

In the GG02 protocol, Alice prepares an ensemble of coherent states $|\alpha = q + ip\rangle$ with probabilities $p_A(\alpha)$ distributed according to Gaussian law,

$$p_A(\alpha) = \frac{1}{2\pi V_A} \exp \left[-\frac{|\alpha|^2}{2V_A} \right] \quad (1)$$

i.e.

$$\rho_A = \int d^2\alpha p_A(\alpha) |\alpha\rangle\langle\alpha| \quad (2)$$

! As the phase-rotated quadrature operator is defined as (see preprint):

$$\hat{x}_\varphi = \hat{a}e^{-i\varphi} + \hat{a}^\dagger e^{i\varphi}, \quad (3)$$

i.e.

$$\hat{q} = \hat{a} + \hat{a}^\dagger, \quad \hat{p} = -i [\hat{a} - \hat{a}^\dagger], \quad (4)$$

the first two moments of each component of α read

$$\langle \hat{q} \rangle = \text{Tr} \hat{q} \rho_A = \int d^2\alpha p_A(\alpha) \underbrace{\langle \alpha | \hat{q} | \alpha \rangle}_{2 \text{Re } \alpha = 2q} = 0, \quad (5a)$$

$$\langle \hat{q}^2 \rangle = \text{Tr} \hat{q}^2 \rho_A = \int d^2\alpha p_A(\alpha) \underbrace{\langle \alpha | \hat{q}^2 | \alpha \rangle}_{4q^2+1} = 4 \underbrace{\int d^2\alpha p_A(\alpha) q^2}_{V_A} + 1 = 4V_A + 1, \quad (5b)$$

$$\langle \hat{p} \rangle = 0, \quad (5c)$$

$$\langle \hat{p}^2 \rangle = 4V_A + 1. \quad (5d)$$

After transmission through a Gaussian channel, which attenuates the coherent amplitude by a factor of \sqrt{T} , where T is the channel transmission, the state transforms as $|\alpha\rangle \mapsto |\sqrt{T}\alpha\rangle \equiv |\tilde{\alpha} = \tilde{q} + i\tilde{p}\rangle$. The ensemble reads

$$\tilde{\rho}_A = \frac{1}{T} \int d^2\tilde{\alpha} p_A(\tilde{\alpha}) |\tilde{\alpha}\rangle\langle\tilde{\alpha}|. \quad (6)$$

Then, Bob performs a measurement described by POVM $\{\hat{\Pi}_x\}$, where the index x parametrizes the measurement outcomes (quadrature values q and p in homodyne detection). The conditional probability that Bob obtains measurement outcome x given that Alice sent the specific coherent state $|\tilde{\alpha}\rangle$ is given by the Born rule:

$$p_B(x|\tilde{\alpha}) = p_A(\tilde{\alpha}) \text{Tr} \left[|\tilde{\alpha}\rangle\langle\tilde{\alpha}| \hat{\Pi}_x \right] = p_A(\tilde{\alpha}) Q_x(\tilde{\alpha}), \quad (7)$$

where $Q_x(\tilde{\alpha})$ is the Q -function of POVM used.

1 Homodyne

If homodyne detection is used

$$Q_x(\tilde{\alpha}) = \frac{1}{\sqrt{2\pi}\sigma_G} \exp\left[-\frac{(x - \overbrace{\langle \hat{x}_\varphi \rangle}^{2\tilde{q}})^2}{2\sigma_x}\right], \quad (8)$$

$$p_B(x = q|\tilde{\alpha}) = p_A(\tilde{\alpha})Q_x(\tilde{\alpha}) \sim \exp\left[-\frac{(x - 2\tilde{q})^2}{2\sigma_x} - \frac{\tilde{q}^2}{2TV_A}\right]. \quad (9)$$

Rewriting exponential's power in Eq.(9) in quadratic form results in

$$\begin{aligned} -\frac{1}{2} \left[\frac{(x - 2\tilde{q})^2}{\sigma_x} + \frac{\tilde{q}^2}{TV_A} \right] &= -\frac{1}{2} \left[\frac{x^2 - 4x\tilde{q} + 4\tilde{q}^2}{\sigma_x} + \frac{\tilde{q}^2}{TV_A} \right] = \\ &= -\frac{1}{2} \left[\frac{x^2}{\sigma_x} - x\tilde{q}\frac{4}{\sigma_x} + \tilde{q}^2 \left(\frac{4}{\sigma_x} + \frac{1}{TV_A} \right) \right] = \\ &= -\frac{1}{2} \begin{pmatrix} \tilde{q} & x \end{pmatrix} \begin{pmatrix} \frac{4}{\sigma_x} + \frac{1}{TV_A} & -\frac{2}{\sigma_x} \\ -\frac{2}{\sigma_x} & \frac{1}{\sigma_x} \end{pmatrix} \begin{pmatrix} \tilde{q} \\ x \end{pmatrix} = \\ &= / \det^{-1} = \sigma_x TV_A / = -\frac{1}{2} \begin{pmatrix} \tilde{q} & x \end{pmatrix} \underbrace{\begin{pmatrix} TV_A & 2TV_A \\ 2TV_A & 4TV_A + \sigma_x \end{pmatrix}^{-1}}_{\Sigma^H} \begin{pmatrix} \tilde{q} \\ x \end{pmatrix}, \end{aligned} \quad (10)$$

Mutual information between Alice and Bob can be calculated as follows [1]

$$I_{AB}^H = \frac{1}{2} \log \frac{\Sigma_{11}^H \Sigma_{22}^H}{\det \Sigma^H} = \frac{1}{2} \log \frac{4TV_A + \sigma_x}{\sigma_x}. \quad (11)$$

This result is fine, as in Ref. [2] modulation variance is defined per quadrature component (see Eq. (8) in Ref.), corresponding to variable change $V_A \mapsto \frac{V_A}{4}$ (see Eq. (5), resulting in formula (50) from Ref.

2 Double homodyne

If double homodyne detection is used, i.e.

$$Q_x(\tilde{\alpha}) = \frac{1}{2\pi\sqrt{\sigma_G^{(1)}\sigma_G^{(2)}}} \exp\left[-\frac{(x_1 - \tilde{q})^2}{\sigma_1} - \frac{(x_2 - \tilde{p})^2}{\sigma_2}\right], \quad (12)$$

$$p_A(\tilde{\alpha})Q_x(\tilde{\alpha}) \sim \exp\left[-\frac{(x_1 - \tilde{q})^2}{\sigma_1} - \frac{\tilde{q}^2}{2TV_A} - \frac{(x_2 - \tilde{p})^2}{\sigma_2} - \frac{\tilde{p}^2}{2TV_A}\right], \quad (13)$$

analogously,

$$\begin{aligned} & -\frac{(x_1 - \tilde{q})^2}{\sigma_1} - \frac{\tilde{q}^2}{2TV_A} = -\frac{1}{2} \begin{pmatrix} \tilde{q} & x_1 \end{pmatrix} \begin{pmatrix} \frac{2}{\sigma_1} + \frac{1}{TV_A} & -\frac{2}{\sigma_1} \\ -\frac{2}{\sigma_1} & \frac{2}{\sigma_1} \end{pmatrix} \begin{pmatrix} \tilde{q} \\ x_1 \end{pmatrix} = \\ & = \left/ \det^{-1} = \frac{\sigma_1 TV_A}{2} \right/ = -\frac{1}{2} \begin{pmatrix} \tilde{q} & x_1 \end{pmatrix} \underbrace{\begin{pmatrix} TV_A & TV_A \\ TV_A & TV_A + \frac{\sigma_1}{2} \end{pmatrix}^{-1}}_{\Sigma^{\text{DH}(1)}} \begin{pmatrix} \tilde{q} \\ x_1 \end{pmatrix}, \end{aligned} \quad (14)$$

$$-\frac{(x_2 - \tilde{p})^2}{\sigma_1} - \frac{\tilde{q}^2}{TV_A} = -\frac{1}{2} \begin{pmatrix} \tilde{p} & x_2 \end{pmatrix} \underbrace{\begin{pmatrix} TV_A & TV_A \\ TV_A & TV_A + \frac{\sigma_2}{2} \end{pmatrix}^{-1}}_{\Sigma^{\text{DH}(2)}} \begin{pmatrix} \tilde{p} \\ x_2 \end{pmatrix}, \quad (15)$$

$$I_{\text{AB}}^{\text{DH}} = \frac{1}{2} \sum_{i=1,2} \log \frac{\Sigma_{11}^{\text{DH}(i)} \Sigma_{22}^{\text{DH}(i)}}{\det \Sigma^{\text{DH}(i)}} = \frac{1}{2} \sum_i \log \frac{2TV_A + \sigma_i}{\sigma_i}. \quad (16)$$

This result is fine by the same reasoning.

3 Channel noise

Channel noise is modeled as independent Gaussian noise of variance ξ , modifying Eq. (6) as

$$\tilde{\rho}_A = \frac{1}{T} \int d^2\alpha' d^2\tilde{\alpha} \exp \left[-\frac{|\tilde{\alpha} - \alpha'|^2}{2\xi} \right] p_A(\tilde{\alpha}) |\tilde{\alpha}\rangle \langle \tilde{\alpha}| \quad (17)$$

$$p_B(x|\alpha) \sim \exp \left[-\frac{(x - 2\tilde{q})^2}{2(\sigma_x + \xi)} - \frac{\tilde{q}^2}{2TV_A} \right], \quad (18)$$

$$\begin{aligned} & -\frac{1}{2} \left[\frac{(x - 2\tilde{q})^2}{\sigma_x + \xi} + \frac{\tilde{q}^2}{TV_A} \right] = \\ & = -\frac{1}{2} \left[\frac{x^2}{\sigma_x + \xi} - x\tilde{q} \frac{4}{\sigma_x + \xi} + \tilde{q}^2 \left(\frac{4}{\sigma_x + \xi} + \frac{1}{TV_A} \right) \right] = \\ & \quad -\frac{1}{2} \begin{pmatrix} \tilde{q} & x \end{pmatrix} \begin{pmatrix} \frac{4}{\sigma_x + \xi} + \frac{1}{TV_A} & \frac{2}{\sigma_x + \xi} \\ \frac{2}{\sigma_x + \xi} & \frac{1}{\sigma_x + \xi} \end{pmatrix} \begin{pmatrix} \tilde{q} \\ x \end{pmatrix} = \\ & = \left/ \det^{-1} = TV_A(\sigma_x + \xi) \right/ = -\frac{1}{2} \begin{pmatrix} \tilde{q} & x \end{pmatrix} \underbrace{\begin{pmatrix} TV_A & 2TV_A \\ 2TV_A & 4TV_A + \sigma_x + \xi \end{pmatrix}^{-1}}_{\Sigma^{\text{H}}} \begin{pmatrix} \tilde{q} \\ x \end{pmatrix}, \end{aligned} \quad (19)$$

resulting in

$$I_{AB}^H = \frac{1}{2} \log \frac{\Sigma_{11}^H \Sigma_{22}^H}{\det \Sigma^H} = \frac{1}{2} \log \frac{TV_A + \sigma_x + \xi}{\sigma_x + \xi} \quad (20)$$

and for double homodyne, we have

$$p_A(\tilde{\alpha}) Q_x(\tilde{\alpha}) \sim \exp \left[-\frac{(x_1 - \tilde{q})^2}{2 \left(\frac{\sigma_1}{2} + \xi \right)} - \frac{\tilde{q}^2}{2TV_A} - \frac{(x_2 - \tilde{p})^2}{2 \left(\frac{\sigma_2}{2} + \xi \right)} - \frac{\tilde{p}^2}{2TV_A} \right], \quad (21)$$

$$\begin{aligned} -\frac{(x_1 - \tilde{q})^2}{2 \left(\frac{\sigma_1}{2} + \xi \right)} - \frac{\tilde{q}^2}{2TV_A} &= -\frac{1}{2} \begin{pmatrix} \tilde{q} & x_1 \end{pmatrix} \begin{pmatrix} \frac{2}{\sigma_1 + 2\xi} + \frac{1}{TV_A} & -\frac{2}{\sigma_1 + 2\xi} \\ -\frac{2}{\sigma_1 + 2\xi} & \frac{2}{\sigma_1 + 2\xi} \end{pmatrix} \begin{pmatrix} \tilde{q} \\ x_1 \end{pmatrix} = \\ &= \left/ \det^{-1} = \frac{TV_A (\sigma_1 + 2\xi)}{2} \right/ = -\frac{1}{2} \begin{pmatrix} \tilde{q} & x_1 \end{pmatrix} \underbrace{\begin{pmatrix} TV_A & TV_A \\ TV_A & TV_A + \frac{\sigma_1}{2} + \xi \end{pmatrix}^{-1}}_{\Sigma^{\text{DH}(1)}} \begin{pmatrix} \tilde{q} \\ x_1 \end{pmatrix}, \end{aligned} \quad (22)$$

$$-\frac{(x_2 - \tilde{p})^2}{\sigma_1} - \frac{\tilde{p}^2}{TV_A} = -\frac{1}{2} \begin{pmatrix} \tilde{p} & x_2 \end{pmatrix} \underbrace{\begin{pmatrix} TV_A & TV_A \\ TV_A & TV_A + \frac{\sigma_2}{2} + \xi \end{pmatrix}^{-1}}_{\Sigma^{\text{DH}(2)}} \begin{pmatrix} \tilde{p} \\ x_2 \end{pmatrix}, \quad (23)$$

$$I_{AB}^{\text{DH}} = \frac{1}{2} \sum_i \log \frac{2TV_A + \sigma_i + 2\xi}{\sigma_i + 2\xi}. \quad (24)$$

4 Equivalency between EB and PM

The goal is to show that the prepared ensemble sent through channel is the same as Eq. (2) after some variable change.

To calculate Holevo information, first, consider two mode squeezed vacuum state (TMSVS), in ket notation written as [3]

$$|\Psi\rangle_{AB} = \sqrt{1 - \lambda^2} \sum_{n=0}^{\infty} (-\lambda)^n |n, n\rangle_{AB}, \quad (25)$$

where $\lambda = \tanh r$, r is the squeezing parameter. Let Alice hold the state $\varsigma_{AB} = |\Psi\rangle_{AB} \langle \Psi|_{AB}$. In the entanglement-based (EB) protocol, Alice measures one mode using a double measurement, which corresponds to a POVM of coherent state projectors $\left\{ \hat{\Pi}_\beta = \frac{|\beta\rangle\langle\beta|}{\pi} \right\}$, and sends the second mode to Bob. The probability that Alice observes double homodyne outcome β is given by the Born rule:

$$p_A^{\text{EB}}(\beta) = \text{Tr } \hat{\Pi}_\beta \varsigma_A, \quad (26)$$

where $\varsigma_A = \text{Tr}_B \varsigma_{AB}$ is the reduced density matrix of Alice's mode. Straightforward calculations

$$\text{Tr}_B \varsigma_{AB} = (1 - \lambda^2) \sum_{n,m} (-\lambda)^{n+m} |n\rangle_A \langle m|_A \underbrace{\text{Tr}_B |n\rangle_B \langle m|_B}_{\delta_{nm}} = (1 - \lambda^2) \sum_n \lambda^{2n} |n\rangle_A \langle n|_A \quad (27)$$

lead to the expression of the reduced matrix as

$$\varsigma_A = (1 - \lambda^2) \sum_n \lambda^{2n} |n\rangle \langle n|. \quad (28)$$

Substituting Eq. (28) into into Eq. (26) yields

$$\begin{aligned} \langle \beta | \varsigma_A | \beta \rangle &= (1 - \lambda)^2 \sum_n \lambda^{2n} \underbrace{|\langle \beta | n \rangle|^2}_{e^{-|\beta|^2} \frac{|\beta|^{2n}}{n!}} = (1 - \lambda)^2 e^{-|\beta|^2} \underbrace{\sum_n \frac{(\lambda |\beta|)^{2n}}{n!}}_{e^{(\lambda |\beta|)^2}} = \\ &= (1 - \lambda)^2 \exp [(\lambda^2 - 1) |\beta|^2], \end{aligned} \quad (29)$$

$$p_A^{\text{EB}}(\beta) = \text{Tr} \hat{\Pi}_\beta \varsigma_A = \frac{(1 - \lambda)^2}{\pi} \exp [(\lambda^2 - 1) |\beta|^2]. \quad (30)$$

After Alice obtains outcome β , second mode before channel transmission reads

$$\begin{aligned} \varsigma_B^\beta &= \frac{\text{Tr}_A [\hat{\Pi}_\beta \otimes \mathbb{I}] \varsigma_{AB}}{p_A^{\text{EB}}(\beta)} = \exp [-(\lambda^2 - 1) |\beta|^2] \sum_{nm} (-\lambda)^{n+m} \underbrace{\langle \beta | n \rangle}_{e^{-\frac{|\beta|^2}{2}} \frac{\beta^* n}{\sqrt{n!}}} \underbrace{\langle m | \beta \rangle}_{e^{-\frac{|\beta|^2}{2}} \frac{\beta^m}{\sqrt{m!}}} |n\rangle_B \langle m|_B = \\ &= \underbrace{\sum_n e^{-\frac{(\lambda |\beta|)^2}{2}} \frac{(-\lambda \beta^*)^n}{\sqrt{n!}} |n\rangle_B}_{|-\lambda \beta^*\rangle} \underbrace{\sum_m e^{-\frac{(\lambda |\beta|)^2}{2}} \frac{(-\lambda \beta)^m}{\sqrt{m!}} \langle m|_B}_{\langle -\lambda \beta^*|} = \\ &= |-\lambda \beta^*\rangle \langle -\lambda \beta^*| \end{aligned} \quad (31)$$

and the ensemble reads

$$\varsigma_B = \int d^2 \beta p_A^{\text{EB}}(\beta) |-\lambda \beta^*\rangle \langle -\lambda \beta^*|, \quad (32)$$

exactly the same as Eq. (2) after substituting

$$\alpha = -\lambda \beta^* \implies |\beta|^2 = \frac{|\alpha|^2}{\lambda^2}, \quad (33)$$

$$\frac{1 - \lambda^2}{\lambda^2} = \frac{1}{2V_A} \implies 2V_A = \frac{\lambda^2}{1 - \lambda^2} \quad (34)$$

It follows that covariance matrix of TMSVS can be used to calculate Holevo information in PM scheme. It is defined as follows

$$\Sigma^{\text{TMSVS}} = \begin{pmatrix} V\mathbb{I} & \sqrt{V^2 - 1}\sigma_Z \\ \sqrt{V^2 - 1}\sigma_Z & V\mathbb{I} \end{pmatrix}, \quad (35)$$

where $V = \cosh 2r$. Relating V to $\lambda = \tanh r$:

$$V = \frac{1 + \lambda^2}{1 - \lambda^2} = 1 + 2 \underbrace{\frac{\lambda^2}{1 - \lambda^2}}_{\text{Eq. (34)}} = 1 + 4V_A. \quad (36)$$

After transmitting through noisy channel,

$$\Sigma_{\text{ABC}} = [\mathbb{I} \oplus \text{BS}] [\Sigma^{\text{TMSVS}} \oplus N\mathbb{I}] [\mathbb{I} \oplus \text{BS}]^T, \quad (37)$$

$$\text{BS} = \begin{pmatrix} \sqrt{T}\mathbb{I} & \sqrt{1-T}\mathbb{I} \\ -\sqrt{1-T}\mathbb{I} & \sqrt{T}\mathbb{I} \end{pmatrix}, \quad N = 1 + \frac{\xi}{1-T}, \quad (38)$$

(see Ref. [2] app. C3) covariance matrix reads

$$\Sigma_{\text{AB}}^{\text{EB}} = \begin{pmatrix} V\mathbb{I} & \sqrt{T(V^2 - 1)}\sigma_Z \\ \sqrt{T(V^2 - 1)}\sigma_Z & [T(V - 1) + 1 + \xi]\mathbb{I} \end{pmatrix}. \quad (39)$$

$$N = 1 + \frac{\xi + \sigma_x}{1-T}?$$

5 Holevo information

References

- [1] Joram Soch et al. *StatProofBook/StatProofBook.github.io: The Book of Statistical Proofs (Version 2023)*. <https://doi.org/10.5281/zenodo.4305949>. Accessed: 2025-07-15. 2024.
- [2] Fabian Laudenbach et al. “Continuous-variable quantum key distribution with Gaussian modulation—the theory of practical implementations”. In: *Advanced Quantum Technologies* 1.1 (2018), p. 1800011.
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