$ determine parameters and other applications
- Algorithm-based fault tolerant technique with Hermite interpolation (in progress)

Thank you