

The truth table for the exclusive or of two propositions is displayed in Table 4.

EXAMPLE 8 Let p and q be the propositions that state “A student can have a salad with dinner” and “A student can have soup with dinner,” respectively. What is $p \oplus q$, the exclusive or of p and q ?

Solution: The exclusive or of p and q is the statement that is true when exactly one of p and q is true. That is, $p \oplus q$ is the statement “A student can have soup or salad, but not both, with dinner.” Note that this is often stated as “A student can have soup or a salad with dinner,” without explicitly stating that taking both is not permitted. ◀

EXAMPLE 9 Express the statement “I will use all my savings to travel to Europe or to buy an electric car” in propositional logic using the statement p : “I will use all my savings to travel to Europe” and the statement q : “I will use all my savings to buy an electric car.”

Solution: To translate this statement, we first note that the or in this statement must be an exclusive or because this student can either use all his or her savings to travel to Europe or use all these savings to buy an electric car, but cannot both go to Europe and buy an electric car. (This is clear because either option requires all his savings.) Hence, this statement can be expressed as $p \oplus q$. ▶

1.1.3 Conditional Statements

We will discuss several other important ways in which propositions can be combined.

Definition 5 Let p and q be propositions. The *conditional statement* $p \rightarrow q$ is the proposition “if p , then q .” The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \rightarrow q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

Assessment ▶

The statement $p \rightarrow q$ is called a conditional statement because $p \rightarrow q$ asserts that q is true on the condition that p holds. A conditional statement is also called an **implication**.

The truth table for the conditional statement $p \rightarrow q$ is shown in Table 5. Note that the statement $p \rightarrow q$ is true when both p and q are true and when p is false (no matter what truth value q has).

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

TABLE 5 The Truth Table for the Conditional Statement $p \rightarrow q$.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

$$(p, q) = (T, F) \Rightarrow p \rightarrow q = F$$

hypothesis
false \Rightarrow Proposition
True

hypothesis
true
 \Rightarrow Proposition
= conclusion

hypothesis sufficient
for conclusion

conclusion
necessary
for hypothesis

① P is suff for q

$\hookrightarrow P := T$

$\Rightarrow q = F$ means

$P \rightarrow q = F$

$\Rightarrow q = T$ means

$P \rightarrow q = T$

$\Rightarrow P \rightarrow q = q$

② q is necessary for P

$\hookrightarrow q := F$

$\Rightarrow P = T$ means $P \rightarrow q = F$

$\Rightarrow P = F$ means $P \rightarrow q = T$

$\Rightarrow P \rightarrow q = \neg P$

③ q unless $\neg P$

\hookrightarrow when $P := F, \neg P = T$

$\Rightarrow P \rightarrow q = T$

otherwise,

$P := T,$

$P \rightarrow q = q$

$T \rightarrow q = q$
 $P \rightarrow T = T$
 $P \rightarrow F = \neg P$

check exercise problems

You might have trouble understanding how "unless" is used in conditional statements unless you read this paragraph carefully.

Because conditional statements play such an essential role in mathematical reasoning, a variety of terminology is used to express $p \rightarrow q$. You will encounter most if not all of the following ways to express this conditional statement:

"if p , then q "

"if p , q "

" p is sufficient for q "

" q if p "

" q when p "

"a necessary condition for p is q "

" q unless $\neg p$ "

" p implies q "

" p only if q "

"a sufficient condition for q is p "

" q whenever p "

" q is necessary for p "

" q follows from p "

" q provided that p "

q depends on p

A useful way to understand the truth value of a conditional statement is to think of an obligation or a contract. For example, the pledge many politicians make when running for office is

"If I am elected, then I will lower taxes."

If the politician is elected, voters would expect this politician to lower taxes. Furthermore, if the politician is not elected, then voters will not have any expectation that this person will lower taxes, although the person may have sufficient influence to cause those in power to lower taxes. It is only when the politician is elected but does not lower taxes that voters can say that the politician has broken the campaign pledge. This last scenario corresponds to the case when p is true but q is false in $p \rightarrow q$.

Similarly, consider a statement that a professor might make:

"If you get 100% on the final, then you will get an A."

If you manage to get 100% on the final, then you would expect to receive an A. If you do not get 100%, you may or may not receive an A depending on other factors. However, if you do get 100%, but the professor does not give you an A, you will feel cheated.

Remark: Because some of the different ways to express the implication p implies q can be confusing, we will provide some extra guidance. To remember that " p only if q " expresses the same thing as "if p , then q ," note that " p only if q " says that p cannot be true when q is not true. That is, the statement is false if p is true, but q is false. When p is false, q may be either true or false, because the statement says nothing about the truth value of q .

For example, suppose your professor tells you

"You can receive an A in the course only if your score on the final is at least 90%."

Then, if you receive an A in the course, then you know that your score on the final is at least 90%. If you do not receive an A, you may or may not have scored at least 90% on the final. Be careful not to use " q only if p " to express $p \rightarrow q$ because this is incorrect. The word "only" plays an essential role here. To see this, note that the truth values of " q only if p " and $p \rightarrow q$ are different when p and q have different truth values. To see why " q is necessary for p " is equivalent to "if p , then q ," observe that " q is necessary for p " means that p cannot be true unless q is true, or that if q is false, then p is false. This is the same as saying that if p is true, then q is true. To see why " p is sufficient for q " is equivalent to "if p , then q ," note that " p is sufficient for q " means if p is true, it must be the case that q is also true. This is the same as saying that if p is true, then q is also true.

To remember that " q unless $\neg p$ " expresses the same conditional statement as "if p , then q ," note that " q unless $\neg p$ " means that if $\neg p$ is false, then q must be true. That is, the statement " q unless $\neg p$ " is false when p is true but q is false, but it is true otherwise. Consequently, " q unless $\neg p$ " and $p \rightarrow q$ always have the same truth value.

We illustrate the translation between conditional statements and English statements in Example 10.

EXAMPLE 10

Extra
Examples

Let p be the statement “Maria learns discrete mathematics” and q the statement “Maria will find a good job.” Express the statement $p \rightarrow q$ as a statement in English.

Solution: From the definition of conditional statements, we see that when p is the statement “Maria learns discrete mathematics” and q is the statement “Maria will find a good job,” $p \rightarrow q$ represents the statement

“If Maria learns discrete mathematics, then she will find a good job.”

There are many other ways to express this conditional statement in English. Among the most natural of these are

“Maria will find a good job when she learns discrete mathematics.”

“For Maria to get a good job, it is sufficient for her to learn discrete mathematics.”

and

“Maria will find a good job unless she does not learn discrete mathematics.”

Note that the way we have defined conditional statements is more general than the meaning attached to such statements in the English language. For instance, the conditional statement in Example 10 and the statement

“If it is sunny, then we will go to the beach”

are statements used in normal language where there is a relationship between the hypothesis and the conclusion. Further, the first of these statements is true unless Maria learns discrete mathematics, but she does not get a good job, and the second is true unless it is indeed sunny, but we do not go to the beach. On the other hand, the statement

“If Juan has a smartphone, then $2 + 3 = 5$ ”

is true from the definition of a conditional statement, because its conclusion is true. (The truth value of the hypothesis does not matter then.) The conditional statement

“If Juan has a smartphone, then $2 + 3 = 6$ ”

is true if Juan does not have a smartphone, even though $2 + 3 = 6$ is false. We would not use these last two conditional statements in natural language (except perhaps in sarcasm), because there is no relationship between the hypothesis and the conclusion in either statement. In mathematical reasoning, we consider conditional statements of a more general sort than we use in English. The mathematical concept of a conditional statement is independent of a cause-and-effect relationship between hypothesis and conclusion. Our definition of a conditional statement specifies its truth values; it is not based on English usage. Propositional language is an artificial language; we only parallel English usage to make it easy to use and remember.

The if-then construction used in many programming languages is different from that used in logic. Most programming languages contain statements such as **if p then S** , where p is a proposition and S is a program segment (one or more statements to be executed). (Although this looks as if it might be a conditional statement, S is not a proposition, but rather is a set of executable instructions.) When execution of a program encounters such a statement, S is executed if p is true, but S is not executed if p is false, as illustrated in Example 11.

$p \rightarrow T = T$

$p \rightarrow F = \neg p$

EXAMPLE 11 What is the value of the variable x after the statement

if $2 + 2 = 4$ then $x := x + 1$

if $x = 0$ before this statement is encountered? (The symbol $:=$ stands for assignment. The statement $x := x + 1$ means the assignment of the value of $x + 1$ to x .)

Solution: Because $2 + 2 = 4$ is true, the assignment statement $x := x + 1$ is executed. Hence, x has the value $0 + 1 = 1$ after this statement is encountered. ◀

CONVERSE, CONTRAPOSITIVE, AND INVERSE We can form some new conditional statements starting with a conditional statement $p \rightarrow q$. In particular, there are three related conditional statements that occur so often that they have special names. The proposition $q \rightarrow p$ is called the **converse** of $p \rightarrow q$. The **contrapositive** of $p \rightarrow q$ is the proposition $\neg q \rightarrow \neg p$. The proposition $\neg p \rightarrow \neg q$ is called the **inverse** of $p \rightarrow q$. We will see that of these three conditional statements formed from $p \rightarrow q$, **only the contrapositive always has the same truth value as $p \rightarrow q$.**

We first show that the contrapositive, $\neg q \rightarrow \neg p$, of a conditional statement $p \rightarrow q$ always has the same truth value as $p \rightarrow q$. To see this, note that the contrapositive is false only when $\neg p$ is false and $\neg q$ is true, that is, only when p is true and q is false. We now show that neither the converse, $q \rightarrow p$, nor the inverse, $\neg p \rightarrow \neg q$, has the same truth value as $p \rightarrow q$ for all possible truth values of p and q . Note that when p is true and q is false, the original conditional statement is false, but the converse and the inverse are both true.

When two compound propositions always have the same truth values, regardless of the truth values of its propositional variables, we call them **equivalent**. Hence, a conditional statement and its contrapositive are equivalent. The converse and the inverse of a conditional statement are also equivalent, as the reader can verify, but neither is equivalent to the original conditional statement. (We will study equivalent propositions in Section 1.3.) Take note that one of the most common logical errors is to assume that the converse or the inverse of a conditional statement is equivalent to this conditional statement.

We illustrate the use of conditional statements in Example 12.

EXAMPLE 12 Find the contrapositive, the converse, and the inverse of the conditional statement

Extra
Examples ▶

“The home team wins whenever it is raining.”

Solution: Because “ q whenever p ” is one of the ways to express the conditional statement $p \rightarrow q$, the original statement can be rewritten as

“If it is raining, then the home team wins.”

Consequently, the contrapositive of this conditional statement is

“If the home team does not win, then it is not raining.”

The converse is

“If the home team wins, then it is raining.”

The inverse is

“If it is not raining, then the home team does not win.”

Only the contrapositive is equivalent to the original statement. ◀

Given $p \rightarrow q$,
inverse : $\neg p \rightarrow \neg q$
converse : $q \rightarrow p$
contrapositive :
 $\neg q \rightarrow \neg p$

inverse = F when
 $\neg p = T, \neg q = F$
 $(p, q) = (F, T)$
converse = F when
 $q = T, p = F$
 $(p, q) = (F, T)$
contr. = F when
 $\neg q = T, \neg p = F$
 $(p, q) = (T, F)$
Remember that the
contrapositive, but
neither the converse or
inverse, of a conditional
statement is equivalent
to it.

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

$p \rightarrow q$	$q \rightarrow p$
T	T
F	T
T	F
T	F

BICONDITIONALS We now introduce another way to combine propositions that expresses that two propositions have the same truth value.

Definition 6

Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition “ p if and only if q .” The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

The truth table for $p \leftrightarrow q$ is shown in Table 6. Note that the statement $p \leftrightarrow q$ is true when both the conditional statements $p \rightarrow q$ and $q \rightarrow p$ are true and is false otherwise. That is why we use the words “if and only if” to express this logical connective and why it is symbolically written by combining the symbols \rightarrow and \leftarrow . There are some other common ways to express $p \leftrightarrow q$:

“ p is necessary and sufficient for q ”
 “if p then q , and conversely”
 “ p iff q .” “ p exactly when q .”

The last way of expressing the biconditional statement $p \leftrightarrow q$ uses the abbreviation “iff” for “if and only if.” Note that $p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$.

EXAMPLE 13

Let p be the statement “You can take the flight,” and let q be the statement “You buy a ticket.” Then $p \leftrightarrow q$ is the statement

“You can take the flight if and only if you buy a ticket.”

*Extra**Examples*

This statement is true if p and q are either both true or both false, that is, if you buy a ticket and can take the flight or if you do not buy a ticket and you cannot take the flight. It is false when p and q have opposite truth values, that is, when you do not buy a ticket, but you can take the flight (such as when you get a free trip) and when you buy a ticket but you cannot take the flight (such as when the airline bumps you). ◀

IMPLICIT USE OF BICONDITIONALS You should be aware that biconditionals are not always explicit in natural language. In particular, the “if and only if” construction used in biconditionals is rarely used in common language. Instead, biconditionals are often expressed using an “if, then” or an “only if” construction. The other part of the “if and only if” is implicit. That is, the converse is implied, but not stated. For example, consider the statement in English “If you finish your meal, then you can have dessert.” What is really meant is “You can have dessert if and only if you finish your meal.” This last statement is logically equivalent to the two statements “If you finish your meal, then you can have dessert” and “You can have dessert only if you finish your meal.” Because of this imprecision in natural language, we need to make an assumption whether a conditional statement in natural language implicitly includes its converse. Because precision is essential in mathematics and in logic, we will always distinguish between the conditional statement $p \rightarrow q$ and the biconditional statement $p \leftrightarrow q$.

$$\begin{aligned} T \leftrightarrow T &= T \\ F \leftrightarrow F &= T \\ T \leftrightarrow F &= F \\ F \leftrightarrow T &= F \end{aligned}$$

1.1.4 Truth Tables of Compound Propositions

Demo

We have now introduced five important logical connectives—conjunction, disjunction, exclusive or, implication, and the biconditional operator—as well as negation. We can use these connectives to build up complicated compound propositions involving any number of propositional variables. We can use truth tables to determine the truth values of these compound propositions, as Example 14 illustrates. We use a separate column to find the truth value of each compound expression that occurs in the compound proposition as it is built up. The truth values of the compound proposition for each combination of truth values of the propositional variables in it is found in the final column of the table.

EXAMPLE 14 Construct the truth table of the compound proposition

$$(p \vee \neg q) \rightarrow (p \wedge q).$$

Solution: Because this truth table involves two propositional variables p and q , there are four rows in this truth table, one for each of the pairs of truth values TT, TF, FT, and FF. The first two columns are used for the truth values of p and q , respectively. In the third column we find the truth value of $\neg q$, needed to find the truth value of $p \vee \neg q$, found in the fourth column. The fifth column gives the truth value of $p \wedge q$. Finally, the truth value of $(p \vee \neg q) \rightarrow (p \wedge q)$ is found in the last column. The resulting truth table is shown in Table 7.

TABLE 7 The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

= q

1.1.5 Precedence of Logical Operators

We can construct compound propositions using the negation operator and the logical operators defined so far. We will generally use parentheses to specify the order in which logical operators in a compound proposition are to be applied. For instance, $(p \vee q) \wedge (\neg r)$ is the conjunction of $p \vee q$ and $\neg r$. However, to reduce the number of parentheses, we specify that the negation operator is applied before all other logical operators. This means that $\neg p \wedge q$ is the conjunction of $\neg p$ and q , namely, $(\neg p) \wedge q$, not the negation of the conjunction of p and q , namely $\neg(p \wedge q)$. (It is generally the case that unary operators that involve only one object precede binary operators.)

Another general rule of precedence is that the conjunction operator takes precedence over the disjunction operator, so that $p \vee q \wedge r$ means $p \vee (q \wedge r)$ rather than $(p \vee q) \wedge r$ and $p \wedge q \vee r$ means $(p \wedge q) \vee r$ rather than $p \wedge (q \vee r)$. Because this rule may be difficult to remember, we will continue to use parentheses so that the order of the disjunction and conjunction operators is clear.

Finally, it is an accepted rule that the conditional and biconditional operators, \rightarrow and \leftrightarrow , have lower precedence than the conjunction and disjunction operators, \wedge and \vee . Consequently, $p \rightarrow q \vee r$ means $p \rightarrow (q \vee r)$ rather than $(p \rightarrow q) \vee r$ and $p \vee q \rightarrow r$ means $(p \vee q) \rightarrow r$ rather than $p \vee (q \rightarrow r)$. We will use parentheses when the order of the conditional operator and biconditional operator is at issue, although the conditional operator has precedence over the biconditional operator. Table 8 displays the precedence levels of the logical operators, \neg , \wedge , \vee , \rightarrow , and \leftrightarrow .

TABLE 8
Precedence of
Logical Operators.

Operator	Precedence
\neg	1
\wedge	2
\vee	3
\rightarrow	4
\leftrightarrow	5