The effective Deuring correspondence

the key to the next generation of isogeny based cryptography?

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Seminario UMI Crittografia e Codici

Why isogenies?

Quantum-safe crypto

- Shortest ciphertexts and public keys for Encryption: SIDH/SIKE
  CSIDH*
- Shortest public key + Signature: SQISign
- Only efficient Non-Interactive Key Exchange: CSIDH*
- Acceptable Threshold Signatures: CSI-FiSh*

*Secure parameter sizes still debated, big impact on performance.

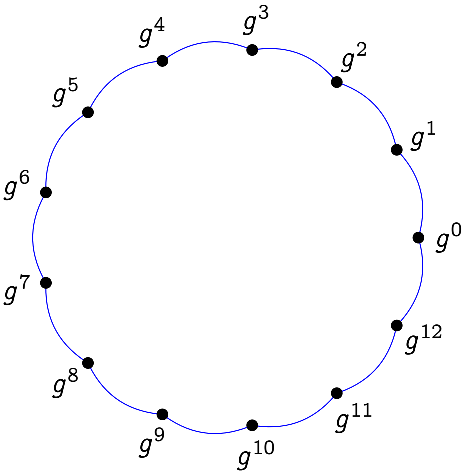
Time-delay crypto (not quantum safe)

- Only efficient alternative to group-based Verifiable Delay Functions: Asiacrypt ’19
- Only known instantiation of Delay Encryption: Eurocrypt ’21

Brief history of isogeny-based cryptography

- **1997** Couveignes introduces the Hard Homogeneous Spaces framework. His work stays unpublished for 10 years.
- **2006** Rostovtsev & Stolbunov independently rediscover Couveignes ideas, suggest isogeny-based Diffie–Hellman as a quantum-resistant primitive.
- **2006-2010** Other isogeny-based protocols by Teske and Charles, Goren & Lauter.
- **2011-2012** D., Jao & Plût introduce SIDH, an efficient post-quantum key exchange inspired by Couveignes, Rostovtsev, Stolbunov, Charles, Goren, Lauter.
- **2017** SIDH is submitted to the NIST competition (with the name SIKE, only isogeny-based candidate).
- **2018** Castryck, Lange, Martindale, Panny & Renes create an efficient variant of the Couveignes–Rostovtsev–Stolbunov protocol, named CSIDH.
- **2019** Isogeny signature craze: SeaSign (D. & Galbraith; Decru, Panny & Vercauteren), CSI-FiSh (Beullens, Kleinjung & Vercauteren), VDF (D., Masson, Petit & Sanso).
- **2020** Isogeny signatures get interesting: SQISign (D., Kohel, Leroux, Petit, Wesolowski). SIKE is an Alternate candidate finalist in NIST's 3rd round.

A cyclic group



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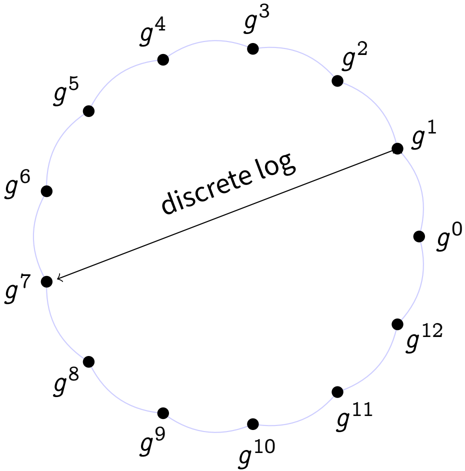

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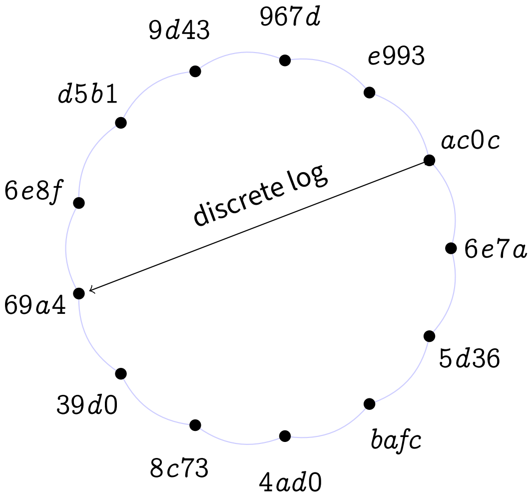

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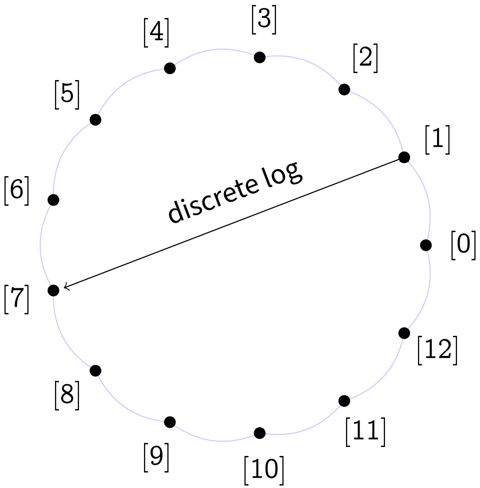


The axioms of a dlog group:

prod: $[a][b] = [a+b]$,

exp: $n[a] = [na]$.

The hard problem:

dlog: $[a] \mapsto a$.

Diffie–Hellman

Alice | **Bob**

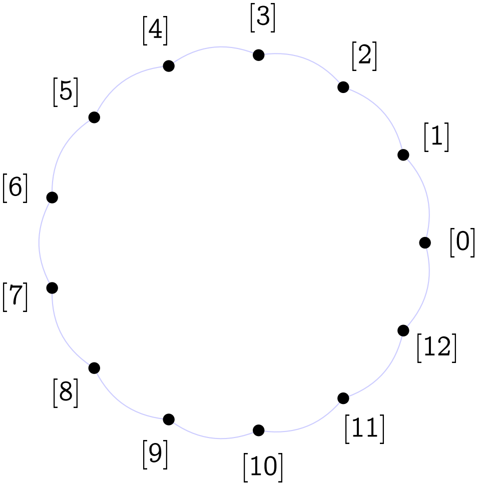
pick random $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ | pick random $b \in (\mathbb{Z}/N\mathbb{Z})^\times$

$$\xrightarrow{\quad [a] \quad}$$

$$\xleftarrow[\quad [b] \quad]{}$$

Shared secret is $a[b] = [ab] = b[a]$

A cyclic group



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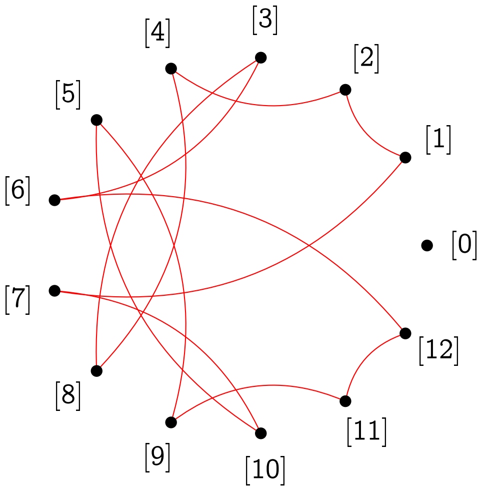

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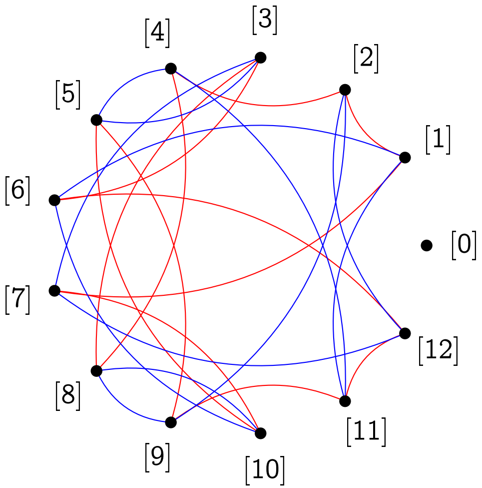

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$[a]$ ——— $2[a]$

$[a]$ ——— $6[a]$

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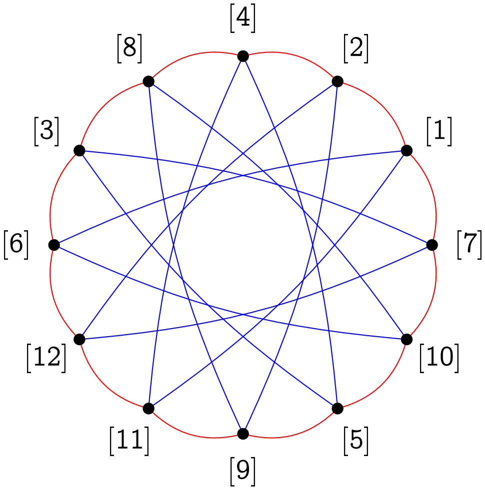

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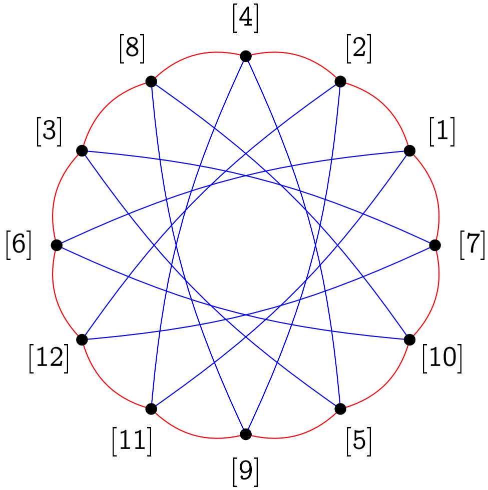

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Automorphism group: $(\mathbb{Z}/13\mathbb{Z})^{\times}.$

Group action

$\mathcal{G} \circlearrowleft \mathcal{E}$: A (finite) set $\mathcal{E}$ acted upon by a group $\mathcal{G}$ freely and transitively:

$$\begin{aligned} * : \mathcal{G} \times \mathcal{E} &\longrightarrow \mathcal{E} \\ \mathfrak{g} * E &\longmapsto E' \end{aligned}$$

Compatibility: $\mathfrak{g}' * (\mathfrak{g} * E) = (\mathfrak{g}'\mathfrak{g}) * E$ for all $\mathfrak{g}, \mathfrak{g}' \in \mathcal{G}$ and $E \in \mathcal{E}$;

Identity: $\mathfrak{e} * E = E$ if and only if $\mathfrak{e} \in \mathcal{G}$ is the identity element;

Regularity: for all $E, E' \in \mathcal{E}$ there exist a unique $\mathfrak{g} \in \mathcal{G}$ such that $\mathfrak{g} * E' = E$.

Cryptographic Group Actions (Alamati, D., Montgomery, Patranabis 2021)

Hard Homogeneous Space (HHS) — Couveignes 1997 (eprint:2006/291)

$\mathcal{G} \circlearrowleft \mathcal{E}$ such that $\mathcal{G}$ is commutative and:

- Evaluating $E' = \mathfrak{g} * E$ is easy;
- Inverting the action is hard.

Example

Let G be a group of order 13, then $(\mathbb{Z}/13\mathbb{Z})^\times \circlearrowleft G$ defined by

$$a * g := g^a$$

is an HHS…

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Example

Let G be a group of order 13, then $(\mathbb{Z}/13\mathbb{Z})^\times \circlearrowright G$ defined by

$$a * g := g^a$$

is an HHS…But

$$g^a \cdot g^b = g^{a+b}$$

has no interpretation as a group action!

Key exchange from group actions

Public parameters:

- A HHS $\mathcal{G} \circlearrowright \mathcal{E}$ of order N (large, but not necessarily prime);
- A starting set element $E_0 \in \mathcal{E}$.

Notation: $[\mathfrak{a}] := \mathfrak{a} * E_0$.

Alice | **Bob**

pick random $\mathfrak{a} \in \mathcal{G}$ | pick random $\mathfrak{b} \in \mathcal{G}$

$$\xrightarrow{\quad[\mathfrak{a}]\quad}$$

$$\xleftarrow[\quad[\mathfrak{b}]\quad]{}$$

Shared secret is $\mathfrak{a}[\mathfrak{b}] = [\mathfrak{a}\mathfrak{b}] = \mathfrak{b}[\mathfrak{a}]$

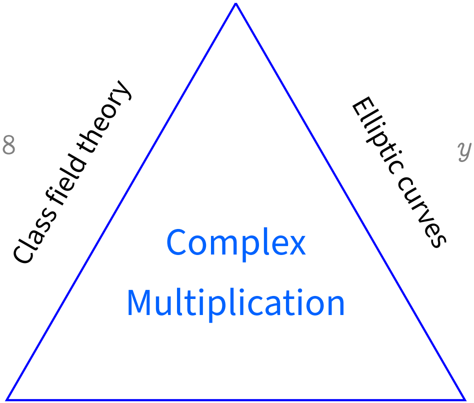
Quantum security

Fact: Shor's algorithm does not apply to Diffie-Hellman protocols from group actions.

Subexponential attack $\exp(\sqrt{\log p \log\log p})$

- Reduction to the hidden shift problem by evaluating the class group action in quantum superposition (subexpoential cost);
- Well known reduction from the hidden shift to the dihedral (non-abelian) hidden subgroup problem;
- Kuperberg's algorithm solves the dHSP with a subexponential number of class group evaluations.
- Recent work suggests that 2^{64}-qbit security is achieved somewhere in $512 < \log p < 2048$.

$H(j) = j - 1728$



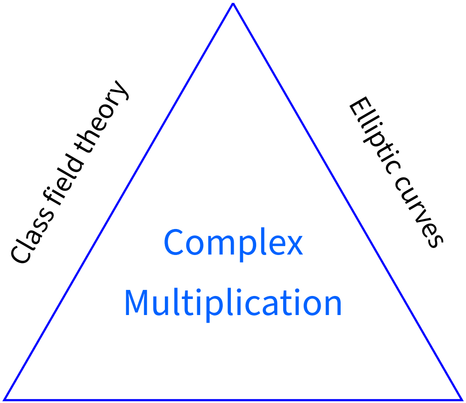
$y^2 = x^3 - ax - b$

Modular functions

$$j(z) = \frac{1}{q} + 744 + 196884q + \cdots$$

Abelian extensions

of $\mathbb{Q}(\sqrt{-D})$



Elliptic curves with

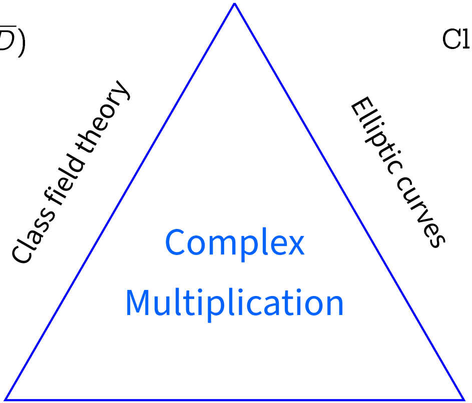
$\mathrm{End}(E) \subset \mathbb{Q}(\sqrt{-D})$

Modular functions

Galois group of $K/\mathbb{Q}(\sqrt{-D})$

$\simeq$

Class group $\mathrm{Cl}(-D)$

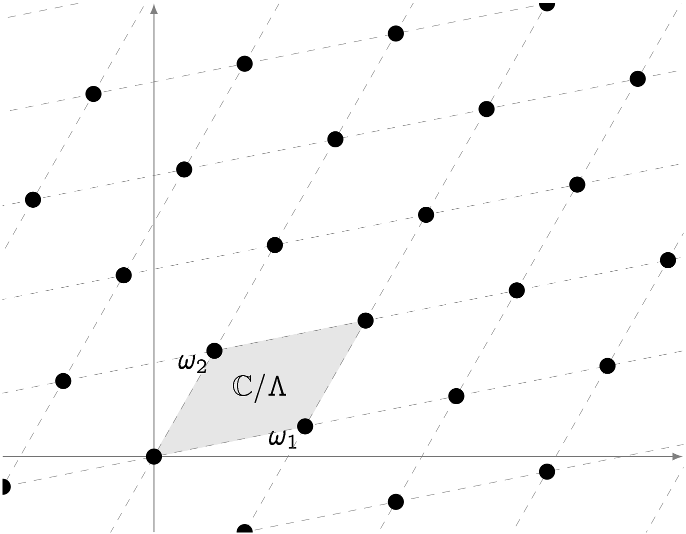


$\mathrm{Cl}(-D)$ acts on set of E s.t.

$\mathrm{End}(E) \subset \mathbb{Q}(\sqrt{-D})$

Modular functions

Complex tori

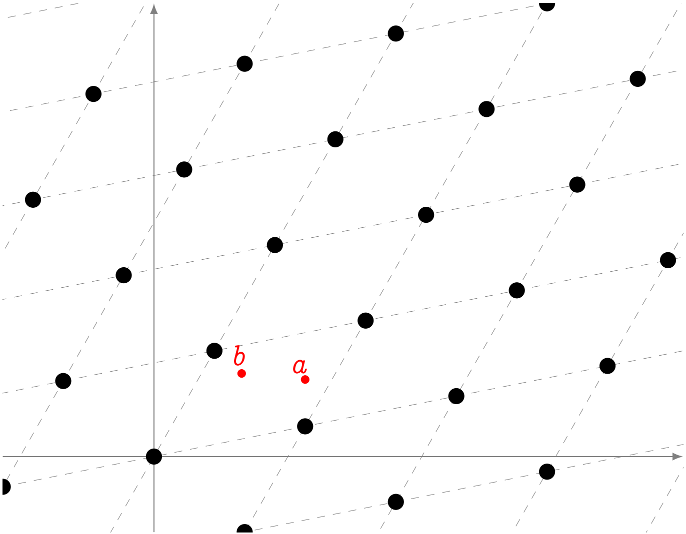


Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent complex numbers. Set

$$\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$$

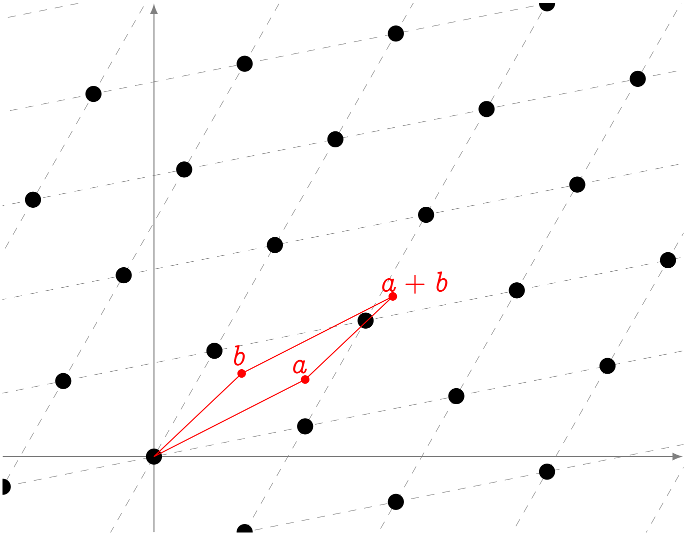
$\mathbb{C}/\Lambda$ is a complex torus.

Complex tori



Addition law induced by addition on $\mathbb{C}$.

Complex tori



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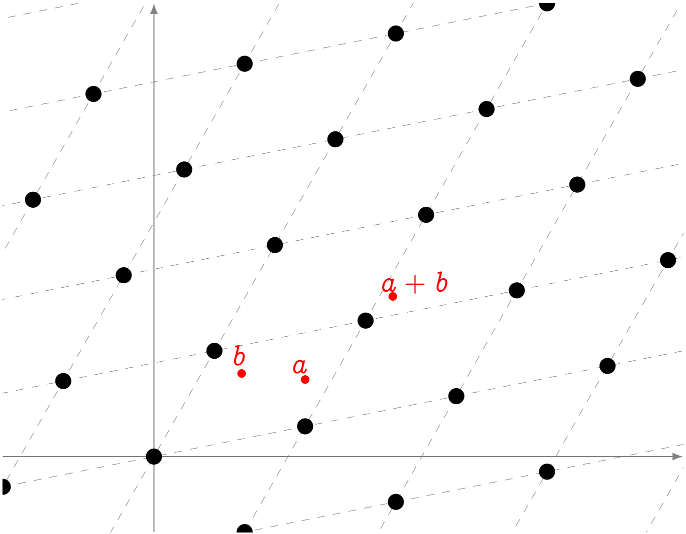

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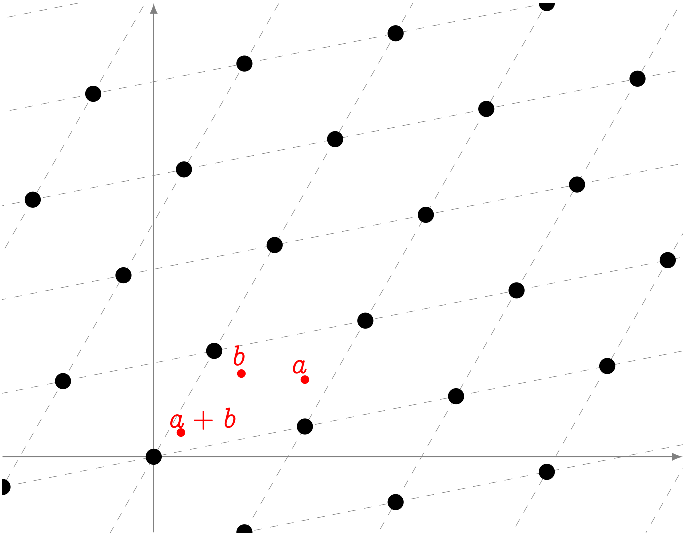

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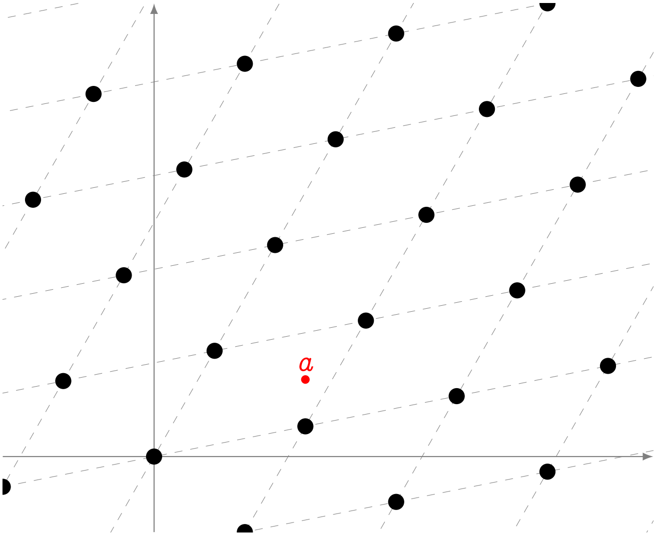
Homotheties



Two lattices are homothetic if there exist $\alpha \in \mathbb{C}$ such that

$$\alpha\Lambda_1 = \Lambda_2$$

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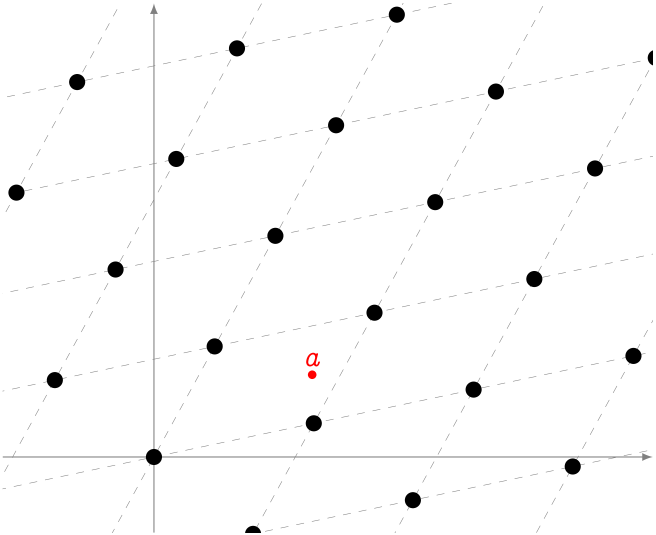

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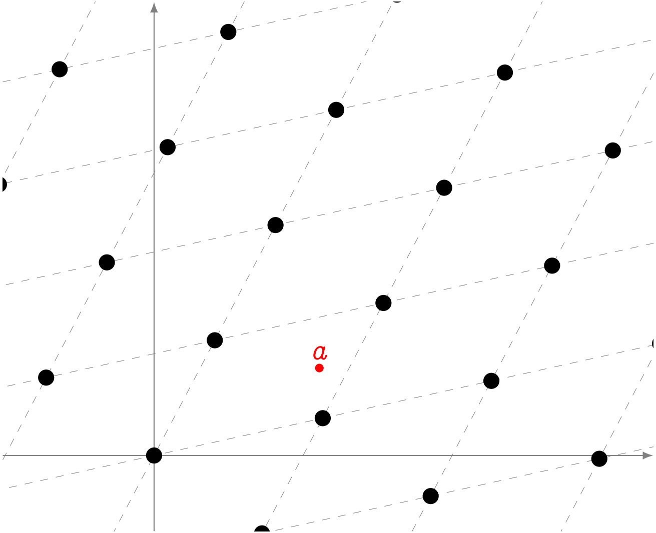

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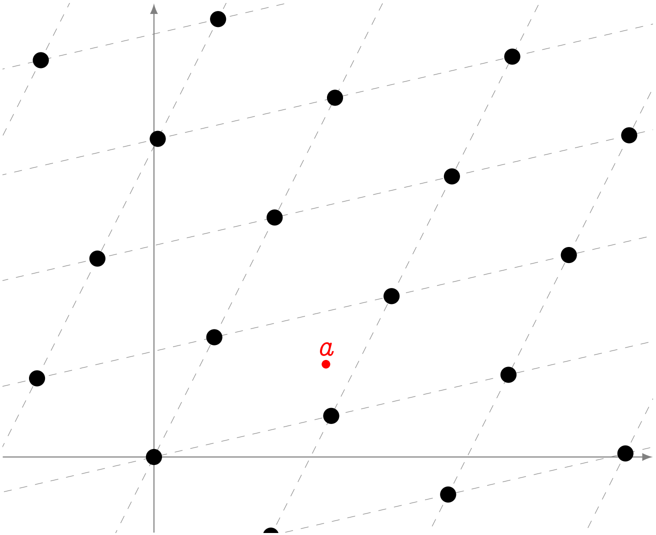

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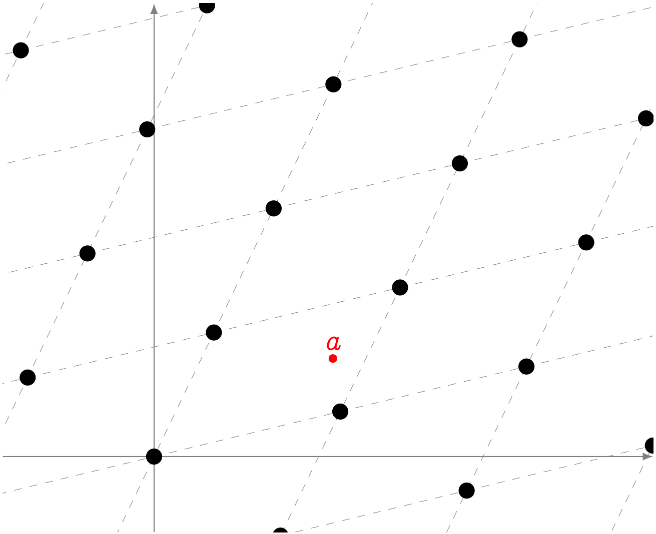

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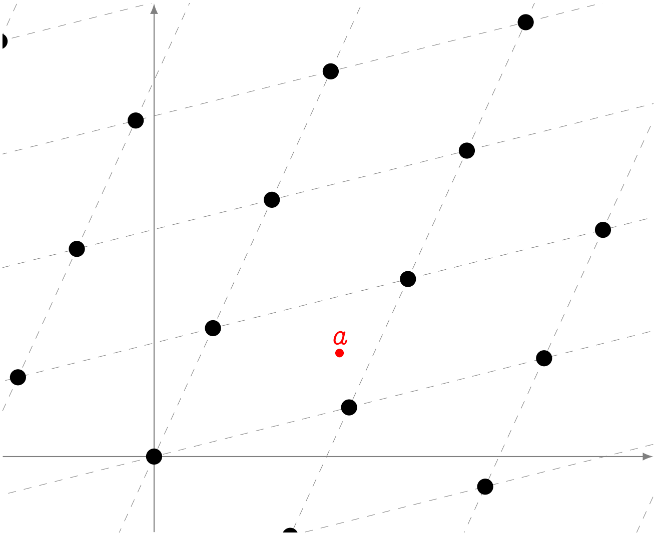

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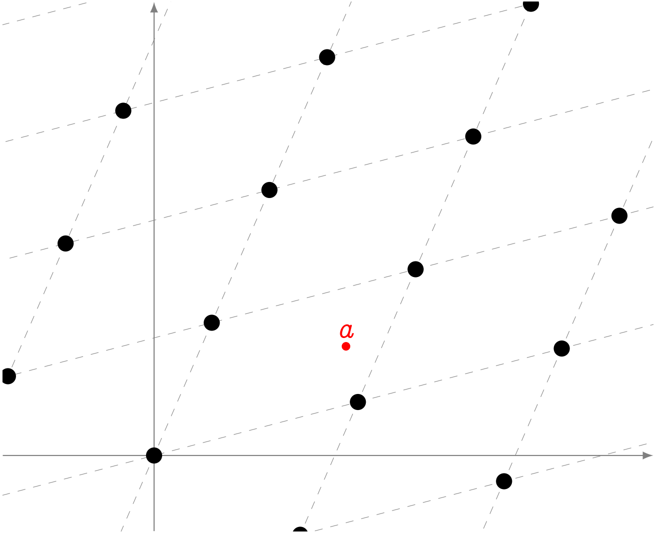

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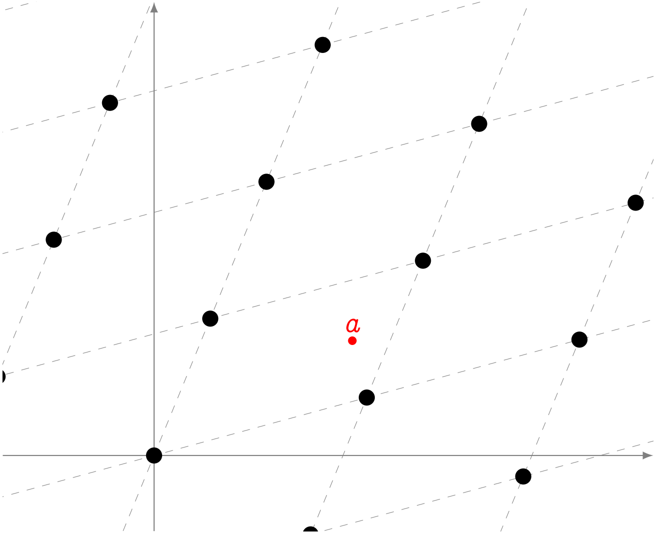

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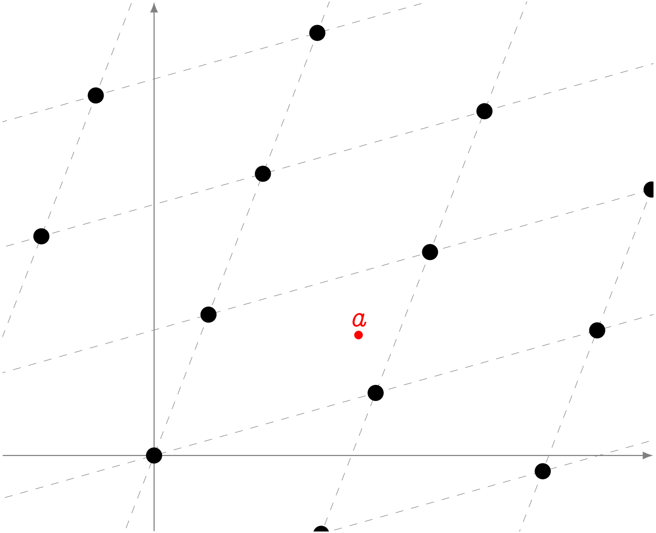

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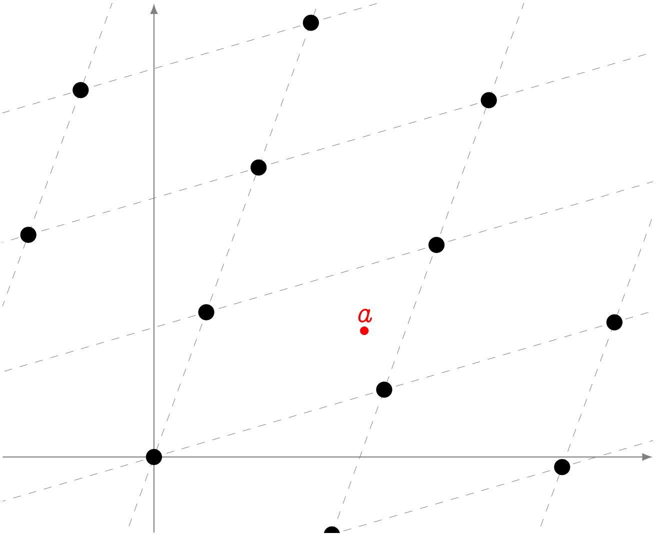

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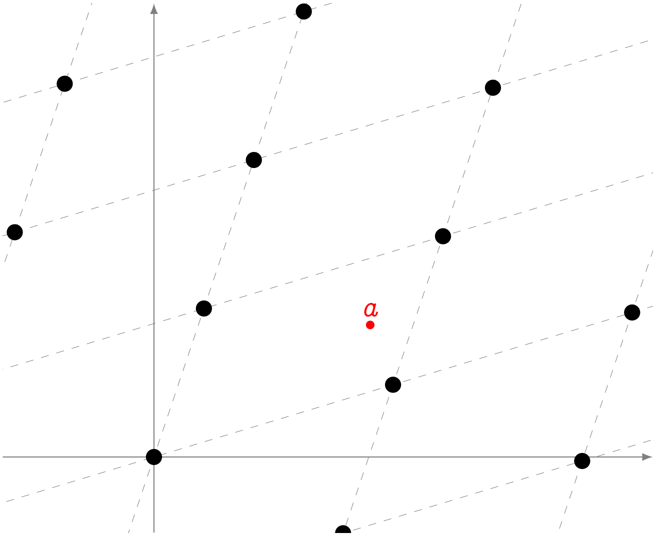

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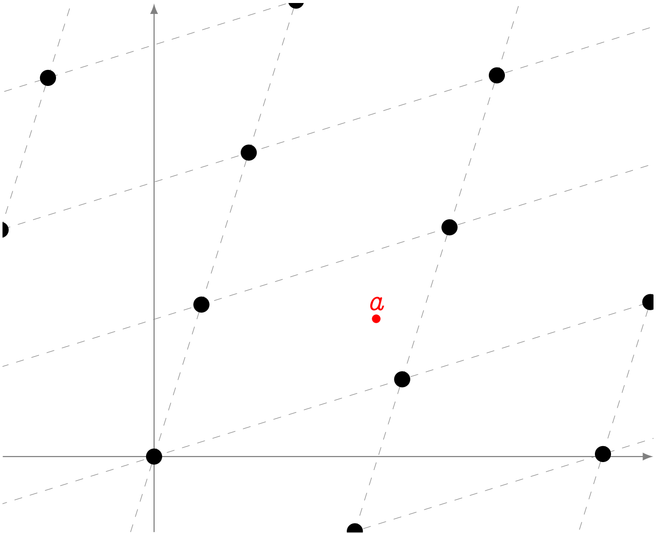

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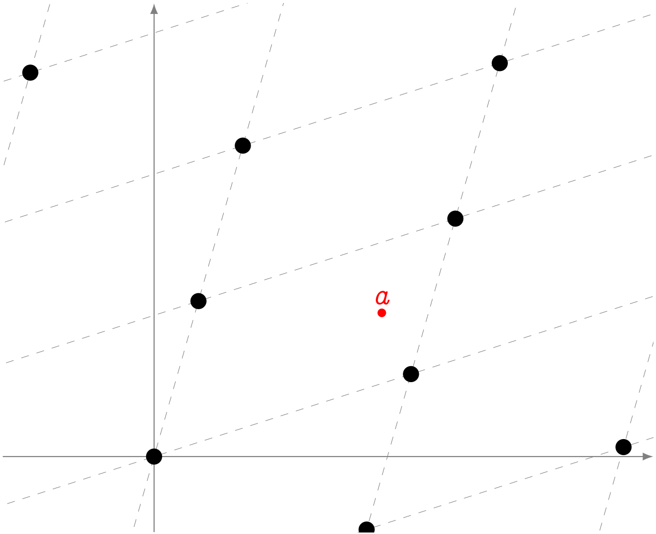

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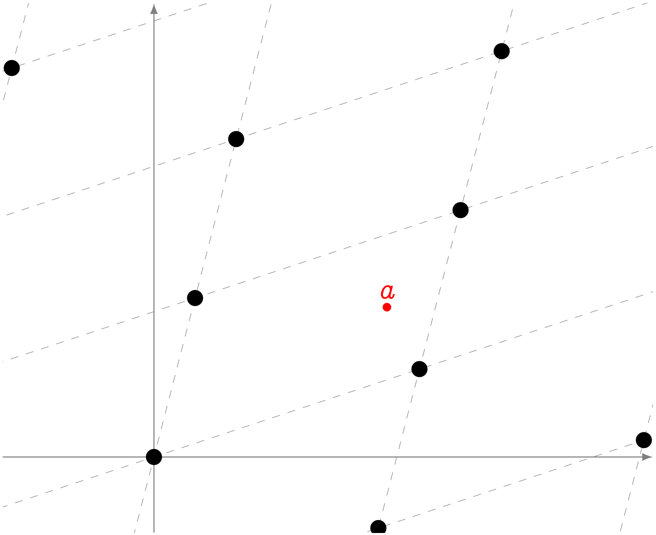

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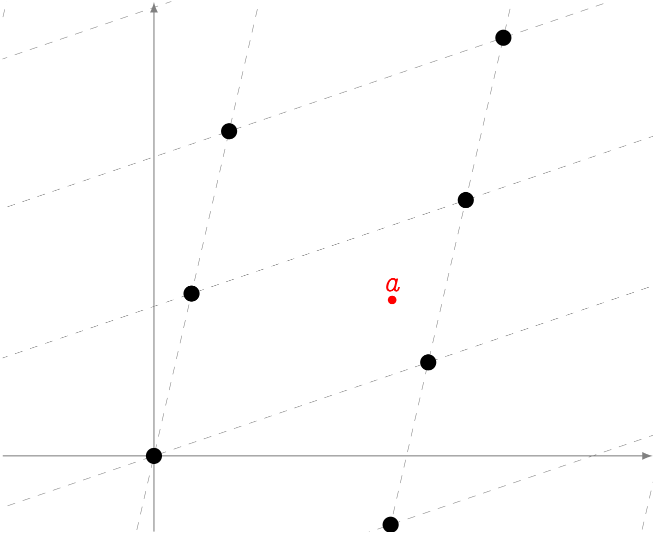

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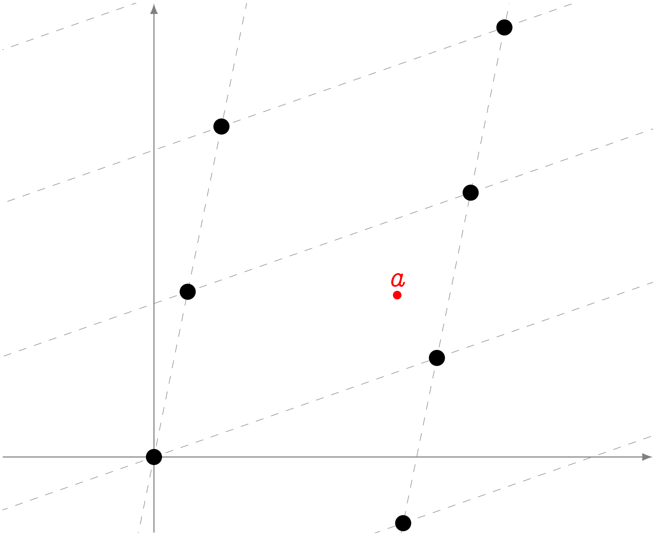

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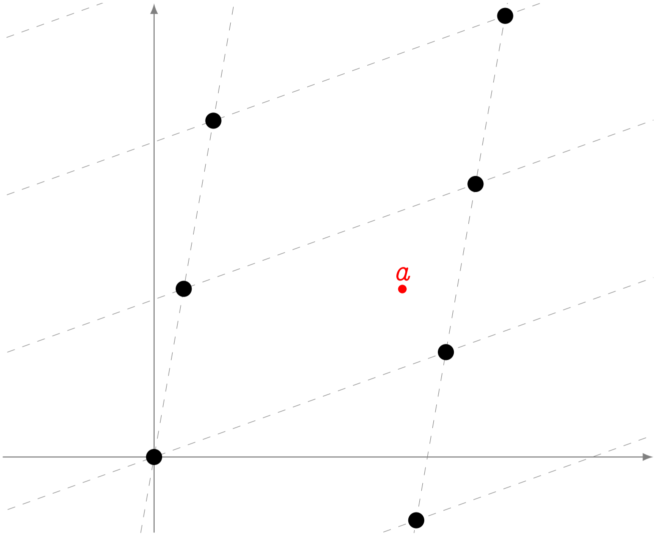

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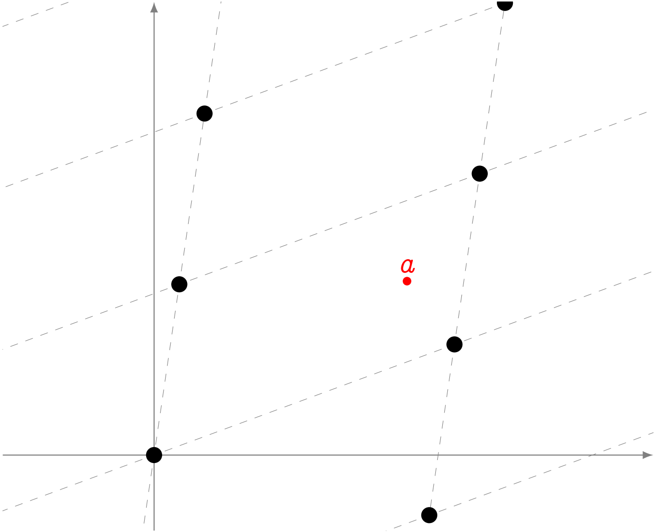

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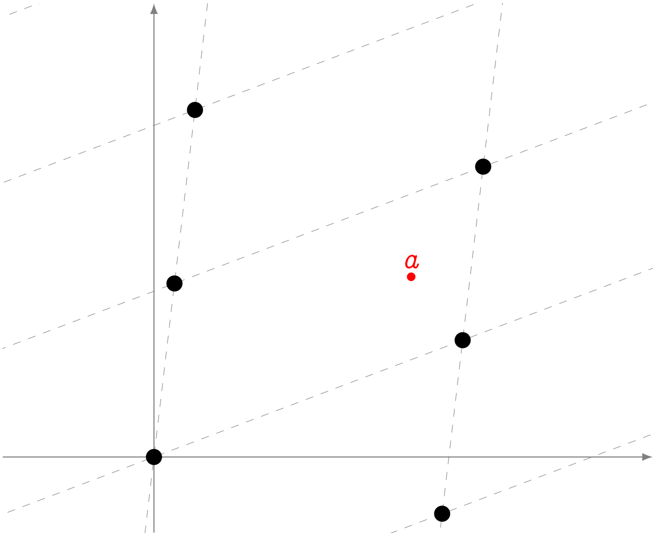

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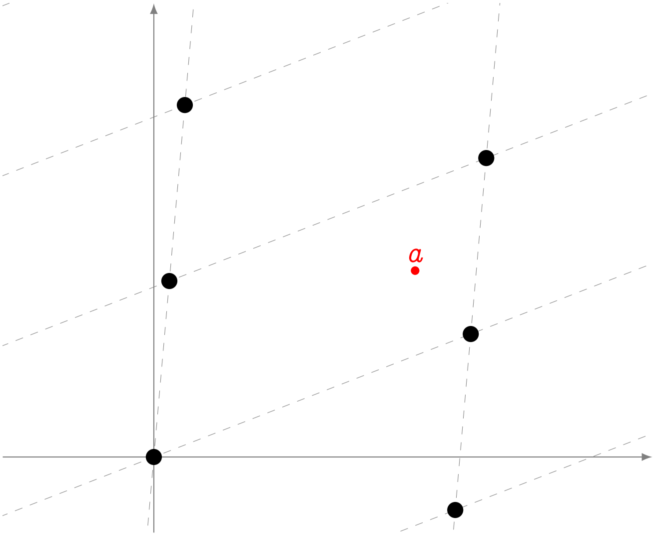

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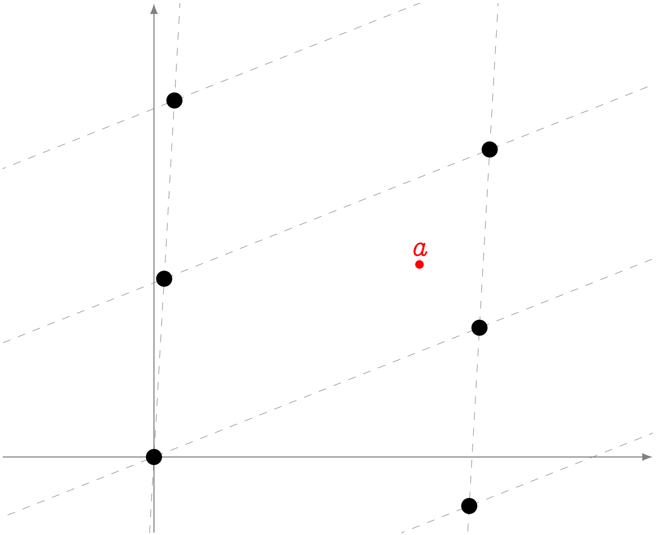

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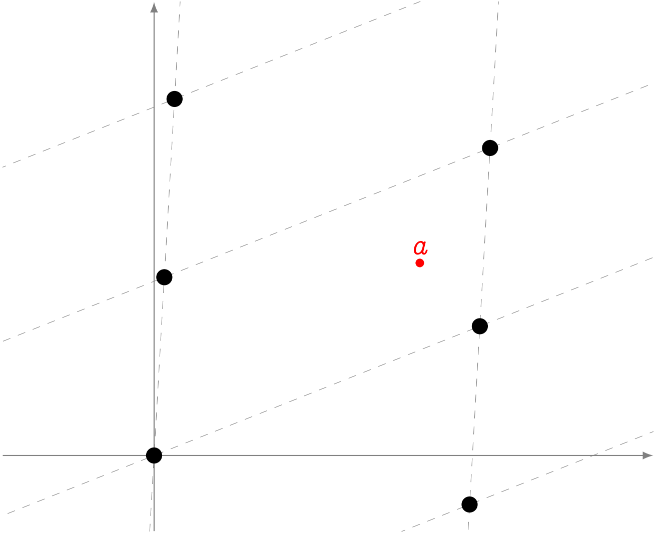

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Uniformization theorem

One to one correspondence: Complex tori $\leftrightarrow$ Elliptic curves over $\mathbb{C}$

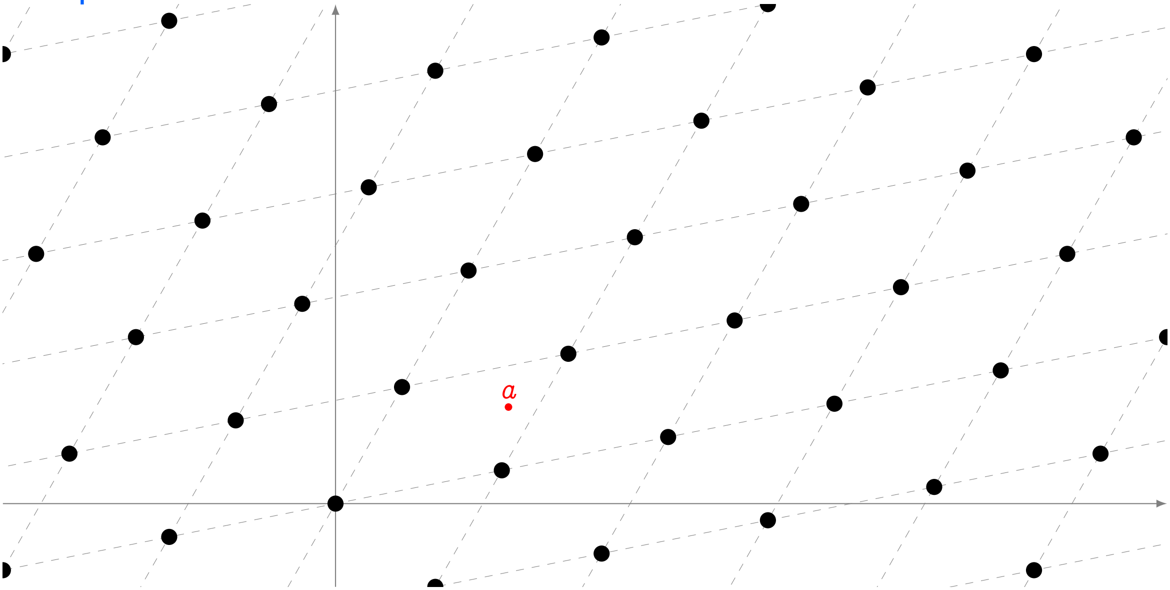
- Isomorphic as Riemann surfaces,
- Isomorphic as groups,
- Homotheties of lattices = Isomorphisms of elliptic curves.

The j-invariant

$$j(E) = 1728\frac{4a^3}{4a^3 - 27b^2}$$

classifies curves/tori up to isomorphism/homothety.

Multiplication



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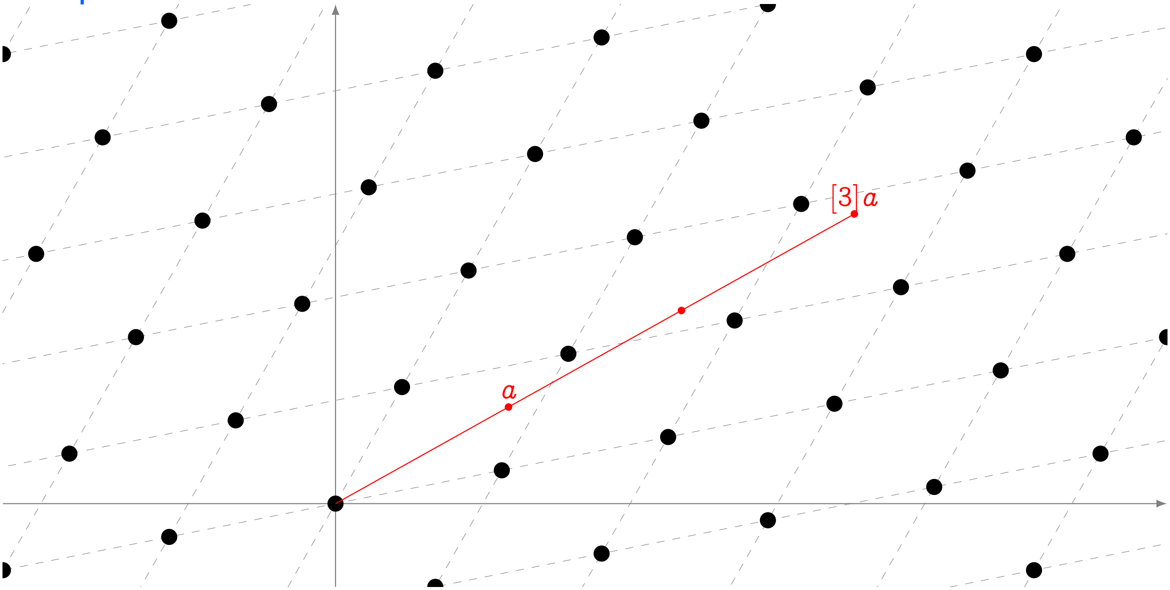

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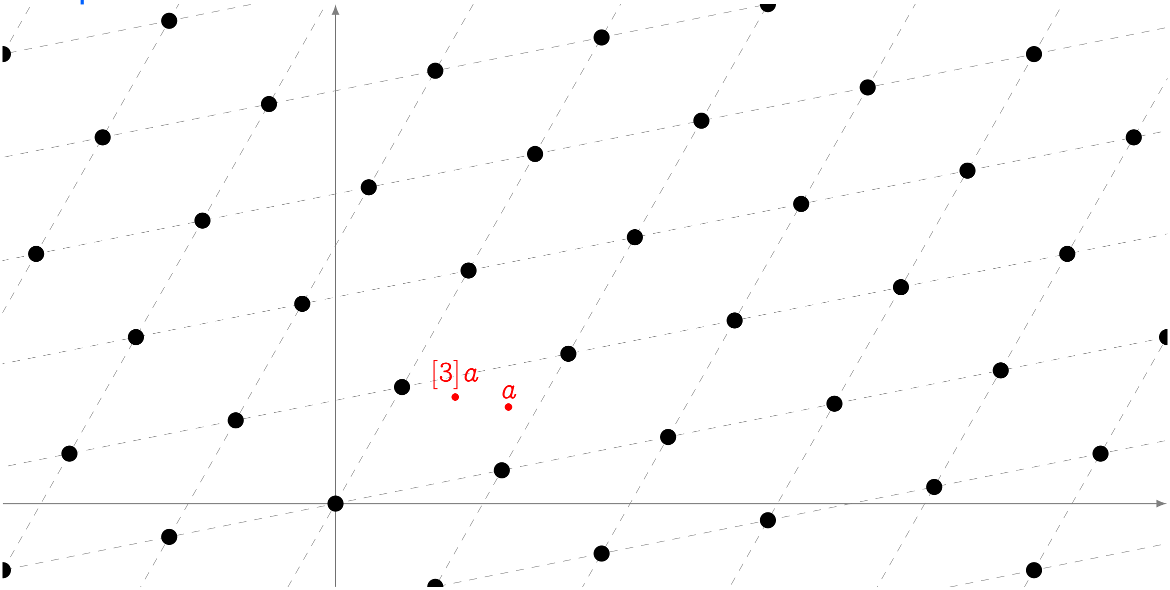
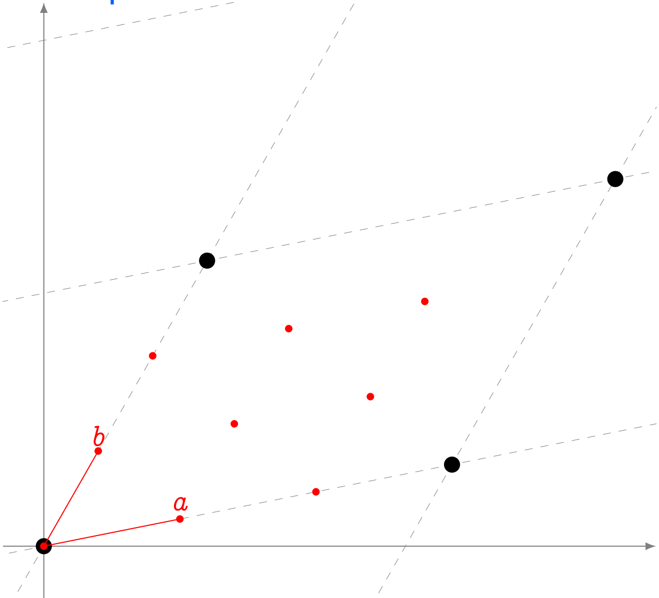


Endomorphisms



Let α be such that $\alpha\Lambda \subset \Lambda$, then

$$\phi_\alpha \ : \ z \mapsto \alpha z \mod \Lambda$$

is an endomorphism of $\mathbb{C}/\Lambda$.

Let ℓ be an integer, the kernel of ϕ_ℓ is:

$$\begin{aligned}(\mathbb{C}/\Lambda)[\ell] &= \langle a, b \rangle \\ &\simeq (\mathbb{Z}/\ell\mathbb{Z})^2\end{aligned}$$

Complex Multiplication (CM)

Endomorphisms form a subring of $\mathbb{C}$: indeed $\alpha\Lambda \subset \Lambda$ and $\beta\Lambda \subset \Lambda$ imply

- $(\alpha + \beta)\Lambda \subset \Lambda$,
- $(\alpha\beta)\Lambda \subset \Lambda$.

Theorem

Let C/Λ be a complex torus, its endomorphism ring is one of:

- *The ring of integers $\mathbb{Z}$,*
- *An order in an imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$.[a]*

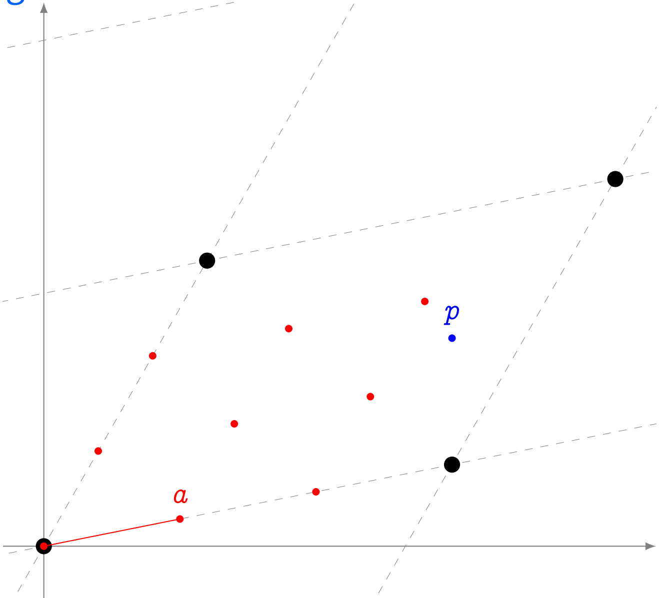
[a] A subring that is a lattice of dimension 2.

Corollary

For any endomorphism ϕ_α there exist integers t, n such that

$$\phi_\alpha^2 - t\phi_\alpha + n = 0.$$

Isogenies



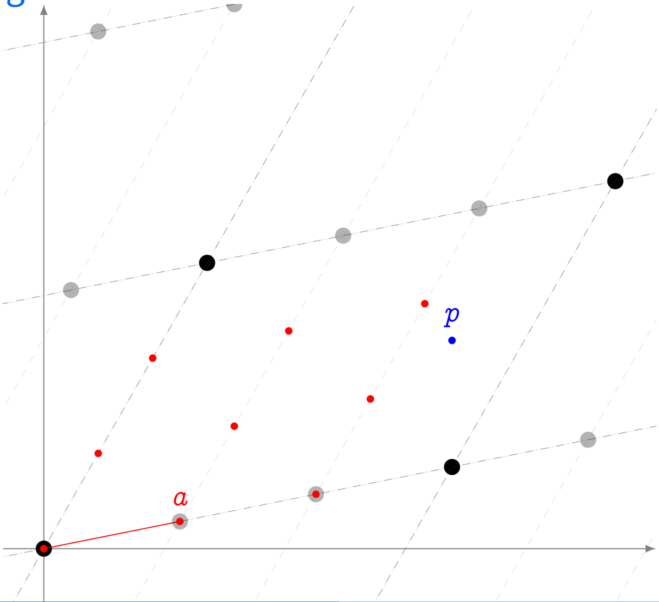
Let $\alpha\Lambda \subset \Lambda'$, the map

$$\begin{aligned}\phi_\alpha \;:\; \mathbb{C}/\Lambda &\to \mathbb{C}/\Lambda' \\ z &\mapsto \alpha z \mod \Lambda'\end{aligned}$$

is a morphism of complex Lie groups.

It is called an isogeny, and it is completely characterized by its kernel $\alpha^{-1}\Lambda'$.

Isogenies



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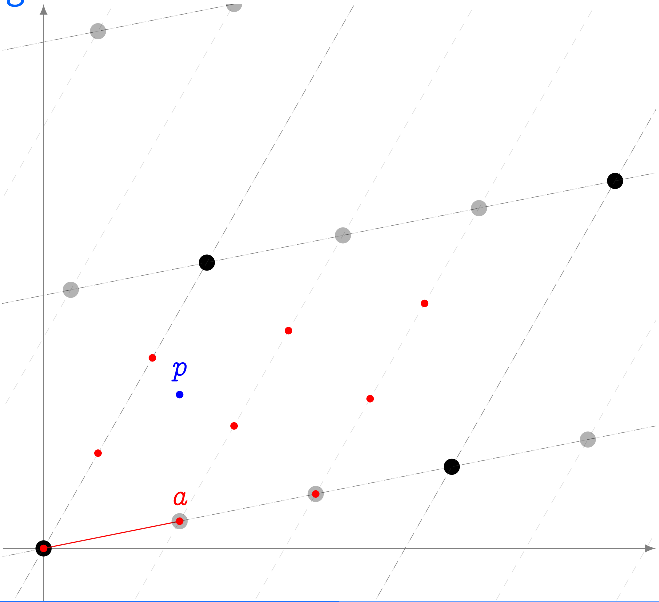

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Isogenies $\leftrightarrow$ ideals

- Let E be an elliptic curve/complex torus with endomorphism ring $\mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$.
- Let $G \subset E(\mathbb{C})$ be a finite subgroup.

Define the kernel ideal

$$\mathrm{Ann}(G) = \{\alpha \in \mathcal{O} \mid \alpha(G) = 0\}.$$

Conversely, given an ideal $\mathfrak{a} \subset \mathcal{O}$, define

$$E[\mathfrak{a}] = \bigcap_{\alpha \in \mathfrak{a}} \ker \alpha.$$

Finally, let $\mathcal{I}(\mathcal{O})$ be the group of (fractional) ideals of $\mathcal{O}$ and let $\mathcal{P}(\mathcal{O})$ be the subgroup of principal ideals, define the class group

$$\mathrm{Cl}(\mathcal{O}) = \mathcal{I}(\mathcal{O})/\mathcal{P}(\mathcal{O}).$$

CM dictionary

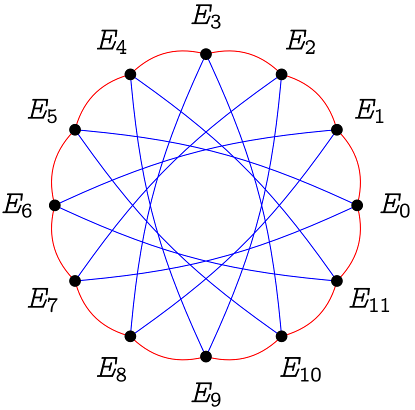
Quadratic imaginary fields	Elliptic curves
Integers of $\mathbb{Q}(\sqrt{-D})$	Endomorphisms of E
Integral ideals of $\mathbb{Q}(\sqrt{-D})$	Isogenies of E
Ideal classes in $\mathrm{Cl}(-D)$	Isogenies $\bullet \rightrightarrows \bullet$
Ideal norm	Isogeny degree
Conjugate ideal	Dual isogeny

The fundamental theorem of CM

- Let E be an elliptic curve with CM by a quadratic imaginary order $\mathcal{O}$.
- Let $\mathfrak{a} \subset \mathcal{O}$ be an integral ideal.
- Denote by $E/E[\mathfrak{a}]$ the image curve of the unique isogeny $\phi_{\mathfrak{a}}$ of kernel $E[\mathfrak{a}]$.

Theorem

The operator $\mathfrak{a} * E := E/E[\mathfrak{a}]$ defines a transitive action of the group of fractional ideals of $\mathcal{O}$ on the (finite) set $\mathcal{E}(\mathcal{O})$ of elliptic curves with complex multiplication by $\mathcal{O}$. The action factors through principal ideals. In other words, the class group $\mathrm{Cl}(\mathcal{O})$ acts regularly on $\mathcal{E}(\mathcal{O})$.



Reduction at $\mathfrak{p}$

Complex multiplication over $\mathbb{C}$ $\sim$ Discrete log in $\mathbb{Q}(e^{2i\pi/N})$

Theorem

Let E be an elliptic curve over a number field L, with CM by an order $\mathcal{O} \subset \mathbb{Q}(\sqrt{-D})$. Let p be a prime split in L, denote by E_p the reduction of E at a place above p, and assume that E_p is non-singular.

- *If $\left(\frac{-D}{p}\right) = 1$ then E_p is said to be ordinary and $\mathrm{End}(E_p) \simeq \mathcal{O}$.*
- *If $\left(\frac{-D}{p}\right) = -1$ then E_p is said to be supersingular and $\mathcal{O} \subsetneq \mathrm{End}(E_p)$.*

Complex multiplication over $\mathbb{F}_p$: Couveignes '06, Rostovtsev–Stolbunov '06, CSIDH '18, …

A partial converse

Deuring's lifting theorem

Let E_p be an elliptic curve in characteristic p, with an endomorphism ω_p which is not trivial. Then there exists an elliptic curve E defined over a number field L, an endomorphism ω of E, and a non-singular reduction of E at a place $\mathfrak{p}$ of L lying above p, such that E_p is isomorphic to $E(\mathfrak{p})$, and ω_p corresponds to $\omega(\mathfrak{p})$ under the isomorphism.

The full endomorphism ring

Theorem (Deuring)

Let E be a supersingular elliptic curve, then

- E is isomorphic to a curve defined over $\mathbb{F}_{p^2}$;
- Every isogeny of E is defined over $\mathbb{F}_{p^2}$;
- Every endomorphism of E is defined over $\mathbb{F}_{p^2}$;
- $\mathrm{End}(E)$ is isomorphic to a maximal order in a quaternion algebra ramified at p and ∞.

In particular:

- If E is defined over $\mathbb{F}_p$, then $\mathrm{End}_{\mathbb{F}_p}(E)$ is strictly contained in $\mathrm{End}(E)$.
- Some endomorphisms do not commute!

An example

The curve of j-invariant 1728

$$E : y^2 = x^3 + x$$

is supersingular over $\mathbb{F}_p$ iff $p = -1 \mod 4$.

Endomorphisms

$\mathrm{End}(E) = \mathbb{Z}\langle\iota, \pi\rangle$, with:

- π the Frobenius endomorphism, s.t. $\pi^2 = -p$;
- ι the map

  $$\iota(x, y) = (-x, iy),$$

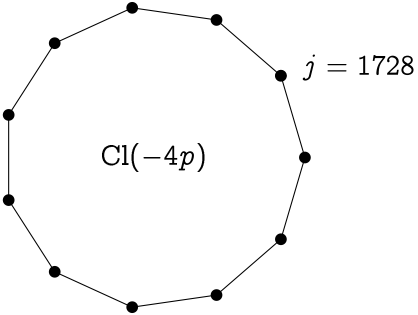
  where $i \in \mathbb{F}_{p^2}$ is a 4-th root of unity. Clearly, $\iota^2 = -1$.

And $\iota\pi = -\pi\iota$.

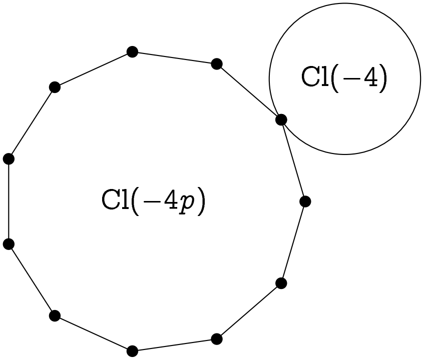
Class group action party

$\bullet\ j = 1728$

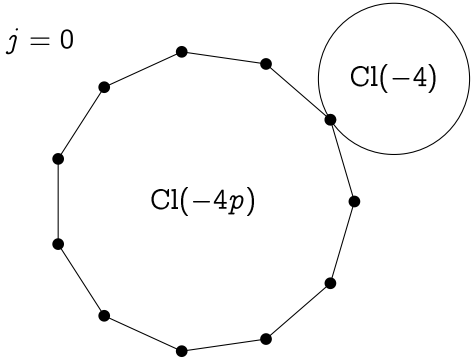
Class group action party



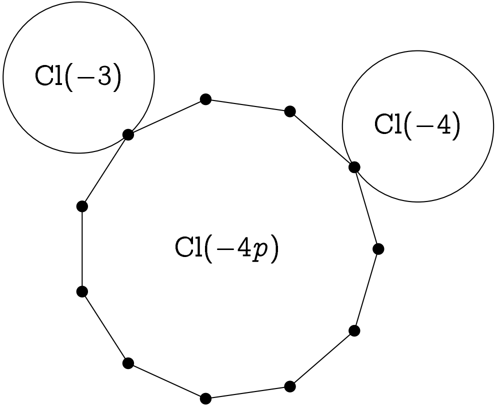
Class group action party



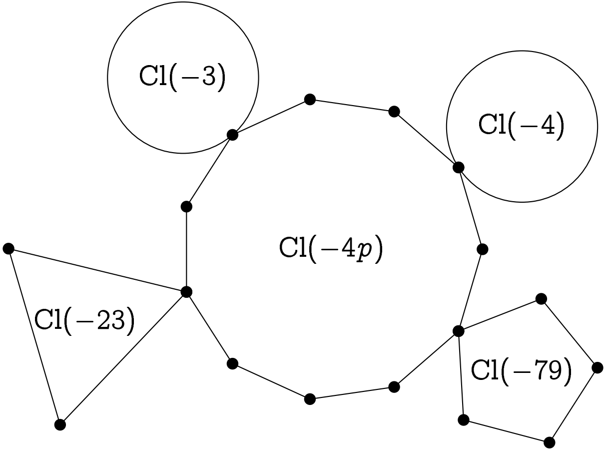
Class group action party



Class group action party



Class group action party



Quaternion algebra?! WTF?[2]

The quaternion algebra $B_{p,\infty}$ is:

- A 4-dimensional $\mathbb{Q}$-vector space with basis $(1, i, j, k)$.
- A non-commutative division algebra[1] $B_{p,\infty} = \mathbb{Q}\langle i, j\rangle$ with the relations:

$$i^2 = a, \quad j^2 = -p, \quad ij = -ji = k,$$

  for some $a < 0$ (depending on p).
- All elements of $B_{p,\infty}$ are quadratic algebraic numbers.
- $B_{p,\infty} \otimes \mathbb{Q}_\ell \simeq \mathcal{M}_{2\times 2}(\mathbb{Q}_\ell)$ for all $\ell \neq p$.
  I.e., endomorphisms restricted to $E[\ell^e]$ are just 2×2 matrices $\bmod \ell^e$.
- $B_{p,\infty} \otimes \mathbb{R}$ is isomorphic to Hamilton's quaternions.
- $B_{p,\infty} \otimes \mathbb{Q}_p$ is a division algebra.

[1] All elements have inverses.
[2] What The Field?

The Deuring correspondence

Let $\mathcal{O}, \mathcal{O}' \subset B_{p,\infty}$ be two maximal orders. They have the same type if there exists α s.t.

$$\mathcal{O} = \alpha\mathcal{O}'\alpha^{-1}.$$

Theorem (Deuring)

Maximal order types of $B_{p,\infty}$ are in one-to-one correspondence with supersingular curves up to Galois conjugation in $\mathbb{F}_{p^2}/\mathbb{F}_p$.

The Deuring correspondence

Two left ideals $\mathfrak{a}, \mathfrak{b} \subset \mathcal{O}$ are in the same class if there exists β s.t. $\mathfrak{a} = \mathfrak{b}\beta$.

An equivalence of categories (Kohel, roughly)

$\{\alpha \in B_{p,\infty} \mid \alpha\mathfrak{a} = \mathfrak{a}\}$
left order

connecting ideal (class)

$\{\alpha \in B_{p,\infty} \mid \mathfrak{a}\alpha = \mathfrak{a}\}$
right order

$$\begin{tikzcd}
\mathcal{O} \arrow[r, "\mathfrak{a}"] & \mathcal{O}' \\
E \arrow[r, "\phi_{\mathfrak{a}}"'] & E'
\end{tikzcd}$$

supersingular curve

isogeny (class)

supersingular curve

Supersingular isogeny graphs

- There is a unique isogeny class of supersingular curves over $\bar{\mathbb{F}}_p$ of size $\approx p/12$.
- The graph of isogenies of degree ℓ is $(\ell + 1)$-regular.
- It is a Ramanujan graphs, i.e., an optimal expander.
- Related to Hecke operators, modular forms, Brandt matrices…

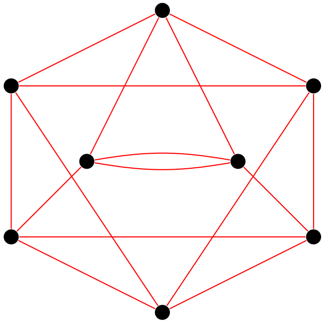


Figure: 3-isogeny graph on $\mathbb{F}_{97^2}$.

Effective correspondences (over finite fields)

Discrete log: $g \xrightarrow{\quad \exp \quad} g^n$

- schoolbook method

Complex multiplication: $E \xrightarrow{\quad \mathfrak{a} \in \mathrm{Cl}(\mathcal{O}) \quad} E'$

- Vélu ’71, Elkies ’92, and many others…

Deuring correspondence: $E \xrightarrow{\quad \mathfrak{a} \subset B_{p,\infty} \quad} E'$

- all of the above,
- Kohel, Lauter, Petit, Tignol ’14 (KLPT),
- D., Kohel, Leroux, Petit, Wesolowski ’20 (part of SQISign).

SQISign: Signatures from the effective Deuring correspondence

$$
\begin{tikzcd}
E_0 \arrow[r, "\psi"] \arrow[d, dashed, "\tau"'] & E_1 \arrow[d, "\varphi"] \\
E_A \arrow[r, "\sigma"] & E_2
\end{tikzcd}
$$

$\longrightarrow$ commitment isogeny (prover)

$\longrightarrow$ challenge isogeny (verifier)

$\longrightarrow$ response isogeny (prover)

$- - - - -$ secret key isogeny

Most compact PQ signature scheme: PK + Signature combined **5 × smaller** than Falcon.

Secret Key (bytes)	Public Key (bytes)	Signature (bytes)	Security
16	64	204	NIST-1

Thank you

https://defeo.lu/

@luca_defeo