Chapter 5

Probabilistic analysis and randomized algorithms

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5 Probabilistic analysis and randomized algorithms

- Explain the difference between probabilistic analysis & randomized algorithms.
  （概率分析与随机算法）

- Present the technique of indicator random variables.
  （用指示(器)随机变量来对算法进行概率分析）

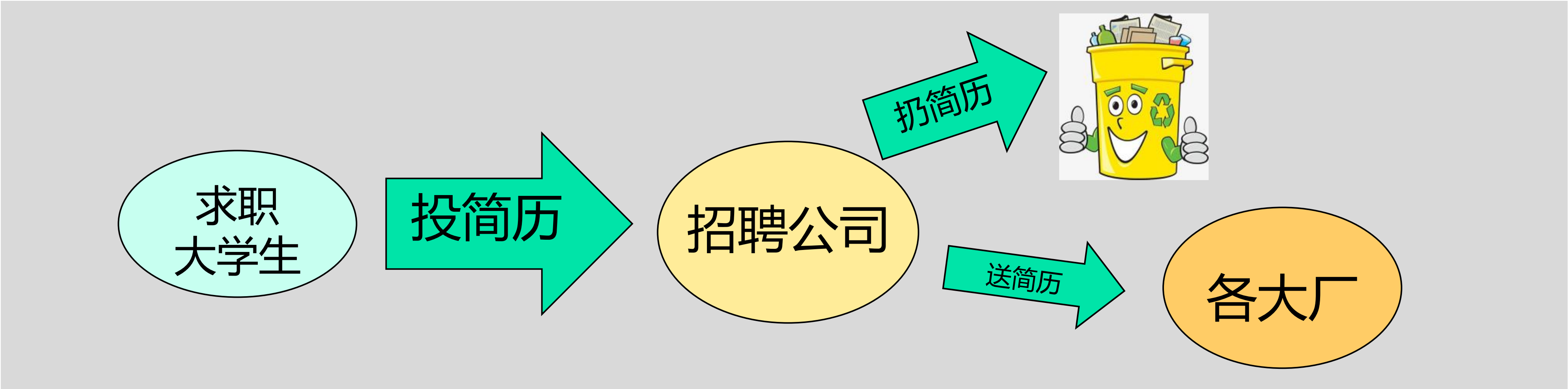
- Give another example of a randomized algorithm.
  （一个随机算法实例）

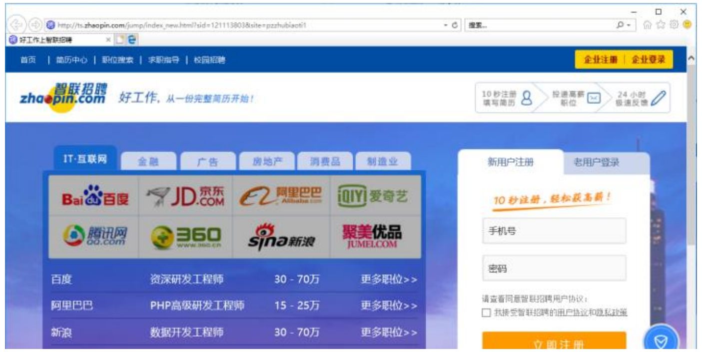
5.1 The hiring problem (雇佣问题，or 助手问题)

Scenario（情景）: hire the best office assistant in a month

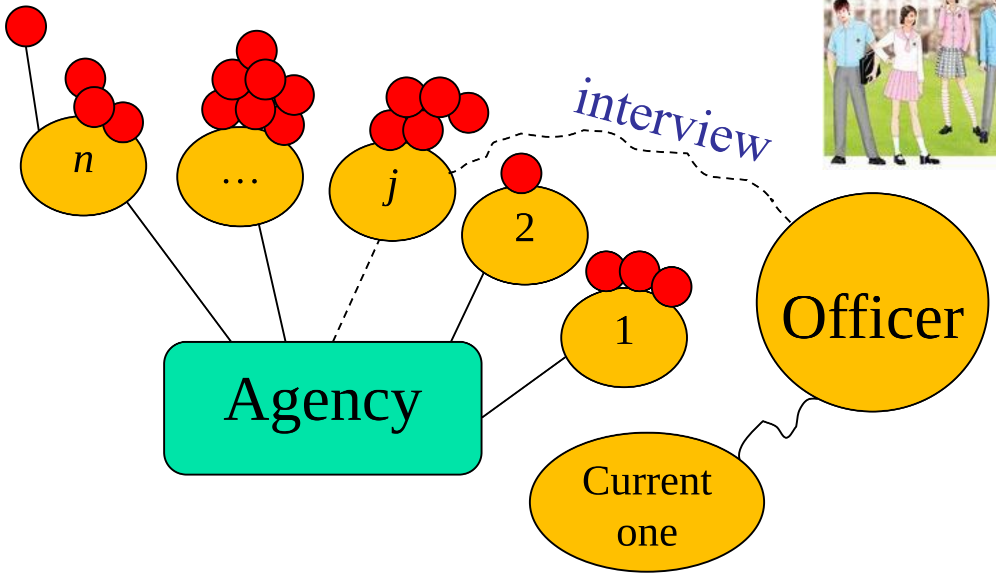
You are using an employment agency to hire a new office assistant.

招聘公司帮你物色办公助理候选人





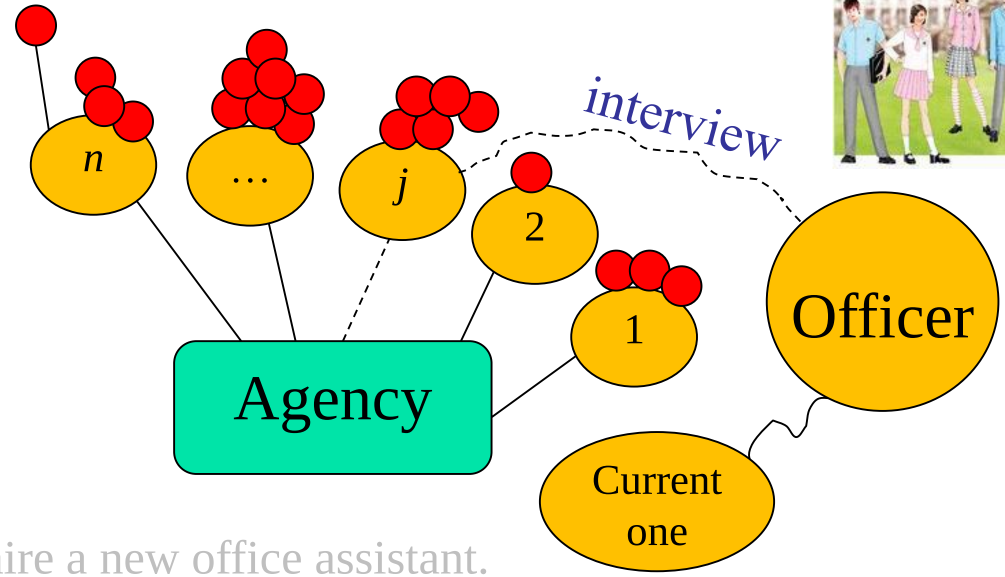
5.1 The hiring problem



Scenario: hire the best office assistant in a month

- You are using an employment agency to hire a new office assistant. （招聘公司帮你物色办公助理候选人）
- The agency sends you one candidate each day.
  (1, 2, ... , n 表示候选人的编号，红色圆圈数表示候选人的工作能力【如果是程序员，则是AC的题目数】)
- You interview the candidate and must immediately decide whether or not to hire him. But if you hire, you must also fire your current one.
- ……

5.1 The hiring problem

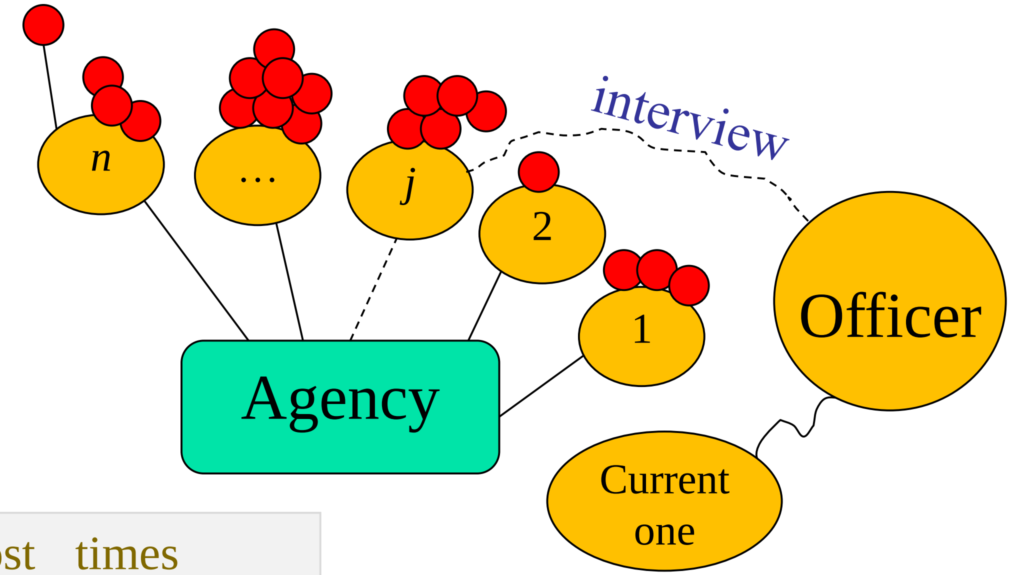


- You are using an employment agency to hire a new office assistant.
- The agency sends you one candidate each day.
- You interview the candidate and must immediately decide whether or not to hire him. But if you hire, you must also fire your current one.
- Cost to interview is 1K per candidate (interview fee paid to agency).
- Cost to hire is 10K per candidate, includes: cost to fire current office assistant + hiring fee paid to agency （面试一个候选人支付招聘公司 1K；临时录用当天面试的候选人需要支付招聘公司10K）

***Goal*: Determine what the price of this strategy will be？**

5.1 The hiring problem

Scenario: Assumes that the candidates are numbered 1 to *n*. Uses a dummy candidate 0 that is worse than all others, so that the first candidate is always hired.



HIRE-ASSISTANT(*n*)	cost	times
best ← 0 // candidate 0 is a least-qualified dummy candidate		
for *i* ← 1 to *n*		
interview candidate *i*	c_i	n
if candidate *i* is better than candidate *best*		
best ← *i*		
hire candidate *i*	c_h	m

Cost: n candidates, we hire m of them, cost is $T(n) = nc_i + mc_h$

5.1 The hiring problem

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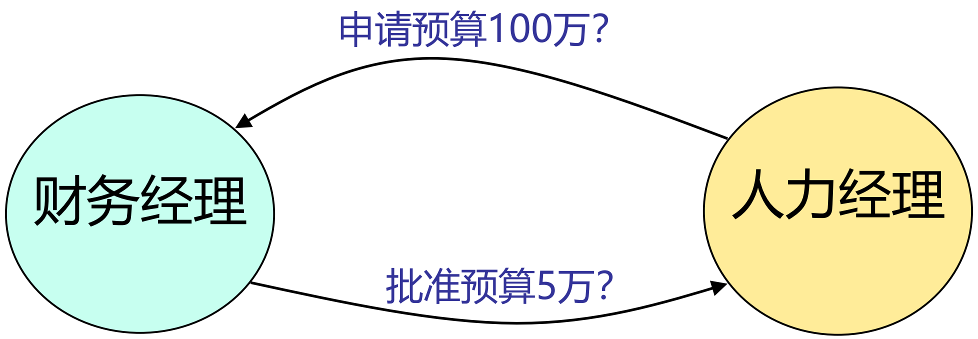


分析算法的人享有双重的幸福：

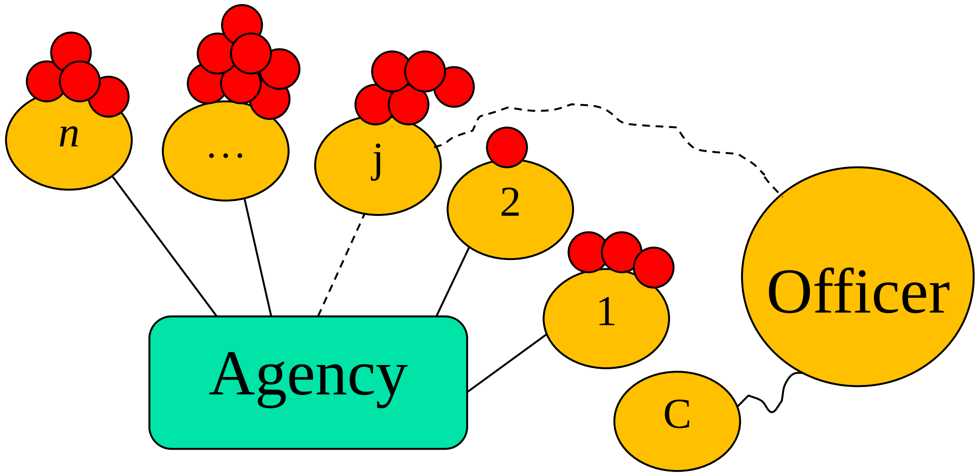
- **一方面，他们能够体验到优雅数学模式纯粹的美，这种模式存在于优美的计算过程之中；**
- **另一方面，当他们的理论使得其他工作能够做得更快、更经济时，他们能够得到实际的褒奖。**

——Donald E. Knuth

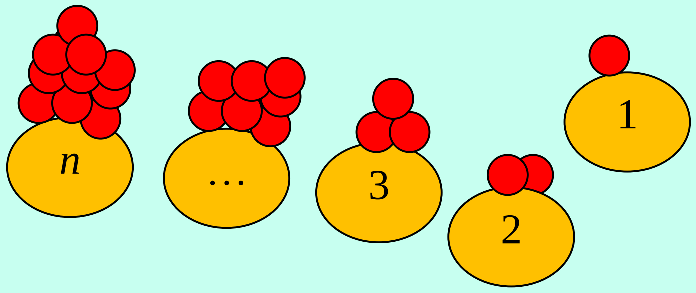
5.1.1 Worst-case analysis

Cost: n candidates, we hire m of them, cost is $T(n) = nc_i + mc_h$

- We focus on analyzing the hiring cost mc_h? ($c_h >> c_i$)
- mc_h varies with each run of the algorithm: it depends on the order in which we interview the candidates.
- Worst-case analysis
  - We hire all n candidates. $T(n) = nc_i + nc_h$? When?
  - The candidates appear in increasing order of qulity.



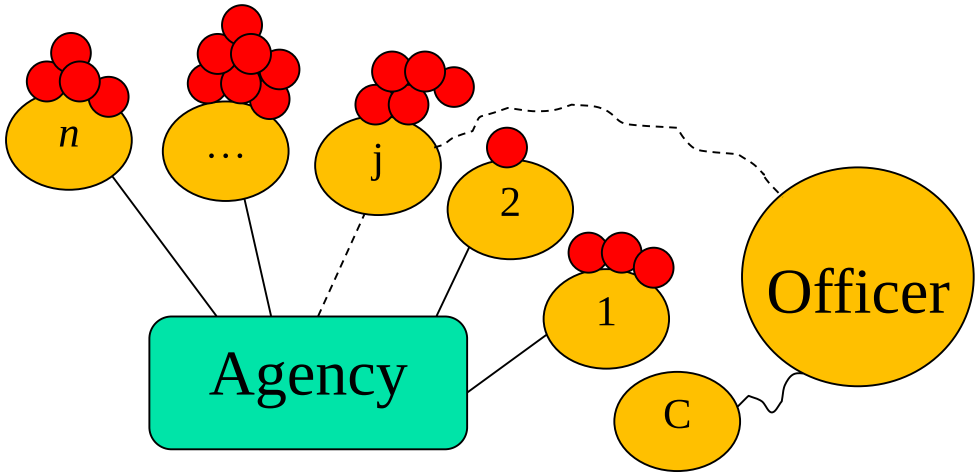
In a month, $T(30) = 30 + 30*10 =$ 330 K

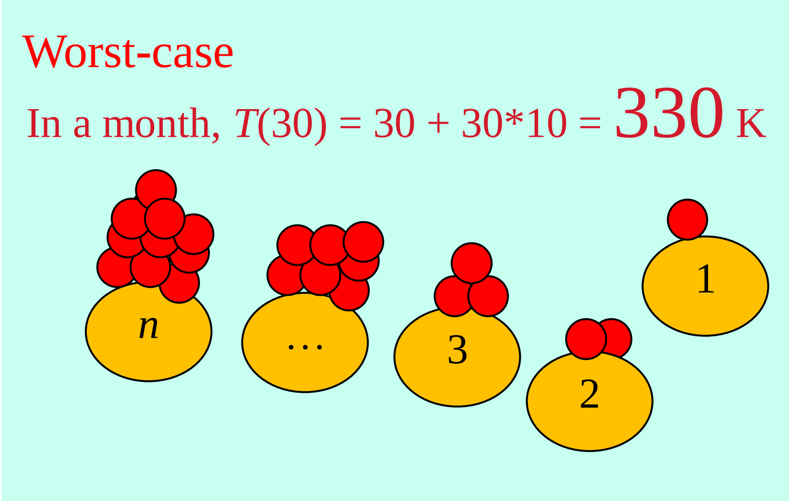


5.1.1 Worst-case analysis

Cost: n candidates, we hire m of them, cost is $T(n) = nc_i + mc_h$

- Worst-case analysis: We hire all n candidates, $T(n) = nc_i + nc_h$
- Best-case ? We hire only one candidate. When?
- **Average-case** ? **The expected number of times we hire new office assistant.**



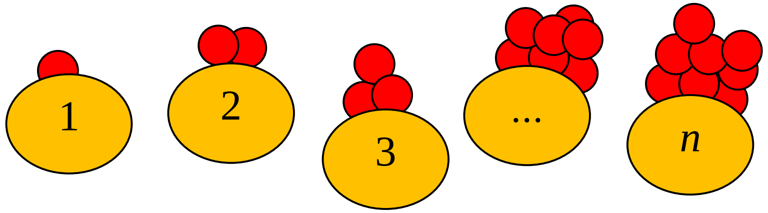


5.1.2 Propbabilistic analysis

Assume that candidates come in a random order:

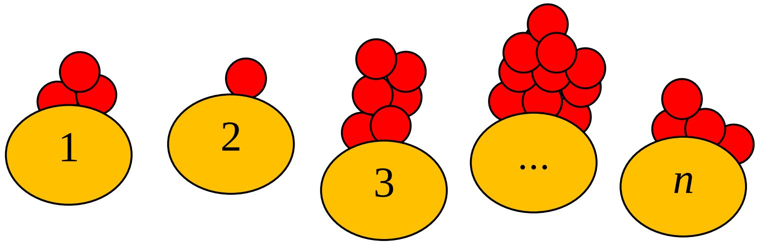
- Assign a rank to each candidate: *rank*(i) is a unique integer in the range 1 to n.
  （对每一个候选人分配一个“名次”）
- The list <*rank*(1), …, *rank*(n)> is a permutation of the candidate numbers <1, …, n>, such as <5, 2, 1, 28, 9, … , 11>
- The ranks form a uniform random permutation: each of the possible $n!$ permutations appears with equal probability

A[1]	A[2]	A[3]	…	A[n]
1	2	3		n



随机打乱数组

A[1]	A[2]	A[3]	…	A[n]
3	1	6		4



5.1.2 Propbabilistic analysis

HIRE-ASSISTANT(*n*)	cost	times
best ← 0 // candidate 0 is a least-qualified dummy candidate		
for *i* ← 1 to *n*		
interview candidate *i*	c_i	n
if candidate *i* is better than candidate *best*		
best ← *i*		
hire candidate *i*	c_h	m

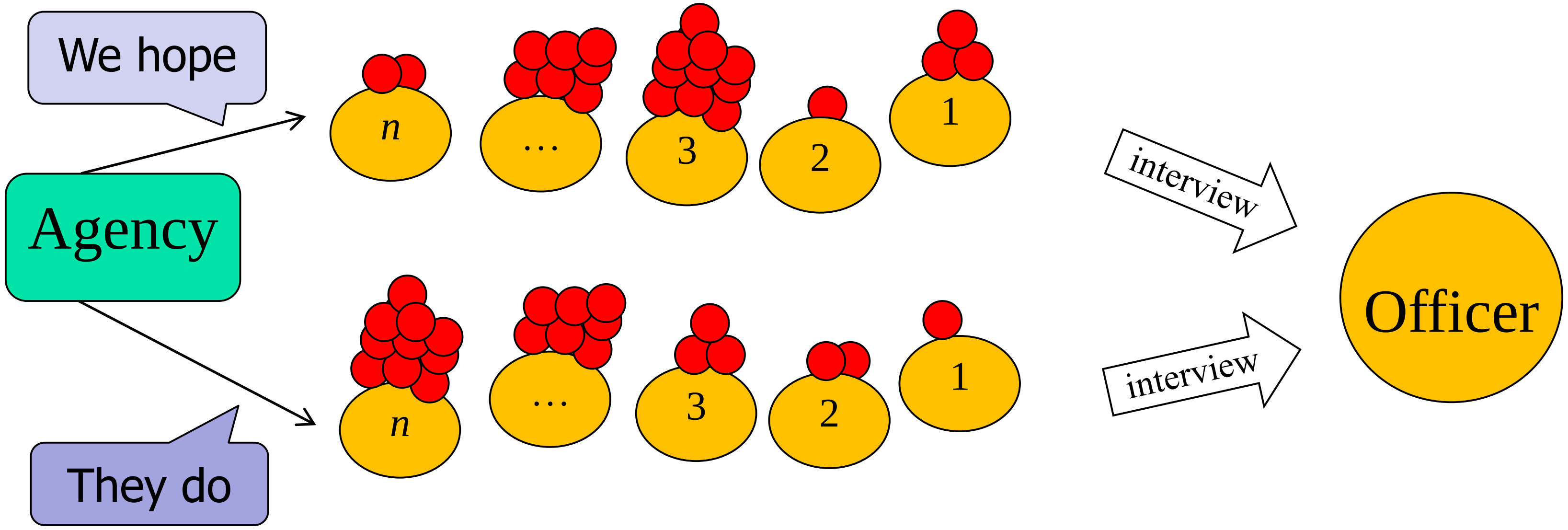
Essential idea of probabilistic analysis: We must use knowledge of, or make assumptions about, the distribution of inputs.（概率分析的本质：需已知或假定输入的概率分布）

- The expectation is taken over this distribution.
  （依据概率分布来求期望，期望值即是平均 hire 的人数）
- Section 5.2 contains a probabilistic analysis of the hiring problem.

5.1.3 Randomized algorithms

Idea

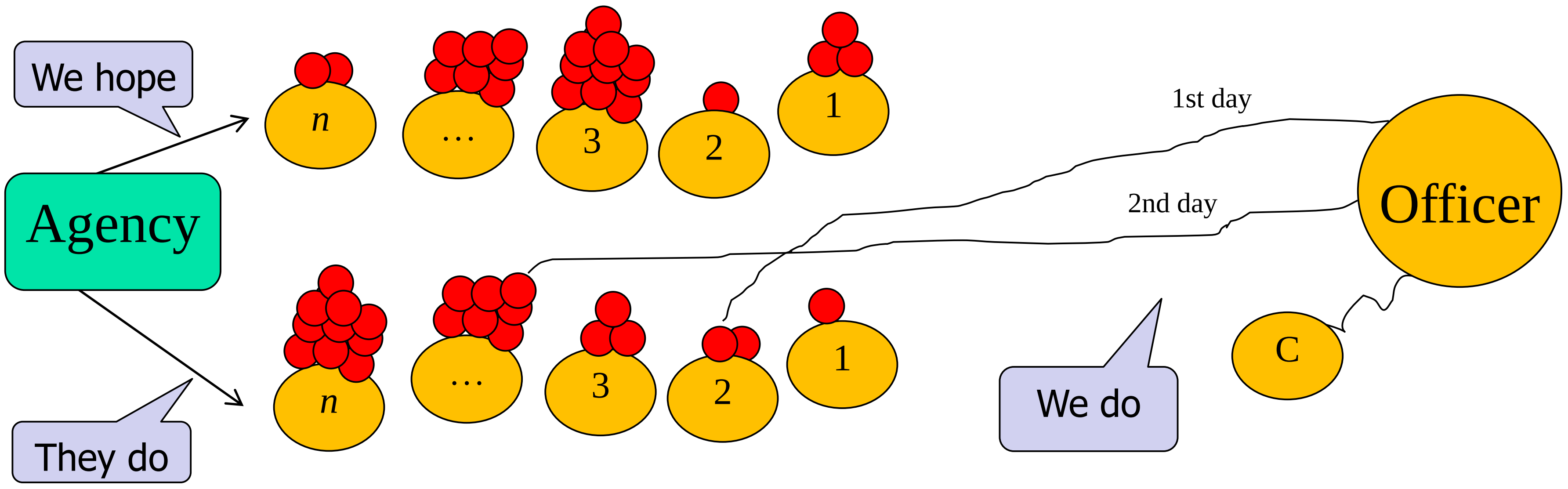
- We might not know the distribution of inputs, or we might not be able to model it computationally.
  （我们不知道输入的分布，也不可能为输入的分布进行可计算的建模）
- Instead, we use randomization within the algorithm in order to impose a distribution on the inputs.
  （在算法中通过对输入进行随机化处理，从而为输入施加一种分布）



5.1.3 Randomized algorithms

For the hiring problem, change the scenario:

- The employment agency sends us a list of all *n* candidates in advance.
  （猎头公司预先提供 *n* 个候选人列表）
- On each day, we randomly choose a candidate from the list to interview.
  （每天，我们随机选取一人进行面试）
- Instead of relying on the candidates being presented to us in a random order, we take control of the process and enforce a random order. (Chap 5.3)
  （无须担心候选人是否被随机提供，我们通过随机运算预处理可以控制候选人的随机顺序）



5.1.3 Randomized algorithms

What makes an algorithms randomized: An algorithm is randomized if its behavior is determined in part by values produced by a *random-number generator*.

（算法随机化：由随机数产生器生成数值……）

- RANDOM(a, b) returns an integer r, where $a \le r \le b$ and each of the b-a+1 possible values of r is equally likely.
- In practice, RANDOM is implemented by a pseudorandom-number generator, which is a deterministic method returning numbers that “look” random and pass statistical tests.

（RANDOM实际上由一个确定的算法〔伪随机产生器〕产生，其结果表面上看上去像是随机数）

Random number generator（随机数产生器）

- Most random number generators generate a sequence of integers by the following recurrence
- X_0= a given integer (seed), $X_{i+1} = aX_i \bmod M$
- For example, for $X_0 = 1$, $a = 5$, $M = 13$, we have

  $X_1 = 5*1\%13 = 5$, $X_2 = 5*5\%13 = 12$, $X_3 = 5*12\%13 = 8$, $X_4 = 5*8\%13 = 1$, …

  Each integer in the sequence lies in the range $[0, M - 1]$.

A probabilistic analysis of the hiring problem

5.2 Indicator random variables（指示随机变量）


Given a sample space and an event A, we define the indicator random variable:

$$\mathrm{I}\{A\} = \begin{cases} 1 & \text{if } A \text{ occur,} \\ 0 & \text{if } A \text{ does not occur.} \end{cases}$$

Lemma

For an event, let $X_A = \mathrm{I}\{A\}$. Then $\mathrm{E}[X_A] = \Pr\{A\}$.

Proof Letting $\sim A$ be the complement of A, we have

$$\begin{aligned} \mathrm{E}[X_A] &= \mathrm{E}[\mathrm{I}\{A\}] \\ &= 1.\Pr\{A\} + 0.\Pr\{\sim A\} \qquad \{\text{definition of expected value}\} \\ &= \Pr\{A\}. \end{aligned}$$

5.2 Indicator random variables

Lemma

For an event A, let $X_A = \mathrm{I}\{A\}$.
Then $\mathrm{E}[X_A] = \Pr\{A\}$.

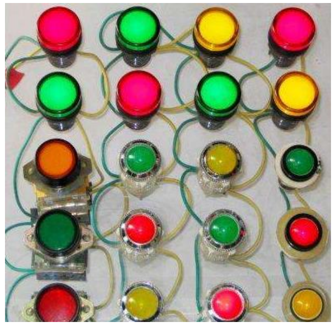
Simple example: Determine the expected number of heads when we flip a fair coin one time. （投一次硬币，正面向上的期望〔平均数〕）

- Sample space is $\{H, T\}$
- $\Pr\{H\} = \Pr\{T\} = 1/2$
- Define indicator random variable $X_H = \mathrm{I}\{H\}$.
  X_H counts the number of heads in one flip.
- Since $\Pr\{H\} = 1/2$, lemma says that $\mathrm{E}[X_H] = 1/2$.

5.2 Indicator random variables

Lemma

For an event A, let $X_A = \mathrm{I}\{A\}$.
Then $\mathrm{E}[X_A] = \Pr\{A\}$.



Slightly more complicated example: Determine the expected number of heads when in n coin flips. （投 n 次硬币，正面向上的期望（平均数））

Let X be a random variable for the number of heads in n flips.
（令随机变量 X 表示投 n 次硬币正面向上的数）

$$\mathrm{E}[X] = \sum_{k=0}^{n} k \cdot \Pr\{X = k\}$$

例：硬币正面向上为1，反面向上为0，投三个硬币（或一个硬币投三次），则有8种情况：

000, 001, 010, 011, 100, 101, 110, 111

0 个硬币为正面

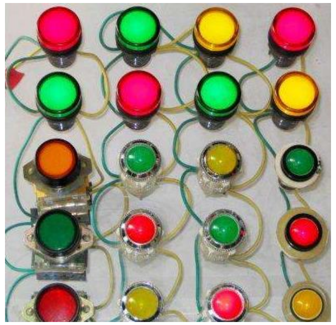
3 个硬币为正面

正面向上的平均次数： E[X] = 0*1/8 + 1*3/8 + 2*3/8 + 3*1/8 = 3/2

5.2 Indicator random variables

Lemma

For an event A, let $X_A = \mathrm{I}\{A\}$.
Then $\mathrm{E}[X_A] = \Pr\{A\}$.



Slightly more complicated example: Determine the expected number of heads when in n coin flips.（投 n 次硬币，正面向上的期望〔平均数〕）

Let X be a random variable for the number of heads in n flips.
（令随机变量 X 表示投 n 次硬币正面向上的数）

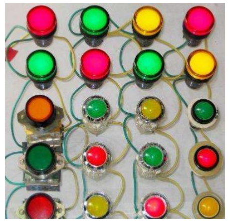
$$\mathrm{E}[X] = \sum_{k=0}^{n} k \cdot \Pr\{X = k\}$$

A slightly more complicated? Yes!

Instead, we'll use indicator random variables ……

5.2 Indicator random variables

Lemma



For an event A, let $X_A = \mathrm{I}\{A\}$. Then $\mathrm{E}[X_A] = \Pr\{A\}$.

Slightly more complicated example: the expected number of heads when in n coin flips.

（投 n 次硬币，正面向上的期望〔平均数〕）

- For $i = 1, \ldots, n$, define $X_i = \mathrm{I}\{\text{the } i\text{th flip results in event } H\}$
- Then, $X = \sum_{i=1}^{n} X_i$
- Lemma says that $\mathrm{E}[X_i] = \Pr\{H\} = 1/2$ for $i = 1, 2, \ldots, n$
- Expected number of heads is

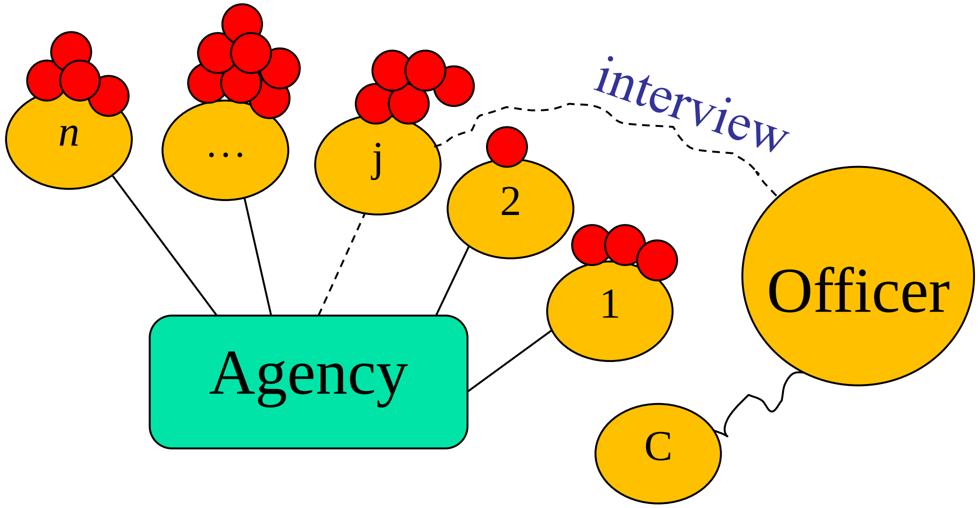
$$\mathrm{E}[X] = \mathrm{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathrm{E}[X_i] = \sum_{i=1}^{n} 1/2 = n/2$$

Analysis of the hiring problem

Average-case: The expected number of times we hire new office assistant.

Assume that the candidates arrive in a random order.

Let X be a random variable that equals the number of times we hire new office assistant.（令随机变量 X 表示雇用新助手的人数）



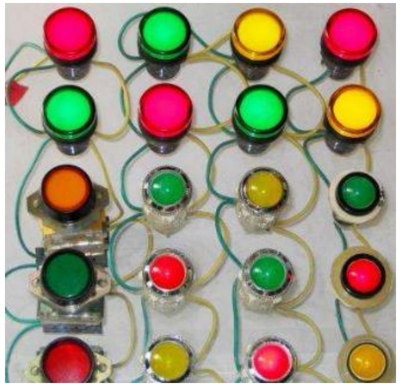
Use a probabilistic analysis

Then $\mathrm{E}[X] = \sum_{i=1}^{n} i \Pr\{X = i\}$

The calculation would be cumbersome（计算烦琐）

Analysis of the hiring problem

Assume that the candidates arrive in a random order.



random variable X = the number of times we hire new office assistant（随机变量 X 表示雇用新助手的人数）

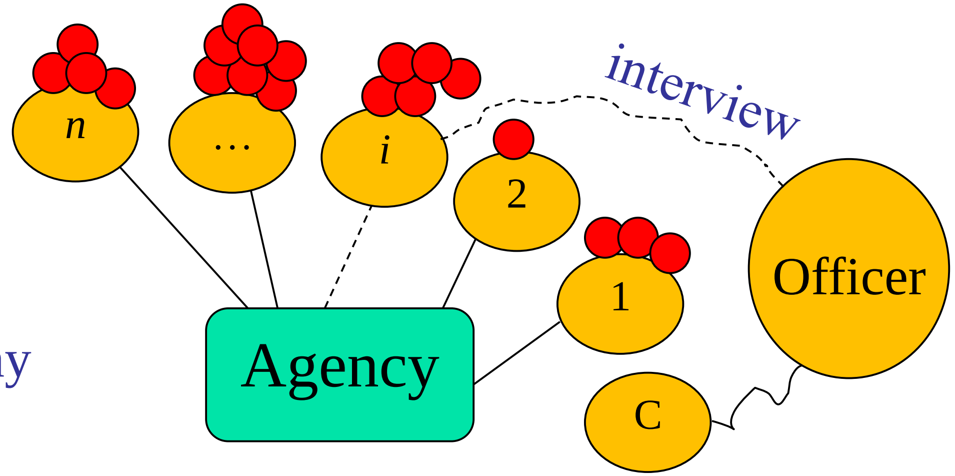
Use indicator random variables

- Define indicator random variables $X_1, X_2, \ldots, X_n$, where
  $X_i = \mathrm{I}\{\text{candidate } i \text{ is hired}\}$
- Useful properties:
  $X = X_1 + X_2 + \ldots + X_n$
  Lemma => $\mathrm{E}[X_i] = \Pr\{\text{candidate } i \text{ is hired}\}$.

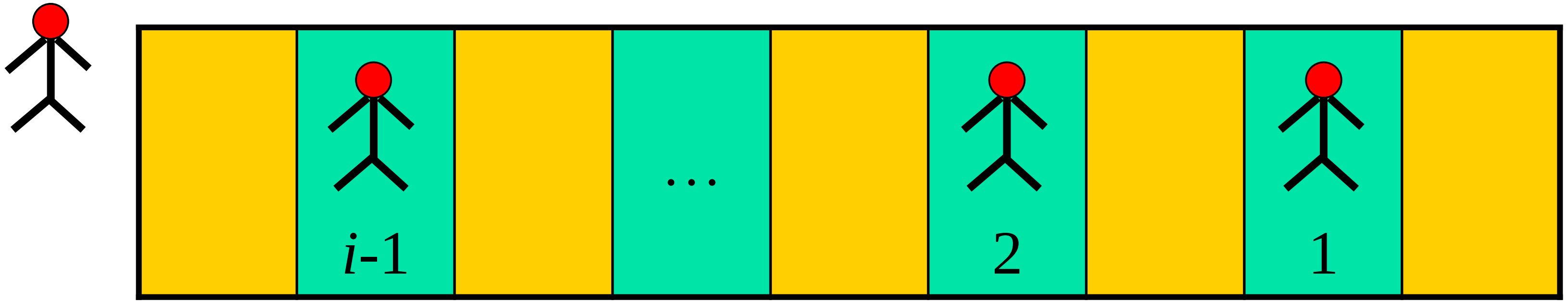
We need to compute $\Pr\{\text{candidate } i \text{ is hired}\}$?

Analysis of the hiring problem

Pr{candidate i is hired}?



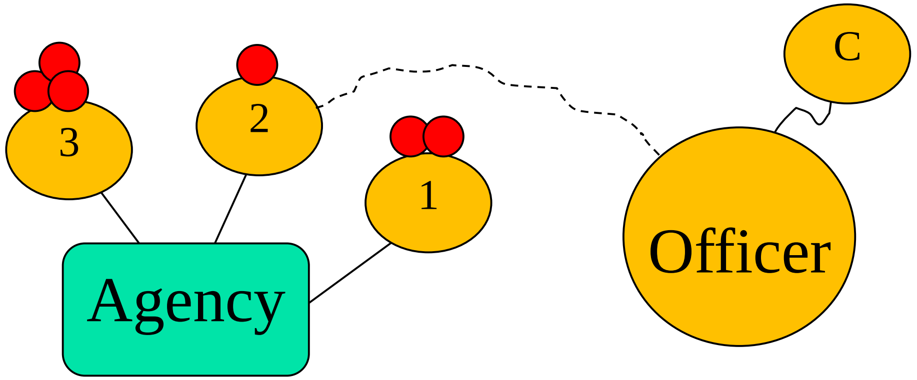
- i is hired ⇔ i is better than each of candidates 1, 2, … , i-1.
- Assumption that the candidates arrive in random order => any one of these candidates is equally likely to be the best one.
  （若候选人随机到来，则每一个候选人为最佳人选的概率相等）
- Thus, $E\{X_i\}$ = Pr{candidate i is the best so far} = $1/i$?



Analysis of the hiring problem

Pr{ candidate i is hired}?

- i is hired ⇔
  i is better than each of candidates 1, 2, … , i-1.
- Thus, $E\{X_i\}$ = Pr{ candidate i is the best so far} = $1/i$?



viewed : viewing , Pr of hired
已面试：待面试，待面试人被雇佣的概率
0 : {1, 2, 3} , 1

1:{2, 3}, 1/3*1
2:{1, 3}, 1/3*1/2
3:{1, 2}, 1/3*0

{1, 2}:3, 1/3*1
{1, 3}:2, 1/3*0
{2, 3}:1, 1/3*0

第一天面试的人的资历可能是1or2or3，每种情况的Pr是1/3；
若是第一种情况，第二天面试的人被雇佣的Pr是1，则条件概率 Pr=1/3*1；
其他情况相似。

Analysis of the hiring problem

HIRE-ASSISTANT(n)	cost	times
best ← 0 // candidate 0 is a least-qualified dummy candidate		
for i ← 1 to n		
interview candidate i	c_i	n
if candidate i is better than candidate *best*		
best ← i		
hire candidate i	c_h	m

$$T(n) = nc_i + mc_h$$

$E\{X_i\} = \Pr\{\text{candidate } i \text{ is the best so far}\} = 1/i$?

Then $$E[X] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} 1/i = \ln n$$

Thus, the expected hiring cost is ------------------------ $nc_i + (\ln n)c_h$,

which is much better than the worst-case cost of --- $nc_i + nc_h$.

$\ln n$ vs n

$(nc_i + mc_h)$: 30+3.4*10 = 64 vs 30+30*10 = 330

(ln30 = 3.4…)

Analysis of the hiring problem

$$\mathrm{E}[X]=\mathrm{E}\left[\sum_{i=1}^{n} X_i\right]=\sum_{i=1}^{n}\mathrm{E}[X_i]=\sum_{i=1}^{n}1/i=\ln n$$

Harmonic series

For positive integers n, the nth ***harmonic number*** is

$$\begin{aligned} H_n &= 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n} \\ &= \sum_{k=1}^{n}\frac{1}{k} \\ &= \ln n + O(1)\,. \end{aligned} \tag{A.7}$$

Thus, the expected hiring cost is ----------------------- $nc_i + (\ln n)c_h$,
which is much better than the worst-case cost of --- $nc_i + nc_h$.

$(nc_i + mc_h)$: 30+3.4*10 = 64 vs 30+30*10 = 330

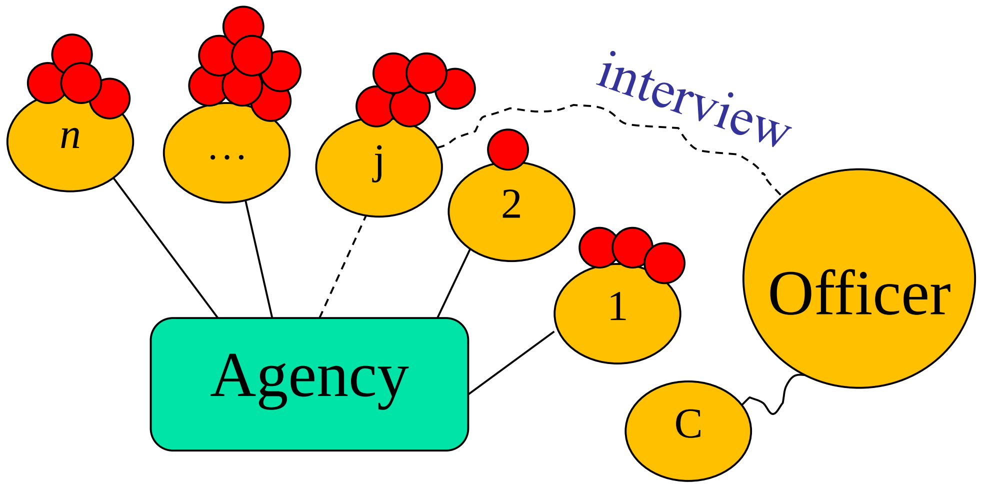
($\ln 30 = 3.4\ldots$)

ln*n* vs *n*

5.3 Randomized algorithms

For the hiring problem,

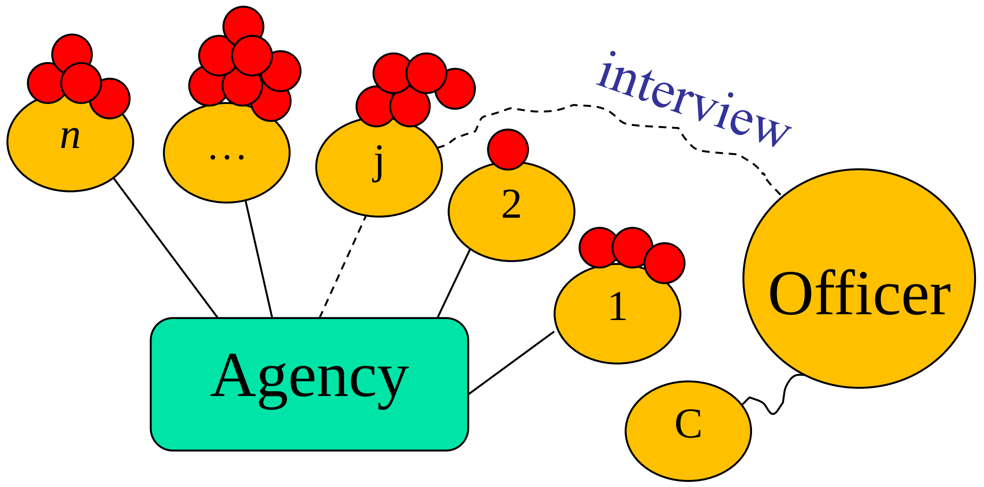
the algorithm is deterministic.



- For any given input, the number of times we hire a new office assistant will always be the same.
  （给定输入，则雇用的人数确定）
- The number of times we hire a new office assistant depends only on the input.
  （雇用的人数〔资源消费〕依赖于输入）
  - Some rank orderings will always produce a high hiring cost. Example: <1, 2, 3, 4, 5, 6>, where each is hired.
  - Some will always produce a low hiring cost. Example: <6, *, *, *, *, *>, where only the first is hired.
  - Some may be in between.

5.3.1 The hiring problem

Instead of always interviewing the candidates in the order presented, we first randomly permuted input.



- The randomization is now in the algorithm, not in the input distribution.
  （在算法中先进行随机化处理，与输入分布无关）
- Given a particular input, we can no longer say what its hiring cost will be. Each time we run the algorithm, we can get a different hiring cost. （算法的运行时间与输入无关）
- No particular input always elicits worst-case behavior.（算法的最坏运算时间不取决于特定的输入）
- Bad behavior occurs only if we get “unlucky” numbers from the random-number generator. （只有当随机数产生器产生很不幸运的数时，算法的运算时间最坏）

Algorithm for randimized hiring problem

RANDOMIZED-HIRE-ASSISTANT(*n*)

Randomly permute the list of candidates

HIRE-ASSISTANT(*n*)

- *Lemma*

The expected hiring cost of RANDOMIZED-HIRE-ASSISTANT is $nc_i + (\ln n)c_h$

Proof

After permuting the input array, we have a situation identical to the probabilistic analysis of deterministic HIRE-ASSISTANT.

（对输入矩阵进行随机置换后，情况同HIRE-ASSISTANT相同）

5.3.2 Randomly permuting an array

Goal: Produce a uniform random permutation (each of the n! permutations is equally likely to be produced),

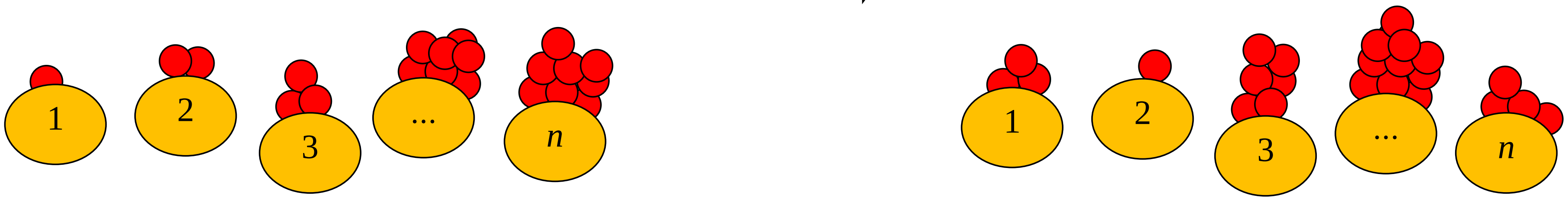
that is, for $A = <1, 2, 3, \dots, n>$,

the numbers of permutation of A is $P_n^n = n!$, each of that is equally likely to be produced.

A[1]	A[2]	A[3]	…	A[n]
1	2	3		n

随机打乱数组

A[1]	A[2]	A[3]	…	A[n]
3	1	6		4



5.3.2 Randomly permuting an array

Applications:

- ✓ 洗牌程序
- ✓ 随机打乱 n 个数，构造测试数据
- ✓ 数据预处理，防止“不好”的输入（比如在OJ上做题时，极端的“不好”数据导致程序超时，可以尝试先把输入随机置换）

5.3.2 Randomly permuting an array

(1) Permute-by-sorting

The method is not very good

- Assume the given array A contains the element 1 through n.
- Assign each element $A[i]$ a random priority $P[i]$, then sort the elements of A according to these priorities. Example
  - If initial array is A = <1, 2, 3, 4>, choose random priorities P = <36, 3, 97, 19>, then produce an array B = <2, 4, 1, 3>

ind	[1]	[2]	[3]	[4]
A value	1	2	3	4

ind	[1]	[2]	[3]	[4]
A value	1	2	3	4
Priorities	36	3	97	19

PERMUTE-BY-SORTING(*A*)

$n = length[A]$
for(i = 1; i <= n; i++)
  $P[i]$ = RANDOM(1, n^3)
sort A, using P as sort keys
return A

We use a range of 1 to n^3 in RANDOM to make it likely that all the priorities in P are unique.(Exercise 5.3-5)

5.3.2 Randomly permuting an array

(2) RANDOMIZE-IN-PLACE

The method is better

RANDOMIZE-IN-PLACE(A, n)

```
RANDOMIZE-IN-PLACE(A, n)
    for(i = 1; i <= n; i++)
        swap(A[i], A[RANDOM(i, n)])
```

1	2	3	…	n
$A(1)$	$A(2)$	$A(3)$	…	$A(n)$

1	2	3	…	i_1	…	n
$A(i_1)$	$A(2)$	$A(3)$	…	$A(1)$	…	$A(n)$

1	2	3	…	i_2	…	n
$A(i_1)$	$A(i_2)$	$A(3)$	…	$A(2)$	…	$A(n)$

Idea:

- In iteration i, choose $A[i]$ randomly from $A[i .. n]$.
- Will never alter $A[i]$ after iteration i. （第 i 次迭代后不再改变 $A[i]$）

Merit：

- It runs in linear time without requiring sorting ($O(n)$).
- It needs fewer random bits (n random numbers in the range 1 to n rather than the range 1 to n^3)（仅需更小范围的随机数产生器）
- No auxiliary array is required. （不需要辅助空间）

5.3.2 Randomly permuting an array

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The method is better

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RANDOMIZE-IN-PLACE(A, n)
    for(i = 1; i <= n; i++)
        swap(A[i], A[RANDOM(i, n)])
```

1	2	3	…	n
$A(1)$	$A(2)$	$A(3)$	…	$A(n)$

1	2	3	…	i_1	…	n
$A(i_1)$	$A(2)$	$A(3)$	…	$A(1)$	…	$A(n)$

1	2	3	…	i_2	…	n
$A(i_1)$	$A(i_2)$	$A(3)$	…	$A(2)$	…	$A(n)$

*** *Correctness:*

- Given a set of n elements, a k-permutation is a sequence containing k of the n elements. There are $n!/(n-k)!$ possible k-permutations ? （给定 n 个元素，从其中任取 k 个元素进行排列，则有 $n!/(n-k)!$ 种不同的 k-排列，或 k-置换 ?）

$$P_n^k = C_n^k \cdot P_k^k = \frac{n!}{k!(n-k)!} \cdot k! = \frac{n!}{(n-k)!}$$

- ***Lemma***

  RANDOMIZE-IN-PLACE computes a uniform random permutation.

  Proof Use a loop invariant:

5.3.2 Randomly permuting an array

(2) RANDOMIZE-IN-PLACE

The method is better

```
RANDOMIZE-IN-PLACE(A, n)
    for(i = 1; i <= n; i++)
        swap(A[i], A[RANDOM(i, n)])
```

1	2	3	…			n
$A(1)$	$A(2)$	$A(3)$	…			$A(n)$

1	2	3	…	i_1	…	n
$A(i_1)$	$A(2)$	$A(3)$	…	$A(1)$	…	$A(n)$

1	2	3	…	i_2	…	n
$A(i_1)$	$A(i_2)$	$A(3)$	…	$A(2)$	…	$A(n)$

- ***Lemma***

  RANDOMIZE-IN-PLACE computes a uniform random permutation.

$$1/\mathrm{P}_n^k = 1/\frac{n!}{(n-k)!} = \frac{(n-k)!}{n!}$$

Proof Use a loop invariant:

Loop invariant: Just prior to the ith iteration of the for loop, for each possible (i-1)-permutation, subarray A[1 .. i-1] contains this (i-1)-permutation with probability (n-i+1)!/n! ?

（第 i 次迭代之前，对 (i-1)-置换，任意一个(i-1)-置换A[1 .. i-1]的概率为(n-i+1)!/n! ?）

5.3.2 Randomly permuting an array

(2) RANDOMIZE-IN-PLACE

The method is better

RANDOMIZE-IN-PLACE(*A*, *n*)
    for(i = 1; i <= n; i++)
        swap($A[i]$, A[RANDOM(i, n)])

1	2	3	…	n
$A(1)$	$A(2)$	$A(3)$	…	$A(n)$

1	2	3	…	i_1	…	n
$A(i_1)$	$A(2)$	$A(3)$	…	$A(1)$	…	$A(n)$

1	2	3	…	i_2	…	n
$A(i_1)$	$A(i_2)$	$A(3)$	…	$A(2)$	…	$A(n)$

- ***Lemma*** RANDOMIZE-IN-PLACE computes a uniform random permutation.

  Proof Use a loop invariant:

  Loop invariant: $A[1 .. i\text{-}1]$ contains each (i-1)-permutation with probability $(n\text{-}i+1)!/n!$.

- Initialization: Just before first iteration, i = 1. Loop invariant says for each possible 0-permutation, subarray $A[1 .. 0]$ contains this 0-permutation with probability $n!/n!$ = 1. $A[1 .. 0]$ is an empty subarray, and a 0-permutation has no elements. So, A[1 .. 0] contains any 0-permutation with probability 1. （空集包含空置换的概率为1）

5.3.2 Randomly permuting an array

(2) RANDOMIZE-IN-PLACE

```
RANDOMIZE-IN-PLACE(A, n)
    for(i = 1; i <= n; i++)
        swap(A[i], A[RANDOM(i, n)])
```

- ***Lemma*** RANDOMIZE-IN-PLACE computes a uniform random permutation.

  Proof Loop invariant: $\Pr\{A[1 .. i\text{-}1]$ contains each $(i\text{-}1)$-permutation$\} = (n-i+1)!/n!$.

- Maintenance: Assume that prior to the ith iteration, $\Pr\{A[1 .. i\text{-}1]$ contains each $(i\text{-}1)$-permutation$\} = (n-i+1)!/n!$, we will show that after the ith iteration, $\Pr(A[1 .. i]$ contains each i-permutation$) = (n-i)!/n!$.

  （第 i 次迭代前，设$(i\text{-}1)$-置换 $A[1 .. i\text{-}1]$ 中，任一置换发生的概率为 $(n-i+1)!/n!$ ，则需证明在第i 次迭代后，任一i-置换 的概率为$(n-i)!/n!$ ）

  Consider a particular i-permutation $R=\langle x_1, x_2, \ldots, x_i\rangle$. It consists of an $(i\text{-}1)$-permutation $R'=\langle x_1, x_2, \ldots, x_{i-1}\rangle$, followed by x_i . （考虑一个特别的 i-置换 R，其前 i-1 个元素组成 $(i\text{-}1)$-置换 R’，最后一个元素为 x_i ）. ……

5.3.2 Randomly permuting an array

(2) RANDOMIZE-IN-PLACE

```
RANDOMIZE-IN-PLACE(A, n)
    for(i = 1; i <= n; i++)
        swap(A[i], A[RANDOM(i, n)])
```

- ***Lemma*** RANDOMIZE-IN-PLACE computes a uniform random permutation.

  Proof Loop invariant: Pr{A[1 .. i-1] contains each (i-1)-permutation} = (n-i+1)!/n! .

- Maintenance: … …

  i-permutation R = $<x_1, x_2, \ldots, x_i> = <x_1, x_2, \ldots, x_{i}> \cup x_i$ = R' $\cup\ x_i$.

  Let E_1 be the event that the algorithm actually puts R' into A[1 .. i-1] . By the loop invariant, Pr{E_1} = (n-i+1)!/n! .

  Let E_2 be the event that the ith iteration puts x_i into A[i].

  We get the i-Permutation R in A[1 .. i] if and only if both E_1 and E_2 occur => the probability that the algorithm produces R in A[1 .. i] is Pr{$E_2 \cap E_1$} = ? …

  （令事件E_1表示算法实际输出 (i-1)-置换R'为A[1 .. i-1]，根据循环不变量， Pr{E_1}=(n-i+1)!/n!，令事件E_2表示第 i 次迭代后输出 A[i] 为x_i，则当且仅当E_1 和 E_2同时发生时我们得到 i-置换 R 为A[1 .. i]，其概率为Pr{$E_2 \cap E_1$} = ? ） …

5.3.2 Randomly permuting an array

(2) RANDOMIZE-IN-PLACE

```
RANDOMIZE-IN-PLACE(A, n)
    for(i = 1; i <= n; i++)
        swap(A[i], A[RANDOM(i, n)])
```

- **_Lemma_** RANDOMIZE-IN-PLACE computes a uniform random permutation.

  Proof Loop invariant: Pr{A[1 .. i-1] contains each (i-1)-permutation} = (n-i+1)!/n! .

- Maintenance:

i	i+1	…	n
$A(j_i)$	$A(j_{i+1})$	…	$A(j_n)$

… …

$\Pr\{E_2 \cap E_1\} = \Pr\{E_2 | E_1\} \Pr\{E_1\}$.

The algorithm choose x_i randomly from the n-i+1 possibilities in $A[i .. n]$ => $\Pr\{E_2|E_1\} = 1/(n-i+1)$? Thus,

$$\Pr\{E_2 \cap E_1\} = \Pr\{E_2 | E_1\}\Pr\{E_1\}$$
$$= \frac{1}{n-i+1} \cdot \frac{(n-i+1)!}{n!} = \frac{(n-i)!}{n!}$$

5.3.2 Randomly permuting an array

(2) RANDOMIZE-IN-PLACE

RANDOMIZE-IN-PLACE(A, n)
  for(i = 1; i <= n; i++)
    swap(A[i], A[RANDOM(i, n)])

- ***Lemma*** RANDOMIZE-IN-PLACE computes a uniform random permutation.

  Proof Loop invariant: $\Pr\{A[1 .. i\text{-}1]$ contains each $(i\text{-}1)$-permutation$\} = (n-i+1)!/n!$.

- Termination:

  At termination, $i = n+1$, it is true prior to the ith iteration, so we conclude that $A[i .. n]$ is a given n-permutation with probability $(n\text{-n})!/n! = 1/n!$.