

2.4 Sequences and Summations

2.4.1 Introduction

Sequences are ordered lists of elements, used in discrete mathematics in many ways. For example, they can be used to represent solutions to certain counting problems, as we will see in Chapter 8. They are also an important data structure in computer science. We will often need to work with sums of terms of sequences in our study of discrete mathematics. This section reviews the use of summation notation, basic properties of summations, and formulas for the sums of terms of some particular types of sequences.

The terms of a sequence can be specified by providing a formula for each term of the sequence. In this section we describe another way to specify the terms of a sequence using a recurrence relation, which expresses each term as a combination of the previous terms. We will introduce one method, known as iteration, for finding a closed formula for the terms of a sequence specified via a recurrence relation. Identifying a sequence when the first few terms are provided is a useful skill when solving problems in discrete mathematics. We will provide some tips, including a useful tool on the Web, for doing so.

2.4.2 Sequences

A sequence is a discrete structure used to represent an ordered list. For example, 1, 2, 3, 5, 8 is a sequence with five terms and 1, 3, 9, 27, 81, ..., 3^n , ... is an infinite sequence.

Definition 1

A *sequence* is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, \dots\}$ or the set $\{1, 2, 3, \dots\}$) to a set S . We use the notation a_n to denote the image of the integer n . We call a_n a *term* of the sequence.

We use the notation $\{a_n\}$ to describe the sequence. (Note that a_n represents an individual term of the sequence $\{a_n\}$. Be aware that the notation $\{a_n\}$ for a sequence conflicts with the notation for a set. However, the context in which we use this notation will always make it clear when we are dealing with sets and when we are dealing with sequences. Moreover, although we have used the letter a in the notation for a sequence, other letters or expressions may be used depending on the sequence under consideration. That is, the choice of the letter a is arbitrary.)

We describe sequences by listing the terms of the sequence in order of increasing subscripts.

EXAMPLE 1 Consider the sequence $\{a_n\}$, where

$$a_n = \frac{1}{n}.$$

The list of the terms of this sequence, beginning with a_1 , namely,

$$a_1, a_2, a_3, a_4, \dots,$$

starts with

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$$



Definition 2

A **geometric progression** is a sequence of the form

$$a, ar, ar^2, \dots, ar^n, \dots$$

where the *initial term* a and the *common ratio* r are real numbers.

Remark: A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$.

EXAMPLE 2

The sequences $\{b_n\}$ with $b_n = (-1)^n$, $\{c_n\}$ with $c_n = 2 \cdot 5^n$, and $\{d_n\}$ with $d_n = 6 \cdot (1/3)^n$ are geometric progressions with initial term and common ratio equal to 1 and -1 , 2 and 5, and 6 and $1/3$, respectively, if we start at $n = 0$. The list of terms $b_0, b_1, b_2, b_3, b_4, \dots$ begins with

$$1, -1, 1, -1, 1, \dots;$$

the list of terms $c_0, c_1, c_2, c_3, c_4, \dots$ begins with

$$2, 10, 50, 250, 1250, \dots;$$

and the list of terms $d_0, d_1, d_2, d_3, d_4, \dots$ begins with

$$6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \dots$$

Definition 3

An **arithmetic progression** is a sequence of the form

$$a, a + d, a + 2d, \dots, a + nd, \dots$$

where the *initial term* a and the *common difference* d are real numbers.

Remark: An arithmetic progression is a discrete analogue of the linear function $f(x) = dx + a$.

EXAMPLE 3

The sequences $\{s_n\}$ with $s_n = -1 + 4n$ and $\{t_n\}$ with $t_n = 7 - 3n$ are both arithmetic progressions with initial terms and common differences equal to -1 and 4, and 7 and -3 , respectively, if we start at $n = 0$. The list of terms $s_0, s_1, s_2, s_3, \dots$ begins with

$$-1, 3, 7, 11, \dots,$$

and the list of terms $t_0, t_1, t_2, t_3, \dots$ begins with

$$7, 4, 1, -2, \dots$$

Sequences of the form a_1, a_2, \dots, a_n are often used in computer science. These finite sequences are also called **strings**. This string is also denoted by $a_1 a_2 \dots a_n$. (Recall that bit strings, which are finite sequences of bits, were introduced in Section 1.1.) The **length** of a string is the number of terms in this string. The **empty string**, denoted by λ , is the string that has no terms. The empty string has length zero.

EXAMPLE 4

The string $abcd$ is a string of length four.

2.4.3 Recurrence Relations


In Examples 1–3 we specified sequences by providing explicit formulas for their terms. There are many other ways to specify a sequence. For example, another way to specify a sequence is to provide one or more initial terms together with a rule for determining subsequent terms from those that precede them.

Definition 4

A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the **previous terms of the sequence**, namely, a_0, a_1, \dots, a_{n-1} , for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer. A sequence is called a **solution of a recurrence relation** if its terms satisfy the recurrence relation. (A recurrence relation is said to *recursively define* a sequence. We will explain this alternative terminology in Chapter 5.)


EXAMPLE 5

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1, 2, 3, \dots$, and suppose that $a_0 = 2$. What are a_1 , a_2 , and a_3 ?

Solution: We see from the recurrence relation that $a_1 = a_0 + 3 = 2 + 3 = 5$. It then follows that $a_2 = 5 + 3 = 8$ and $a_3 = 8 + 3 = 11$. 

EXAMPLE 6

Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \dots$, and suppose that $a_0 = 3$ and $a_1 = 5$. What are a_2 and a_3 ?

Solution: We see from the recurrence relation that $a_2 = a_1 - a_0 = 5 - 3 = 2$ and $a_3 = a_2 - a_1 = 2 - 5 = -3$. We can find a_4 , a_5 , and each successive term in a similar way. 

The **initial conditions** for a recursively defined sequence **specify the terms that precede the first term** where the recurrence relation takes effect. For instance, the initial condition in Example 5 is $a_0 = 2$, and the initial conditions in Example 6 are $a_0 = 3$ and $a_1 = 5$. Using mathematical induction, a proof technique introduced in Chapter 5, it can be shown that a recurrence relation together with its initial conditions determines a unique solution.

Next, we define a particularly useful sequence defined by a recurrence relation, known as the **Fibonacci sequence**, after the Italian mathematician Fibonacci who was born in the 12th century (see Chapter 5 for his biography). We will study this sequence in depth in Chapters 5 and 8, where we will see why it is important for many applications, including modeling the population growth of rabbits. Fibonacci numbers occur naturally in the structures of plants and animals, such as in the arrangement of sunflower seeds in a seed head and in the shell of the chambered nautilus.

Hop along to Chapter 8 to learn how to find a formula for the Fibonacci numbers.



Definition 5

The **Fibonacci sequence**, f_0, f_1, f_2, \dots , is defined by the initial conditions $f_0 = 0, f_1 = 1$, and the recurrence relation

$$f_n = f_{n-1} + f_{n-2}$$

for $n = 2, 3, 4, \dots$.

EXAMPLE 7 Find the Fibonacci numbers f_2, f_3, f_4, f_5 , and f_6 .

Solution: The recurrence relation for the Fibonacci sequence tells us that we find successive terms by adding the previous two terms. Because the initial conditions tell us that $f_0 = 0$ and $f_1 = 1$, using the recurrence relation in the definition we find that

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$

$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$

$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$

$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$

$$f_6 = f_5 + f_4 = 5 + 3 = 8. \quad \blacktriangleleft$$

EXAMPLE 8 Suppose that $\{a_n\}$ is the sequence of integers defined by $a_n = n!$, the value of the factorial function at the integer n , where $n = 1, 2, 3, \dots$. Because $n! = n((n-1)(n-2)\dots 2 \cdot 1) = n(n-1)! = na_{n-1}$, we see that the sequence of factorials satisfies the recurrence relation $a_n = na_{n-1}$, together with the initial condition $a_1 = 1$. \blacktriangleleft

We say that we have solved the recurrence relation together with the initial conditions when we find an explicit formula, called a **closed formula**, for the terms of the sequence.

EXAMPLE 9 Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n , is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \dots$. Answer the same question where $a_n = 2^n$ and where $a_n = 5$.

Solution: Suppose that $a_n = 3n$ for every nonnegative integer n . Then, for $n \geq 2$, we see that $2a_{n-1} - a_{n-2} = 2(3(n-1)) - 3(n-2) = 3n = a_n$. Therefore, $\{a_n\}$, where $a_n = 3n$, is a solution of the recurrence relation.

Suppose that $a_n = 2^n$ for every nonnegative integer n . Note that $a_0 = 1$, $a_1 = 2$, and $a_2 = 4$. Because $2a_1 - a_0 = 2 \cdot 2 - 1 = 3 \neq a_2$, we see that $\{a_n\}$, where $a_n = 2^n$, is not a solution of the recurrence relation.

Suppose that $a_n = 5$ for every nonnegative integer n . Then for $n \geq 2$, we see that $a_n = 2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n$. Therefore, $\{a_n\}$, where $a_n = 5$, is a solution of the recurrence relation. \blacktriangleleft

Many methods have been developed for solving recurrence relations. Here, we will introduce a straightforward method known as iteration via several examples. In Chapter 8 we will study recurrence relations in depth. In that chapter we will show how recurrence relations can be used to solve counting problems and we will introduce several powerful methods that can be used to solve many different recurrence relations.

EXAMPLE 10 Solve the recurrence relation and initial condition in Example 5.

Solution: We can **successively** apply the recurrence relation in Example 5, starting with the initial condition $a_1 = 2$, and working upward until we reach a_n to **deduce a closed formula** for the sequence. We see that

$$a_2 = 2 + 3$$

$$a_3 = (2 + 3) + 3 = 2 + 3 \cdot 2$$

$$a_4 = (2 + 2 \cdot 3) + 3 = 2 + 3 \cdot 3$$

$$\vdots$$

$$a_n = a_{n-1} + 3 = (2 + 3 \cdot (n-2)) + 3 = 2 + 3(n-1).$$

We can also **successively apply** the recurrence relation in Example 5, starting with the term a_n and working downward until we reach the initial condition $a_1 = 2$ to deduce this same formula. The steps are

$$\begin{aligned} a_n &= a_{n-1} + 3 \\ &= (a_{n-2} + 3) + 3 = a_{n-2} + 3 \cdot 2 \\ &= (a_{n-3} + 3) + 3 \cdot 2 = a_{n-3} + 3 \cdot 3 \\ &\vdots \\ &= a_2 + 3(n-2) = (a_1 + 3) + 3(n-2) = 2 + 3(n-1). \end{aligned}$$

At each iteration of the recurrence relation, we obtain the next term in the sequence by adding 3 to the previous term. We obtain the n th term after $n - 1$ iterations of the recurrence relation. Hence, we have added $3(n - 1)$ to the initial term $a_0 = 2$ to obtain a_n . This gives us the closed formula $a_n = 2 + 3(n - 1)$. Note that this sequence is an arithmetic progression. ◀

The technique used in Example 10 is called **iteration**. We have iterated, or repeatedly used, the recurrence relation. The first approach is called **forward substitution**—we found successive terms beginning with the initial condition and ending with a_n . The second approach is called **backward substitution**, because we began with a_n and iterated to express it in terms of falling terms of the sequence until we found it in terms of a_1 . Note that when we use iteration, we essentially guess a formula for the terms of the sequence. To prove that our guess is correct, we need to use mathematical induction, a technique we discuss in Chapter 5.

In Chapter 8 we will show that recurrence relations can be used to model a wide variety of problems. We provide one such example here, showing how to use a recurrence relation to find compound interest.

EXAMPLE 11

Compound Interest Suppose that a person deposits \$10,000 in a savings account at a bank yielding 11% per year with interest compounded annually. How much will be in the account after 30 years?

Extra
Examples ▶

Solution: To solve this problem, let P_n denote the amount in the account after n years. Because the amount in the account after n years equals the amount in the account after $n - 1$ years plus interest for the n th year, we see that the sequence $\{P_n\}$ satisfies the recurrence relation

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)P_{n-1}.$$

The initial condition is $P_0 = 10,000$.

We can use an iterative approach to find a formula for P_n . Note that

$$\begin{aligned} P_1 &= (1.11)P_0 \\ P_2 &= (1.11)P_1 = (1.11)^2P_0 \\ P_3 &= (1.11)P_2 = (1.11)^3P_0 \\ &\vdots \\ P_n &= (1.11)P_{n-1} = (1.11)^nP_0. \end{aligned}$$

When we insert the initial condition $P_0 = 10,000$, the formula $P_n = (1.11)^n 10,000$ is obtained.

Inserting $n = 30$ into the formula $P_n = (1.11)^n 10,000$ shows that after 30 years the account contains

$$P_{30} = (1.11)^{30} 10,000 = \$228,922.97. \quad \blacktriangleleft$$

2.4.4 Special Integer Sequences

A common problem in discrete mathematics is finding a closed formula, a recurrence relation, or some other type of general rule for constructing the terms of a sequence. Sometimes only a few terms of a sequence solving a problem are known; the goal is to identify the sequence. Even though the initial terms of a sequence do not determine the entire sequence (after all, there are infinitely many different sequences that start with any finite set of initial terms), knowing the first few terms may help you make an educated conjecture about the identity of your sequence. Once you have made this conjecture, you can try to verify that you have the correct sequence.

When trying to deduce a possible formula, recurrence relation, or some other type of rule for the terms of a sequence when given the initial terms, try to find a pattern in these terms. You might also see whether you can determine how a term might have been produced from those preceding it. There are many questions you could ask, but some of the more useful are:

- ▶ Are there runs of the same value? That is, does the same value occur many times in a row?
- ▶ Are terms obtained from previous terms by adding the same amount or an amount that depends on the position in the sequence?
- ▶ Are terms obtained from previous terms by multiplying by a particular amount?
- ▶ Are terms obtained by combining previous terms in a certain way?
- ▶ Are there cycles among the terms?

EXAMPLE 12 Find formulae for the sequences with the following first five terms: (a) 1, $1/2$, $1/4$, $1/8$, $1/16$ (b) 1, 3, 5, 7, 9 (c) 1, -1 , 1, -1 , 1.

Extra Examples ▶

Solution: (a) We recognize that the denominators are powers of 2. The sequence with $a_n = 1/2^n$, $n = 0, 1, 2, \dots$ is a possible match. This proposed sequence is a geometric progression with $a = 1$ and $r = 1/2$.

(b) We note that each term is obtained by adding 2 to the previous term. The sequence with $a_n = 2n + 1$, $n = 0, 1, 2, \dots$ is a possible match. This proposed sequence is an arithmetic progression with $a = 1$ and $d = 2$.

(c) The terms alternate between 1 and -1 . The sequence with $a_n = (-1)^n$, $n = 0, 1, 2, \dots$ is a possible match. This proposed sequence is a geometric progression with $a = 1$ and $r = -1$. ◀

Examples 13–15 illustrate how we can analyze sequences to find how the terms are constructed.

EXAMPLE 13 How can we produce the terms of a sequence if the first 10 terms are 1, 2, 2, 3, 3, 3, 4, 4, 4, 4?

Solution: In this sequence, the integer 1 appears once, the integer 2 appears twice, the integer 3 appears three times, and the integer 4 appears four times. A reasonable rule for generating this sequence is that the integer n appears exactly n times, so the next five terms of the sequence would all be 5, the following six terms would all be 6, and so on. The sequence generated this way is a possible match. ◀

EXAMPLE 14 How can we produce the terms of a sequence if the first 10 terms are 5, 11, 17, 23, 29, 35, 41, 47, 53, 59?

Solution: Note that each of the first 10 terms of this sequence after the first is obtained by adding 6 to the previous term. (We could see this by noticing that the difference between consecutive terms is 6.) Consequently, the n th term could be produced by starting with 5 and adding 6 a total

TABLE 1 Some Useful Sequences.	
<i>n</i> th Term	First 10 Terms
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...

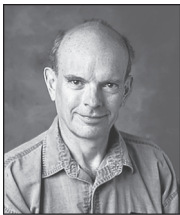
of $n - 1$ times; that is, a reasonable guess is that the n th term is $5 + 6(n - 1) = 6n - 1$. (This is an arithmetic progression with $a = 5$ and $d = 6$.)

EXAMPLE 15 How can we produce the terms of a sequence if the first 10 terms are 1, 3, 4, 7, 11, 18, 29, 47, 76, 123?

Solution: Observe that each successive term of this sequence, starting with the third term, is the sum of the two previous terms. That is, $4 = 3 + 1$, $7 = 4 + 3$, $11 = 7 + 4$, and so on. Consequently, if L_n is the n th term of this sequence, we guess that the sequence is determined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$ with initial conditions $L_1 = 1$ and $L_2 = 3$ (the same recurrence relation as the Fibonacci sequence, but with different initial conditions). This sequence is known as the **Lucas sequence**, after the French mathematician François Édouard Lucas. Lucas studied this sequence and the Fibonacci sequence in the nineteenth century.

Another useful technique for finding a rule for generating the terms of a sequence is to compare the terms of a sequence of interest with the terms of a well-known integer sequence, such as terms of an arithmetic progression, terms of a geometric progression, perfect squares, perfect cubes, and so on. The first 10 terms of some sequences you may want to keep in mind are displayed in Table 1. Note that we have listed these sequences so that the terms of each sequence grow faster than those in the preceding sequence in the list. The rates of growth of these terms will be studied in Section 3.2.

Links




Courtesy of Neil Sloane

NEIL SLOANE (BORN 1939) Neil Sloane studied mathematics and electrical engineering at the University of Melbourne on a scholarship from the Australian state telephone company. He mastered many telephone-related jobs, such as erecting telephone poles, in his summer work. After graduating, he designed minimal-cost telephone networks in Australia. In 1962 he came to the United States and studied electrical engineering at Cornell University. His Ph.D. thesis was on what are now called neural networks. He took a job at Bell Labs in 1969, working in many areas, including network design, coding theory, and sphere packing. He moved to AT&T Labs in 1996 when it was split off from Bell Labs, working there until his retirement in 2012. One of his favorite problems is the **kissing problem** (a name he coined), which asks how many spheres can be arranged in n dimensions so that they all touch a central sphere of the same size. (In two dimensions the answer is 6, because 6 pennies can be placed so that they touch a central penny. In three dimensions, 12 billiard balls can be placed so that they touch a central billiard ball. Two billiard balls that just touch are said to “kiss,” giving rise to the terminology “kissing problem” and “kissing number.”) Sloane, together with Andrew Odlyzko, showed that in 8 and 24 dimensions, the optimal kissing numbers are, respectively, 240 and 196,560. The kissing number is known in dimensions 1, 2, 3, 4, 8, and 24, but not in any other dimensions. Sloane’s books include *Sphere Packings, Lattices and Groups*, 3d ed., with John Conway; *The Theory of Error-Correcting Codes* with Jessie MacWilliams; *The Encyclopedia of Integer Sequences* with Simon Plouffe (which has grown into the popular OEIS website); and *The Rock-Climbing Guide to New Jersey Crags* with Paul Nick. The last book demonstrates his interest in rock climbing; it includes more than 50 climbing sites in New Jersey.

EXAMPLE 16

Conjecture a simple formula for a_n if the first 10 terms of the sequence $\{a_n\}$ are 1, 7, 25, 79, 241, 727, 2185, 6559, 19681, 59047.

Solution: To attack this problem, we begin by looking at the difference of consecutive terms, but we do not see a pattern. When we form the ratio of consecutive terms to see whether each term is a multiple of the previous term, we find that this ratio, although not a constant, is close to 3. So it is reasonable to suspect that the terms of this sequence are generated by a formula involving 3^n . Comparing these terms with the corresponding terms of the sequence $\{3^n\}$, we notice that the n th term is 2 less than the corresponding power of 3. We see that $a_n = 3^n - 2$ for $1 \leq n \leq 10$ and conjecture that this formula holds for all n . 

We will see throughout this text that integer sequences appear in a wide range of contexts in discrete mathematics. Sequences we have encountered or will encounter include the sequence of prime numbers (Chapter 4), the number of ways to order n discrete objects (Chapter 6), the number of moves required to solve the Tower of Hanoi puzzle with n disks (Chapter 8), and the number of rabbits on an island after n months (Chapter 8).

Integer sequences appear in an amazingly wide range of subject areas besides discrete mathematics, including biology, engineering, chemistry, and physics, as well as in puzzles. An amazing database of over 250,000 different integer sequences (as of 2017) can be found in the *On-Line Encyclopedia of Integer Sequences (OEIS)*. This database was originated by Neil Sloane in 1964 and is now maintained by the OEIS Foundation. The last printed version of this database was published in 1995 ([SIPI95]); the current encyclopedia would occupy more than 900 volumes of the size of the 1995 book with more than 10,000 new submissions a year. You can use a program on the OEIS website to find sequences from the encyclopedia that match initial terms you provide, if there is a match. For instance, when you enter 1, 1, 2, 3, 5, 8, OEIS displays a page that identifies these numbers as successive terms of the Fibonacci sequence, provides the recurrence relation that generates this sequence, lists an extensive set of comments about the many ways the Fibonacci sequence arises including references, and displays information about quite a few other sequences that begin with these same terms.

Check out the puzzles at the OEIS site.

Links 

2.4.5 Summations

Next, we consider the addition of the terms of a sequence. For this we introduce **summation notation**. We begin by describing the notation used to express the sum of the terms

$$a_m, a_{m+1}, \dots, a_n$$

from the sequence $\{a_n\}$. We use the notation

$$\sum_{j=m}^n a_j, \quad \sum_{j=m}^n a_j, \quad \text{or} \quad \sum_{m \leq j \leq n} a_j$$

(read as the sum from $j = m$ to $j = n$ of a_j) to represent

$$a_m + a_{m+1} + \dots + a_n.$$

Here, the variable j is called the **index of summation**, and the choice of the letter j as the variable is arbitrary; that is, we could have used any other letter, such as i or k . Or, in notation,

$$\sum_{j=m}^n a_j = \sum_{i=m}^n a_i = \sum_{k=m}^n a_k.$$

Here, the index of summation runs through all integers starting with its **lower limit** m and ending with its **upper limit** n . A large uppercase Greek letter sigma, \sum , is used to denote summation.

The usual laws for arithmetic apply to summations. For example, when a and b are real numbers, we have $\sum_{j=1}^n (ax_j + by_j) = a \sum_{j=1}^n x_j + b \sum_{j=1}^n y_j$, where x_1, x_2, \dots, x_n and y_1, y_2, \dots, y_n are real numbers. (We do not present a formal proof of this identity here. Such a proof can be constructed using mathematical induction, a proof method we introduce in Chapter 5. The proof also uses the commutative and associative laws for addition and the distributive law of multiplication over addition.)

We give some examples of summation notation.

EXAMPLE 17 Use summation notation to express the sum of the first 100 terms of the sequence $\{a_j\}$, where $a_j = 1/j$ for $j = 1, 2, 3, \dots$.


Extra Examples 

Solution: The lower limit for the index of summation is 1, and the upper limit is 100. We write this sum as

$$\sum_{j=1}^{100} \frac{1}{j}.$$



EXAMPLE 18 What is the value of $\sum_{j=1}^5 j^2$?

Solution: We have

$$\begin{aligned} \sum_{j=1}^5 j^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 \\ &= 1 + 4 + 9 + 16 + 25 \\ &= 55. \end{aligned}$$


EXAMPLE 19 What is the value of $\sum_{k=4}^8 (-1)^k$?

Solution: We have

$$\begin{aligned} \sum_{k=4}^8 (-1)^k &= (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8 \\ &= 1 + (-1) + 1 + (-1) + 1 \\ &= 1. \end{aligned}$$


Sometimes it is useful to shift the index of summation in a sum. This is often done when two sums need to be added but their indices of summation do not match. When shifting an index of summation, it is important to make the appropriate changes in the corresponding summand. This is illustrated by Example 20.

EXAMPLE 20 Suppose we have the sum

$$\sum_{j=1}^5 j^2$$

Extra Examples

but want the index of summation to run between 0 and 4 rather than from 1 to 5. To do this, we let $k = j - 1$. Then the new summation index runs from 0 (because $k = 1 - 0 = 0$ when $j = 1$) to 4 (because $k = 5 - 1 = 4$ when $j = 5$), and the term j^2 becomes $(k + 1)^2$. Hence,

$$\sum_{j=1}^5 j^2 = \sum_{k=0}^4 (k+1)^2.$$

It is easily checked that both sums are $1 + 4 + 9 + 16 + 25 = 55$. ◀

Sums of terms of geometric progressions commonly arise (such sums are called **geometric series**). Theorem 1 gives us a formula for the sum of terms of a geometric progression.

THEOREM 1

If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^n ar^j = \begin{cases} \frac{ar^{n+1} - a}{r - 1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1. \end{cases}$$

Proof: Let

$$S_n = \sum_{j=0}^n ar^j.$$

To compute S , first multiply both sides of the equality by r and then manipulate the resulting sum as follows:

$$\begin{aligned} rS_n &= r \sum_{j=0}^n ar^j && \text{substituting summation formula for } S \\ &= \sum_{j=0}^n ar^{j+1} && \text{by the distributive property} \\ &= \sum_{k=1}^{n+1} ar^k && \text{shifting the index of summation, with } k = j + 1 \\ &= \left(\sum_{k=0}^n ar^k \right) + (ar^{n+1} - a) && \text{removing } k = n + 1 \text{ term and adding } k = 0 \text{ term} \\ &= S_n + (ar^{n+1} - a) && \text{substituting } S \text{ for summation formula} \end{aligned}$$

From these equalities, we see that

$$rS_n = S_n + (ar^{n+1} - a).$$

Solving for S_n shows that if $r \neq 1$, then

$$S_n = \frac{ar^{n+1} - a}{r - 1}.$$

If $r = 1$, then the $S_n = \sum_{j=0}^n ar^j = \sum_{j=0}^n a = (n+1)a$. ◀

EXAMPLE 21 Double summations arise in many contexts (as in the analysis of nested loops in computer programs). An example of a double summation is

$$\sum_{i=1}^4 \sum_{j=1}^3 ij.$$

To evaluate the double sum, first expand the inner summation and then continue by computing the outer summation:

$$\begin{aligned} \sum_{i=1}^4 \sum_{j=1}^3 ij &= \sum_{i=1}^4 (i + 2i + 3i) \\ &= \sum_{i=1}^4 6i \\ &= 6 + 12 + 18 + 24 = 60. \end{aligned}$$

We can also use summation notation to add all values of a function, or terms of an indexed set, where the index of summation runs over all values in a set. That is, we write

$$\sum_{s \in S} f(s)$$

to represent the sum of the values $f(s)$, for all members s of S .

EXAMPLE 22 What is the value of $\sum_{s \in \{0,2,4\}} s$?

Solution: Because $\sum_{s \in \{0,2,4\}} s$ represents the sum of the values of s for all the members of the set $\{0, 2, 4\}$, it follows that

$$\sum_{s \in \{0,2,4\}} s = 0 + 2 + 4 = 6.$$

Certain sums arise repeatedly throughout discrete mathematics. Having a collection of formulae for such sums can be useful; Table 2 provides a small table of formulae for commonly occurring sums.

We derived the first formula in this table in Theorem 1. The next three formulae give us the sum of the first n positive integers, the sum of their squares, and the sum of their cubes. These three formulae can be derived in many different ways (for example, see Exercises 37 and 38). Also note that each of these formulae, once known, can be proved using mathematical induction, the subject of Section 5.1. The last two formulae in the table involve infinite series and will be discussed shortly.

Example 23 illustrates how the formulae in Table 2 can be useful.

EXAMPLE 23 Find $\sum_{k=50}^{100} k^2$.

Solution: First note that because $\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2$, we have

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2.$$

TABLE 2 Some Useful Summation Formulae.

Sum	Closed Form
$\sum_{k=0}^n ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^n k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^n k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^n k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, x < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, x < 1$	$\frac{1}{(1-x)^2}$

Using the formula $\sum_{k=1}^n k^2 = n(n+1)(2n+1)/6$ from Table 2 (and proved in Exercise 38), we see that

$$\sum_{k=50}^{100} k^2 = \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6} = 338,350 - 40,425 = 297,925.$$

SOME INFINITE SERIES Although most of the summations in this book are finite sums, infinite series are important in some parts of discrete mathematics. Infinite series are usually studied in a course in calculus and even the definition of these series requires the use of calculus, but sometimes they arise in discrete mathematics, because discrete mathematics deals with infinite collections of discrete elements. In particular, in our future studies in discrete mathematics, we will find the closed forms for the infinite series in Examples 24 and 25 to be quite useful.

EXAMPLE 24 (Requires calculus) Let x be a real number with $|x| < 1$. Find $\sum_{n=0}^{\infty} x^n$.

Extra Examples ➤

Solution: By Theorem 1 with $a = 1$ and $r = x$ we see that $\sum_{n=0}^k x^n = \frac{x^{k+1} - 1}{x - 1}$. Because $|x| < 1$, x^{k+1} approaches 0 as k approaches infinity. It follows that

$$\sum_{n=0}^{\infty} x^n = \lim_{k \rightarrow \infty} \frac{x^{k+1} - 1}{x - 1} = \frac{0 - 1}{x - 1} = \frac{1}{1 - x}.$$

We can produce new summation formulae by differentiating or integrating existing formulae.

EXAMPLE 25 (Requires calculus) Differentiating both sides of the equation

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x},$$

from Example 24 we find that

$$\sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}.$$

(This differentiation is valid for $|x| < 1$ by a theorem about infinite series.)

Exercises

- Find these terms of the sequence $\{a_n\}$, where $a_n = 2 \cdot (-3)^n + 5^n$.
 a) a_0 b) a_1 c) a_4 d) a_5
- What is the term a_8 of the sequence $\{a_n\}$ if a_n equals
 a) 2^{n-1} ? b) 7 ?
 c) $1 + (-1)^n$? d) $-(-2)^n$?
- What are the terms a_0, a_1, a_2 , and a_3 of the sequence $\{a_n\}$, where a_n equals
 a) $2^n + 1$? b) $(n+1)^{n+1}$?
 c) $\lfloor n/2 \rfloor$? d) $\lfloor n/2 \rfloor + \lceil n/2 \rceil$?
- What are the terms a_0, a_1, a_2 , and a_3 of the sequence $\{a_n\}$, where a_n equals
 a) $(-2)^n$? b) 3 ?
 c) $7 + 4^n$? d) $2^n + (-2)^n$?
- List the first 10 terms of each of these sequences.
 - the sequence that begins with 2 and in which each successive term is 3 more than the preceding term
 - the sequence that lists each positive integer three times, in increasing order
 - the sequence that lists the odd positive integers in increasing order, listing each odd integer twice
 - the sequence whose n th term is $n! - 2^n$
 - the sequence that begins with 3, where each succeeding term is twice the preceding term
 - the sequence whose first term is 2, second term is 4, and each succeeding term is the sum of the two preceding terms
 - the sequence whose n th term is the number of bits in the binary expansion of the number n (defined in Section 4.2)
 - the sequence where the n th term is the number of letters in the English word for the index n
- List the first 10 terms of each of these sequences.
 - the sequence obtained by starting with 10 and obtaining each term by subtracting 3 from the previous term
 - the sequence whose n th term is the sum of the first n positive integers
 - the sequence whose n th term is $3^n - 2^n$
 - the sequence whose n th term is $\lfloor \sqrt{n} \rfloor$
 - the sequence whose first two terms are 1 and 5 and each succeeding term is the sum of the two previous terms
 - the sequence whose n th term is the largest integer whose binary expansion (defined in Section 4.2) has n bits (Write your answer in decimal notation.)
 - the sequence whose terms are constructed sequentially as follows: start with 1, then add 1, then multiply by 1, then add 2, then multiply by 2, and so on
 - the sequence whose n th term is the largest integer k such that $k! \leq n$
- Find at least three different sequences beginning with the terms 1, 2, 4 whose terms are generated by a simple formula or rule.
- Find at least three different sequences beginning with the terms 3, 5, 7 whose terms are generated by a simple formula or rule.
- Find the first five terms of the sequence defined by each of these recurrence relations and initial conditions.
 - $a_n = 6a_{n-1}, a_0 = 2$
 - $a_n = a_{n-1}^2, a_1 = 2$
 - $a_n = a_{n-1} + 3a_{n-2}, a_0 = 1, a_1 = 2$
 - $a_n = na_{n-1} + n^2a_{n-2}, a_0 = 1, a_1 = 1$
 - $a_n = a_{n-1} + a_{n-3}, a_0 = 1, a_1 = 2, a_2 = 0$
- Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.
 - $a_n = -2a_{n-1}, a_0 = -1$
 - $a_n = a_{n-1} - a_{n-2}, a_0 = 2, a_1 = -1$
 - $a_n = 3a_{n-1}^2, a_0 = 1$
 - $a_n = na_{n-1} + a_{n-2}^2, a_0 = -1, a_1 = 0$
 - $a_n = a_{n-1} - a_{n-2} + a_{n-3}, a_0 = 1, a_1 = 1, a_2 = 2$
- Let $a_n = 2^n + 5 \cdot 3^n$ for $n = 0, 1, 2, \dots$.
 - Find a_0, a_1, a_2, a_3 , and a_4 .
 - Show that $a_2 = 5a_1 - 6a_0, a_3 = 5a_2 - 6a_1$, and $a_4 = 5a_3 - 6a_2$.
 - Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers n with $n \geq 2$.
- Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = -3a_{n-1} + 4a_{n-2}$ if
 - $a_n = 0$. b) $a_n = 1$.
 - $a_n = (-4)^n$. d) $a_n = 2(-4)^n + 3$.
- Is the sequence $\{a_n\}$ a solution of the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ if
 - $a_n = 0$? b) $a_n = 1$?
 - $a_n = 2^n$? d) $a_n = 4^n$?
 - $a_n = n4^n$? f) $a_n = 2 \cdot 4^n + 3n4^n$?
 - $a_n = (-4)^n$? h) $a_n = n^24^n$?
- For each of these sequences find a recurrence relation satisfied by this sequence. (The answers are not unique)

because there are infinitely many different recurrence relations satisfied by any sequence.)

- a) $a_n = 3$ b) $a_n = 2n$
 c) $a_n = 2n + 3$ d) $a_n = 5^n$
 e) $a_n = n^2$ f) $a_n = n^2 + n$
 g) $a_n = n + (-1)^n$ h) $a_n = n!$

15. Show that the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if

- a) $a_n = -n + 2$.
 b) $a_n = 5(-1)^n - n + 2$.
 c) $a_n = 3(-1)^n + 2^n - n + 2$.
 d) $a_n = 7 \cdot 2^n - n + 2$.

16. Find the solution to each of these recurrence relations with the given initial conditions. Use an iterative approach such as that used in Example 10.

- a) $a_n = -a_{n-1}$, $a_0 = 5$
 b) $a_n = a_{n-1} + 3$, $a_0 = 1$
 c) $a_n = a_{n-1} - n$, $a_0 = 4$
 d) $a_n = 2a_{n-1} - 3$, $a_0 = -1$
 e) $a_n = (n+1)a_{n-1}$, $a_0 = 2$
 f) $a_n = 2na_{n-1}$, $a_0 = 3$
 g) $a_n = -a_{n-1} + n - 1$, $a_0 = 7$

17. Find the solution to each of these recurrence relations and initial conditions. Use an iterative approach such as that used in Example 10.

- a) $a_n = 3a_{n-1}$, $a_0 = 2$
 b) $a_n = a_{n-1} + 2$, $a_0 = 3$
 c) $a_n = a_{n-1} + n$, $a_0 = 1$
 d) $a_n = a_{n-1} + 2n + 3$, $a_0 = 4$
 e) $a_n = 2a_{n-1} - 1$, $a_0 = 1$
 f) $a_n = 3a_{n-1} + 1$, $a_0 = 1$
 g) $a_n = na_{n-1}$, $a_0 = 5$
 h) $a_n = 2na_{n-1}$, $a_0 = 1$

18. A person deposits \$1000 in an account that yields 9% interest compounded annually.

- a) Set up a recurrence relation for the amount in the account at the end of n years.
 b) Find an explicit formula for the amount in the account at the end of n years.
 c) How much money will the account contain after 100 years?

19. Suppose that the number of bacteria in a colony triples every hour.

- a) Set up a recurrence relation for the number of bacteria after n hours have elapsed.
 b) If 100 bacteria are used to begin a new colony, how many bacteria will be in the colony in 10 hours?

20. Assume that the population of the world in 2017 was 7.6 billion and is growing at the rate of 1.12% a year.

- a) Set up a recurrence relation for the population of the world n years after 2017.
 b) Find an explicit formula for the population of the world n years after 2017.
 c) What will the population of the world be in 2050?

21. A factory makes custom sports cars at an increasing rate. In the first month only one car is made, in the second month two cars are made, and so on, with n cars made in the n th month.

- a) Set up a recurrence relation for the number of cars produced in the first n months by this factory.
 b) How many cars are produced in the first year?
 c) Find an explicit formula for the number of cars produced in the first n months by this factory.

22. An employee joined a company in 2017 with a starting salary of \$50,000. Every year this employee receives a raise of \$1000 plus 5% of the salary of the previous year.

- a) Set up a recurrence relation for the salary of this employee n years after 2017.
 b) What will the salary of this employee be in 2025?
 c) Find an explicit formula for the salary of this employee n years after 2017.

23. Find a recurrence relation for the balance $B(k)$ owed at the end of k months on a loan of \$5000 at a rate of 7% if a payment of \$100 is made each month. [Hint: Express $B(k)$ in terms of $B(k-1)$; the monthly interest is $(0.07/12)B(k-1)$.]

24. a) Find a recurrence relation for the balance $B(k)$ owed at the end of k months on a loan at a rate of r if a payment P is made on the loan each month. [Hint: Express $B(k)$ in terms of $B(k-1)$ and note that the monthly interest rate is $r/12$.]
 b) Determine what the monthly payment P should be so that the loan is paid off after T months.

25. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

- a) 1, 0, 1, 1, 0, 0, 1, 1, 1, 0, 0, 0, 1, ...
 b) 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, 8, 8, ...
 c) 1, 0, 2, 0, 4, 0, 8, 0, 16, 0, ...
 d) 3, 6, 12, 24, 48, 96, 192, ...
 e) 15, 8, 1, -6, -13, -20, -27, ...
 f) 3, 5, 8, 12, 17, 23, 30, 38, 47, ...
 g) 2, 16, 54, 128, 250, 432, 686, ...
 h) 2, 3, 7, 25, 121, 721, 5041, 40321, ...

26. For each of these lists of integers, provide a simple formula or rule that generates the terms of an integer sequence that begins with the given list. Assuming that your formula or rule is correct, determine the next three terms of the sequence.

- a) 3, 6, 11, 18, 27, 38, 51, 66, 83, 102, ...
 b) 7, 11, 15, 19, 23, 27, 31, 35, 39, 43, ...
 c) 1, 10, 11, 100, 101, 110, 111, 1000, 1001, 1010, 1011, ...
 d) 1, 2, 2, 2, 3, 3, 3, 3, 3, 5, 5, 5, 5, 5, ...
 e) 0, 2, 8, 26, 80, 242, 728, 2186, 6560, 19682, ...
 f) 1, 3, 15, 105, 945, 10395, 135135, 2027025, 34459425, ...
 g) 1, 0, 0, 1, 1, 1, 0, 0, 0, 0, 1, 1, 1, 1, ...
 h) 2, 4, 16, 256, 65536, 4294967296, ...

- **27. Show that if a_n denotes the n th positive integer that is not a perfect square, then $a_n = n + \{\sqrt{n}\}$, where $\{x\}$ denotes the integer closest to the real number x .

Links ➤

Links ➤

- *28. Let a_n be the n th term of the sequence 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, 6, 6, 6, 6, 6, ..., constructed by including the integer k exactly k times. Show that $a_n = \lfloor \sqrt{2n} + \frac{1}{2} \rfloor$.
29. What are the values of these sums?
- a) $\sum_{k=1}^5 (k+1)$ b) $\sum_{j=0}^4 (-2)^j$
- c) $\sum_{i=1}^{10} 3$ d) $\sum_{j=0}^8 (2^{j+1} - 2^j)$
30. What are the values of these sums, where $S = \{1, 3, 5, 7\}$?
- a) $\sum_{j \in S} j$ b) $\sum_{j \in S} j^2$
- c) $\sum_{j \in S} (1/j)$ d) $\sum_{j \in S} 1$
31. What is the value of each of these sums of terms of a geometric progression?
- a) $\sum_{j=0}^8 3 \cdot 2^j$ b) $\sum_{j=1}^8 2^j$
- c) $\sum_{j=2}^8 (-3)^j$ d) $\sum_{j=0}^8 2 \cdot (-3)^j$
32. Find the value of each of these sums.
- a) $\sum_{j=0}^8 (1 + (-1)^j)$ b) $\sum_{j=0}^8 (3^j - 2^j)$
- c) $\sum_{j=0}^8 (2 \cdot 3^j + 3 \cdot 2^j)$ d) $\sum_{j=0}^8 (2^{j+1} - 2^j)$
33. Compute each of these double sums.
- a) $\sum_{i=1}^2 \sum_{j=1}^3 (i+j)$ b) $\sum_{i=0}^2 \sum_{j=0}^3 (2i+3j)$
- c) $\sum_{i=1}^3 \sum_{j=0}^2 i$ d) $\sum_{i=0}^2 \sum_{j=1}^3 ij$
34. Compute each of these double sums.
- a) $\sum_{i=1}^3 \sum_{j=1}^2 (i-j)$ b) $\sum_{i=0}^3 \sum_{j=0}^2 (3i+2j)$
- c) $\sum_{i=1}^3 \sum_{j=0}^2 j$ d) $\sum_{i=0}^2 \sum_{j=0}^3 i^2 j^3$
35. Show that $\sum_{j=1}^n (a_j - a_{j-1}) = a_n - a_0$, where a_0, a_1, \dots, a_n is a sequence of real numbers. This type of sum is called **telescoping**.
36. Use the identity $1/(k(k+1)) = 1/k - 1/(k+1)$ and Exercise 35 to compute $\sum_{k=1}^n 1/(k(k+1))$.
37. Sum both sides of the identity $k^2 - (k-1)^2 = 2k - 1$ from $k = 1$ to $k = n$ and use Exercise 35 to find
- a) a formula for $\sum_{k=1}^n (2k - 1)$ (the sum of the first n odd natural numbers).
- b) a formula for $\sum_{k=1}^n k$.
- *38. Use the technique given in Exercise 35, together with the result of Exercise 37b, to derive the formula for $\sum_{k=1}^n k^2$ given in Table 2. [Hint: Take $a_k = k^3$ in the telescoping sum in Exercise 35.]
39. Find $\sum_{k=100}^{200} k$. (Use Table 2.)
40. Find $\sum_{k=99}^{200} k^3$. (Use Table 2.)
41. Find $\sum_{k=10}^{20} k^2(k-3)$. (Use Table 2.)
42. Find $\sum_{k=10}^{20} (k-1)(2k^2+1)$. (Use Table 2.)
- *43. Find a formula for $\sum_{k=0}^m \lfloor \sqrt{k} \rfloor$, when m is a positive integer.
- *44. Find a formula for $\sum_{k=0}^m \lfloor \sqrt[3]{k} \rfloor$, when m is a positive integer.
- There is also a special notation for products. The product of a_m, a_{m+1}, \dots, a_n is represented by $\prod_{j=m}^n a_j$, read as the product from $j = m$ to $j = n$ of a_j .
45. What are the values of the following products?
- a) $\prod_{i=0}^{10} i$ b) $\prod_{i=5}^8 i$
- c) $\prod_{i=1}^{100} (-1)^i$ d) $\prod_{i=1}^{10} 2$
- Recall that the value of the factorial function at a positive integer n , denoted by $n!$, is the product of the positive integers from 1 to n , inclusive. Also, we specify that $0! = 1$.
46. Express $n!$ using product notation.
47. Find $\sum_{j=0}^4 j!$.
48. Find $\prod_{j=0}^4 j!$.

2.5 Cardinality of Sets

2.5.1 Introduction

In Definition 4 of Section 2.1 we defined the cardinality of a finite set as the number of elements in the set. We use the cardinalities of finite sets to tell us when they have the same size, or when one is bigger than the other. In this section we extend this notion to infinite sets. That is, we will define what it means for two infinite sets to have the same cardinality, providing us with a way to measure the relative sizes of infinite sets.

We will be particularly interested in countably infinite sets, which are sets with the same cardinality as the set of positive integers. We will establish the surprising result that the set of