# Chapter 2

# Divide-and-conquer algorithms

The *divide-and-conquer* strategy solves a problem by:

1. Breaking it into *subproblems* that are themselves smaller instances of the same type of problem
2. Recursively solving these subproblems
3. Appropriately combining their answers

The real work is done piecemeal, in three different places: in the partitioning of problems into subproblems; at the very tail end of the recursion, when the subproblems are so small that they are solved outright; and in the gluing together of partial answers. These are held together and coordinated by the algorithm's core recursive structure.

As an introductory example, we'll see how this technique yields a new algorithm for multiplying numbers, one that is much more efficient than the method we all learned in elementary school!

## 2.1 Multiplication

The mathematician Carl Friedrich Gauss (1777–1855) once noticed that although the product of two complex numbers

$$(a + bi)(c + di) = ac - bd + (bc + ad)i$$

seems to involve *four* real-number multiplications, it can in fact be done with just *three*: $ac$, $bd$, and $(a + b)(c + d)$, since

$$bc + ad = (a + b)(c + d) - ac - bd.$$

In our big-$O$ way of thinking, reducing the number of multiplications from four to three seems wasted ingenuity. But this modest improvement becomes very significant *when applied recursively*.

Let's move away from complex numbers and see how this helps with regular multiplication. Suppose $x$ and $y$ are two $n$-bit integers, and assume for convenience that $n$ is a power of $2$ (the more general case is hardly any different). As a first step toward multiplying $x$ and $y$, split each of them into their left and right halves, which are $n/2$ bits long:

$$x = \boxed{x_L}\boxed{x_R} = 2^{n/2}x_L + x_R$$
$$y = \boxed{y_L}\boxed{y_R} = 2^{n/2}y_L + y_R.$$

For instance, if $x = 10110110_2$ (the subscript 2 means "binary") then $x_L = 1011_2$, $x_R = 0110_2$, and $x = 1011_2 \times 2^4 + 0110_2$. The product of $x$ and $y$ can then be rewritten as

$$xy \;=\; (2^{n/2}x_L + x_R)(2^{n/2}y_L + y_R) \;=\; 2^n\, x_L y_L \;+\; 2^{n/2}\,(x_L y_R + x_R y_L) \;+\; x_R y_R.$$

We will compute $xy$ via the expression on the right. The additions take linear time, as do the multiplications by powers of 2 (which are merely left-shifts). The significant operations are the four $n/2$-bit multiplications, $x_L y_L,\ x_L y_R,\ x_R y_L,\ x_R y_R$; these we can handle by four recursive calls. Thus our method for multiplying $n$-bit numbers starts by making recursive calls to multiply these four pairs of $n/2$-bit numbers (four subproblems of half the size), and then evaluates the preceding expression in $O(n)$ time. Writing $T(n)$ for the overall running time on $n$-bit inputs, we get the *recurrence relation*

$$T(n) \;=\; 4T(n/2) + O(n).$$

We will soon see general strategies for solving such equations. In the meantime, this particular one works out to $O(n^2)$, the same running time as the traditional grade-school multiplication technique. So we have a radically new algorithm, but we haven't yet made any progress in efficiency. How can our method be sped up?

This is where Gauss's trick comes to mind. Although the expression for $xy$ seems to demand four $n/2$-bit multiplications, as before just three will do: $x_L y_L, x_R y_R$, and $(x_L + x_R)(y_L + y_R)$, since $x_L y_R + x_R y_L = (x_L + x_R)(y_L + y_R) - x_L y_L - x_R y_R$. The resulting algorithm, shown in Figure 2.1, has an improved running time of[1]

$$T(n) \;=\; 3T(n/2) + O(n).$$

The point is that now the constant factor improvement, from 4 to 3, occurs *at every level of the recursion*, and this compounding effect leads to a dramatically lower time bound of $O(n^{1.59})$.

This running time can be derived by looking at the algorithm's pattern of recursive calls, which form a tree structure, as in Figure 2.2. Let's try to understand the shape of this tree. At each successive level of recursion the subproblems get halved in size. At the $(\log_2 n)^{\text{th}}$ level, the subproblems get down to size 1, and so the recursion ends. Therefore, the height of the tree is $\log_2 n$. The branching factor is 3—each problem recursively produces three smaller ones—with the result that at depth $k$ in the tree there are $3^k$ subproblems, each of size $n/2^k$.

For each subproblem, a linear amount of work is done in identifying further subproblems and combining their answers. Therefore the total time spent at depth $k$ in the tree is

$$3^k \times O\left(\frac{n}{2^k}\right) = \left(\frac{3}{2}\right)^k \times O(n).$$

[1] Actually, the recurrence should read

$$T(n) \;\leq\; 3T(n/2 + 1) + O(n)$$

since the numbers $(x_L + x_R)$ and $(y_L + y_R)$ could be $n/2 + 1$ bits long. The one we're using is simpler to deal with and can be seen to imply exactly the same big-$O$ running time.

**Figure 2.1** A divide-and-conquer algorithm for integer multiplication.

```
function multiply(x, y)
Input:  Positive integers x and y, in binary
Output:  Their product

n = max(size of x, size of y)
if n = 1:  return xy

x_L, x_R = leftmost ⌈n/2⌉, rightmost ⌊n/2⌋ bits of x
y_L, y_R = leftmost ⌈n/2⌉, rightmost ⌊n/2⌋ bits of y

P_1 = multiply(x_L, y_L)
P_2 = multiply(x_R, y_R)
P_3 = multiply(x_L + x_R, y_L + y_R)
return P_1 × 2^n + (P_3 − P_1 − P_2) × 2^(n/2) + P_2
```

At the very top level, when $k = 0$, this works out to $O(n)$. At the bottom, when $k = \log_2 n$, it is $O(3^{\log_2 n})$, which can be rewritten as $O(n^{\log_2 3})$ (do you see why?). Between these two endpoints, the work done increases *geometrically* from $O(n)$ to $O(n^{\log_2 3})$, by a factor of $3/2$ per level. The sum of any increasing geometric series is, within a constant factor, simply the last term of the series: such is the rapidity of the increase (Exercise 0.2). Therefore the overall running time is $O(n^{\log_2 3})$, which is about $O(n^{1.59})$.

In the absence of Gauss's trick, the recursion tree would have the same height, but the branching factor would be $4$. There would be $4^{\log_2 n} = n^2$ leaves, and therefore the running time would be at least this much. In divide-and-conquer algorithms, the number of subproblems translates into the branching factor of the recursion tree; small changes in this coefficient can have a big impact on running time.

A practical note: it generally does not make sense to recurse all the way down to $1$ bit. For most processors, 16- or 32-bit multiplication is a single operation, so by the time the numbers get into this range they should be handed over to the built-in procedure.

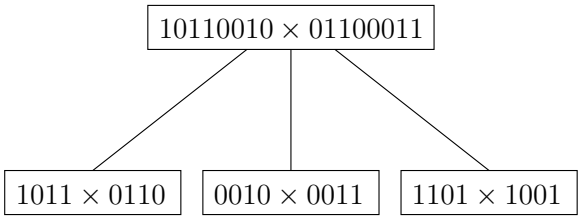
Finally, the eternal question: *Can we do better?* It turns out that even faster algorithms for multiplying numbers exist, based on another important divide-and-conquer algorithm: the fast Fourier transform, to be explained in Section 2.6.

## 2.2 Recurrence relations

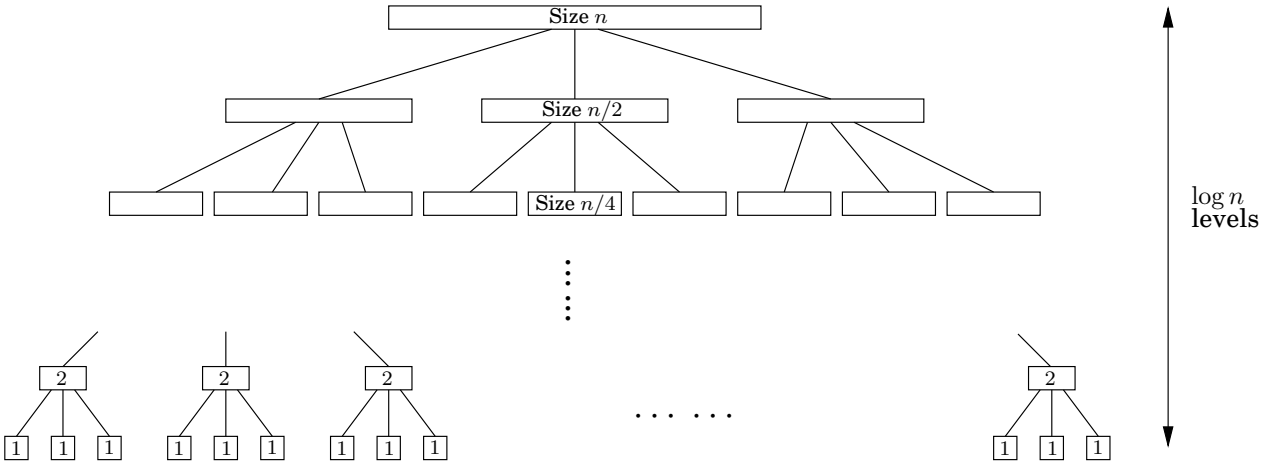
Divide-and-conquer algorithms often follow a generic pattern: they tackle a problem of size $n$ by recursively solving, say, $a$ subproblems of size $n/b$ and then combining these answers in $O(n^d)$ time, for some $a, b, d > 0$ (in the multiplication algorithm, $a = 3$, $b = 2$, and $d = 1$). Their running time can therefore be captured by the equation $T(n) = aT(\lceil n/b \rceil) + O(n^d)$. We next derive a closed-form solution to this general recurrence so that we no longer have to solve it explicitly in each new instance.

**Figure 2.2** Divide-and-conquer integer multiplication. (a) Each problem is divided into three subproblems. (b) The levels of recursion.

(a)




(b)




**Master theorem**[2] *If* $T(n) = aT(\lceil n/b \rceil) + O(n^d)$ *for some constants* $a > 0$, $b > 1$, *and* $d \geq 0$, *then*

$$T(n) = \begin{cases} O(n^d) & \text{if } d > \log_b a \\ O(n^d \log n) & \text{if } d = \log_b a \\ O(n^{\log_b a}) & \text{if } d < \log_b a \,. \end{cases}$$

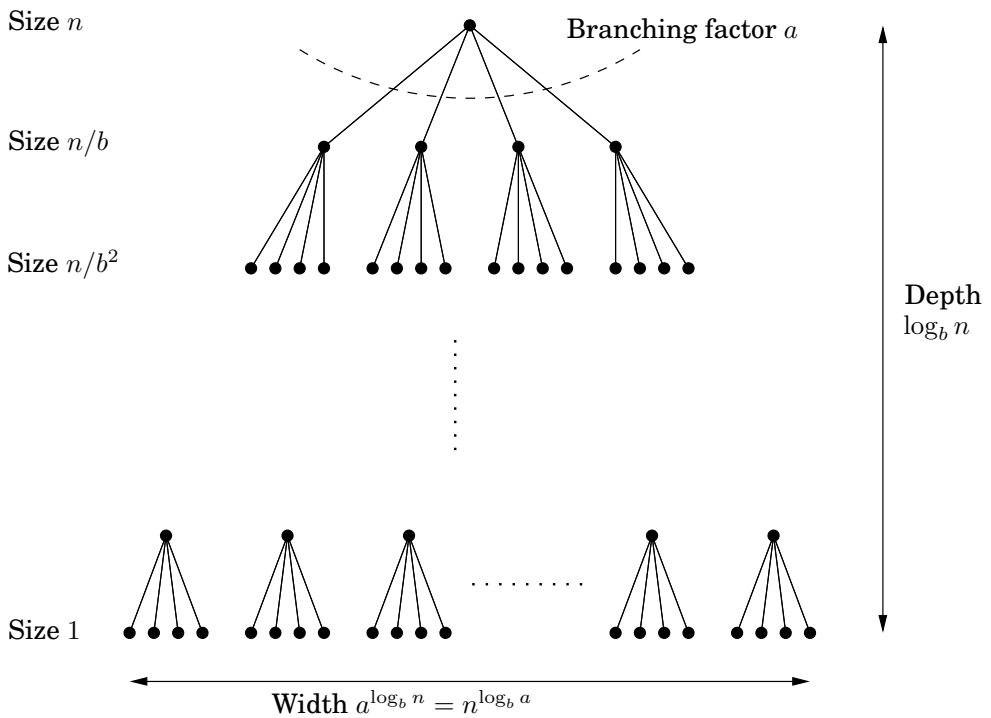
This single theorem tells us the running times of most of the divide-and-conquer procedures we are likely to use.

*Proof.* To prove the claim, let's start by assuming for the sake of convenience that $n$ is a power of $b$. This will not influence the final bound in any important way—after all, $n$ is at most a multiplicative factor of $b$ away from some power of $b$ (Exercise 2.2)—and it will allow us to ignore the rounding effect in $\lceil n/b \rceil$.

Next, notice that the size of the subproblems decreases by a factor of $b$ with each level of recursion, and therefore reaches the base case after $\log_b n$ levels. This is the height of the recursion tree. Its branching factor is $a$, so the $k$th level of the tree is made up of $a^k$

[2]There are even more general results of this type, but we will not be needing them.

**Figure 2.3** Each problem of size $n$ is divided into $a$ subproblems of size $n/b$.




subproblems, each of size $n/b^k$ (Figure 2.3). The total work done at this level is

$$a^k \times O\left(\frac{n}{b^k}\right)^d \;=\; O(n^d) \times \left(\frac{a}{b^d}\right)^k.$$

As $k$ goes from $0$ (the root) to $\log_b n$ (the leaves), these numbers form a geometric series with ratio $a/b^d$. Finding the sum of such a series in big-$O$ notation is easy (Exercise 0.2), and comes down to three cases.

1. *The ratio is less than* $1$.

   Then the series is decreasing, and its sum is just given by its first term, $O(n^d)$.

2. *The ratio is greater than* $1$.

   The series is increasing and its sum is given by its last term, $O(n^{\log_b a})$:

$$n^d \left(\frac{a}{b^d}\right)^{\log_b n} \;=\; n^d \left(\frac{a^{\log_b n}}{(b^{\log_b n})^d}\right) \;=\; a^{\log_b n} \;=\; a^{(\log_a n)(\log_b a)} \;=\; n^{\log_b a}.$$

3. *The ratio is exactly* $1$.

   In this case all $O(\log n)$ terms of the series are equal to $O(n^d)$.

These cases translate directly into the three contingencies in the theorem statement. ∎