hSets in Homotopy type theory

April 13, 2021

Homotopy type theory

Towards a new practical foundation for mathematics.

- Modern ((higher) categorical) mathematics
- Formalization
- Computational interpretation

Closer to mathematical practice, inherent treatment of equivalences.

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Towards a new design of proof assistants:
Proof assistant with a clear (denotational) semantics,
guiding the addition of new features (e.g. cubical agda)
HoTT library allows us to experiment using hacks.

Challenges

pre-HoTT:

Sets as Types no quotients (setoids), no unique choice (in Coq), ...

Types as Sets not fully abstract $\rightarrow$ Groupoid model

Towards a more symmetric treatment.

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Towards a more symmetric treatment.

Formalization of discrete mathematics: four color theorem, Feit Thompson, ... computational interpretation was crucial.
Can this be extended to non-discrete types?
Computation with quotients, Circle, ...

Some motivation for constructions inside HoTT

To keep track of isomorphisms we generalize sets to
groupoids (proof relevant equivalence relations)
2-groupoids (add coherence conditions for associativity),
. . . , weak ∞-groupoids

Topos theory

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Parabel

Four blind men, who had been blind from birth, wanted to know what an elephant was like, so they asked an elephant-driver for information. He led them to an elephant, and invited them to examine it, so one man felt the elephant's leg, another its trunk, another its tail and the fourth its ear. Then they attempted to describe the elephant to one another. The first man said, 'The elephant is like a tree'. 'No,' said the second, 'the elephant is like a snake'. 'Nonsense' said the third, 'the elephant is like a broom'. 'You are all wrong,' said the fourth, 'the elephant is like a fan'. And so they went on arguing amongst themselves, while the elephant stood watching them quietly.

Topos theory

A topos is like:

- a semantics for intuitionistic formal systems
  model of intuitionistic higher order logic/type theory.
- a category of sheaves on a site (forcing)
- a category with finite limits and power-objects
- a generalized space

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Free topos as a foundation of mathematics (Lambek and Scott):
Proposal to reconcile formalism, platonism and intuitionism

Higher topos theory

Combine these two generalizations of sets.

A higher topos is (represented by):

a model category which is Quillen equivalent to simplicial $Sh(C)_S$ for some model ∞-site (C, S)

Less precisely:

- a generalized space (presented by homotopy types)
- a place for abstract homotopy theory
- a place for abstract algebraic topology
- a semantics for Martin-Löf type theory with univalence and higher inductive types (Shulman/Cisinski).

Type theory

Type theory is another elephant

- a foundation for constructive mathematics
  an abstract set theory ($\Pi\Sigma$).
- a calculus for proofs
- an abstract programming language
- a system for developing computer proofs

Homotopy Type Theory

The homotopical interpretation of type theory

- types as homotopy types of spaces
- dependent types as fibrations (continuous families of types)
- identity types as path spaces

(homotopy type) theory = homotopy (type theory)

Truncation

Higher inductive definition:

```
Inductive minus1Trunc (A : Type) : Type :=
  | min1 : A → minus1Trunc A
  | min1_path : forall (x y: minus1Trunc A), x = y
```

Reflection into the mere propositions

Awodey, Bauer [ ]-types as internal language for regular cats

Theorem

epi-mono factorization. Set is a regular category.

n-connected-*n*-truncated-factorization

space of factorizations is contractible

Unique choice

```
Definition hexists {X} (P:X→ Type):=minus1Trunc (sig P).

Definition atmost1P {X} (P:X→ Type):=
    (forall x1 x2 :X, P x1 → P x2 → (x1 = x2 )).

Definition hunique {X} (P:X→ Type):=(hexists P) * (atmost1P P).

Lemma iota {X} (P:X→ Type):
  (forall x, IsHProp (P x)) → (hunique P) → sig P.
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Direct from elimination principle of truncations.

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Direct from elimination principle of truncations.

In Coq we cannot escape Prop because we want program extraction.
Exact completion: freely add quotients to a category.
Similarly: Consider setoids $(T, \equiv)$ type with equivalence relation.
Spiwack: Setoids in Coq give a quasi-topos (topos without AC!).

Quotients

Towards sets in homotopy type theory.

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Voevodsky: univalence provides (impredicative) quotient types.
Consider the type of equivalence classes.
Requires small power type: $A : U \vdash (A \to Prop_U) : U$.
Dependents only on propositional univalence:
equivalent propositions are equal

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Quotients can also be defined as a higher inductive type

```
Inductive Quot (A : Type) (R:rel A) : Type :=
  | quot : A → Quot A
  | quot_path : forall x y, (R x y), quot x = quot y
  | _ :isset (Quot A).
```

Truncated colimit.
We verified the universal properties of quotients (exactness).

Modelling set theory

pretopos: extensive exact category
(extensive: good coproducts, exact: good quotients)
ΠW-pretopos: pretopos with Π and W-types.

Theorem (Rijke,S)

hSets is a ΠW-pretopos ('constructive set theory').

Assuming AC, and thus classical logic and impredicativity,
we have a well-pointed boolean elementary topos with choice
'Lawvere structural set theory'.

Predicativity (some context)

Impredicative definition as specification (minimal such that)
Predicative definitions give a construction

Avoid Russell's paradox $Set : Set$.
Distinguish small and large objects.
By propositions as types, Prop(=Set) will not be small.
Martin-Löf type theory is predicative.

Proof theoretic strength of IZF equals ZF, but CZF is much lower

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Shulman: fits well with higher dimensional mathematics
Usually the $(n+1)$-category of all small n-categories is not small.
Why should we expect it for $n = -1$: the poset of all truth values.

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Voevodsky's resizing rules are a way of adding impredicativity to type theory. When phrased carefully, holds in many models.

Predicativity: Large subobject classifier

In predicative topos theory: no subobject classifier/power set.
The subobject classifier lives in a higher universe.

$$\begin{tikzcd} X \arrow[r, "!"] \arrow[d, hook, "\alpha"] & 1 \arrow[d, "\mathsf{True}"'] \\ A \arrow[r, "P"] & \mathsf{hProp_i} \end{tikzcd}$$

With propositional univalence, hProp classifies monos into A.

$$A, X : U_i \qquad hProp_i := \Sigma_{B:U_i} isprop(B) \quad hProp_i : U_{i+1}$$

Equivalence between predicates and subsets.
Coq supplies universe polymorphism. Check that there is some way to satisfy the constraints.
This correspondence is the crucial property of a topos.

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Sanity check: epis are surjective (by universe polymorphism).

Object classifier

$Fam(A) := \{(X, \alpha) \mid X : Type, \alpha : X \to A\}$ (slice cat)
$Fam(A) \cong A \to Type$
(Grothendieck construction, using univalence)

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$\mathrm{Type}_\bullet = \{(B, x) \mid B : \mathrm{Type}, x : B\}$
Classifies all maps into A + group action of isomorphisms.
Crucial construction in ∞-toposes.
Grothendieck universes from set theory by universal property

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Accident: $hProp_\bullet \equiv 1$?

Object classifier

Theorem (Rijke/S)

Assuming funext, TFAE

1. *Univalence*
2. *Object classifier*
3. *Descent: Homotopy colimits defined by HITs are well behaved.*

$2 \leftrightarrow 3$ is fundamental in higher topos theory.
Translate statement and give a direct proof.
Not in the HoTT library yet. Probably in Egbert's agda lib.

Conclusion

Quick overview of fragments of the HoTT-library.
Have fun with the exercises. Submit a PR if you have proved something fun.