Part VI

Graph Algorithms (III)

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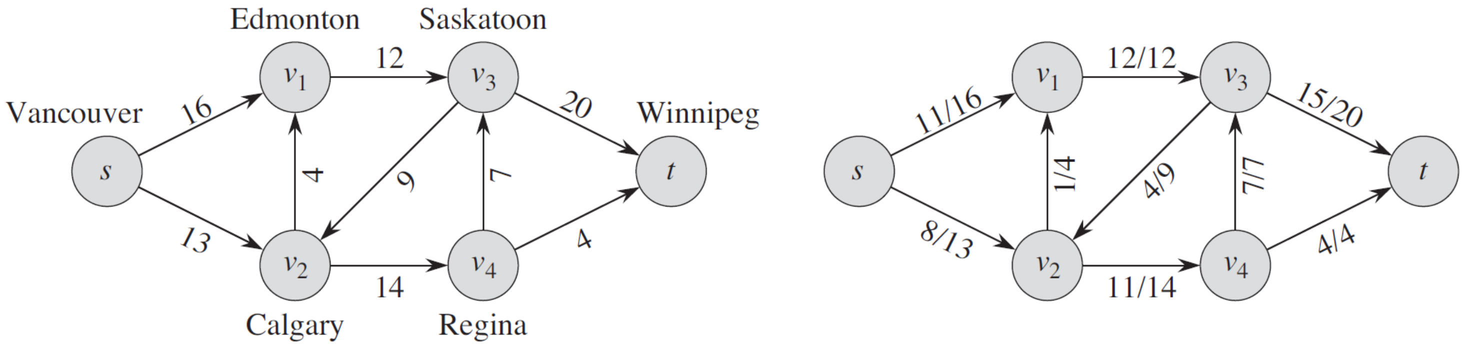
Graph Algorithms

- Elementary Graph Algorithms
  - Representations of Graphs
  - BFS, DFS
  - Sort Topologically
- Single-Source Shortest Paths
  - Finding shortest paths from a given source vertex to all other vertices (BF, TS, DJ)
  - Relaxation
- All-Pairs Shortest Paths

  Computing shortest paths between every pair of vertices
- Maximum Flow

26 Maximum Flow

Imagine a material coursing through a system from a source, where the material is produced, to a sink, where it is consumed. The source produces the material at some steady rate, and the sink consumes the material at the same rate.



(a)　　　　(b)

capacity network（容量网络）　　flow network（流网络）

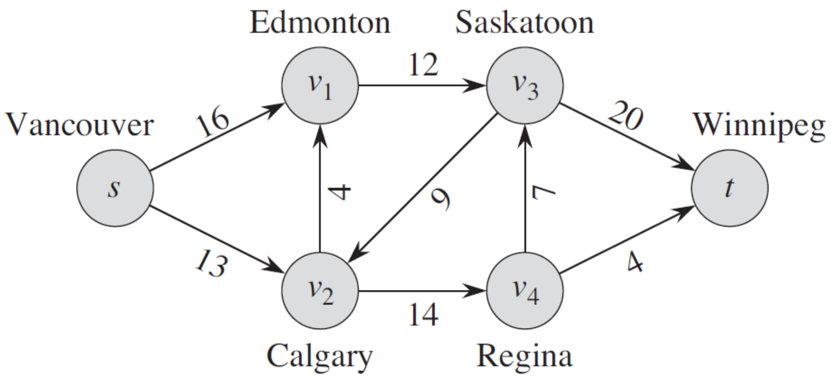
是流为0的一个初始流网络

最大流：又称为网络的最大流量问题，或最小分割问题。

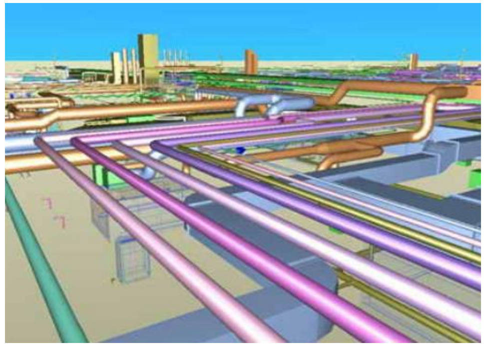
26 Maximum Flow

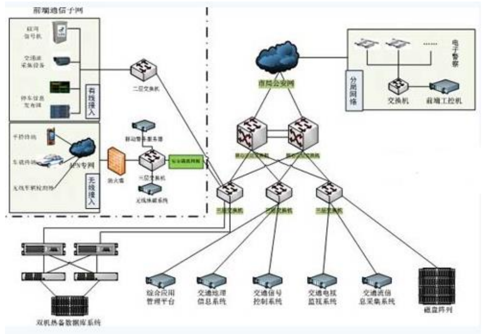
Flow networks

- Liquids flowing through pipes （石油、水等管道网络）
- Parts through assembly lines （装配流水线的产品流）
- Current through electrical networks （电流）
- Cars through highway traffic networks （交通流）
- Information through communication networks （通信网络）



capacity network（容量网络）





26 Maximum Flow

Basic definition

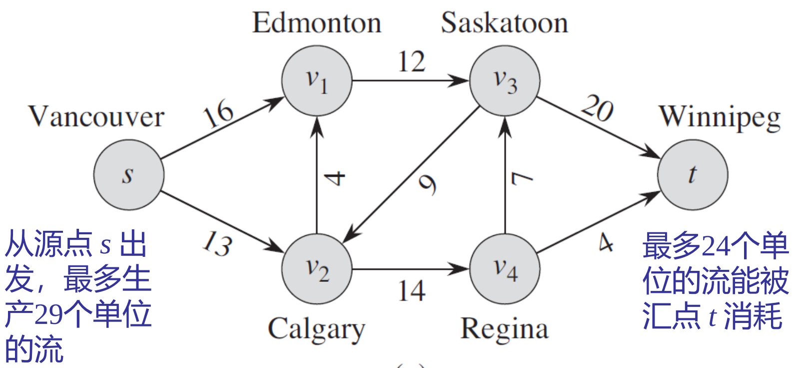
1. Flow networks G
2. Flow f
3. Maximum-flow $f_{\max}$
4. Residual networks G_f -- (残留网络)
5. Residual capacity $c_f(u, v)$ of G_f ------------------------------------ (顶点间的残留容量)
6. Augmenting path p -- (增广路径)
7. Residual capacity $c_f(p)$ of p --------------------------------------- (路径上的残留容量)
8. Cut (S, T) and its net flow $f(S, T)$ and capacity $c(S, T)$ -------- (分割及其流与容量)
9. Max-flow min-cut -- (最大流最小割)

26 Maximum Flow

- **Capacity**: a maximum rate at which the material can flow through the conduit.
  容量：管道能通过的最大 rate（单位时间的容量）
- **Flow conservation**: the rate at which material enters a vertex must equal the rate at which it leaves the vertex.
  流守恒：流入一个顶点 material ＝ 流出该顶点的 material
- **Maximum-flow problem**: we wish to compute the greatest rate at which we can ship material from the source to the sink without violating any capacity constraints.
  最大流问题：在容量允许的情况，从源点到汇点能 ship 的最大容量（单位时间情况下，即 rate）

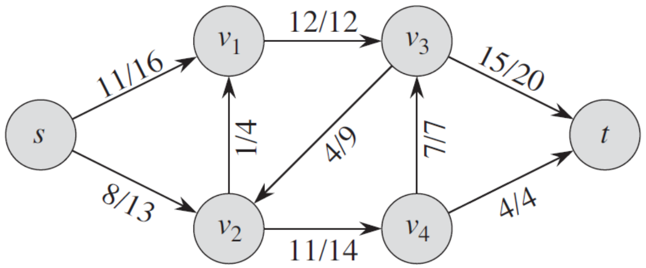
此网络的最大流（量）是多少?

19 是一个被允许的网络流。是否达到最大?



(a)

capacity network（容量网络）



(b)

flow network（流网络）

26.1 Flow networks

Flow networks and flows

- A flow network $G = (V, E)$ is a directed graph in which each edge $(u, v) \in E$ has a nonnegative capacity $c(u, v) \geq 0$. If $(u, v) \notin E$, $c(u, v) = 0$.

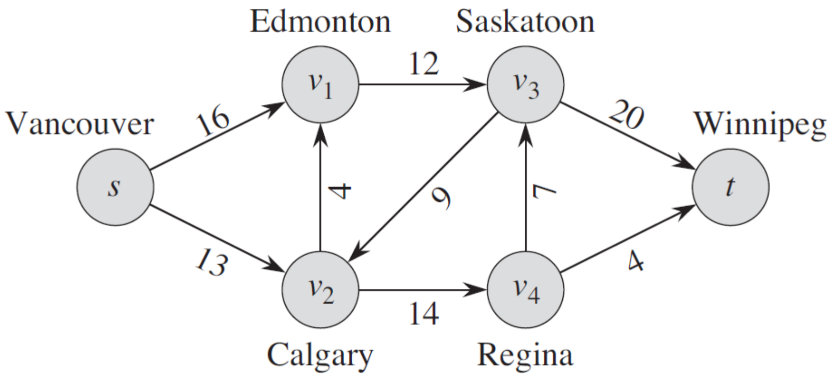
- Each vertex lies on some path from the source to the sink.
- ***source*** s （源点，生产点）
- ***sink*** t （汇点，消费点）
- A ***flow*** in G is

  a real-valued function

  $f : V \times V \to R$ that satisfies

  The following two properties:

  …



capacity network（容量网络）

是流为 0 的一个初始流网络

26.1 Flow networks

Flow networks and flows

A ***flow*** $f: V \times V \rightarrow R$ that satisfies The following two properties:

(1) Capacity constraint: For all $u, v \in V$, we require

$$0 \leq f(u, v) \leq c(u, v) \ .$$

容量限制：两个顶点之间的流不大于两个顶点的之间的容量

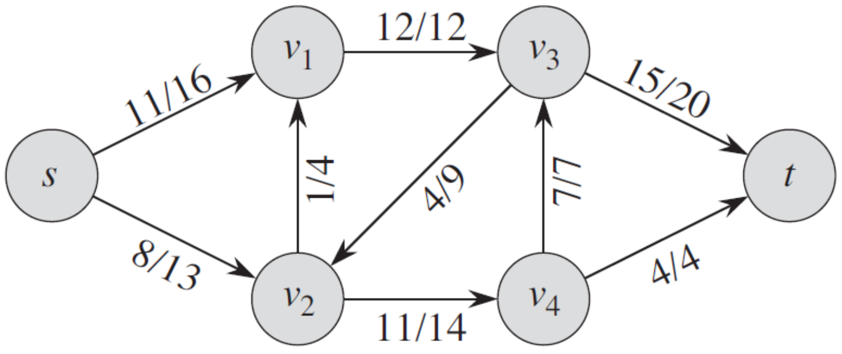
(2) Flow conservation: For all $u \in V - \{s, t\}$, we require

$$\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v) \ .$$

流守恒：流入顶点 u 的流等于流出顶点 u 的流

" flow in equals flow out. "

When $(u, v) \,!\!\in E$, there can be no flow from u to v, and $f(u, v) = 0$.



flow network（流网络）

26.1 Flow networks

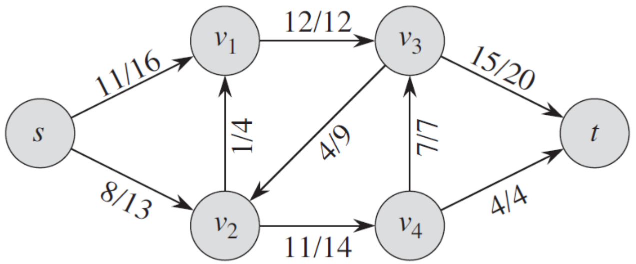
Flow networks and flows

- $f(u, v)$: the flow from vertex u to v.
- The value $|f|$ of a flow f of network is defined as

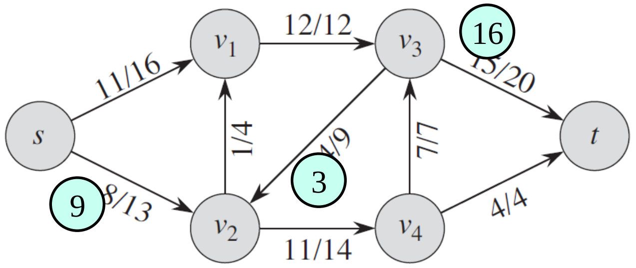
$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s) \, ,$$

that is, the total flow out of the source minus the flow into the source.

网络流 = 总流量 = 流出源点的流减去流入源点的流



$|f|$ is 19. Can the flow be increased?

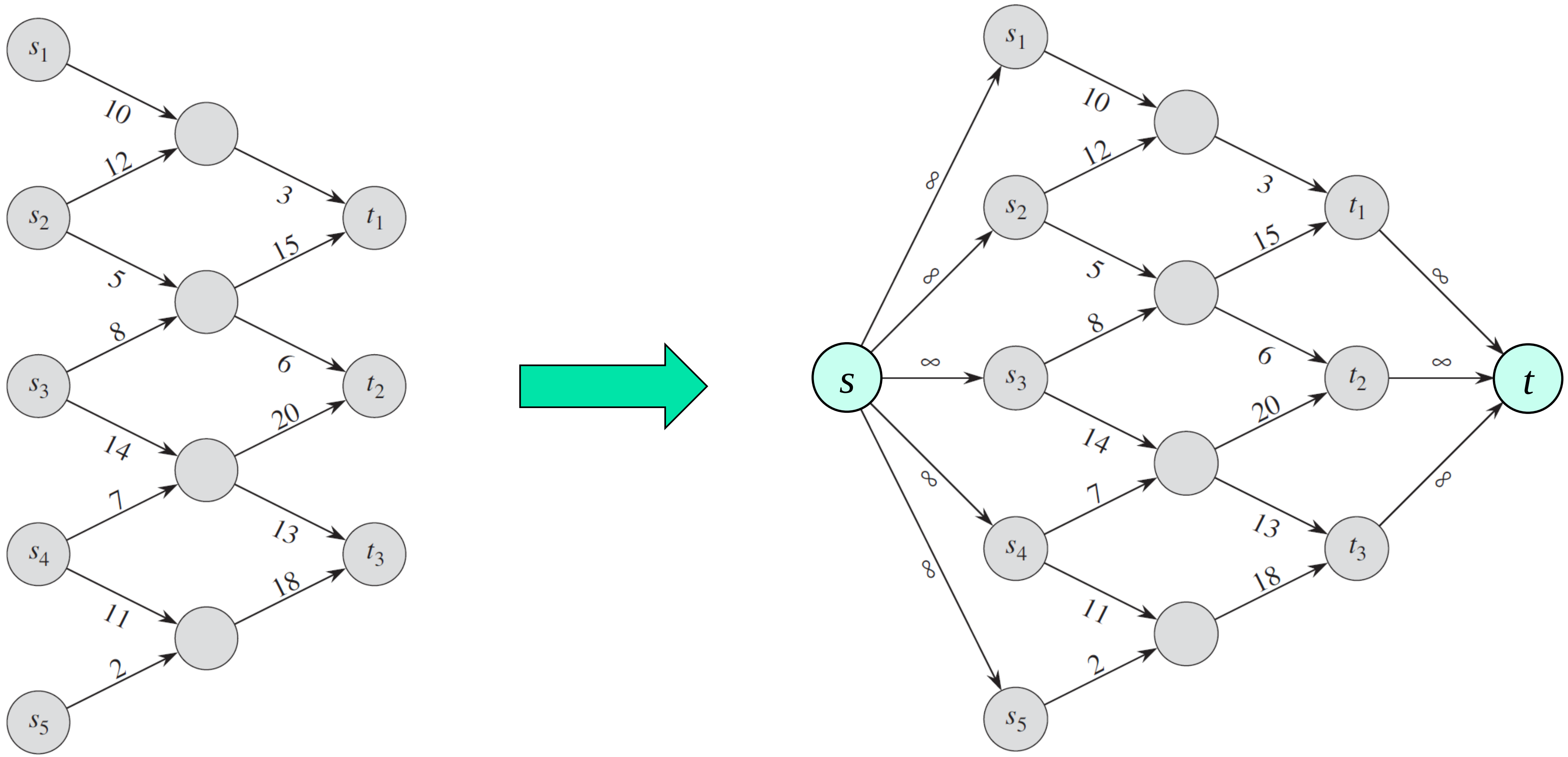


s 到 v_2，增加1个到 v_2 的输入流，v_2 的输出流不变，其他地方到 v_2 的输入流就得减少1个（只能是 v_3 到 v_2）；v_3 的输出流减少了，可以让 v_3 到 t 增加1（保持 v_3 的流守恒）。此时网络流增加 1。

- **Maximum-flow problem**: we are given a flow network G with source s and sink t , and we wish to find a flow of maximum value.

最大流：又称为网络的最大流量问题，或最小分割问题。

Networks with multiple sources and sinks



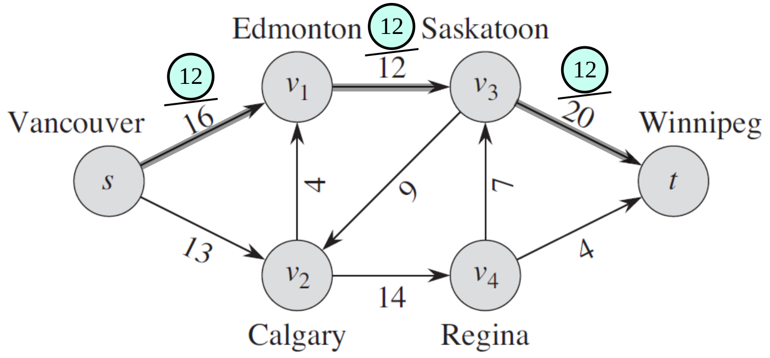
(a) (b)

26.2 The Ford-Fulkerson method

A “method” rather than an “algorithm”.（福特-福克森方法，由Ford 和Fulkerson于1956年提出的方法）

The Ford-Fulkerson method depends on three important ideas:

- residual networks（剩余网络（残留网络、残差网络），核心思想：存在一些边，在其上还能增加额外流，这些边就构成了残留网络的增广路径【增益路径】）
- augmenting paths（增益路径，增广路径）
- cuts（割、分割、切割）




```
FORD-FULKERSON-METHOD(G, s, t)
initialize flow f to 0
while there exists an augmenting path p in the residual network G_f
    augment flow f along p
return f
```

26.2 The Ford-Fulkerson method

Residual networks

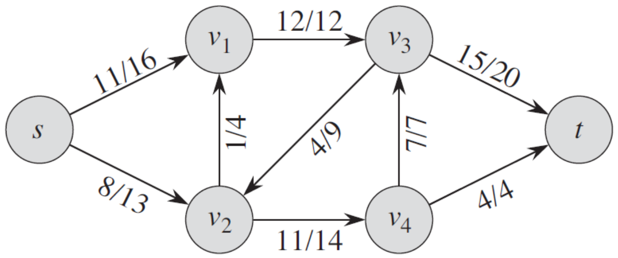
residual capacity

(残留容量：是双向的)

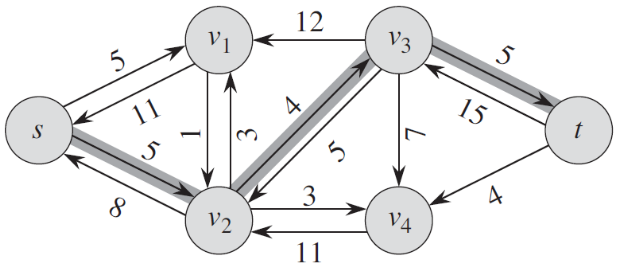
边上还能增加的额外流

反向流最多抵消正向流

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E , \\ f(v,u) & \text{if } (v,u) \in E , \\ 0 & \text{otherwise} . \end{cases}$$



(a)



(b)

Example: let $u \leftarrow s$, $v \leftarrow v_1$, there are $c(u, v) = 16$ and $f(u, v) = 11$, then we can increase $f(u, v)$ by up to $c_f(u, v) = 5$ units before we exceed the capacity constraint on edge (u, v). We also wish to allow an algorithm to return up to 11 units of flow from v to u, and hence $c_f(v, u) = 11$.

26.2 The Ford-Fulkerson method

Residual networks

- residual capacity

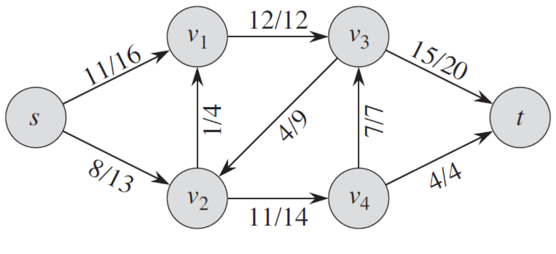
$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E , \\ f(v,u) & \text{if } (v,u) \in E , \\ 0 & \text{otherwise} . \end{cases}$$

- residual network: $G_f = (V, E_f)$, where

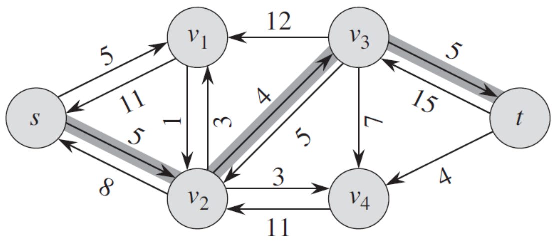
$$E_f = \{(u,v) \in V \times V : c_f(u,v) > 0\}$$

残留网络：顶点跟流网络一样，边（残留边）的权值为流网络的残留容量

流网络



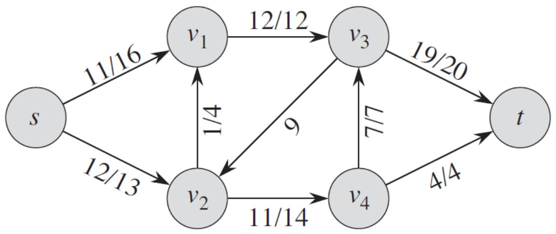
(a)



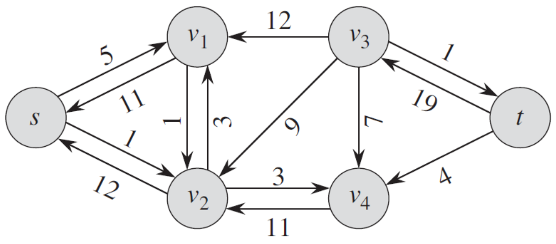
(b)

(a)的残留网络（粗线是某一条增广路径）

在(a)图中，沿着其残留网络的增广路径上增加流以后新的流网络



(c)



(d)

(c)的残留网络

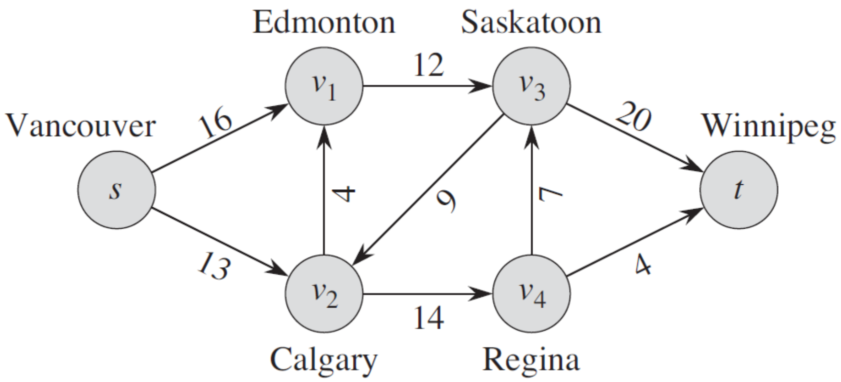
26.2 The Ford-Fulkerson method

Residual networks

- residual capacity

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \,, \\ f(v,u) & \text{if } (v,u) \in E \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

- 只有容量（初始流为零）的网络，其残留网络就是其自身。如下图是容量网络（初始流为零的流网络），也是残留网络。



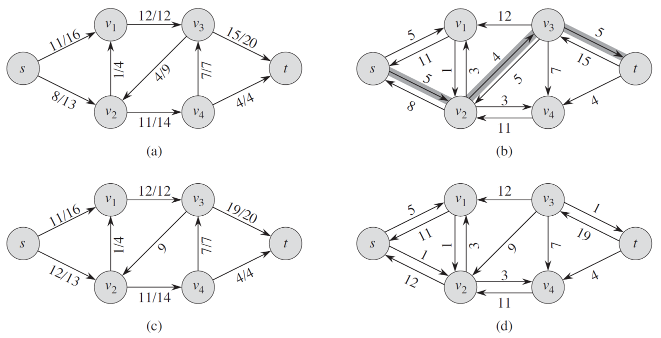
26.2 The Ford-Fulkerson method

Residual networks

Lemma 26.1 Let $G = (V, E)$ be a flow network, and let f be a flow in G. Let G_f be the residual network of G induced by f, and let f' be a flow in G_f. Then the flow sum $f + f'$ ($f \uparrow f'$) defined by equation (26.4) is a flow in G with value

$$|f + f'| = |f| + |f'|.$$

$$(f_1 + f_2)(u, v) = f_1(u, v) + f_2(u, v) \quad \ldots\ldots \ (26.4)$$



$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E, \\ f(v,u) & \text{if } (v,u) \in E, \\ 0 & \text{otherwise}. \end{cases}$$

已知流网络 G 及其上面的一个流 f，G_f 是由 f 诱导产生的残留网络，f' 是 G_f 上的一个流，则如式(26.4)定义的“流和”为 G 上的流。

Proof Verify that the **capacity constraints**, **flow conservation** are obeyed ……

Capacity constraint: For all u, v, we have $f(u, v) \le c(u, v)$

Flow conservation: For all $u \in V - \{s, t\}$, flow in equals flow out

Ideaf ,

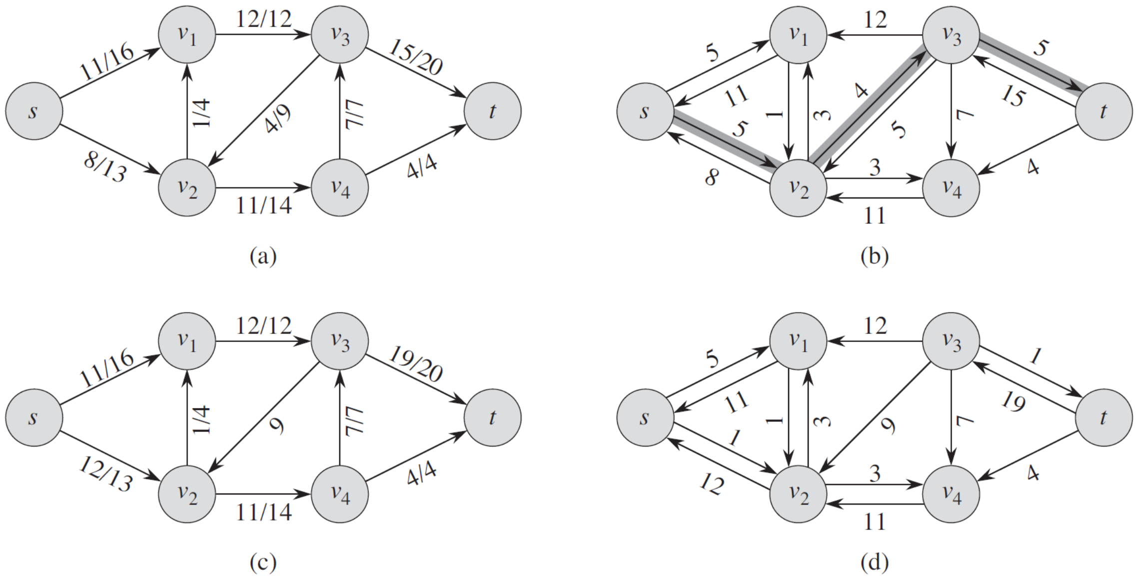
capacity constraints: 根据“残留容量+容量限制”的定义来证明。

flow conservation: 根据“流守恒”的定义来证明（流 f 和 f' 都是流，都满足流守恒，相加自然满足流守恒），即 $\sum_{v \in V}(f(u,v) + f'(u,v)) = \sum_{v \in V} f(u,v) + \sum_{v \in V} f'(u,v) = \sum_{v \in V} f(v,u) + \sum_{v \in V} f'(v,u) = \ldots$ ，即，流出=流入。

26.2 The Ford-Fulkerson method

Residual networks

How to find a flow f' in G_f?



An **augmenting path** p is a simple path from s to t in the residual network G_f.

需要找到残留网络 G_f 上的的增广路径 p

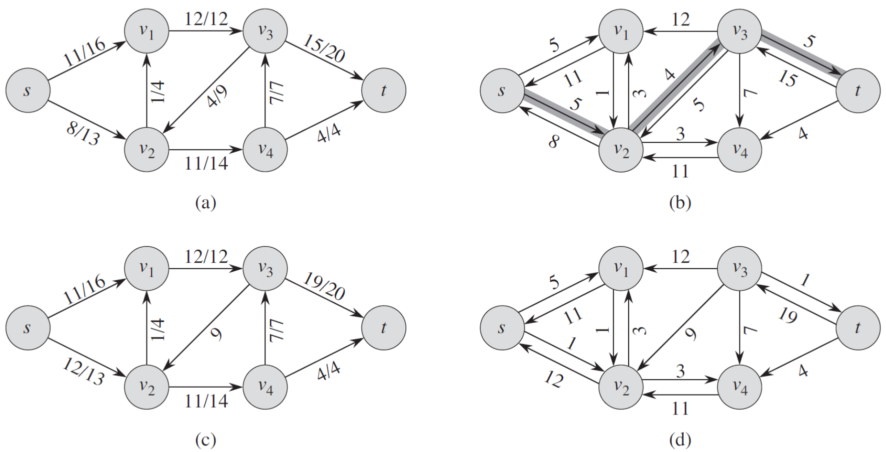
26.2 The Ford-Fulkerson method

Augmenting paths（增广路径）

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E\,, \\ f(v,u) & \text{if } (v,u) \in E\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

- Residual capacity $c_f(u, v)$ of G_f （顶点 (u, v) 间的残留容量）
- An **augmenting path** p is a simple path from s to t in the residual network G_f .（增广路径 p: G_f 中从 s 到 t 的简单路径）
- ***residual capacity*** of p : the maximum amount by which we can increase the flow on each edge in the augmenting path p .
  （增广路径 p 上的残留容量 $c_f(p)$。容量最小的那条边 (u, v)，也称为关键边。）
  $c_f(p) = \min\{c_f(u, v) : (u, v) \text{ is on } p\}$.
- **Lemma 26.2** Let $G = (V, E)$ be a flow network, let f be a flow in G, and let p be an augmenting path in G_f. Define a function f_p : $V \times V \rightarrow R$ by

$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \text{ is on } p\,, \\ 0 & \text{otherwise}\,. \end{cases}$$



Then, f_p is a flow in G_f with value $|f_p| = c_f(p) > 0$.

增广路径上的残留容量 $c_f(p) \rightarrow f_p$ 是残留网络上的流

Proof : verify two properties of flow…

26.2 The Ford-Fulkerson method

Augmenting paths（增广路径）

$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \text{ is on } p \text{ ,} \\ 0 & \text{otherwise .} \end{cases}$$

Corollary 26.3 Let $G = (V, E)$ be a flow network, let f be a flow in G, and let p be an augmenting path in G_f. Then the function $(f \uparrow f_p): V \times V \rightarrow R$, is a flow in G with value $|f| + |f_p| > |f|$.

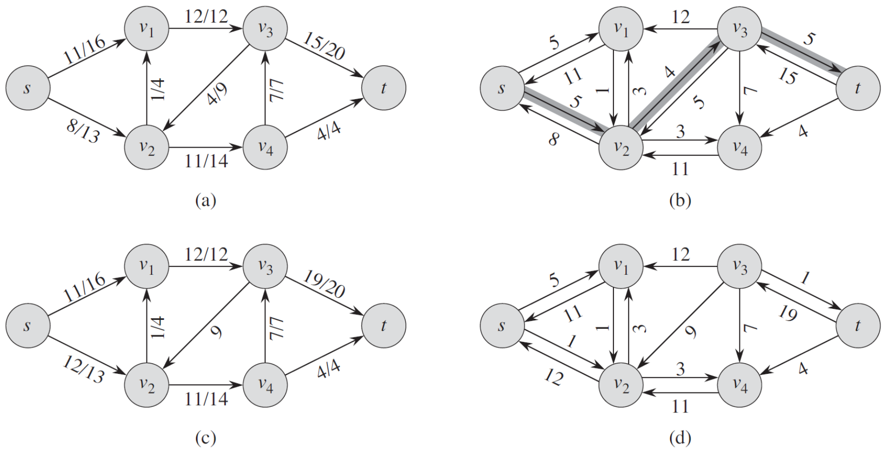
f_p是残留网络某一条增广路径 p 上的残留容量，则 $|f| + |f_p| > |f|$ 是 G 上的流。

***Proof* :**

Immediately, from Lemmas 26.2 and 26.1,

Lemmas 26.2: f_p is a flow in G_f,

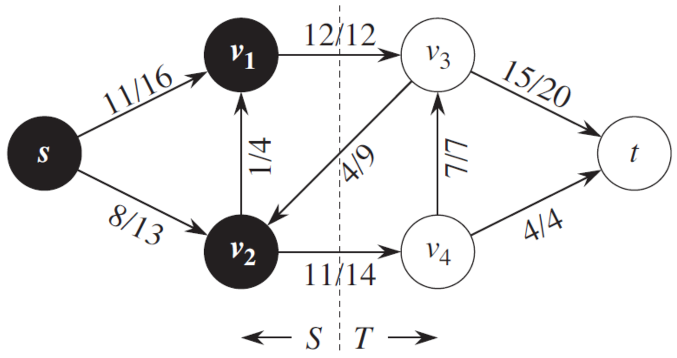
Lemmas 26.1: $f + f_p$ is a flow in G .



26.2 The Ford-Fulkerson method

Cuts of flow networks

- A ***cut*** (S, T) of flow network $G = (V, E)$ is a partition of V into S and $T = V - S$ such that $s \in S$ and $t \in T$ (source is in S, sink is in T).
  分割：把顶点集分成两个子集，源点 s 和汇点 t 分别属于两个子集。
- If f is a flow, then the ***net flow*** across the cut (S, T) is defined to be $f(S, T)$.
  分割的净流 $$f(S,T) = \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u)$$
- The ***capacity*** of the cut (S, T) is $c(S, T)$.
  分割的容量 $$c(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v)$$
  注意：没有反向容量的概念
- A ***minimum cut*** of a network is a cut whose capacity is minimum over all cuts of the network.
  一个网络的最小分割是网络中具有最小容量的分割



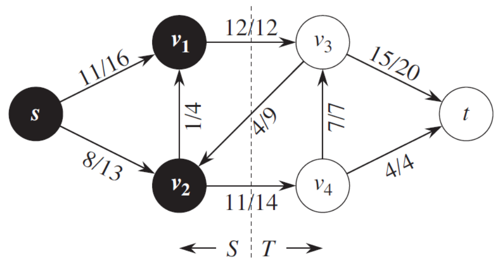
$f(S, T) = 19$

$c(S, T) = 26$

26.2 The Ford-Fulkerson method

Cuts of flow networks

Lemma 26.4 Let f be a flow in a flow network G, and let (S, T) be any cut of G. Then the net flow across (S, T) is $f(S, T) = |f|$. 任意分割的净流都相等



Proof …略（根据流的定义与流守恒性质来证明，证明略）

流的定义

$$|f| = \sum_{v\in V} f(s,v) - \sum_{v\in V} f(v,s)$$

切割的净流定义

$$f(S,T) = \sum_{u\in S}\sum_{v\in T} f(u,v) - \sum_{u\in S}\sum_{v\in T} f(v,u)$$

流守恒性质

$$\sum_{v\in V} f(v,u) = \sum_{v\in V} f(u,v) \quad \left(\sum_{v\in V} f(u,v) - \sum_{v\in V} f(v,u) = 0 \right)$$

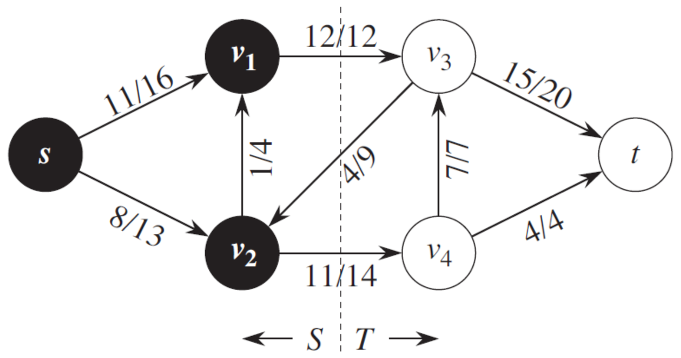
26.2 The Ford-Fulkerson method

Cuts of flow networks

- **Lemma 26.4** Let f be a flow in a flow network G, and let (S, T) be any cut of G. Then the net flow across (S, T) is $f(S, T) = |f|$.
- **Corollary 26.5** The value of any flow f in a flow network G is bounded from above by the capacity of any cut of G. 任意分割的流都不会超过任意分割的容量

Proof 很显然。根据切割的净流与容量的定义来证明。

$$\begin{aligned}
|f| &= f(S,T) \\
&= \sum_{u\in S}\sum_{v\in T} f(u,v) - \sum_{u\in S}\sum_{v\in T} f(v,u) \\
&\le \sum_{u\in S}\sum_{v\in T} f(u,v) \\
&\le \sum_{u\in S}\sum_{v\in T} c(u,v) \\
&= c(S,T)\,.
\end{aligned}$$



26.2 The Ford-Fulkerson method

Cuts of flow networks

Theorem 26.6: (Max-flow min-cut theorem) （最大流最小割）

If f is a flow, then the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) of G.

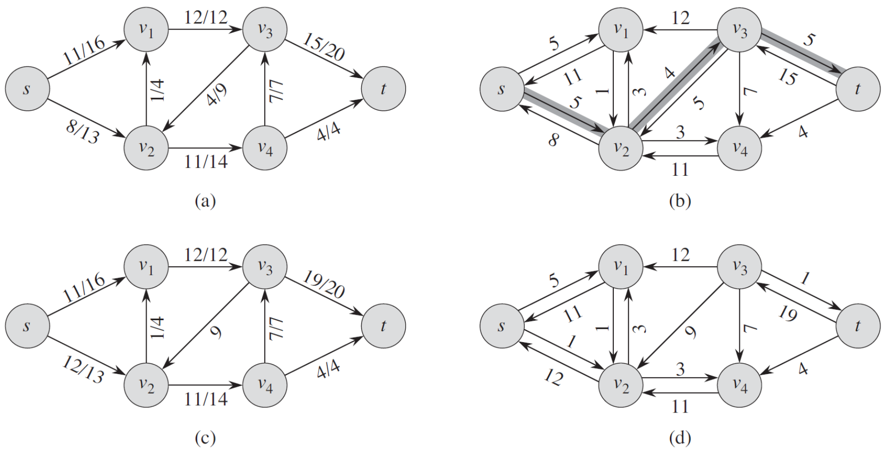
***Proof** …*

证明思路:

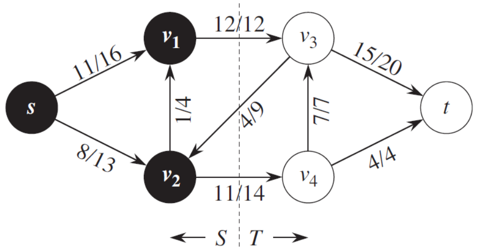
1=>2

2=>3

3=>1



(a) (b) (c) (d)



26.2 The Ford-Fulkerson method

Cuts of flow networks

Theorem 26.6: (Max-flow min-cut theorem) （最大流最小割）

If f is a flow, then the following conditions are equivalent:

1. f is a maximum flow in G.
2. The residual network G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) of G.

```
FORD-FULKERSON-METHOD(G, s, t)
initialize flow f to 0
while there exists an augmenting path p in the residual network G_f
    augment flow f along p
return f
```

最大流求解算法:

① 流网络比较小时：穷举出所有切割(cut)，求出最小cut。

② 流网络比较大时：求残留网络，找增广路径（求路径上的残留容量），在流网络中沿增广路径压入残留（剩余）容量，如 F-F 方法。

26.2 The Ford-Fulkerson method

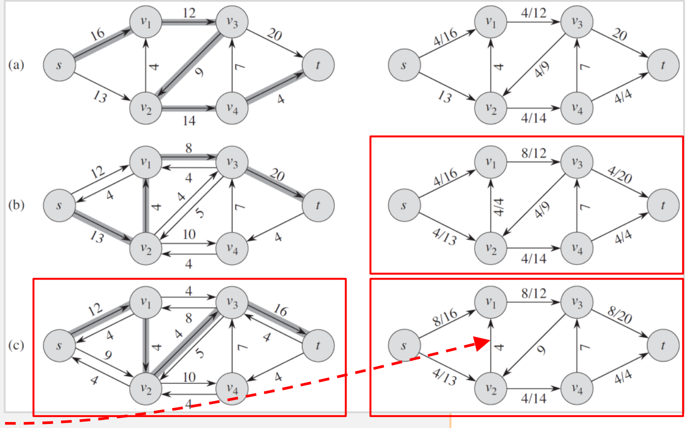
The basic Ford-Fulkerson algorithm

FORD-FULKERSON(G, s, t)

for each edge $(u, v) \in E$
    $f[u, v] \leftarrow 0$
while there exists a path p from s to t in the residual network G_f
    $c_f(p) \leftarrow \min\{c_f(u, v) : (u, v) \text{ is in } p\}$
    for each edge (u, v) in p
        if $(u, v) \in E$
            $f[u, v] \leftarrow f[u, v] + c_f(p)$
        else $f[v, u] \leftarrow f[v, u] - c_f(p)$

FORD-FULKERSON-METHOD(G, s, t)

initialize flow f to 0
while there exists an augmenting path p in the residual network G_f
    augment flow f along p
return f



(b)→(c)的解释:

图左是残留网络，右是压入残留容量后的流网络。

图(b)右，流网络；得到其残留网络图(c)左，残留网络上有增广路径p，其残留容量为4；在其原流网络“图(b)右”沿路径p压入残留容量，特别地$(u, v) = (v_1, v_2)\ !\in E$，因此，反向减少流（等价于正向抵消流），即$(v_2, v_1)$的新流$f[v_2, v_1] \leftarrow 4 - 4 = 0$。

算法：求残留网络G_f，找增广路径p，求路径上的残留容量 $c_f(p)$，在流网络中沿增广路径在每条边上压入残留（剩余）容量。

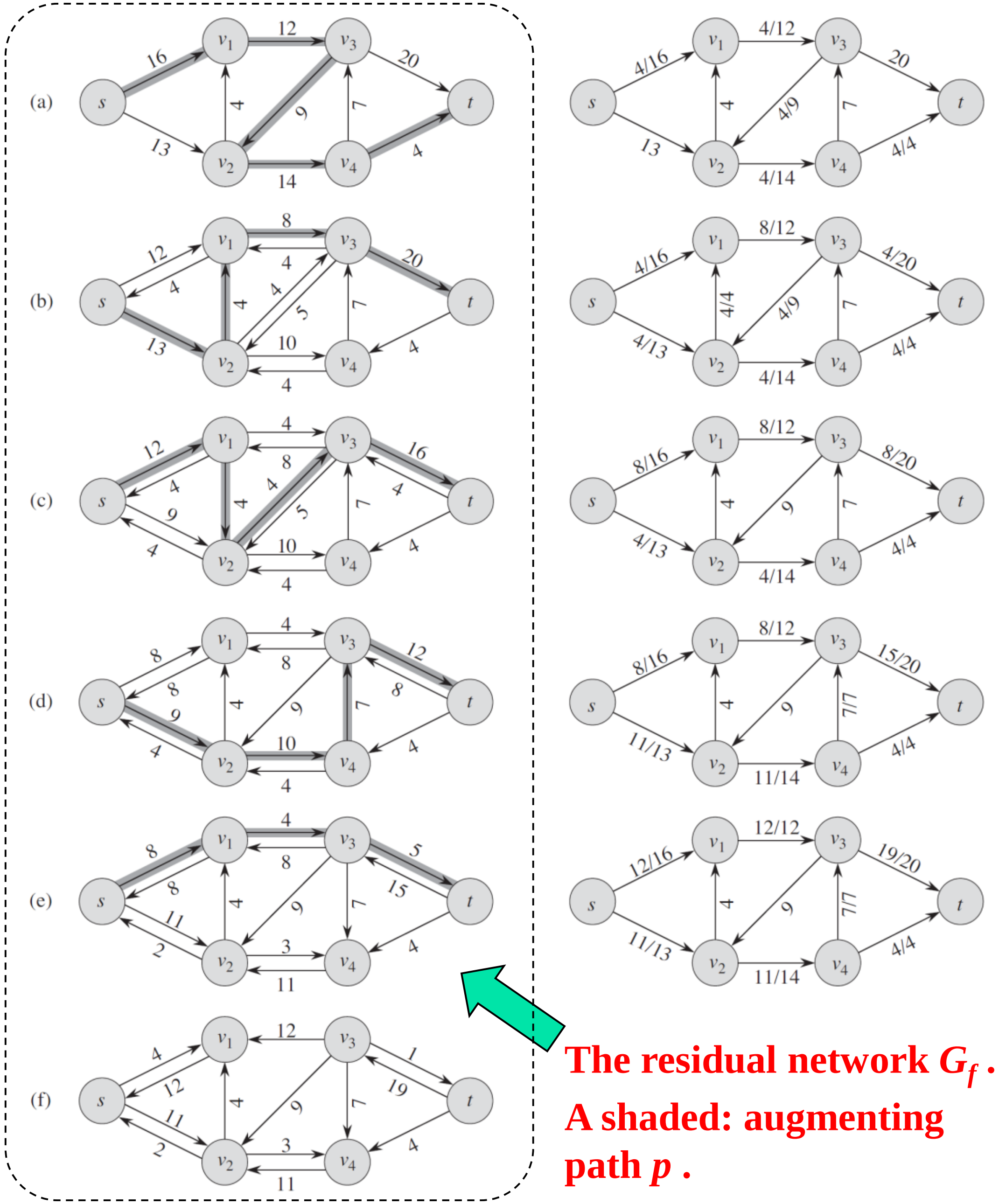
Ford-Fulkerson Algorithm

虚线框里的是 residual network

1. 初始，图(a)-Left，流 f 为0，增广路径的残留容量为 4；
2. 图(a)-Right，沿增广路径可压入流4，图中的流为4；
3. 图(b)-Left是图(a)-Right的残留网络，图(b)-Left的一个增广路径的残留容量是4；
4. 在流网络图(a)-Right的基础上，沿图(b)-Left的增广路径可压入流4，得到图(b)-Right；
5. ……

图(f)中不存在增广路径（不能再增加流，因此图(e)-Right中的流就是最大流。

增广路径选取方法决定了计算效率。

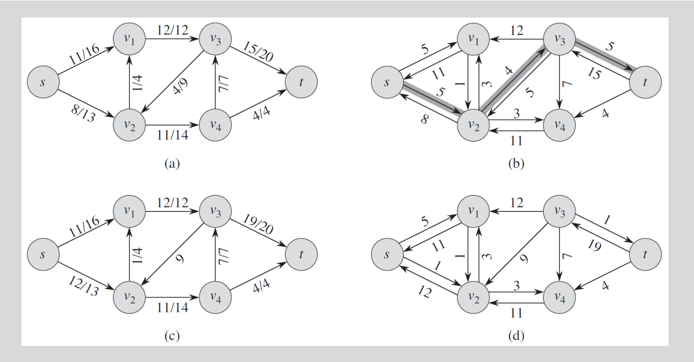


The residual network G_f .

A shaded: augmenting path p .

26.2 The Ford-Fulkerson method

Analysis of Ford-Fulkerson



FORD-FULKERSON(G, s, t) $O(E\cdot f^*)$

for each edge $(u, v) \in E$
    $f[u, v] \leftarrow 0$
while there exists a path p from s to t in the residual network G_f
    $c_f(p) \leftarrow \min\{c_f(u, v) : (u, v) \text{ is in } p\}$
    for each edge (u, v) in p
        if $(u, v) \in E$
            $f[u, v] \leftarrow f[u, v] + c_f(p)$
        else $f[v, u] \leftarrow f[v, u] - c_f(p)$

When the capacities are integral and the optimal flow value f^* is small, the running time of the F-F algorithm is good.

容量是整数，且最大流 f^* 较小时，F-F 算法有效

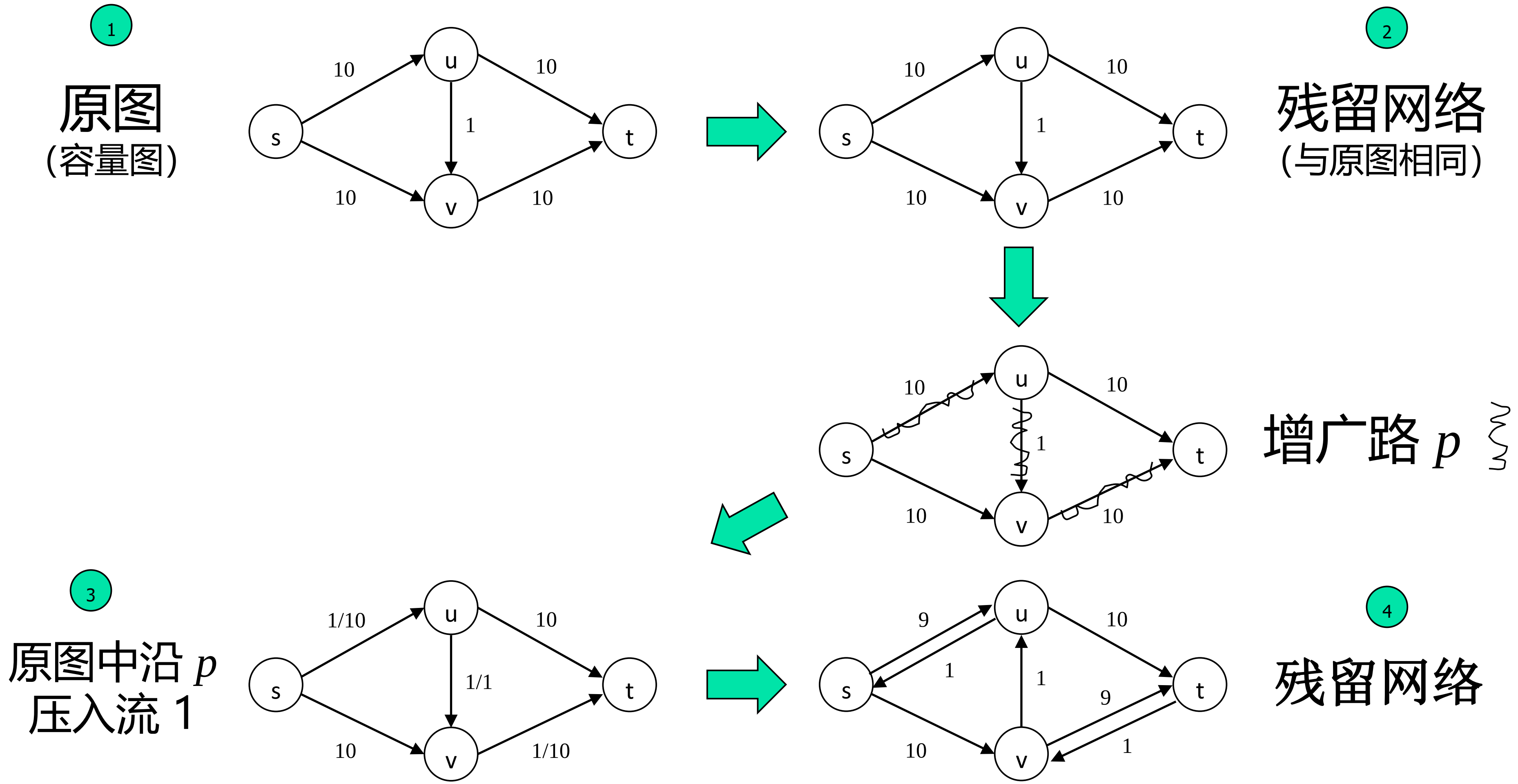
设最大流为 f^*：

每次找到增广路径，流至少增加 1，流从 0 增加到 f^*，时间为 $O(f^*)$；

每次找增广路径与给边增加流的操作，时间为 $O(E+V) = O(E)$；

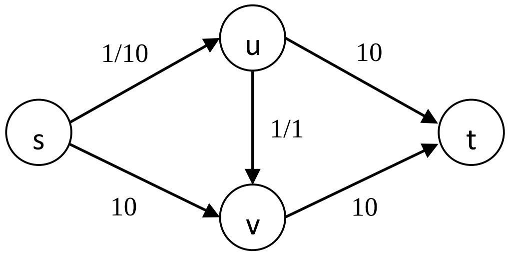
总的时间， $O(E \cdot f^*)$

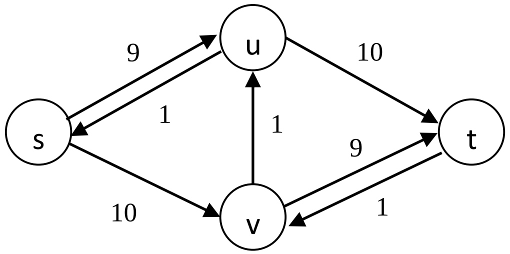
If f^* is large? an example...



3

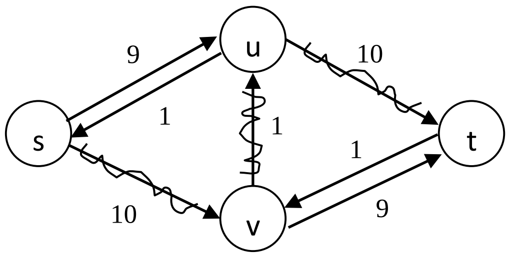
原图中沿 p 压入流 1





4

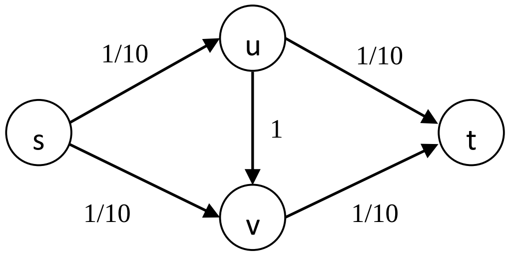
残留网络

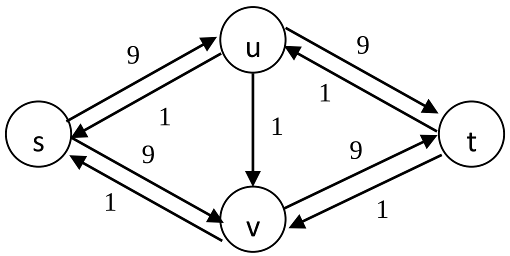


增广路 p

5

上一流图中沿 p 压入流 1



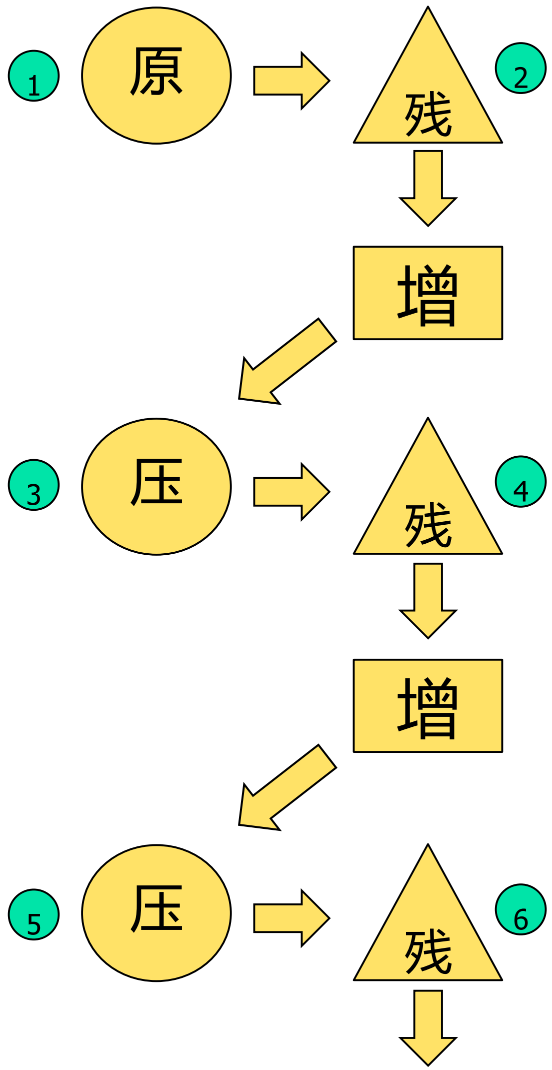


6

残留网络

If the optimal flow value f^* is large, the F-F algorithm is not good.

当最大流 f^* 较大时，F-F 算法的效率低



26.2 The Ford-Fulkerson method

```
FORD-FULKERSON(G, s, t)
1  for each edge (u, v) ∈ E
2      f[u, v] ← 0
3  while there exists a path p from s to t in the residual network G_f
4      c_f(p) ← min {c_f(u, v) : (u, v) is in p}
5      for each edge (u, v) in p
6          if (u, v) ∈ E
7              f[u, v] ← f[u, v] + c_f(p)
8          else f[v, u] ← f[v, u] - c_f(p)
```

$O(\ E\cdot f^*\ )$

The Edmonds-Karp algorithm

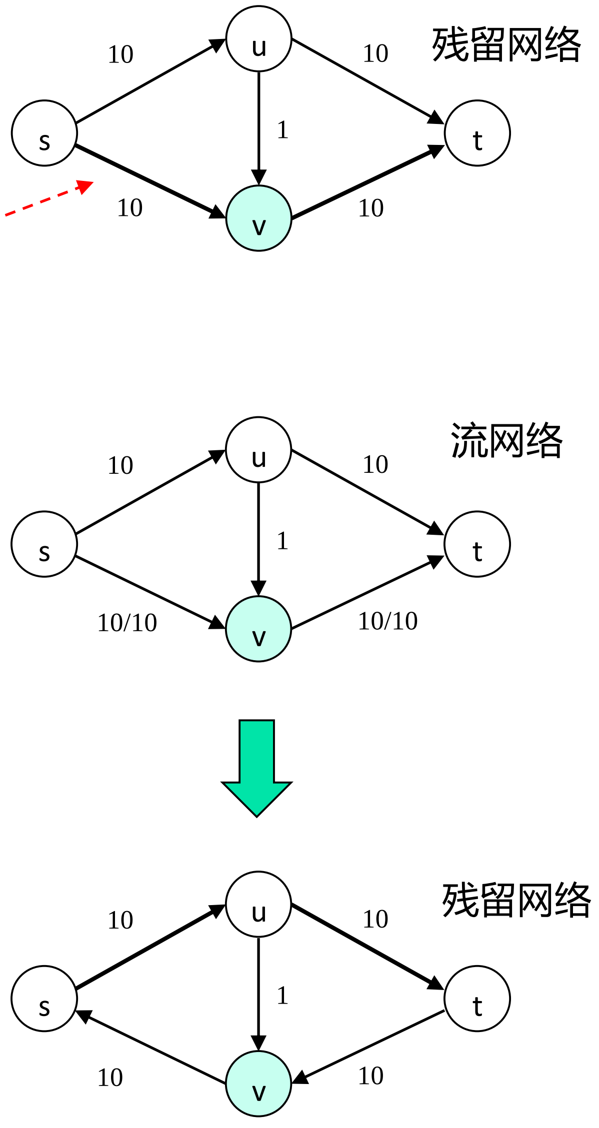
We can improve the bound on F-F by finding the augmenting path p in line 3 with **a breadth-first search**. That is, we choose p as a shortest path from s to t in the residual network, where each edge has unit distance (weight). We call the F-F method so implemented the **Edmonds-Karp algorithm**. The E-K algorithm runs in $O(VE^2)$ time. **Proof* …?

F-F 方法中使用 BFS 寻找增广路径，F-F 方法就被称为 E-K 算法

26.2 The Ford-Fulkerson method → EDMONDS-KARP Algorithm

EDMONDS-KARP(G, s, t) $O(V \cdot E^2)$

for each edge $(u, v) \in E$
  $f[u, v] \leftarrow 0$
while there exists a path p from s to t in the residual network G_f **(using BFS)**
  $c_f(p) \leftarrow \min \{c_f(u, v) : (u, v)$ is in $p\}$
  for each edge (u, v) in p
    if $(u, v) \in E$
      $f[u, v] \leftarrow f[u, v] + c_f(p)$
    else $f[v, u] \leftarrow f[v, u] - c_f(p)$



*证明思想：关键边（增广路径 p 上的最小容量边）。

沿着 p 增加流一次，关键边消失；边 (u, v) 最多$O(V)$次作为关键边（源点 s 到顶点 v 的最短路径随着流增加而单调增加）；共E条边；E-K算法执行中的关键边数量 $O(V \cdot E)$（关键边全部消失，不再有增广路径，最大流找到）。每次找增广路径和给边增加流的操作，时间为 $O(E)$。总时间 $O(V \cdot E^2)$。

该算法最初由 Yefim Dinitz 于1970年发表，并由 Jack Edmonds 和Richard Karp 于1972年独立发表。E-K算法实际就是F-F算法的一种改进（或一种具体实现），因此，F-F称为方法！

26.2 The Ford-Fulkerson method

Idea:

- residual networks
- augmenting paths
- Cuts

Method: The F-F method

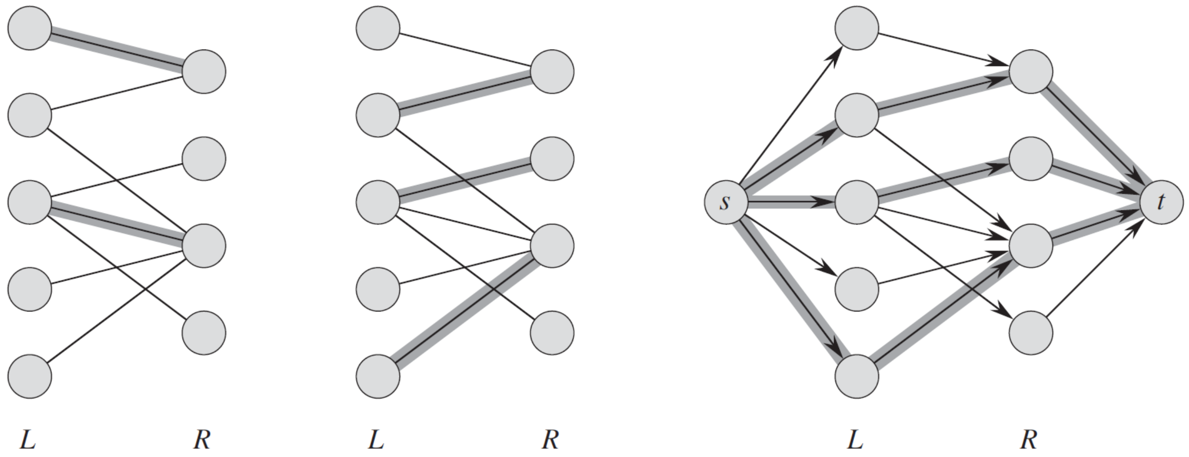
Algorithm: Edmonds-Karp

Code: Your job!

```
EDMONDS-KARP(G, s, t)
for each edge (u, v) ∈ E
    f[u, v] ← 0
while there exists a path p from s to t in the residual network G_f (using BFS)
    c_f(p) ← min {c_f(u, v) : (u, v) is in p}
    for each edge (u, v) in p
        if (u, v) ∈ E
            f[u, v] ← f[u, v] + c_f(p)
        else f[v, u] ← f[v, u] - c_f(p)
```

- 对原网络 G 构造残留网络 G_f；
- 在残留网络 G_f 上寻找增广路径 p (用BFS)，求 p 上的残留容量 $c_f(p)$；
- 在原网络上沿路径 p 增加流 $c_f(p)$.

26.3 Maximum bipartite matching



(a) (b) (c)

Practical applications

- L : machines ; R : tasks
- L : students ; R : scholarships
- L : students ; R : mentors
- L : students ; R : companies
- L : gentlemen ; R : ladies
- …

*Chapter 26.4 Push-relabel algorithms

压入重标记

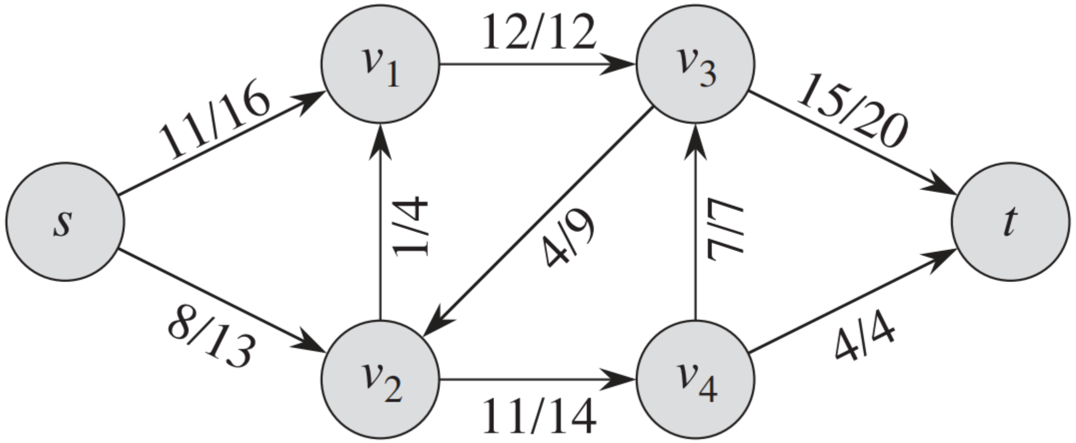
*Chapter 26.5 The relabel-to-front algorithm

重标记与前移算法

*Chapter 29 Linear Programming

最大流可以表示为线性规划问题（用线性规划来解最大流问题）

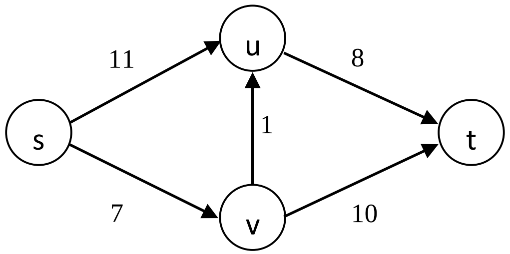
Exercise

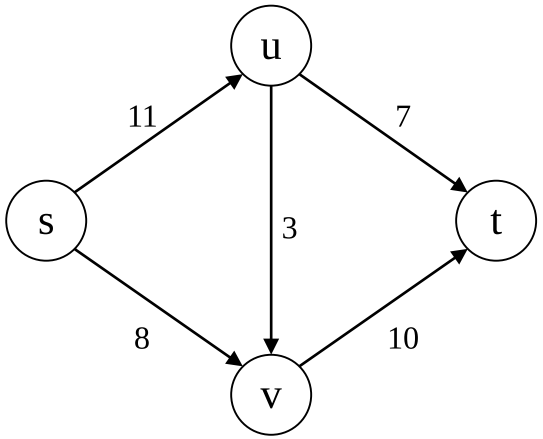


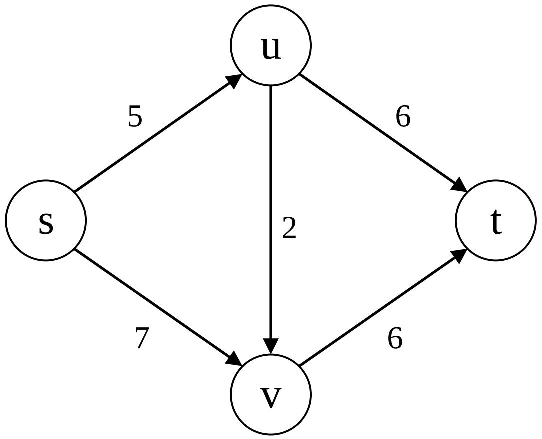
1. What is the flow across the cut $(S, T) = (\{s, v_2, v_4\}, \{v_1, v_3, t\})$?
   What is the capacity of this cut?
2. What is the minimum cut to the figure?
   What is the maximum flow?

Exercise

下面三个图，最大流（最小分割）分别是什么?

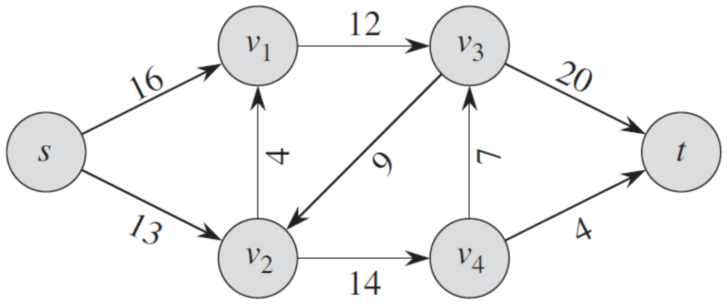






Exercise

采用E-K算法，画出左图的最大流求解过程。



EDMONDS-KARP(G, s, t)

for each edge $(u, v) \in E$

  $f[u, v] \leftarrow 0$

while there exists a path p from s to t in the residual network G_f **(using BFS)**

  $c_f(p) \leftarrow \min \{c_f(u, v) : (u, v) \text{ is in } p\}$

  for each edge (u, v) in p

    if $(u, v) \in E$

      $f[u, v] \leftarrow f[u, v] + c_f(p)$

    else $f[u, v] \leftarrow f[u, v] - c_f(p)$