An entangled state is one which contains more information than its substates looked at separately. Entropy provides one tool which can be used to quantify entanglement, although other entanglement measures exist. If the overall system is pure, the entropy of one subsystem can be used to measure its degree of entanglement with the other subsystems. Mathematically we can formalise it as following:

Consider two non-interacting Hilbert spaces: $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$ and let $|\psi\rangle_A$ and $|\psi\rangle_B$ be vectors from the Hilbert spaces respectively. These vectors can be expanded as:

$|\psi\rangle_A = \sum_i c_i|i\rangle$

$|\psi\rangle_B = \sum_j c_j|j\rangle$

Now it is possible to procude a state, $|\psi\rangle_{AB} = \sum_{ij} c_{ij}|i\rangle \otimes |j\rangle$, which is mixed state consisting of vectores from non-interacting Hilber spaces. If, by some means, we can separate the components coming from the different spaces from this mixed state, we can express the mixed state in a prodcut state as: $|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B = \sum_i c_i \sum_j c_j|i\rangle \otimes |j\rangle$.

If the above decomposition cannot be done, namely $c_{ij} \neq c_i c_j$, the state in question is an entangled state.

On of the famouse example of entanlged states is the set of Bell states. In it the states are maximally entangled. In the [0,1] normalized von Neumann definition, the Bell states have 0 entropy while the component states have an entropy of 1. Following is how it works:

We use the Hilbert spaces of complex numbers, $\mathcal{H}_{\mathcal{A}}, \mathcal{H}_{\mathcal{B}} = \mathbb{C}^2$. So our tensor product space is $\mathcal{H}_{\mathcal{AB}} = \mathbb{C}^4$. This is an easy calculation since the component Hilbert spaces are small and identical.

Now, consider this Bell state: $|\varphi^+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\end{bmatrix}$. Where the bases are: $|0\rangle = \begin{bmatrix}0\\1\end{bmatrix} and |1\rangle = \begin{bmatrix}1\\0\end{bmatrix}$.

Performing the outer product and summing, we get in the $\mathbb{C}^4$ algebra:

$$|\varphi^+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

The von Neumann entropy is defined in terms of the density operator in the following manner: $\mathcal{S}(\rho) = -\mathrm{tr}(\rho \log_2 \rho)$. And, the von Neumann entropy of entanglement can be calculated by calculating the entropy of the partial trace of the density operator:

$\mathcal{E}(\rho) = \mathcal{S}(\mathrm{tr}_B\rho) = \mathcal{S}(\mathrm{tr}_A\rho)$

The density operator would thus be

$$\rho = |\varphi^+\rangle\langle\varphi^+| = \frac{1}{2}\begin{bmatrix}1\\0\\0\\1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 0 & 0 & 1\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\1 & 0 & 0 & 1\end{bmatrix}.$$

A partial trace over $\mathcal{H}_A or \mathcal{H}_B$ returns a 2×2 matrix. In our case, $\mathrm{tr}_A = \mathrm{tr}_B = \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$.

Using the definition of the entropy of entanglement, we get:

$$\mathcal{S}(\rho) = -\mathrm{tr}\begin{bmatrix}\frac{1}{2}\log_2(\frac{1}{2}) & 0\log_2 0\\0\log_2 0 & \frac{1}{2}\log_2(\frac{1}{2})\end{bmatrix} = 1$$

In the python code attempts, we check try to implement the above.