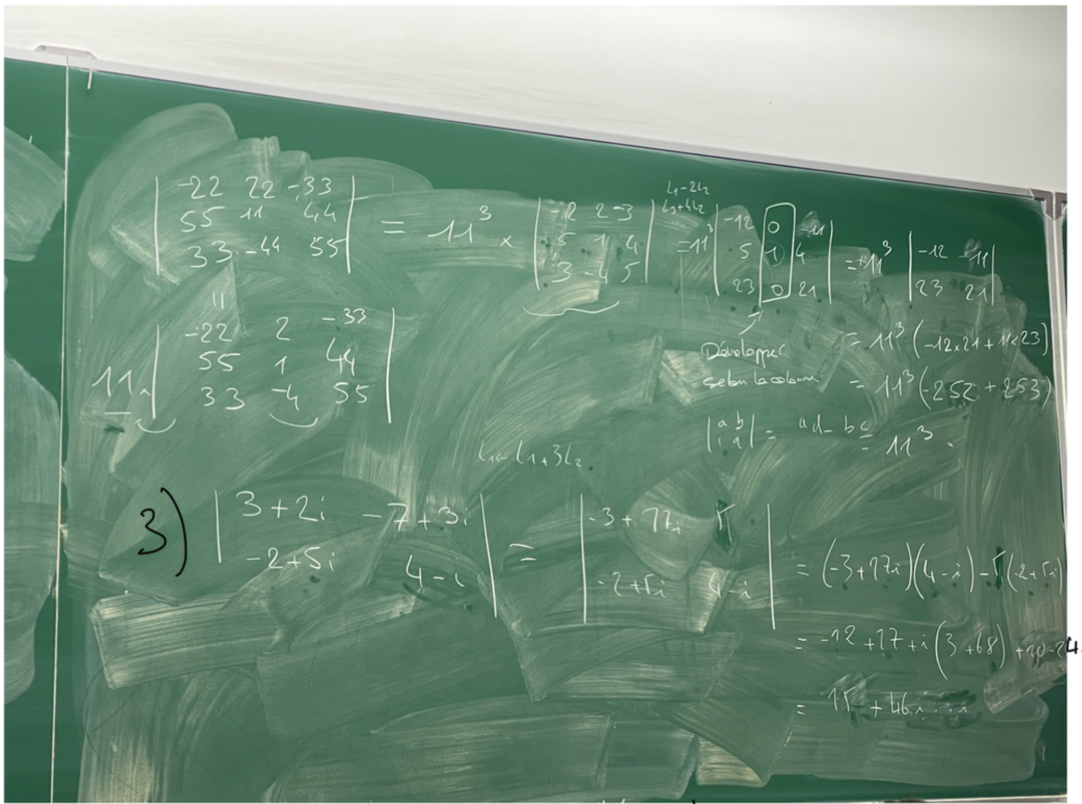
Exercice 1

1) $\begin{vmatrix} 3 & -1 & 2 \\ -2 & 2 & 3 \\ -1 & -3 & 1 \end{vmatrix} = 6+3+12+27-2+4$
$= 50$

2)



4) $\begin{vmatrix} 1+i & 2-i & 2 \\ i & 1+2i & 1+i \\ 2+i & 3+i & 3-i \end{vmatrix} \underset{=}{L_1 - L_2} \begin{vmatrix} 1 & 1-3i & 1-i \\ i & 1+2i & 1+i \\ 2+i & 3+i & 3-i \end{vmatrix}$

$L_2 - iL_1 \rightarrow = \begin{vmatrix} 1 & 1-3i & 1-i \\ 0 & 1+2i-i-3 & 0 \\ 2+i & 3+i & 3-i \end{vmatrix}$

$\underline{\text{Dvt } 2^e \text{ ligne}} = (-2+i) \begin{vmatrix} 1 & 1-i \\ 2+i & 3-i \end{vmatrix}$

$= (-2+i)\left[(3-i) - (1-i)(2+i)\right]$
$= (-2+i)\left[0 - 0i\right] = 0$

5) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \end{vmatrix} \underset{=}{L_1 \leftarrow L_1+L_2+L_3} \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \end{vmatrix}$

$$\begin{vmatrix} 2c & 2c & c-a-b \end{vmatrix} \qquad \begin{vmatrix} 2c & 2c & c-a-b \end{vmatrix}$$

$$\longrightarrow = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

$$= (a+b+c)^3$$

Exercice 2

1) $$\begin{vmatrix} 2-x & 3 & 2 \\ 1 & 1-x & -1 \\ -1 & 1 & 3-x \end{vmatrix} = \begin{vmatrix} 2-x & 3 & 2 \\ 1 & 1-x & -1 \\ 0 & 2-x & 2-x \end{vmatrix} = (2-x) \begin{vmatrix} 2-x & 3 & 2 \\ 1 & 1-x & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= (2-x) \begin{vmatrix} 2-x & 1 & 2 \\ 1 & 2-x & -1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (2-x) \begin{vmatrix} 2-x & 1 \\ 1 & 2-x \end{vmatrix}$$

$$= (2-x)\left[(2-x)^2 - 1\right] = \boxed{(2-x)(3-x)(1-x)}$$

2) $$\begin{vmatrix} 1-x & 2 & 3 \\ 2 & 3-x & 1 \\ 3 & 1 & 2-x \end{vmatrix} \overset{L_1 \leftarrow L_1+L_2+L_3}{=} \begin{vmatrix} 6-x & 6-x & 6-x \\ 2 & 3-x & 1 \\ 3 & 1 & 2-x \end{vmatrix}$$

$$= (6-x) \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3-x & 1 \\ 3 & 1 & 2-x \end{vmatrix}$$

$$= (6-x) \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1-x & -1 \\ 3 & -2 & -1-x \end{vmatrix}$$

$$= (6-x)\left[-(1-x)(1+x) - 2\right]$$

$$= (6-x)(x^2 - 1 - 2)$$

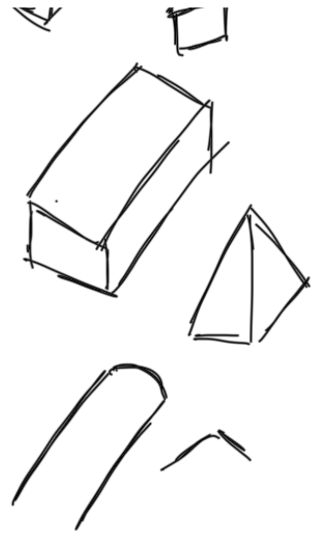
$$= (6-x)(x-\sqrt{3})(x+\sqrt{3})$$

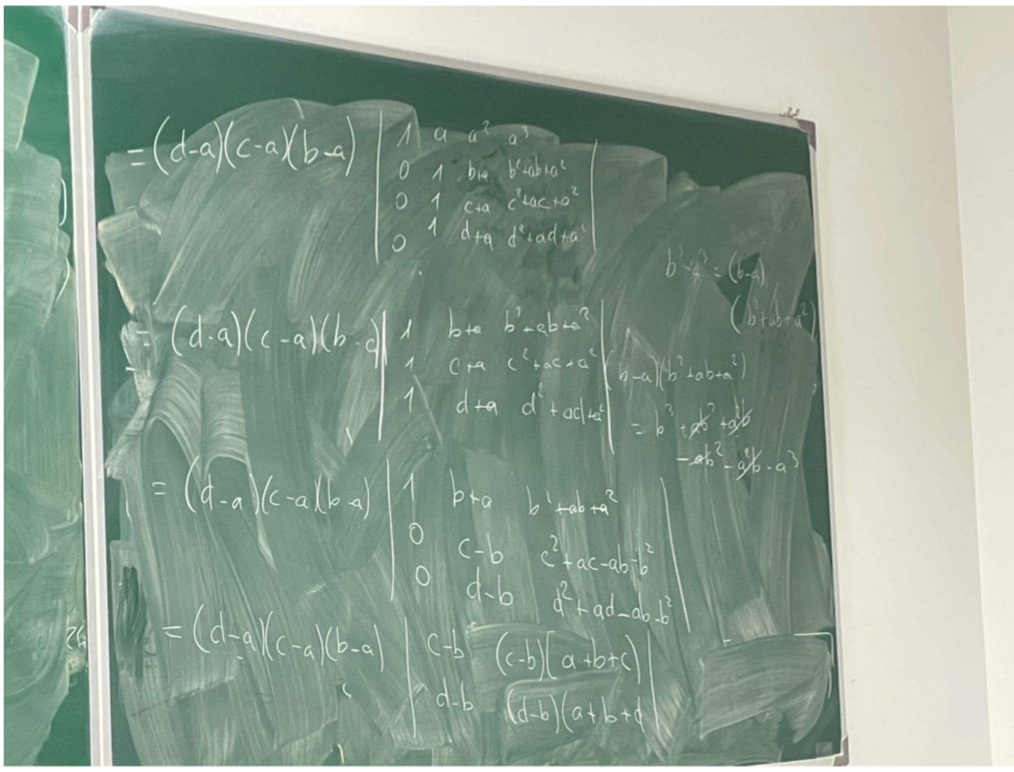
3) $$\begin{vmatrix} 2 & x & -4 \\ -x & y & 2x \\ x & 2y & -y \end{vmatrix} = \begin{vmatrix} 2 & x & 0 \\ -x & y & 0 \\ x & 2y & 2x-y \end{vmatrix} = (2x-y) \begin{vmatrix} 2 & x \\ -x & y \end{vmatrix}$$

$$= (2x-y)(2y+x^2)$$

4) $$\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & b-a & b^2-a^2 & b^3-a^3 \\ 0 & c-a & c^2-a^2 & c^3-a^3 \\ 0 & d-a & d^2-a^2 & d^3-a^3 \end{vmatrix}$$

$$= (d-a)(c-a)(b-a)\begin{vmatrix} 1 & a & a^2 & a^3 \\ 0 & 1 & b+a & \\ 0 & 1 & c+a & \\ 0 & 1 & d+a & \end{vmatrix}$$





Exercice 3

$$A = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\chi_A = \det(A - XI_3)$$

$$= \begin{vmatrix} -1-x & 1 & 1 \\ 1 & -1-x & 1 \\ 1 & 1 & -1-x \end{vmatrix}$$

$$= \begin{vmatrix} 1-x & 1-x & 1-x \\ 1 & -1-x & 1 \\ 1 & 1 & -1-x \end{vmatrix} = (1-x)\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-x & 1 \\ 1 & 1 & -1-x \end{vmatrix} = (1-x)\begin{vmatrix} 1 & 1 & 1 \\ 0 & -2-x & 0 \\ 0 & 0 & -2-x \end{vmatrix}$$

$$= -(1-x)(x+2)^2$$

$$Sp(A) = \{1, -2\}$$

Ker $A+2I$

$$A + 2I_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \leftarrow \text{rg } 1$$

donc $\mathrm{Ker}(A+2I)$ dim 2

$$(A+2I_3)v = 0$$

$v = \begin{pmatrix} \\ y \\ z \end{pmatrix}$ $\qquad (x+y+z=0)$

$\rightarrow E_{-2} = \{x+y+z=0\} \rightarrow$ Base $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$A - \lambda I$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

$E_1 \quad E_{-2}$

$$\underbrace{P^{-1}AP}_{} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$1 \quad -2$

$$B = \begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix}$$

- $\chi_B(\lambda) = \det(B - \lambda I_3) = \begin{vmatrix} -\lambda & -2 & 0 \\ 1 & -\lambda & -1 \\ 0 & 2 & -\lambda \end{vmatrix}$ $L_3 \leftarrow L_3 + L_1$

$$= \begin{vmatrix} -\lambda & -2 & 0 \\ 1 & -\lambda & -1 \\ -\lambda & 0 & -\lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} -\lambda & -2 & 0 \\ 1 & -\lambda & -1 \\ 1 & 0 & 1 \end{vmatrix} \quad C_1 \leftarrow C_1 - C_3$$

$$= -\lambda \begin{vmatrix} -\lambda & -2 & 0 \\ 2 & -\lambda & -1 \\ 0 & 0 & 1 \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & -2 \\ 2 & -\lambda \end{vmatrix}$$

$$= -\lambda(\lambda^2 + 4)$$

$$= -\lambda(\lambda - 2i)(\lambda + 2i)$$

Pas diagonalisable sur $\mathbb{R}$,
mais sur $\mathbb{C}$

- base de $\mathrm{Ker}(B)$

$$\begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0$$

$\dim(\mathrm{Ker}(B)) = 1$

Base de $(\mathrm{Ker}(B)) = \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right)$

- Base de $\mathrm{Ker}(B - 2iI_3)$

$\begin{pmatrix} -2i & -2 & 0 \\ \end{pmatrix} \qquad \begin{pmatrix} 1 & -2i & -1 \end{pmatrix}$

$$\begin{pmatrix} -2i & & \\ 1 & -2i & -1 \\ 0 & 2 & -2i \end{pmatrix} \Longleftrightarrow \begin{pmatrix} -2i & -2 & 0 \\ 0 & 2 & -2i \end{pmatrix}$$

$$\underset{\Longleftrightarrow}{L_2 \leftarrow L_2 + 2i L_1} \begin{pmatrix} 1 & -2i & -1 \\ 0 & 2 & -2i \\ 0 & 2 & -2i \end{pmatrix}$$

$$\Longleftrightarrow \begin{pmatrix} 1 & -2i & -1 \\ 0 & 2 & -2i \end{pmatrix}$$

$$\underset{\Longleftrightarrow}{L_1 \leftarrow L_1 + iL_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -2i \end{pmatrix}$$

$$\begin{cases} x + z = 0 \\ y - iz = 0 \end{cases} \Longleftrightarrow \begin{cases} x = -z \\ y = iz \end{cases}$$

$$\forall v \in \mathrm{Ker}(B - 2i I_3),\ v = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ iz \\ z \end{pmatrix} = z \begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix}$$

$x, y, z \in \mathbb{R}$

$$\text{Base de } \mathrm{Ker}(B - 2i I_3) = \left(\begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix} \right)$$

Ex 6 : $A = \begin{pmatrix} \pi & 4 & 2 \\ 0 & \pi & 1 \\ 0 & 0 & \pi \end{pmatrix}$ n'est pas diagonalisable

$\chi_A = (\pi - x)^3$ $\quad Sp(A) = \{\pi\}$ $\quad \begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & \pi \end{pmatrix}$

Si A étant diagonalisable $\exists P$ $\quad P^{-1}AP = \pi I_3$

$$A = P(\pi I_3)P^{-1}$$
$$= \pi P P^{-1}$$
$$= \pi I_3$$

Impossible

$$A = P^{-1} M P \quad I_3$$

$$A^n = \underbrace{P^{-1} M \overbrace{P P^{-1}} M \dots M P}_{n \text{ fois}}$$

$$= P^{-1} M^n P$$

Exercice 5

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 2 & 4 & 2 \\ -1 & 0 & 3 \end{pmatrix}$$

$$\chi_A = \det \begin{pmatrix} 3-x & 0 & -1 \\ 2 & 4-x & 2 \\ -1 & 0 & 3-x \end{pmatrix}$$

dev. selon C_2

$$= (4-x) \begin{vmatrix} 3-x & -1 \\ -1 & 3-x \end{vmatrix}$$

$$= (4-x)\left[(3-x)^2 - 1\right]$$

$$= (4-x)^2 (2-x)$$

$$Sp(A) = \{2, 4\}$$

$\mathrm{Ker}(A - 2I_3)$

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 2 & 2 \\ -1 & 0 & 1 \end{pmatrix} \leftarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\leftarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$$\begin{cases} x - z = 0 \\ y + 2z = 0 \end{cases}$$

Base : $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

$\mathrm{Ker}(A - 4I_3)$

$$\begin{pmatrix} -1 & 0 & -1 \\ 2 & 0 & 2 \\ -1 & 0 & -1 \end{pmatrix} \leftarrow \{ x + z = 0$$

$\leftarrow$ Diagonalisable.

$$\rightarrow \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

$$P = \begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \leftarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 2 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$\leftarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 2 & 1 & 0 \end{array}\right)$$

$$\leftarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 & 1/2 \\ 0 & 2 & 1 & 2 & 1 & 0 \end{array}\right)$$

$$\leftarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 1 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right).$$

$$P^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & -2 \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

$$A = P\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}P^{-1}$$

$$A^n = P\begin{pmatrix} 2^n & 0 & 0 \\ 0 & 4^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}P^{-1}$$

$$A^n = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 2^n & 0 & 0 \\ 0 & 4^n & 0 \\ 0 & 0 & 4^n \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & -2 \end{pmatrix}.$$

$$A^n = \frac{1}{2}\begin{pmatrix} 2^n & 4^n & 0 \\ -2^{n+1} & 0 & 4^n \\ -2^n & 4^n & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & -2 \end{pmatrix}.$$

$$= \frac{1}{2}\begin{pmatrix} 2^n+4^n & 0 & 4^n-2^n \\ -2^{n+1}+2\times 4^n & 2\times 4^n & 2^{n+1}-2\times 4^n \\ -2^n+4^n & 0 & 2^n+4^n \end{pmatrix}.$$