In the GG02 protocol, Alice prepares an ensemble of coherent states $|\alpha = q + ip\rangle$ with probabilities $p_{\mathrm{A}}(\alpha)$ distributed according to Gaussian law,

$$p_{\mathrm{A}}(\alpha) = \frac{1}{\sqrt{\pi V_{\mathrm{A}}}} \exp\left[-\frac{|\alpha|^2}{2V_{\mathrm{A}}}\right] \tag{1}$$

i.e.

$$\rho_{\mathrm{A}} = \int \mathrm{d}^2\alpha \, p_{\mathrm{A}}(\alpha) \, |\alpha\rangle\langle\alpha| \tag{2}$$

! As the phase-rotated quadrature operator is defined as (see preprint):

$$\hat{x}_\varphi = \hat{a}e^{-i\varphi} + \hat{a}^\dagger e^{i\varphi}, \tag{3}$$

i.e.

$$\hat{q} = \hat{a} + \hat{a}^\dagger, \quad \hat{p} = -i\left[\hat{a} - \hat{a}^\dagger\right], \tag{4}$$

the first two moments of each component of α read

$$\langle\hat{q}\rangle = \mathrm{Tr}\,\hat{q}\rho_{\mathrm{A}} = \int \mathrm{d}^2\alpha \, p_{\mathrm{A}}(\alpha) \underbrace{\langle\alpha|\,\hat{q}\,|\alpha\rangle}_{2\,\mathrm{Re}\,\alpha = 2q} = 0, \tag{5a}$$

$$\langle\hat{q}^2\rangle = \mathrm{Tr}\,\hat{q}^2\rho_{\mathrm{A}} = \int \mathrm{d}^2\alpha \, p_{\mathrm{A}}(\alpha) \underbrace{\langle\alpha|\,\hat{q}^2\,|\alpha\rangle}_{4q^2+1} = 4\underbrace{\int \mathrm{d}^2\alpha \, p_{\mathrm{A}}(\alpha) q^2}_{V_{\mathrm{A}}} + 1 = 4V_{\mathrm{A}} + 1, \tag{5b}$$

$$\langle\hat{p}\rangle = 0, \tag{5c}$$

$$\langle\hat{p}^2\rangle = 4V_{\mathrm{A}} + 1. \tag{5d}$$

After transmission through a Gaussian channel, which attenuates the coherent amplitude by a factor of $\sqrt{T}$, where T is the channel transmission, the state transforms as $|\alpha\rangle \mapsto \left|\sqrt{T}\alpha\right\rangle \equiv |\tilde{\alpha} = \tilde{q} + i\tilde{p}\rangle$. The ensemble reads

$$\tilde{\rho}_{\mathrm{A}} = \frac{1}{T} \int \mathrm{d}^2\tilde{\alpha} \, p_{\mathrm{A}}(\tilde{\alpha}) \, |\tilde{\alpha}\rangle\langle\tilde{\alpha}| \,. \tag{6}$$

Then, Bob performs a measurement described by POVM $\{\hat{\Pi}_x\}$, where the index x parametrizes the measurement outcomes (quadrature values q and p in homodyne detection). The conditional probability that Bob obtains measurement outcome x given that Alice sent the specific coherent state $|\tilde{\alpha}\rangle$ is given by the Born rule:

$$p_{\mathrm{B}}(x|\tilde{\alpha}) = p_{\mathrm{A}}(\tilde{\alpha}) \, \mathrm{Tr}\left[|\tilde{\alpha}\rangle\langle\tilde{\alpha}|\,\hat{\Pi}_x\right] = p_{\mathrm{A}}(\tilde{\alpha}) Q_x(\tilde{\alpha}), \tag{7}$$

where $Q_x(\tilde{\alpha})$ is the Q-function of POVM used.

1 Homodyne

If homodyne detection is used

$$Q_x(\tilde{\alpha}) = \frac{1}{\sqrt{2\pi\sigma_G}} \exp\Bigg[-\frac{(x - \overbrace{\langle\hat{x}_\varphi\rangle}^{2\tilde{q}})^2}{2\sigma_x}\Bigg], \tag{8}$$

$$p_{\mathrm{B}}(x = q|\tilde{\alpha}) = p_{\mathrm{A}}(\tilde{\alpha}) Q_x(\tilde{\alpha}) \sim \exp\left[-\frac{(x - 2\tilde{q})^2}{2\sigma_x} - \frac{\tilde{q}^2}{2TV_{\mathrm{A}}}\right]. \tag{9}$$

Rewriting exponential's power in Eq.(9) in quadratic form results in

$$\begin{aligned} -\frac{1}{2}\left[\frac{(x-2\tilde{q})^2}{\sigma_x} + \frac{\tilde{q}^2}{TV_{\mathrm{A}}}\right] &= -\frac{1}{2}\left[\frac{x^2 - 4x\tilde{q} + 4\tilde{q}^2}{\sigma_x} + \frac{\tilde{q}^2}{TV_{\mathrm{A}}}\right] = \\ &= -\frac{1}{2}\left[\frac{x^2}{\sigma_x} - x\tilde{q}\frac{4}{\sigma_x} + \tilde{q}^2\left(\frac{4}{\sigma_x} + \frac{1}{TV_{\mathrm{A}}}\right)\right] = \\ &= -\frac{1}{2}\begin{pmatrix}\tilde{q} & x\end{pmatrix}\begin{pmatrix}\frac{4}{\sigma_x} + \frac{1}{TV_{\mathrm{A}}} & -\frac{2}{\sigma_x} \\ -\frac{2}{\sigma_x} & \frac{1}{\sigma_x}\end{pmatrix}\begin{pmatrix}\tilde{q} \\ x\end{pmatrix} = \\ = /\det{}^{-1} = \sigma_x TV_{\mathrm{A}}/ = -\frac{1}{2}\begin{pmatrix}\tilde{q} & x\end{pmatrix} &\underbrace{\begin{pmatrix}TV_{\mathrm{A}} & 2TV_{\mathrm{A}} \\ 2TV_{\mathrm{A}} & 4TV_{\mathrm{A}} + \sigma_x\end{pmatrix}^{-1}}_{\Sigma^{\mathrm{HOM}}}\begin{pmatrix}\tilde{q} \\ x\end{pmatrix}, \end{aligned} \tag{10}$$

Mutual information between Alice and Bob can be calculated as follows [1]

$$\boxed{I_{\mathrm{AB}}^{\mathrm{HOM}} = \frac{1}{2}\log\frac{\Sigma_{11}^{\mathrm{HOM}}\Sigma_{22}^{\mathrm{HOM}}}{\det\Sigma^{\mathrm{HOM}}} = \frac{1}{2}\log\frac{4TV_{\mathrm{A}} + \sigma_x}{\sigma_x}.} \tag{11}$$

This result is fine, as in Ref. [2] $\hat{q}$ and $\hat{p}$ are defined with division by 2 (see Eq. (10) in Ref.), and division by 2 corresponds to variable change $V_{\mathrm{A}} \mapsto \frac{V_{\mathrm{A}}}{4}$ (see Eq. (5), resulting in formula (50) from Ref.

2 Double homodyne

If double homodyne detection is used, i.e.

$$Q_x(\tilde{\alpha}) = \frac{1}{2\pi\sqrt{\sigma_G^{(1)}\sigma_G^{(2)}}} \exp\left[-\frac{(x_1 - \tilde{q})^2}{\sigma_1} - \frac{(x_2 - \tilde{p})^2}{\sigma_2}\right], \tag{12}$$

$$p_{\mathrm{A}}(\tilde{\alpha}) Q_x(\tilde{\alpha}) \sim \exp\left[-\frac{(x_1 - \tilde{q})^2}{\sigma_1} - \frac{\tilde{q}^2}{2TV_{\mathrm{A}}} - \frac{(x_2 - \tilde{p})^2}{\sigma_2} - \frac{\tilde{p}^2}{2TV_{\mathrm{A}}}\right], \tag{13}$$

analogously,

$$-\frac{(x_1-\tilde{q})^2}{\sigma_1}-\frac{\tilde{q}^2}{2TV_\mathrm{A}} = -\frac{1}{2}\begin{pmatrix}\tilde{q} & x_1\end{pmatrix}\begin{pmatrix}\frac{2}{\sigma_1}+\frac{1}{TV_\mathrm{A}} & -\frac{2}{\sigma_1}\\ -\frac{2}{\sigma_1} & \frac{2}{\sigma_1}\end{pmatrix}\begin{pmatrix}\tilde{q}\\ x_1\end{pmatrix} =$$

$$= \Big/ \det{}^{-1} = \frac{\sigma_1 TV_\mathrm{A}}{2} \Big/ = -\frac{1}{2}\begin{pmatrix}\tilde{q} & x_1\end{pmatrix}\underbrace{\begin{pmatrix}TV_\mathrm{A} & TV_\mathrm{A}\\ TV_\mathrm{A} & TV_\mathrm{A}+\frac{\sigma_1}{2}\end{pmatrix}^{-1}}_{\Sigma^{\mathrm{DH}^{(1)}}}\begin{pmatrix}\tilde{q}\\ x_1\end{pmatrix}, \tag{14}$$

$$-\frac{(x_2-\tilde{p})^2}{\sigma_1}-\frac{\tilde{q}^2}{TV_\mathrm{A}} = -\frac{1}{2}\begin{pmatrix}\tilde{p} & x_2\end{pmatrix}\underbrace{\begin{pmatrix}TV_\mathrm{A} & TV_\mathrm{A}\\ TV_\mathrm{A} & TV_\mathrm{A}+\frac{\sigma_2}{2}\end{pmatrix}^{-1}}_{\Sigma^{\mathrm{DH}^{(2)}}}\begin{pmatrix}\tilde{p}\\ x_2\end{pmatrix}, \tag{15}$$

$$\boxed{I_\mathrm{AB}^\mathrm{DH} = \frac{1}{2}\sum_{i=1,2}\log\frac{\Sigma_{11}^{\mathrm{DH}^{(i)}}\Sigma_{22}^{\mathrm{DH}^{(i)}}}{\det\Sigma^{\mathrm{DH}^{(i)}}} = \frac{1}{2}\sum_i \log\frac{2TV_\mathrm{A}+\sigma_i}{\sigma_i}.} \tag{16}$$

This result is fine by the same reasoning.

3 Channel noise

Channel noise of variance ξ could be modeled in Eq. (9) as

$$p_\mathrm{B}(x|\alpha) = p_\mathrm{A}(\tilde{\alpha})Q_x(\tilde{\alpha}) \sim \exp\left[-\frac{(x-2\tilde{q})^2}{2\sigma_x}-\frac{\tilde{q}^2}{2(TV_\mathrm{A}+\xi)}\right], \tag{17}$$

resulting in

$$I_\mathrm{AB}^\mathrm{HOM} = \frac{1}{2}\log\frac{\Sigma_{11}^\mathrm{HOM}\Sigma_{22}^\mathrm{HOM}}{\det\Sigma^\mathrm{HOM}} = \frac{1}{2}\log\frac{TV_\mathrm{A}+\sigma_x+\xi}{\sigma_x+\xi}. \tag{18}$$

With channel noise of variance ξ:

$$I_\mathrm{AB}^\mathrm{DH} = \frac{1}{2}\sum_i \log\frac{2TV_\mathrm{A}+\sigma_i+\xi}{\sigma_i+\xi}. \tag{19}$$

Just add to covariance matrix?

References

[1] Joram Soch et al. *StatProofBook/StatProofBook.github.io: The Book of Statistical Proofs (Version 2023)*. `https://doi.org/10.5281/zenodo.4305949`. Accessed: 2025-07-15. 2024.

[2] Fabian Laudenbach et al. "Continuous-variable quantum key distribution with Gaussian modulation—the theory of practical implementations". In: *Advanced Quantum Technologies* 1.1 (2018), p. 1800011.