Algorithm Analysis

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What is an Algorithm?

Definition

"A clearly specified set of simple instructions to be followed to solve a problem."

Important properties

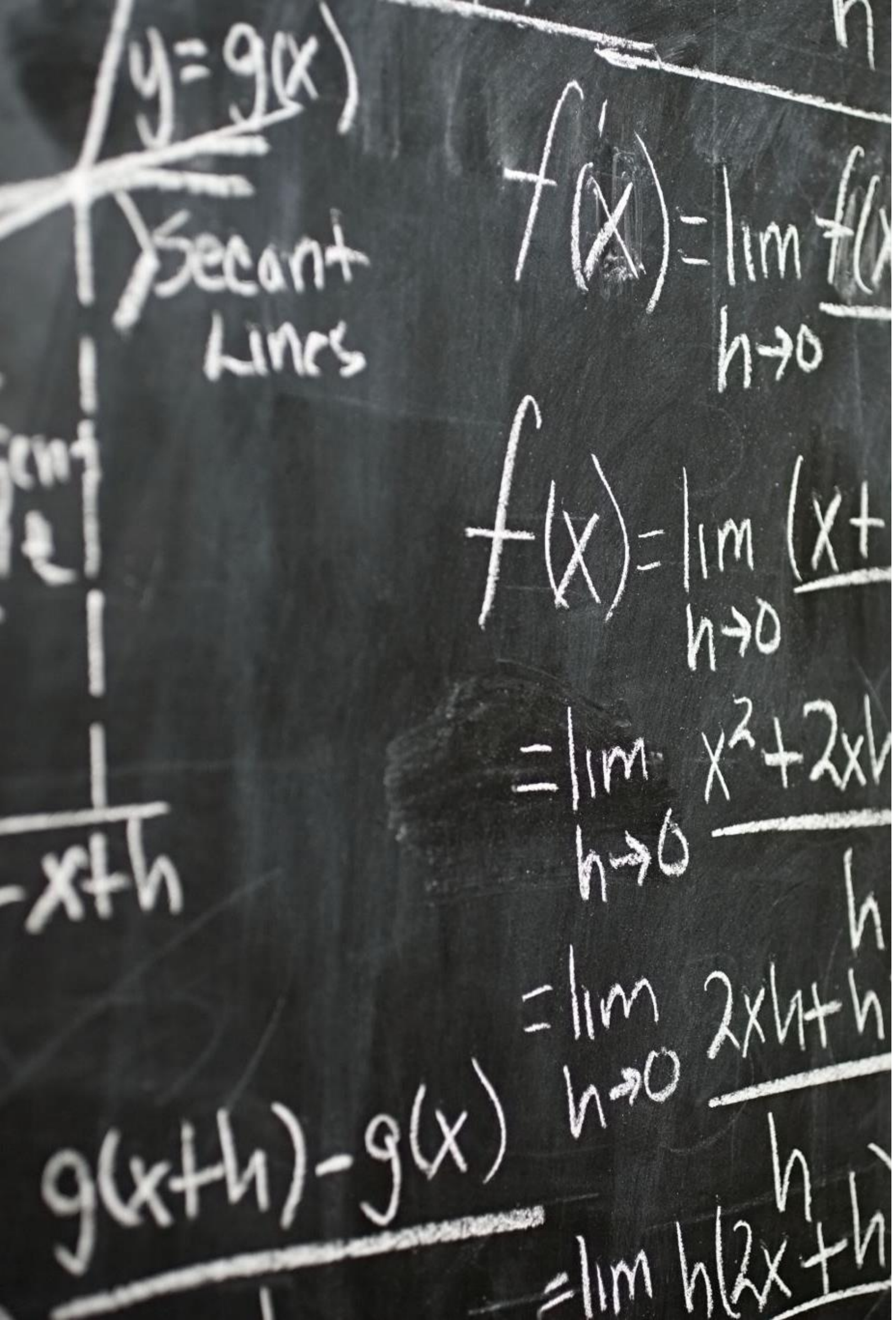
- Correctness
- **Run time** **- time complexity**
- Memory requirement - space complexity

How do we assess/compare time complexity?

Issue: Runtime and memory requirements are machine/programming language/compiler dependent!

Approach

1. Assume that each instruction takes one time unit.
2. Define the problem size (typically called N) and look at the relative growth rate when N becomes large.
3. We typically analyze the worst-case behavior.



Mathematical Definitions

Big-Oh
(upper bound = worst case, i.e. the growth in not more than f)

$\boldsymbol{T(N) = O(f(N))}$ **if there is a** $\boldsymbol{c}$ **for which** $\boldsymbol{T(N) \leq cf(N)}$
for $\boldsymbol{N > n_0}$

Most common

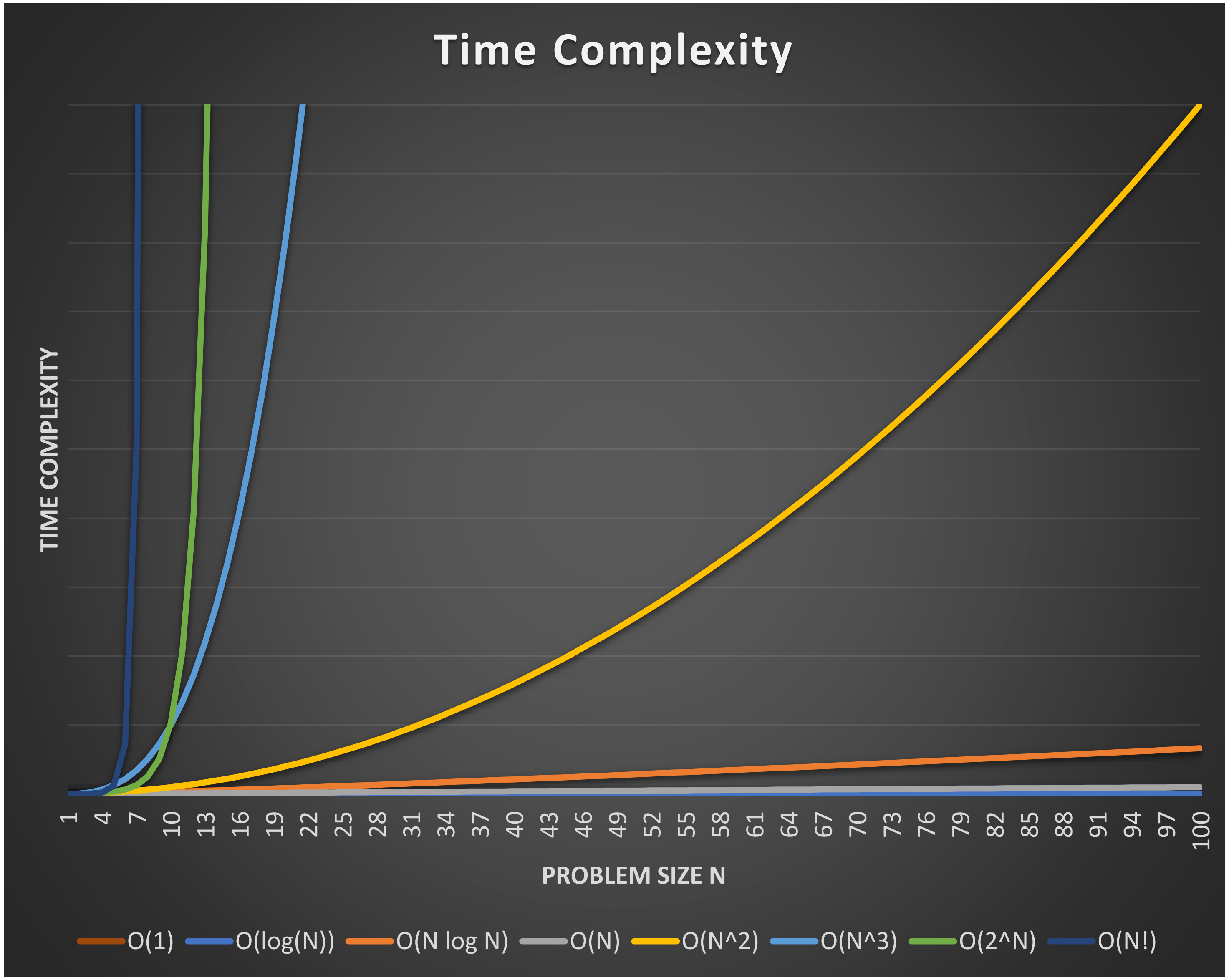
Omega
(lower bound, i.e. the growth in not less than g)

$T(N) = \Omega(g(N))$ if there is a c for which $T(N) \geq cg(N)$
for $N > n_0$

Theta
(exact growth rate is h)

$T(N) = \Theta(h(N))$ if $T(N) = O(h(N))$ and $\mathrm{T(N)} = \Omega(h(N))$

Comparison of different complexity functions f



Some Rules

1. Ignore constants

$$f(N) = 5 + 3N^2 \rightarrow O(N^2)$$

2. Only keep the highest degree since it grows the fastest with N

$$f(N) = N^2 + N^3 \rightarrow O(N^3)$$

3. Sequence: $O(f(N)) + O(g(N)) = O(\max[f(N), g(N)])$ $\rightarrow$ the worst part counts.

4. Nested loops: $O(f(N)) * O(g(N)) = O(f(N)g(N))$ $\rightarrow$ do $g(N)$ $f(N)$ times.

For code this means

- Only loops really count: k nested loops over N $\rightarrow O(N^k)$
- Consecutive code blocks: only the most complex block counts.
- Conditional code blocks: count the most complex path (worst case)

Some Useful Series

$$\sum_{i=0}^{N} 2^i = 2^{N+1} - 1$$

$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2} \approx \frac{N^2}{2}$$

$$\sum_{i=1}^{N} i^k \approx \frac{N^{k+1}}{|k+1|} \; for\; k \neq -1$$

$$\sum_{i=1}^{N} \frac{1}{i} \approx \log_e N$$

Example: Bubble Sort

```
// n is the length of the arr
void bubbleSort(int arr[], int n)
{
    int i, j;
    for (i = 0; i < n - 1; ++i)

        // Last i elements are already
        // in place
        for (j = 0; j < n - i - 1; ++j)
            if (arr[j] > arr[j + 1])
                swap(arr[j], arr[j + 1]);
}
```

Worst-case time complexity?

Worst-case space complexity?

Example: Bubble Sort

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            if (arr[j] > arr[j + 1])
                swap(arr[j], arr[j + 1]);
}
```

Worst-case time complexity?

i = 1 : n – 1 iterations in j-loop
i = 2 : n – 2
i = 3 : n – 3
…
i = n - 2 : n – (n - 2) – 1 = 1
i = n - 1 : n – (n - 1) – 1 = 0

$$\sum_{i=0}^{n-1} i = \frac{(n-1)(n-1+1)}{2} = \frac{n(n-1)}{2} = \frac{n^2}{2} - \frac{n}{2} \rightarrow O(n^2)$$

Worst-case space complexity?

Only memory for i and j is needed $\rightarrow O(1)$

Example: Linear Search

```
template <typename Comparable>
int linearSearch(
  const vector<Comparable> & a,
  const Comparable & x )
{
    for(int i = 0; i < a.size(); ++i)
    {
        if( a[ i ] == x )
            return i;
    }
    return -1
}
```

Find the index of a value in an array/vector.

Time complexity?

Example: Linear Search

```
template <typename Comparable>
int linearSearch(
  const vector<Comparable> & a,
  const Comparable & x )
{
    for(int i = 0; i < a.size(); ++i)
    {
        if( a[ i ] == x )
            return i;
    }
    return -1
}
```

Time complexity?

1. Problem size n = size of a.
2. Worse case is if we do not find the element x. We have to iterate n times $\rightarrow O(n)$

Example: Binary Search

```
template <typename Comparable>
int binarySearch(
  const vector<Comparable> & a,
  const Comparable & x )
{
    int low = 0, high = a.size( ) - 1;

    while( low <= high )
    {
        int mid = ( low + high ) / 2;

        if( a[ mid ] < x )
            low = mid + 1;
        else if( a[ mid ] > x )
            high = mid - 1;
        else
            return mid;   // Found
    }
    return -1
}
```

Find the index of a value in a presorted array/vector.

Time complexity?

Example: Binary Search

```
template <typename Comparable>
int binarySearch( const vector<Comparable> & a,
                  const Comparable & x )
{
   int low = 0, high = a.size( ) - 1;

   while( low <= high )
   {
      int mid = ( low + high ) / 2;

      if( a[ mid ] < x )
         low = mid + 1;
      else if( a[ mid ] > x )
         high = mid - 1;
      else
         return mid;   // Found
   }
   return -1
}
```

Time complexity?

1. Problem size n = size of a.
2. Worse case is if we do not find the element x.
3. In every iteration, high-low halves so we get:
$$n, \frac{n}{2}, \frac{n}{4}, \frac{n}{8}, \ldots\ 1$$
4. How many steps (k) does it take to go form n to 1?
$$\frac{n}{2^k} = 1$$
$$n = 2^k$$
$$k = \log_2(n) = \frac{\log(n)}{\log(2)}$$

$\rightarrow O(\log n)$

General rule: If every iteration halves the problem size, then we have a logarithmic time complexity.

Example With More Complicated Problem Size

- Given is a list of numbers:

  6, 88, 12, 5, 6 ,7, 93, 8

- Check which of another set of numbers is in the list. E.g.,

  7, 0, 12

O(?)

Example With More Complicated Problem Size

- Given is a list of numbers:

  $\underbrace{6, 88, 12, 5, 6, 7, 93, 8}_{n\ =\ 8}$

- Check which of another set of numbers is in the list. E.g.,

  $\underbrace{7, 0, 12}_{m\ =\ 3}$

Algorithm: Check m times n elements (worst case for linear search)

$$O(n \times m)$$

If $n > m$ then we can also say:

$$O(n^2)$$

Note: Sorting the first list and then using binary search would be a better idea if both n and m are very large.

Some Final Words on Complexity

- Complexity analysis points to the part of the code (algorithms and data structures) that would benefit from **algorithmic optimization** the most.
- **Space complexity** analysis looks at how the memory need grows with n.

Note:

- Better complexity does not mean faster! Complexity analysis ignores potentially large constants and factors in the **runtime.**
- **Big-Oh looks at the worst case.** Average case analysis is typically much harder to do.
- **Assumption** that each operation takes the same amount of time is very strong. Memory access is different for cash memory vs hard drive.
- Complexity analysis looks at the **algorithm** and bad implementations (e.g., copying arrays unnecessarily) may lead to worse run time.