Lecture 10

CSE250 - Circuits and Electronics

Maximum power Transfer Theorem

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Maximum Power Transfer

- In many practical situations, a circuit is designed to provide power to a load. There are applications in areas such as communications and amplification where it is desirable to **maximize the power delivered to an antenna (load) and a speaker** respectively.

- When designing a circuit, it is often important to be able to answer the question, *"What load should be applied to a system or what driving circuitry for a particular load should be designed to ensure that the load is receiving maximum power from the system or from the circuit respectively?"*

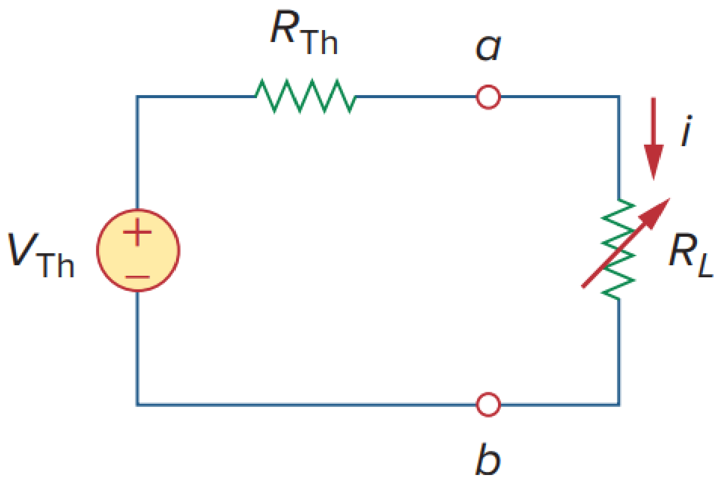
- Given a system with known internal losses the **Thevenin equivalent is useful in finding the maximum power a linear circuit can deliver to a load**. **We assume that we can adjust the load resistance R_L.**

- Maximum power is transferred to the load when the load resistance **equals** the Thevenin resistance as seen from the load ($R_L = R_{Th}$). This is known as the *Maximum Power Transfer Theorem.*

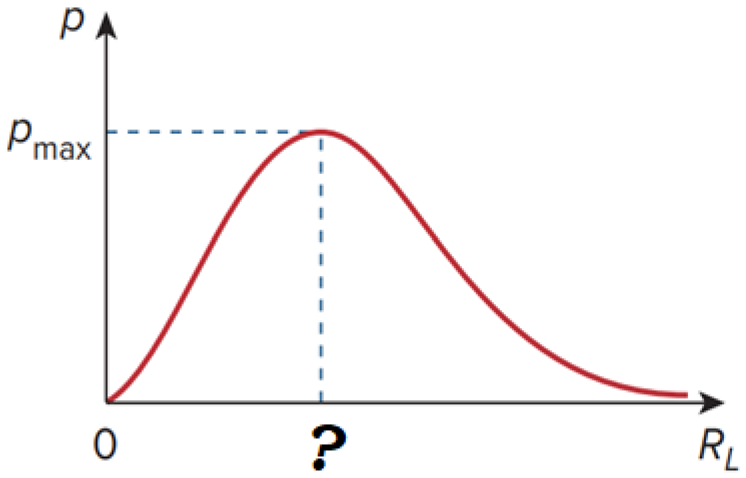
Graphically

- Given any linear two terminal circuit, it can be reduced to a Thevenin equivalent as shown. Power delivered to the load by the Thevenin equivalent circuit is then,

$$p = i^2R = \left(\frac{V_{Th}}{R_{Th} + R_L}\right)^2 R_L$$



- For a given circuit, V_{Th} and R_{Th} are fixed. By varying the load resistance R_L, the power delivered to the load varies as sketched in the figure.



- Notice that the power is small for small or large values of R_L but maximum for some value of R_L between 0 and ∞.

- Let's now see mathematically that this maximum power occurs when R_L is equal to R_{Th}.

Mathematically

If $y = \frac{u}{v}$

where ‘u’ and ‘v’ are differential function of ‘x’

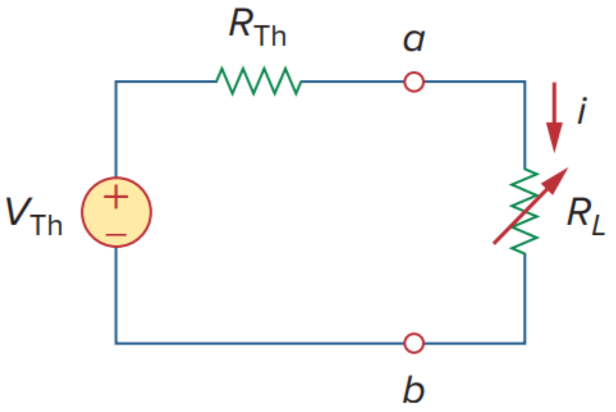
$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right)$$

$$\frac{dy}{dx} = \frac{v\frac{d}{dx}(u) - u\frac{d}{dx}(v)}{v^2}$$

- The Thevenin equivalent circuit for a load R_L is shown below. The load current is, $i = \frac{V_{Th}}{R_{Th} + R_L}$.

Power delivered to the load is,

$$p = i^2R = \left(\frac{V_{Th}}{R_{Th} + R_L}\right)^2 R_L = V^2{}_{Th}\left[\frac{R_L}{(R_{Th} + R_L)^2}\right] ----- (i)$$



- Differentiating with respect to R_L,

$$\frac{dp}{dR_L} = V^2{}_{Th}\left[\frac{(R_{Th} + R_L)^2 \frac{d}{dR_L}(R_L) - R_L \frac{d}{dR_L}\{(R_{Th} + R_L)^2\}}{(R_{Th} + R_L)^4}\right] = V^2{}_{Th}\left[\frac{(R_{Th} + R_L)^2 - 2R_L(R_{Th} + R_L)}{(R_{Th} + R_L)^4}\right]$$

- Setting $\frac{dp}{dR_L}$ to zero will lead to the condition for maximum power transfer to the load.

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Condition to P_{max} transfer & P_{max}

- For maxima/minima,

$$\frac{dp}{dR_L} = 0 = V^2{}_{Th}\left[\frac{R_{Th}^2 + 2R_{Th}R_L + R_L^2 - 2R_LR_{Th} - 2R_L^2}{(R_{Th} + R_L)^4}\right]$$

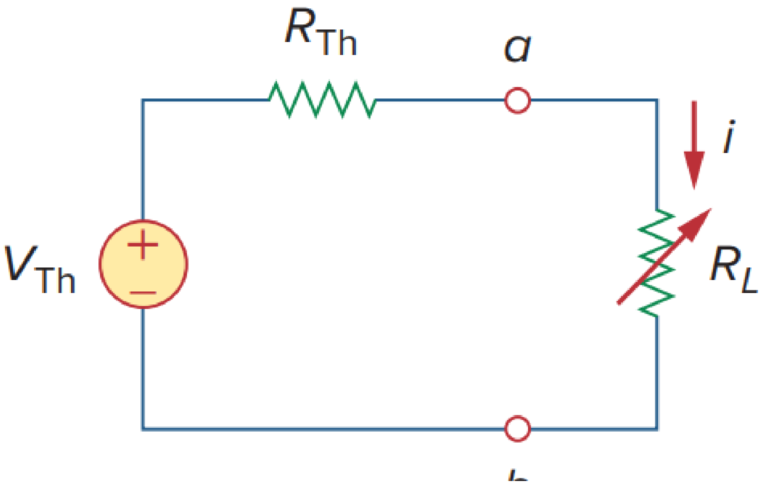
$$\Rightarrow R_{Th}^2 + 2R_{Th}R_L + R_L^2 - 2R_LR_{Th} - 2R_L^2 = 0$$

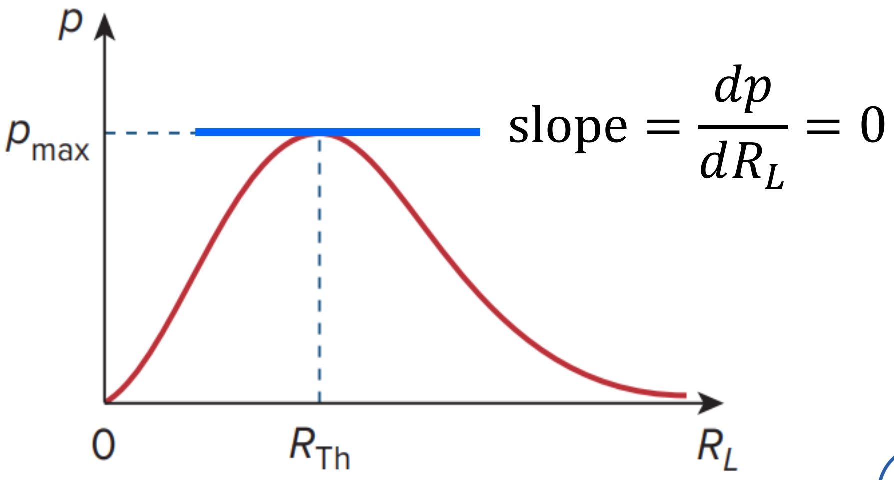
$$\Rightarrow R_{Th}^2 = R_L^2$$

$$\Rightarrow \boldsymbol{R_L = R_{Th}}$$

Substituting in (i) in the previous slide,

$$\boxed{p_{max} = \frac{V_{Th}^2}{4R_{Th}}}$$





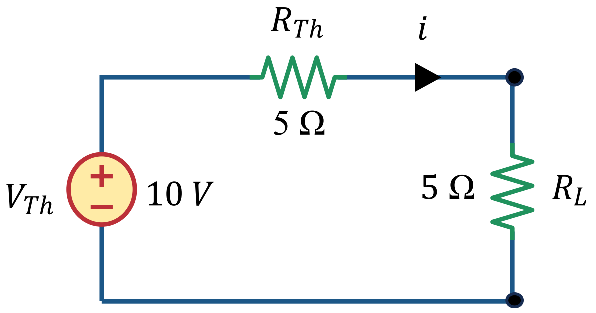
Intuition of maximum power

$$p_{max} = \frac{V_{Th}^2}{4R_{Th}}$$

You may think that, why $R_L = R_{Th}$ gives the maximum power?

If I increase R_L it should give me more power as $P_{R_L} = i^2 R_L$

Let's take an example

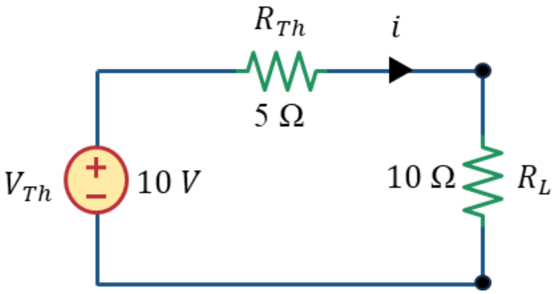


$$i = \frac{10}{5+5} = 1\ A$$

$$P_{R_L} = i^2 R_L = 1^2 \times 5 = 5\ W$$

This is the maximum power R_L can have as $R_{Th} = R_L$

Now, let us increase R_L with hope to increase its power. Let's set $R_L = 10\ \Omega$



$$i = \frac{10}{10+5} = 0.67\ A$$

$$P_{R_L} = i^2 R_L = 0.67^2 \times 10 = 4.489\ W$$

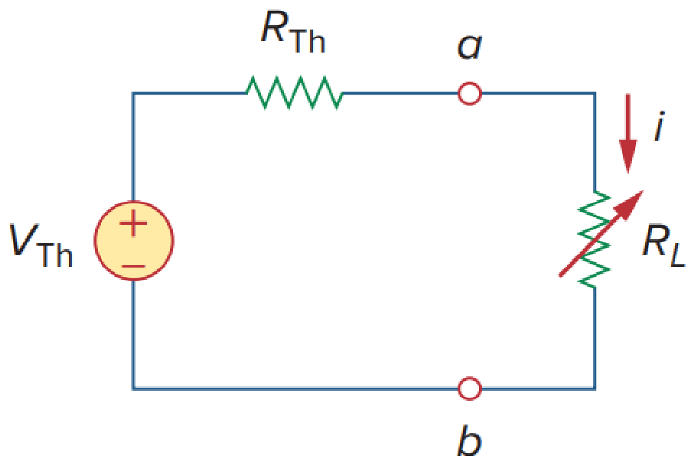
You see, even though we increased R_L, overall current dropped. Which essentially decreased the power of R_L

Efficiency of Maximum Power Transfer

We can calculate the efficiency of maximum power transfer, η_{Max} using following formula.

$$\boxed{\eta_{Max} = \frac{P_{max}}{P_s} \times 100\%}$$

P_{max} is the maximum amount of power transferred to the load and P_s is the amount of power generated by the source.



The amount of power generated by the source

$$P_s = I^2 R_{Th} + I^2 R_L$$

$$\Rightarrow P_s = 2I^2 R_{Th}$$

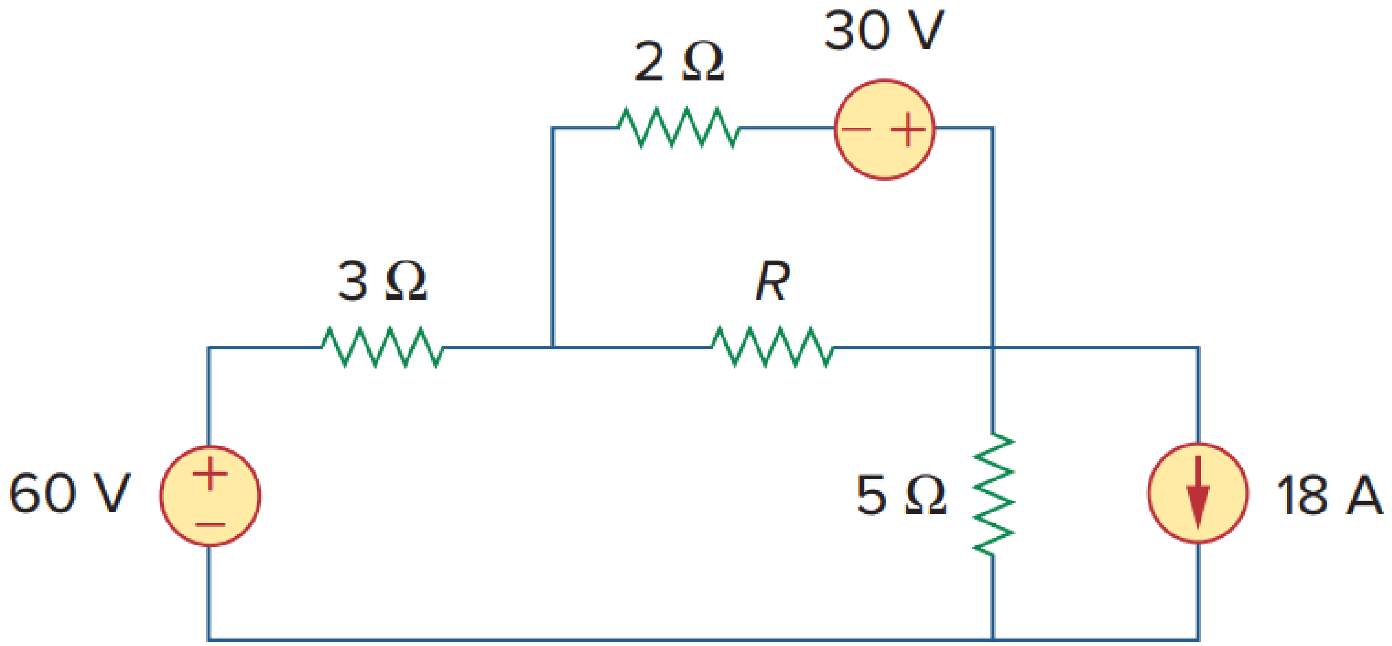
$$\Rightarrow P_s = 2\left(\frac{V_{Th}}{2R_{Th}}\right)^2 R_{Th}$$

$$\boxed{P_s = \frac{V_{Th}^2}{2R_{Th}}}$$

$$\boxed{\eta_{Max} = \frac{P_{max}}{P_s} \times 100\% = \frac{\frac{V_{Th}^2}{4R_{Th}}}{\frac{V_{Th}^2}{2R_{Th}}} \times 100\% = \frac{1}{2} \times 100\% = 50\%}$$

Problem 1

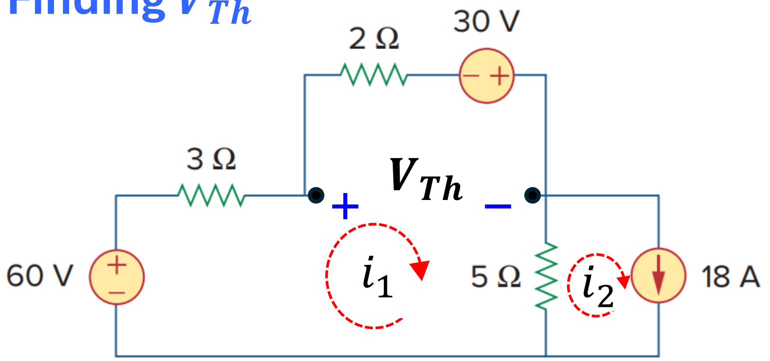
- Find the maximum power that can be delivered to the resistor R. Calculate the power efficiency at the maximum power point.



Ans: $\boldsymbol{V_{Th} = 6\,V}$; $\boldsymbol{R_{Th} = 1.6\,\Omega}$; $\boldsymbol{p_{max} = 5.625\,W}$; $\eta = \mathbf{50}\,\%$

Solution to Problem 1

Finding V_{Th}



Let's use mesh analysis to find the V_{Th}

Applying KVL at mesh 2,

$$i_2 = 18\,A \quad \text{........... (i)}$$

Applying KVL at mesh 1,

$$-60 + 3i_1 + 2i_1 - 30 + 5(i_1 - i_2) = 0$$

$$\Rightarrow -60 + 3i_1 + 2i_1 - 30 + 5i_1 - 5i_2 = 0$$

$$\Rightarrow 10i_1 - 5i_2 = 90 \quad \text{........... (ii)}$$

Solving (i) and (ii),

$$\boxed{\begin{aligned} i_1 &= 18\,A \\ i_2 &= 18\,A \end{aligned}}$$

Now,

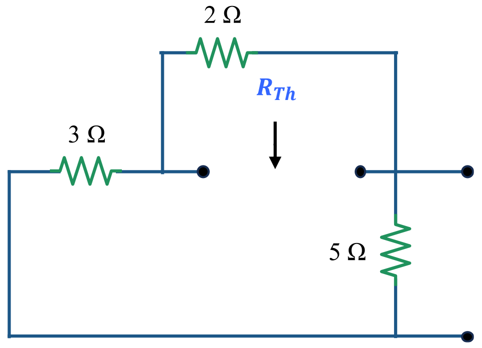
$$V_{Th} = 2i_1 - 30$$

$$\Rightarrow V_{Th} = 2 \times 18 - 30$$

$$\mathbf{V_{Th} = 6\,V}$$

Finding R_{Th}

At first, let's deactivate all the Independent Sources. As there is no dependent sources, we simply use series-parallel combination to find the equivalent resistance seen from the load terminal.



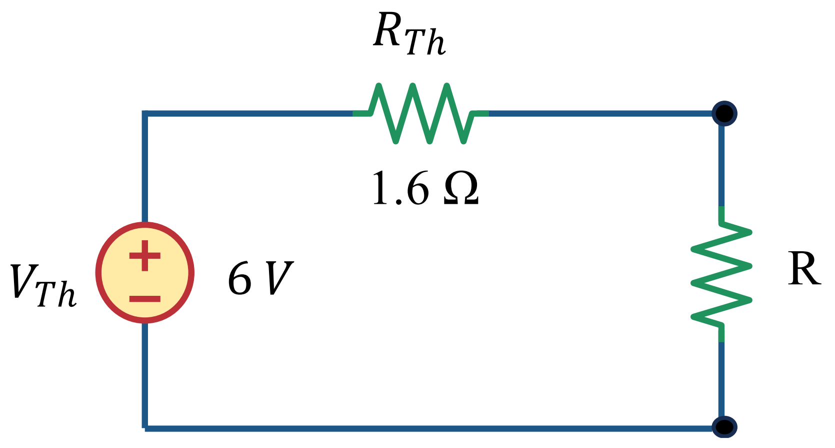
$$R_{Th} = (3 + 5)\ ||\ 2 = 8\ ||\ 2$$

$$\Rightarrow R_{Th} = \frac{8 \times 2}{8 + 2} = \mathbf{1.6\,\Omega}$$

See the next page for Thevenin circuit

Solution to Problem 1 (Continued)

Thevenin circuit becomes,



So, maximum power that can be delivered to the resistor R is,

$$p_{max} = \frac{V_{Th}^2}{4R_{Th}}$$

$$\Rightarrow p_{max} = \frac{6^2}{4 \times 1.6}$$

$$\Rightarrow \boldsymbol{p_{max} = 5.625\ W}$$

Power efficiency at the maximum power point

$$\eta_{Max} = \frac{P_{max}}{P_s} \times 100\%$$

where,

$$p_{max} = 5.625\ W$$

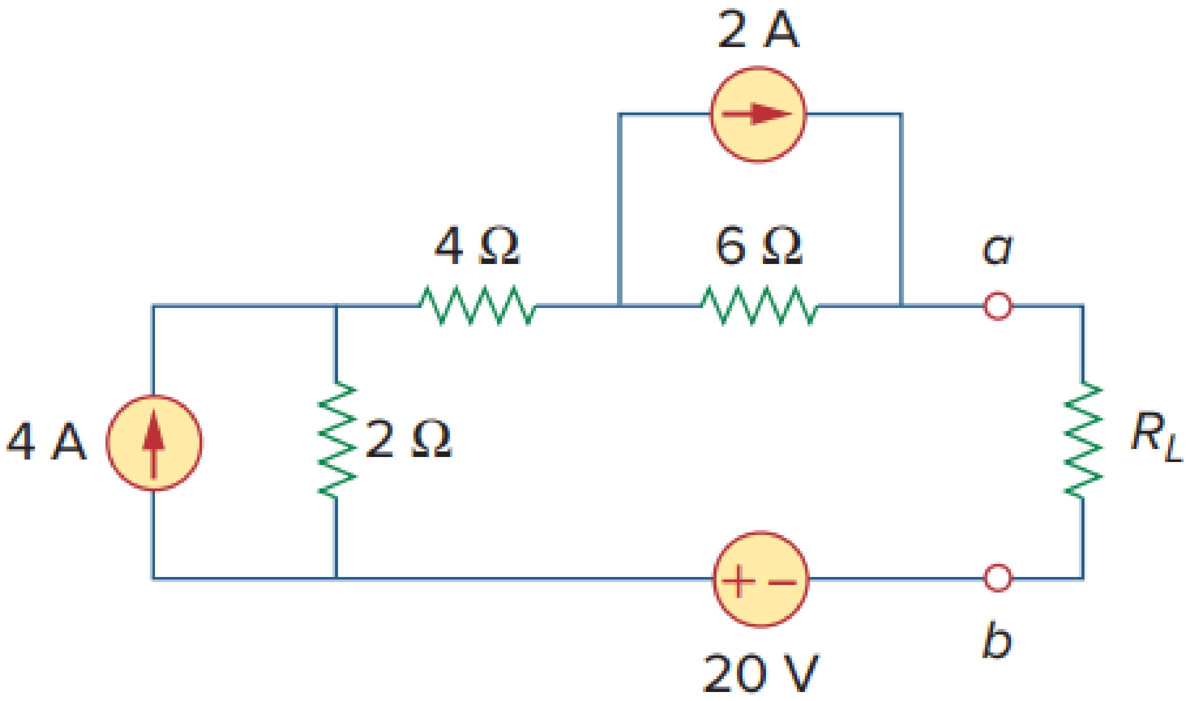
$$P_s = \frac{V_{Th}^2}{2R_{Th}} = \frac{6^2}{2 \times 1.6} = 11.25\ W$$

So,

$$\eta_{Max} = \frac{5.625}{11.25} \times 100\% = \mathbf{50\%}$$

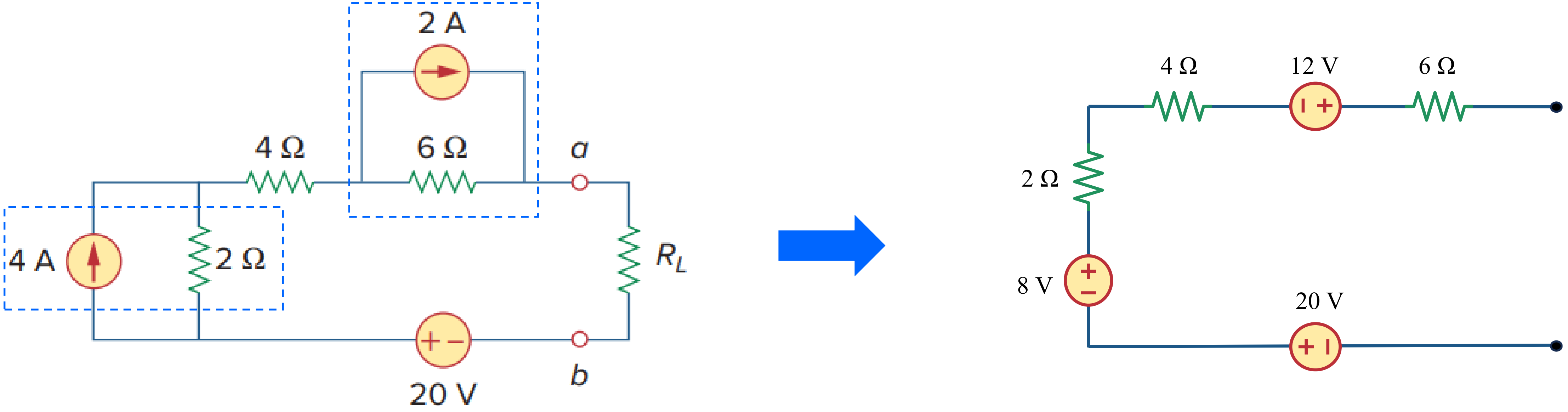
Problem 2

(a) For the circuit in the figure below, obtain the Thevenin equivalent at terminals $a - b$.
(b) Calculate the current if $R_L = 8\ \Omega$.
(c) Find R_L for maximum power deliverable to R_L.
(d) Determine that maximum power.



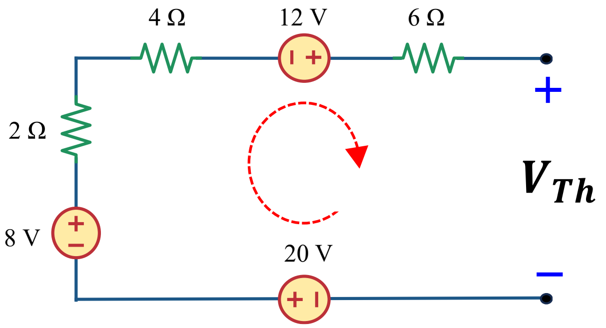
Ans: $\boldsymbol{V_{Th} = 40\ V;\ R_{Th} = 12\ \Omega;\ p_{max} = 33.33\ W}$

Let us, source transform to simplify the circuit



Solution to Problem 2

Finding V_{Th}



Let's use mesh analysis to find the V_{Th}

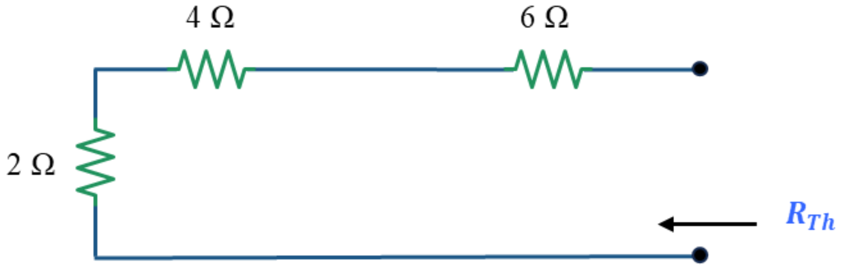
Applying KVL at the mesh,

$$-20 - 8 - 12 + V_{Th} = 0$$

$$\Rightarrow \boldsymbol{V_{Th} = 40\ V}$$

Finding R_{Th}

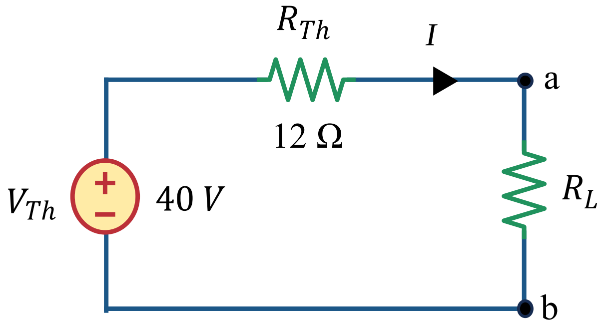
At first, let's deactivate all the Independent Sources. As there is no dependent sources, we simply use series-parallel combination to find the equivalent resistance seen from the load terminal.



$$R_{Th} = 2 + 4 + 6 = \mathbf{12\ \Omega}$$

Thevenin circuit becomes,

(a)



(b) If $R_L = 8\ \Omega$, the current can be found,

$$-40 + IR_{Th} + IR_L = 0$$

$$\Rightarrow -40 + I \times 12 + I \times 8 = 0$$

$$\Rightarrow \boldsymbol{I = 2\ A}$$

(c) R_L for maximum power deliverable to R_L

$$R_L = R_{Th} = \mathbf{12\ \Omega}$$

(d) maximum power that can be delivered to the resistor R_L is,

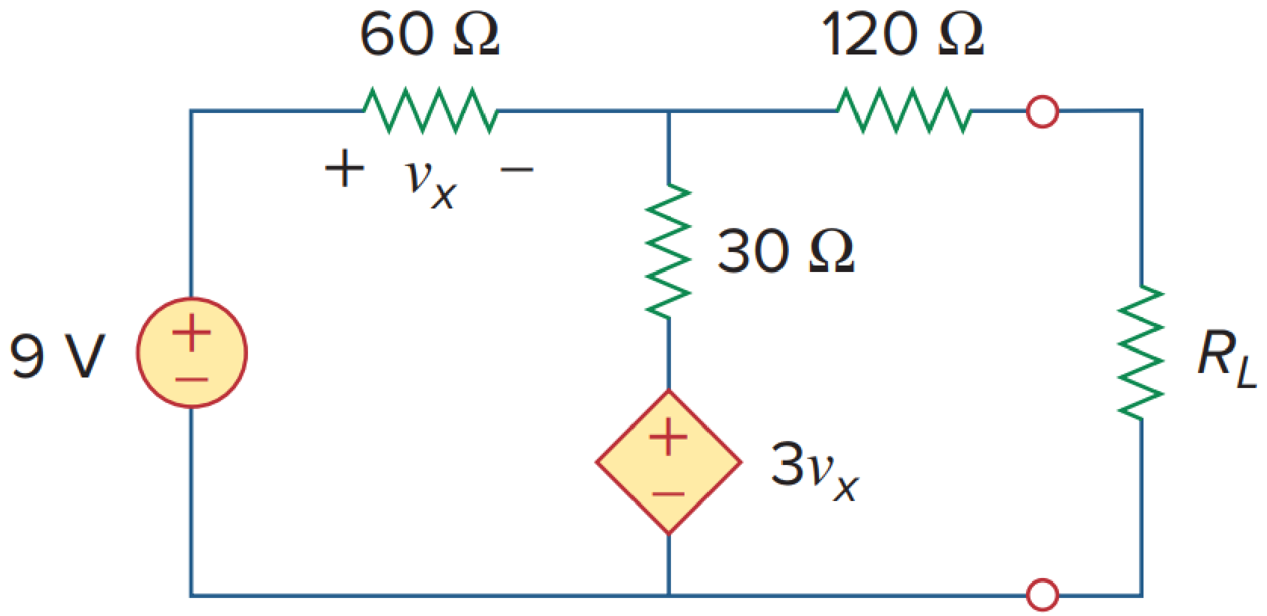
$$p_{max} = \frac{V_{Th}^2}{4R_{Th}}$$

$$\Rightarrow p_{max} = \frac{40^2}{4 \times 12}$$

$$\Rightarrow \boldsymbol{p_{max} = 33.33\ W}$$

Problem 3

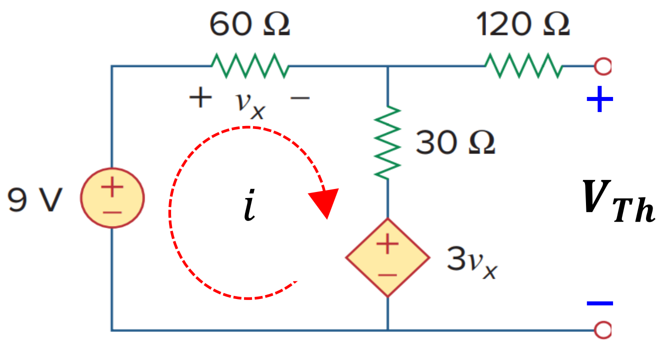
- Determine the value of R_L that will draw the maximum power from the rest of the circuit. Calculate the maximum power.



Ans: $\boldsymbol{R_L = 126.67\ \Omega;\ p_{max} = 96.71\ mW}$

Solution to Problem 3

Finding V_{Th}



Let's use mesh analysis to find the V_{Th}
From the circuit,

$$v_x = 60i$$

$$V_{Th} = V_{30\Omega} + 3v_x$$

Applying KVL at the mesh,

$$-9 + 60i + 30i + 3v_x = 0$$

$$\Rightarrow -9 + 60i + 30i + 3 \times 60i = 0$$

$$\Rightarrow i = 0.0333\ A$$

So,

$$V_{Th} = 30i + 3v_x$$

$$\Rightarrow V_{TH} = 30i + 3 \times 60i$$

$$\Rightarrow V_{Th} = 210i$$

$$= V_{Th} = 210 \times 0.03333$$

$$\Rightarrow \boldsymbol{V_{Th} = 7\ V}$$

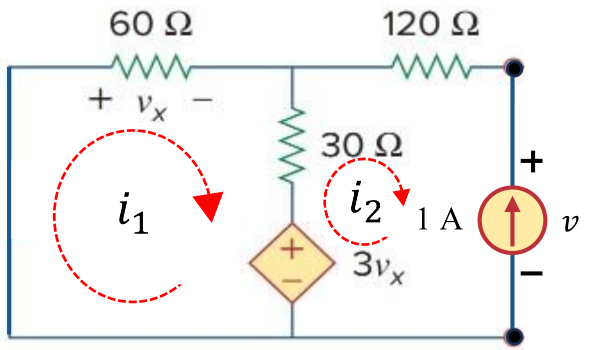
Finding R_{Th}

At first, let's deactivate all the Independent Sources. As there is dependent source, we need to use a known current source across the terminal a-b and find out the voltage across the terminal.

Let's do the second type and apply 1 A at terminal a-b

See the next page

Solution to Problem 2



Let's use mesh analysis to find the v
From the circuit,

$$v_x = 60i_1$$

$$i_2 = -1\ A$$

Applying KVL at the mesh,

$$60i_1 + 30(i_1 - i_2) + 3v_x = 0$$

$$\Rightarrow 60i_1 + 30i_1 - 30i_2 + 3 \times 60i_1 = 0$$

$$\Rightarrow 270i_1 - 30(-1) = 0$$

$$\Rightarrow i_1 = -0.111\ A$$

Now, Applying KVL in mesh 2,

$$-3v_x + 30(i_2 - i_1) + 120i_2 + v = 0$$

$$\Rightarrow -3 \times 60i_1 + 30i_2 - 30i_1 + 120i_2 + v = 0$$

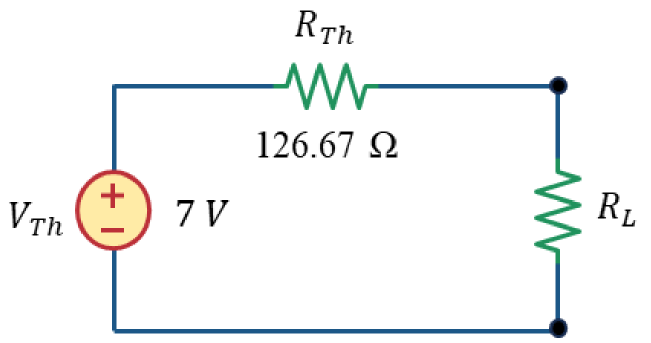
$$\Rightarrow v = 210i_1 - 150i_2$$

$$\Rightarrow v = 210(-0.111) - 150(-1)$$

$$\Rightarrow \boldsymbol{v = 126.67\ V}$$

So,

$$R_{Th} = \frac{v}{1\ A} = \frac{126.67}{1} = \boldsymbol{126.67\ \Omega}$$



R_L for maximum power deliverable to R_L

$$R_L = R_{Th} = \boldsymbol{126.67\ \Omega}$$

Maximum power that can be delivered to the resistor R_L is,

$$p_{max} = \frac{V_{Th}^2}{4R_{Th}}$$

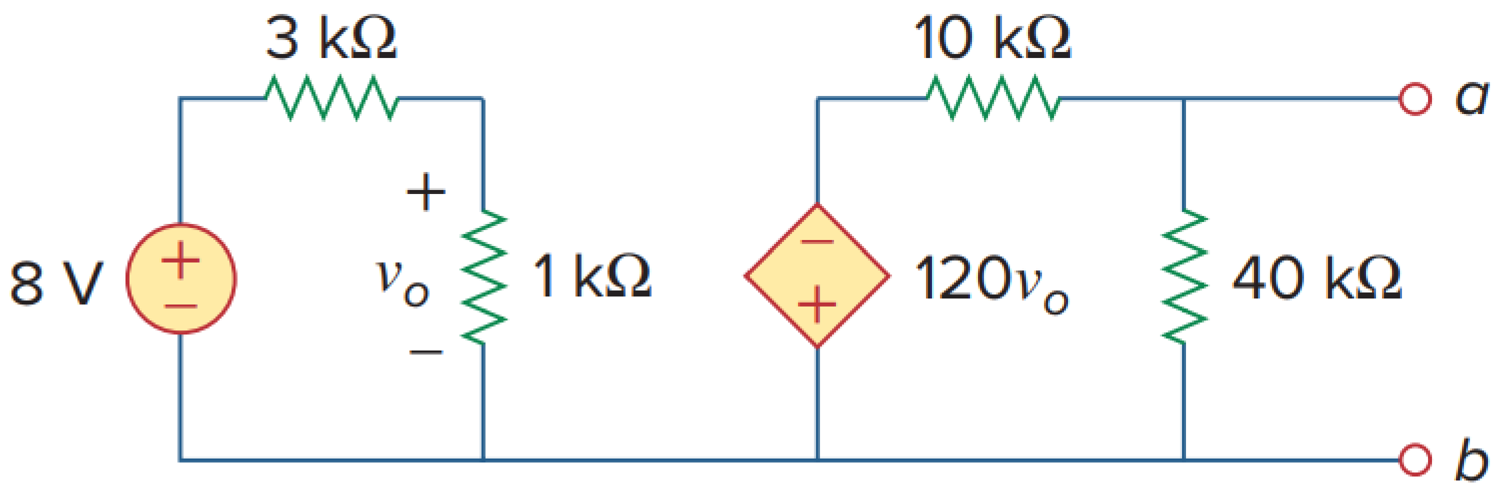
$$\Rightarrow p_{max} = \frac{7^2}{4 \times 126.67}$$

$$\Rightarrow \boldsymbol{p_{max} = 0.0967\ W}$$

$$\Rightarrow \boldsymbol{p_{max} = 96.70\ mW}$$

Problem 4

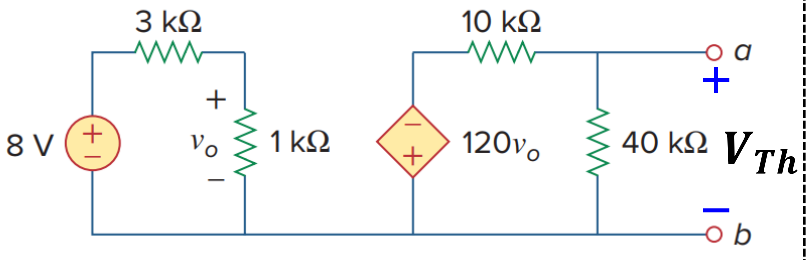
- What resistor connected across terminals will absorb maximum power from the circuit? What is that power?



Ans: $\boldsymbol{V_{Th} = -192\,V;\ R_{Th} = 8\,k\Omega;\ p_{max} = 1.152\,W}$

Solution to Problem 4

Finding V_{Th}



From the left portion of the circuit, applying voltage divider rule

$$v_o = 8 \times \frac{1}{3+1} = 2\ V$$

From the right portion of the circuit, applying voltage divider rule

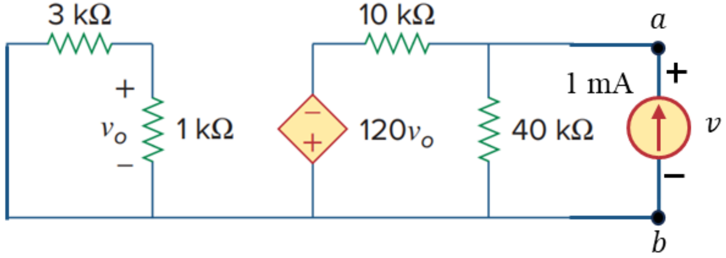
$$V_{Th} = (-120v_o) \times \frac{40}{10+40}$$

$$\Rightarrow V_{Th} = -120 \times 2 \times \frac{40}{10+40}$$

$$\Rightarrow \boldsymbol{V_{Th} = -192\ V}$$

Finding R_{Th}

At first, let's deactivate all the Independent Sources. As there is dependent source, we need to use a known current source across the terminal a-b and find out the voltage across the terminal.



From the left portion of the circuit,

$$v_o = 0\ V$$

From the right portion of the circuit,

$$120v_o = 0\ V$$

So, the voltage across the $(40\ \text{k}\Omega\ ||\ 10\ \text{k}\Omega)$ is the voltage v

$$v = 1 \times (40\ ||\ 10)$$

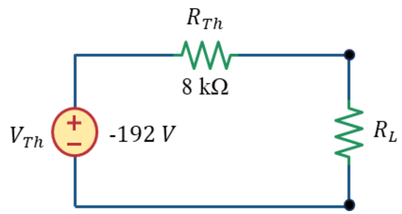
$$\Rightarrow v = 1 \times \frac{40 \times 10}{40 + 10} = 8\ V$$

So,

$$R_{Th} = \frac{v}{1\ mA} = \frac{8\ V}{1\ mA} = \boldsymbol{8\ k\Omega}$$

R_L for maximum power deliverable to R_L

$$R_L = R_{Th} = \boldsymbol{8\ k\Omega}$$



Maximum power that can be delivered to the resistor R_L is,

$$p_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(-192)^2}{4 \times (8 \times 1000)} = \boldsymbol{1.152\ W}$$

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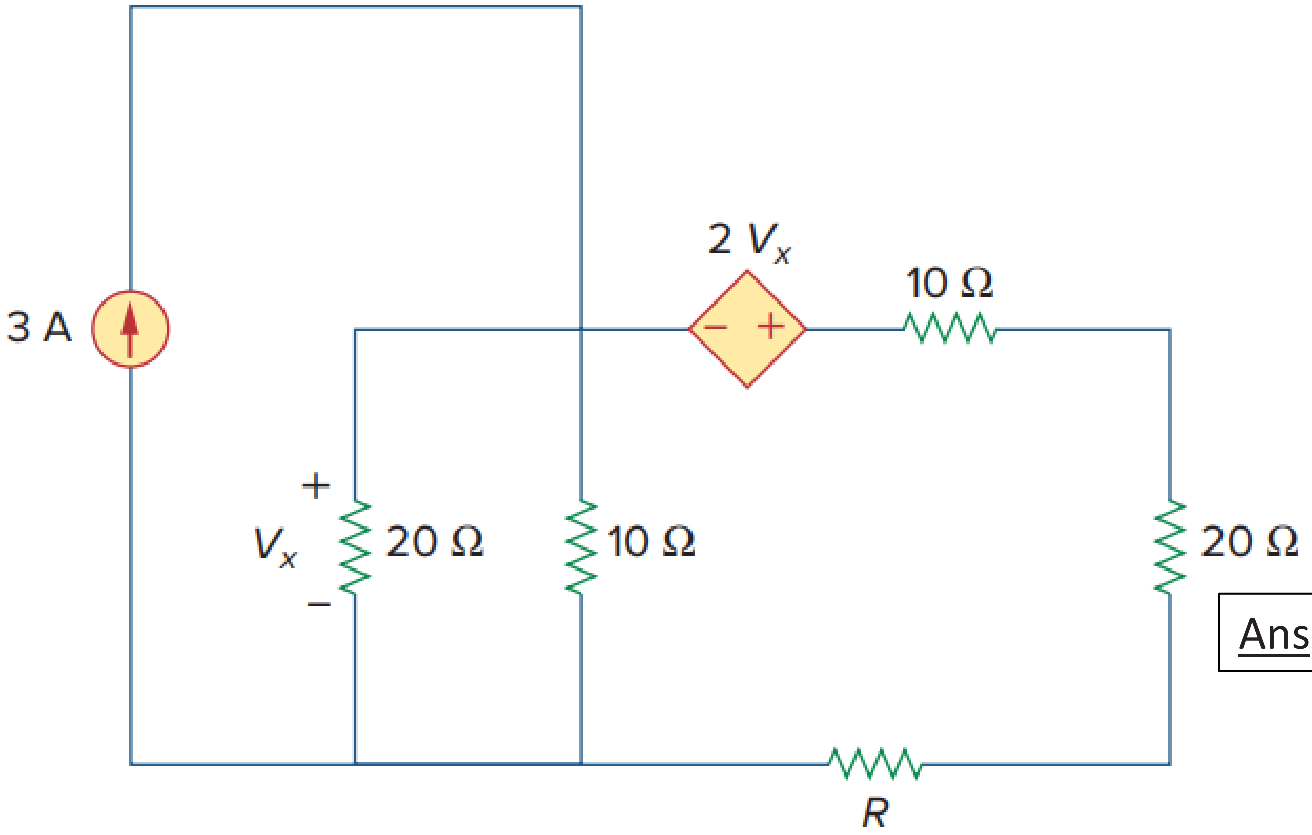
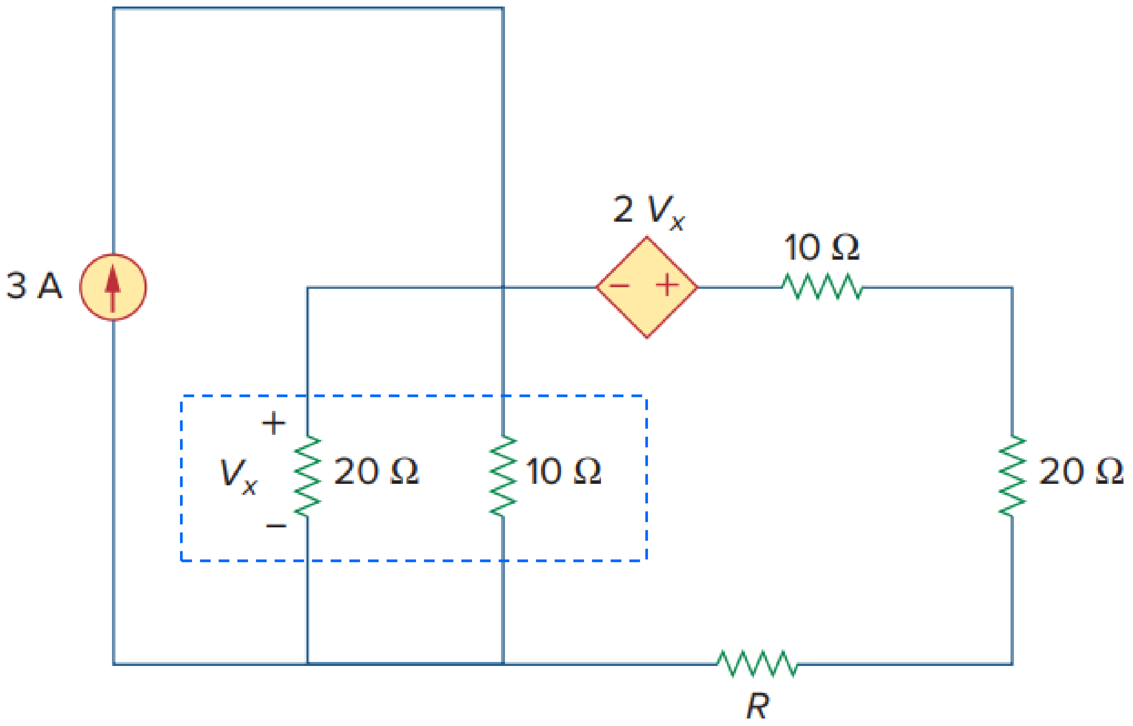
Problem 5

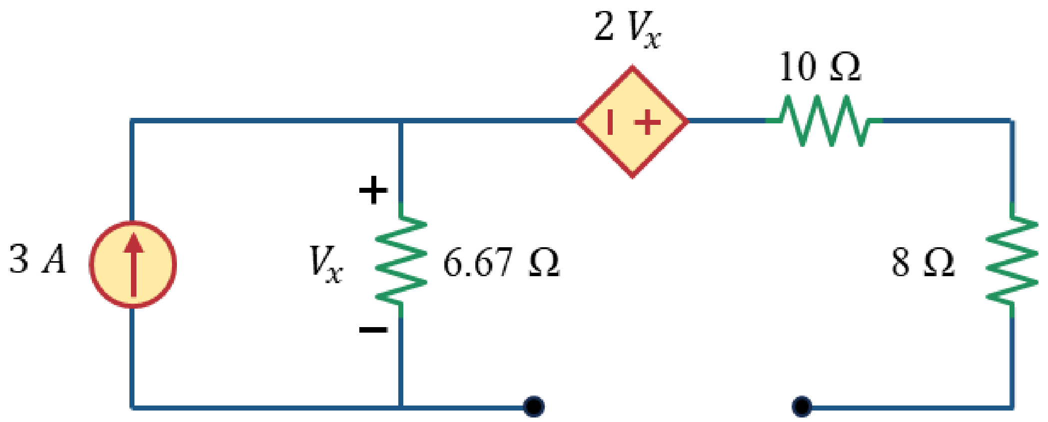
- Determine the maximum power delivered to the variable resistor R shown.



Ans: $V_{Th} = -60\ V;\ R_{Th} = 50\ \Omega; p_{max} = 18\ W$

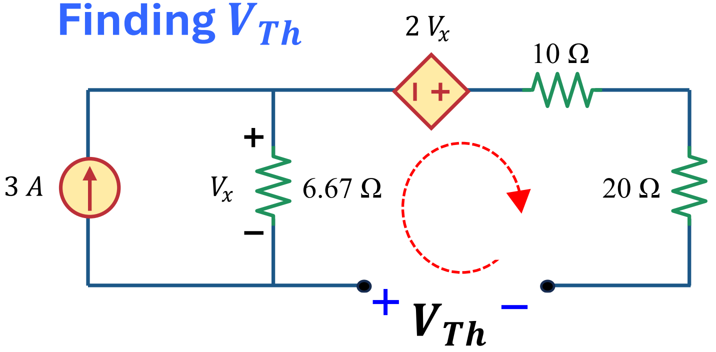
Let us simplify the circuit and make the load terminal open





Solution to Problem 2

Finding V_{Th}



The right mesh is open. So, no current will flow in the right mesh

$$V_x = 3 \times 6.67 = 20V$$

Applying KVL in the right mesh. (No current will flow through the 10 **Ω** and 8 Ω

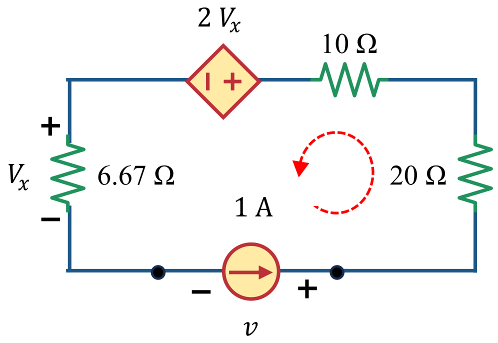
$$-V_x - 2V_x - V_{Th} = 0$$

$$\Rightarrow -20 - 2 \times 20 - V_{Th} = 0$$

$$\Rightarrow \boldsymbol{V_{Th} = -60\ V}$$

Finding R_{Th}

At first, let's deactivate all the Independent Sources. As there is dependent source, we need to use a known current source across the terminal a-b and find out the voltage across the terminal.



From the circuit,

$$V_x = 1 \times 6.67 = 6.67\ V$$

Applying KVL in the single loop

$$-v + 20 \times 1 + 10 \times 1 + 2V_x + 6.67 \times 1 = 0$$

$$-v + 2 \times 6.67 + 36.67 = 0$$

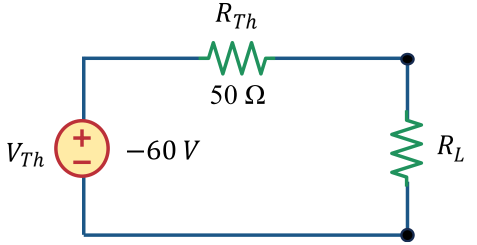
$$\Rightarrow v = 50\ V$$

So,

$$R_{Th} = \frac{v}{1\ A} = \frac{50\ V}{1\ A} = \boldsymbol{50\ \Omega}$$

R_L for maximum power deliverable to R_L

$$R_L = R_{Th} = \mathbf{50\ \Omega}$$

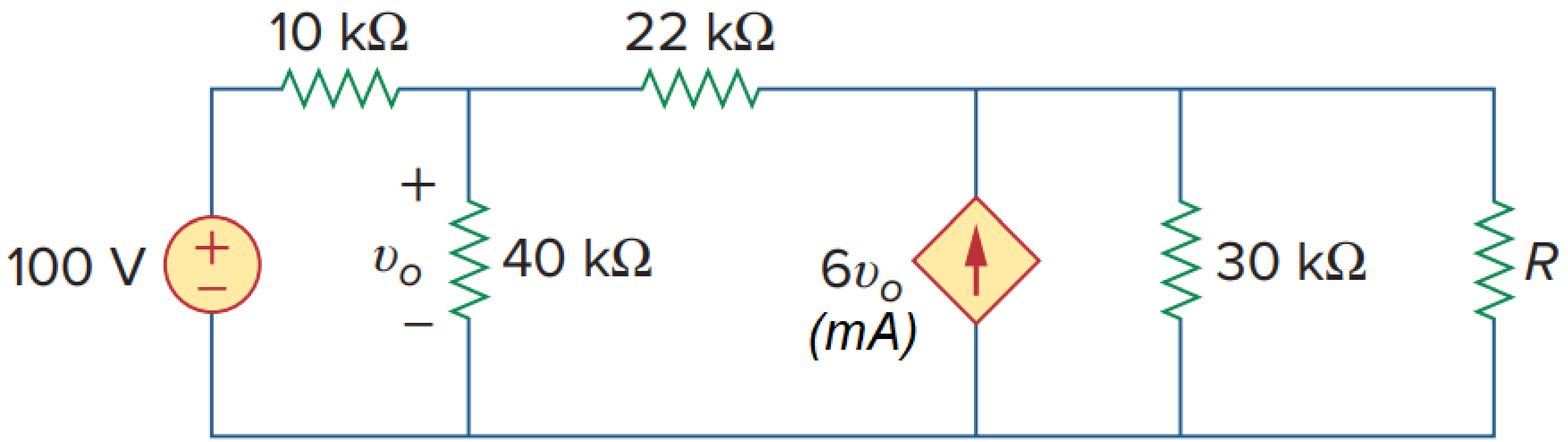


Maximum power that can be delivered to the resistor R_L is,

$$p_{max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{(-60)^2}{4 \times 50} = \boldsymbol{18\ W}$$

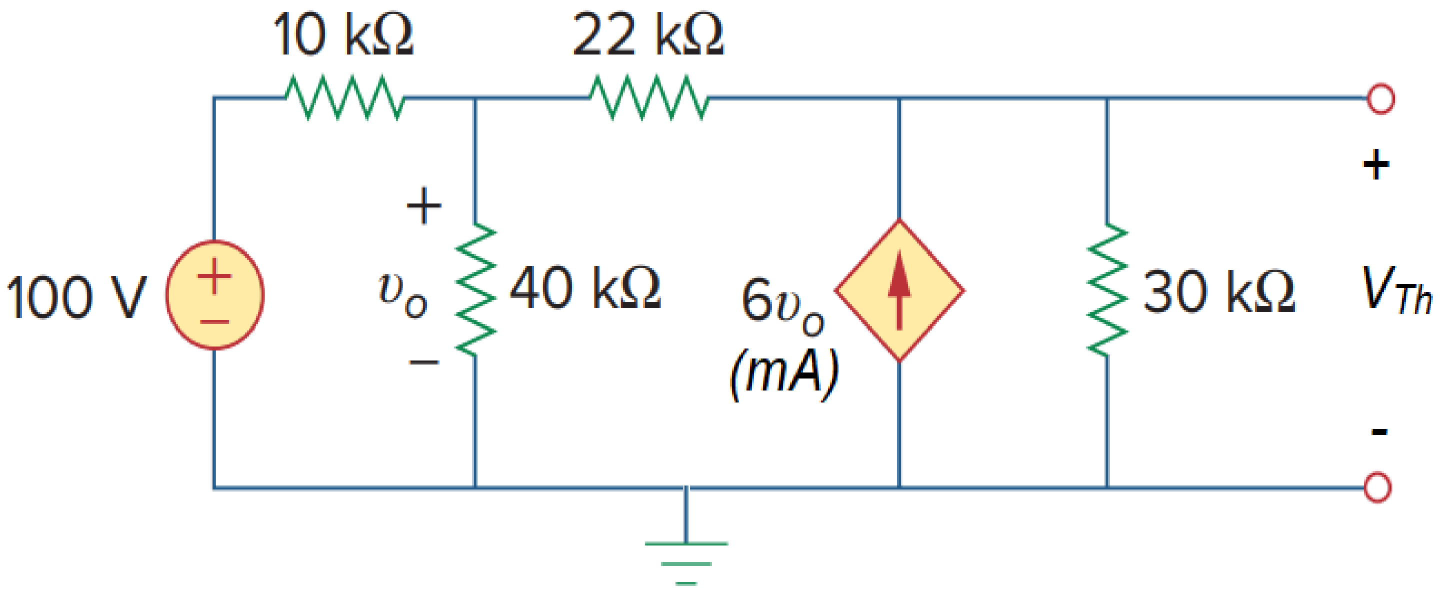
Problem 6

- Find the maximum power that can be delivered to the resistor *R*.



Ans: $V_{Th} = -231.304\,V;\ R_{Th} = -650\,\Omega;\ p_{max} = \infty\ (Theoretically)$

Problem 6: finding V_{Th}



To find P_{max}, we have to first find V_{Th} and R_{Th}.

Let's use nodal analysis to find the V_{Th}.

KCL at node v_0,

$$\frac{v_0 - 100}{10} + \frac{v_0}{40} + \frac{v_0 - V_{Th}}{22} = 0$$

$$\Rightarrow \quad 75v_0 - 20V_{Th} = 4400 \;----(i)$$

KCL at node V_{Th},

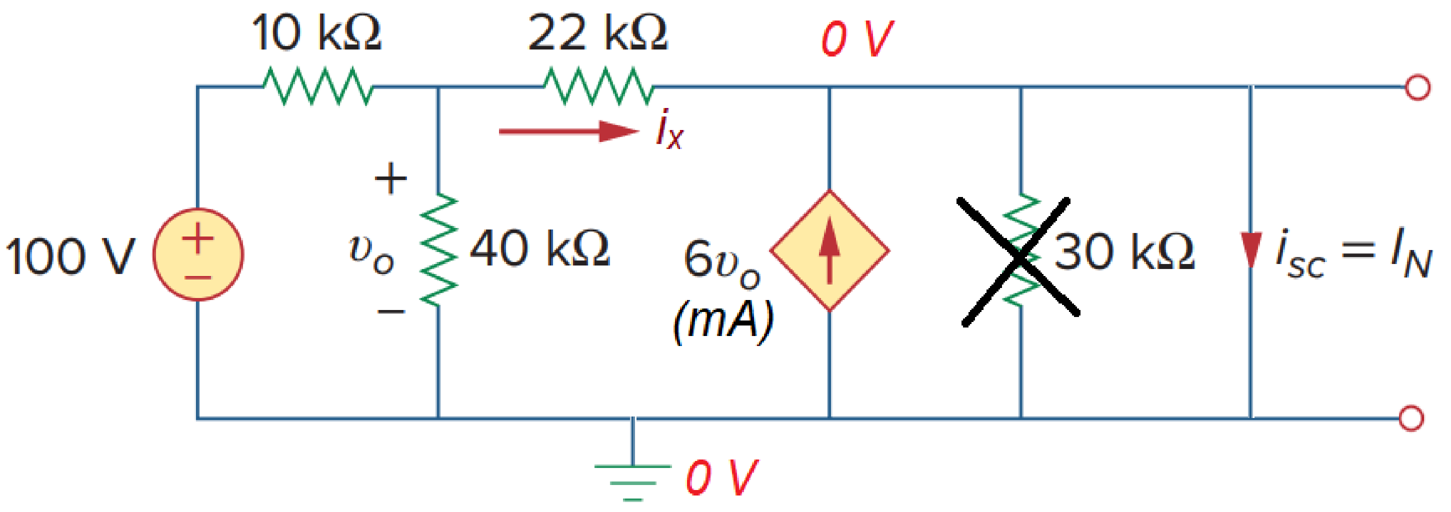
$$\frac{V_{Th} - v_0}{22} + \frac{V_{Th}}{30} = 6v_0$$

$$\Rightarrow \quad 1995v_0 - 26V_{Th} = 0 \;-----(ii)$$

Solving (i) and (ii),

$$V_{Th} = -231.304\,V$$

Problem 6 : finding R_{Th}



As $V_{Th} \neq 0$, let's use $R_{Th} = \frac{V_{Th}}{I_N}$ to determine the Thevenin equivalent resistance. The load terminals have been short circuited as shown in the figure.
Upon short circuiting the terminals $a-b$, the 10 Ω is shorted out. The whole circuit to the left of the dependent source is shorted with respect to it. As a result, the $6v_0$ current supplied by the dependent source will only flow through the short circuit.

Let's use nodal analysis to solve for the current i_x going towards the short circuit though the 22 $k\Omega$ resistor.

KCL at node v_0,

$$\frac{v_0 - 100}{10} + \frac{v_0}{40} + \frac{v_0 - 0}{22} = 0$$

$$\Rightarrow \quad 75v_0 = 4400$$

$$\Rightarrow \quad v_0 = 58.667\ V$$

$$\Rightarrow \quad i_x = \frac{v_0 - 0}{22} = 2.667\ mA$$

So,

$$I_N = i_x + 6v_0 = 354.669\ mA$$

$$R_{Th} = \frac{V_{Th}}{I_N} = -650\ \Omega$$

Problem 6 : finding P_{max}

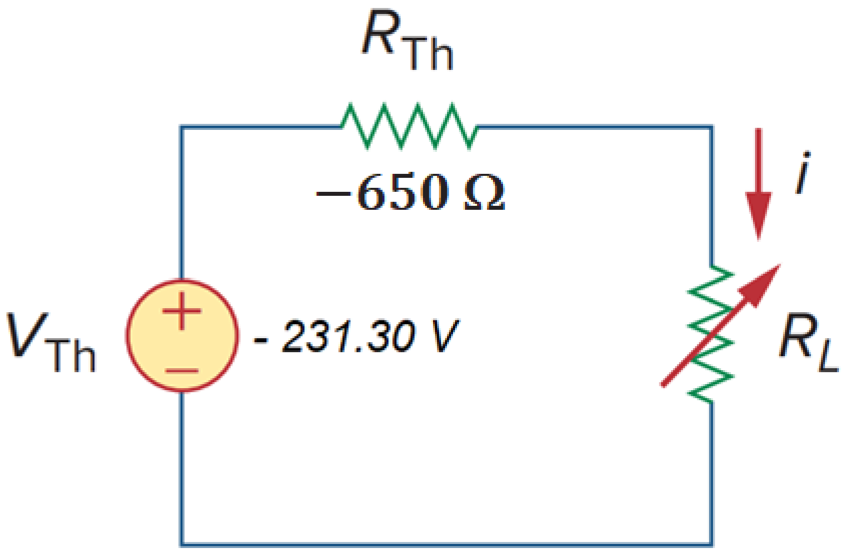
From the previous slides,

$$V_{Th} = -231.30\,V; \qquad R_{Th} = -650\,\Omega$$

What does a negative Thevenin resistance mean!

Negative Thevenin resistance is a part of the circuit model. The conversion of an actual circuit to a Thevenin equivalent is a mechanism for solving circuit problems and does not mean that the Thevenin equivalent circuit replaces the real circuit in all aspects.

Again, negative resistance means an active circuit. This means the circuit is trying to deliver infinite power to the load (assuming the load is practical, that is, $R_L > 0$).



So, the correct answer is,

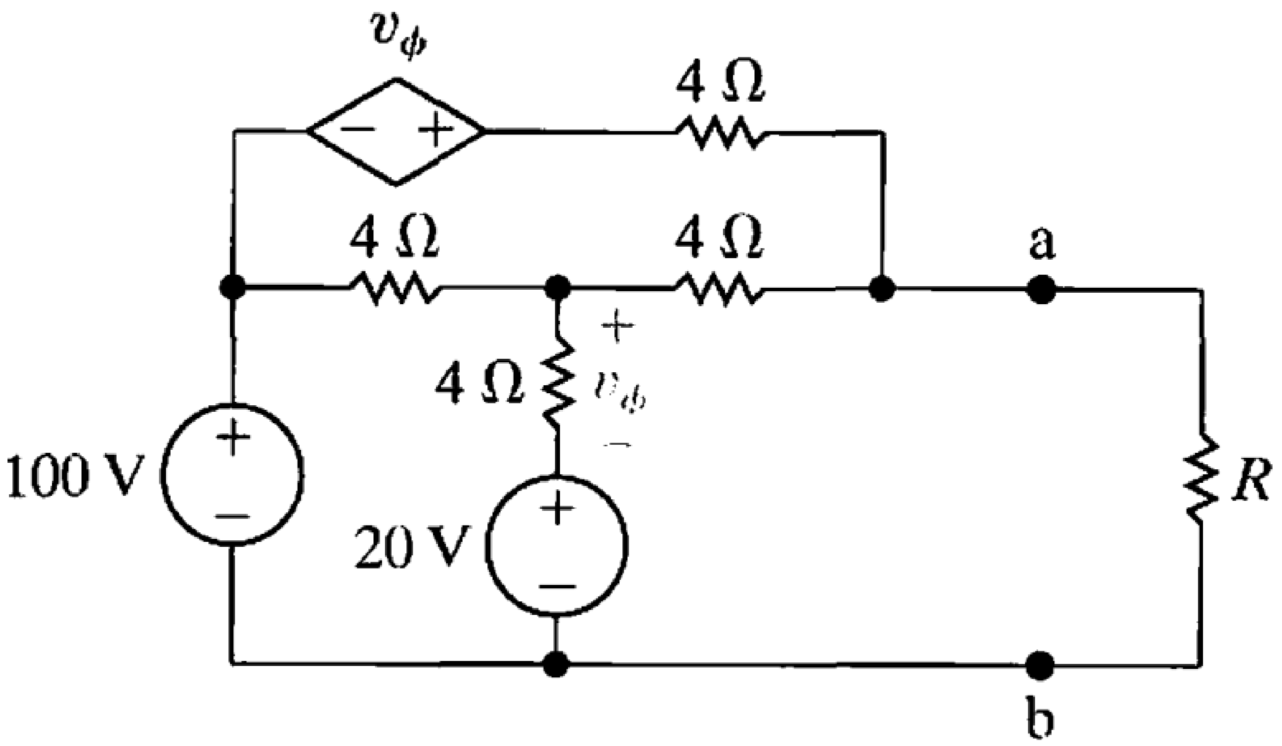
$$i = \frac{V_{Th}}{R_{Th} + R_L} = \frac{-231.30}{-650 + 650} = \infty$$

$$p_{max} = i^2 R_L = \infty\ (theoretically)$$

Problem 7

i. Find the value of R that enables the circuit shown to deliver maximum power to the terminals $a - b$.

ii. Find the maximum power delivered to R.

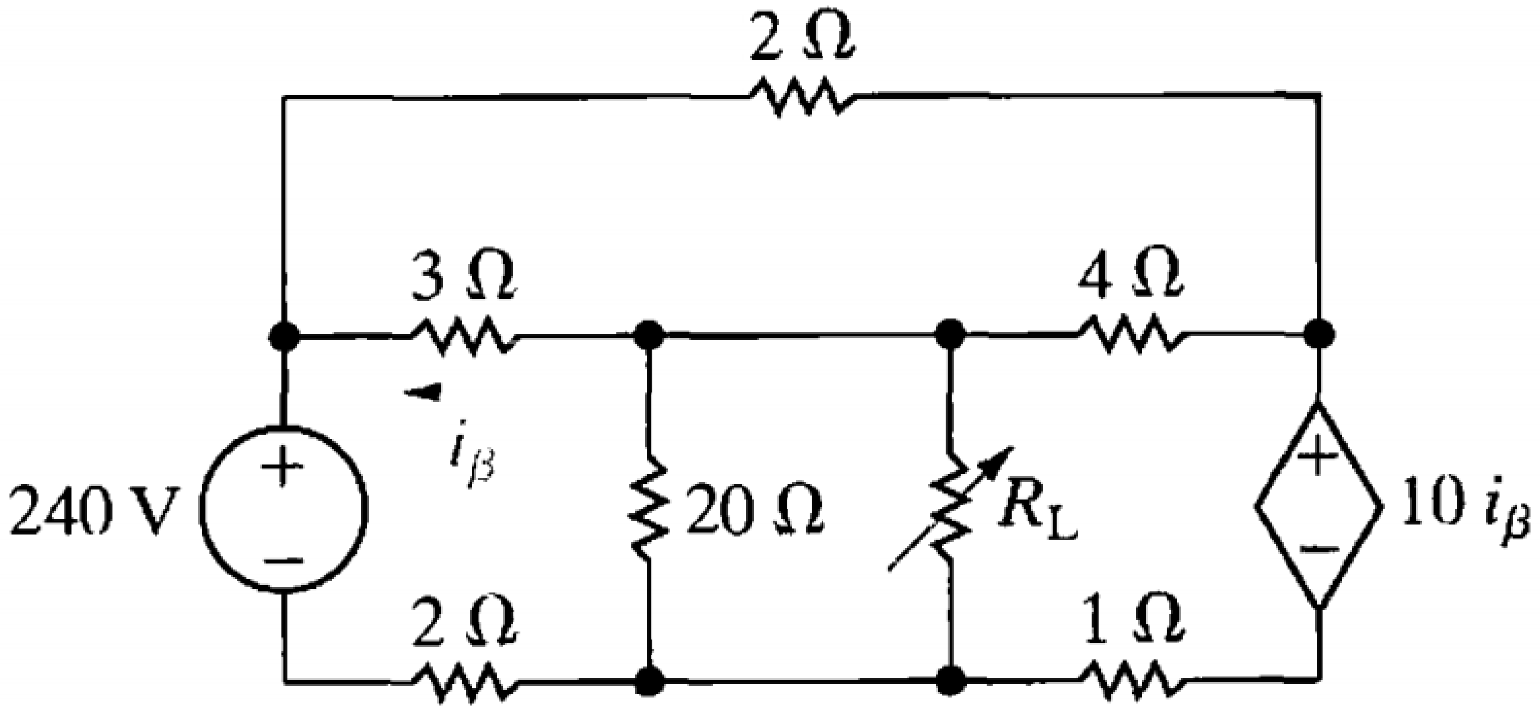


Ans: $(i)\ \mathbf{3\ \Omega};\ (ii)\ \boldsymbol{V_{Th} = 120\ V};\ \boldsymbol{P_{max} = 1.2\ kW}$

Problem 8

i. Find the value of R_L that enables the circuit shown to deliver maximum power to the load (R_L).

ii. Find the maximum power delivered to R_L.



Ans: $(i)\ \mathbf{6}\ \Omega\ (ii)\ \boldsymbol{P_{max}} = \mathbf{24}\ \boldsymbol{W}$

Practice Problems

- Additional recommended practice problems: here
- Other suggested problems from the textbook: here

Thank you for your attention

Course Outline: broad themes