Session 1: Financial Networks and Contagion

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Course Outline

- 31 May, 09:00h - 12:00h – Financial Networks Introduction
- 31 May, 13:00h - 16:00h – Financial Networks Theory
- 01 June, 08:00h - 12:00h – Introduction to Agent-Based Modeling
- 01 June, 13:00h - 15:00h – Applications of Agent-Based Modeling

Prologue - The Case of Iceland

"Fellow Icelanders,

[. . .] *The entire world is experiencing a major economic crisis, which can be likened in its effects on the world's banking systems, to an* **economic natural disaster**. [. . .] *In such circumstances every nation thinks, of course, first and foremost of its* **own interests**. *Even the biggest economies in the world are facing a close struggle with the effects of the crisis.*

[. . .] *In recent weeks the world's financial system has been subject to devastating shocks. Some of the biggest investment banks in the world have become the victims and capital in the markets has in reality dried up. The effects have been that large international banks have stopped financing other banks and complete* **lack of confidence** *has developed in business between banks.*

[. . .] *God bless Iceland"*

– Icelandic Prime Minister Geir Haarde in a TV speech on October 6, 2008, addressing the state of the Icelandic banks.

Prologue - Consequences of the Financial Crisis

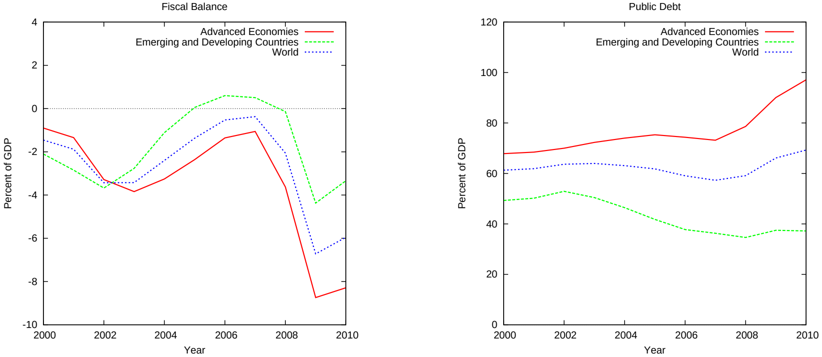

Figure: Impact of the financial crisis on fiscal balances and public debt. Source: IMF (2010).

Prologue - Consequences of the Financial Crisis

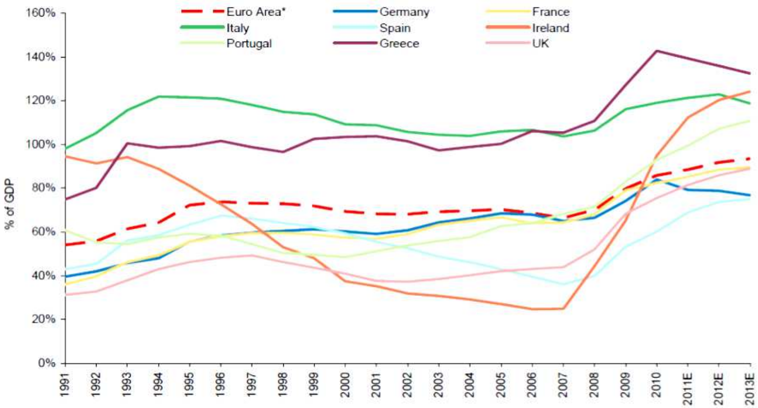

Figure: Decomposition of EU Government Debt. Source: Keynote by Ricardo Caballero, FMS 2012 Conference.

Motivation - A Challenge for Economic Modelling

A challenge for academia...

- The financial system has become increasingly **interconnected**, **heterogeneous** and **complex**

  $\Rightarrow$ particularly thorny to achieve with representative agent models

- **Confidence** is a key aspect of financial (in-)stability

  $\Rightarrow$ not easy to understand with rational expectations

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...and policy makers alike

- Before the crisis: focus on **individual** financial institutions
  $\Rightarrow$ Their complex interactions have to be taken into account
- **Systemic risk** is highly dynamic
  $\Rightarrow$ How do structure and dynamics interact?
- Political challenge: **Global Financial Governance**

Motivation - Living in an Interconnected World

Four key developments:

1. Direct linkages (interbank loans, repos, CDS, etc.) amongst financial intermediaries lead to **counterparty risk**

Motivation - Living in an Interconnected World

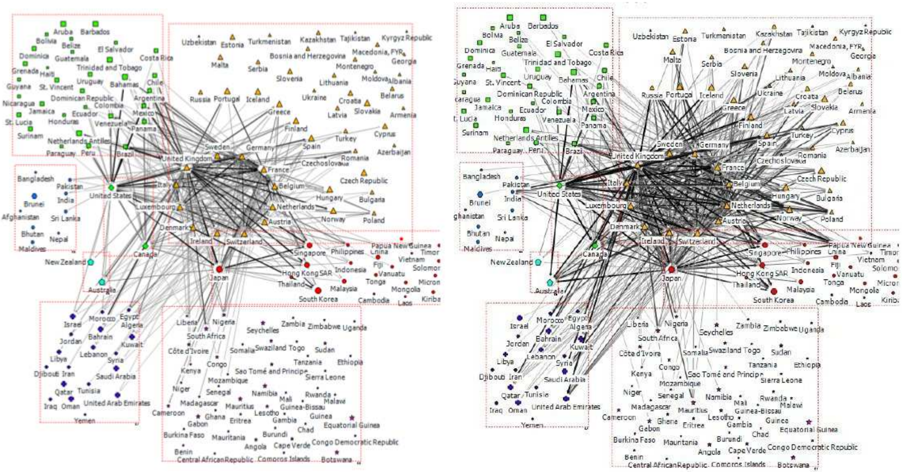

Figure: Interconnectedness of the international banking network in 1980 (left) and 2007 (right). Source: Minoiu and Reyes (2011) using BIS data.

The financial system changed over the last few years

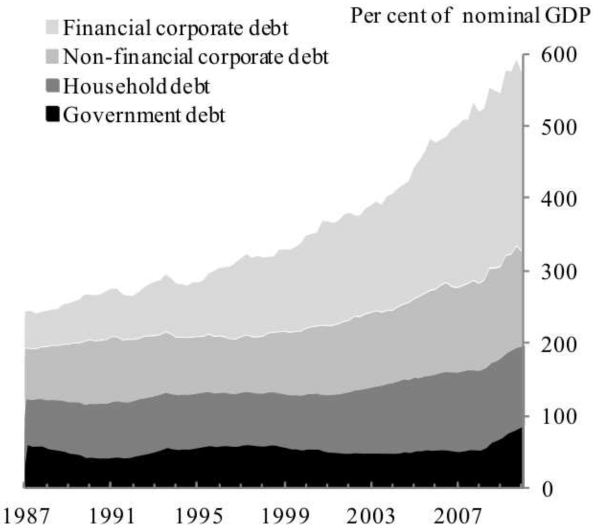

Figure: Decomposition of UK debt. Source: Gai, Haldane and Kapadia (2011).

Motivation - Living in an Interconnected World

Four key developments:

1. Direct linkages (interbank loans, repos, CDS, etc.) amongst financial intermediaries lead to **counterparty risk**

2. Increasing indirect linkages (asset commonality) induce a risk of **fire-sales**

Motivation - Living in an Interconnected World

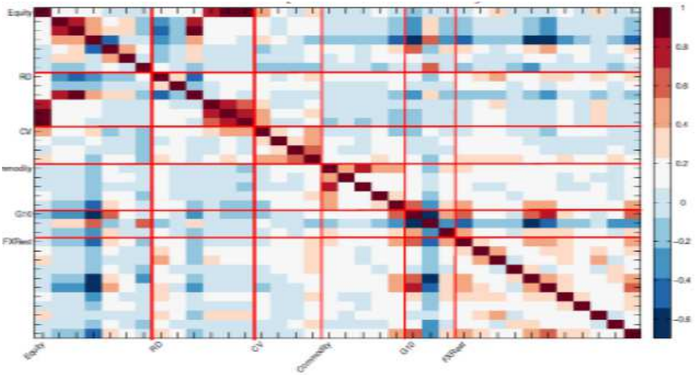

Figure: Correlation of world assets before the crisis. Source: Keynote by Ricardo Caballero, FMS 2012

Motivation - Living in an Interconnected World

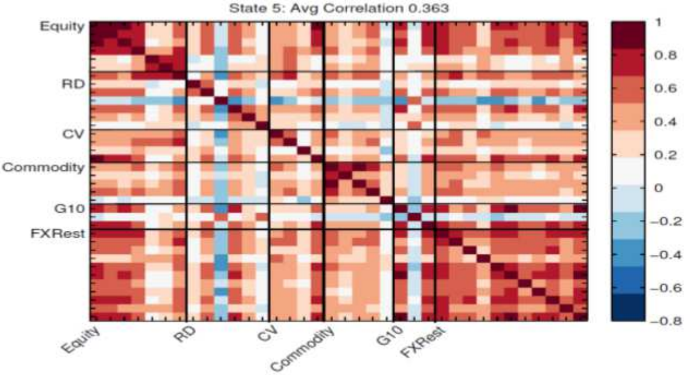

Figure: Correlation of world assets after the crisis. Source: Keynote by Ricardo Caballero, FMS 2012

Motivation - Living in an Interconnected World

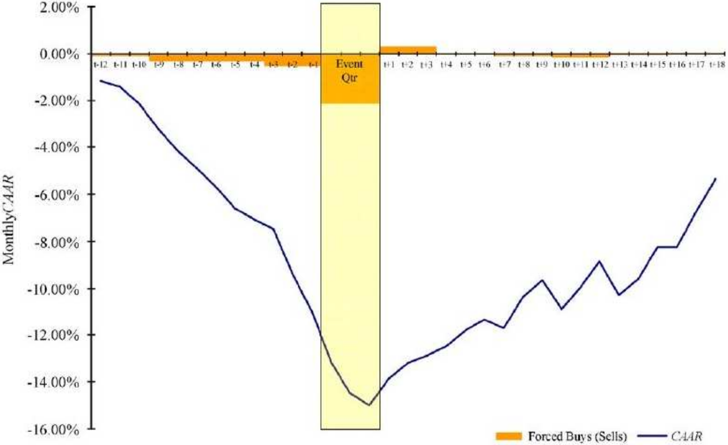

Figure: Cumulative average abnormal returns around mutual fund fire sales. Source: Coval and Stafford (2007).

Motivation - Living in an Interconnected World

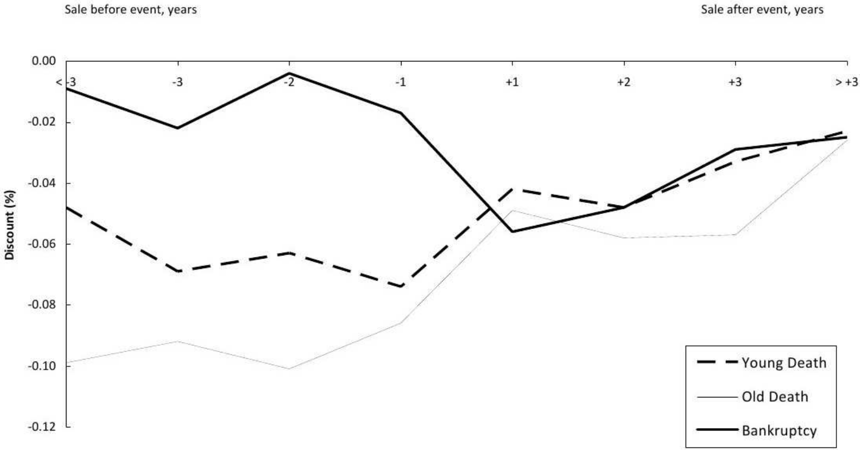

Figure: Forced Sales Discounts and Time Between Sale and Event. Source: Campbell, Giglio and Pathak (2012).

Motivation - Living in an Interconnected World

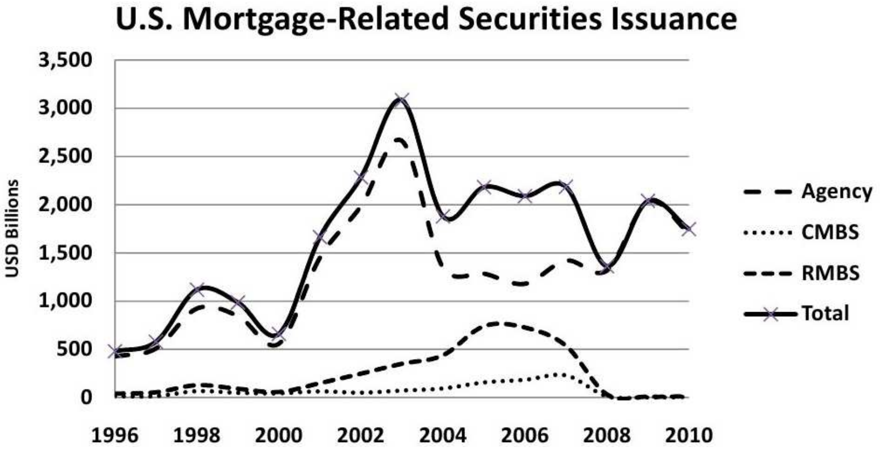

Figure: U.S. Mortgage-Related Securities Issuance. Source: Gorton and Metrick (2010).

Motivation - Living in an Interconnected World

Four key developments:

1. Direct linkages (interbank loans, repos, CDS, etc.) amongst financial intermediaries lead to **counterparty risk**
2. Increasing indirect linkages (asset commonality) induce a risk of **fire-sales**
3. The financial system has become more **concentrated**

Motivation - Living in an Interconnected World

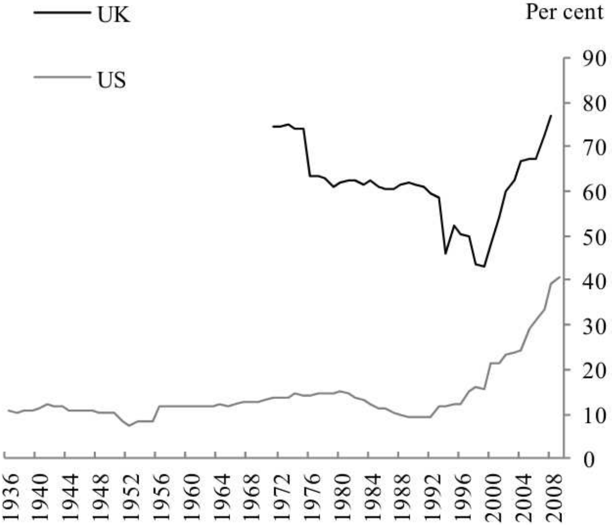

Figure: Concentration of the UK and US banking system. Source: Gai, Haldane and Kapadia (2011).

Motivation - Living in an Interconnected World

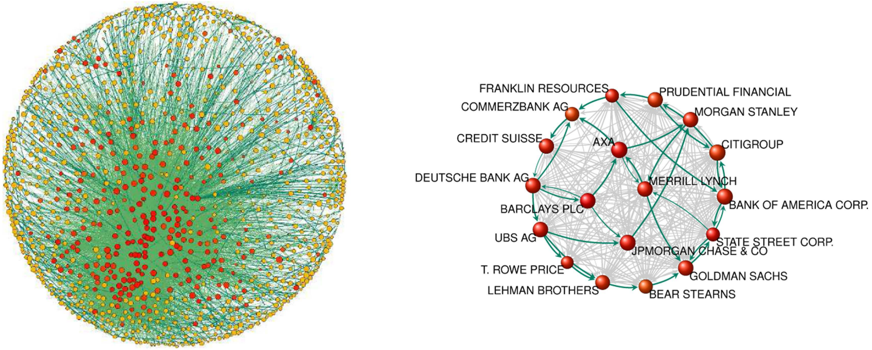

Figure: The Network of Global Corporate Control. Source. Vitali, Glattfelder, and Battiston (2012).

Motivation - Living in an Interconnected World

Four key developments:

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4. The financial system has become increasingly **opaque**

Motivation - Living in an Interconnected World

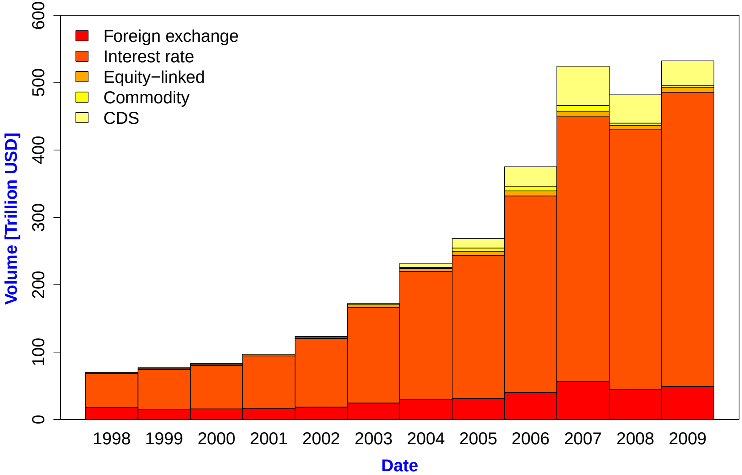


Figure: Global over-the-counter derivatives markets, notional amounts of contracts outstanding. Source: IMF

What is Network Theory?

System: set of interacting or interdependent components forming an integrated whole.

There are many ways of studying a system:

- Study the single components, -nodes- (e.g. computers, humans).
- Study the way of interactions between components -links- (e.g. wireless, emails, communication, interaction).
- Study the pattern of interactions between components -nodes plus links- (e.g. the internet, society).

Network theory focuses on the last aspect.

The pattern of interactions can have a huge effect on the behaviour of the system
$\Rightarrow$ Relevant also for **equilibria in Agent-Based Models**.

Basic Definitions – Network Examples

What is a network?

- A network is a collection of vertices (nodes) joined by edges (links).
- More mathematically: a network g is a set N of nodes and a set E of edges with $e_{ij} = \{i, j\} \in E$ for $i, j \in V$.
- It does not matter which vertex gets which label.
- Notation: $e_{ij} \equiv (i, j) \equiv i : j \equiv g_{ij}$.
- Networks can also be directed (non-symmetric adjacency matrix), weighted (positive real adjacency matrix), bipartite (two types of nodes), or even multi-layered (many types of links, possibly different types of nodes).

Basic Definitions – Network Examples

Network	Node	Link
Internet	Computer	Cable or wireless connection
Social networks	Individual	Friendship
Food web	Species	Predation
Neural Network	Neurons	Synapses
Transport networks	Stop	Street
Financial networks	Banks	Loans or Asset cross-holdings

Basic Definitions – Network Examples

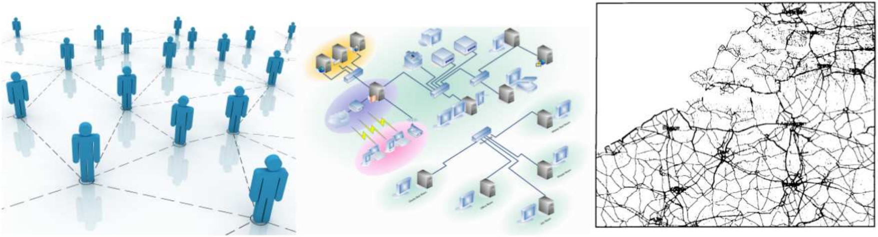


Figure: Three examples of networks.

Basic Definitions – Network Examples

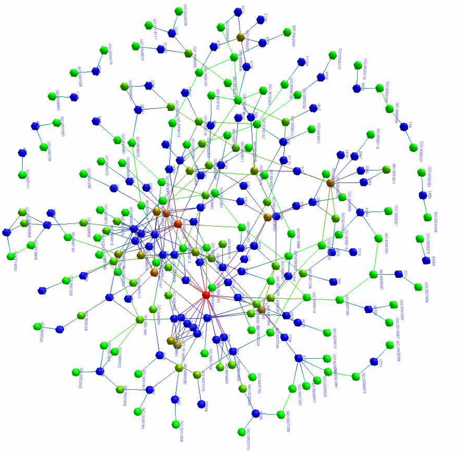


Figure: A network of gene functioning.

Basic Definitions – Network Examples

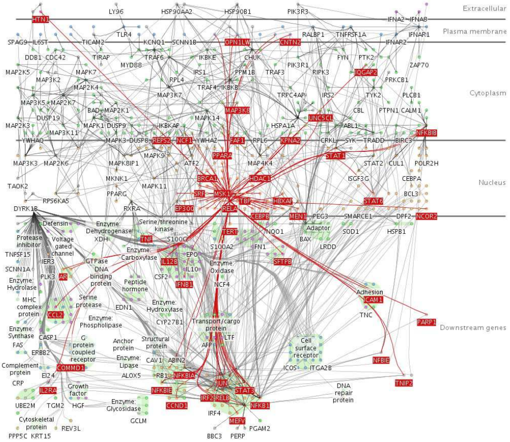


Figure: Cerebral map as a network.

Basic Definitions – Network Examples

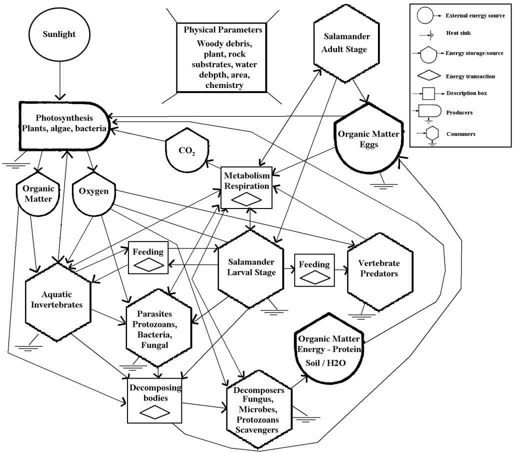


Figure: A food web as a network.

Basic Definitions – Network Examples

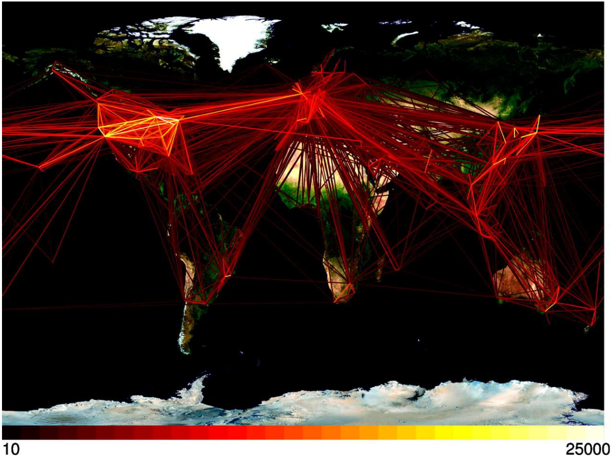


Figure: The network of international air traffic.

Basic Definitions – Network Examples

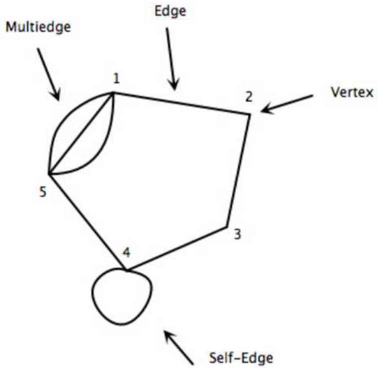


Figure: A stylized representation of a network with multi- and self-edges.

Basic Definitions – Network Representation

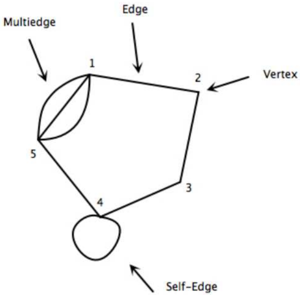
How to represent a network?

- The easiest representation of a network is the adjacency matrix g:

$$g_{ij} = \begin{cases} 1 & \text{if } e_{ij} \in E \\ 0 & \text{otherwise} \end{cases}$$

- If A is symmetric and $diag(g) = 0$ there is no self edge.
- A multi-edge is represented by $g_{ij} = n, n \in \mathbb{N}$.
- Self-edges are represented by $g_{ii} = 2$

Basic Definitions – Network Representation



Corresponding adjacency matrix:

$$g = \begin{bmatrix} 0 & 1 & 0 & 0 & 3 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 3 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Basic Definitions – Weighted networks

Weighted and unweighted networks

- Until now: edges are simply on/off connections between vertices
- In many cases, however, it makes sense to represent edges that have a *strength*, *weight*, or *value* to them.
- Example 1 – Internet: Edges can have weights that represent the amount of data flowing through a given connection.
- Example 2 – Friendship: Edges can have weights representing frequency of contact between actors or duration of their friendship.
- Example 3 – Financial network: Edges have weights representing the amount of lending between two banks.

Representation

- As before, but now $g_{ij} \in \mathbb{R}$ representing the weights.

Basic Definitions – Weighted networks

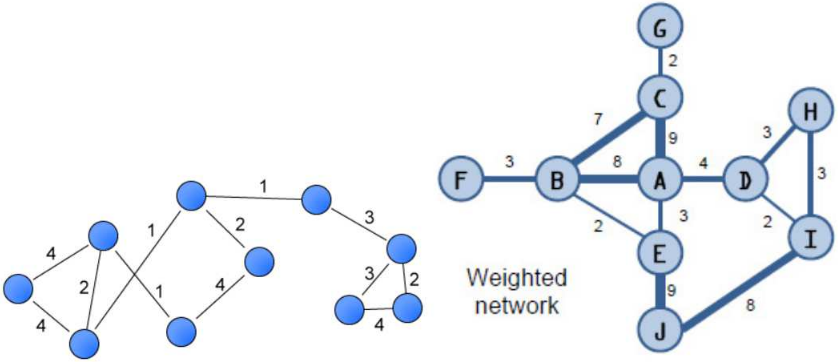


Figure: Examples of weighted networks.

Basic Definitions – Directed networks

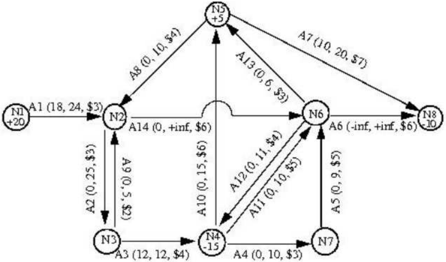
Directed and undirected networks

- Until now: edges are undirected, adjacency matrix is *symmetric*
- In many cases, however, it makes sense to represent edges that have a *direction*.
- Example 1 – Internet: Hyperlinks go from one website to another.
- Example 2 – Citation networks: Papers (=nodes) citing each other (=edges).
- Example 3 – Financial network: It is important to know who is lender and who is borrower in a transaction.

Representation

- As before, but now g_{ij} indicates a link *from* i *to* j. g_{ij} is no longer symmetric.

Basic Definitions – Directed Networks



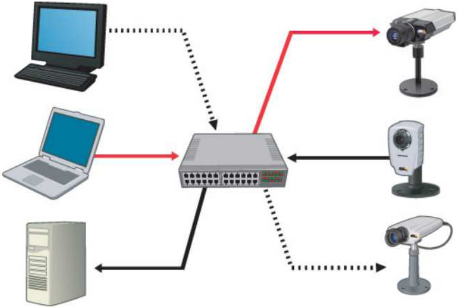
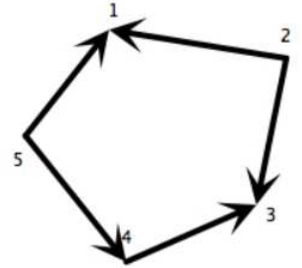


Figure: Two examples of directed networks.

Basic Definitions – Network Representation



Corresponding adjacency matrix:

$$g = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basic Network Properties - Cycles and Hypergraphs

- A **cycle** in a directed network is a closed loop of edges with the arrows on each of the edges pointing the same way around the loop.

- **Hypergraph**, network in which links join more than two vertices at a time, e.g representations of families in a higher community of people, different categories of banks in a interbank network.

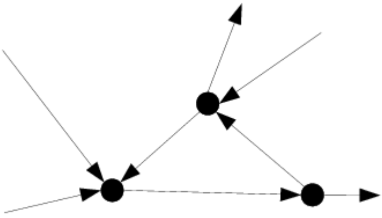


Figure: A cycle in a directed network

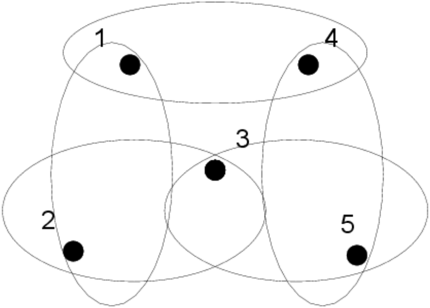


Figure: A hypergraph

Bipartite Networks

Bipartite Networks

- In such a network there are two kind of vertices, one representing the original vertices and the other representing the groups to which they belong.
- The edges in a bipartite network run only between vertices of unlike types.
- Example: Film Network, an edge might connect actors with films in which they appeared.

Bipartite Networks Representation

- The equivalent of an adjacency matrix for a bipartite network is a rectangular matrix called incidence matrix.
- If n is the number of participants of the network and m is the number of groups, than the incidence matrix **B** is a mxn matrix having elements g_{ij} such that

$$g_{ij} = \begin{cases} 1 & \text{if vertex } j \text{ belongs to group } i, \\ 0 & \textit{otherwise}. \end{cases}$$

Bipartite Networks – Network Representation

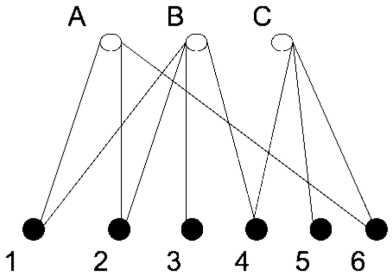


Figure: Example of a bipartite network

Basic Network Properties – Neighborhoods

- A network g is a set of nodes $N = \{1, \dots, n\}$ (with $n < \infty$) and a set of links $g_{ij} \in \{0, 1\}$ between the nodes
- The set of all possible networks on n nodes is denoted by $\mathcal{G}$
- The set of neighbors of node i is referred to as $N_i(g) = \{j \in N | g_{ij} = 1\}$, and $\eta_i(g) = |N_i(g)|$ are the number of neighbors of node i
- The d-neighborhood of i is defined as $N_i^d(g)$:

$$N_i^1(g) = N_i(g) \quad , \quad N_i^k(g) = N_i^{k-1}(g) \cup \left(\bigcup_{j \in N_i^{k-1}} N_j(g) \right)$$

Basic Network Properties – Partitions

- Define a partition $N_1(g), N_2(g), \ldots, N_{n-1}(g)$ of the network such that two nodes belong to the same group iff they have the same number of links: $\eta_i(g) = \eta_j(g)$. Then, a network is:
  1. *regular* if every node has the same number of links: $\eta_i(g) = \eta$
  2. *complete* if every node has degree $\eta_i(g) = n - 1$
  3. *empty* if every node has degree $\eta_i(g) = 0$

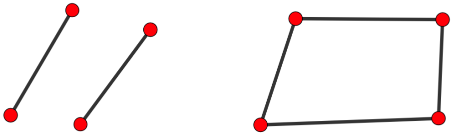


Figure: Regular networks for $n = 4$; $\eta = 1$ (left), $\eta = 2$ (right)

Basic Network Properties – Core-periphery

- In a core-periphery network, there are two groups of nodes: $N_1(g)$ and $N_k(g)$, with $k > |N_k(g)|$

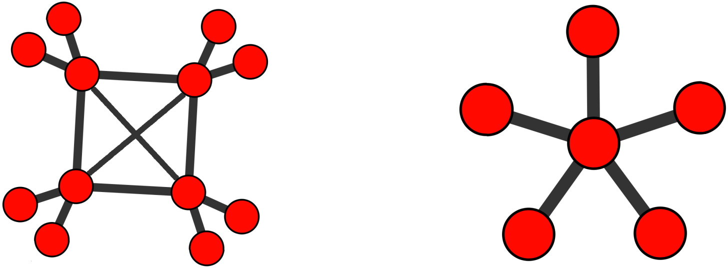


Figure: A core-periphery network with $k = 5$ (left), and $k = 1$ (right)

- There exists a plethora of different networks $\Rightarrow$ **How to categorize them?**

Basic Network Properties - Additional Definitions

- **Clique**, maximal subset of vertices in an undirected network such that every member of the set is connected by an edge to every other.
- **K-plex** of size n, maximal subset of n vertices within a network such that each vertex is connected to at least $n - k$ of the others.
- **K-core**, maximal subset of vertices such that each is connected to at least k others in the subset.
- **K-component**, maximal subset of vertices such that each is reachable from each of the others by at least k vertex-independent paths.

Basic Network Properties - Paths

- **Paths**, any sequence of vertices such that every consecutive pair of vertices in the sequence is connected by an edge in the network.
- **length of a path**, number of edges traversed along the path (not the number of vertices).
- **geodesic path**, also called shortest path, is a path between two vertices such that no shorter path exists.
- The length of a geodesic path, often called the **geodesic distance** or shortest distance, is the shortest network distance between the vertices in question. In mathematical terms, the geodesic distance between vertices i and j is the smallest value of r such that $[g^r]_{ij} > 0$.
- The **diameter** of a graph is the length of the longest geodesic path between any pair of vertices in the network for which a path actually exists.

Basic Network Properties - Paths Examples

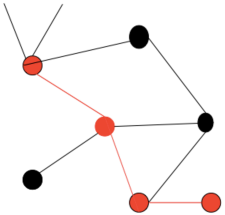


Figure: A path of length three

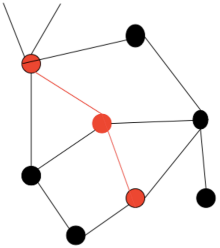


Figure: A geodesic path of length two

Basic Network Properties - Triads

- A **triad** is a group of three vertices linked in the network.
- A triad is **transitive** whenever $i \to j$ and $j \to k$ then also $i \to k$.
- For each vertex the **clustering coefficient** is the fraction of transitive triads the vertex is involved in over the total number of triads in the network. Hence clustering measure the probability that two nodes having a common neighbour are neighbour themselves.
- Formally

$$c_i = \frac{\sum_j \sum_k g_{jk}}{g_i(g_i - 1)}$$

for all j, k that are directly connected to i.

Basic Network Properties - Components

- If there is a path from every vertex in a network to every other the network is **connected**, otherwise it is **disconnected**.
- Subgroups in a disconnected network are called **components**.
- A **component** is a subset of the vertices of a network such that there exists at least one path from each member of that subset to each other member, and such that no other vertex in the network can be added to the subset while preserving this property.
- Components are divided according to whether the vertices in the subset are reachable among them via direct or only via undirected edges. In the first case the sub-network is defined as **strongly connected** (SCC) and in the second case it is **weakly connected** (WCC).
- Most real world networks are composed by one largest WCC and one or more much smaller WCCs that are disconnected from the largest one.

Basic Network Properties - Components Example

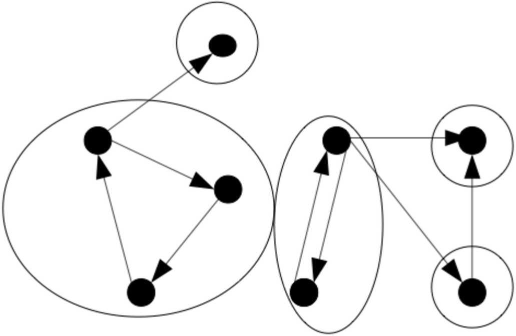


Figure: This network has two weakly connected components of four vertices each, and five strongly connected components (Newmann 2004).

Network Measures - Degree

- **Degree**, The degree of a vertex i in a network is the number of edges connected to it.

- For an undirected graph of n vertices the degree can be written in terms of the adjacency matrix as

$$k_i = \sum_{j=1}^{n} g_{ij}$$

- The mean degree c of a vertex in an undirected graph is

$$c = \frac{1}{n} \sum_{i=1}^{n} k_i$$

- The maximum number of edges in a simple graph (one with no multiedges or self-edges) is $\binom{n}{2} = \frac{1}{2} n(n-1)$.

Network Measures - Degree distribution

- The degree of a node i is the number of direct connections: $\eta_i(g) = |N_i(g)|$
- The degree distribution of a network is a set P where $P(k) = |N_k(g)|/n$ is the fraction of nodes with degree k. Thus, $P(k) \geq 0 \forall k$, and $\sum_{k=0}^{n-1} P(k) = 1$
- The average degree of network g is given by:

$$\hat{\eta}(g) = \sum_{k=0}^{n-1} P(k)k = \sum_{i \in N} \frac{\eta_i(g)}{n}$$

Network Measures - In- and Out-degree

- The in-degree is the number of ingoing edges connected to a vertex:

$$k_i^{in} = \sum_{j=1}^{n} g_{ji}$$

- The out-degree is the number of outgoing edges:

$$k_i^{out} = \sum_{j=1}^{n} g_{ij}.$$

- The number of edges m in a directed network equals the total number of ingoing ends of edges at all vertices, which equals the total number of outgoing ends of edges:

$$m = \sum_{i=1}^{n} k_i^{in} = \sum_{j=1}^{n} k_j^{out} = \sum_{ij} g_{ij}$$

- The mean in-degree c_{in} and the mean out-degree c_{out} of every directed network are equal

Network Measures - Density

- The density ρ of a graph is the fraction of these edges that actually present to the maximum number of possible edges:

$$\rho = \frac{m}{\binom{n}{2}} = \frac{2m}{n(n-1)}$$

Notice that the density is strictly in the range $0 \leq \rho \leq 1$.

Network Measures – Variance and Range

- Also the variation and range of a network is interesting.
- The variation is given as:

$$var(g) = \sum_{k=0}^{n-1} P(k)|\hat{\eta} - k|^2$$

Q: What is $\lim_{n\to\infty} var(g)$ for a star and regular network?

- While the range is defined as:

$$R(g) = \max_{i\in N} \eta_i(g) - \min_{j\in N} \eta_j(g)$$

Q: What is $\lim_{n\to\infty} R(g)$ for a star and regular network?

Network Measures – Centralities I

- A measure for the *"prominence"* is given by various centrality measures
- Degree centrality captures the prominence relative to other nodes in terms of degree:

$$C_d(i;g) = \frac{\eta_i(g)}{n-1}$$

- Centrality can also be defined for an entire network:

$$C_d(g) = \frac{\sum_{i=1}^{n} \left[C_d(i^*;g) - C_d(i;g) \right]}{\max_{g' \in G} \left[\sum_{i=1}^{n} \left[C_d(i^*;g') - C_d(i;g') \right] \right]}$$

where i^* is the node with the highest degree centrality in g

Network Measures – Centralities II

- Minimum degree in any component is 1, while maximum degree is $n-1$
$\Rightarrow C_d(i;g) = \eta_i(g)/(n-1) = (n-2)(n-1)/(n-1)$ which yields:

$$C_d(g) = \frac{\sum_{i=1}^{n} \left[C_d(i^*;g) - C_d(i;g) \right]}{n-2}$$

- Another centrality measure is based on proximity. Total distance of node i to all other nodes in g is

$$\sum_{j \neq i} d(i,j;g)$$

- The closeness centrality is then defined as:

$$C_c(i;g) = \frac{n-1}{\sum_{j \neq i} d(i,j;g)}$$

Q: What is the closeness centrality of a star and a cycle?

Network Measures - Centralities III

- How important is a node i in terms of connecting two other nodes j and k?
- Let $P_i(k,j)$ denote the number of shortest paths between k and j that contain i, and $P(k,j)$ be the total number of shortest paths between k and j
- The betweenness centrality of a node is defined as:

$$C_B^i(g) = \sum_{k \neq j : i \notin \{k,j\}} \frac{P_i(k,j)/P(k,j)}{(n-1)(n-2)/2}$$

Network Measures - Centrality IV

- **Eigenvector centrality**, instead of awarding vertices just one point for each neighbour, eigenvector centrality gives each vertex a score proportional to the sum of the scores of its neighbours.

$$x_i = k_1^{-1} \sum_j A_{ij} x_j,$$

where k_1 is the largest eigenvalue of A and x_j is the centrality of node j.

- **PageRank**, The centrality obtained by own neighbours is proportional to their centrality divided by their out-degree. Vertices that point to many others pass only a small amount of centrality on to each of those others, even if their own centrality is high. In mathematical terms this is defined by

$$x_i = \alpha \sum_j A_{ij} \frac{x_j}{k_j^{out}} + \beta,$$

where α and β are positive constants.

Geodesic, distance and eccentricity

- A **geodesic** is the shortest path between two nodes.
- The **geodesic distance** (d_{ij}) is then the length of a shortest path between node i and node j.
- The **average shortest path** is the mean distance separating vertex i from all other vertices belonging to the same component, that is: $\hat{d}_i = \frac{\sum_{j \neq i} d_{ij}}{n-1}$.
- The **average path length** is then computed by taking the average of the $\hat{d}_i$ across all nodes i:
$$\hat{d} = \frac{\sum_{i-1}^{n} \hat{d}_i}{n}$$
- Similarly the **eccentricity** of each vertex having at least one outgoing arc is the furthest distance to any other node in the network.
- The **diameter** of a network is the maximum eccentricity across all nodes (min 1, maximum n-1).

Transitivity

- **Transitivity**, When there is a tie from i to j, and also from j to h, then there is also a tie from i to h.
- A measure of transitivity is the (global) **transitivity index**, defined as the ratio

  $$\text{Transitivity Index} = \frac{\text{Nr Transitive triads}}{\text{Nr Potentially transitive triads}}$$

  This is also called **clustering index** (insert here other definitions of clustering index).
- For random graphs, the expected value of the transitivity index is close to the density of the graph.

Local Clustering Coefficient

- **Local clustering coefficient**, clustering coefficient for a single vertex. For vertex i we have

$$C_i = \frac{\text{(number of pairs of neighbours of } i \text{ that are connected)}}{\text{(number of pairs of neighbours of } i\text{)}}$$

- Example: in social networks the local clustering represents the average probability that a pair of i's friends are friends of one another.

- Redundancy (R_i), redundancy of a vertex i is the mean number of connections from a neighbour of i to other neighbours of i.

Loops and Reciprocity

- In networks the number of paths of length r that start and end at the same vertex i are called **loops**.
- Another important measure in directed networks is **reciprocity**. Reciprocity measures the frequency of loops of length two.
- Notice that the product of the adjacency matrix elements $A_{ij}A_{ji}$ is 1 if and only if there is an edge from i to j and an edge from j to i and zero otherwise.
- An expression of reciprocity is then

$$r = \frac{1}{m}\sum_{ij} A_{ij}A_{ji} = \frac{1}{m} TrA^2,$$

where m is the number of edges in the network.

Similarity

- **Similarity**, Actual number of common neighbours two vertexes have minus the expected number they would have if they chose their neighbours at random:

$$\sum_k g_{ik}g_{jk} - \frac{k_i k_j}{n} = \sum_k (g_{ik} - \langle g_i \rangle)(g_{jk} - \langle g_j \rangle),$$

  where $\langle g_i \rangle$ denotes the mean $\frac{\sum_k g_{ik}}{n}$ of the elements of the ith row of the adjacency matrix.
- This measure is n times the covariance $cov(g_i, g_j)$ of the two rows of the adjacency matrix. This measure can be normalized to the variance of the set to get a correlation coefficient.
- **Standard Pearson** coefficient of similarity

$$r_{ij} = \frac{cov(g_i, g_j)}{\sigma_i \sigma_j},$$

  where σ_i^2 and σ_j^2 are the variances of either set. Notice that $-1 \leq r_{ij} \leq 1$.

Network typologies

- **Complete network**, graph where each agent has a direct connection to every other agent (hence each node has $n-1$ direct links and all the links are reciprocal).
- **Classical random network**, each node is connected or not with independent probabilities p and $1-p$.

Network typologies - Random Graph

- The network is composed of n vertices and m edges. The connection between vertices is made at random.
- The random graph represents a probability distribution over a set of possible networks.
- The random model $G(n, m)$ is correctly defined as a probability distribution $P(G)$ over all graphs G in which $P(G) = 1/\Omega$ for simple graphs with n vertices and m edges, where Ω is the total number of such graphs.

Network typologies - Properties of Random Graphs

- The properties of a random graph indicate the average properties of the ensemble
- **Diameter**, diameter $l(G)$ of a graph G averaged over the ensemble:

$$\langle l \rangle = \sum_G P(G) l(G) = \frac{1}{\Omega} \sum_G l(G).$$

- **Average degree**, $\langle k \rangle = 2m/n$.

Network typologies - Erdos-Renyi Model

- An alternative specification of random networks is obtained specifying the *probability* of edges between vertices: $G(n, p)$.

- $G(n, p)$ is the ensemble of networks with n vertices in which each simple graph G appears with probability

$$P(G) = p^m (1-p)^{\binom{n}{2} - m}$$

- The mean value of m is

$$\langle m \rangle = \sum_{m=0}^{\binom{n}{2}} mP(m) = \binom{n}{2} p.$$

- The mean degree is

$$\langle k \rangle = \sum_{m=0}^{\binom{n}{2}} \frac{2m}{n} P(m) = \frac{2}{n} \binom{n}{2} p = (n-1)p$$

Network typologies - Small World Effect

- **Small world effect**, This effect occurs when most nodes are not neighbours of other nodes, however, they can be reached from every other node with a small number of steps.
- Specifically, a network presenting the small world effect is one in which the average distance between two randomly chosen nodes is proportional to the logarithm of the number of nodes, namely:

$$L \propto Log(N)$$

- Small-world networks tend to have an high clustering coefficient, explained by the presence of cliques, sub-networks which have connections between almost any two nodes within them (resulting from high clustering coefficient).
- In addition most pairs of nodes are connected by at least one short path (low mean shortest path length). Over abundance of hubs (high degree nodes).
- The degree distribution is fat-tailed towards high degree values (normally power law distribution).

Network typologies - Scale-free Networks

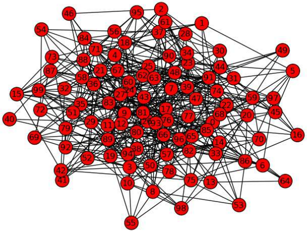
- A binomial (or Poisson for a large n) distribution for network where each node is connected or not with independent probabilities p and $1 - p$ tend to exhibit a distribution that approximately follows a power law:

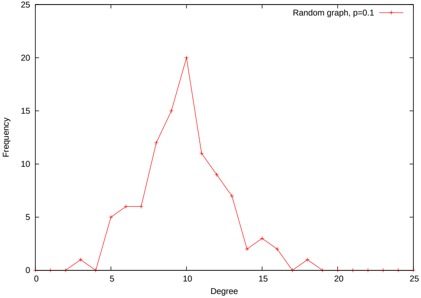
$$P(k) = k^{-\alpha}, \text{for some } \alpha < 1$$

- Such networks are called **scale-free** networks.

Constructing networks - the static case

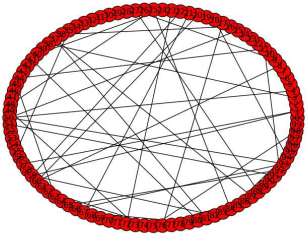
- In a static network all nodes are established at the same time and all links are drawn according to some probabilistic distribution
- Example: Poisson random graph model
  - Randomly select a network from the set of all possible networks with n nodes
  - Alternatively, fix the number of links to be m and pick a network from the set of all possible networks with m links
  - If the probability that two given nodes are connected by a link is p, there will be $pn(n-1)/2$ links in the network

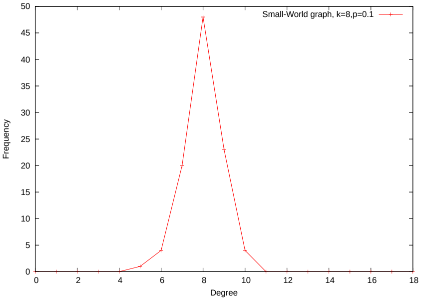




Constructing networks - the static case

- In many real-world social networks, large clustering is found:
  - Networks of co-authorship: 0.496 for computer science, 0.45 for physics, 0.15 for mathematics, and 0.193 for economics
  - Movie actor network: 0.79 costarring
  - WWW: 0.1078
- Q: What is the clustering coefficient and average path length of a random network with n nodes?
- Networks with high clustering, but small diameter are called small-world networks





Constructing networks - the static case

- To generate random networks with a given degree distribution, one can use the configuration model
- A degree sequence is a list of the degrees of the nodes in a network: $(d_1, d_2, \ldots, d_n)$
- The proportion of nodes that have degree d in this sequence is $P^n = 1/n(\#\{i|d_i = d\})$
- Now construct a sequence where node 1 is listed d_1 times, node 2 is listed d_2 times, etc:

$$\underbrace{1, 1, \ldots, 1}_{d_1 \text{ entries}} \quad ; \quad \underbrace{2, 2, \ldots, 2}_{d_2 \text{ entries}} \quad ; \quad \ldots \quad ; \quad \underbrace{n, n, \ldots, n}_{d_n \text{ entries}}$$

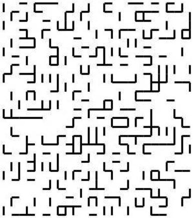
- Randomly pick two elements of the sequence and form a link between the two nodes, delete the entries from the sequence and repeat

Constructing networks - the static case

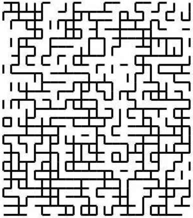
- However, note the following:
  1. It is possible to have more than one link between two nodes (multi-graphs)
  2. Self-links are possible and can even occur multiple times
  3. The sum of the degrees needs to be even
- The problem of multi-graph generation can be addressed by removing all multiple edges from a multi-graph and then show that the resulting graph has a similar degree distribution as the original one

Percolation

- Consider a regular network with the links being present with probability p and absent with probability $1 - p$
- For small p, only small clusters of connected nodes will be present
- At critical probability p_c (percolation threshold) a percolating cluster of nodes connected by edges appears



p=0.315



p=0.525

Figure: Illustration of bond percolation in $2D$.

Percolation

- Percolation probability P is the probability that a given node belongs to the infinite cluster is given by:

$$P = P_p(|C| = \infty) = 1 - \sum_{s<\infty} P_p(|C| = s)$$

- The average cluster size is given by:

$$\langle s \rangle = E_p(|C|) = \sum_{s=1}^{\infty} s P_p(|C| = s)$$

- The cluster size distribution can be defined as the probability of a node to be at the left hand end of a cluster of size s:

$$n_s = \frac{1}{s} P_p(|C| = s)$$

Generating functions

- The generating function of a random graph is defined as:

$$G_0(x) = \sum_{k=0}^{\infty} P(k)x^k$$

- The degree distribution $P(k)$ can be obtained as:

$$P(k) = \frac{1}{k!} \left. \frac{d^k G_0}{dx^k} \right|_{x=0}$$

- Generating functions can be used to describe the degree distribution of the first neighbors of a randomly selected node:
  - A randomly selected edge reaches a node with degree k with probability proportional to $kP(k)$
  - In addition, the generating function will contain a term x^{k-1} because the edge through which the node was reached has to be discounted

Generating functions

- Distribution of outgoing links is generated by the function:

$$G_1(x) = \frac{\sum_k kP(k)x^{k-1}}{\sum_k kP(k)} = \frac{1}{\langle k \rangle} G_0'(x)$$

- The average number of first neighbors is equal to the average degree of the graph:

$$z = \langle k \rangle = \sum_k kP(k) = G_0'(1)$$

- The generating function for the size distributions of clusters reached by following an arbitrary link is given by:

$$H_1(x) = \frac{\sum_k kP(k)\left[H_1(x)\right]^k}{\sum_k kP(k)} = xG_1(H_1(x))$$

Generating functions

- Starting from a randomly chosen node, there is one cluster at the end of each adjoining link
- Hence, the generating function for the size of the whole cluster is:

$$H_0(x) = x \sum_k P(k) \left[H_1(x)\right]^k = xG_0(H_1(x))$$

- The average cluster size is then given by:

$$\langle s \rangle = H_0'(1) = 1 + \frac{G_0'(1)}{1 - G_1''(1)}$$

which diverges when $G_1'(1) = 1$

- When a giant cluster is present, $H_0(x)$ generates the probability distribution of finite clusters

Generating functions

- In this situation $H_0(1) = 1 - S$ where S is the size of the giant cluster. It can then be shown that:

$$S = 1 - G_0(u)$$

where u is the smallest non-negative real solution of $G_1(u) = u$

- From the average number of neighbors, the average path length can be calculated:

$$z_m = \left[G_1'(1)\right]^{m-1} G_0'(1) = \left[\frac{z_2}{z_1}\right]^{m-1} z_1$$

where z_1 and z_2 is the number of first and second neighbors

Generating networks - assembly, growth and evolution

- Contrary to random and small-world models, we now turn to models where the number of nodes changes
- The focus shifts from network properties to network dynamics
- At the heart of the scale free behavior of many real-world networks are two mechanisms:
  1. Growth: starting with a small number (m_0) of initial nodes, add a new node with $m \leq m_0$ links at each step in the iteration
  2. Preferential attachment: when linking the new nodes to the existing ones, the probability of having a link is proportional to the degree of the existing node

Generating networks - assembly, growth and evolution

- Continuum approach to SF networks: degree k_i of node i will increase every time a new node enters the system and links to i
- The probability of this happening is $\Pi(k_i)$:

$$\frac{\partial k_i}{\partial t} = m\Pi(k_i) = m\frac{k_i}{\sum_{j=1}^{N-1} k_j}$$

- The sum in the denominator goes over all nodes except the newly introduced one: $\sum_{j=1}^{N-1} k_j = 2mt - m$ and hence:

$$\frac{\partial k_i}{\partial t} = \frac{k_i}{2t}$$

- With the initial condition $k_i(t_i) = m$, the solution is:

$$k_i(t) = m\left(\frac{t}{t_i}\right)^{\beta} \quad , \quad \text{with } \beta = \frac{1}{2}$$

Generating networks - assembly, growth and evolution

- The probability that a node has a degree $k_i(t)$ smaller k, $P(k_i(t) < k)$ is:

$$P(k_i(t) < k) = P\left(t_i > \frac{m^{1/\beta} t}{k^{1/\beta}}\right)$$

- When nodes are added at equal time intervals, t_i has a constant probability density:

$$P(t_i) = \frac{1}{m_0 + t}$$

which yields:

$$P\left(t_i > \frac{m^{1/\beta} t}{k^{1/\beta}}\right) = 1 - \frac{m^{1/\beta} t}{k^{1/\beta}(t + m_0)}$$

- Then, the degree distribution can be obtained as:

$$P(k) = \frac{\partial P(k_i(t) < k)}{\partial k} = \frac{2m^{1/\beta} t}{m_0 + t} \frac{1}{k^{1/\beta+1}}$$

Generating networks - assembly, growth and evolution

- Alternatively, we can use the master equation approach that studies the probability $p(k, t_i, t)$ that at time t, node i introduced at time t_i has degree k
- When a new node with m edges enters the system, the degree of node i increases by 1 with probability $m\Pi(k) = k/2t$, otherwise it stays the same
- This yields a master equation:

$$p(k, t_i, t+1) = \frac{k-1}{2t} p(k-1, t_i, t) + \left(1 \frac{k}{2t}\right) p(k, t_i, t)$$

- Which gives a degree distribution:

$$P(k) = \lim_{t \to \infty} \frac{1}{t} \left(\sum_{t_i} p(k, t_i, t) \right)$$

Generating networks - assembly, growth and evolution

- The degree distribution is defined as the result of a recursive relation:

$$P(k) = \begin{cases} \frac{k-1}{k-2} P(k-1) & \text{for} \quad k \geq m+1 \\ 2/(m+2) & \text{for} \quad k = m \end{cases}$$

which gives:

$$P(k) = \frac{2m(m+1)}{k(k+1)(k+2)}$$

- This degree distribution is numerically very close to the continuum method result

Motivation

- Recap: Interconnections of financial intermediaries have consequences for financial stability.
- So far, only **single-layer financial networks** have been introduced.
- Literature considers mostly single-layer financial networks (e.g. interbank networks, CDS networks).
- However, financial intermediaries are connected through a multitude of financial instruments

  $\rightarrow$ **multi-layer network**.

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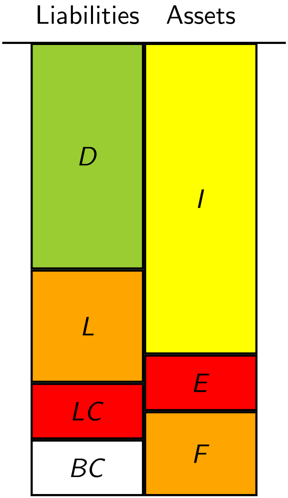
  $\rightarrow$ **multi-layer network**.

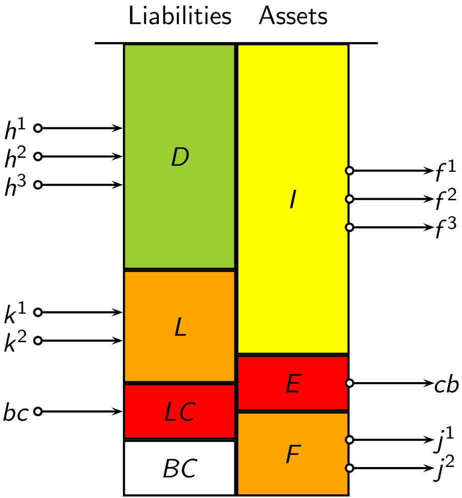
- Model of the financial system: dynamic, weighted, multi-layer network of the financial system to analyse **systemic risk**
- Other applications: international trade network (see e.g. Barigozzi, Fagiolo, and Garlaschelli (2010))

A Simple Bank Balance Sheet



I: investments
E: excess reserves
F: financial assets
D: deposits
L: financial liabilities
LC: central bank credit
BC: banking capital

A Simple Bank Balance Sheet - Multilayer Perspective



Model outline

- Two ingredients: (i) Framework to describe multi-layer networks; (ii) Simple economic model to describe agent behavior and dynamics

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- Weighted networks can be transformed into multigraphs (e.g. Newman (2002))
- The size of the giant component can be caclulated on multigraphs (e.g. Janson and Luczak (2008), Molloy and Reed (1998))
- Develop a framework using **generating functions**

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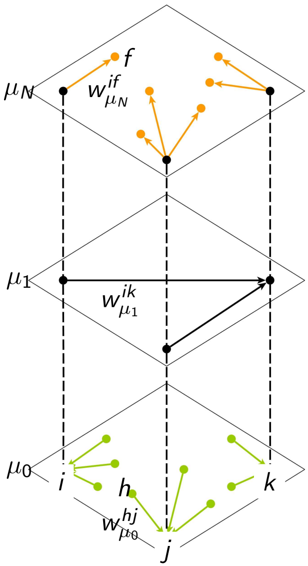
- Develop a framework using **generating functions**

- Portfolio optimization to describe dynamics of the system
- Improving on agent behavior by using cash-flow management / microeconomic theory

Overview of the multi-layer banking system



i, j, k banks

f firms, h households

μ_i network layers

w_{μ}^{ij} link weights

Intermezzo: What is Systemic Risk?

Definition by impact

- FSB definition: *"a risk of disruption to financial services that is (i) caused by an impairment of all or parts of the financial system and (ii) has the potential to have serious negative consequences for the real economy."*

Definition by cause

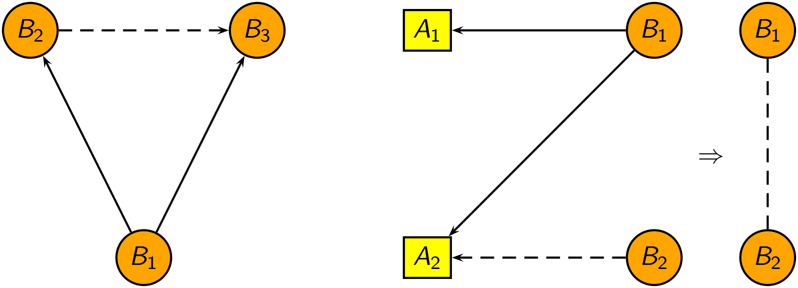


Figure: Left: direct connections (counterparty risk, contagion). Right: indirect connections (common shocks, fire-sales).

Model outline

- nodes i, j, k; $\mu = 0, \dots, N$ network layer; $m = 1, \dots, M$ households; $n = 1, \dots, N_A$ assets; weights w; creation and destruction c^i_μ, d^i_μ
- Simplest possible bank balance sheet:

$$D^i = I^i$$

- Network structure:

$$\begin{aligned} D^i &= \sum_{\{m|m:i\in\mu_0\}} w^{mi}_{\mu_0} \\ I^i &= \sum_{\{n|n:i\in\mu_N\}} w^{in}_{\mu_N} \end{aligned}$$

- Destruction $d^i_0 = D^i$ and creation $c^i_1 = I^i$

Model outline

- Inter-layer constraint

$$c^i_\mu = d^i_{\mu-1}$$

- Intra-layer constraint:

$$\left(\sum_{\{j|j:i\in\mu\}} w^{ji}_\mu - \sum_{\{j|j:i\in\mu\}} w^{ij}_\mu \right) + (c^i_\mu - d^i_\mu) = 0$$

- Slightly less embarassing bank balance sheet:

$$D^i + L^i = I^i + F^i$$

where L^i are interbank loans, while F^i are interbank assets

Model outline

- Two data generating processes: ρ_I, $(\sigma^2)_I$; ρ_F, $(\sigma^2)_F$
- Portfolio approach:

$$\lambda = \frac{I}{I+F} = \frac{\rho_I (\sigma^2)_F}{\rho_I (\sigma^2)_F + \rho_F (\sigma^2)_I} \in [0,1]$$

- Degree of a node i in layer μ:

$$\deg(i,\mu) = |j|i : j \in \mu|$$

- Intra- and inter-layer constraints:

$$\begin{aligned} w_{ij} &= \frac{1}{\deg(i,\mu)} \left(c^i_\mu - d^i_\mu \right) \\ d^i_\mu &= (1-\lambda) c^i_\mu \end{aligned}$$

- **Analytical solution** for limiting cases, **simulations** for real-world data

Credit shocks affect capital directly

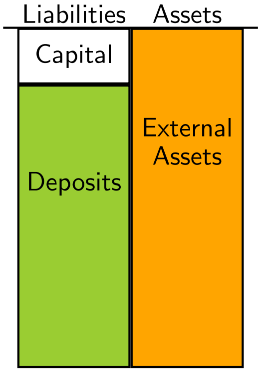


Figure: A credit shock acting on a simplified bank balance sheet. Left: before the shock. Right: after the shock (the size of the shock is shown as a cross-hatched region).

While funding shocks lead to fire sales

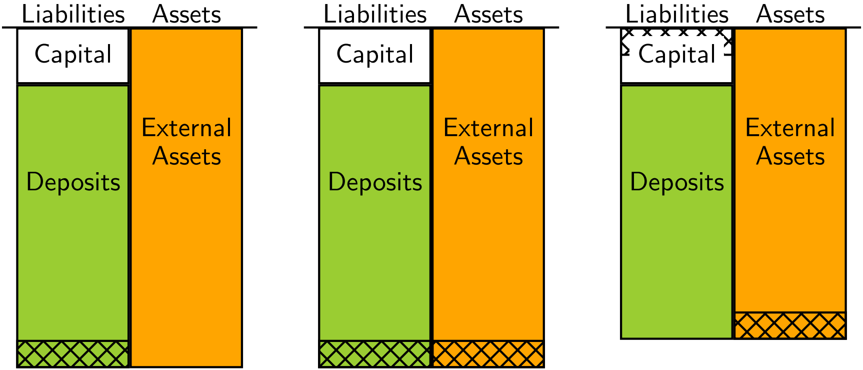


Figure: A funding shock triggering a fire sale on a simplified bank balance sheet. Left: when the funding shock hits (the size of the shock is shown as a cross-hatched region). Center: The shock is compensated by selling off assets. Right: This triggers a fire-sale that further reduces asset value and has to be compensated by banking capital.

Both types of shocks lead to shortfalls

- A simple bank balance sheet with capital:

$$C^i + D^i = I^i$$

- The shortfall depositors suffer in the case of a credit shock is given by:

$$X^i = \Delta I^i - C^i \quad \text{for } \Delta I^i > C^i$$

- A shock leads to reweighting:

$$\widehat{w}^{ji}_{\mu_0} = \left(1 - \frac{X^i}{\widetilde{D}^i}\right) \widetilde{w}^{ji}_{\mu_0}$$

- A similar result can be obtained when some depositors withdraw and a **fire sale** is ignited $\Rightarrow$ Funding shocks $\equiv$ credit shocks

Disease transmission resembles financial contagion

- Newman (2002) describes the probability that a disease is transmitted across two nodes i (infected) and j (susceptible) in a discrete time step to be:

$$T^{ij} = 1 - \left(1 - r^{ij}\right)^{\tau^i}$$

where T^{ij} is the transmissibility, r^{ij} the rate of contact, and τ^i the infection time of node i.

- The average transmissibility is defined as:

$$T = \langle T^{ij} \rangle = 1 - \int_0^{\infty} \sum_{\tau=0}^{\infty} P(r)P(\tau)(1-r)^{\tau} \quad , \quad \text{with } 0 \leq T \leq 1$$

where T gives the probability that an edge transmits a disease (is "occupied"). $\Rightarrow$ **bond percolation**

Disease transmission resembles financial contagion

- The percolation problem can be solved using **generating functions** (for tree-like graphs!):

$$G_0(x) = \sum_{k=0}^{\infty} p_k x^k$$

where p_k is the normalized probability that a randomly chosen vertex has degree k.

- It can be obtained via:

$$p_k = \frac{1}{k!} \left. \frac{\mathrm{d}^k G_0}{\mathrm{d}x^k} \right|_{x=0}$$

with mean $z = \langle k \rangle = \sum_k k p_k = G_0'(1)$.

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- Following a randomly chosen edge, the vertex at the end of the edge has degree distribution:

$$\frac{\sum_k k p_k x^k}{\sum_k k p_k} = x\frac{G_0'(x)}{G_0'(1)} \Rightarrow G_1(x) = \frac{1}{z} G_0'(x)$$

- The probability that a vertex has exactly m of the k emerging edges occupied is given as:

$$\binom{k}{m} T^m (1-T)^{k-m}$$

and hence the probability distribution of m is generated by:

$$G_0(x;T) \sum_{m=0}^{\infty} \sum_{k=m}^{\infty} p_k \binom{k}{m} T^m (1-T)^{k-m} x^m = G_0(1+(x-1)T)$$

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- The distribution $P_s(T)$ of outbreaks of size s is given by:

$$H_0(x; T) = \sum_{s=0}^{\infty} P_S(T)x^s$$

- Cluster of occupied vertices is obtained via a recurrence relation:

$$H_1(x; T) = xG_1(H_1(x; T); T)$$

and the size of a cluster reachable from a randomly chosen vertex is:

$$H_0(x; T) = xG_0(H_1(x; T); T)$$

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- The mean outbreak size $\langle s \rangle$ is given as:

$$\begin{aligned}
\langle s \rangle &= H_0'(1;T) = 1 + G_0'(1;T)H_1'(1;T) \\
\langle s \rangle &= 1 + \frac{G_0'(1;T)}{1 - G_1'(1;T)} = 1 + \frac{TG_0'(1)}{1 - TG_1'(1)}
\end{aligned}$$

- This equation diverges for $TG_1'(1) = 1$ and hence the critical transmissibility is given as:

$$T_c = \frac{1}{G_1'(1)} = \frac{G_0'(1)}{G_0''(1)} = \frac{\sum_k kp_k}{\sum_k k(k-1)p_k}$$

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- Now, following Gai and Kapadia (2010), we obtain the transmissibility for financial contagion.

- Use a simple bank balance sheet:

$$(1-\phi)F^i + qI^i - L^i - D^i = C^i > 0$$

where ϕ is the fraction of bank i's neighbors that failed

- For $F^i \neq 0$, rearranging yields:

$$\phi < \frac{C^i - (1-q)I^i}{F^i}$$

- Interbank assets are distributed according to the economic assumptions and a loss can spread beyond the initial default, if:

$$C^i - (1-q)I^i < X^i \quad \text{with } X^i = w_\mu^{ij} F^i$$

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- The probability that a bank is vulnerable is given as v_j
- The generating function for the joint degree distribution of vulnerable banks is then given as:

$$G_0(x, y) = \sum_{j,k} v_j \cdot p_{jk} \cdot x^j \cdot y^k$$

- And the probability $H_1(y)$ for reaching a vulnerable cluster of given size is:

$$H_1(y) = \Pr\left[\text{reach a safe bank}\right] + y \sum_{j,k} v_j \cdot r_{jk} \cdot \left[H_1(y)\right]^k$$

- Which yields:

$$\langle s \rangle = G_0(1) + \frac{G_0'(1) G_1(1)}{1 - G_1'(1)}$$

and diverges for $G_1'(1) = 1$