Department of Computer Science and Engineering (CSE)
BRAC University

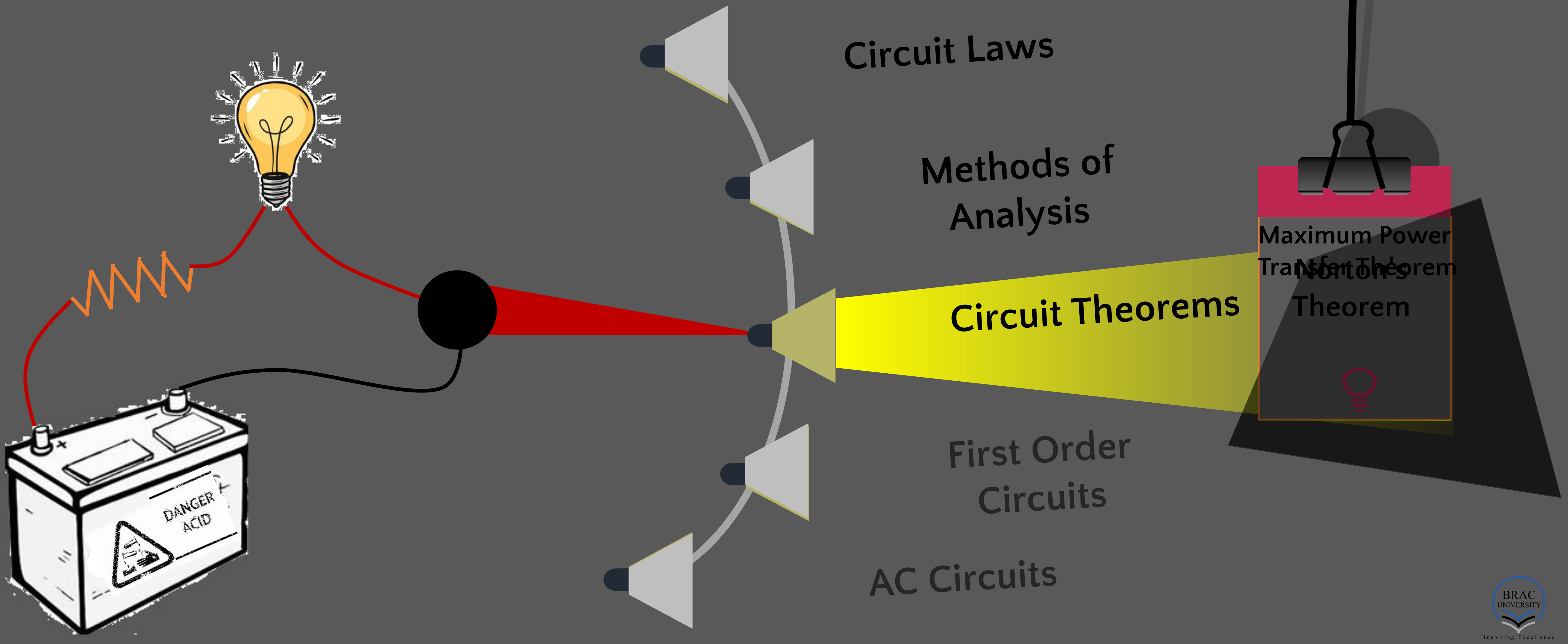
Fall 2023

CSE250 - Circuits and Electronics

Maximum Power Transfer Theorem

Kaniz Fatema Supti, Adjunct Lecturer
Department of Computer Science and Engineering (CSE)
BRAC University

Course Outline: broad themes



Maximum Power Transfer

- In many practical situations, a circuit is designed to provide power to a load. There are applications in areas such as communications and amplification where it is desirable to maximize the power delivered to an antenna (load) and a speaker respectively.

- When designing a circuit, it is often important to be able to answer the question, *"What load should be applied to a system or what driving circuitry for a particular load should be designed to ensure that the load is receiving maximum power from the system or from the circuit respectively?"*

- Given a system with known internal losses the Thevenin equivalent is useful in finding the maximum power a linear circuit can deliver to a load. We assume that we can adjust the load resistance R_L.

- Maximum power is transferred to the load when the load resistance equals the Thevenin resistance as seen from the load ($R_L = R_{Th}$). This is known as the *Maximum Power Transfer Theorem.*

Graphically

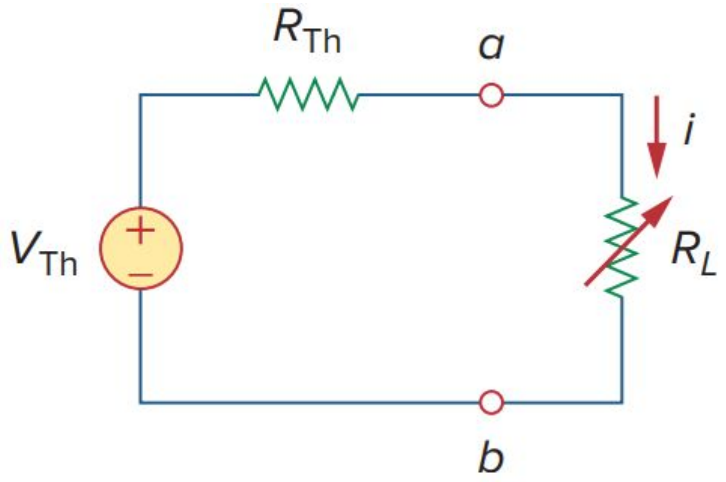
- Given any linear two terminal circuit, it can be reduced to a Thevenin equivalent as shown. Power delivered to the load by the Thevenin equivalent circuit is then,

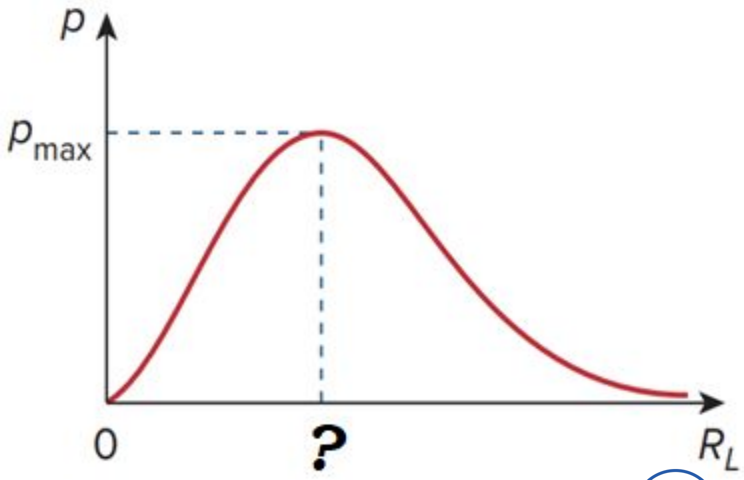
$$p = i^2 R = \left(\frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L$$

- For a given circuit, V_{Th} and R_{Th} are fixed. By varying the load resistance R_L, the power delivered to the load varies as sketched in the figure.

- Notice that the power is small for small or large values of R_L but maximum for some value of R_L between 0 and ∞.

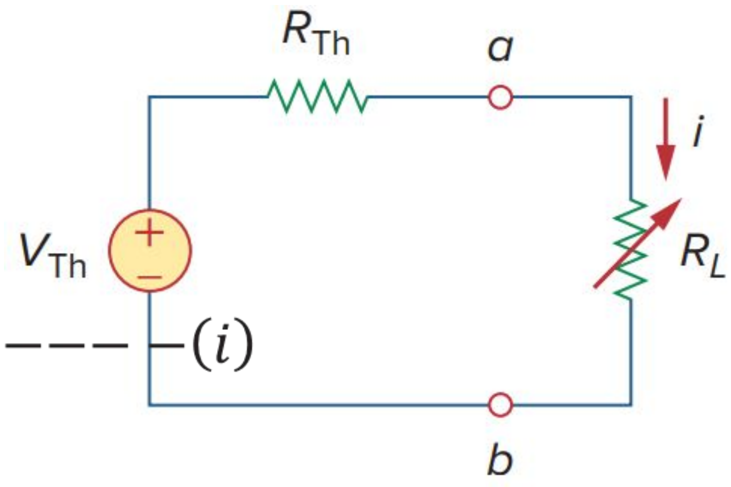
- Let's now see mathematically that this maximum power occurs when R_L is equal to R_{Th}.





Mathematically

- The Thevenin equivalent circuit for a load R_L is shown below. The load current is, $i = \frac{V_{Th}}{R_{Th} + R_L}$.



- Power delivered to the load is,

$$p = i^2 R = \left(\frac{V_{Th}}{R_{Th} + R_L}\right)^2 R_L = V^2{}_{Th}\left[\frac{R_L}{(R_{Th} + R_L)^2}\right] ----(i)$$

- Differentiating with respect to R_L,

$$\frac{dp}{dR_L} = V^2{}_{Th}\left[\frac{(R_{Th} + R_L)^2 \frac{d}{dR_L}(R_L) - R_L \frac{d}{dR_L}\{(R_{Th} + R_L)^2\}}{(R_{Th} + R_L)^4}\right] = V^2{}_{Th}\left[\frac{(R_{Th} + R_L)^2 - 2R_L\,(R_{Th} + R_L)^2}{(R_{Th} + R_L)^4}\right]$$

- Setting $\frac{dp}{dR_L}$ to zero will lead to the condition for maximum power transfer to the load.

Condition to P_{max} transfer & P_{max}

- For maxima/minima,

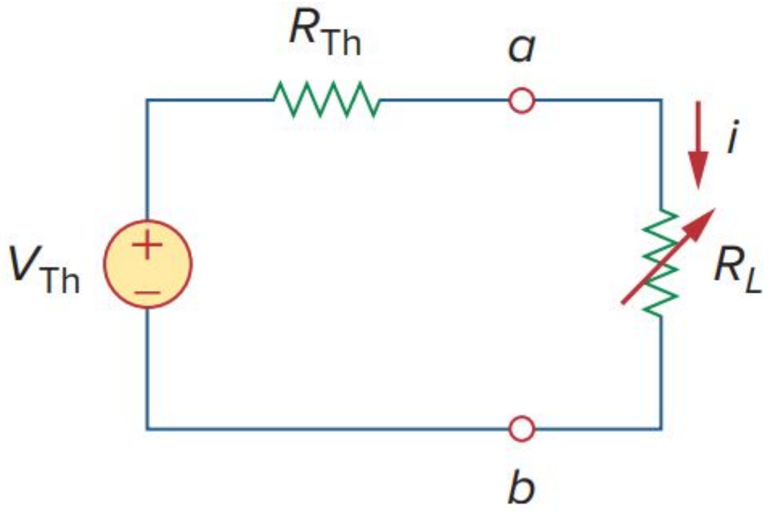
$$\frac{dp}{dR_L} = 0 = V^2{}_{Th}\left[\frac{(R_{Th} + R_L - 2R_L)}{(R_{Th} + R_L)^3}\right]$$

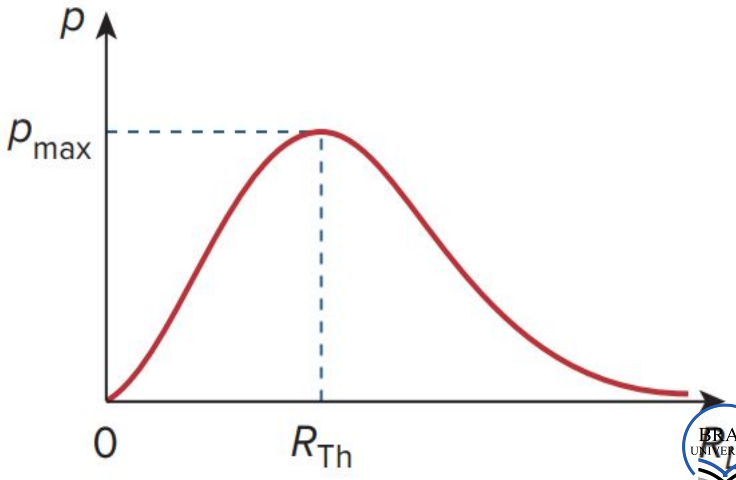
$$\Rightarrow R_{Th} + R_L - 2R_L = 0$$

$$\Rightarrow R_L = R_{Th}$$

- Substituting in (i) in the previous slide,

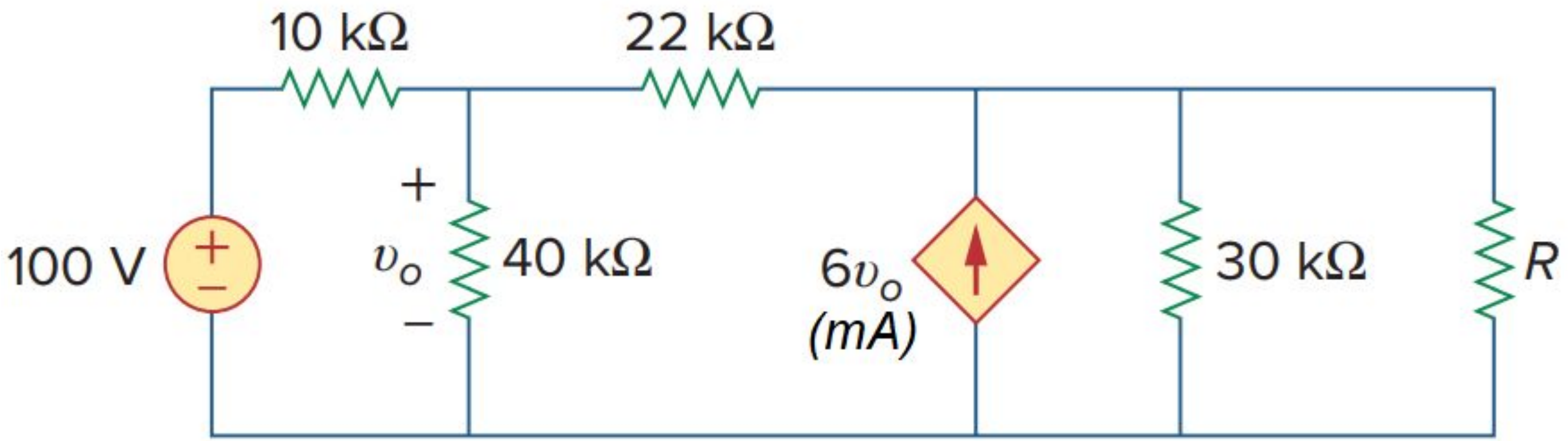
$$p_{max} = \frac{V_{Th}^2}{4R_{Th}}$$





Example 1

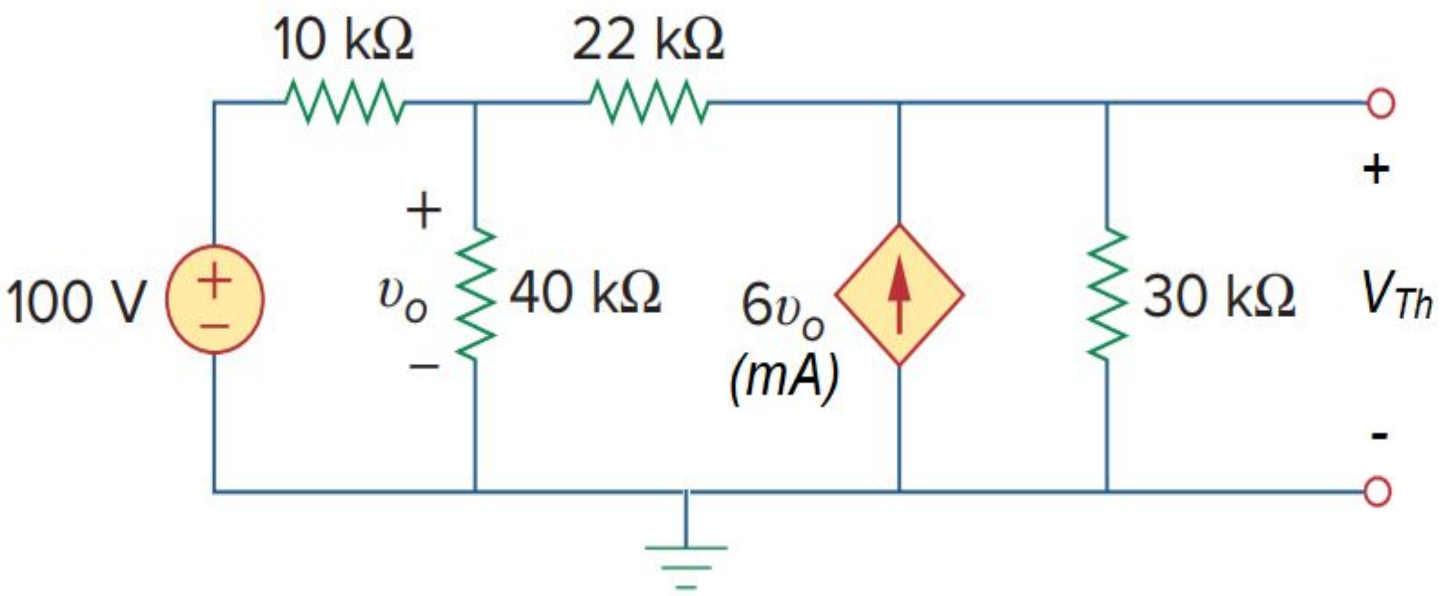
- Find the maximum power that can be delivered to the resistor *R*.



Ans: $V_{Th} = -231.304\,V;\ R_{Th} = -650\ \Omega; p_{max} = \infty\ (Theoretically)$

* *See solution in the next slide if necessary*

Example 1: finding V_{Th}



To find P_{max}, we have to first find V_{Th} and R_{Th}.

Let's use nodal analysis to find the V_{Th}.

KCL at node v_0,

$$\frac{v_0 - 100}{10} + \frac{v_0}{40} + \frac{v_0 - V_{Th}}{22} = 0$$

$$\Rightarrow \quad 75v_0 - 20V_{Th} = 4400 \;----(i)$$

KCL at node V_{Th},

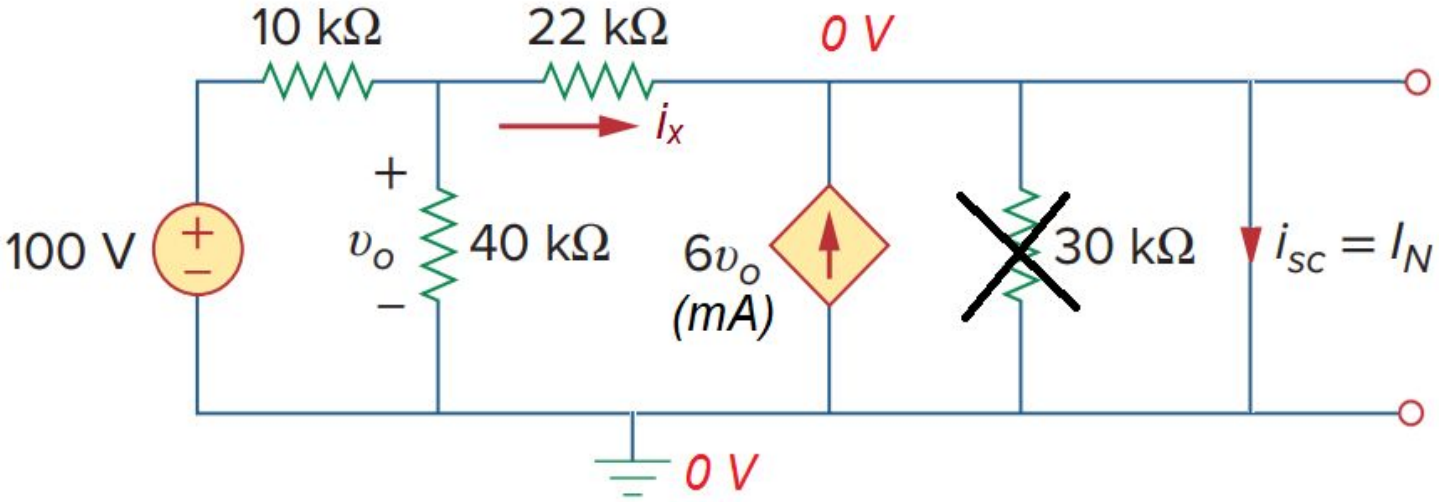
$$\frac{V_{Th} - v_0}{22} + \frac{V_{Th}}{30} = 6v_0$$

$$\Rightarrow \quad 1995v_0 - 26V_{Th} = 0 \;-----(ii)$$

Solving (i) and (ii),

$$V_{Th} = -231.304\ V$$

Example 1: finding R_{Th}



As $V_{Th} \neq 0$, let's use $R_{Th} = \frac{V_{Th}}{I_N}$ to determine the Thevenin equivalent resistance. The load terminals have been short circuited as shown in the figure.
Upon short circuiting the terminals $a - b$, the 10 Ω is shorted out. The whole circuit to the left of the dependent source is shorted with respect to it. As a result, the $6v_0$ current supplied by the dependent source will only flow through the short circuit.

Let's use nodal analysis to solve for the current i_x going towards the short circuit though the 22 $k\Omega$ resistor.

KCL at node v_0,

$$\frac{v_0 - 100}{10} + \frac{v_0}{40} + \frac{v_0 - 0}{22} = 0$$
$$\Rightarrow \quad 75v_0 = 4400$$
$$\Rightarrow \quad v_0 = 58.667\ V$$
$$\Rightarrow \quad i_x = \frac{v_0 - 0}{22} = 2.667\ mA$$

So,

$$I_N = i_x + 6v_0 = 354.669\ mA$$
$$R_{Th} = \frac{V_{Th}}{I_N} = -650\ \Omega$$

Example 1: finding P_{max}

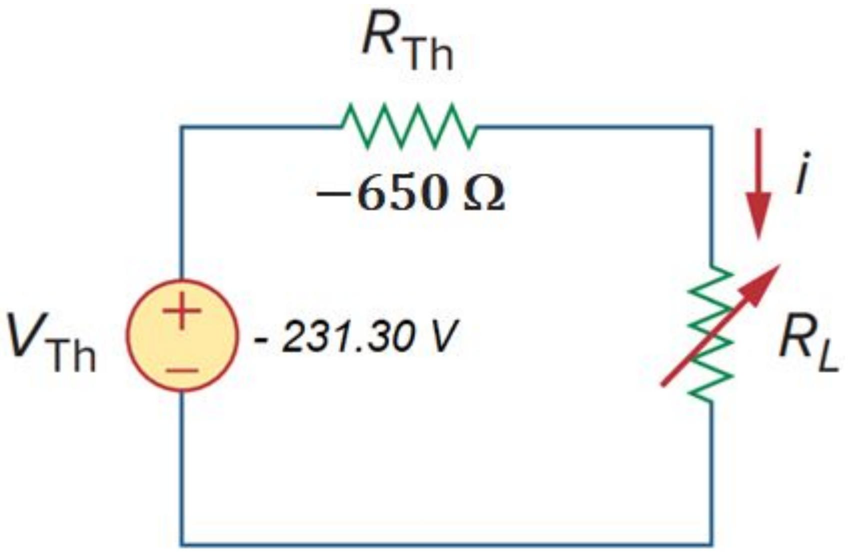
From the previous slides,

$$V_{Th} = -231.30\,V; \qquad R_{Th} = -650\,\Omega$$

What does a negative Thevenin resistance mean!

Negative Thevenin resistance is a part of the circuit model. The conversion of an actual circuit to a Thevenin equivalent is a mechanism for solving circuit problems and does not mean that the Thevenin equivalent circuit replaces the real circuit in all aspects.

Again, negative resistance means an active circuit. This means the circuit is trying to deliver infinite power to the load (assuming the load is practical, that is, $R_L > 0$).



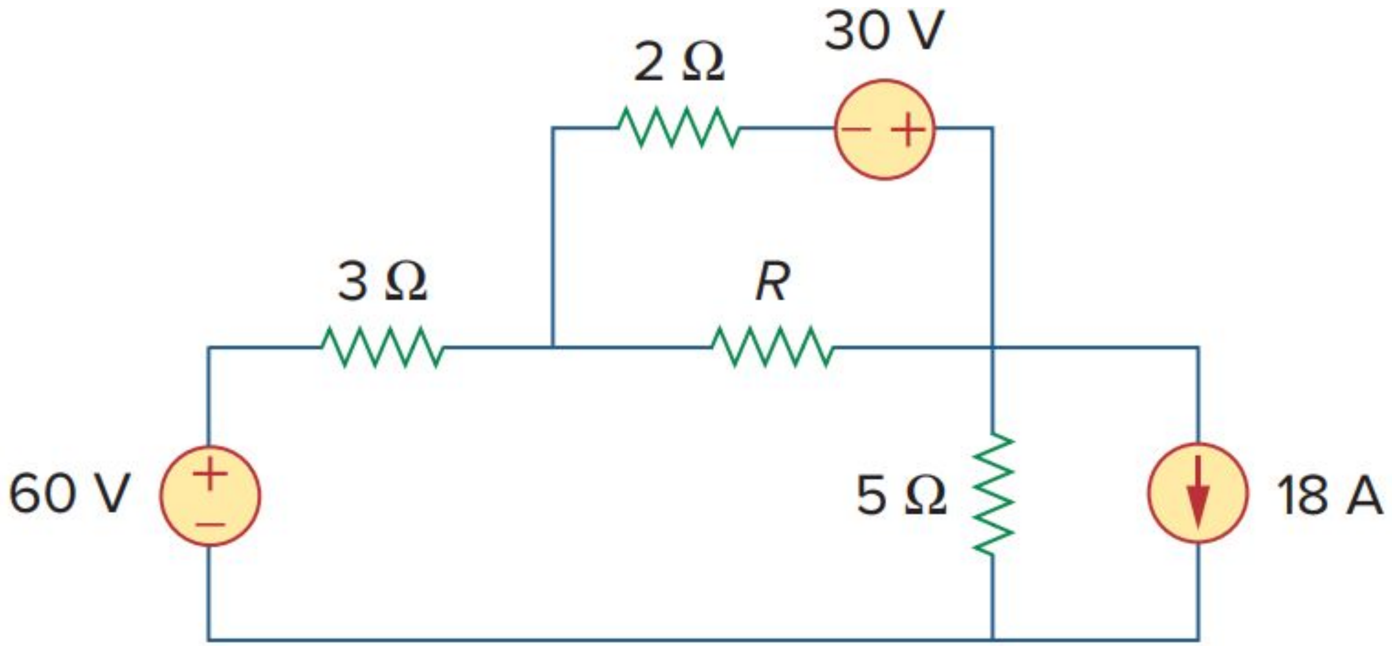
So, the correct answer is,

$$i = \frac{V_{Th}}{R_{Th} + R_L} = \frac{-231.30}{-650 + 650} = \infty$$

$$p_{max} = i^2 R_L = \infty\ (theoretically)$$

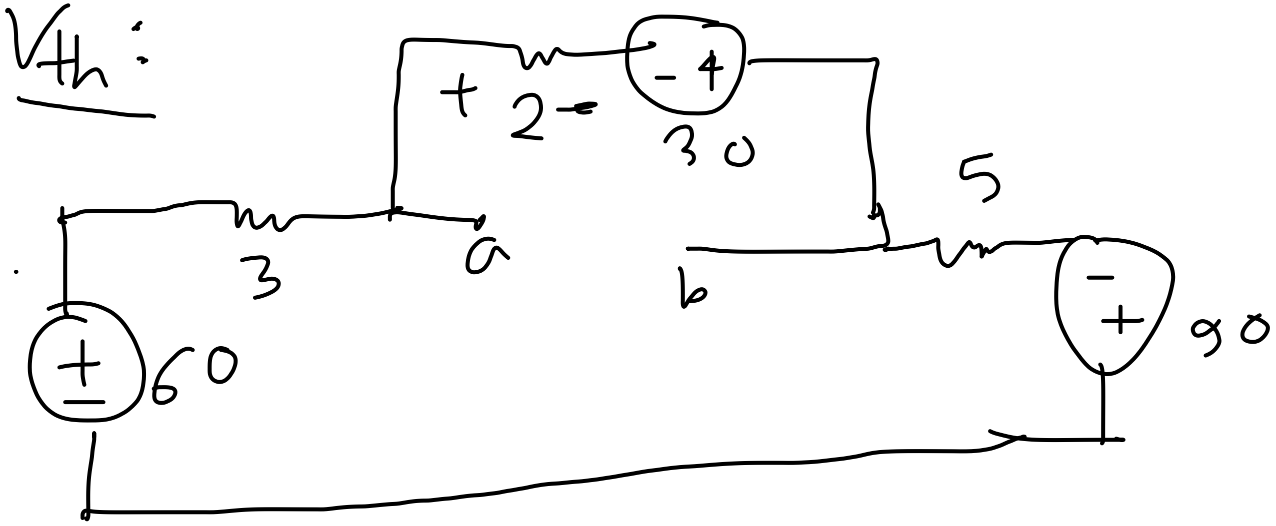
Problem 1

- Find the maximum power that can be delivered to the resistor R. Calculate the power efficiency at the maximum power point.



Ans: $\boldsymbol{V_{Th} = 6\,V}$; $\boldsymbol{R_{Th} = 1.6\,\Omega}$; $\boldsymbol{p_{max} = 5.625\,W}$; $\eta = \mathbf{50\,\%}$

Vth:



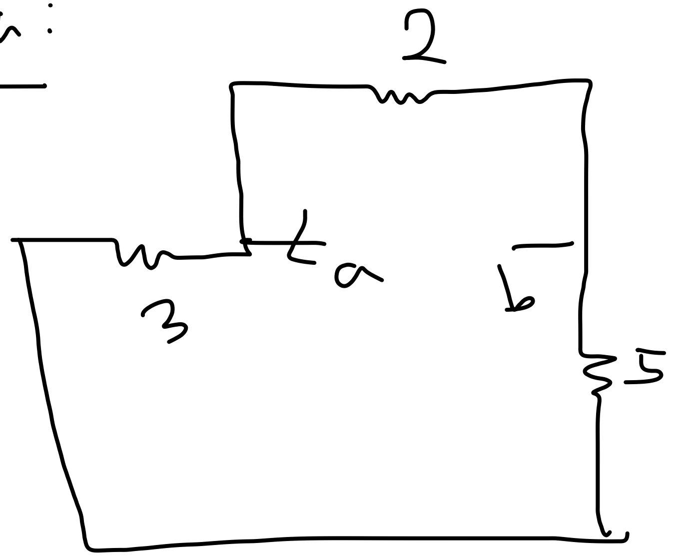
$-60+10I-30-90=0$

$\therefore I=18A$

So, $V_{th} = 30-2\times18$

$= 6V$

Rth:



$R_{th} = 2\|8$

$= 1.6\,\Omega$

So, $P_{max} = \frac{V_{th}^2}{4R_{th}} = 5.625W$

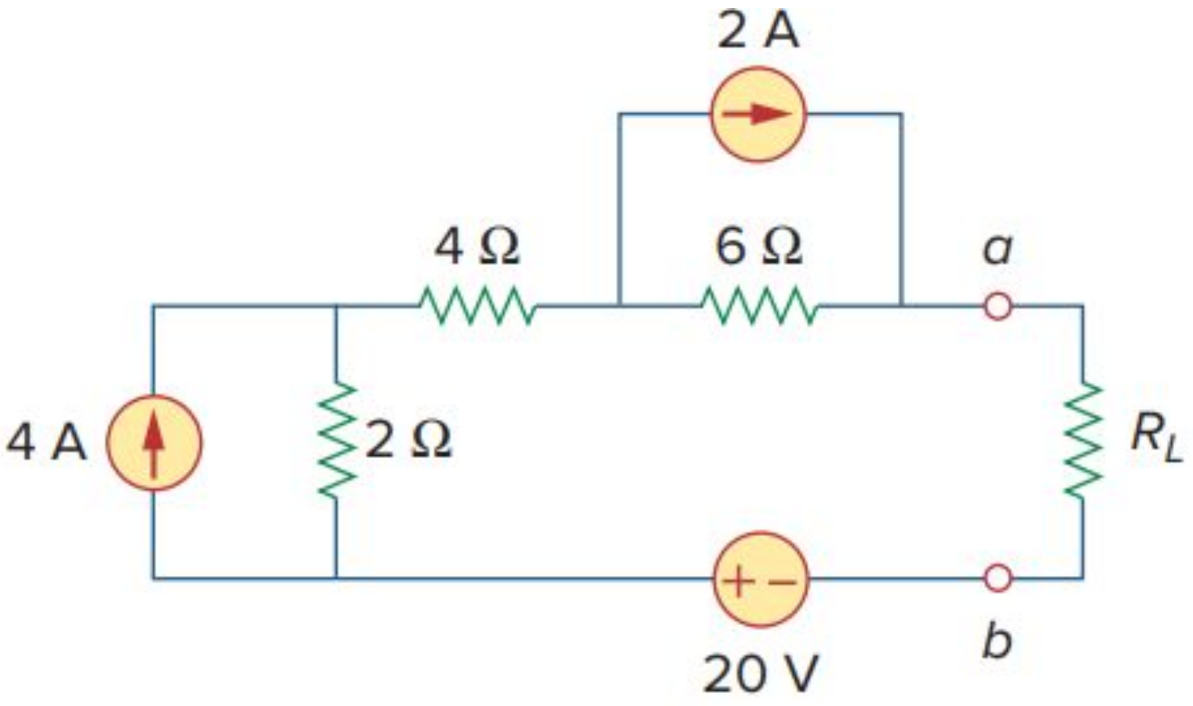
$I_1 = \frac{6}{2\times1.6} = 1.875$

$\eta = \frac{P_{out}}{P_{in}} = \frac{5.625\times100\%}{6\times I_1}$

$= 50\%$

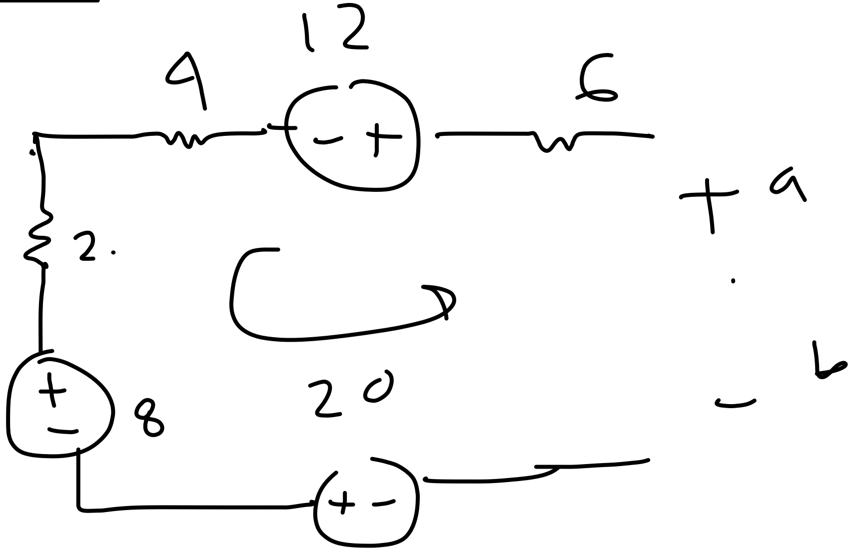
Problem 2

(a) For the circuit in the figure below, obtain the Thevenin equivalent at terminals $a - b$.
(b) Calculate the current if $R_L = 8\ \Omega$.
(c) Find R_L for maximum power deliverable to R_L.
(d) Determine that maximum power.



Ans: $V_{Th} = 40\ V$; $R_{Th} = 12\ \Omega$; $p_{max} = 33.33\ W$

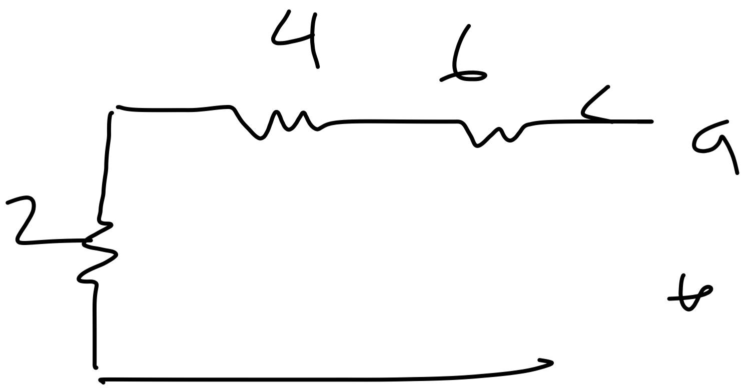
V_{th}:



$-V_{ab} + 20 + 8 + 12 = 0$

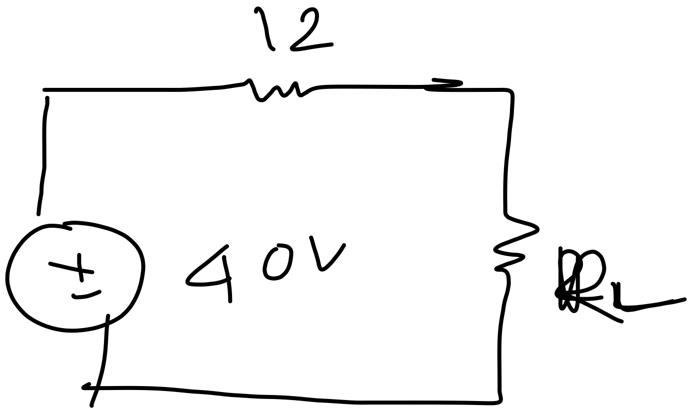
$\therefore V_{th} = 40\,V$

R_{th}:



$R_{ab} = R_{th}$

$= 12$



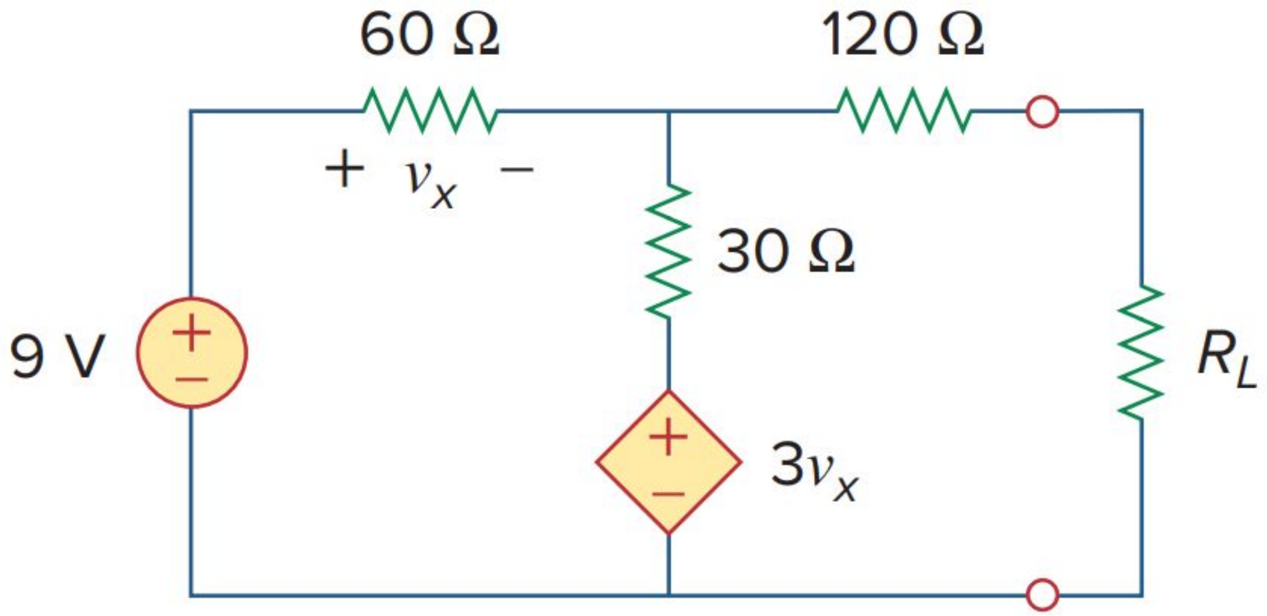
(b) $R_L = 8\Omega$, then, $I = \dfrac{40}{12+8} = 2A$

(c) for P_{max}, $R_L = 12\Omega$

(d) $P_{max} = \dfrac{40^2}{4 \times 12} = 33.33W$

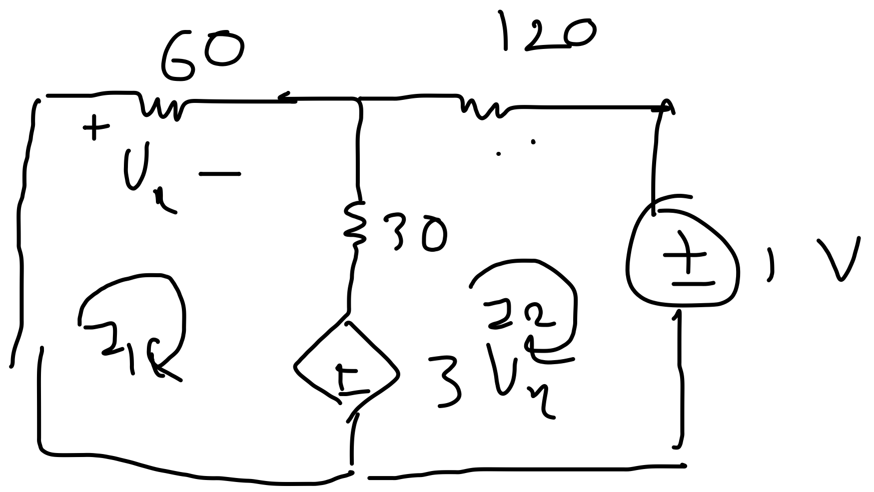
Problem 3

- Determine the value of R_L that will draw the maximum power from the rest of the circuit. Calculate the maximum power.



Ans: $\boldsymbol{R_L = 126.67\ \Omega;\ p_{max} = 96.71\ mW}$

R_{th}: $V_x = 60 I_1$



I_1

$90 I_1 - 30 I_2 + 3V_x = 0$

$270 I_1 - 30 I_2 = 0$

I_2

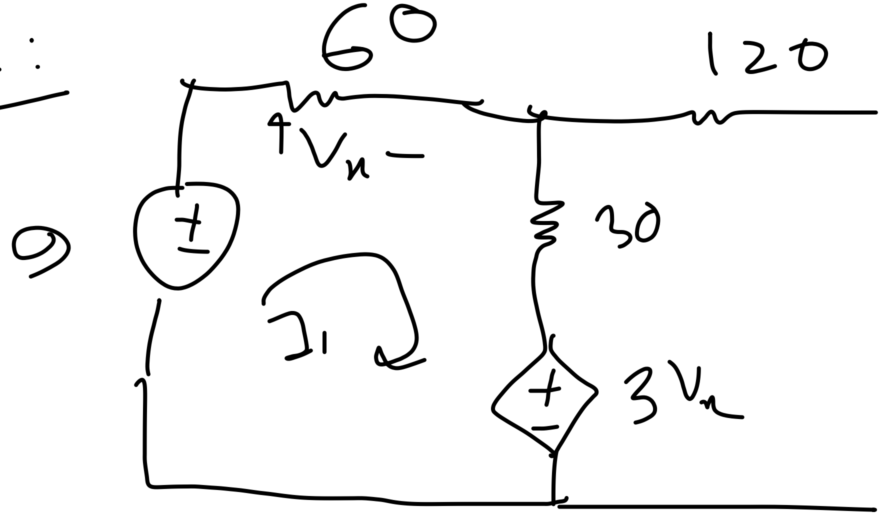
$150 I_2 - 30 I_1 + 1 - 3V_x = 0$

$150 I_2 - 30 I_1 - 3 \times 60 I_1 = -1$

$I_2 = -3/380$

$$R_{th} = -\frac{1}{I_2} = -\frac{1}{-3/380} = 126.67$$

V_{th}:



$$90 I_1 + 3V_x = 9$$

$$(90 + 3\times 60) I_1 = 9$$

$$I_1 = \frac{1}{30}$$

So, $V_{th} = V_{30} + 3V_x$

$$= 30\times\frac{1}{30} + 3\times 60\times\frac{1}{30}$$

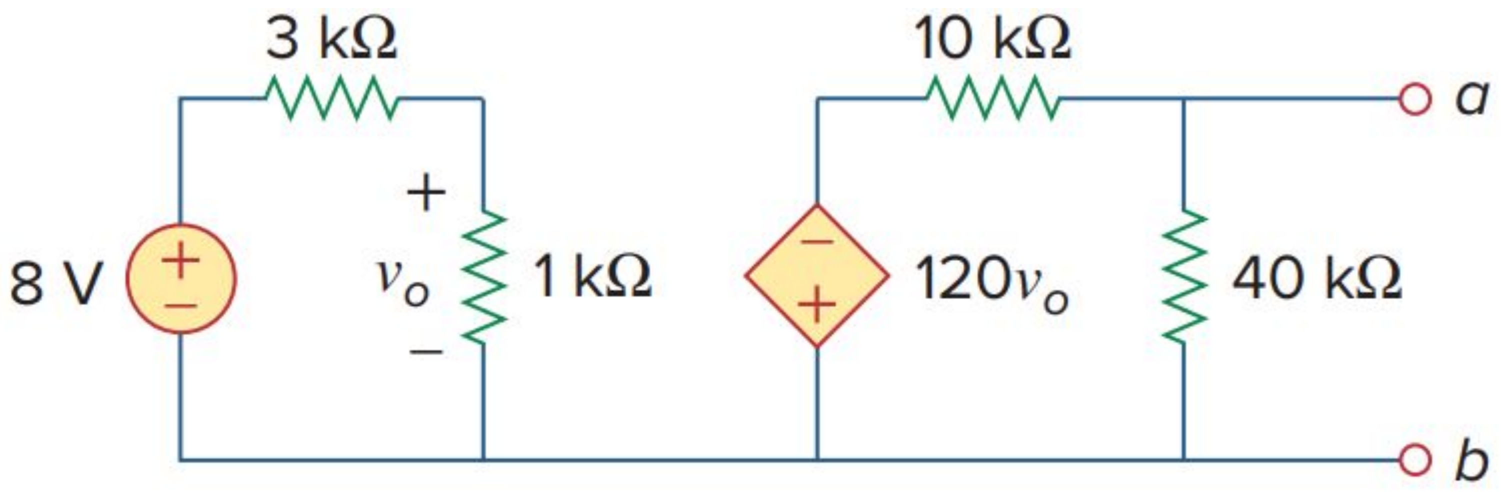
$$= 7V$$

$$\therefore P_{max} = \frac{7}{4\times R_{th}}$$

$$= 0.0967\ W$$

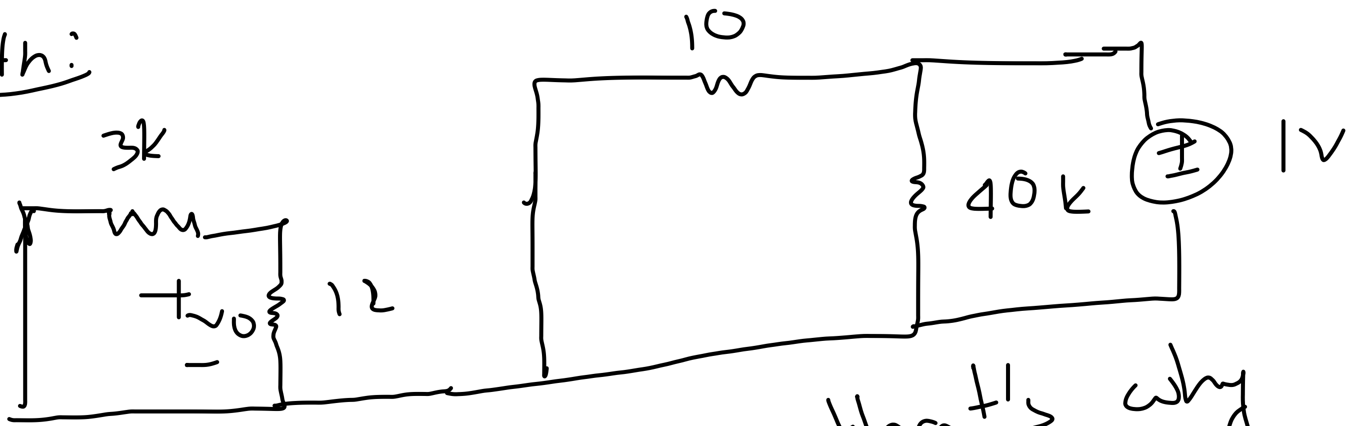
Problem 4

- What resistor connected across terminals will absorb maximum power from the circuit? What is that power?



Ans: $V_{Th} = -192\ V$; $R_{Th} = 8\ k\Omega$; $p_{max} = 1.152\ W$

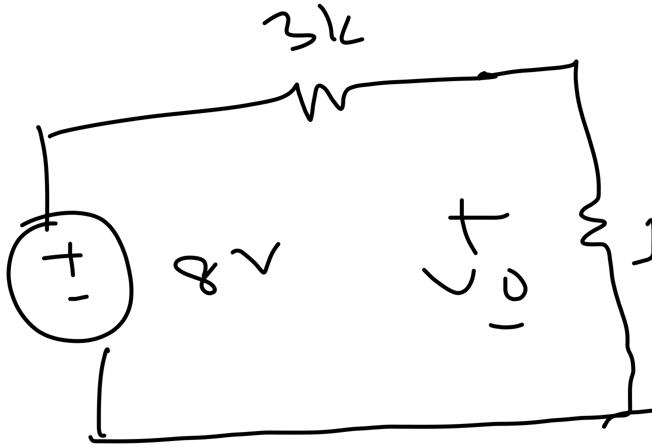
Rth:

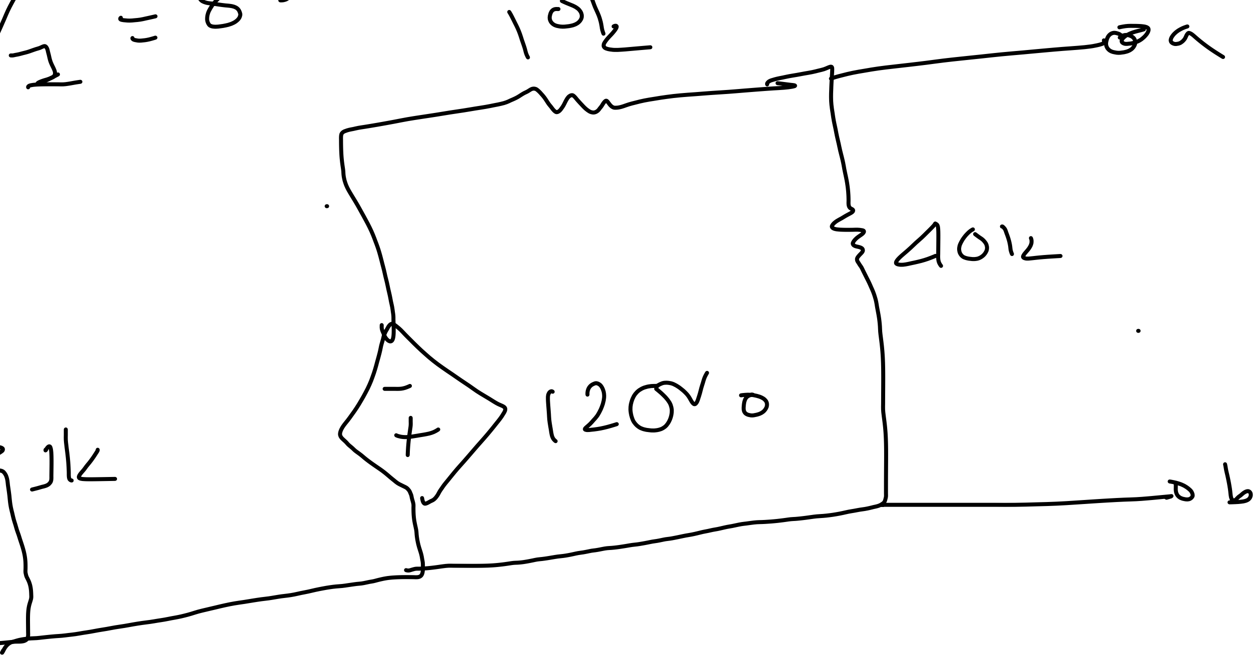


$V_o = 0$ So, $120 V_o = 0$, that's why short circuit.

$I = \dfrac{1}{40 \parallel 10}$ $\quad R_{th} = 1/I = 8\,\Omega$

Vth:





$$V_o = \frac{1}{3+1} \times 8 = 2V$$

$$P_{max} = \frac{V^2}{4R_{th}}$$

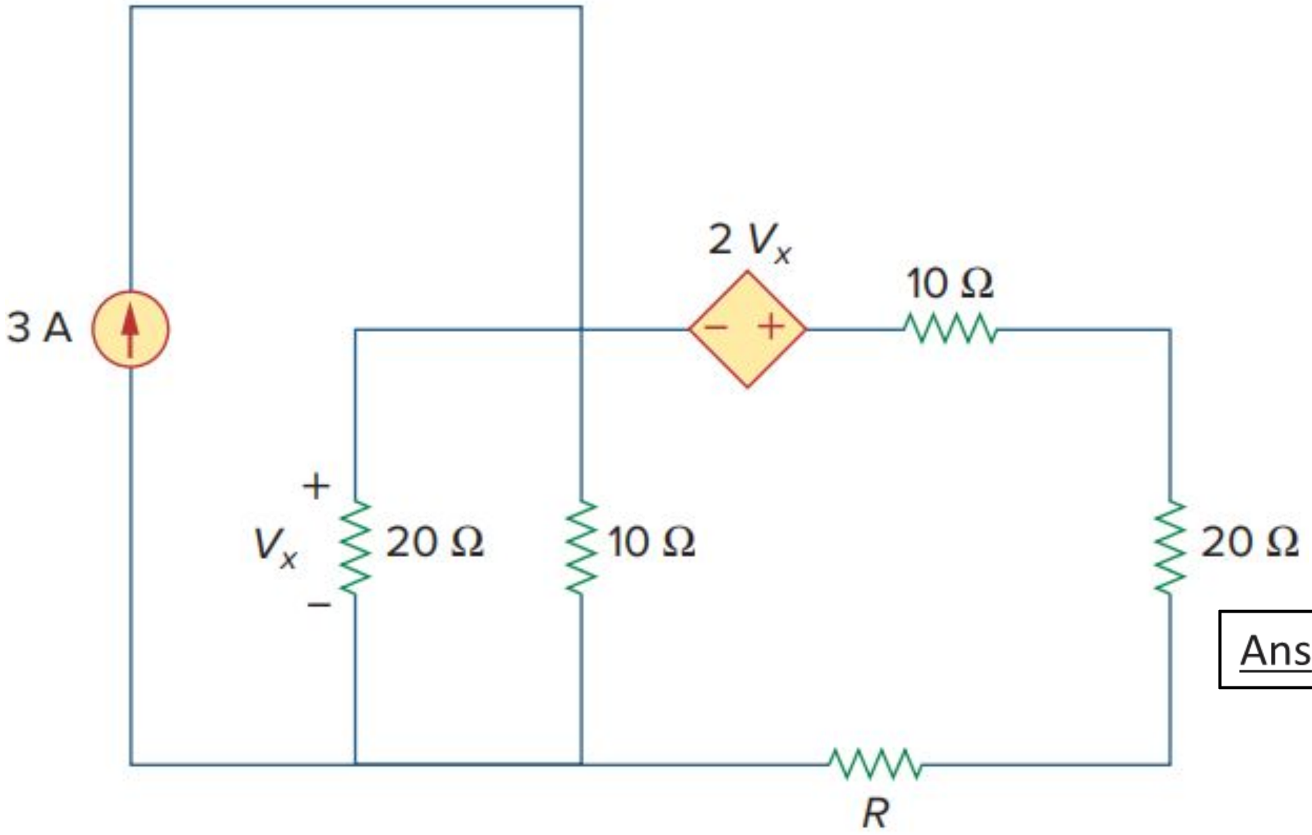
$$V_{ab} = -\frac{40}{40+10} \times 120 V_o$$

$$= -\frac{40}{50} \times 120 \times 2$$

$$= -192V$$

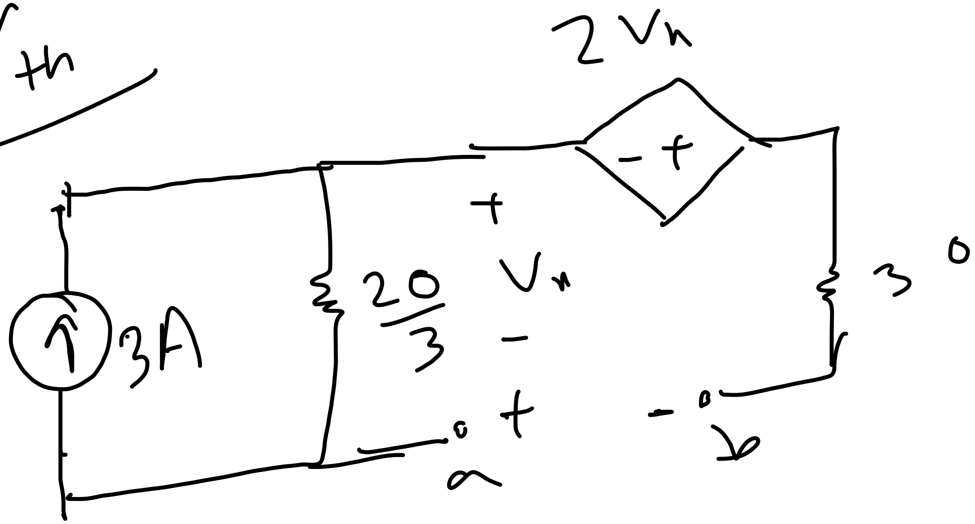
Problem 5

- Determine the maximum power delivered to the variable resistor R shown.



Ans: $\boldsymbol{V_{Th} = -60\,V;\ R_{Th} = 50\,\Omega; p_{max} = 18\,W}$

V_{th}



$V_x = 20V$

$-V_{th} = V_x + 2V_x + \cancel{30I} \to 0$

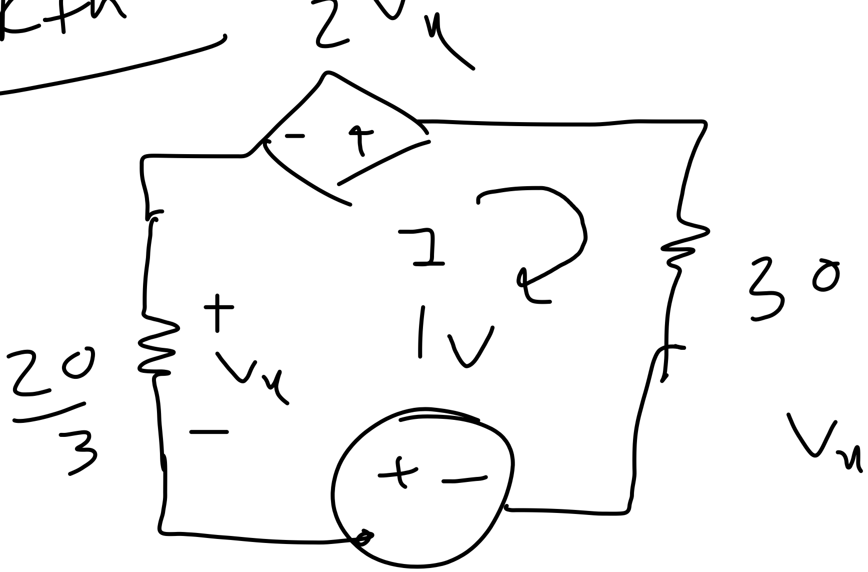
$= 3V_x$

$= 3 \times 20$

$= 60$

$V_{th} = -60V$

R_{th}:



$V_x = -\frac{20}{3} I$

$\left(\frac{20}{3} + 30\right) I - 2V_x = 1$

$\left(\frac{20}{3} + 30 + \frac{40}{3}\right) I = 1$

$\therefore \quad I = 1/50$

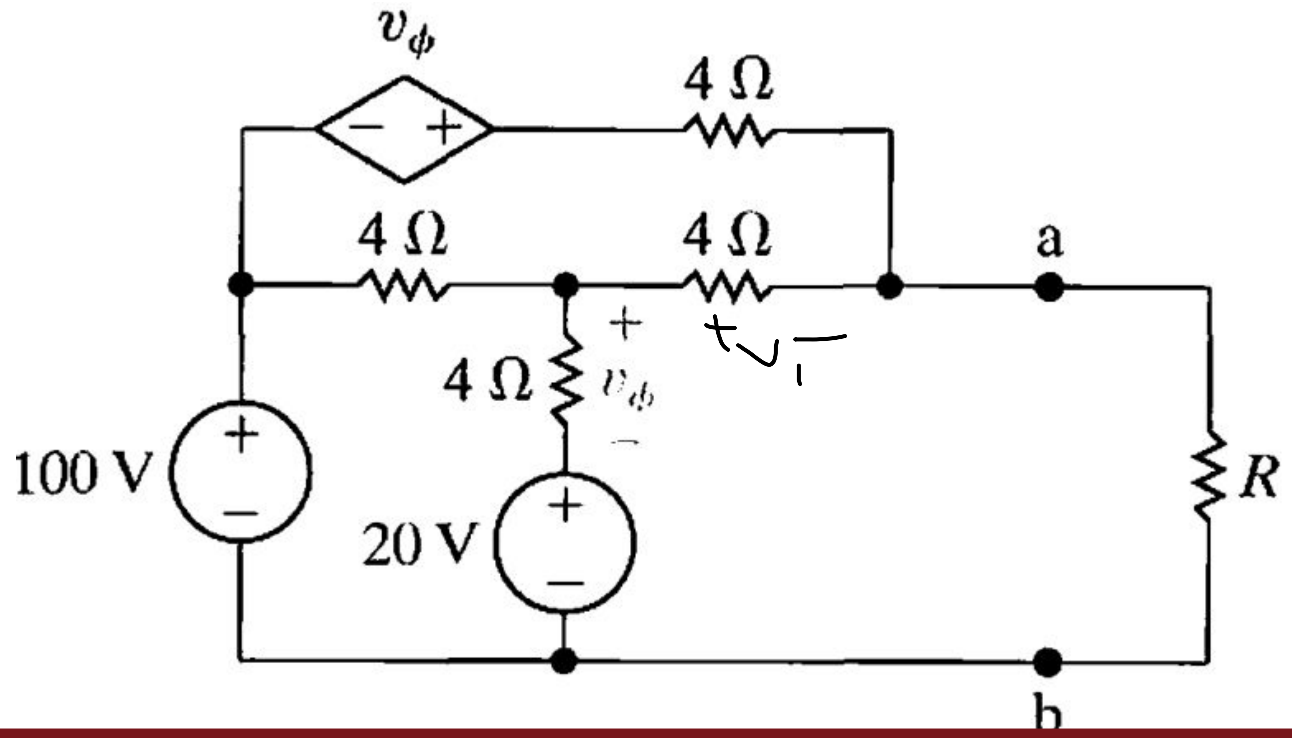
So, $R_{th} = \frac{V}{I} = 50\ \Omega$

So, $P_{max} = \frac{V_{th}^2}{4R_{th}} = \frac{(-60)^2}{4 \times 50}$

Problem 6

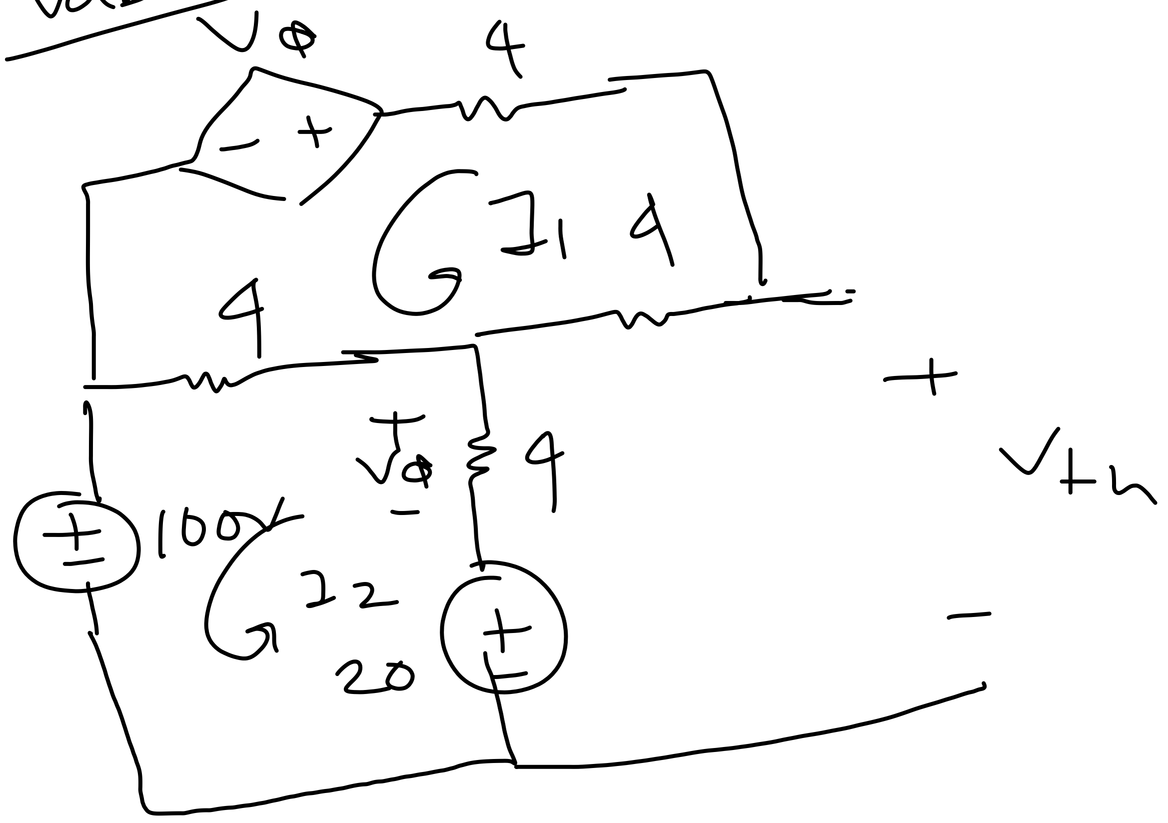
i. Find the value of R that enables the circuit shown to deliver maximum power to the terminals $a - b$.

ii. Find the maximum power delivered to R.



Ans: $(i)\ 3\ \Omega\ (ii)\ \boldsymbol{V_{Th} = 60\ V;\ P_{max} = 1.2\ kW}$

Vab



$$V_\phi = -4I_2$$

$$8I_2 - 4I_1 + 100 - 20 = 0$$

$$8I_2 - 4I_1 = -80 \quad (1)$$

$$12I_1 - 4I_2 + V_\phi = 0$$

$$12I_1 - 4I_2 + (-4I_2) = 0$$

$$12I_1 - 8I_2 = 0$$

$$I_1 = -10$$

$$I_2 = -15$$

So,

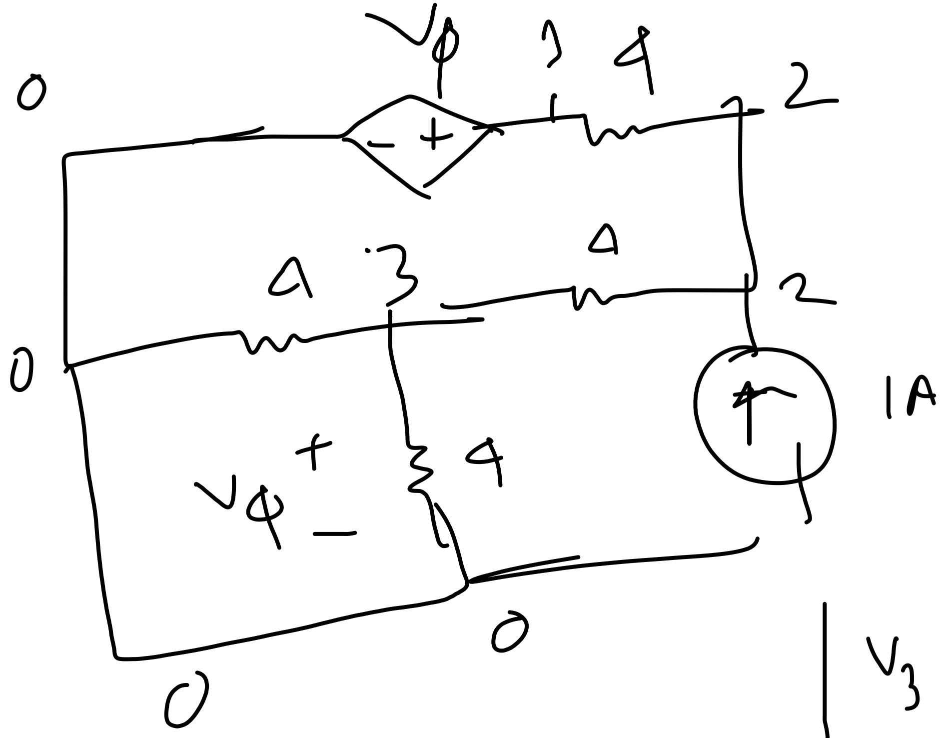
$$V_1 = -10\times4 = -40$$

$$V_2 = -(-15)\times4$$

$$= 60$$

$$-V_1 + V_\phi + 20 = V_{th}$$

$$V_{th} = 40 + 60 + 20$$



$$V_3 = V_\phi$$

$$V_1 = V_\phi = V_3$$

$$V_2\left(\frac{1}{4}+\frac{1}{4}\right) - \frac{V_3}{4} - \frac{V_1}{4} - 1 = 0$$

$$V_2 \cdot \frac{2}{4} - V_3 \cdot \frac{2}{4} = 1$$

$$V_3\left(\frac{1}{4}+\frac{1}{4}+\frac{1}{4}\right) - \frac{V_2}{4} = 0$$

$$V_2 = 3V$$

$$V_1 = 1V$$

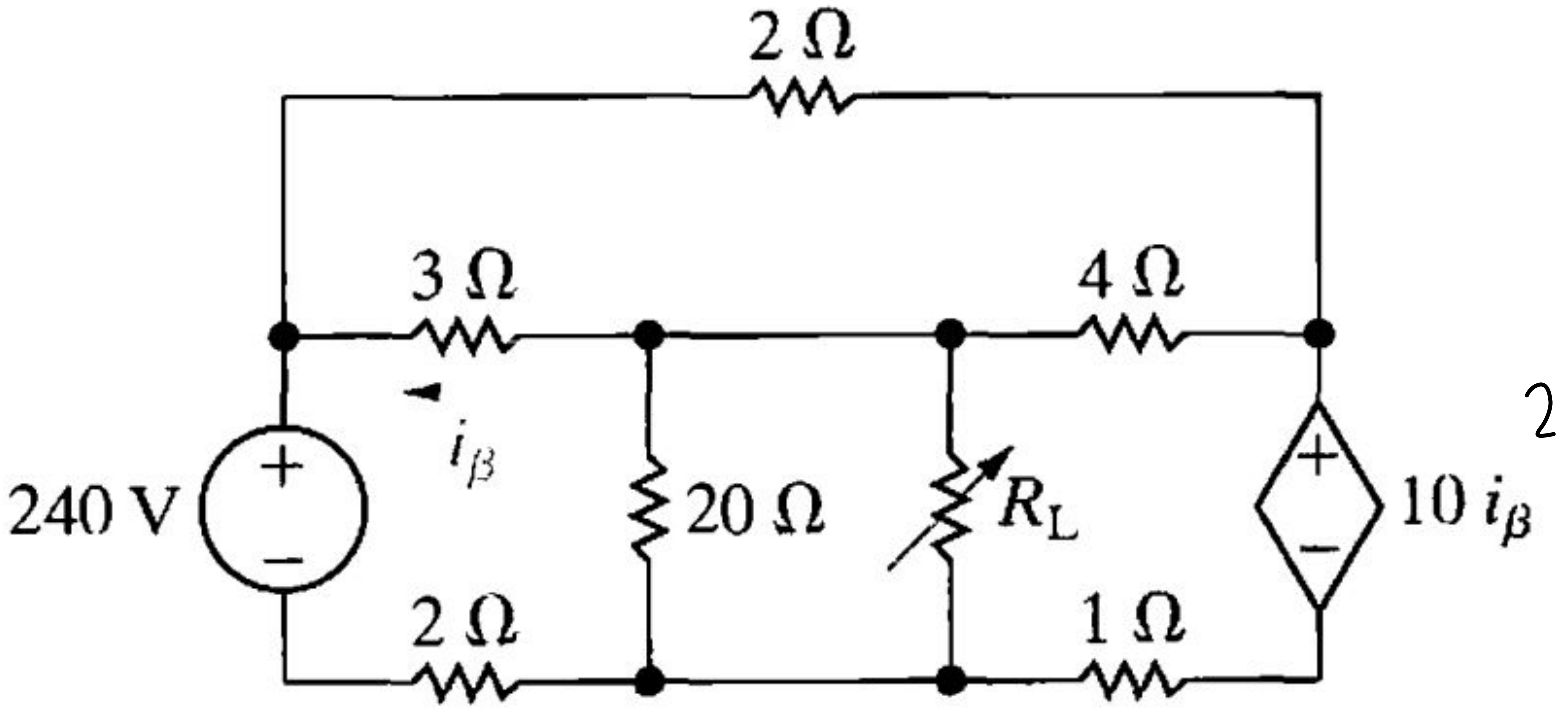
$$R_{th} = \frac{3}{1}$$

$$= 3\Omega$$

Problem 7

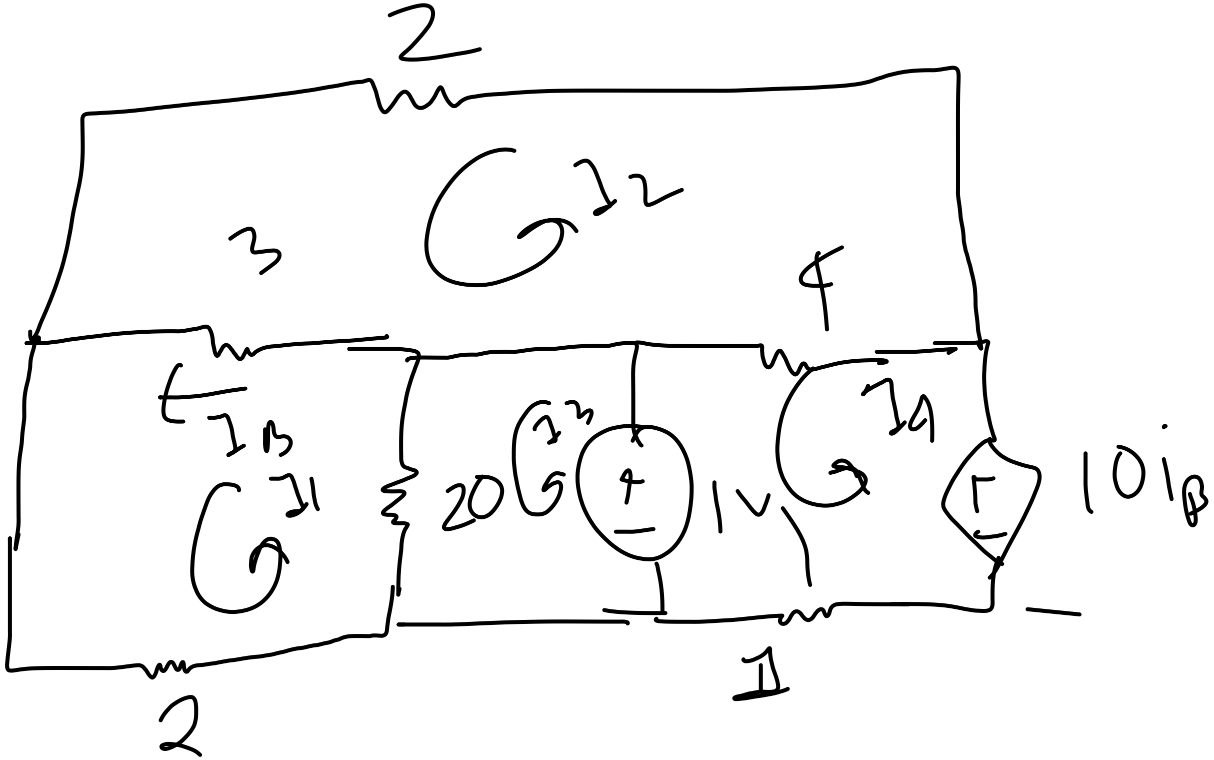
i. Find the value of R_L that enables the circuit shown to deliver maximum power to the load (R_L).

ii. Find the maximum power delivered to R_L.



$20(I_1 - I_2)$

Ans: (*i*) **6 Ω** (*ii*) $P_{max} = 24\ W$



$I_1 - I_2 = I_B$

$25I_1 - 3I_2 - 20I_3 = 0$ — I

$20I_3 - 20I_1 = 1$ — II

$5I_4 - 4I_2 + 1 = 10I_B$

$5I_4 - 4I_2 - 10(I_1 - I_2) = -1$

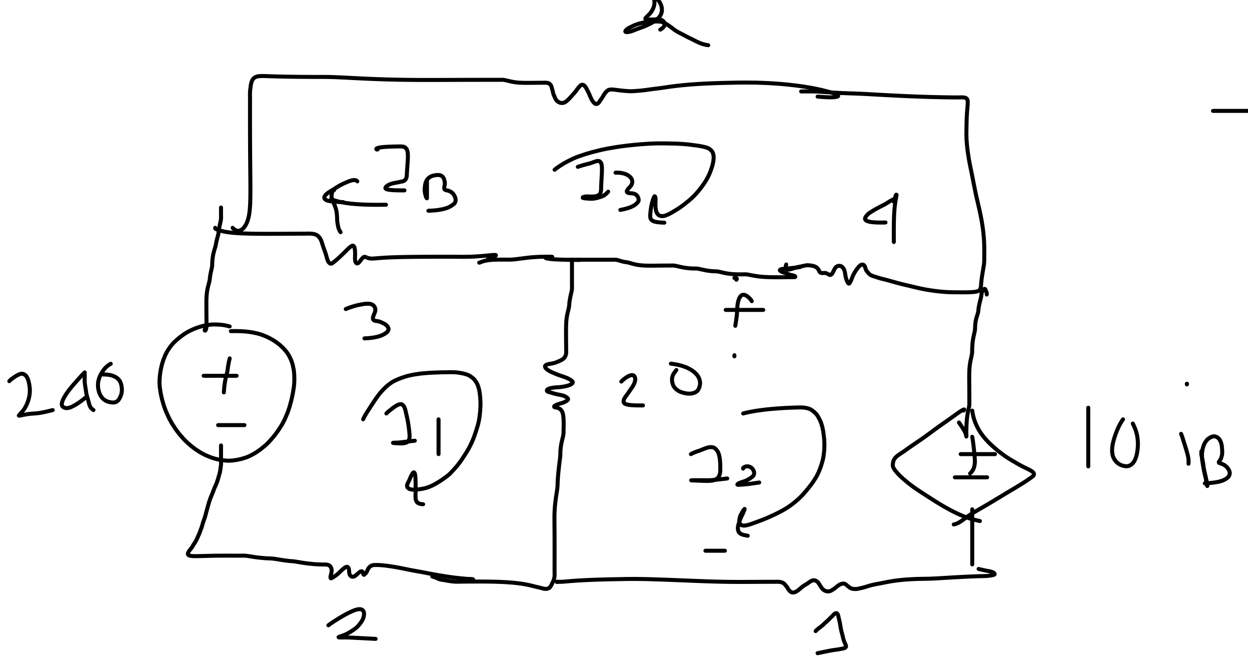
$5I_4 - 4I_2 - 10I_1 + 10I_2 = -1$ — III

$9I_2 - 3I_1 - 4I_4 = 0$ — IV

$I_3 = 11/30$

$I_4 = 1/5$

V_{th}



$I_B = I_3 - I_1$

$R_{th} = \dfrac{1}{I_3 - I_4}$

$= \dfrac{1}{1/6}$

$= 6\,\Omega$

$25I_1 - 3I_3 - 20I_2 = 240$

$25I_2 - 20I_1 - 4I_3 + 10i_B = 0$

$25I_2 - 20I_1 - 4I_3 + 10(I_3 - I_1) = 0$

Practice Problems

So,

$$-30I_1 + 25I_2 + 6I_3 = 0$$

- Additional recommended practice problems: here
- Other suggested problems from the textbook: here

$$9I_3 - 3I_1 - 4I_2 = 0$$

$$I_1 = \frac{498}{5}, \quad I_2 = \frac{504}{5}, \quad I_3 = 78$$

$$V_{th} = 20(I_2 - I_1) = -24\ V$$

So, $$P = \frac{V_{th}^2}{4R_{th}} = 24\ W$$

Thank you for your attention

Course Outline: broad themes