Analysis I und Lineare Algebra für Ingenieurwissenschaften

Hausaufgabe 10 - Al-Maweri 13

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Inhaltsverzeichnis

Aufgabe 1

a)

z.z.:

$$\forall n \in \mathbb{N}, \sum_{k=1}^{n} 2k^3 = \frac{n^2(n+1)^2}{2}$$

IA: Es gilt für $n = 0$

$$\sum_{k=1}^{0} 2k^3 = 0 = \frac{0^2(0+1)^2}{2}$$

IV: Es gelte für ein festes beliebiges $n \in \mathbb{N}$

$$\sum_{k=1}^{n} 2k^3 = \frac{n^2(n+1)^2}{2}$$

IB: Dann gilt

$$\sum_{k=1}^{n+1} 2k^3 = \frac{(n+1)^2(n+2)^2}{2}$$

IS: Es gilt

$$\begin{aligned}
\sum_{k=1}^{n+1} 2k^3 &= \sum_{k=1}^{n} 2k^3 + 2(n+1)^3 \\
&= \frac{n^2(n+1)^2}{2} + 2(n+1)^3 \\
&= \frac{n^2(n+1)^2 + 4(n+1)^3}{2} \\
&= \frac{n^2(n+1)^2 + (4n+4)(n+1)^2}{2} \\
&= \frac{(n^2+4n+4)(n+1)^2}{2} \\
&= \frac{(n+2)^2(n+1)^2}{2}
\end{aligned}$$

Damit gilt für alle $n \in \mathbb{N}$

$$\sum_{k=1}^{n} 2k^3 = \frac{n^2(n+1)^2}{2}$$

b)

Es gilt

$$
\begin{aligned}
\int_0^a 2x^3 dx &= \lim_{n\to\infty} \sum_{i=1}^{n} 2\left((i-1)\frac{a}{n}\right)^3 \frac{a}{n} \\
&= \lim_{n\to\infty} \left(\frac{a}{n}\right)^4 \sum_{i=1}^{n} 2(i-1)^3 \\
&\overset{i=k+1}{=} \lim_{n\to\infty} \left(\frac{a}{n}\right)^4 \sum_{k=0}^{n} 2k^3 \\
&= \lim_{n\to\infty} \left(\frac{a}{n}\right)^4 \sum_{k=1}^{n} 2k^3 \\
&= \lim_{n\to\infty} \left(\frac{a}{n}\right)^4 \cdot \frac{n^2(n+1)^2}{2} \\
&= \lim_{n\to\infty} \frac{a^4(n^4+2n^3+n^2)}{2n^4} \\
&= \frac{a^4}{2} \lim_{n\to\infty} \frac{n^4+2n^3+n^2}{n^4} \overset{\text{GWS}}{=} \frac{a^4}{2}
\end{aligned}
$$

Aufgabe 2

a)

$$\int (3x^{-2} + 4e^{-3x})dx = -3x^{-1} - \frac{4}{3}e^{-3x} + c, \forall c \in \mathbb{R}$$

b)

$$
\begin{aligned}
\int_1^e x\ln(x)dx &= \frac{1}{2}x^2 \cdot \ln(x) - \int \frac{1}{2}x^2 \cdot \frac{1}{x} dx \\
&= \frac{1}{2}x^2 \cdot \ln(x) - \int \frac{1}{2} x dx \\
&= \frac{1}{2}x^2 \cdot \ln(x) - \frac{1}{4}x^2 \Big|_1^e \\
&= \left(\frac{1}{2}e^2 \cdot 1 - \frac{1}{4}e^2\right) - \left(\frac{1}{2} \cdot 0 - \frac{1}{4}\right) \\
&= \frac{1}{4}e^2 + \frac{1}{4}
\end{aligned}
$$

c)

$$\begin{aligned}\int e^x \sin(x)dx &= e^x \sin(x) - \int e^x \cos(x)dx \\ &= e^x \sin(x) - e^x \cos(x) - \int e^x \sin(x)dx \\ \Leftrightarrow 2\int e^x \sin(x)dx &= e^x \sin(x) - e^x \cos(x) + c \\ \Leftrightarrow \int e^x \sin(x)dx &= \frac{1}{2}e^x(\sin(x) - \cos(x)) + \frac{1}{2}c \forall c \in \mathbb{R}\end{aligned}$$

Aufgabe 3

a)

$$\begin{aligned}\int 2\cot(x)dx &= 2\int \cot(x)dx \\ &= \int \frac{\cos(x)}{\sin(x)}dx\end{aligned}$$

Substituiere $u = \sin(x) \rightarrow \frac{du}{dx} = \cos(x) \rightarrow dx = \frac{1}{\cos(x)}du$:

$$= \int \frac{1}{u}du = \ln(u)$$

Ruecksubstitution von $u = \sin(x) \rightarrow \ln(\sin(x))$

$$\int 2\cot(x)dx = 2\ln|\sin(x)| + c, c \in \mathbb{R}$$

b)

$$\int x^3 e^{-x^4+1}dx$$

Substituiere $u = -x^4 + 1 \rightarrow \frac{du}{dx} = -4x^3 \rightarrow dx = -\frac{1}{4x^3}du$:

$$= -\frac{1}{4}\int e^u dx = -\frac{1}{4}e^u$$

Ruecksubstitution von $u = -x^4 + 1 \rightarrow -\frac{1}{4}e^{-x^4+1}$

$$\int x^3 e^{-x^4+1}dx = -\frac{e^{-x^4+1}}{4} + c, c \in \mathbb{R}$$

c)

$$\int \frac{1+\ln(x)}{x^x} dx$$

Substitution:

$$\begin{aligned}
u &= \frac{1}{x^x} = x^{-x} = e^{-x\ln(x)} \\
\frac{du}{dx} &= e^{-x\ln(x)}(-\ln(x) - x\frac{1}{x}) \\
&= e^{-x\ln(x)}(-1 - \ln(x)) \\
&= \frac{-(\ln(x)+1)}{x^x} \\
\Leftrightarrow dx &= \frac{x^x}{-\ln(x)-1} du :
\end{aligned}$$

Damit gilt

$$\int \frac{1+\ln(x)}{x^x} dx = -\int du = -u$$

Ruecksubstitution von $u = \frac{1}{x^x}$:

$$\int \frac{1+\ln(x)}{x^x} dx = -\frac{1}{x^x} + c, c \in \mathbb{R}$$