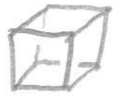
Models of HoTT

(simplicial & cubical)

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0. Introduction & Overview
1. What is a model of type theory?
2. Presheaf models

——— Exercises

3. The simplicial model of HoTT
4. Cubical models and open questions

——— Exercises

0. Introduction & overview

Model of type theory:

- Interpretation of language that turns judgmental equality into actual equality. Slogan

There are many frameworks for models, sometimes equivalent, other times differing in some aspects:

- collection of types over context Γ seen as set or groupoid or category.
- split vs. non-split
- contextual / democratic
- algebraic or categorical notion

  category of models — bicategory of models
- terms or context projections (display maps) as primitive

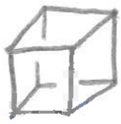
We will work with categories with families.

Model of HoTT: Just a model of type theory with elements witnessing function extensionality + univalence.

History of models of HoTT:

Set model
- univalent universe of propositions

Groupoid model Hofmann-Streicher
- univalence for (strict) sets



Simplicial model Voevodsky
- full univalence
- non-constructive
- models spaces/ ∞-groupoids

⋮

Inverse diagram models Shulman

⋮

Model for injective fibrations in simplicial presheaves

Cubical models Coquand + coworkers
- full univalence
- often does not model spaces/ ∞-groupoids
- computational

- BCH model
- connection-based models
  - unknown if can model spaces
- Cartesian-based models
  - can model spaces (equivariant model)

HoTT and ∞-toposes

We want to "interpret" HoTT in ∞-toposes.

But models of HoTT are not (directly) ∞-categories!

↳ They are 1-categories with homotopical structure, <u>presenting</u> an ∞-category.

Like model categories!

$$\begin{tikzcd} \text{models of HoTT} \arrow[r, "\text{present}^*"] & \infty\text{-toposes} \\ \text{HoTT} \arrow[u, "\text{interprets into}"] & \end{tikzcd}$$

So to interpret HoTT into a given ∞-topos $\mathcal{X}$, first need to find suitable <u>presentation</u> of $\mathcal{X}$.

Shulman found such a presentation for every ∞-topos!

So far, this is a classical (non-constructive) story.

1. What is a model of type theory?

Don't want to bother with modelling "named variables".

↳ Abstract over contexts as lists of hypotheses $x_1 : A_1, \dots, x_n : A_n$ by modelling contexts + substitutions as a <u>category</u> $\mathcal{C}$.

$$\Delta \xrightarrow{\ \sigma\ } \Gamma$$

$$[y_1 : B_1, \dots, y_m : B_m] \qquad [x_1 : A_1, \dots, x_n : A_n]$$

$$x_1 = t_1[y_1, \dots, y_m]$$
$$\vdots$$
$$x_n = t_n[y_1, \dots, y_m]$$

Every context Γ has a <u>set of types</u> $Ty(\Gamma)$.
We can <u>substitute</u> types: $A \in Ty(\Gamma) \rightsquigarrow A[\sigma] \in Ty(\Delta)$.
↳ Ty is a <u>presheaf</u> over C.

Same for terms Tm,
but they additionally depend on a type.

Excursion: Presheaves and discrete fibrations

Def Presheaves over category $\mathcal{C}$ are functors $\mathcal{C}^{op} \to Set$.

Notation: $\widehat{\mathcal{C}} = \mathrm{Presheaf}(\mathcal{C})$

Traditional definition

Grothendieck construction*

*restricted version

$$\int : [\mathcal{C}^{op}, Set] \xrightarrow[\text{ff}]{\text{fully faithful}} \{\text{category over } \mathcal{C}\}$$

$$F \longmapsto \begin{array}{c} \int F \\ \downarrow \\ \mathcal{C} \end{array} \quad \text{"category of elements"} \qquad \begin{array}{c} \mathcal{E} \\ \downarrow \\ \mathcal{C} \end{array}$$

We regard $\mathcal{E}$ as "displayed" over $\mathcal{C}$:

- $\mathcal{E}_0(A)$ set for $A \in \mathcal{C}_0$
- $\mathcal{E}_1(X, Y, f)$ set for
  $X \in \mathcal{E}_0(A)$
  $Y \in \mathcal{E}_0(B)$
  $f \in \mathcal{C}_1(A, B)$

$(\int F)_0(A) = F(A)$

$(\int F)_1(x, y, f) = [F(f)(y) = x]$

That way, we can strictly reindex categories over a base.

The essential image of $\int$ are the discrete fibrations:

$$\begin{tikzcd} ? \arrow[r, dashed, "u"] \arrow[d] & Y & \mathcal{E} \\ A \arrow[r, "f"] & B & \mathcal{C} \end{tikzcd}$$

$\forall f \in \mathcal{C}_1(A, B), Y \in \mathcal{E}_0(B)$ have unique lift $X \in \mathcal{E}_0(A)$, $u \in \mathcal{E}_1(f, X, Y)$.

We use both interchangeably.

Often, discrete fibrations are a better model for presheaves $\widehat{\mathcal{C}}$, than $\mathcal{C}^{op} \to Set$!

Categories with families

Definition

A <u>category with families</u> consists of: cwf

- Category $\mathcal{C}$
- Terminal object $1 \in \mathcal{C}$
- $\mathrm{Ty} \in \widehat{\mathcal{C}}$
- $\mathrm{Tm} \in \widehat{\int \mathrm{Ty}}$
- <u>Context extension:</u>
  for $\Gamma \in \mathcal{C}$ and $A \in \mathrm{Ty}(\Gamma)$, a representation of:
  $$(\mathcal{C} \downarrow \Gamma)^{op} \longrightarrow \mathrm{Set}$$
  $$\Delta \xrightarrow{\sigma} \Gamma \longmapsto \mathrm{Tm}(\Delta, A[\sigma])$$
  $\Leftrightarrow$ a terminal object $(\Gamma.A, p_A, q_A)$
  in the category of tuples (Δ, σ, t) where:
  - $\Delta \in \mathcal{C}$
  - $\Delta \xrightarrow{\sigma} \Gamma$
  - $t \in \mathrm{Tm}(\Delta, A[\sigma])$ $\square$

$$\begin{tikzcd} \Gamma.A \arrow[d] \\ \Gamma \end{tikzcd}$$

$\Gamma.A$ context extension

context projection / display map for A

$q_A \in \mathrm{Tm}(\Gamma.A, A[p_A])$
generic term, last variable

$$\begin{tikzcd} \Delta \arrow[rr, dashed, "\langle \sigma, t \rangle"] \arrow[dr, "\sigma"'] & & \Gamma.A \arrow[dl, "p_A"] \\ & \Gamma & \end{tikzcd}$$

For algebraic notion of model:

- context extension is structure, strictly preserved by morphisms of models
- obtain <u>category</u> of models
- "Syntax" $=_{def}$ initial object

Context extension is really a property of the presheaf Tm!

Alternative definition (cwf)

$$\begin{tikzcd}
\mathrm{Tm} \arrow[d] \\
\mathrm{Ty} \arrow[u, bend right=60, "\dashv\ \text{context extension}"'] \arrow[d] \\
\mathcal{C} \arrow[d] \\
1 \arrow[u, bend right=60, "\dashv\ \text{terminal object}"']
\end{tikzcd}$$

discrete fibrations

Have right adjoints to indicated projections

Notation for adjunctions:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, bend right, "\text{right adjoint}"'] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, bend right, "\text{left adjoint}"']
\end{tikzcd}$$

$\square$

Exercise: show that this means the same as the first definition.

Type formers

Dependent sums

For $\Gamma \in \mathcal{C}$, $A \in Ty(\Gamma)$, $B \in Ty(\Gamma . A)$,
have $\Sigma(A, B) \in Ty(\Gamma)$ with a bijection

$$\underset{\eta}{Tm(\Gamma, \Sigma(A,B))} \underset{\pi_1, \pi_2}{\overset{\text{pair}}{\rightleftarrows}} \underset{\beta}{\left\{ \left(\begin{array}{l} a \in Tm(\Gamma, A), \\ b \in Tm(\Gamma, B[\langle id, a \rangle]) \end{array} \right) \right\}},$$

natural in Γ.

Dependent products

For $\Gamma \in \mathcal{C}$, $A \in Ty(\Gamma)$, $B \in Ty(\Gamma . A)$,
have $\Pi(A, B) \in Ty(\Gamma)$ with a bijection

$$\underset{\eta}{Tm(\Gamma, \Pi(A,B))} \underset{\text{app}}{\overset{\lambda}{\rightleftarrows}} \underset{\beta}{Tm(\Gamma . A, B)},$$

natural in Γ.

Exercise: identity types

Adjoint perspective on Σ/Π

$$\begin{tikzcd} \Gamma.\Sigma(A,B) \simeq \Gamma.A.B \arrow[d] \arrow[dr] & \\ \Gamma & \Gamma.A \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} \Gamma.\Sigma(A,B) \arrow[r, dashed] \arrow[dr] & \Delta \arrow[d, "\sigma"] \\ & \Gamma \end{tikzcd} \quad \underset{\text{natural}}{\simeq} \quad \begin{tikzcd} \Gamma.A.B \arrow[r, dashed] \arrow[dr] & \Delta.A[\sigma] \arrow[d] \\ & \Gamma.A \end{tikzcd}$$

$$\begin{tikzcd} \Delta \arrow[r, dashed] \arrow[dr] & \Gamma.\Pi(A,B) \arrow[d] \\ & \Gamma \end{tikzcd} \quad \underset{\text{natural}}{\simeq} \quad \begin{tikzcd} \Delta.A[\sigma] \arrow[r, dashed] \arrow[dr] & \Gamma.A.B \arrow[d] \\ & \Gamma.A \end{tikzcd}$$

$$\Sigma_A \dashv p_A^* \dashv \Pi_A$$

after switching to "display map" presentation.

Caveat: Σ_A / Π_A only defined on types / display maps!

$$\mathrm{Tm}(\Gamma, A) \simeq \left\{ \begin{tikzcd} \Gamma.A \arrow[d] \\ \Gamma \arrow[u, dashed, bend left] \end{tikzcd} \text{ section} \right\}$$

Lifting perspective on Id

$$\begin{tikzcd} \Gamma.A \arrow[r, "d"] \arrow[d, "\mathrm{refl}_A"'] & \Gamma.A.A.\mathrm{Id}_A.C \arrow[d, "p_C"] \\ \Gamma.A.A.\mathrm{Id}_A \arrow[r, equal] \arrow[ur, dashed, "J_{C,d}"] & \Gamma.A.A.\mathrm{Id}_A \end{tikzcd}$$

$C \in \mathrm{Ty}(\Gamma.A.A.\mathrm{Id}_A)$ motive

$d \in \mathrm{Tm}(\Gamma.A, \cancel{C[\langle p_A, q, \ldots]} \; C[\mathrm{refl}_A])$ witness

Universe

$\mathcal{U} \in Ty(\Gamma)$, $El \in Ty(\Gamma.\mathcal{U})$
natural in Γ

Equivalently:
$\mathcal{U} \in Ty(1)$, $El \in Ty(1.\mathcal{U})$

$A \in Tm(\Gamma, \mathcal{U}) \longmapsto El[\langle id_\Gamma, A\rangle]$
decodes elements of $\mathcal{U}$ into types.

$Ty_{\mathcal{U}} \in \widehat{\mathcal{C}}$ $\qquad$ $Tm_{\mathcal{U}} \in \widehat{\int Ty_{\mathcal{U}}}$

$Ty_{\mathcal{U}}(\Gamma) = Tm(\Gamma, \mathcal{U})$ $\qquad$ $Tm_{\mathcal{U}}(A) = Tm(\Gamma, El[\langle id_\Gamma, A\rangle])$

defines <u>cwf structure on $\mathcal{C}$ induced by $\mathcal{U}$.</u>

$\mathcal{U}$ closed under type formers means:
(1) induced cwf structure has type formers, $\quad Ty_{\mathcal{U}}$
(2) map of cwf structures preserves type formers. $\quad Ty_{\mathcal{U}} \to Ty$

Can abstractly define cumulative hierachy using morphisms of cwf structures.

Axioms (FunExt + univalence)

Witnessed by a term of the type for the axiom (naturally in the context).

Examples

- Set as cwf:

  $Ty(\Gamma) = \Gamma \to Set$

  $Tm(\Gamma, A) = (x : \Gamma) \to A(x)$

- Groupoids as cwf:

  $Ty(\Gamma) = \Gamma \to Gpd$

  $$Tm(\Gamma, A) = \left\{ \begin{tikzcd} \int A \arrow[d] \\ \Gamma \arrow[u, bend left, dashed] \end{tikzcd} \right\}$$

  Split fibration model.
  There is also the (cloven) fibration model: pseudofunctors $\Gamma \to Gpd$.

- Cube category $\square$:

  $Ty(X) = 1$

  $Tm(X, *) = \square(X, I)$

  for interval I generating $\square$

- For a cwf $\mathcal{C}$, the <u>core</u> $\mathcal{C}_{fib}$ has
  - objects $Ty(1)$
  - maps $1.A \dashrightarrow 1.B$
  - $Ty_{fib}(A) = Ty(1.A)$

- For a cwf $\mathcal{C}$, and $\Gamma \in \mathcal{C}$, the slice $\mathcal{C} \downarrow \Gamma$ inherits a cwf structure.

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