

1. Show that: $0+1+2+\dots+n = \frac{n(n+1)}{2}$

$$\text{i}) 0+1+2+\dots+n = \frac{n(n+1)}{2} \quad [\text{for } n]$$

$$\text{ii}) 0+1+2+\dots+(n+1) = \frac{(n+1)(n+1+1)}{2} \quad [\text{for } (n+1)]$$

$$\therefore 0+1+2+\dots+n+(n+1) = \frac{(n+1)(n+1+1)}{2}$$

$$\Rightarrow \frac{n(n+1)}{2} + (n+1) = \frac{(n+1)(n+2)}{2}$$

$$\Rightarrow \frac{n(n+1)}{2} + \frac{2(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

$$\Rightarrow \frac{(n+2)(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

$$\therefore \frac{(n+1)(n+2)}{2} = \frac{(n+1)(n+2)}{2}$$

Proven

$$2 \cdot n > 2 \quad n+1 < n^2$$

Base case:

$$\text{let } n = n_0 = 2$$

$$\begin{array}{ll} n_0 > 2 & n_0 + 1 < n_0^2 \\ \Rightarrow 2 > 2 & \Rightarrow 2 + 1 < 2^2 \\ & \Rightarrow 3 < 4 \end{array}$$

for $(n+1)$ prove $(n+1) + 1 < (n+1)^2$

• rewrite as: $(n+1)^2 > (n+1) + 1$

$$\Rightarrow (n+1)^2 > n+2$$

LHS:

$$(n+1)^2$$

$$= n^2 + 2n + 1$$

$$= n^2 + 2n + 1$$

$$= (n+1) + 2n + 1$$

$$= n+1 + 2n + 1$$

$$= 3n + 2$$

$$[3n > n]$$

$$= n+2$$

proven

3] $n > 4$, $\nabla \rightarrow n \quad n! > n^2$

$P(n): n! > n^2$

Base case: $n=4$, $4! = 24 > 16 = 4^2$

$4! > 4^2$ Base proved

Inductive Step:

Let $P(n=m)$ is true,

i.e. for an $m \in \mathbb{N}$, $m! > m^2$ ————— ①

Using ①, we have to prove that, $P(n=m+1)$ is also true.

i.e. use " " " " " , $(m+1)! > (m+1)^2$ ————— ②

To prove ②,

L.H.S of ② = $(m+1)!$

$$= (m+1)m!$$

$$= m \cdot m! + m!$$

$$> 24m + m^2 \quad [\text{From ①, } m^2 < m! \text{ and from base, } m \geq 4 \\ \Rightarrow 24m > 24]$$

$$= m^2 + 2m + 22m$$

$$> m^2 + 2m + 1 \quad [\because 22m > 1]$$

$$\Rightarrow (m+1)^2$$

∴ Shown

$$4. F_1 + F_3 + F_5 + \dots + F_{2n-1} = F_{2n} - 1 \quad \text{for } n \geq 1$$

let $n = n_0 = 1$

$$F_{2(1)-1} = F_{2(1)} - 1$$

$$\Rightarrow F_1 = F_2 - 1$$

$$\Rightarrow 1 = 2 - 1$$

$\Rightarrow 1 = 1$ proven

for $(n+1)$

$$(F_1 + F_3 + F_5 + \dots + F_{2n-1}) + F_{2(n+1)-1} = F_{2(n+1)} - 1$$

$$\Rightarrow (F_1 + F_3 + \dots + F_{2n-1}) + F_{2n+2-1} = F_{2n+2} - 1$$

$$\Rightarrow F_{2n}-1 + F_{2n+1} = F_{2n+2} - 1$$

$$\Rightarrow F_{2n} + F_{2n+1} - 1 = F_{2n+2} - 1$$

In fibonacci series : $F_1 + F_2 = F_3$

$$F_2 + F_3 = F_4$$

.....

$$\Rightarrow (F_{2n} + F_{2n+1}) - 1 = F_{2n+2} - 1$$

$$\Rightarrow F_{2n+2} - 1 = F_{2n+2} - 1 \quad \text{proven}$$

s-

$$(1-x)(1+x + x^2 + x^3 + \dots + x^n) = 1 - x^{n+1}$$

for $n=0$

$$(1-x)(1-x^0) = 1 - x^{0+1}$$

$$\Rightarrow (1-x)(1) = 1-x$$

$$\Rightarrow (1-x) = 1-x$$

proven

let $n > 0$

$$(1-x)(x^0 + x^1 + x^2 + \dots + x^n + x^{n+1}) = 1 - x^{(n+1)+1}$$

$$\Rightarrow (1-x)(x^0 + x^1 + x^2 + \dots + x^n) + (1-x)(x^{n+1}) = 1 - x^{n+2}$$

$$\Rightarrow (1-x^{n+1}) + (1-x)(x^{n+1})$$

$$\Rightarrow 1 - x^{n+1} + x^{n+1} - x^{1+(n+1)}$$

$$\Rightarrow 1 - x^{n+2}$$

proven

$$6. \quad 3 \cdot 2 + 3 \cdot 2^2 + 3 \cdot 2^3 + \dots + 3 \cdot 2^n$$

$$\Rightarrow 6 \cdot 2^0 + 6 \cdot 2^1 + 6 \cdot 2^2 + \dots + 6 \cdot 2^{n-1}$$

we know $\sum_{i=0}^n a \cdot r^i = a \cdot \frac{1-r^{n+1}}{1-r}$

$$6 \cdot 2^0 + 6 \cdot 2^1 + 6 \cdot 2^2 + \dots + 6 \cdot 2^{n-1} = \sum_{i=0}^{n-1} 6 \cdot 2^i = 6 \cdot \frac{1-2^{(n-1)+1}}{1-2}$$

$$= 6 \cdot \frac{1-2^n}{(-1)}$$

$$= 6(2^n - 1)$$

$$7. \quad 2 + 10 + 50 + \dots + 1250$$

$$= 2 + 2 \cdot 5 + 2 \cdot 5^2 + \dots + 2 \cdot 5^4$$

$$= 2 \cdot \frac{(1-5^{4+1})}{1-5} = 1562$$

$$8. \quad a_0 = 0, \quad a_1 = 2, \quad a_n = 4(a_{n-1} - a_{n-2}) \text{ for } n \geq 2$$

$$b_n = n \cdot 2^n = a_n$$

$$\text{for } n=0$$

$$b_0 = 0 \cdot 2^{(0)} = 0 = a_0$$

$$\text{for } n=1$$

$$b_1 = 1 \cdot 2^{(1)} = 2 = a_1$$

$$\begin{aligned} a_n &= 4(a_{n-1} - a_{n-2}) & a_{n-1} &= b_{n-1} = (n-1)2^{(n-1)} \\ &= 4((n-1)2^{n-1} - (n-2)2^{n-2}) & &= (n-1)2^{n-1} \end{aligned}$$

$$= 4(n \cdot 2^{n-1} - 2^{n-1} - n \cdot 2^{n-2} + 2^{n-2+1}) \quad | \quad a_{n-2} = b_{n-2} = (n-2)2^{n-2}$$

$$= 4(n \cdot 2^{n-1} - n \cdot 2^{n-2} - 2^{n-1} + 2^{n-1})$$

$$= 4(n \cdot 2^{n-1} - n \cdot 2^{n-2})$$

$$= 2^2 4(n) (2^{n-1} - 2^{n-2})$$

$$= 4(n) (2^{n-2+1} - 2^{n-2})$$

$$= 4(n) (2^{n-2} \cdot 2 - 2^{n-2})$$

$$= 4(n) 2^{n-2} (2-1) = 4n \cdot 2^{n-2} = 2^2 \cdot n \cdot 2^{n-2} = n \cdot 2^n = b_n$$

proven

$$q. \quad a_0 = 1, a_1 = 1, a_n = 2 \cdot a_{n-1} + 3 \cdot a_{n-2} \quad \text{for } n \geq 2$$

$$b_n = \frac{1}{2} \cdot 3^n + \frac{1}{2} \cdot (-1)^n$$

$$a_n = b_n$$

$n=0$

$$b_0 = \frac{1}{2} (3^0 + (-1)^0) = \frac{1}{2} (1+1) = 1 = a_0$$

$$b_1 = \frac{1}{2} (3^1 + (-1)^1) = \frac{1}{2} (3-1) = 1 = a_1$$

$$a_n = 2 \cdot a_{n-1} + 3 \cdot a_{n-2}$$

$$a_{n-1} = b_{n-1} = \frac{1}{2} \cdot 3^{n-1} + \frac{1}{2} \cdot (-1)^{n-1}$$

$$= \frac{1}{2} \cdot 3^{n-1} + \frac{1}{2} \cdot (-1)^n \cdot (-1)$$

$$= \frac{1}{2} \cdot 3^{n-1} + \frac{1}{2} \cdot (-1)^n \cdot (-1)$$

$$a_{n-2} = b_{n-2} = \frac{1}{2} \cdot 3^{n-2} + \frac{1}{2} \cdot (-1)^{n-2}$$

$$= \frac{1}{2} \cdot 3^{n-2} + \frac{1}{2} \cdot (-1)^n \cdot (-1)$$

$$= \frac{1}{2} \cdot 3^{n-2} + \frac{1}{2} \cdot (-1)^n$$

$$\begin{aligned}
 a_n &= 2 \left(\frac{1}{2} \cdot 3^{n-1} + \frac{1}{2} (-1)^n (-1) \right) + 3 \left(\frac{1}{2} 3^{n-2} + \frac{1}{2} (-1)^n \right) \\
 &= 2 \left(\frac{1}{2} \cdot 3^{n-1} - \frac{1}{2} (-1)^n \right) + 3 \left(\frac{1}{2} 3^{n-2} + \frac{1}{2} (-1)^n \right) \\
 &= 2 \left(\frac{1}{2} (3^{n-1} - (-1)^n) \right) + 3 \left(\frac{1}{2} (3^{n-2} + (-1)^n) \right) \\
 &= 3^{n-1} - (-1)^n + 3 \left(\frac{3^{n-2}}{2} + \frac{(-1)^n}{2} \right) \\
 &= 3^{n-1} - (-1)^n + \frac{3^{n-2+1}}{2} + \frac{3}{2} (-1)^n \\
 &= 3^{n-1} + \frac{3^{n-1}}{2} + \frac{3}{2} (-1)^n - (-1)^n \\
 &= 3^{n-1} + \frac{3^{n-1}}{2} + \frac{3}{2} (-1)^n - (-1)^n \\
 &= 3^{n-1} \left(1 + \frac{1}{2} \right) + (-1)^n \left(\frac{3}{2} - 1 \right) \\
 &= 3^n - 3^{-1} \left(\frac{3}{2} \right) + \left(\frac{1}{2} \right) (-1)^n \\
 &= 3^n \cdot \frac{1}{3} \left(\frac{3}{2} \right) + \left(\frac{1}{2} \right) (-1)^n
 \end{aligned}$$

$$a_n = \frac{1}{2} \cdot 3^n + (-1)^n = b_n$$

proven

10.

$$F(3) = 2F(2) + 3F(1) + F(0) = 10 + 9 + 2 = 21$$

$$F(4) = 2F(3) + 3F(2) + F(1) = 42 + 15 + 3 = 60$$

$$F(5) = 2F(4) + 3F(3) + F(2) = 120 + 63 + 5 = 188$$

$$11. T(n) = 2T(n-1) + 1$$

$$= 2(2T(n-2) + 1) + 1$$

$$= 4T(n-2) + 2 + 1$$

$$= 4(2T(n-3) + 1) + 2 + 1$$

$$= 8T(n-3) + 4 + 2 + 1$$

$$\frac{T(n)}{2} = 2^3 T(n-3) + 2^{3-1} + 2^{2-1} + 2^{1-1}$$

$$\therefore T(n) = 2^i T(n-i) + 2^{i-1} + 2^{i-2} + \dots + 2 + 1$$

Let $i = n - 2$

$$\therefore T(n) = 2^{n-2} T(n-(n-2)) + 2^{n-3} + 2^{n-4} + 2^{n-5} + \dots + 2 + 1$$

$$= 2^{n-2} T(\frac{n}{2}) + 2^{n-3} + 2^{n-4} + 2^{n-5} + \dots + 2 + 1$$

$$= 2^{n-2} (2T(1) + 1) + 2^{n-3} + 2^{n-4} + \dots + 2 + 1$$

$$= 2^{n-1} T(1) + 2^{n-2} + 2^{n-3} + \dots + 2 + 1$$

$$T(n) = 2^{n-1} \cdot (1) + 2^{n-2} + 2^{n-3} + \dots + 2 + 1$$

$$\therefore T(n) = \sum_{i=0}^{n-1} 2^i = 1 + 2 + 4 + 8 + \dots + 2^{n-1}$$

$$2T(n) = 2 + 4 + 8 + 16 + \dots + 2^n$$

$$2T(n) - T(n) = T(n)$$

$$\Rightarrow (2 + 4 + 8 + 16 + \dots + 2^n) - (1 + 2 + 4 + 8 + \dots + 2^{n-1})$$

$$\Rightarrow (2 + 4 + 8 + 16 + \dots + 2^{n-1} + 2^n)$$

$$-(1 + 2 + 4 + 8 + 16 + \dots + 2^{n-1})$$

$$= 2^n - 1 = T(n)$$

$$\begin{aligned}
 12. T(n) &= cT(n-1) + f(n) \\
 &= c(cT(n-2) + f(n-1)) + f(n) \\
 &= c^2T(n-2) + cf(n-1) + f(n) \\
 &= c^2(cT(n-3) + f(n-2)) + cf(n-1) + f(n) \\
 &= c^3T(n-3) + c^2f(n-2) + cf(n-1) + f(n) \\
 &= c^3(cT(n-4) + f(n-3)) + c^2f(n-2) + cf(n-1) + f(n) \\
 &= c^4T(n-4) + c^3f(n-3) + c^2f(n-2) + cf(n-1) + f(n)
 \end{aligned}$$

$$T(n) = c^i T(n-i) + \sum_{j=0}^{i-1} c^j f(n-j)$$

let $i = n-k \Rightarrow k = n-i$
we get

$$T(n) = c^{n-k} T(k) + \sum_{j=0}^{n-k-1} c^j f(n-j)$$

$$= \sum_{j=0}^{n-k-1} c^j f(n-j) + c^{n-k} T(k)$$

$$= f(n) + c f(n-1) + c^2 f(n-2) + \dots + c^{n-k-1} f(k+1) + c^{n-k} \cdot f(k)$$

$$T(n) = \sum_{j=0}^{n-k} c^j f(n-j) = \sum_{j=k}^n c^{n-j} f(j)$$

$$13. \quad T(n) = 2T\left(\frac{n}{2}\right) + n \quad T(1) = 1$$

$$T(n) = 2\left(2T\left(\frac{n}{4}\right) + \frac{n}{2}\right) + n$$

$$= 4T\left(\frac{n}{4}\right) + n + n$$

$$= 4\left(2T\left(\frac{n}{8}\right) + \frac{n}{4}\right) + 2n$$

$$= 8T\left(\frac{n}{8}\right) + 3n$$

$$= 8\left(2T\left(\frac{n}{16}\right) + \frac{n}{8}\right) + 3n$$

$$T(n) = 16T\left(\frac{n}{16}\right) + 4n$$

$$T(n) = 2^k \cdot \left(\frac{n}{2^k}\right) + kn$$

$$\text{let } n = 2^k$$

$$\therefore T(n) = n \cdot T(1) + \frac{\log n}{\log 2} \cdot n$$

$$T(n) = n \cdot (1) + n \cdot \frac{\log n}{\log 2}$$

$$T(1) = 1$$

$$\begin{cases} n = 2^k \\ \Rightarrow \log n = k \log 2 \\ \Rightarrow \frac{\log n}{\log 2} = k \end{cases}$$

$$14. T(n) = 2T\left(\frac{n}{2}\right) + 1, \quad T(1) = 1$$

$$T(n) = 2T\left(\frac{n}{2}\right) + 1$$

$$= 2\left(2T\left(\frac{n}{4}\right) + 1\right) + 1$$

$$= 4T\left(\frac{n}{4}\right) + 2 + 1$$

$$= 4\left(2T\left(\frac{n}{8}\right) + 1\right) + 3$$

$$= 8T\left(\frac{n}{8}\right) + 4 + 3 = 8T\left(\frac{n}{8}\right) + 7$$

$$T(n) = 8\left(2T\left(\frac{n}{16}\right) + 1\right) + 7 = 16T\left(\frac{n}{16}\right) + 15$$

$$T(n) = 2^k T\left(\frac{n}{2^k}\right) + 2^k - 1$$

$$\text{let } 2^k = n$$

$$\therefore T(n) = n T\left(\frac{n}{n}\right) + n - 1$$

$$= n \cdot T(1) + n - 1 \quad | T(1) = 1$$

$$T(n) = n + n - 1$$

15.

$$\begin{aligned}
 T(n) &= T(n-1) + n , \quad T(1)=1 \\
 &= T(n-2) + (n-1) + n \\
 &= T(n-3) + (n-2) + (n-1) + n \\
 &\vdots \\
 &= T(n-k) + (n-k+1) + (n-k+2) + \dots + (n-1) + n
 \end{aligned}$$

$$T(n) = T(n-k) + (n-k+1) + (n-k+2) + \dots + (n-1) + n$$

$$\text{let } k = n-1$$

$$\begin{aligned}
 T(n) &= T(n-(n-1)) + (n-(n-1)+1) + \dots + (n-1) + n \\
 &= T(1) + (2) + (3) + \dots + (n-1) + n \\
 &= 1 + 2 + 3 + \dots + n
 \end{aligned}$$

$$\therefore T(n) = \frac{n(n+1)}{2}$$

