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6.046 Problem 5-2

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State the decision version of this problem, which we will call NRDP, and prove that NRDP is NP-complete using a reduction from a problem that we have already shown to be NP-complete.

The decision version of this problem is as follows:

"Does there exist a set of S edges such that for each v_i, v_j, there exist $R_{i,j}$ disjoint paths, of cost at most k?"

1 Proving this problem is in NP

If we were given a certificate for this problem, we can verify it in polynomial time by just checking for each path that there are actually $R_{i,j}$ distinct paths. There are V vertices we have to check, and for each vertex, the sum of path lengths could be up to V vertices long. Since the paths are disjoint, for any vertex, there can not be more than v vertices involved in all of the paths, for that would involve the paths sharing a vertex. Even in this case, the problem is polynomial in V; it would be O(V^2) in the worst case.

2 Reduction Phase

We will reduce from Hamiltonian Cycle in order to prove this problem is NP-complete. Hamiltonian Cycle is actually a special case of NRDP: It is the case where the matrix $R_{i,j}$ has zeros on the diagonal and 2 for all remaining pairs of vertices. In other words:

$$R_{i,j} = \begin{cases} 0, if\ i = j \\ 2, if\ i \neq j \end{cases}$$

Let k here equal V. Also, let the integer weights be unit; that is, all edges are of weight one. By definition, if we have a Hamiltonian Cycle, we have two paths whose sum of weights, if the weights are unit, is precisely V.

2.1 Is the Reduction possible in polynomial time?

These changes can be made in polynomial time; we are simply constructing the matrix $R_{i,j}$ and setting k = V. The construction of the matrix takes constant time per entry, but since the matrix is of size VxV, it takes V^2 time to fill in the matrix, then constant time to set k=V. Thus, the reduction is polynomial time in V.

2.2 Reduction Correctness Part One: If "yes" to Hamiltonian Cycle, "yes" to NRDP

In the case that we have "yes" to Hamiltonian Cycle, we have "yes" to the special case of NRDP. The reasoning is as follows.

If we have a Hamiltonian Cycle, then for any arbitrary vertex s, and another arbitrary vertex t, there exists a cyclic path through s to t, and from t back to s, that passes through all vertices exactly once.

In that case, examine these two parts of the cycle as individual paths; that is, first examine the path from s to t, then examine the path from t back to s. By the constraint of the definition of Hamiltonian Cycle, given above, these paths must in fact be disjoint paths, because if they shared any vertex other than their start and end points, the cycle would not be Hamiltonian as that would involve visiting at least one vertex more than once. So for any vertices s and t, we have two disjoint paths between them if we know the graph has a Hamiltonian Cycle.

Given this fact, the proof of correctness for this section naturally follows; If we do indeed have a Hamiltonian Cycle in our graph, then given any arbitrary vertex s, and another arbitrary vertex t, there are two disjoint paths that lead from s to t, and by construction, since we set $R_{i,j} = 2$ for i $\neq$ j, we have satisfied our NRDP problem instance as well. Thus, "yes" in Hamiltonian Cycle means "yes" in this particular construction of NRDP.

As a final note, in this case, the set of edges S would actually contain all edges.

2.3 Reduction Correctness Part Two: If "yes" to special NRDP, "yes" to Hamiltonian Cycle

If we have yes to this special case of NRDP, then we have found that for all members in S, we have found two disjoint paths from any arbitrary vertex s to another arbitrary vertex t.

Referring back to some points made in Part One of the reduction correctness proof, recall that all weights are unit, and that we have two paths from vertex s to vertex t. If these are disjoint paths, then their sum is at most V.

By the construction of our graph requiring two such paths for every vertex, the sum of the two paths for any given vertex pair sums to precisely V.

Now consider the meaning of this. Since all weights are unit, for us to have paths that sum to weight V, we must have visited V vertices. By the restriction imposed by the definition of the NRDP problem, these paths are disjoint as well.

To visit V disjoint vertices, we must visit each vertex once on our way from s to t if we take the two paths together. By definition, these two paths together create a Hamiltonian Cycle; they are disjoint paths that visit V vertices in total, that start and end at the same locations. If we take one path then take the "reverse" of the other, we will travel a cyclic path from S to T and then from T to S that touches all vertices.