Chapter 4

Recurrences(1)

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4 Recurrences

- **Difficulty of Algorithms Research**
  - **Model 建模**（树、图、链表、转移方程…？）
  - **Specify 描述**
  - **Correctness 正确性**
    - **Verify 验证**
    - **Proof 证明**
  - **Complex 复杂度 （Efficiency 有效性）**
    - **Feasible actually，实际可行**

Design 设计

Correctness Analysis
正确性分析

Computing Analysis
可计算性分析

- **Recurrence is a basic method to analyze algorithm**

4 Recurrences

Algorithms analysis

INSERTION-SORT(*A*)	cost	times
for($j = 2$; $j <= length[A]$; j++)	c_1	n
{ $key = A[j]$	c_2	n-1
// Insert $A[j]$ into the sorted sequence $A[1 .. j\text{-}1]$	0	n-1
$i = j$-1	c_4	n-1
while($i > 0$ && $A[i] > key$)	c_5	$\sum_{j=2}^{n} t_j$
{ $A[i+1] = A[i]$	c_6	$\sum_{j=2}^{n} (t_j - 1)$
$i = i$-1	c_7	$\sum_{j=2}^{n} (t_j - 1)$
}		
$A[i+1] = key$	c_8	n-1
}		

sum

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1)$$

4 Recurrences

Algorithms analysis

	cost	times
MERGE-SORT(*A*, *p*, *r*)	$T(n)$	1
if $p < r$		
$q \leftarrow \lfloor (p+r)/2 \rfloor$		
MERGE-SORT(*A*, *p*, *q*)	$T(n/2)$	1
MERGE-SORT(*A*, *q*+1, *r*)	$T(n/2)$	1
MERGE(*A*, *p*, *q*, *r*)	n	

recursion

$$T(n) = \begin{cases} 1 & , \text{if } n = 1 \\ 2T(n/2) + n & , \text{if } n > 1 \end{cases} \qquad (4.1)$$

4 Recurrences

Algorithms analysis

	cost	times
f(n)	$T(n)$	1
return f(n-1) + f(n-2);	$T(n-1)+T(n-2)$	1

```
f(n)
{
   if(n <= 2)
      return 1;
   else
      return f(n-1) + f(n-2);
}
```

recursion

$$T(n)=\begin{cases}1 & \text{, if } n\le 2\\ T(n-1)+T(n-2) & \text{, if } n>2\end{cases}$$

4 Recurrences

E Zexal的二叉树（签到）

时间限制：1000ms 内存限制：65536kb

通过率：200/209 (95.69%) 正确率：200/596 (33.56%)

题目

知识点：树，数论，dp，递归（都可以做）

上学期我们学习了二叉树，也都知道3个结点的二叉树有5种， 现给你二叉树的结点个数n，要你输出不同形态二叉树的种数。

输入

第一个数为一个整数$n(n <= 30)$

输出

对于每组数据，输出一行，不同形态二叉树的种数。

输入样例

```
3
```

输出样例

```
5
```

Solutions ?

4 Recurrences

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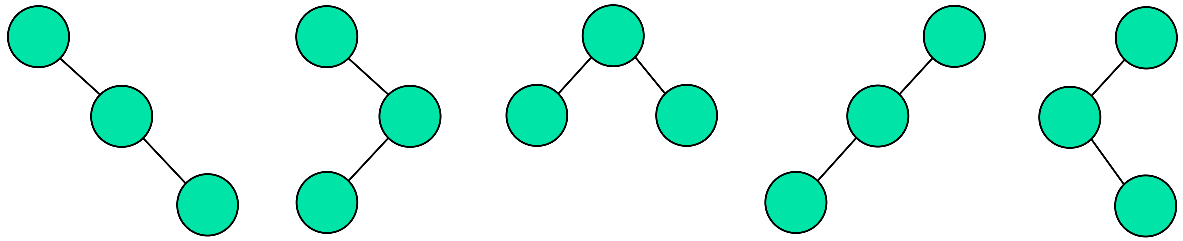
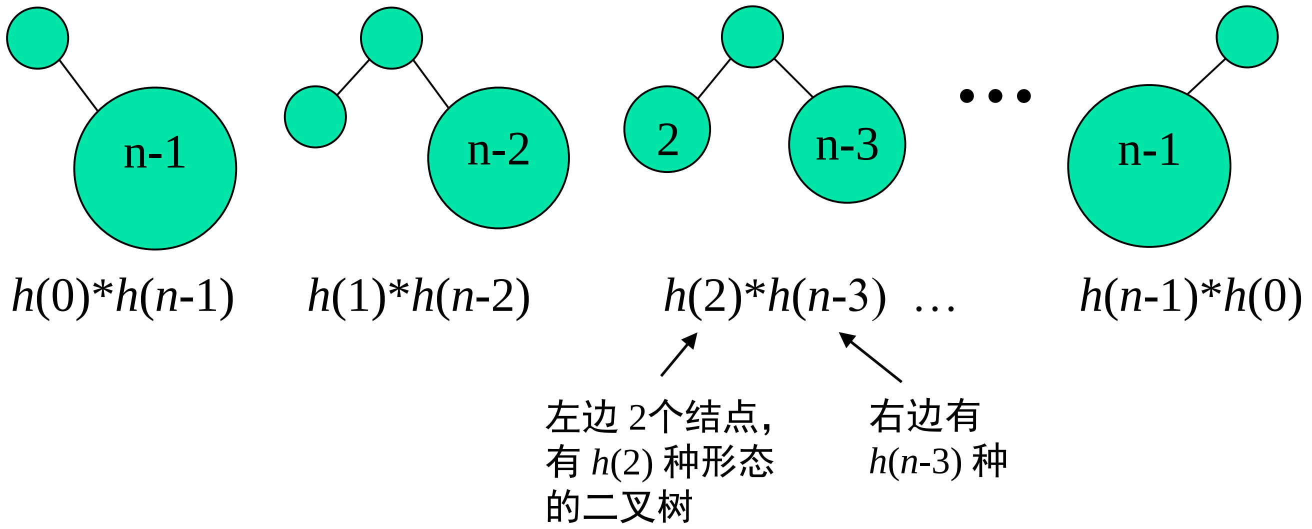
Solutions:

$n = 3$，五种情况, $h(3) = 5$



n,



4 Recurrences

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第一个数为一个整数$n(n <= 30)$

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Algorithms design and analysis

recursion

$h(n) = h(0)*h(n-1) + h(1)*h(n-2) + \ldots + h(n-1)*h(0)$

另一种递归式:

$h(n) = ((4*n-2)/(n+1))*h(n-1)$

该递推关系的解为: $h(n) = C(2n, n)/(n+1)$

其中，规定 $h(0) = 1$

catalan数，卡特兰数，是一个常出现在各种计数问题中的数列，以比利时的数学家欧仁-查理-卡特兰命名。

4 Recurrences

A recurrence is an equation or inequality in terms of

- one or more base cases, and
- itself, with smaller arguments.

Examples:

(1) $T(n) = \begin{cases} 1 & \text{, if } n{=}1, \\ T(n-1)+1 & \text{, if } n>1. \end{cases}$

Solution: $T(n) = n$.

(2) $T(n) = \begin{cases} 1 & \text{, if } n{=}1, \\ 2T(n/2)+n, & \text{if } n>1. \end{cases}$

Solution: $T(n) = n\lg n + n$.

(3) $T(n) = \begin{cases} 0 & \text{, if } n{=}2, \\ T(\sqrt{n})+1 & \text{, if } n>2. \end{cases}$

Solution: $T(n) = \lg\lg n$.

(4) $T(n) = \begin{cases} 1 & \text{, if } n{=}1, \\ T(n/3)+T(2n/3)+n, & \text{if } n>1. \end{cases}$

Solution: $T(n) = \Theta(n\lg n)$

4 Recurrences

How to obtain asymptotic "Θ" or "O" bounds on the recurrencesolution?

- Substitution method（置换法）: guesses a bound and then use mathematical induction to prove our guess correct.
- Iteration method（迭代法）: converts the recurrence into a summation and then relies on techniques for bounding summations to solve the recurrence.
- Recursion-tree method (a kind of iteration method)
- Master method（主方法，主定理，母函数法）: provides bounds for recurrences of the form $T(n) = aT(n/b)+f(n)$, where $a \geq 1$, $b > 1$, and $f(n)$ is a given function.

Technicalities

In practice, we neglect certain technical details when we state and solve recurrences. (忽略技术细节)

(1) Assumption of integer arguments to functions（输入n为整数）

- Normally, $T(n)$ is only defined when n is an integer
- Example, the worst-case running time of MERGE-SORT

$$T(n)=\begin{cases}\Theta(1) & \text{, if } n=1, \\ T(\lceil n/2\rceil)+T(\lfloor n/2\rfloor)+\Theta(n) & \text{, if } n>1.\end{cases} \qquad (4.2)$$

$$\Updownarrow$$

$$T(n)=\begin{cases}\Theta(1) & \text{, if } n=1, \\ T(n/2)+T(n/2)+\Theta(n) & \text{, if } n>1.\end{cases} \qquad (4.2)$$

Technicalities

(2) Ignore boundary conditions

- Omit statements of the boundary conditions of recurrences, assume that $T(n)$ is constant for small n , that is $T(n) = \Theta(1)$ for sufficiently small n.
  （n 较小时，$T(n)$为常数，因为问题可以直接求解）

- Example, state recurrence (4.1)

$$T(n) = \begin{cases} 1 & , \text{if } n = 1 \\ 2T(n/2) + n & , \text{if } n > 1 \end{cases} \qquad (4.1)$$

as $T(n) = 2T(n/2) + n$, (Omit $T(1) = 1$) (4.3)

without explicitly giving values for small n.

The reason is that although changing the value of $T(1)$ changes the solution to the recurrence, the order of growth is unchanged.（改变边界值，可能改变递归式的解，但不改变解的函数增长率，即，$T(1)$为常数c，不用考虑常数为具体的什么值）

Technicalities

neglect certain technical details

(1) Assumption of integer arguments to functions

(2) Ignore boundary conditions

These details don’t affect the asymptotic bounds of many recurrences encountered in the analysis of algorithms

（忽略细节不会影响算法的渐近分析）

4.1 The substitution method

Key points

(1) guessing the form of the solution;

(2) using mathematical induction to show the solution works.

Example: $T(n)=\begin{cases}1 & \text{, if } n=1,\\ 2T(n/2)+n, & \text{if } n>1.\end{cases}$

(1) Guess: $T(n)=n\lg n+n$

(2) Induction

Basic: $n=1\Rightarrow n\lg n+n=1=T(n)=T(1)$

Inductive step: Inductive hypothesis is that $T(k)=k\lg k+k$ for all $k<n$.

Use the inductive hypothesis of $T(n/2)$ to prove $T(n)$

$$
\begin{aligned}
T(n) &= 2T(n/2)+n \\
&= 2((n/2)\lg(n/2)+(n/2))+n \quad \text{(by inductive hypothesis)} \\
&= n\lg(n/2)+n+n = n(\lg n-\lg 2)+2n = n\lg n+n.
\end{aligned}
$$

4.1 The substitution method

Key points

(1) guessing the form of the solution;

(2) using mathematical induction to show the solution works.

Example: $T(n) = \begin{cases} 1 & \text{, if } n = 1, \\ 2T(n/2) + n, & \text{if } n > 1. \end{cases}$

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(2) Induction

Basic: $n = 1 \Rightarrow n\lg n + n = 1 = T(n)$

Inductive step: Inductive hypothesis is that $T(k) = k\lg k + k$ for all $k < n$.

Use the inductive hypothesis of $T(n/2)$ to prove $T(n)$.

The method is powerful, but it can be applied only in cases when it is easy to guess the form of the answer.

4.1 The substitution method

The substitution method is powerful, but it can be applied only in cases when it is easy to guess the form of the answer.

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recursion

$$h(n) = h(0)*h(n-1) + h(1)*h(n-2) + \ldots + h(n-1)*h(0)$$

另一种递归式(其中 $n >= 2$)：

$$h(n) = ((4*n-2)/(n+1))*h(n-1)$$

这个就不好猜!

$$h(n) = C(2n, n)/(n+1)$$

$$(n = 1, 2, 3, \ldots)$$

习题：证明该公式。

4.1 The substitution method

- The substitution method can be used to establish either upper (O) or lower bounds (Ω) on a recurrence.
- example, determining an upper bound on the recurrence

$$T(n) = 2T(\lfloor n/2 \rfloor) + n, \qquad (4.4)$$

(1) Guessing that the solution is $T(n) = O(n\lg n)$.

(2) Proving $T(n) \le cn\lg n$ for a some constant $c > 0$.

- Assume that this bound holds for $n/2$, that is, $T(\lfloor n/2 \rfloor) \le c\lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)$. Substituting into the recurrence yields

$$\begin{aligned} T(n) &= 2T(\lfloor n/2 \rfloor) + n \le 2(c\lfloor n/2 \rfloor \lg(\lfloor n/2 \rfloor)) + n \\ &\le cn\lg(n/2) + n = cn\lg n - cn\lg 2 + n = cn\lg n - cn + n \le cn\lg n, \end{aligned}$$

where the last step holds as long as $c \ge 1$.

4.1 The substitution method

- Mathematical induction now requires us to show that our solution holds for the boundary conditions (some small n). （数学归纳法要求答案满足边界条件。）
- Typically, the boundary conditions are suitable as base cases for the inductive proof. （典型地，对于归纳法证明，边界条件与递归方程的基本情况一般等同。）

$$T(n) = 2T(\lfloor n/2 \rfloor) + n, \qquad (4.4)$$

$$T(n) = \begin{cases} 1 & , \text{if } n = 1 \\ 2T(n/2) + n & , \text{if } n > 1 \end{cases} \qquad (4.1)$$

$$T(n) = O(n \lg n) \quad , \quad T(n) \le cn \lg n$$

- This requirement can sometimes lead to problems.
- Assume that $T(1) = 1$ is the sole boundary condition of the recurrence. Then, we can't choose c large enough, since $T(1) \le c*1*\lg 1 = 0$, which is at odds with $T(1) = 1$. The case of our inductive proof fails to hold. （归纳证明结果与边界条件矛盾，即归纳证明失败？因为归纳证明过程的依据是从边界条件为真出发来进行证明，但现在验证解并不满足边界条件。）

(1)To extend boundary conditions

$$T(n) = 2T(\lfloor n/2 \rfloor) + n$$
$$T(n) = O(n\lg n) \quad , \; T(n) \le cn\lg n$$

$$T(n) = \begin{cases} 1 & , \text{ if } n = 1 \\ 2T(n/2) + n & , \text{ if } n > 1 \end{cases} \qquad (4.1)$$

An inductive hypothesis inconsistent with specific boundary condition, How to overcome the difficulty?（如何克服归纳结果与边界条件不一致的问题？）

- Asymptotic notation only requires us to prove $T(n) \le cn\lg n$ for $n \ge n_0$, where n_0 is a constant.
- Remove the difficult boundary condition $T(1) = 1$.
- Impose $T(2)$ and $T(3)$ as boundary conditions for the inductive proof.
- From the recurrence, we derive $T(2) = 2*T(1) + 2 = 4$, $T(3) = 2*T(1) + 3 = 5$.
- The inductive proof that $T(n) \le cn\lg n$ can now be completed by choosing any $c \ge 2$ so that $T(2) = 4 \le c*2*\lg 2$ and $T(3) = 5 \le c*3*\lg 3$.

4.1.1 Making a good guess

- Unfortunately, there is no general way to guess the correct solutions to recurrences.（猜想不是一种方法）

- Guessing a solution takes experience and, occasionally, creativity. （why we study the course? It's a training for us to get experience, to catch occasion, to have creativity.)

- Fortunately, though, there are some heuristics (recursion trees) that can help you become a good guesser.

4.1.1 Making a good guess

(1) If a recurrence is similar to one you have seen before, then guessing a similar solution is reasonable. For example,

$$T(n) = 2T(\lfloor n/2 \rfloor + 23) + n, \quad T(n) = ?$$

4.1.1 Making a good guess

(1) If a recurrence is similar to one you have seen before, then guessing a similar solution is reasonable. For example,

$$T(n) = 2T(\lfloor n/2 \rfloor + 23) + n, \quad T(n) = ?$$

- which looks difficult because of the added "23".
- Intuitively, this additional term cannot substantially affect the solution to the recurrence. （该附加项不会从本质上影响递归解）
- When n is large, the difference between $T(n/2)$ and $T(n/2 + 23)$ is not that large. Consequently, we make the guess that $T(n) = O(n\lg n)$, which you can verify as correct by using the substitution method.

4.1.1 Making a good guess

(2) Another way to make a good guess is to prove loose upper and lower bounds on the recurrence and then reduce the range of uncertainty. （寻找松的上、下渐近界，范围缩小，逐步逼近）

For example

$$T(n) = 2T(\lfloor n/2 \rfloor) + n \qquad (4.4)$$

- we might start with a lower bound of $T(n) = \Omega(n)$, since we have the term n in the recurrence,
- and we can prove an initial upper bound $T(n) = O(n^2)$.
- Then, we can gradually lower the upper bound and raise the lower bound until we converge on the correct, asymptotically tight solution of $T(n) = \Theta(n\lg n)$.

4.1.2 Subtleties

- Sometimes, guess correctly, but somehow the math doesn't seem to work out in the induction.

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1.$$

For example,
Guess the solution is $O(n)$, then try to show that $T(n) \le cn$ for an appropriate constant c. Substituting .. , then

$$\begin{aligned} T(n) &= T(n/2) + T(n/2) + 1 \\ &\le c \cdot n/2 + c \cdot n/2 + 1 = cn + 1, \end{aligned}$$

which does not imply $T(n) \le cn$ for any choice of c.

4.1.2 Subtleties

- Sometimes, guess correctly, but somehow the math doesn't seem to work out in the induction.

  For example $$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1.$$

  guess $T(n) \le cn$, then $T(n) \le c\lfloor n/2 \rfloor + c\lceil n/2 \rceil + 1 = cn + 1$,

  contradiction.

- Usually, it is that the inductive assumption isn't strong enough to prove the detailed bound. How to overcome?

  (归纳假设条件不强)

(2)Subtracting a lower-order term

$$T(n) = T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + 1 \qquad \text{Solution: } T(n) = O(n)$$

- try a larger guess $T(n) = O(n^2)$, which can work.
- But the guess that the solution is $T(n) = O(n)$ is correct.
- Intuitively, our guess is nearly right: we're only off by the constant 1, a lower-order term.
- Nevertheless, mathematical induction doesn't work!
- Subtracting a lower-order term from our previous guess. New guess is $T(n) \le cn - b$, where $b \ge 0$ is constant, then

$$T(n) \le (c\lfloor n/2 \rfloor - b) + (c\lceil n/2 \rceil - b) + 1 = cn - 2b + 1 = cn - b - b + 1 \le cn - b \ ,$$

as long as $b \ge 1$. As before, the constant c must be chosen large enough to handle the boundary conditions.

4.1.2 Subtleties

- Most people find the idea of subtracting a lower-order term counterintuitive. （违反直觉）
- After all, if the math doesn't work out, shouldn't we be increasing our guess?
- The key to understand this step is to remember that we are using mathematical induction: we can prove something stronger for a given value by assuming something stronger for smaller values.（假设更强的条件，可证明更强的结论）

4.1.3 (3)Avoiding pitfalls (陷阱)

It is easy to err in the use of asymptotic notation.

For example, in the recurrence (4.4)

$$T(n) = 2T(\lfloor n/2 \rfloor) + n \qquad (4.4)$$

we can falsely prove $T(n) = O(n)$ by guessing $T(k) \le ck$ and then arguing

$$T(n) = 2T(\lfloor n/2 \rfloor) + n \le 2(c\lfloor n/2 \rfloor) + n$$

$$\le cn + n$$

$$= O(n)\ .$$

Wrong!!!

since c is a constant. The error is that we haven't proved the exact form of the inductive hypothesis, that is, that $T(n) \le cn$.

4.1.4 Changing variables

algebraic manipulation: sometimes solute an unknown recurrence similar to one you have seen before.

Example, $T(n) = 2T(\lfloor \sqrt{n} \rfloor) + \lg n,$

which looks difficult. Simplify the recurrence with a change of variables. For convenience, we shall not worry about rounding off values, such as $\sqrt{n}$ to be integers.

Let $m = \lg n$, then $T(2^m) = 2T(2^{m/2}) + m$.

Thus rename $S(m) = T(2^m)$ => $S(m) = 2S(m/2) + m$,

which is very much like recurrence (4.4) and has the same solution: $S(m) = O(m\lg m)$. Changing back from $S(m)$ to $T(n)$, we obtain $T(n) = T(2^m) = S(m) = O(m\lg m) = O(\lg n \lg\lg n)$.

4.1 The substitution method: Notes

(1) To extend boundary conditions

(2) Subtracting a lower-order term

(3) Avoiding pitfalls

4.2 The iteration method

- Substitution: It is difficult to come up with a good guess
- The iteration method
  - doesn't require us to guess the answer
  - may require more algebra (迭代法对代数能力的要求较高)
  - to expand (iterate) the recurrence and express it as a summation of terms, and the initial conditions
  - to evaluate summations. (不断迭代展开为级数，并求和)

For example, $T(n) = 3T(\lfloor n/4 \rfloor) + n$

$$\begin{aligned}
T(n) &= n + 3T(\lfloor n/4 \rfloor) \\
&= n + 3\ (\lfloor n/4 \rfloor + 3T(\lfloor n/16 \rfloor)) \\
&= n + 3(\lfloor n/4 \rfloor + 3(\lfloor n/16 \rfloor + 3T(\lfloor n/64 \rfloor))) \\
&= n + 3\lfloor n/4 \rfloor + 9\lfloor n/16 \rfloor + 27T(\lfloor n/64 \rfloor),
\end{aligned}$$

4.2 The iteration method

$$
\begin{aligned}
T(n) &= n + 3T(\lfloor n/4 \rfloor) = n + 3\,(\lfloor n/4 \rfloor + 3T(\lfloor n/16 \rfloor)) \\
&= n + 3(\lfloor n/4 \rfloor + 3(\lfloor n/16 \rfloor + 3T(\lfloor n/64 \rfloor))) \\
&= n + 3\lfloor n/4 \rfloor + 9\lfloor n/16 \rfloor + 27T(\lfloor n/64 \rfloor),
\end{aligned}
$$

How far must we iterate the recurrence?

- The ith term in the series is $3^i T(\lfloor n/4^i \rfloor)$
- The iteration halts when $\lfloor n/4^i \rfloor = 1$. By continuing the iteration until this point

and using the bound $\lfloor n/4^i \rfloor \le n/4^i$, we get a decreasing geometric series:

$$
\begin{aligned}
T(n) &\le n + 3n/4 + 9n/16 + 27n/64 + \ldots + 3^i T(n/4^i) \\
&\le n\sum_{i=0}^{\infty}(3/4)^i + \Theta(n^{\log_4 3}) = 4n + o(n) = O(n).
\end{aligned}
$$

$$(n/4^i = 1 \;\Rightarrow\; 4^i = n \;\Rightarrow\; i = \log_4 n \;\Rightarrow\; 3^i = 3^{\log_4 n} = n^{\log_4 3} = o(n))$$

4.2 The iteration method

- The iteration method usually leads to lots of algebra. It can be a challenge. The key points:
  - the number of times the recurrence needs to be iterated to reach the boundary condition,（迭代次数）
  - and the sum of the terms arising from each level of the iteration process.（级数求和）

- Sometimes, in the process of iterating a recurrence, you can guess the solution without working out all the math. Then, the iteration can be abandoned in favor of the substitution method, which usually requires less algebra.

  （在展开递归式为迭代求和的过程中，有时只需要部分展开，然后根据其规律来猜想递归式的解，接着用置换法进行证明。）

4.2 The iteration method

- When a recurrence contains floor and ceiling functions, the math can become especially complicated.

- Often, it helps to assume that the recurrence is defined only on exact powers of a number. （通常，为简化运算，假定递归方程的参数为数的幂）

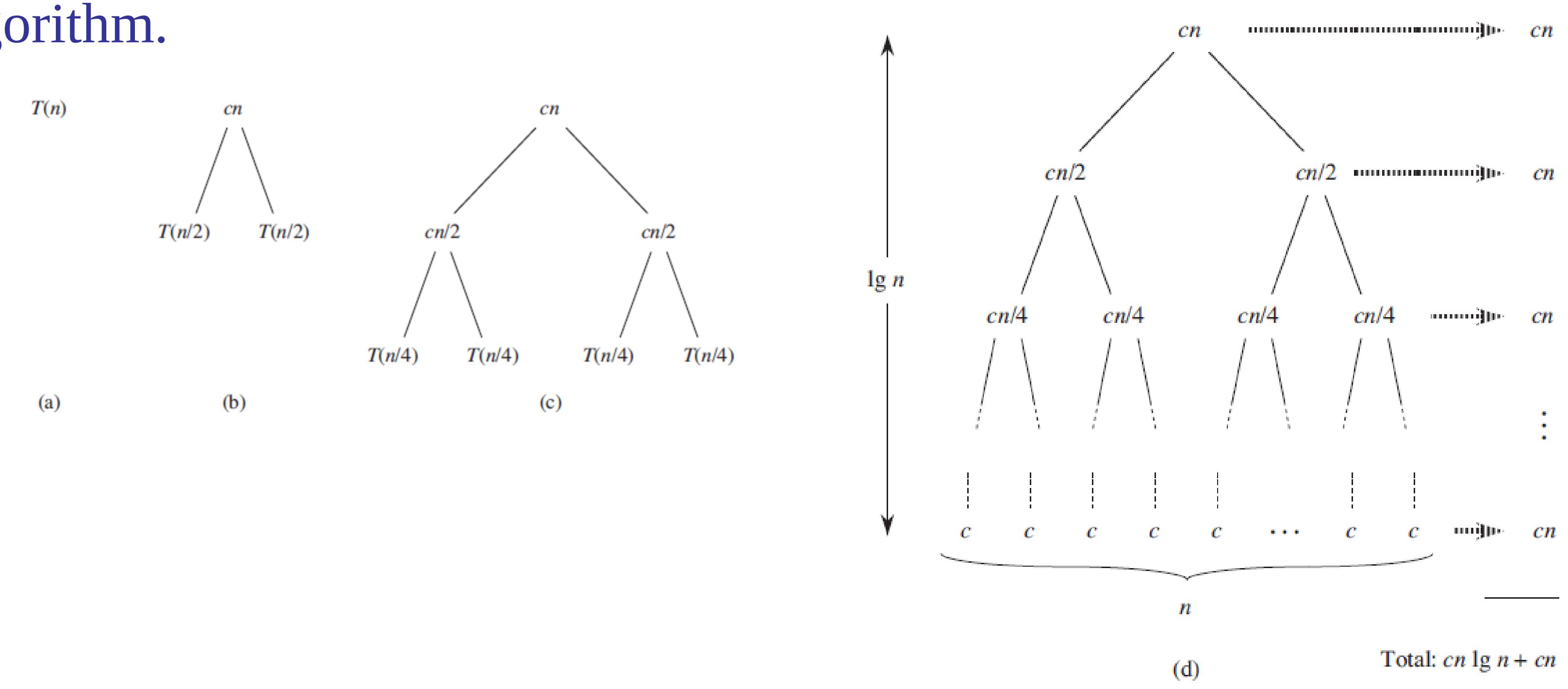
Example, $T(n) = 3T(\lfloor n/4 \rfloor) + n$

if we had assumed that $n = 4^k$ for some integer k, the floor functions could have been conveniently omitted.

* 4.3 The recursion-tree method

- Drawing out a recursion tree, is a straightforward way to devise a good guess, and to show the iteration method intuitively.（画递归树可以从直观上表示迭代法，也有助于猜想递归式的解）
- Recursion trees are particularly useful when the recurrence describes the running time of a divide-and-conquer algorithm.

$$T(n) = 2T(n/2) + n$$



小结

1. 一种重要的设计方法：分治法（以归并排序为例）
2. 算法分析的数学基础：函数渐近分析（几种符号）
3. 递归分析方法与递归方程求解：置换法、迭代法、递归树方法

Exercises

(1) $T(n) = 3T(\lfloor n/3 \rfloor + 3) + n$ 解是什么?

(2) $h(n) = h(0)*h(n\text{-}1) + h(1)*h(\text{n-}2) + \ldots + h(n\text{-}1)*h(0)$

另一种递归式: $h(n) = ((4*n\text{-}2)/(n+1))*h(n\text{-}1)$, ($其中\ n >= 2, h(0) = 1$)

证明: $h(n) = C(2n, n)/(n+1)$, $(n = 1, 2, 3, \ldots)$

(3) Show that for any real constants a and b , where $b > 0$, $(n + a)^b = \Theta(n^b)$.

(4) Is $2^{n+1} = O(2^n)$? Is $2^{2n} = O(2^n)$?

Exercises

The substitution method is powerful, but it can be applied only in cases when it is easy to guess the form of the answer.

E Zexal的二叉树（签到）

时间限制：1000ms 内存限制：65536kb

通过率：200/209 (95.69%) 正确率：200/596 (33.56%)

题目

知识点：树，数论，dp，递归（都可以做）

上学期我们学习了二叉树，也都知道3个结点的二叉树有5种， 现给你二叉树的结点个数n，要你输出不同形态二叉树的种数。

输入

第一个数为一个整数$n(n <= 30)$

输出

对于每组数据，输出一行，不同形态二叉树的种数。

输入样例

```
3
```

输出样例

```
5
```

recursion

$$h(n) = h(0)*h(n-1) + h(1)*h(n-2) + \ldots + h(n-1)*h(0)$$

另一种递归式(其中 $n >= 2$) :

$$h(n) = ((4*n-2)/(n+1))*h(n-1)$$

这个就不好猜!

$$h(n) = C(2n, n)/(n+1)$$

$$(n = 1, 2, 3, \ldots)$$

习题：证明该公式。