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6.046 Problem 5-5
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1 Show that the DC algorithm is k-competitive even in this tree structure. (The good news is that you can use the same potential function as we used in class. The analysis will be just a bit more involved, but it will all work out!)

First, let us define some useful variables:

Let l be the number of robots who saw a request.

Let k be the total number of robots.

Let d be the distance the robot in OPT travels to the request.

Let d_i be the distance traveled by all moving robots in a given phase in the DC algorithm.

We will use the potential function given in lecture, that is:

$$\Phi_i = k * M_{min} + \sum_{DC_{robots}} pairwise\ \ distances \tag{1}$$

Let us first examine M_{min}.

In a given phase, the distance between the robots in the minimum matching between OPT and DC goes down for one pair of robots, the pair that arrived at the request, since they now perfectly match each other, and potentially increases for every other moving robot moving away from their match who remains stationary in the OPT solution.

To express this concretely in an equation:

$$\begin{aligned} \Delta M_{min} &\leq -d_i + (l-1)d_i \\ &= (l-2)d_i \end{aligned} \tag{2}$$

Now, let us consider the four types of cases for robot movement:

1) Moving robots vs Moving robots

2) Non-Moving robots vs Non-Moving robots

3) Non-moving robots vs the moving robot blocking their view

4) Non-moving robots vs the other moving robots who aren't the particular robot blocking their view

In case 1, since the movers are all approaching the same point, the distance between them is decreasing.

In case 2, since there is no motion, the distance stays the same.

In case 3, since the blocking robot is currently moving away from the stationary robot towards a request the stationary cannot see, the distance between those two is increasing.

In case 4, since all the other robots are moving toward the blocking robot, the distance between them and the robot behind the block is decreasing; they're moving towards each other, and since the blocker is moving away, they have to be moving toward the blocked robot.

Now, we will describe concrete equations that model these cases. Consider a single phase of movement.

There are l robots currently moving toward each other, with the pairwise distances between all of them decreasing. There are $\binom{l}{2}$ pairs within these robots, all of whom will move a distance d_i towards each other (so 2* d_i distance reduced per pair). Then the distance reduced in total is:

$$\begin{aligned}&\binom{l}{2} * d_i * 2\\ &= \frac{l^2 - l}{2} * 2d_i\\ &= (l^2 - l) * d_i \end{aligned} \tag{3}$$

for Case 1.

Case two has no motion, so we ignore that.

Now, for cases 3 and 4. The distance between a single non-mover versus its blocking robot increases by d_i in this phase.

However, the distance between that same non-mover and all other moving robots decreases by d_i.

We then have to consider this same effect for all (k-l) non-moving robots.

If we describe that concretely in an equation, we get:

$$-d_i(l-2)(k-l) \tag{4}$$

We skipped a little bit of detail in the above equation; we combined the one positive increase from Case 3 with all of the negatives from Case 4. Basically, we did not show that d(l-2) is d(l-1) + d.

If we combine all of the results for these four cases, we obtain:

$$\sum_{DC_{robots}} = -(l^2 - l) * d_i - d_i(l-2)(k-l) \tag{5}$$

Now we can finally go back to our potential function and plug in these evaluations for M_{min} and $\sum_{DC_{robots}}$.

Recall that

$$\Phi_i = k * M_{min} + \sum_{DC_{robots}} pairwise \;\; distances. \tag{6}$$

If we plug in all our evaluations, then we find that the change in our potential for a single phase is:

$$\Delta\Phi_i = k * [d_i(l-2) + (l^2 - l)d_i] - d(k-l)(l-2) \tag{7}$$

Simplifying the above:

$$\begin{aligned}\Delta\Phi_i &= k * d_i(l-2) + (l^2 - l)d_i - d(k-l)(l-2) \\ &= dl(l-2) - dl(l-1) \\ &= dl(l-2-l-1) \\ &= -dl \qquad (8)\end{aligned}$$

for a given phase.

Now note that l is necessarily upper bounded at any time by k, so we can write:

$$\Delta\Phi_i \leq -d_i k \qquad (9)$$

Finally, note that $\sum_i d_i$ = d. Plugging this in and summing over all phases i, we finally obtain:

$$\sum_i \Delta\Phi_i \leq dk \qquad (10)$$

This is precisely the result we wanted, which proves that DC is k-competitive.