Math 6620: Perturbation Methods

Perturbation methods are used to approximate solutions of problems which involve a small parameter, often denoted by ε. To say 'ε is small' means that its magnitude is much less than one. e.g. $\varepsilon = .1, .02, 10^{-6}, \ldots$

<u>Notation:</u> $\varepsilon \ll 1$

implies that ε is positive ($0<\varepsilon\ll 1$)

Write $|\varepsilon| \ll 1$ if ε is allowed to be negative.

<u>Examples:</u>

1. Approximate the roots of
$$\varepsilon(x^3+2)+x^2+1=0.$$

2. $$\varepsilon x' = x+y \quad , \; x(0)=A$$
$$y' = x+\varepsilon y^2 \quad , \; y(0)=B.$$

3. $$mx''+kx+\varepsilon x^3=0 \quad \text{(weakly nonlinear spring)}$$
$$x(0)=1, \; x'(0)=0$$

Small parameters appear in many physical and biological models. Perturbation methods are applicable only if the problem is solvable when $\varepsilon=0$, as in the above examples.

Polynomials

Example: Approximate the roots of

$$x^2 + \varepsilon x - 1 = 0, \text{ where } \varepsilon \ll 1.$$

Exact Roots: $\boxed{x = -\frac{\varepsilon}{2} \pm \sqrt{1 + \varepsilon^2/4}}$

Taylor expand in powers of ε. $\left(\sqrt{1+t} = 1 + \frac{t}{2} + \cdots\right)$

$$x = -\frac{\varepsilon}{2} \pm \left(1 + \frac{\varepsilon^2/4}{2} + \cdots\right) = \pm 1 - \frac{\varepsilon}{2} \pm \varepsilon^2/8 + \cdots$$

$$\boxed{\begin{aligned} x^{(1)} &= 1 - \frac{\varepsilon}{2} + \varepsilon^2/8 + \cdots \\ x^{(2)} &= -1 - \frac{\varepsilon}{2} - \varepsilon^2/8 + \cdots \end{aligned}}$$

Perturbation Method

Assume that x can be expanded in powers of ε, and write $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$.

Then, plug the expansion into the equation and successively determine $x_0, x_1, x_2, \ldots$ until the desired accuracy is achieved.

$$x^2 + \varepsilon x - 1 = (x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 + \varepsilon(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) - 1 = 0$$

Ignore ε^3, and smaller terms.

$$(x_0^2 + \varepsilon^2 x_1^2 + 2\varepsilon x_0 x_1 + 2\varepsilon^2 x_0 x_2 + \cdots) + (\varepsilon x_0 + \varepsilon^2 x_1 + \cdots) - 1 = 0$$

$$\boxed{(x_0^2 - 1) + \varepsilon(2x_0 x_1 + x_0) + \varepsilon^2(x_1^2 + 2x_0 x_2 + x_1) + \cdots = 0}$$

Equate the coefficient of each power of ε to 0 to determine $x_0, x_1, x_2, \ldots$

<u>Leading Order Approximation</u>: $x_0^2 - 1 = 0$
(corresponds to $\varepsilon = 0$)

$$\boxed{x_0 = \pm 1}$$

<u>First order Correction</u>: $2\cancel{x_0}x_1 + \cancel{x_0} = 0$

$$\boxed{x_1 = -\tfrac{1}{2}}$$

<u>Second Order Correction</u>: $x_1^2 + 2x_0x_2 + x_1 = 0$

$$\tfrac{1}{4} \pm 2x_2 - \tfrac{1}{2} = 0$$

$$\boxed{x_2 = \pm \tfrac{1}{8}}$$

$x_0 = +1 \Rightarrow x_2 = +1/8$
$x_0 = -1 \Rightarrow x_2 = -1/8$

Then, $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$

$$\boxed{x = \pm 1 - \varepsilon/2 \pm \varepsilon^2/8 + \cdots}$$

Agrees with the Taylor expansion of the exact solution.

Consider the root $x^{(1)} = 1 - \frac{\varepsilon}{2} + \varepsilon^2/8 + \cdots$.

Suppose $\varepsilon = 0.1. \Rightarrow x^{(1)} = .9512492197250 4$

	# of terms	approximation of $x^{(1)}$	|Error|
ε^0:	1	1	.04875
ε^1:	2	.95	.00125
ε^2:	3	.95125	7.8×10^{-7}
ε^3:	4	.95125	7.8×10^{-7} ($x_3 = 0$)
ε^4:	5	.951249921875	9.75×10^{-10}

$$x^{(1)} = 1 - \frac{\varepsilon}{2} + \frac{\varepsilon^2}{8} - \frac{\varepsilon^4}{128} + \cdots \quad (\underline{x_3 = 0})$$

The above approach may fail.

Example: $(1-\varepsilon)x^2 - 2x + 1 = 0, \quad \varepsilon \ll 1$

Naively try $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$

(This is called the 'naive' expansion. It may be used as a blind first attempt since it is easiest and it often works. If it fails the way in which it fails may indicate how to proceed next.)

$$(1-\varepsilon)(x_0 + \varepsilon x_1 + \cdots)^2 - 2(x_0 + \varepsilon x_1 + \cdots) + 1 = 0$$

$$(1-\varepsilon)(x_0^2 + 2\varepsilon x_0 x_1 + \cdots) - 2(x_0 + \varepsilon x_1 + \cdots) + 1 = 0$$

$$(x_0^2 - 2x_0 + 1) + \varepsilon(2x_0 x_1 - x_0^2 - 2x_1) + \cdots = 0$$

Leading Order: $x_0^2 - 2x_0 + 1 = 0$

$$(x_0 - 1)^2 = 0$$

$x_0 = 1$ double root

First Order: $2x_0 x_1 - x_0^2 - 2x_1 = 0$

$$2x_1 - 1 - 2x_1 = 0$$

$-1 \neq 0$ contradiction

The naive expansion fails. Why?

Exact Roots: $x = \dfrac{1 \pm \sqrt{\varepsilon}}{1-\varepsilon}$

The exact roots cannot be expanded in powers of ε. However, they can be expanded in powers of $\sqrt{\varepsilon}$.

$$x = (1 \pm \sqrt{\varepsilon})\left(\frac{1}{1-\varepsilon}\right) = (1 \pm \sqrt{\varepsilon})(1 + \varepsilon + \varepsilon^2 + \cdots) = 1 \pm \sqrt{\varepsilon} + \varepsilon + \cdots$$

$$x = 1 \pm \sqrt{\varepsilon} + \varepsilon + \cdots$$

$\Rightarrow$ Need to expand x as $x = x_0 + \varepsilon^{1/2} x_1 + \varepsilon x_2 + \varepsilon^{3/2} x_3 + \cdots$.

If the exact roots are unknown, how do we know to expand x in powers of $\sqrt{\varepsilon}$?

Try a more general expansion.

$$X = X_0 + \delta_1(\varepsilon) X_1 + \delta_2(\varepsilon) X_2 + \cdots,$$

where $\delta_1 \ll 1$, $\delta_{i+1} \ll \delta_i$, $\delta_i(\varepsilon) \to 0$ as $\varepsilon \to 0$, and $X_i \neq 0$.

The $\delta_i(\varepsilon)$'s are called Gauge Functions (or Scaling Functions), and are to be determined as part of the solution process.

We have $(1-\varepsilon)x^2 - 2x + 1 = 0$.

To find a 3-term expansion of the roots, we must determine X_0, X_1, X_2, δ_1, and δ_2.

$$(1-\varepsilon)(X_0 + \delta_1 X_1 + \delta_2 X_2 + \cdots)^2 - 2(X_0 + \delta_1 X_1 + \delta_2 X_2 + \cdots) + 1 = 0$$

$$(1-\varepsilon)(X_0^2 + \delta_1^2 X_1^2 + \delta_2^2 X_2^2 + 2\delta_1 X_0 X_1 + 2\delta_2 X_0 X_2 + 2\delta_1\delta_2 X_1 X_2 + \cdots)$$
$$- 2(X_0 + \delta_1 X_1 + \delta_2 X_2 + \cdots) + 1 = 0$$

$$(X_0^2 + \delta_1^2 X_1^2 + \delta_2^2 X_2^2 + 2\delta_1 X_0 X_1 + 2\delta_2 X_0 X_2 + 2\delta_1\delta_2 X_1 X_2 + \cdots)$$
$$-(\varepsilon X_0^2 + \varepsilon\delta_1^2 X_1^2 + \varepsilon\delta_2^2 X_2^2 + 2\varepsilon\delta_1 X_0 X_1 + 2\varepsilon\delta_2 X_0 X_2 + 2\varepsilon\delta_1\delta_2 X_1 X_2 + \cdots)$$
$$- 2(X_0 + \delta_1 X_1 + \delta_2 X_2 + \cdots) + 1 = 0$$

Leading Order: $X_0^2 - 2X_0 + 1 = 0$

$$(X_0 - 1)^2 = 0$$

$$\boxed{X_0 = 1, 1}$$

$$(\cancel{1} + \delta_1^2 X_1^2 + \delta_2^2 X_2^2 + \cancel{2\delta_1 X_1} + \cancel{2\delta_2 X_2} + 2\delta_1\delta_2 X_1 X_2 + \cdots)$$
$$-(\varepsilon + \varepsilon\delta_1^2 X_1^2 + \varepsilon\delta_2^2 X_2^2 + 2\varepsilon\delta_1 X_1 + 2\varepsilon\delta_2 X_2 + 2\varepsilon\delta_1\delta_2 X_1 X_2 + \cdots)$$
$$- 2(\cancel{1} + \cancel{\delta_1 X_1} + \cancel{\delta_2 X_2} + \cdots) \cancel{+1} = 0$$

$$\delta_1^2 X_1^2 + \delta_2^2 X_2^2 + 2\delta_1\delta_2 X_1 X_2 - \varepsilon - \varepsilon\delta_1^2 X_1^2 - \varepsilon\delta_2^2 X_2^2$$
$$- 2\varepsilon\delta_1 X_1 - 2\varepsilon\delta_2 X_2 - 2\varepsilon\delta_1\delta_2 X_1 X_2 + \cdots = 0 \qquad \otimes$$

Determine the first order correction (X_1 and δ_1).

Possible First Order Terms: $\underbrace{\delta_1^2 X_1^2}_{①} - \underbrace{\varepsilon}_{②} = 0$

All other terms in $\otimes$ are smaller in magnitude. In general, there may be several terms here. Some or all of them may be part of the first order correction. $\delta_1(\varepsilon)$ must be determined so that the appropriate terms balance.

① and ② may be comparable in magnitude, or one may dominate the other.

Suppose ① ≪ ② $\Longrightarrow$ $-\varepsilon = 0$ ✗ $(\varepsilon > 0)$

Suppose ② ≪ ① $\Longrightarrow$ $\delta_1^2 X_1^2 = 0 \Longrightarrow X_1 = 0$ ✗ $(X_1 \neq 0)$

$\Longrightarrow$ Neither term can dominate the other.

Therefore, ① and ② must balance. (i.e. be of the same magnitude)

$$\Longrightarrow \delta_1^2 X_1^2 - \varepsilon = 0$$

$$\Longrightarrow \boxed{\begin{aligned}\delta_1(\varepsilon) &= \varepsilon^{1/2}\\ X_1 &= \pm 1\end{aligned}}$$

This choice is not unique. e.g. $\delta_1 = 2\varepsilon^{1/2}$, $X_1 = \pm\frac{1}{2}$ works also. It is only required that $\delta_1 X_1 = \varepsilon^{1/2}$, but it is convenient ~~to~~ pick the δ_i's as powers of ε.

$$\cancel{\varepsilon} + \delta_2^2 X_2^2 \pm 2\varepsilon^{1/2}\delta_2 X_2 - \cancel{\varepsilon} - \varepsilon^2 - \varepsilon\delta_2^2 X_2^2 \mp 2\varepsilon^{3/2} - 2\varepsilon\delta_2 X_2 \mp 2\varepsilon^{3/2}\delta_2 X_2 + \cdots = 0$$

$$\delta_2^2 X_2^2 \pm 2\varepsilon^{1/2}\delta_2 X_2 - \varepsilon^2 - \varepsilon\delta_2^2 X_2^2 \mp 2\varepsilon^{3/2} - 2\varepsilon\delta_2 X_2 \mp 2\varepsilon^{3/2}\delta_2 X_2 + \cdots = 0$$

Recall: $\delta_2 \ll \delta_1 = \varepsilon^{1/2}$

Possible Second Order Terms: $\pm 2\varepsilon^{1/2}\delta_2 X_2 \mp 2\varepsilon^{3/2} = 0$

As above, both terms are nonzero, so they must balance.

$$\Rightarrow \quad 2\varepsilon^{1/2}\delta_2 X_2 = 2\varepsilon^{3/2}$$

$$\delta_2 X_2 = \varepsilon$$

Choose $\boxed{\begin{array}{c}\delta_2(\varepsilon) = \varepsilon \\ X_2 = 1\end{array}}$

Then, $x = x_0 + \delta_1 X_1 + \delta_2 X_2 + \cdots$

$$\boxed{x = 1 \pm \varepsilon^{1/2} + \varepsilon + \cdots}$$

This agrees with the Taylor expansion of the exact roots.

<u>Note:</u> Asymptotic expansions and gauge functions are not unique. The above roots can be approximated by other expansions just as well.

There will be examples when we discuss theory.

Singular Problems

Example: $\varepsilon x^2 - 2x + 1 = 0$; $0 < \varepsilon \ll 1$

The problem is singular since ε is the coefficient of the largest power of x. The leading order equation $(-2x+1=0)$ is linear and yields only one root.

(In the previous example, the leading order equation gave only one root also, but it was a double root corresponding to a quadratic equation)

This is analogous to singular ODEs, where ε is the coefficient of the highest order derivative.

e.g. $$\varepsilon x'' - 2x' + x = 0$$
$$x(0) = A$$
$$x(L) = B$$

The leading order equation $(-2x' + x = 0)$ is a 1^{st} order ODE and the solution cannot satisfy both boundary conditions.

$$\varepsilon x^2 - 2x + 1 = 0$$

Try the naive expansion

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x^2 + \cdots$$

Keep terms up to ε^2: $\varepsilon(x_0^2 + 2\varepsilon x_0 x_1 + \cdots) - 2(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + 1 = 0$

ε^0: $-2x_0 + 1 = 0$

$$x_0 = \frac{1}{2}$$

ε^1: $x_0^2 - 2x_1 = 0$

$$x_1 = \frac{x_0^2}{2} \Rightarrow x_1 = \frac{1}{8}$$

ε^2: $2x_0x_1 - 2x_2 = 0$

$$x_2 = x_0 x_1 \Rightarrow x_2 = \frac{1}{16}$$

This gives only one root,

$$\boxed{x^{(1)} = \frac{1}{2} + \frac{\varepsilon}{8} + \frac{\varepsilon^2}{16} + \cdots}\,.$$

If we try $x = x_0 + \delta_1(\varepsilon)x_1 + \delta_2(\varepsilon)x_2 + \cdots$, we'd find that $\delta_n = \varepsilon^n$, and get the same root as found above.

Problem: We assumed that $\varepsilon x^2 \to 0$ as $\varepsilon \to 0$, but it is possible that the root $x \to \infty$ as $\varepsilon \to 0$, in which case εx^2 does not necessarily approach 0 as $\varepsilon \to 0$.

Exact Roots: $x = \dfrac{1 \pm \sqrt{1-\varepsilon}}{\varepsilon} \sim \dfrac{1}{\varepsilon}\left[1 \pm (1 - \varepsilon/2 + \cdots)\right]$

$$x^{(1)} \sim \frac{1}{\varepsilon}\left[1 - 1 + \varepsilon/2 + \cdots\right] \sim \frac{1}{2} + \cdots$$

$$x^{(2)} \sim \frac{1}{\varepsilon}\left[1 + 1 - \varepsilon/2 + \cdots\right] \sim \frac{2}{\varepsilon} - \frac{1}{2} + \cdots$$

$\varepsilon x^{(2)^2} \to \infty$ as $\varepsilon \to 0$.

Try a more general expansion

$$x = \delta_0(\varepsilon)x_0 + \delta_1(\varepsilon)x_1 + \delta_2(\varepsilon)x_2 + \cdots$$

$\delta_{i+1} \ll \delta_i$

$\delta_i(\varepsilon) \to 0$ as $\varepsilon \to 0$

$x_i \neq 0$

$$\varepsilon x^2 - 2x + 1 = 0$$

$$\varepsilon(\delta_0^2 x_0^2 + \delta_1^2 x_1^2 + \delta_2^2 x_2^2 + 2\delta_0\delta_1 x_0 x_1 + 2\delta_0\delta_2 x_0 x_2 + 2\delta_1\delta_2 x_1 x_2 + \cdots)$$
$$-2(\delta_0 x_0 + \delta_1 x_1 + \delta_2 x_2 + \cdots) + 1 = 0 \quad \circledast$$

Now the leading order is not necessarily $O(1)$.

Possible Leading Order Terms: $\underset{①}{\varepsilon\delta_0^2 x_0^2} - \underset{②}{2\delta_0 x_0} + \underset{③}{1} = 0$

Balance: determine the dominant terms here.

— Suppose ① and ③ dominate

$\Rightarrow \varepsilon\delta_0^2 x_0^2 + 1 = 0$

$\delta_0 = \varepsilon^{-\frac{1}{2}}$, $x_0 = \pm i$

But then, the equation becomes

$x_0^2 - \frac{2x_0}{\varepsilon^{1/2}} + 1 = 0$

② dominates

CONTRADICTION

— Suppose ② and ③ dominate

$\Rightarrow -2\delta_0 x_0 + 1 = 0$

$\delta_0 = 1$, $x_0 = \frac{1}{2}$

This gives the naive expansion, and yields the same root as found above.

— Suppose ① and ② dominate

$\Rightarrow \varepsilon\delta_0^2 x_0^2 - 2\delta_0 x_0 = 0$

$\varepsilon\delta_0 x_0 - 2 = 0$

①, ② $= O(\frac{1}{\varepsilon})$ ✓

③ $= O(1)$

$\delta_0 = \frac{1}{\varepsilon}$ $x_0 = 2$

Terms ① and ② balance.

Then, equation $\circledast$ becomes

$$\varepsilon\left(\frac{4}{\varepsilon^2} + \delta_1^2 X_1^2 + \delta_2^2 X_2^2 + \frac{4}{\varepsilon}\delta_1 X_1 + \frac{4}{\varepsilon}\delta_2 X_2 + 2\delta_1\delta_2 X_1 X_2 + \cdots\right)$$
$$- 2\left(\frac{2}{\varepsilon} + \delta_1 X_1 + \delta_2 X_2 + \cdots\right) + 1 = 0$$

Possible Next Order Terms: $4\delta_1 X_1 - 2\delta_1 X_1 + 1 = 0$

$$2\delta_1 X_1 + 1 = 0$$
① ②

① and ② must balance $\Longrightarrow$ $\delta_1 = 1$ $X_1 = -\frac{1}{2}$

Then, equation $\circledast$ becomes

$$\varepsilon\left(\frac{1}{4} + \delta_2^2 X_2^2 \cancel{- \frac{2}{\varepsilon}} + \frac{4}{\varepsilon}\delta_2 X_2 - \delta_2 X_2 + \cdots\right) - 2\left(\cancel{-\frac{1}{2}} + \delta_2 X_2 + \cdots\right) \cancel{+1} = 0$$

Possible Next Order Terms: $\frac{\varepsilon}{4} + 4\delta_2 X_2 - 2\delta_2 X_2 = 0$

$$\frac{\varepsilon}{4} + 2\delta_2 X_2 = 0$$
$$\Longrightarrow \delta_2 = \varepsilon \quad X_2 = -\frac{1}{8}$$

$$\Longrightarrow \quad X = \delta_0 X_0 + \delta_1 X_1 + \delta_2 X_2 + \cdots$$

$$\boxed{X = \frac{2}{\varepsilon} - \frac{1}{2} - \varepsilon/8 + \cdots}$$

The first term is 'big'.

The result agrees with the Taylor expansion of the exact root $X = \frac{1 + \sqrt{1-\varepsilon}}{\varepsilon}$.

Rescaling

Recall: $\varepsilon x^2 - 2x + 1 = 0$

$$x^{(1)} = \underbrace{1/2}_{\delta_0 = 1} + \varepsilon/8 + \varepsilon^2/16 + \cdots$$

$$x^{(2)} = \underbrace{2/\varepsilon}_{\delta_0 = 1/\varepsilon} - \tfrac{1}{2} - \varepsilon/8 + \cdots$$

Rescaling is an alternate method for determining the leading order gauge function δ_0.

Idea: Make a change of variable to convert the singular equation into a regular equation.

Example: $\varepsilon x^2 - 2x + 1 = 0$

Let $x = \varepsilon^\alpha t$. $\Rightarrow$ $\varepsilon^{1+2\alpha} t^2 - 2\varepsilon^\alpha t + 1 = 0$

$$\underbrace{t^2}_{①} - \underbrace{2\varepsilon^{-(1+\alpha)} t}_{②} + \underbrace{\varepsilon^{-(1+2\alpha)}}_{③} = 0$$

Pick α so that term ① balances with one of the other terms.

① ~ ③ $\Rightarrow \alpha = -\frac{1}{2} \Rightarrow t^2 - 2\varepsilon^{-1/2} t + 1 = 0$

② dominates
contradiction

① ~ ② $\Rightarrow \alpha = -1 \Rightarrow t^2 - 2t + \varepsilon = 0$ (regular equation)

① and ② balance.

We have $t^2 - 2t + \varepsilon = 0$, where $X = t/\varepsilon$.

Try the naive expansion: $t = t_0 + \varepsilon t_1 + \varepsilon^2 t_2 + \cdots$

$$\left(t_0^2 + 2\varepsilon t_0 t_1 + \varepsilon^2 t_1^2 + 2\varepsilon^2 t_0 t_2 + \cdots\right) - 2\left(t_0 + \varepsilon t_1 + \varepsilon^2 t_2 + \cdots\right) + \varepsilon = 0$$

ε^0: $t_0^2 - 2t_0 = 0$

$$\boxed{t_0 = 2}$$

(The solution $t_0 = 0$ corresponds to the smaller root $X = \frac{1}{2} + \frac{\varepsilon}{8} + \cdots$, since $t = \varepsilon X = \underbrace{0}_{t_0 = 0} + \frac{\varepsilon}{2} + \varepsilon^2/8 + \cdots$.)

ε^1: $2t_0 t_1 - 2t_1 + 1 = 0$

$$2t_1 + 1 = 0 \Rightarrow \boxed{t_1 = -1/2}$$

ε^2: $t_1^2 + 2t_0 t_2 - 2t_2 = 0$

$$\frac{1}{4} + 2t_2 = 0 \Rightarrow \boxed{t_2 = -\frac{1}{8}}$$

$$\Rightarrow \boxed{t = 2 - \varepsilon/2 - \varepsilon^2/8 + \cdots}$$

Then,

$$\boxed{X = t/\varepsilon = 2/\varepsilon - 1/2 - \varepsilon/8 + \cdots}$$

Notes: 1. The naive expansion will not work in general. It may be necessary to expand t as

$$t = t_0 + \delta_1 t_1 + \delta_2 t_2 + \cdots.$$

2. Rescaling is also useful when x is small.

e.g. $X = \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \Rightarrow X = \varepsilon t$

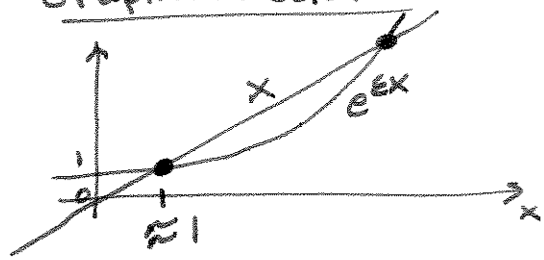
$\alpha = 1$

Transcendental Equations

(trig functions
exponentials
logarithms
etc.)

Example: $e^{\varepsilon x} = x, \ \varepsilon \ll 1$

Graphical Solution



$\varepsilon = 0 \Rightarrow x = 1$

Find a three-term asymptotic approximation of the smaller of the two solutions.

$$x \approx 1 \Rightarrow \varepsilon x \ll 1 \Rightarrow e^{\varepsilon x} = 1 + \varepsilon x + \frac{(\varepsilon x)^2}{2} + \cdots$$

$$e^{\varepsilon x} = x$$

$$1 + \varepsilon x + \frac{(\varepsilon x)^2}{2} + \cdots = x \quad \text{(polynomial)}$$

Naive expansion: $x = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$

Keep terms up to $O(\varepsilon^2)$: $\cancel{1} + \varepsilon(1 + \varepsilon x_1 + \cdots) + \frac{\varepsilon^2}{2}(1 + \cdots) + \cdots = \cancel{1} + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$

ε^1: $1 = x_1$

ε^2: $x_1 + \frac{1}{2} = x_2 \Rightarrow x_2 = \frac{3}{2}$

$$\Rightarrow \boxed{x \sim 1 + \varepsilon + \varepsilon^2 \frac{3}{2} + \cdots}$$

Note: Perturbation methods are not useful for finding the larger solution. Since εx is not necessarily small, $e^{\varepsilon x}$ is not accurately approximated by a few terms of the Taylor series. The general expansion $x = \delta_0 x_0 + \delta_1 x_1 + \cdots$ yields an implicit relation between ε and δ_0 at leading order, $e^{\varepsilon \delta_0 x_0} = \delta_0 x_0$, which is the same as the original equation. No progress has been made.

Theory / Definitions / Terminology / Notation

Asymptotic Relations

1. $f(\varepsilon) \ll g(\varepsilon)$ "f is much less than g as $\varepsilon \to 0$",

or equivalently,

$g(\varepsilon) \gg f(\varepsilon)$ "g is much greater than f as $\varepsilon \to 0$",

$$\text{if} \quad \frac{f(\varepsilon)}{g(\varepsilon)} \to 0 \text{ as } \varepsilon \to 0.$$

e.g. $f(\varepsilon) = \varepsilon^2$, $g(\varepsilon) = \sin\varepsilon$ $\quad \dfrac{\varepsilon^2}{\sin\varepsilon} \to 0$ as $\varepsilon \to 0 \Rightarrow \varepsilon^2 \ll \sin\varepsilon$

---

2. $f(\varepsilon) \sim g(\varepsilon)$ "f is asymptotic to g as $\varepsilon \to 0$"

$$\text{if} \quad \frac{f(\varepsilon)}{g(\varepsilon)} \to 1 \text{ as } \varepsilon \to 0.$$

e.g. $f(\varepsilon) = \sin\varepsilon$, $g(\varepsilon) = \ln(1+\varepsilon)$ $\quad \dfrac{\sin\varepsilon}{\ln(1+\varepsilon)} \to 1$ as $\varepsilon \to 0$

$\Rightarrow \sin\varepsilon \sim \ln(1+\varepsilon)$

If $X = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \cdots$, then

$$X \sim X_0$$
$$X \sim X_0 + \varepsilon X_1$$
$$\vdots$$
$$X \sim \sum_{n=0}^{N} \varepsilon^n X_n \quad \text{for each } N = 0, 1, 2, \ldots$$

Order Symbols

1. $f(\varepsilon) = O(g(\varepsilon))$ "f is big Oh of g as $\varepsilon \to 0$"

if $\dfrac{f(\varepsilon)}{g(\varepsilon)}$ is bounded as $\varepsilon \to 0$.

(though $\lim\limits_{\varepsilon \to 0} \dfrac{f(\varepsilon)}{g(\varepsilon)}$ may not exist)

e.g. $f(\varepsilon) = \cos\frac{1}{\varepsilon}$ $\quad \left|\dfrac{f(\varepsilon)}{g(\varepsilon)}\right| = \left|\cos\frac{1}{\varepsilon}\right| \le 1$

$g(\varepsilon) = 1$ $\quad \Rightarrow \cos\frac{1}{\varepsilon} = O(1)$

Strict Definition: $f(\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to 0$ if there exists constants K and ε_0 such that

$$|f(\varepsilon)| \le K|g(\varepsilon)| \text{ for } 0 < \varepsilon < \varepsilon_0.$$

---

2. $f(\varepsilon) = o(g(\varepsilon))$ "f is little oh of g as $\varepsilon \to 0$"

if $\dfrac{f(\varepsilon)}{g(\varepsilon)} \to 0$ as $\varepsilon \to 0$.

e.g. $f(\varepsilon) = \varepsilon\cos\frac{1}{\varepsilon}$ $\quad \dfrac{f(\varepsilon)}{g(\varepsilon)} \to 0$ as $\varepsilon \to 0$

$g(\varepsilon) = 1$ $\quad \Rightarrow \varepsilon\cos\frac{1}{\varepsilon} = o(1)$

Strict Definition: $f(\varepsilon) = o(g(\varepsilon))$ as $\varepsilon \to 0$ if for any $k > 0$, there exists $\varepsilon_0(k) > 0$ such that

$$|f(\varepsilon)| \le k|g(\varepsilon)| \text{ for } 0 < \varepsilon < \varepsilon_0(k).$$

Note:

$f(\varepsilon) = o(g(\varepsilon)) \Rightarrow$ f is much less than g as $\varepsilon \to 0$.

$f(\varepsilon) = O(g(\varepsilon)) \Longrightarrow$ f is either much less than g, or 'comparable' to g as $\varepsilon \to 0$.

i) $f(\varepsilon) = o(g(\varepsilon))$ and $f(\varepsilon) \ll g(\varepsilon)$ are equivalent statements.

ii) $f(\varepsilon) = o(g(\varepsilon)) \Longrightarrow f(\varepsilon) = O(g(\varepsilon))$

$f(\varepsilon) = O(g(\varepsilon)) \not\Longrightarrow f(\varepsilon) = o(g(\varepsilon))$

(little oh implies big Oh, but big Oh does not imply little oh)

Examples:

1. $f(\varepsilon) = O(f(\varepsilon))$ $\quad \frac{f(\varepsilon)}{f(\varepsilon)} \to 1$ as $\varepsilon \to 0$

   $f(\varepsilon) \neq o(f(\varepsilon))$

2. $\varepsilon^n = O(\varepsilon^m)$ for $n \geq m$

   $\varepsilon^n = o(\varepsilon^m)$ for $n > m$

3. $\sin\varepsilon = O(\tan\varepsilon) = O(\ln(1+\varepsilon)) = O(\varepsilon)$

4. $\cos\varepsilon \sim 1 - \varepsilon^2/2! + \varepsilon^4/4!$
$$
\begin{aligned}
&= O(1) \\
&= 1 + O(\varepsilon^2) \\
&\neq 1 + o(\varepsilon^2) \\
&= 1 + O(\varepsilon) \\
&= 1 + o(\varepsilon) \\
&= 1 - \varepsilon^2/2! + O(\varepsilon^4) \\
&= O(\tfrac{1}{\varepsilon})
\end{aligned}
$$

$\cos\varepsilon - 1 = O(\varepsilon^2)$

Formally, it is correct to write

$$\sin\varepsilon = \varepsilon - \varepsilon^3/3! + \cdots = \varepsilon + O(\varepsilon)$$
$$= \varepsilon + o(\varepsilon)$$
$$= \varepsilon + O(1)$$
$$\vdots$$

However, it is prefferable to use the sharpest and most informative estimate, $\sin\varepsilon = \varepsilon + O(\varepsilon^3)$.

↑ Indicates the exact order of the next term.

$f(\varepsilon) = O(g(\varepsilon))$ is the 'best' estimate of the behavior of $f(\varepsilon)$ as $\varepsilon \to 0$ if it is also true that $g(\varepsilon) = O(f(\varepsilon))$, or equivalently, if $f(\varepsilon) \neq o(g(\varepsilon))$. That is, if $f(\varepsilon)$ and $g(\varepsilon)$ are of the same order.

Properties

i) constants: The order symbols O and o are insensitive to multiplicative constants.

e.g. $K\varepsilon = O(\varepsilon)$ for any constant K

$-100^{100}\sin\varepsilon = O(\varepsilon)$

---

ii) addition/subtraction: If $f(\varepsilon) = O(g(\varepsilon))$, then

$$O(f(\varepsilon)) \pm O(g(\varepsilon)) = O(g(\varepsilon)).$$

e.g. $f(\varepsilon) = \varepsilon^3$, $g(\varepsilon) = \varepsilon^2$: $O(\varepsilon^2) \pm O(\varepsilon^3) = O(\varepsilon^2)$

$$O(\varepsilon^m) \pm O(\varepsilon^n) = O(\varepsilon^m) \text{ for } m \le n.$$

---

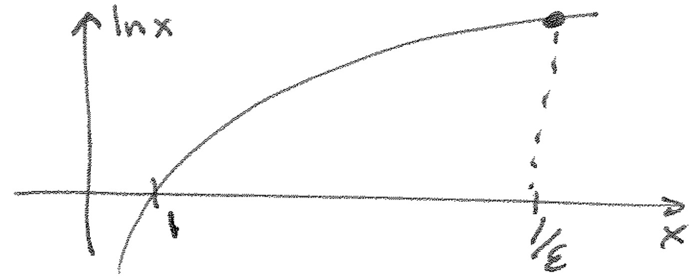
iii) multiplication: $O(f(\varepsilon)) \cdot O(g(\varepsilon)) = O(f(\varepsilon)g(\varepsilon))$

e.g. $O(\varepsilon^2) \cdot O(\varepsilon^3) = O(\varepsilon^5)$

Natural Logarithms

Gauge functions may involve powers of $\ln\frac{1}{\varepsilon}$ $(= -\ln\varepsilon)$.

Consider $\ln\frac{1}{\varepsilon}$ for $\varepsilon \ll 1$.



Facts:

① $\left(\ln\frac{1}{\varepsilon}\right)^{\gamma} \gg 1$ for $\gamma > 0$.

Verify: $\dfrac{1}{\left(\ln\frac{1}{\varepsilon}\right)^{\gamma}} \to 0$ as $\varepsilon \to 0$. ✓

② $\varepsilon^{\alpha}\left(\ln\frac{1}{\varepsilon}\right)^{\beta} \ll 1$ for $\alpha > 0$ and all β.

Verify: It can be shown that $\dfrac{\varepsilon^{\alpha}\left(\ln\frac{1}{\varepsilon}\right)^{\beta}}{1} \to 0$ as $\varepsilon \to 0$.

Notes:

1. ② can be written as $\left(\ln\frac{1}{\varepsilon}\right)^{\beta} \ll 1/\varepsilon^{\alpha}$.

   Though $\left(\ln\frac{1}{\varepsilon}\right)^{\beta}$ may be much greater than one, it is much less than any negative power of ε.

2. ① and ② $\Longrightarrow$ $\boxed{\varepsilon^{\alpha}\left(\ln\frac{1}{\varepsilon}\right)^{\beta} \ll 1 \ll \left(\ln\frac{1}{\varepsilon}\right)^{\gamma}, \quad \alpha, \gamma > 0, \text{ all } \beta}$

   Though any positive power of $\left(\ln\frac{1}{\varepsilon}\right)$ is much greater than one, if multiplied by any positive power of ε, the product is much less than one.

Suppose we wish to arrange several terms of the form $\varepsilon^{\alpha}(\ln\frac{1}{\varepsilon})^{\beta}$ $(\alpha, \beta \in \mathbb{R})$ from largest to smallest as $\varepsilon \to 0$. The ordering is dominated by the powers α of ε, while the powers β of $\ln\frac{1}{\varepsilon}$ have only a secondary effect.

Example: Order the following quantities from largest to smallest as $\varepsilon \to 0$.

$$\varepsilon\ln\tfrac{1}{\varepsilon},\ \varepsilon^2\ln\tfrac{1}{\varepsilon},\ 1,\ \ln\tfrac{1}{\varepsilon},\ \frac{(\ln\frac{1}{\varepsilon})^{100}}{\varepsilon^{1/2}},\ (\ln\tfrac{1}{\varepsilon})^2,\ \varepsilon,\ \varepsilon^{-1/2},\ \varepsilon(\ln\tfrac{1}{\varepsilon})^{-5}$$

First, group terms which have the same value of α. These groups are ordered according to increasing α.
(smallest $\alpha \Rightarrow$ largest terms)

$\alpha = -\frac{1}{2}$: $\varepsilon^{-1/2}(\ln\frac{1}{\varepsilon})^{100}$, $\varepsilon^{-1/2}$ ← largest terms

$\alpha = 0$: 1, $\ln\frac{1}{\varepsilon}$, $(\ln\frac{1}{\varepsilon})^2$

$\alpha = 1$: $\varepsilon\ln\frac{1}{\varepsilon}$, ε, $\varepsilon(\ln\frac{1}{\varepsilon})^{-5}$

$\alpha = 2$: $\varepsilon^2\ln\frac{1}{\varepsilon}$ ← smallest term

Then, the terms are ordered within groups according to decreasing β.

$$\boxed{\underbrace{\overset{\beta=100}{\varepsilon^{-\frac{1}{2}}(\ln\tfrac{1}{\varepsilon})^{100}} \gg \overset{\beta=0}{\varepsilon^{-1/2}}}_{\alpha=-\frac{1}{2}} \gg \underbrace{\overset{\beta=2}{(\ln\tfrac{1}{\varepsilon})^2} \gg \overset{\beta=1}{\ln\tfrac{1}{\varepsilon}} \gg \overset{\beta=0}{1}}_{\alpha=0} \gg \underbrace{\overset{\beta=1}{\varepsilon\ln\tfrac{1}{\varepsilon}} \gg \overset{\beta=0}{\varepsilon} \gg \overset{\beta=-5}{\varepsilon(\ln\tfrac{1}{\varepsilon})^{-5}}}_{\alpha=1} \gg \underbrace{\overset{\beta=1}{\varepsilon^2\ln\tfrac{1}{\varepsilon}}}_{\alpha=2}}$$

Gauge Functions

Definition: A sequence of gauge functions is a sequence $\{g_n(\varepsilon)\}$ in which $g_n(\varepsilon) = o(g_{n-1}(\varepsilon))$ for each n.

Such a sequence is called an asymptotic sequence.

Examples:

1. Naive Expansion: $\{g_n(\varepsilon)\} = \{\varepsilon^n\}_{n=0,1,\ldots}$
2. $\{g_n(\varepsilon)\} = \{\sin \varepsilon^n\}_{n=0,1,\ldots} = \{\sin 1, \sin\varepsilon, \sin\varepsilon^2, \ldots\}$
3. $\{g_n(\varepsilon)\} = \{\varepsilon^{-\frac{1}{2}}(\ln\frac{1}{\varepsilon})^{100}, \varepsilon^{-\frac{1}{2}}, (\ln\frac{1}{\varepsilon})^2, \ln\frac{1}{\varepsilon}, 1, \ldots\}$
   (from previous example)

We expanded roots of polynomials as

$$x = \delta_0(\varepsilon)x_0 + \delta_1(\varepsilon)x_1 + \delta_2(\varepsilon)x_2 + \cdots.$$

$\{\delta_n(\varepsilon)\}$ is the gauge sequence and x_n are scalar coefficients.

More generally, we may expand functions $f(x;\varepsilon)$ which involve a small parameter ε in terms of gauge functions $\{g_n(\varepsilon)\}_{n=1,2,\ldots}$.

$$f(x;\varepsilon) = \underbrace{g_1(\varepsilon)f_1(x) + g_2(\varepsilon)f_2(x) + \cdots + g_N(\varepsilon)f_N(x)}_{F_N(x;\varepsilon)} + O(g_{N+1}(\varepsilon))$$

(OR $o(g_N(\varepsilon))$)

$$\boxed{f(x;\varepsilon) = F_N(x;\varepsilon) + O(g_{N+1}(\varepsilon))}$$

Now the coefficients $f_n(x)$ of the gauge functions are functions, rather than scalars.

Notes:

1. $F_N(x;\varepsilon)$ is called an <u>N-term asymptotic expansion</u> of $f(x;\varepsilon)$ as $\varepsilon \to 0$. We write $f(x;\varepsilon) \sim F_N(x;\varepsilon)$ and say that $f(x;\varepsilon)$ 'is asymptotic to' $F_N(x;\varepsilon)$.

2. A function (or scalar) cannot be expanded in terms of any gauge sequence. The gauge functions must be chosen, or determined, appropriately.

3. When approximating a function with an asymptotic expansion, the choice of gauge functions is <u>not</u> unique. However, once a gauge sequence is specified, the coefficients are uniquely determined.

Example:

$$\sin(2\varepsilon) \sim 2\varepsilon - \frac{4}{3}\varepsilon^3 + \frac{4}{15}\varepsilon^5$$

This is called the '<u>Natural Expansion</u>'. The gauge functions are powers of ε and it agrees with the Taylor series.

$$\sin(2\varepsilon) \sim 2\tan\varepsilon - 2\tan^3\varepsilon - 2\tan^5\varepsilon$$

$$\sin(2\varepsilon) \sim 2\ln(1+\varepsilon) + \ln(1+\varepsilon^2) - 2\ln(1+\varepsilon^3)$$

$$\sin(2\varepsilon) \sim 6\left(\frac{\varepsilon}{3+2\varepsilon^2}\right) - \frac{378}{3}\left(\frac{\varepsilon}{3+2\varepsilon^2}\right)^5$$

The right-hand-sides are all asymptotic to each other. If expanded as Taylor Series in powers of ε, the four expansions will agree up to $O(\varepsilon^5)$.

4. An asymptotic expansion does not uniquely determine a function since many functions may be represented by the same asymptotic expansion.

e.g. $f_1(\varepsilon) = \dfrac{1}{1-\varepsilon} = 1 + \varepsilon + \varepsilon^2 + \cdots$

$f_2(\varepsilon) = \dfrac{1}{1-\varepsilon} + e^{-1/\varepsilon} = 1 + \varepsilon + \varepsilon^2 + \cdots$

The addition of the term $e^{-1/\varepsilon}$ does not affect the expansion, why?

In terms of the gauge sequence $\{\varepsilon^n\}$, the unique expansion of $e^{-1/\varepsilon}$ is 0. Since $e^{-1/\varepsilon} \ll \varepsilon^n$ for all n, the coefficient of each gauge function is zero.

Transcendentally Small Terms (TST)

Given a sequence of gauge functions $\{g_n(\varepsilon)\}$, a transcendentally small term is a term which is much less than each gauge function in the sequence in the limit $\varepsilon \to 0$, and therefore, its unique expansion in terms of $\{g_n(\varepsilon)\}$ is 0.

Examples:

1. ε is transcendentally small with respect to the gauge sequence $\{(\ln \frac{1}{\varepsilon})^{-n}\}$ since $\varepsilon \ll (\ln\frac{1}{\varepsilon})^{-n}$ for each n.
2. More commonly, $e^{-1/\varepsilon}$ is transcendentally small with respect to the gauge sequence $\{\varepsilon^n\}$ since $e^{-1/\varepsilon} \ll \varepsilon^n$ for each n.

Asymptotic Expansions of Functions

Recall: A function $f(x;\varepsilon)$ may be asymptotically expanded as

$$f(x;\varepsilon) = \underbrace{g_1(\varepsilon) f_1(x) + \cdots + g_N(\varepsilon) f_N(x)}_{F_N(x;\varepsilon)} + O(g_{N+1}(\varepsilon))$$

Our definitions of 'big Oh' and 'little oh' must be generalized for functions.

Definition (big Oh): For fixed x, $f(x;\varepsilon) = O(g(\varepsilon))$ as $\varepsilon \to 0$ if there exists numbers $K(x)$ and $\varepsilon_0(x)$ such that

$$|f(x;\varepsilon)| \le K(x)\,|g(\varepsilon)| \quad \text{for} \quad 0 < \varepsilon < \varepsilon_0(x).$$

A similar modification must be made for 'little oh'.

In general, K and ε_0 depend on x.

i.e. If a single value of ε_0 and K work for all $x \in I$.

Definition: If K and ε_0 can be chosen independent of x for all x in some interval I, then the ordering is said to be uniform in I.

The idea of uniform ordering is similar to that of uniform continuity and uniform convergence.

Example: $f(x;\varepsilon) = \frac{\varepsilon}{x}$, $0 < x < 2$; $0 < \varepsilon \ll 1$

$$|f(x;\varepsilon)| = \frac{\varepsilon}{x} \leq \underset{\uparrow\, K(x)}{\frac{1}{x}} \cdot \underset{\uparrow\, g(\varepsilon)}{\varepsilon} \quad \text{for all } \varepsilon$$

ε_0 is arbitrary

$$\Rightarrow \boxed{f(x;\varepsilon) = O(\varepsilon) \text{ for each } x \in (0,2)}$$

However, K cannot be chosen independent of x.

That is, no single value of K will work for all $x \in (0,2)$.

The smaller is x, the larger must be K.

$$\Rightarrow \boxed{f(x;\varepsilon) = O(\varepsilon), \text{ but not uniformly, on } (0,2)}$$

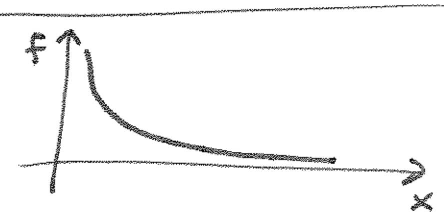
Now consider the interval $(1,2)$ $\qquad \frac{1}{2} < \frac{1}{x} < 1$ for $x \in (1,2)$

$$|f(x;\varepsilon)| = \frac{\varepsilon}{x} \leq \underset{\uparrow\, K=1}{1} \cdot \varepsilon \quad \text{for all } \varepsilon \text{ and all } x \in (1,2).$$

$$\Rightarrow \boxed{f(x;\varepsilon) = O(\varepsilon) \text{ uniformly on } (1,2)}$$

Analogy: Uniform Continuity

$f(x;\varepsilon) = \frac{\varepsilon}{x}$



- $f(x;\varepsilon)$ is continuous, but not uniformly, on $(0,2)$
- $f(x;\varepsilon)$ is uniformly continuous on $(1,2)$

Example: $f(x;\varepsilon) = e^{\varepsilon x}$

- $f(x;\varepsilon) = O(1)$ uniformly on any bounded interval (a,b).
- $f(x;\varepsilon) = O(1)$, but not uniformly, on (a,∞).

Uniformly Valid Expansions

Definition: The asymptotic expansion $f(x;\varepsilon) \sim g_1(\varepsilon) f_1(x) + \cdots + g_N(\varepsilon) f_N(x)$ is valid for fixed x if each term is much less than the preceding terms, $g_n(\varepsilon) f_n(\varepsilon) = o\left(g_{n-1}(\varepsilon) f_{n-1}(x)\right)$ for each $n = 2, \ldots, N$.

Definition: If the ordering in the above definition is uniform on an interval I, then the expansion is uniformly valid on I.

e.g. The expansion $1 + \frac{\varepsilon}{x} + \frac{\varepsilon^2}{x^2} + \cdots$ is uniformly valid on the interval $(1,2)$, but not on the interval $(0,2)$ since $\frac{\varepsilon}{x}$ is not $o(1)$ if x is sufficiently small. Though we consider the limit $\varepsilon \to 0$, there will always be values of x in the interval $(0,2)$ which are less than ε.

Differentiation of Asymptotic Expansions

It is often necessary to differentiate asymptotic expansions, though differentiation is not always a permissible operation.

e.g. $f(x;\varepsilon) \sim x + \varepsilon\sqrt{x} + \varepsilon^2 x$

The ordering is uniformly valid on any bounded interval (a,b)

However, $f'(x;\varepsilon) \not\sim 1 + \frac{\varepsilon}{2\sqrt{x}} + \varepsilon^2$ is not uniformly valid on any interval which contains the point 0, or has 0 as an endpoint.

Suppose we have $f(x;\varepsilon) \sim g_1(\varepsilon) f_1(x) + \cdots + g_N(\varepsilon) f_N(x)$. We may proceed under the assumption that differentiation is permissible and write

$$f'(x;\varepsilon) \sim g_1(\varepsilon) f_1'(x) + \cdots + g_N(\varepsilon) f_N'(x).$$

If the assumption is not valid, it will lead to nonuniformities in the expansion. After the f_j's are determined the validity of the expansion must be checked. If the solution expansion is not uniformly valid over the interval of interest, the problem is singular and requires special treatment.

Region of Nonuniformity: Region of the x-axis where an asymptotic expansion is not valid.

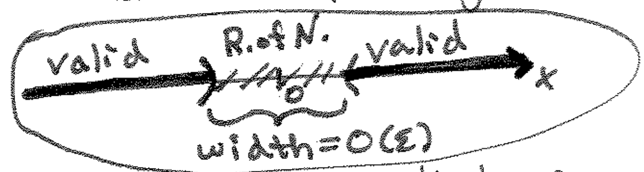
When approximating a solution of a differential equation, it is desirable that the solution expansion be uniformly valid over the entire domain of the problem. It is important to recognize if an asymptotic expansion is not uniform over the interval of interest, and to identify where on the x-axis the nonuniformities occur.

Examples: ① $\dfrac{1}{1-\varepsilon/x} = \sum_{n=0}^{\infty}\left(\dfrac{\varepsilon}{x}\right)^n = 1 + \dfrac{\varepsilon}{x} + \dfrac{\varepsilon^2}{x^2} + \cdots, \quad 0<\varepsilon \ll 1.$

The expansion is valid provided each term is much less than the preceding terms,

i.e. $\left|\dfrac{\varepsilon^{n+1}}{x^{n+1}}\right| \ll \left|\dfrac{\varepsilon^n}{x^n}\right|$

$\Rightarrow$ $|x| \gg \varepsilon$ (Region of Uniformity)



The expansion is valid for $|x| \gg \varepsilon$, and not valid if $|x|$ is comparable to or less than ε. $\Longrightarrow$ Region of Nonuniformity : $\boxed{x = O(\varepsilon)}$

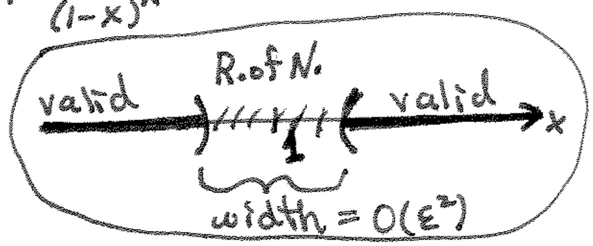
---

② $f(x;\varepsilon) = 1 + \dfrac{\varepsilon^2}{1-x} + \dfrac{\varepsilon^4}{(1-x)^2} + \cdots + \dfrac{\varepsilon^{2n}}{(1-x)^n} + \cdots$

The expansion is valid when

$$\left|\frac{\varepsilon^{2(n+1)}}{(1-x)^{n+1}}\right| \ll \left|\frac{\varepsilon^{2n}}{(1-x)^n}\right|$$

$\Rightarrow$ $|1-x| \gg \varepsilon^2$



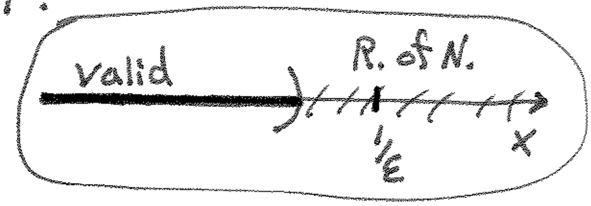
Region of Nonuniformity : $\boxed{|1-x| = O(\varepsilon^2), \text{ or equivalently, } x = 1 + O(\varepsilon^2)}$

---

③ $f(t;\varepsilon) \sim 1 + 2\varepsilon^2 t + \varepsilon^3 t^2$

Valid when $1 \gg |2\varepsilon^2 t|$ and $|2\varepsilon^2 t| \gg |\varepsilon^3 t^3|$.

$\Rightarrow$ $t \ll \dfrac{1}{\varepsilon^2}$ $\qquad\Rightarrow$ $t \ll \dfrac{1}{\varepsilon}$



The expansion is valid over the intersection of these two regions, $t \ll \dfrac{1}{\varepsilon}$.

The expansion is not valid when t is comparable to or greater than $\frac{1}{\varepsilon}$, or equivalently when $1/t = O(\varepsilon)$.

Not valid when $t \gtrsim \frac{1}{\varepsilon}$ $\Rightarrow \frac{1}{t} \lesssim \varepsilon$

Region of Nonuniformity : $\boxed{\dfrac{1}{t} = O(\varepsilon)}$

Convergent Series vs. Asymptotic Series

Consider the series $\sum_{n=0}^{\infty} f_n(\varepsilon)$.

The series is <u>convergent</u> if

$$\lim_{n\to\infty}\left|\frac{f_{n+1}(\varepsilon)}{f_n(\varepsilon)}\right| = K < 1 \quad \left(\begin{array}{c}\text{ratio}\\ \text{test}\end{array}\right)$$

The series is <u>asymptotic</u> if

$$\lim_{\varepsilon\to 0}\left|\frac{f_{n+1}(\varepsilon)}{f_n(\varepsilon)}\right| = 0 \text{ for each } n = 0, 1, \ldots$$

<u>Convergent</u>: Fix ε and let $n\to\infty$

<u>Asymptotic</u>: Fix n and let $\varepsilon \to 0$

Convergent Series are not necessarily Asymptotic, and Asymptotic Series are not necessalily convergent

---

<u>Example</u>: <u>Convergent / Not Asymptotic</u>

Consider $\sum_{n=1}^{\infty}\frac{\varepsilon}{n!} = \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{6} + \cdots$

$$\left|\frac{f_{n+1}(\varepsilon)}{f_n(\varepsilon)}\right| = \left|\frac{\varepsilon/(n+1)!}{\varepsilon/n!}\right| = \frac{1}{n+1}$$

$\nearrow$ 0 as $n\to\infty$ <u>Convergent</u>

$\searrow$ $\frac{1}{n+1}$ as $\varepsilon\to 0$ <u>NOT Asymptotic</u>

<u>Example</u>: <u>Convergent/Asymptotic</u>

Consider $\sum_{n=1}^{\infty}\frac{\varepsilon^n}{n!} = \varepsilon + \varepsilon^2/2 + \varepsilon^3/6 + \cdots$

$$\left|\frac{f_{n+1}(\varepsilon)}{f_n(\varepsilon)}\right| = \left|\frac{\varepsilon^{n+1}/(n+1)!}{\varepsilon^n/n!}\right| = \frac{\varepsilon}{n+1}$$

$\nearrow$ 0 as $n\to\infty$ <u>Convergent</u>

$\searrow$ 0 as $\varepsilon\to 0$ <u>Asymptotic</u>

Example: Not Convergent / Asymptotic

Consider the integral $f(\varepsilon) = \int_0^\infty \frac{e^{-t}}{1+\varepsilon t} dt$; $0 < \varepsilon \ll 1$.

Integrate by Parts

$$\left.\begin{aligned} &\int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du \\ &u = \frac{1}{1+\varepsilon t} \quad v = -e^{-t} \\ &du = \frac{-\varepsilon\,dt}{(1+\varepsilon t)^2} \quad dv = e^{-t}dt \end{aligned}\right\} \Rightarrow f(\varepsilon) = 1 - \varepsilon\int_0^\infty \frac{e^{-t}}{(1+\varepsilon t)^2}dt$$

Integrate by Parts again $\Rightarrow f(\varepsilon) = 1 - \varepsilon + 2\varepsilon^2 \int_0^\infty \frac{e^{-t}}{(1+\varepsilon t)^3}dt$

⋮

Repeat Indefinitely

⋮

$$\Rightarrow f(\varepsilon) = 1 - \varepsilon + 2!\varepsilon^2 - 3!\varepsilon^3 + \cdots$$

$$\boxed{f(\varepsilon) = \int_0^\infty \frac{e^{-t}}{1+\varepsilon t}dt = \sum_{n=0}^{\infty} (-1)^n n!\,\varepsilon^n}$$

$$\left|\frac{f_{n+1}(\varepsilon)}{f_n(\varepsilon)}\right| = \left|\frac{(n+1)!\,\varepsilon^{n+1}}{n!\,\varepsilon^n}\right| = (n+1)\varepsilon \begin{cases} \to \infty \text{ as } n \to \infty & \text{Diverges} \\ \to 0 \text{ as } \varepsilon \to 0 & \text{Asymptotic} \end{cases}$$

Though the series diverges, a few terms give a good approximation of the integral. For example, consider three terms with $\varepsilon = 0.1$, $f(\varepsilon) \sim 1 - \varepsilon + 2\varepsilon^2 - 6\varepsilon^3 = .914$, while the exact value of the integral is $\int_0^\infty \frac{e^{-t}}{1+\varepsilon t}dt = .9156$.

Note: The expansion is not valid if too many terms are included and accuracy is lost.

Many asymptotic expansions do not converge, but it does not pose a problem. Typically, 2 or 3 terms provide a reasonable and sufficiently accurate approximation, whether the corresponding infinite series converges or not. Consequently, convergence is not a concern.

Regular ODEs

Regular ODEs are those which are not singular (to be discussed later). The naive expansion is usually (but not always) sufficient for such problems.

Example: Consider the Initial Value Problem

(nonlinear) $2yy' - y^2 + \varepsilon\sqrt{y} = 0, \quad t > 0 \quad ; \quad y(0) = 1$

Find a two-term asymptotic approximation of the solution.

Naive Expansion: $y(t;\varepsilon) \sim y_0(t) + \varepsilon y_1(t) + O(\varepsilon^2)$

$$2(y_0+\varepsilon y_1)(y_0'+\varepsilon y_1') - (y_0+\varepsilon y_1)^2 + \varepsilon\sqrt{y_0} + O(\varepsilon^2) = 0$$

Keep terms up to $O(\varepsilon)$:

$$2(y_0y_0' + \varepsilon y_0y_1' + \varepsilon y_1y_0') - (y_0^2 + 2\varepsilon y_0y_1) + \varepsilon\sqrt{y_0} + O(\varepsilon^2) = 0$$

$$(2y_0y_0' - y_0^2) + \varepsilon\left(2y_0y_1' + 2y_1y_0' - 2y_0y_1 + \sqrt{y_0}\right) + O(\varepsilon^2) = 0$$

$O(1)$: $2y_0y_0' - y_0^2 = 0$

$$y_0(2y_0' - y_0) = 0$$

$O(\varepsilon)$: $2y_0y_1' + 2y_1y_0' - 2y_0y_1 + \sqrt{y_0} = 0$

$$y_1' + \frac{y_0' - y_0}{y_0}\, y_1 = -\frac{1}{2}\, y_0^{-1/2}$$

First-order linear ODE for y_1. $\Rightarrow$ Use Integrating Factor Method.

Initial Condition: $y(0) \sim y_0(0) + \varepsilon y_1(0) \sim 1$

$O(1)$: $y_0(0) = 1$

$O(\varepsilon)$: $y_1(0) = 0$

$\underline{O(1)}$: $\underline{y_0(2y_0' - y_0) = 0 \,;\; y_0(0) = 1}$

$y_0 = 0$ — Can't satisfy the initial condition

OR $y_0' = \frac{y_0}{2}$

$\Rightarrow y_0(t) = c\,e^{t/2}$ $\Rightarrow$ $y_0(t) = e^{t/2}$

$y_0(0) = 1 \Rightarrow c = 1$

$\underline{O(\varepsilon)}$: $\underline{y_1' - \frac{y_0' - y_0}{y_0} y_1 = -\frac{1}{2} y_0^{-1/2} \,;\; y_1(0) = 0}$

$$y_1' - \frac{1}{2} y_1 = -\frac{1}{2} e^{-t/4}$$

Integrating Factor: $\mu(t) = e^{-t/2}$

$$\Rightarrow e^{-t/2} y_1 = \int -\frac{1}{2} e^{-t/4} \cdot e^{-t/2} dt = -\frac{1}{2} \int e^{-3t/4} dt = \frac{2}{3} e^{-3t/4} + C$$

$$y_1 = Ce^{t/2} + \frac{2}{3} e^{-t/4} \Rightarrow y_1(t) = -\frac{2}{3}\left(e^{t/2} - e^{-t/4}\right)$$

$$y_1(0) = 0 \Rightarrow C = -\frac{2}{3}$$

Then,

$$\boxed{y(t) \sim e^{t/2} - \frac{2}{3}\varepsilon\left(e^{t/2} - e^{-t/4}\right)}$$

Though both terms of the expansion $\to \infty$ as $t \to \infty$, they increase at the same rate and the expansion is well-ordered for all $t \geq 0$. Thus, the expansion is uniformly valid over the interval of interest $(0 \leq t < \infty)$. Notice that the expansion is not uniformly valid on $(-\infty, \infty)$ since the second term $\to \infty$, while the first term $\to 0$ as $t \to -\infty$.

Though the ODE $2yy' - y^2 + \varepsilon\sqrt{y} = 0$ is nonlinear, it is separable and can be solved exactly.

Taylor expansion for $\varepsilon \ll 1$.

$$y(t) = \left[(1-\varepsilon)e^{3/4\,t} + \varepsilon\right]^{2/3} \sim e^{t/2} - \frac{2}{3}\varepsilon\left(e^{t/2} - e^{-t/4}\right)$$

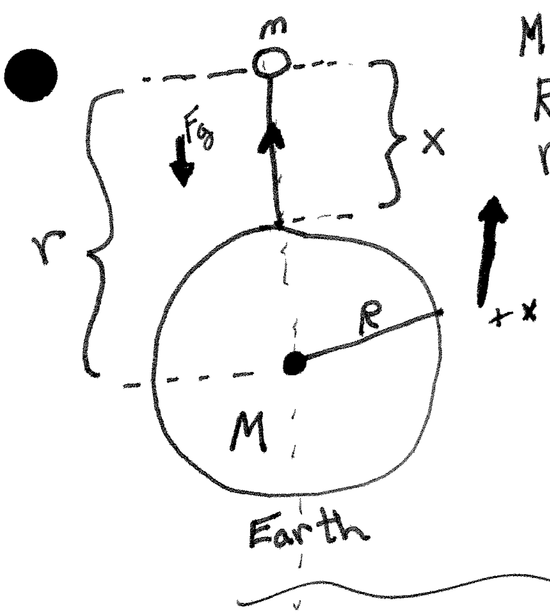
The Taylor expansion of the exact solution agrees with our asymptotic approximation of the solution.

Physical Applications

- Small parameters don't just magically appear in mathematical models of physical phenomena. Instead reasonable assumptions must be made and variables must be scaled appropriately to arrive at small parameters. The following two examples demonstrate how a small parameter may be introduced into a mathematical model.

Example: Fire a projectile from ground level straight upwards with a moderate initial velocity V_0. Describe the motion of the projectile while allowing earth's gravitational field to be altitude dependent.

Model:



x = position of the projectile (distance above earth's surface)

a = acceleration of the projectile

m = mass of the projectile

M = mass of the earth

R = radius of the earth ($R \approx 4000$ miles)

$r = R + x$ = distance between the projectile's center of mass and that of the earth.

F_g = gravitational force exerted by the earth on the projectile

G = gravitational constant

g_0 = gravitation acceleration at the surface of the earth ($x=0$). ($g_0 \approx 32 \frac{ft}{s^2}$)

Newton's Laws

$F_g = ma$

$F_g = -\frac{GMm}{r^2}$

$F_g = \cancel{m}a = \frac{-GM\cancel{m}}{r^2}$

At $x=0$, $a = \frac{-GM}{R^2} = -g_0$

$GM = g_0 R^2$

$$a = \frac{d^2x}{dt^2} = \frac{-GM}{(R+x)^2} = \frac{-g_0 R^2}{(R+x)^2}$$

$$\boxed{\frac{d^2x}{dt^2} = \frac{-g_0}{(1+x/R)^2}\ ; \quad x(0)=0, \quad x'(0) = V_0}$$

We may introduce a small parameter into the model by assuming that the projectile's initial velocity V_0 is sufficiently small.

(What is meant by sufficiently small will be revealed during the course of the analysis.)

If V_0 is sufficiently small, the projectile's height x will be small in comparison to earth's radius. (i.e. $x/R \ll 1$).

It can be assumed that x/R is a small quantity, but we cannot set $\varepsilon = x/R$ since ε is a parameter, whereas x is a variable. Nor can we set $\varepsilon = 1/R$ since R is not necessarily a large quantity. The magnitude of R depends on which units it is measured in. For example, if R is measured in terms of lightyears, it will be a small quantity, thus making $\varepsilon = \frac{1}{R}$ large.

In general, two quantities can be compared only if they have the same units. $\varepsilon = 1/R$ has units of **1/length** and cannot be compared to 1. That is, it is <u>not</u> permissible to write $\varepsilon = \frac{1}{R} << 1$.

(*) To say $\varepsilon << 1$ requires that ε be dimensionless.

When applying perturbation methods, it is common to 'nondimensionalize' a problem so that all quantities are dimensionless, and thus can be compared to 1 and/or each other.

Nondimensionalization

To be able to compare the various quantities appearing in a model, it is necessary to 'nondimensionalize' the model. That is, to scale the variables in such a way that the model involves only dimensionless quantities.

In the projectile example, the variables are

space x (dependent)
time t (independent).

<u>Idea:</u> Scale x by a length X_* which is characteristic of its magnitude, and treat t similarly.

Let $y = x/X_*$ and $\tau = t/t_*$,

where X_* and t_* are characteristic length and time scales associated with the problem, respectively. Appropriate choices for X_* and t_* will be determined.

Transform the model into y and τ variables.

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \frac{1}{t_*}\frac{d}{d\tau}$$

$$\frac{d^2}{dt^2} = \frac{1}{t_*^2}\frac{d^2}{d\tau^2}$$

Initial Conditions

$$x(t=0) = y(\tau=0) = 0$$

$$\frac{dx}{dt}(0) = \frac{X_*}{t_*}\frac{dy}{d\tau}(0) = V_0$$

$$\frac{dy}{d\tau}(0) = \frac{V_0 t_*}{X_*}$$

$$\frac{d^2x}{d\tau^2} = \frac{-g_0}{(1+x/R)^2}$$

$$\frac{X_*}{t_*^2}\frac{d^2y}{d\tau^2} = \frac{-g_0}{(1+\frac{X_*}{R}y)^2}$$

$$\boxed{\frac{d^2y}{d\tau^2} = \frac{-g_0 t_*^2}{X_*(1+\frac{X_*}{R}y)^2}}$$

$\frac{g_0 t_*^2}{X_*}$ is dimensionless

$\frac{X_*}{R}$ is dimensionless

To implement our assumption $\frac{x}{R} \ll 1$, we pick X_* to be a length which is 'characteristic' of the magnitude of x, in which case

$\varepsilon = \frac{X_*}{R} \ll 1$ is an appropriate small parameter.

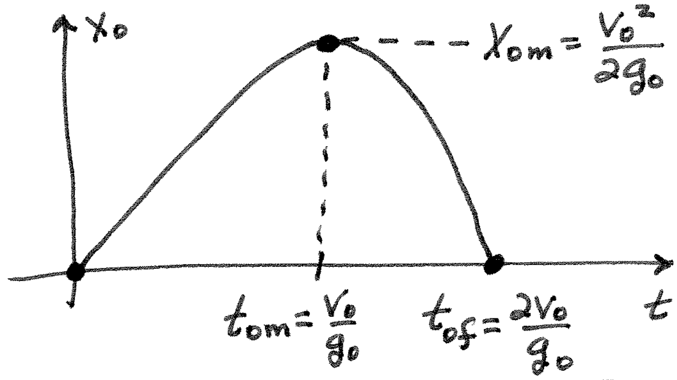
Characteristic Length (X_*) and Time (t_*) Scales

- Consider the familiar 'unperturbed' ($\varepsilon=0$) problem corresponding to the approximation of a constant gravitational force ($F_g = -mg_0$ for all X).

$$\frac{d^2x_0}{dt^2} = -g_0 \; ; \quad x_0(0)=0, \quad x_0'(0)=V_0$$

$$\frac{dx_0}{dt} = V_0 - g_0 t$$

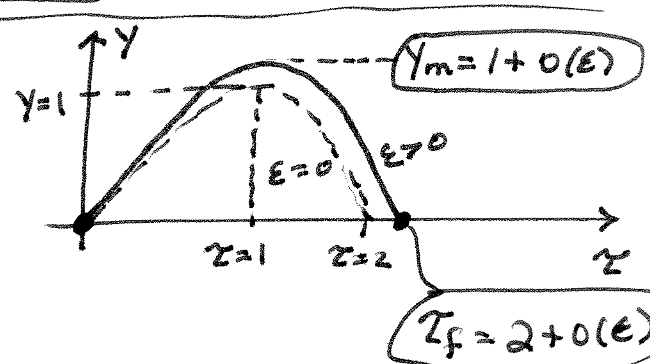
$$x_0 = V_0 t - g_0 t^2/2$$



It is not the only choice, but it is reasonable to pick $\boxed{X_* = X_{om} = \frac{V_0^2}{2g_0}}$ and $\boxed{t_* = t_{om} = \frac{V_0}{g_0}}$.

- Perturbed Problem:

$$y, \tau = O(1)$$



Dimensionless Equations

$$\frac{d^2y}{d\tau^2} = \frac{-g_0 t_*^2}{X_*\left(1+\frac{X_*}{R}y\right)^2} \qquad \frac{g_0 t_*^2}{X_*} = \frac{g_0 \frac{V_0^2}{g_0^2}}{\frac{V_0^2}{2g_0}} = 2$$

$$\boxed{\frac{d^2y}{d\tau^2} = \frac{-2}{(1+\varepsilon y)^2}} \qquad \boxed{\varepsilon = \frac{X_*}{R} = \frac{V_0^2}{2g_0 R}}$$

- Initial Conditions: $\boxed{y(0)=0}$

$$\frac{dy}{d\tau}(0) = \frac{t_* V_0}{X_*} = \frac{\frac{V_0}{g_0}V_0}{V_0^2/2g_0} = 2$$

$$\boxed{\frac{dy}{d\tau}(0) = 2}$$

Notes:

1. $\varepsilon = \dfrac{V_0^2}{2 g_0 R}$ is dimensionless : $[\varepsilon] = \dfrac{ft^2/s^2}{ft/s^2 \cdot ft} = 1$

2. The asymptotic solution expansion will yield a reasonable approximation provided that $\varepsilon = \dfrac{V_0^2}{2g_0R} << 1$, or equivalently, when

$$\boxed{V_0 << \sqrt{2g_0R} = \text{escape velocity} \approx 25{,}000 \ \frac{\text{miles}}{\text{hour}}}$$

(initial velocity required to escape earth's gravitational field)

We have $\dfrac{d^2y}{d\tau^2} = \dfrac{-2}{(1+\varepsilon y)^2}$; $\begin{array}{l} y(0) = 0 \\ y'(0) = 2 \end{array}$.

Find an asymptotic expansion of the solution which is accurate to $O(\varepsilon^2)$.

It is convenient to expand the nonlinearity in the ODE before expanding y.

$$\frac{1}{(1+\varepsilon y)^2} = \frac{1}{1+2\varepsilon y+\varepsilon^2y^2} \overset{\text{Taylor expansion}}{=} 1-(2\varepsilon y+\varepsilon^2y^2)+(2\varepsilon y+\varepsilon^2y^2)^2 + O(\varepsilon^3)$$

$$= 1 - 2\varepsilon y - \varepsilon^2 y^2 + 4\varepsilon^2 y^2 + O(\varepsilon^3)$$

$$\frac{1}{(1+\varepsilon y)^2} = 1 - 2\varepsilon y + 3\varepsilon^2 y^2 + O(\varepsilon^3)$$

$$\Rightarrow \quad \boxed{\frac{d^2y}{d\tau^2} = -2\left(1-2\varepsilon+3\varepsilon^2y^2\right)+O(\varepsilon^3) \ ; \quad \begin{array}{l} y(0) = 0 \\ y'(0) = 2 \end{array}}$$

$$\frac{d^2y}{d\tau^2} = -2\left(1-2\varepsilon y+3\varepsilon^2 y^2\right)+O(\varepsilon^3) \;;\quad \begin{matrix} y(0)=0 \\ y'(0)=2 \end{matrix}$$

Naive Expansion: $y(\tau;\varepsilon) = y_0(\tau)+\varepsilon y_1(\tau)+\varepsilon^2 y_2(\tau)+O(\varepsilon^3)$

$$\Rightarrow \left(y_0+\varepsilon y_1+\varepsilon^2 y_2\right)'' = -2\left[1-2\varepsilon(y_0+\varepsilon y_1)+3\varepsilon^2 y_0^2\right]+O(\varepsilon^3)$$

$$y_0''+\varepsilon y_1''+\varepsilon^2 y_2'' = -2+4\varepsilon y_0+\varepsilon^2\left(4y_1-6y_0^2\right)+O(\varepsilon^3)$$

$O(1)$: $\boxed{y_0''=-2}$

$O(\varepsilon)$: $\boxed{y_1''=4y_0}$

$O(\varepsilon^2)$: $\boxed{y_2''=4y_1-6y_0^2}$

Initial Conditions

$$y(0)=y_0(0)+\varepsilon y_1(0)+\varepsilon^2 y_2(0)+O(\varepsilon^3)=0$$

$$\Rightarrow \boxed{y_0(0)=y_1(0)=y_2(0)=0}$$

$$y'(0)=y_0'(0)+\varepsilon y_1'(0)+\varepsilon^2 y_2'(0)+O(\varepsilon^3)=2$$

$$\Rightarrow \boxed{y_0'(0)=2 \;;\; y_1'(0)=y_2'(0)=0}$$

$O(1)$: $y_0''=-2 \;;\; \begin{matrix} y_0(0)=0 \\ y_0'(0)=2 \end{matrix} \Rightarrow y_0(\tau)=\tau(2-\tau)$

$O(\varepsilon)$: $y_1''=4y_0=4\tau(2-\tau) \;;\; \begin{matrix} y_1(0)=0 \\ y_1'(0)=0 \end{matrix} \Rightarrow y_1(\tau)=\frac{\tau^3}{3}(4-\tau)$

$O(\varepsilon^2)$: $y_2''=4y_1-6y_0^2=-\frac{22}{3}\tau^4+\frac{88}{3}\tau^3-24\tau^2 \;;\; \begin{matrix} y_2(0)=0 \\ y_2'(0)=0 \end{matrix}$

$$\Rightarrow y_2(\tau)=-\frac{\tau^4}{45}\left(11\tau^2-66\tau+90\right)$$

$$\Rightarrow \boxed{y(\tau;\varepsilon)=\tau(2-\tau)+\varepsilon\frac{\tau^3}{3}(4-\tau)-\varepsilon^2\frac{\tau^4}{45}\left(11\tau^2-66\tau+90\right)+O(\varepsilon^3)}$$

$$Y(\tau;\varepsilon) = \underbrace{\tau(2-\tau)}_{n=0} + \underbrace{\varepsilon\frac{\tau^3}{3}(4-\tau)}_{n=1} - \underbrace{\varepsilon^2\frac{\tau^4}{45}(11\tau^2-66\tau+90)}_{n=2} + O(\varepsilon^3)$$

As $\tau\to\infty$, $\quad \sim -\tau^2 \qquad \sim -\frac{1}{3}\varepsilon\tau^4 \qquad \sim -\frac{11}{45}\varepsilon^2\tau^6$

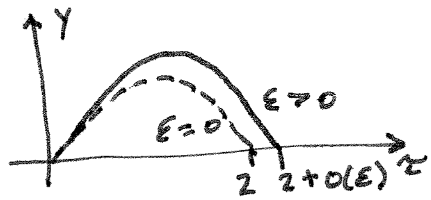
$$\Rightarrow \quad Y_n(\tau) \sim k_n\,\varepsilon^n\,\tau^{2(n+1)} \text{ as } \tau\to\infty, \quad n=0,1,\ldots$$

Notice that the magnitude of the $(n+1)^{st}$ term increases faster than the n^{th} term as $\tau\to\infty$. This suggests the possibility that the expansion may not be uniformly valid over the relevant range of τ.

The expansion is valid when $\left|\frac{Y_{n+1}}{Y_n}\right| \ll 1$ for each $n=0,1,\ldots$.

$$\left|\frac{\varepsilon^{n+1}\tau^{2(n+2)}}{\varepsilon^n\tau^{2(n+1)}}\right| = \varepsilon\tau^2 \ll 1$$

$$\Rightarrow \boxed{\tau \ll \frac{1}{\sqrt{\varepsilon}}}.$$



The domain of the problem is $0\le\tau\le 2+O(\varepsilon) = O(1)$.

Since $\tau = O(1) \ll \frac{1}{\sqrt{\varepsilon}}$, the expansion is uniformly valid of the interval of interest.

Finally, convert back to the original dimensional quantities.

$$\frac{2X_*}{t_*} = V_0, \quad \frac{X_*}{t_*^2} = \frac{g_0}{2}, \quad \varepsilon = \frac{V_0^2}{2g_0R} = \frac{X_*}{R}$$

$$Y \sim \tau(2-\tau) + \varepsilon\frac{\tau^3}{3}(4-\tau)+\cdots$$

$$\frac{X}{X_*} \sim \frac{t}{t_*}\left(2-\frac{t}{t_*}\right) + \frac{X_*}{R}\frac{1}{3}\left(\frac{t}{t_*}\right)^3\left(4-t/t_*\right)+\cdots$$

$$X \sim \left(t\cdot\frac{2X_*}{t_*} - t^2\cdot\frac{X_*}{t_*^2}\right) + \frac{t^3}{3R}\left(\frac{X_*}{t_*^2}\right)\left(4\frac{X_*}{t_*} - t\frac{X_*}{t_*^2}\right)+\cdots$$

$$\sim \left(V_0t - g_0t^2/2\right) + \frac{t^3}{3R}\frac{g_0}{2}\left(2V_0 - g_0t/2\right)+\cdots$$

$$\boxed{X(t) \sim \left(V_0t - g_0t^2/2\right) + \frac{g_0t^3}{6R}\left(2V_0 - g_0t/2\right)+\cdots}$$

Initial / Boundary Conditions

Suppose we expand y as

$$y = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \cdots .$$

1. Unperturbed Conditions: e.g. $y(0) = 2$

$$y(0) = y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \cdots = 2$$

$O(1)$: $\boxed{y_0(0) = 2}$

$O(\varepsilon^n)$: $\boxed{y_n(0) = 0,\ n = 1, 2, \ldots}$

2. Perturbed Conditions: e.g. $y(1) = 3 - 2\varepsilon$

$$y(1) = y_0(1) + \varepsilon y_1(1) + \varepsilon^2 y_2(1) + \cdots = 3 - 2\varepsilon$$

$O(1)$: $\boxed{y_0(1) = 3}$

$O(\varepsilon)$: $\boxed{y_1(1) = -2}$

$O(\varepsilon^n)$: $\boxed{y_n(1) = 0,\ n = 2, 3, \ldots}$

3. Perturbed Boundary: e.g. $y(1-\varepsilon) = 2$

Taylor expand $y(1-\varepsilon)$ about 1: $\quad y(1-\varepsilon) = y(1) - \varepsilon y'(1) + \frac{\varepsilon^2}{2} y''(1) + \cdots = 2$

Expand y: $\quad \left(y_0(1) + \varepsilon y_1(1) + \varepsilon^2 y_2(1)\right) - \varepsilon\left(y_0'(1) + \varepsilon y_1'(1)\right) + \frac{\varepsilon^2}{2} y_0''(1) + \cdots = 2$

$O(1)$: $\boxed{y_0(1) = 2}$ ← BC for y_0

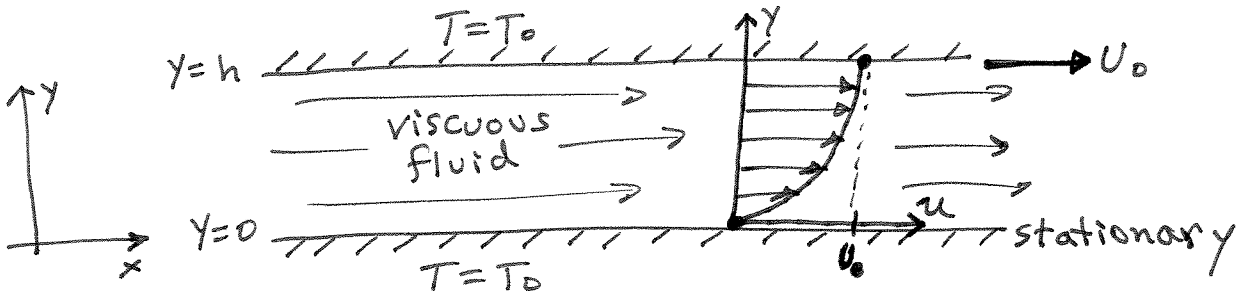
$O(\varepsilon)$: $\boxed{y_1(1) - y_0'(1) = 0}$ ← BC for y_1

$O(\varepsilon^2)$: $\boxed{y_2(1) - y_1'(1) + \frac{1}{2} y_0''(1) = 0}$ ← BC for y_2

$\vdots$

Example: Steady ($\frac{\partial}{\partial t} = 0$) Planar Couette Flow with Variable Viscosity.

Consider a channel of fluid confined between two plates.



The problem is independent of x.

U_0 = speed of the upper plate (m/s)

T_0 = ambient temperature ($^\circ K$)

h = height of the channel (m)

The upper plate moves with velocity U_0, while the lower plate is stationary.

Governing Equations

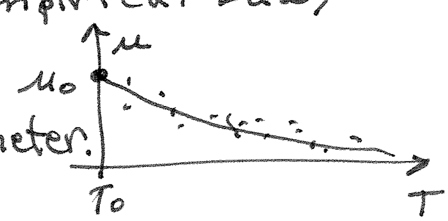
① $$\boxed{\frac{d}{dy}\left(\mu \frac{du}{dy}\right) = 0}$$ (conservation of momentum)

μ = viscosity ($\frac{kg}{ms}$)

② $$\boxed{K \frac{d^2 T}{dy^2} + \mu \left(\frac{du}{dy}\right)^2 = 0}$$ (conservation of energy)

K = coefficient of thermal conductivity ($\frac{J}{^\circ K\, m\, s}$)

③ $$\boxed{\mu = \mu_0 e^{-\beta \frac{T - T_0}{T_0}}}$$ (Empirical Law)



β = dimensionless data-fit parameter.

($\beta \geq 0$)

No Slip Boundary Conditions

- It is assumed that the fluid is 'attached' to the plates. On a molecular scale, fluid particles become entangled in the microstructure of the plates and does not flow relative to the plates at their surface.

$$\Longrightarrow \boxed{\begin{array}{l} u(0) = 0 \\ u(h) = U_0 \end{array}} \quad (4)$$

Also, $\boxed{T(0) = T(h) = T_0}$ (5)

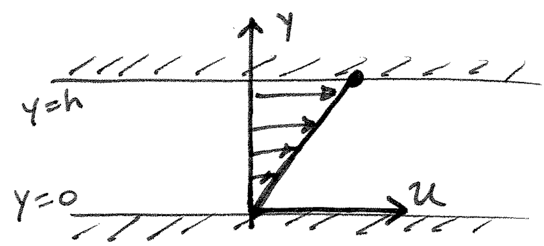
$u(y)$ and $T(y)$ are determined from (1) – (3) subject to (4) and (5).

Goal: Determine the effects of a temperature dependent viscosity as opposed to a constant viscosity.

- Consider a constant viscosity: $\mu = \mu_0$ $(\beta = 0)$

$$(1) \Longrightarrow \frac{d^2u}{dy^2} = 0 \; ; \quad \begin{array}{l} u(0) = 0 \\ u(h) = U_0 \end{array}$$

$$\Longrightarrow \boxed{u(y) = U_0 \frac{y}{h}}$$



The fluid velocity increases linearly from the lower plate $(u=0)$ to the upper plate $(u = U_0)$.

- Now consider a temperature dependent viscosity as described by equation (3). The system cannot be solved exactly, but a perturbation method can be used if we assume that the viscosity is relatively small. Though our results will apply only to low-viscosity fluids, they will give a good indication of the effects of a variable viscosity.

The problem may be nondimensionalized as follows.

Let $\boxed{\eta = \frac{y}{h}}$ $\boxed{V = \frac{u}{U_0}}$ $\boxed{\theta = \frac{T-T_0}{T_0}}$ $T = T_0(1+\theta)$

$\Rightarrow$ $0 \le \eta \le 1$ $0 \le V \le 1$ $0 \le \theta < \infty$

---

③ $\mu = \mu_0 e^{-\beta \frac{T-T_0}{T_0}} \Longrightarrow \boxed{\mu = \mu_0 e^{-\beta\theta}}$

---

① $\frac{d}{dy}\left(\mu \frac{du}{dy}\right) = 0 \Longrightarrow \frac{1}{\cancel{h}}\frac{d}{d\eta}\left(\cancel{\mu_0} e^{-\beta\theta} \cdot \frac{\cancel{V_0}}{\cancel{h}}\frac{dV}{d\eta}\right) = 0$

$$\boxed{\frac{d}{d\eta}\left(e^{-\beta\theta}\frac{dV}{d\eta}\right) = 0} \quad 1'$$

---

② $K\frac{d^2T}{dy^2} + \mu\left(\frac{du}{dy}\right)^2 = 0 \Longrightarrow K\frac{T_0}{\cancel{h^2}}\frac{d^2\theta}{d\eta^2} + \mu_0 e^{-\beta\theta}\left(\frac{U_0}{\cancel{h}}\frac{dV}{d\eta}\right)^2 = 0$

$$\frac{d^2\theta}{d\eta^2} + \frac{\mu_0 U_0^2}{KT_0} e^{-\beta\theta}\left(\frac{dV}{d\eta}\right)^2 = 0$$

Let $\boxed{\varepsilon = \frac{\mu_0 U_0^2}{KT_0}} \Longrightarrow \boxed{\frac{d^2\theta}{d\eta^2} + \varepsilon e^{-\beta\theta}\left(\frac{dV}{d\eta}\right)^2 = 0}$ $2'$

$[\varepsilon] = \frac{\frac{kg}{\cancel{m s}} \cdot \frac{m^2}{s^2}}{\frac{J}{\cancel{K m s}} \cdot \cancel{K}} = \frac{kg\, m^2/s^2}{J} = \frac{N \cdot m}{J} = \frac{J}{J} = 1$

Boundary Conditions:

$\left.\begin{aligned} \theta(\eta=0) &= \frac{T(y=0)-T_0}{T_0} = \frac{T_0 - T_0}{T_0} = 0 \\ \theta(\eta=1) &= \frac{T(y=h)-T_0}{T_0} = \frac{T_0-T_0}{T_0} = 0 \end{aligned}\right\} \Rightarrow \boxed{\theta(0) = \theta(1) = 0}$ $4'$

$\left.\begin{aligned} V(\eta=0) &= \frac{u(y=0)}{U_0} = \frac{0}{U_0} = 0 \\ V(\eta=1) &= \frac{u(y=h)}{U_0} = \frac{V_0}{U_0} = 1 \end{aligned}\right\} \Rightarrow \boxed{\begin{aligned} V(0) &= 0 \\ V(1) &= 1 \end{aligned}}$ $5'$

Solve $1'$ and $2'$ subject to $4'$ and $5'$.

Though ε is not necessarily a small parameter, it can be assumed that $\varepsilon \ll 1$ while keeping in mind that the solution expansion will be valid only when the assumption is satisfied.

$\varepsilon = \dfrac{\mu_0 V_0^2}{k T_0}$ $\quad$ $\varepsilon \ll 1$ corresponds to a relatively small viscosity (μ_0), so our solution will be valid only for low viscosity fluids, such as air.

($\varepsilon \ll 1$ also corresponds to a relatively small upper plate velocity (V_0), a relatively large thermal conductivity (k), or a relatively large ambient temperature (T_0))

We have

1′ $\boxed{(e^{-\beta\theta} V')' = 0}$ $\quad$ $\boxed{V(0) = 0,\ V(1) = 1}$ 4′

3′ $\boxed{\theta'' + \varepsilon e^{-\beta\theta}(V')^2 = 0}$ $\quad$ $\boxed{\theta(0) = \theta(1) = 0}$ 5′

Naive Expansion:

$$V(\eta;\varepsilon) = V_0(\eta) + \varepsilon V_1(\eta) + \cdots$$
$$\theta(\eta;\varepsilon) = \theta_0(\eta) + \varepsilon \theta_1(\eta) + \cdots$$

1′ (to $O(\varepsilon)$) $\Rightarrow$

$$\Big(\underbrace{e^{-\beta(\theta_0+\varepsilon\theta_1)}}(V_0' + \varepsilon V_1')\Big)' = 0$$

$$= e^{-\beta\theta_0} e^{-\varepsilon\beta\theta_1} \sim e^{-\beta\theta_0}(1 - \varepsilon\beta\theta_1)$$

$$\left(e^{-\beta\theta_0}(1-\varepsilon\beta\theta_1)(V_0' + \varepsilon V_1')\right)' = 0$$

$$\boxed{\left(e^{-\beta\theta_0}(V_0' + \varepsilon V_1' - \varepsilon\beta\theta_1 V_0')\right)' = 0}$$

3′ (to $O(\varepsilon)$) $\Rightarrow$ $\boxed{(\theta_0'' + \varepsilon\theta_1'') + \varepsilon e^{-\beta\theta_0}(V_0')^2 = 0}$

Boundary Conditions:

$V(0) = V_0(0) + \varepsilon V_1(0) + \cdots = 0$

$V(1) = V_0(1) + \varepsilon V_1(1) + \cdots = 1$

$\Longrightarrow$ $\boxed{V_0(0) = 0,\ V_0(1) = 1}$ $\quad$ $\boxed{V_1(0) = 0,\ V_1(1) = 0}$

$\theta(0) = \theta_0(0) + \varepsilon\theta_1(0) + \cdots = 0$

$\theta(1) = \theta_0(1) + \varepsilon\theta_1(1) + \cdots = 0$

$\Longrightarrow$ $\boxed{\theta_0(0) = 0,\ \theta_0(1) = 0}$ $\quad$ $\boxed{\theta_1(0) = 0,\ \theta_1(1) = 0}$

$O(1)$: The leading order problem corresponds to an inviscid ($\mu=0$) fluid.

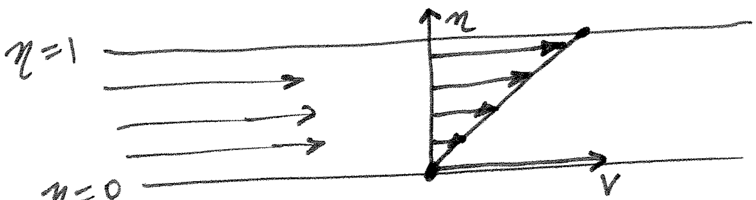
$$(e^{-\beta\theta_0} V_0')' = 0 \;;\; V_0(0)=0,\; V_0(1)=1$$

$$\theta_0'' = 0 \;;\; \theta_0(0)=\theta_0(1)=0$$

Clearly, $\boxed{\theta_0(\eta) \equiv 0}$

Then, $(e^{-\beta\theta_0} V_0')' = V_0'' = 0$ $\quad V_0(0) = B = 0$, $\; V_0(1) = A = 1$

$$V_0(\eta) = A\eta + B \quad \Rightarrow \quad \boxed{V_0(\eta) = \eta}$$



Inviscid fluids exhibit a linear increase in velocity in going from the upper to the lower plate, as was the case for a constant viscocity.

$O(\varepsilon)$: $\left(e^{-\beta\theta_0}(V_1' - \beta\theta_1 V_0')\right)' = 0$

$\theta_0 = 0$, $V_0' = 1$ $\Rightarrow$ $V_1'' - \beta\theta_1' = 0$ $\quad V_1(0) = V_1(1) = 0$

$$\theta_1'' + e^{-\beta\theta_0}(V_0')^2 = 0$$

$\theta_0 = 0$, $V_0' = 1$ $\Rightarrow$ $\theta_1'' + 1 = 0$ $\quad \theta_1(0) = \theta_1(1) = 0$

$\theta_1'' = -1$ $\qquad \theta_1(0) = B = 0$

$\theta_1' = -\eta + A$ $\qquad \theta_1(1) = -\frac{1}{2} + A = 0$

$\qquad A = \frac{1}{2}$

$\theta_1 = -\eta^2/2 + A\eta + B$

$$\Rightarrow \boxed{\theta_1(\eta) = \frac{\eta}{2}(1-\eta)}$$

$V_1'' = \beta\theta_1' = \beta(-\eta + \tfrac{1}{2})$

$V_1' = \beta(-\eta^2/2 + \eta/2 + A)$

$V_1 = \beta(-\eta^3/6 + \eta^2/4 + A\eta + B)$

$V_1(0) = \beta B = 0$

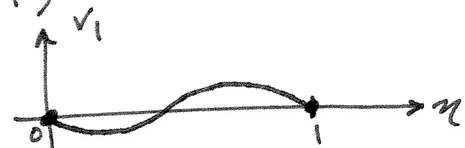
$\underline{B = 0}$

$V_1(1) = \beta(-\tfrac{1}{6} + \tfrac{1}{4} + A) = 0$

$\underline{A = -\tfrac{1}{12}}$

$$V_1 = \beta\left(-\frac{\eta^3}{6} + \frac{\eta^2}{4} - \frac{\eta}{12}\right) = -\frac{\beta\eta}{12}(2\eta^2 - 3\eta + 1)$$

$$\boxed{V_1(\eta) = -\frac{\beta\eta}{12}(2\eta^2 - 3\eta + 1)}$$

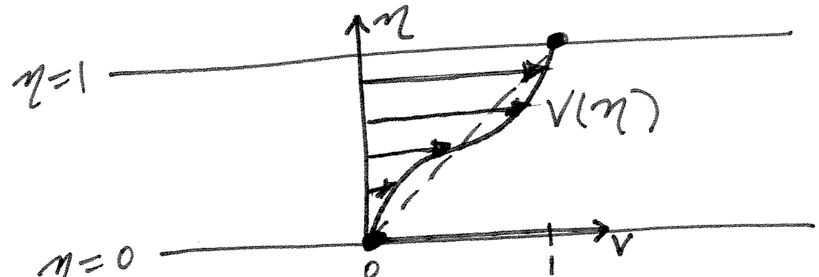


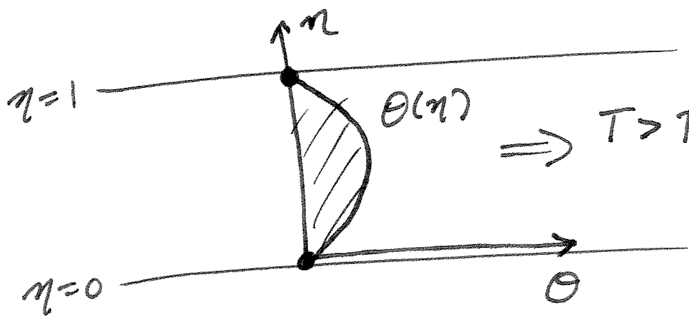
$V_1 < 0$ near the lower plate
$V_1 > 0$ near the upper plate

Then,
$$\boxed{\begin{aligned} V(\eta) &\sim \eta - \varepsilon\frac{\beta\eta}{12}(2\eta^2 - 3\eta + 1) \\ \theta(\eta) &\sim \varepsilon\frac{\eta}{2}(1-\eta) \end{aligned}}$$

The expansion for V becomes disordered as $\eta \to \infty$, but here η is bounded ($0 \le \eta \le 1$).

Finally, convert back to the original dimensional quantities.





$\Rightarrow$ $T > T_0$ within the channel.

Singular ODEs

Singular problems are ones in which an expansion of the form

$$y(x;\varepsilon) = y_0(x) + g_1(\varepsilon)y_1(x) + g_2(\varepsilon)y_2(x) + \cdots$$

fails to yield a uniformly valid solution expansion over the entire domain of the problem.

Examples:

1. Boundary Value Problem

$$\varepsilon y'' + y' + 1 = 0; \quad \begin{matrix} y(0)=0 \\ y(1)=1 \end{matrix}$$

Boundary Layer Problem

Expand y: $\varepsilon(y_0'' + g_1 y_1'' + \cdots) + (y_0' + g_1 y_1' + \cdots) + 1 = 0$

$O(1)$: $y_0' + 1 = 0$ $\qquad y_0(0) = -0 + c = 0$, $c = 0$

$y_0 = -x + c$ $\qquad y_0(1) = -1 + c = 1$, $c = 2$

Not Possible

The leading order ($\varepsilon = 0$) problem is overdetermined in that there are 2 boundary conditions for a first order ODE. The constant of integration cannot be determined to satisfy both conditions.

---

2. Initial Value Problem

$$x'' + \varepsilon x' + x = 0, \quad t > 0; \quad \begin{matrix} x(0)=1 \\ x'(0)=0 \end{matrix}$$

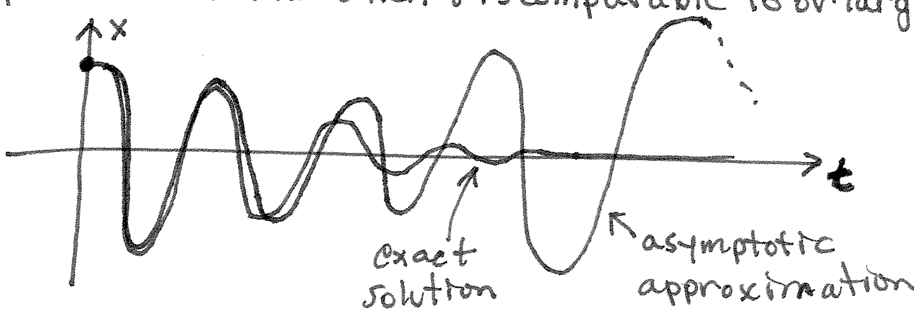
(Weakly Damped oscillator)

Naive Expansion $\Rightarrow$ $\boxed{x(t) \sim \cos t - \frac{\varepsilon}{2}\left[t\cos t - \sin t\right] + \cdots}$

Valid only when $t \ll \frac{1}{\varepsilon}$

The expansion breaks down when t is comparable to or larger than $\frac{1}{\varepsilon}$.

Multiple Scales Problem



As t gets large, accuracy is lost in both the amplitude and the frequency.

Boundary Layer Problems

Example: $\varepsilon y'' + (1+\varepsilon)y' + y = 0,\ 0<x<1;\ \begin{matrix} y(0)=0 \\ y(1)=1 \end{matrix};\ 0<\varepsilon \ll 1$

Naive Expansion:

$$\left.\begin{aligned} y(x;\varepsilon) &= y_0(x) + \varepsilon y_1(x) + \cdots \\ y'(x;\varepsilon) &= y_0'(x) + \varepsilon y_1'(x) + \cdots \\ y''(x;\varepsilon) &= y_0''(x) + \varepsilon y_1''(x) + \cdots \end{aligned}\right\}$$

The naive expansion assumes that y and its derivatives are O(1) quantities.

Leading Order ($\varepsilon = 0$): $y_0' + y_0 = 0;\ \begin{matrix} y_0(0)=0 \\ y_0(1)=1 \end{matrix}$

$y_0(x) = Ce^{-x}$ $\quad$ $y_0(0) = C = 0$

$y_0(1) = Ce^{-1} = 1$ overdetermined

$C = e$

Exact Solution: $\boxed{y(x) = \dfrac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}}$ $\qquad y'(x) = \dfrac{-e^{-x} + \frac{1}{\varepsilon}e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}$

Note: $e^{-x/\varepsilon}$ is the term that causes the difficulty.

$e^{-x/\varepsilon}$ is a TST when $x = O(1)$, but $e^{-x/\varepsilon} = O(1)$ when $x = O(\varepsilon)$.

When $x = O(1)$, $y \sim \dfrac{e^{-x} - TST}{e^{-1} - TST} \sim e^{1-x}$ $\quad$ (Satisfies $y(1)=1$)

When $x = O(\varepsilon)$, $y \sim \dfrac{1 - e^{-x/\varepsilon}}{e^{-1} - TST} \sim e(1 - e^{-x/\varepsilon})$ $\quad$ (Satisfies $y(0)=0$)

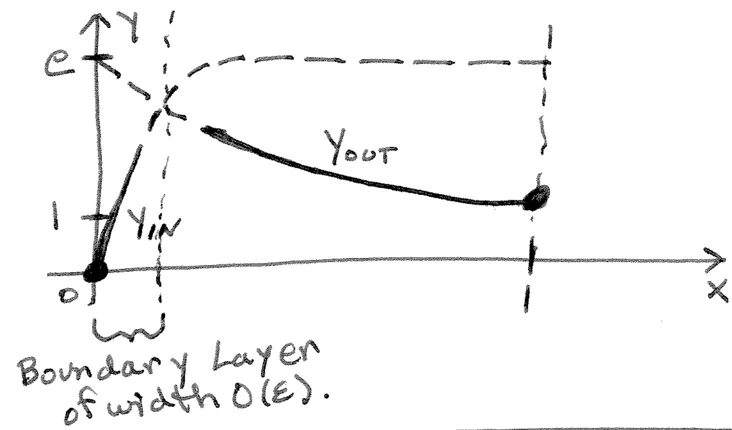
We get two different leading order approximations, depending on the magnitude of x.

The naive expansion cannot capture the effects of the term $e^{-x/\varepsilon}$ since it assumes that y and its derivatives are O(1) quantities. Though $y = O(1)$ for all $x \in [0,1]$, differentiation of the exact solution reveals that $y' = O(\frac{1}{\varepsilon})$ and $y'' = O(\frac{1}{\varepsilon^2})$ when $x = O(\varepsilon)$. Thus, differentiation of the naive expansion is not a valid operation in the vicinity of $x=0$, where derivatives are large.

Plot the two solution expansions found above.

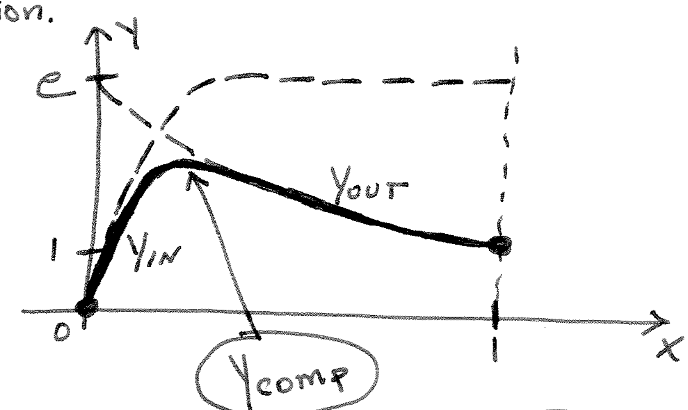
$O(1)$: $y \sim y_{OUT} = e^{1-x}$ (Outer Solution)

$O(\varepsilon)$: $y \sim y_{IN} = e(1-e^{-x/\varepsilon})$ (Inner Solution)

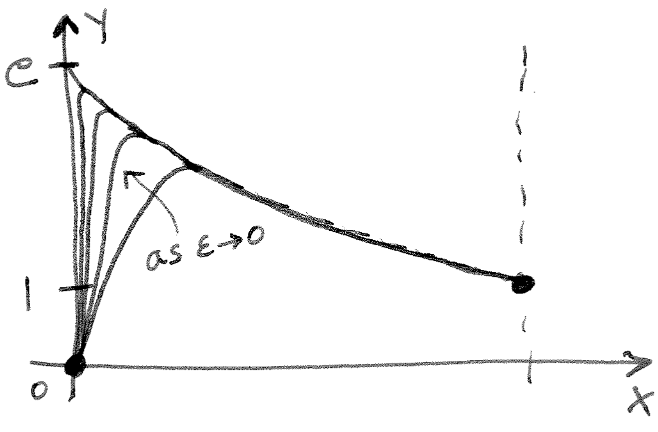


Method of Matched Asymptotic Expansions

Idea: Determine expansions for y_{OUT} and y_{IN}, and combine them in such a way to obtain a uniformly valid 'composite' solution expansion.



Note: The width of the boundary layer $\to 0$ as $\varepsilon \to 0$.



Example: Find a leading order composite solution expansion for the following boundary value problem.

$$\varepsilon y'' + (1+\varepsilon) y' + y = 0 \,; \quad \begin{matrix} y(0) = 0 \\ y(1) = 1 \end{matrix} \,; \quad 0 < \varepsilon \ll 1$$

Outer Solution: The naive expansion gives the outer solution.

$$y_{OUT} \sim Y_0(x) + \varepsilon Y_1(x) + \cdots \,; \quad y_{OUT}(1) = 1$$

y_{OUT} satisfies the boundary condition at $x=1$.

Leading Order: $Y_0' + Y_0 = 0 \,; \; Y_0(1) = 1$

$$\Rightarrow Y_0(x) = e^{1-x}$$

To leading order, $\boxed{y_{OUT} \sim e^{1-x}}$

Inner Solution: The naive expansion fails near $x=0$ since it assumes that y' and y'' are $O(1)$ quantities everywhere, when in fact $y' = O(\frac{1}{\varepsilon})$ and $y'' = O(\frac{1}{\varepsilon^2})$ for $x = O(\varepsilon)$.

To accommodate the large derivatives, we may **RESCALE** the independent variable x (i.e. stretch space) so that the derivatives of y with repect to the new independent variable are $O(1)$ quantities when x is near 0.

To determine the boundary layer solution expansion Y_{IN}, introduce a 'Stretching Variable (ξ)' to stretch the boundary layer so that derivatives are $O(1)$ there.

For a boundary layer located at $x = x_0$ and of thickness $\delta(\varepsilon) \ll 1$, the appropriate stretching transformation is

$$\boxed{\xi = \frac{x - x_0}{\delta(\varepsilon)}}.$$

It usually suffices to use $\delta(\varepsilon) = \varepsilon^{\alpha}$, $\alpha > 0$. (i.e. $\xi = \frac{x - x_0}{\varepsilon^{\alpha}}$)

For the above example, let $\xi = \frac{x}{\delta(\varepsilon)}$. ($x_0 = 0$)

$$y' = \frac{dy}{dx} = \frac{d\xi}{dx}\frac{dy}{d\xi} = \frac{1}{\delta}\frac{dy}{d\xi} = \frac{1}{\delta}y_{\xi}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{1}{\delta^2}y_{\xi\xi}$$

$$\varepsilon y'' + (1+\varepsilon)y' + y = 0$$

$$\varepsilon\left(\frac{1}{\delta^2}y_{\xi\xi}\right) + (1+\varepsilon)\left(\frac{1}{\delta}y_{\xi}\right) + y = 0$$

Dominant Terms: $\frac{\varepsilon}{\delta^2}y_{\xi\xi} + \frac{1}{\delta}y_{\xi} = 0$

$$\Rightarrow \delta = \varepsilon$$

$$y_{\xi\xi} + (1+\varepsilon)y_{\xi} + \varepsilon y = 0$$

Expand y: $Y_{IN} \sim y_0(\xi) + \varepsilon y_1(\xi) + \cdots$; $Y_{IN}(0) = 0$

The Inner Solution satisfies the boundary condition at $x=0$.

Leading Order: $y_{0\xi\xi} + y_{0\xi} = 0$; $y_0(0) = 0$

$$y_0 = A + Be^{-\xi}$$

$$y_0(0) = A + B = 0$$
$$B = -A$$
$$\Rightarrow y_0(\xi) = A(1 - e^{-\xi})$$

$$\boxed{Y_{IN} \sim A(1 - e^{-\xi})}$$

To leading order, we have

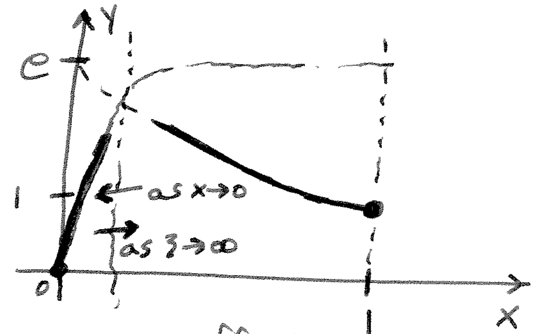
$$\boxed{\begin{array}{l} Y_{OUT} \sim Y_0(x) = e^{1-x} \text{ for } x = O(1) \\ Y_{IN} \sim y_0(\xi) = A(1-e^{-\xi}) \text{ for } x = O(\varepsilon),\ \xi = \frac{x}{\varepsilon} \end{array}}$$

The constant A is determined so that Y_{OUT} and Y_{IN} '<u>match</u>'.

For leading order approximations, it is sufficient to use

<u>Primitive Matching</u>.

$$\boxed{\lim_{\xi\to\infty} y_0(\xi) = \lim_{x\to 0} Y_0(x)}$$



For now, we'll consider only leading order solutions. Matching higher-order expansions is more difficult.

$$\lim_{\xi\to\infty} y_0(\xi) = \lim_{\xi\to\infty} A(1-e^{-\xi}) = A$$

$$\lim_{x\to 0} Y_0(x) = \lim_{x\to 0} e^{1-x} = e$$

$A = e$

This is called the <u>Common Part</u> (CP).

$$\Rightarrow \quad \begin{array}{l} Y_{OUT} \sim e^{1-x} \\ Y_{IN} \sim e(1-e^{-\xi}) \end{array}$$

<u>Note</u>: Y_{OUT} is determined from a first order ODE $\Rightarrow$ 1 const. of integration.
Y_{IN} is <u>determined</u> from a second order ODE $\Rightarrow$ 2 consts. of integration.

There are a total of 3 integration constants. Two of them are determined by the boundary conditions, and the third is chosen so that Y_{OUT} and Y_{IN} 'match'.

Finally, combine y_{OUT} and y_{IN} to get a single smooth uniformly valid solution expansion for y (called the 'Composite Expansion').

$$\boxed{y_{comp} = y_{OUT} + \underbrace{y_{IN} - \begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}}_{\text{Boundary Layer Correction}}}$$

For the above example, the leading order composite solution expansion is

$$y_{comp} = y_{OUT} + y_{IN} - C.P.$$

$$\sim e^{1-x} + e(1-e^{-\xi}) - e = \underline{e^{1-x} - e^{1-\xi}}$$

$$\xi = \frac{x}{\varepsilon} \implies \boxed{y_{comp}(x) = e^{1-x} - e^{1-x/\varepsilon}} \begin{pmatrix}\text{Uniformly Valid}\\ \text{on } (0,1).\end{pmatrix}$$

$$\text{OR} \quad y_{comp}(x) = e(e^{-x} - e^{-x/\varepsilon}).$$

<u>Note</u>: When $x = O(1)$, $y_{IN} \sim C.P. = e \implies y_{comp} \sim y_{OUT}$.

When $x = O(\varepsilon)$, $y_{OUT} \sim C.P. = e \implies y_{comp} \sim y_{IN}$.

<u>Note</u>: The coefficients of the expansion now depend on ε.

i.e. $y(x;\varepsilon) = y_0(x;\varepsilon) + \varepsilon\, y_1(x;\varepsilon) + \varepsilon^2 y_2(x;\varepsilon) + \cdots$.

<u>Note</u>: y_{comp} is the same as the exact solution $\left(y = \dfrac{e^{-x} - e^{-x/\varepsilon}}{e^{-1} - e^{-1/\varepsilon}}\right)$, except for the term $e^{-1/\varepsilon}$.

$e^{-1/\varepsilon}$ is a TST and cannot be captured by an expansion in powers of ε. Consequently, the composite solution does not satisfy the boundary conditions exactly.

$$y_{comp}(0) = 0 \ \checkmark$$

$$y_{comp}(1) = 1 - e^{1-\frac{1}{\varepsilon}} = 1 + TST \ \begin{pmatrix}\text{close}\\ \text{enough}\end{pmatrix}$$

Example: $\varepsilon y'' - (1+\varepsilon) y' + y = 0$; $\begin{matrix} y(0) = 0 \\ y(1) = 1 \end{matrix}$; $0 < \varepsilon \ll 1$

Find a uniformly valid leading order composite expansion for y.

Outer Solution: $y_{OUT} = Y_0(x) + \varepsilon Y_1(x) + \cdots$

Leading Order: $-Y_0' + Y_0 = 0$

$$\boxed{Y_0(x) = C e^{x}}$$

The boundary layer may occur at either boundary ($x=0$ or $x=1$). The appropriate boundary condition for $Y_0(x)$ depends on the location of the boundary layer.

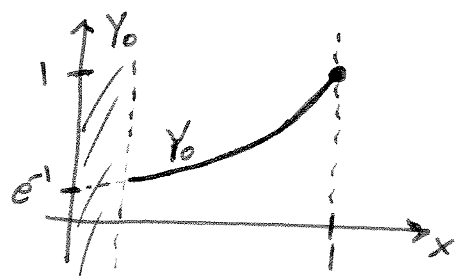
Boundary Layer at $x=0 \implies Y_0(1) = 1$

Boundary Layer at $x=1 \implies Y_0(0) = 0$

For now, we can determine the location of the boundary layer by trial and error. That is, we can randomly pick one boundary and attempt to find a matchable inner solution. If no such solution can be found, try the other boundary.

Check for a Boundary Layer at $x=0$

In this case $Y_0(1) = 1 \implies C = e^{-1} \implies \boxed{Y_0(x) = e^{-(1-x)}}$



Inner Solution: (at $x=0$)

Let $\boxed{\xi = \frac{x}{\varepsilon}}$.

$$\varepsilon y'' - (1+\varepsilon)y' + y = 0$$

$$\varepsilon\left(\frac{1}{\varepsilon^2} y_{\xi\xi}\right) - (1+\varepsilon)\left(\frac{1}{\varepsilon} y_\xi\right) + y = 0$$

$$\boxed{y_{\xi\xi} - (1+\varepsilon) y_\xi + \varepsilon y = 0}$$

Expand y: $y_{IN} = y_0(\xi) + \varepsilon y_1(\xi) + \ldots$

Leading order: $y_{0\xi\xi} - y_{0\xi} = 0$; $y_0(0) = 0$

$$y_0 = A + Be^{\xi}$$

$$y_0(0) = 0 \implies \boxed{y_0(\xi) = A(1 - e^{\xi})}$$

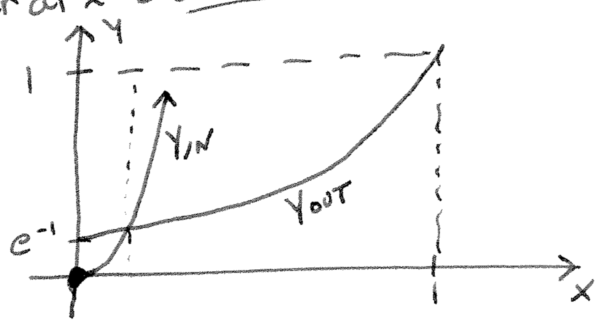
Primitive Matching:

$$\lim_{x\to 0} Y_0(x) = \lim_{\xi\to\infty} y_0(\xi)$$

$$\lim_{x\to 0} e^{-(1-x)} = \lim_{\xi\to\infty} A(1-e^{\xi})$$

$$\cancel{e^{-1} = A(1-\infty)}$$

A boundary layer at $x=0$ does not yield a matchable inner solution.



Try a boundary layer at $x = 1$.

Try a Boundary Layer at $x=1$

Outer Solution: $Y_o(x) = Ce^x$; $Y_o(0)=0$

$Y_o(0) = C = 0 \Rightarrow \boxed{Y_o(x) = 0}$

Inner Solution: Let $\boxed{\xi = \frac{x-1}{\varepsilon}}$. $\quad x=1 \Rightarrow \xi = 0$

$Y_{IN}(x=1) = Y_{IN}(\xi=0) = 1$

$$\varepsilon y'' - (1+\varepsilon)y' + y = 0$$

$$\varepsilon\left(\frac{1}{\varepsilon^2}Y_{\xi\xi}\right) - (1+\varepsilon)\left(\frac{1}{\varepsilon}Y_\xi\right) + Y = 0$$

$$Y_{\xi\xi} - (1+\varepsilon)Y_\xi + \varepsilon Y = 0$$

Expand y: $Y_{IN} = y_0(\xi) + \varepsilon y_1(\xi) + \cdots$; $Y_{IN}(0) = 1$

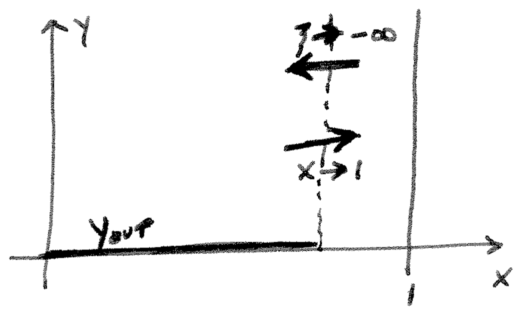
Leading order: $y_{0\xi\xi} - y_{0\xi} = 0$; $y_0(0) = 1$

$y_0(\xi) = A + Be^\xi$

$y_0(0) = 1 \Rightarrow \boxed{y_0(\xi) = A + (1-A)e^\xi}$

Primitive Matching

$$\boxed{\lim_{x\to 1} Y_o(x) = \lim_{\xi\to -\infty} y_o(\xi)}$$



$$\lim_{x\to 1} 0 = \lim_{\xi\to -\infty} A + (1-A)e^\xi$$

$$\boxed{0 = A}$$

Common Part

To leading order, we have

$$Y_{OUT} \sim Y_0(x) = 0$$

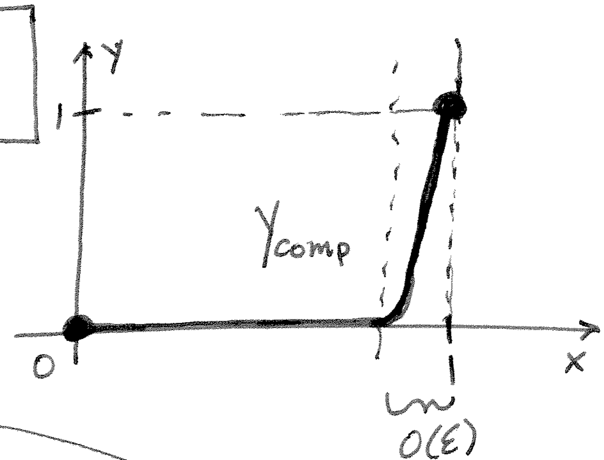
$$Y_{IN} \sim y_0(\xi) = e^{\xi}$$

$$\text{Common Part} \sim 0$$

Then, $Y_{comp} = Y_{OUT} + Y_{IN} - \left(\begin{matrix}\text{Common}\\ \text{Part}\end{matrix}\right)$

$$\sim 0 + e^{\xi} - 0 = e^{\xi} = e^{\frac{x-1}{\varepsilon}}$$

$$\boxed{Y_{comp}(x) \sim e^{-\frac{1}{\varepsilon}(1-x)}}$$



Exact Solution: $\boxed{Y_{ex} = \dfrac{e^{x} - e^{x/\varepsilon}}{e - e^{1/\varepsilon}}}$

To leading order, $Y_{ex} \sim \dfrac{e^{x} - e^{x/\varepsilon}}{-e^{1/\varepsilon}} = -e^{-\frac{1}{\varepsilon}+x} + e^{-\frac{1}{\varepsilon}(1-x)}$ TST

$$Y_{ex} \sim e^{-\frac{1}{\varepsilon}(1-x)}$$

The composite expansion agrees with the leading order approximation of the exact solution.

Summary of the Method of Matched Asymptotic Expansions

Steps:

Outer Solution

1. Naive Expansion: $Y_{OUT} = Y_0(x) + \varepsilon Y_1(x) + \cdots$

Inner Solution

2. Rescale the ODE by stretching the spatial coordinate.

$$\xi = \frac{x - x_0}{\delta(\varepsilon)}$$

x_0 = boundary layer location
$\delta(\varepsilon)$ = boundary layer thickness

3. Naive Expansion: $Y_{IN} = y_0(\xi) + \varepsilon y_1(\xi) + \cdots$

Composite Solution

4. Match Y_{OUT} and Y_{IN}.

The Common Part is determined here.

5. $Y_{comp} = Y_{OUT} + Y_{IN} - \begin{pmatrix} \text{Common} \\ \text{Part} \end{pmatrix}$

General Theory

In the two previous examples, we considered

$$\varepsilon y'' \pm (1+\varepsilon) y' + y = 0 \; ; \; \begin{matrix} y(0)=0 \\ y(1)=1 \end{matrix} \; ; \; 0<\varepsilon \ll 1 .$$

$+ \Rightarrow$ Boundary Layer of thickness $O(\varepsilon)$ at the left boundary $(x=0)$

$- \Rightarrow$ Boundary Layer of thickness $O(\varepsilon)$ at the right boundary $(x=1)$

It is true in general that the problem

$$\varepsilon y'' + a y' + b y = 0 \; ; \; \begin{matrix} y(x_1)=A \\ y(x_2)=B \end{matrix} \; ; \; 0<\varepsilon \ll 1$$

has a single boundary layer of thickness $O(\varepsilon)$

at the left boundary $(x=x_1)$ if $a>0$, and

at the right boundary $(x=x_2)$ if $a<0$.

Note: Any interval (x_1, x_2) can be transformed into $(0,1)$ by making a linear change of independent variable $t = \dfrac{x-x_1}{x_2-x_1}$, so for convenience we'll restrict ourselves to the interval $(0,1)$.

More generally, we may consider singular, second order, homogeneous, linear ODEs with variable coefficients.

General Form:
$$\boxed{\varepsilon y'' + a(x) y' + b(x) y = 0 \; ; \; \begin{matrix} y(0)=A \\ y(1)=B \end{matrix} \; ; \; 0<\varepsilon \ll 1 .}$$

where $a(x)$ and $b(x)$ are smooth functions.

We'll see that

i) if $a(x)>0$ on $[0,1]$, then there is a single boundary layer of thickness $O(\varepsilon)$ at the left boundary $(x=0)$

ii) if $a(x)>0$ on $[0,1]$, then there is a single boundary layer of thickness $O(\varepsilon)$ at the right boundary $(x=1)$

iii) if $a(x_0)=0$ for some $x_0 \in [0,1]$, then there may be

a) a boundary layer on both sides ($x=0$ and $x=1$)

b) an interior layer at $x=x_0$ $(0<x_0<1)$

c) a boundary layer of thickness $O(\varepsilon^\alpha)$, $\alpha \neq 1$

b) nonunique composite expansions.

Consider

$$\varepsilon y'' + a(x)y' + b(x)y = 0\;; \quad \begin{matrix} y(0)=A \\ y(1)=B \end{matrix}\;; \quad 0<\varepsilon \ll 1,$$

where $a(x)$ and $b(x)$ are smooth functions.

For now, assume that $a(x) \neq 0$ on the interval $[0,1]$.
(i.e. either $a(x)>0$ or $a(x)<0$ for all $x\in[0,1]$)

Suppose that the boundary layer is located at $x = x_0$, and denote the opposite boundary by $x = x_1$.

Outer Solution

Expand y: $y_{OUT} = Y_0(x) + \varepsilon Y_1(x) + \cdots, \quad y_{OUT}(x_1) = \underset{(A \text{ or } B)}{?}$

$$\Rightarrow \varepsilon(Y_0'' + \varepsilon Y_1'' + \cdots) + a(x)(Y_0' + \varepsilon Y_1' + \cdots) + b(x)(Y_0 + \varepsilon Y_1 + \cdots) = 0$$

Leading Order: $\quad a(x)Y_0' + b(x)Y_0 = 0$

$$\frac{dY_0}{Y_0} = -\frac{b(x)}{a(x)}\,dx \qquad (a(x)\neq 0)$$

Integrate both sides from x_1 to x.

$$\int_{Y_0(x_1)}^{Y_0(x)} \frac{dY_0}{Y_0} = -\int_{x_1}^{x} \frac{b(s)}{a(s)}\,ds$$

$$\ln|Y_0(x)| - \ln|Y_0(x_1)| = -\int_{x_1}^{x} \frac{b(s)}{a(s)}\,ds$$

$$\Longrightarrow \boxed{Y_0(x) = Y_0(x_1)\, e^{-\int_{x_1}^{x} \frac{b(s)}{a(s)}\,ds}}$$

Try a boundary layer at $x = 0$.

Outer Solution: $x_1 = 1$, $Y_0(1) = B$ $\Rightarrow$ $\boxed{Y_0(x) = B\, e^{\int_x^1 \frac{b(s)}{a(s)}\, ds}}$

Inner Solution: Stretch near $x=0$. $\xi = \frac{x}{\varepsilon}$ $\quad y' = \frac{1}{\varepsilon} Y_\xi$, $\quad y'' = \frac{1}{\varepsilon^2} Y_{\xi\xi}$ $\quad x = \varepsilon \xi$

$$\varepsilon y'' + a(x) y' + b(x) y = 0$$

$$\varepsilon\left(\frac{1}{\varepsilon^2} Y_{\xi\xi}\right) + a(\varepsilon\xi)\left(\frac{1}{\varepsilon} Y_\xi\right) + b(\varepsilon\xi) Y = 0$$

$$Y_{\xi\xi} + a(\varepsilon\xi) Y_\xi + \varepsilon\, b(\varepsilon\xi)\, Y = 0$$

Taylor expand $a(\varepsilon\xi)$ and $b(\varepsilon\xi)$:

$$a(\varepsilon\xi) = a(0) + \varepsilon\xi\, a'(0) + \frac{\varepsilon^2\xi^2}{2} a''(0) + \cdots$$

$$b(\varepsilon\xi) = b(0) + \varepsilon\xi\, b'(0) + \frac{\varepsilon^2\xi^2}{2} b''(0) + \cdots$$

$$\Rightarrow \quad Y_{\xi\xi} + \left(a(0) + \varepsilon\xi\, a'(0) + \cdots\right) Y_\xi + \varepsilon\left(b(0) + \varepsilon\xi\, b'(0) + \cdots\right) Y = 0$$

Expand Y: $Y_{IN} = y_0(\xi) + \varepsilon y_1(\xi) + \cdots$, $\quad Y_{IN}(0) = A$

Leading order: $y_{0\xi\xi} + a(0)\, y_{0\xi} = 0$, $\quad y_0(0) = A$

$$\Rightarrow \quad y_0 = C_1 + C_2 e^{-a(0)\xi}$$

$$y_0(0) = C_1 + C_2 = A$$

$$C_2 = A - C_1$$

$$\Rightarrow \quad \boxed{y_0(\xi) = C_1 + (A - C_1)\, e^{-a(0)\xi}}$$

To leading order, we have

$$Y_{OUT} \sim Y_0(x) = Be^{\int_x^1 \frac{b(s)}{a(s)} ds}$$

$$Y_{IN} \sim y_0(\xi) = C_1 + (A - C_1)e^{-a(0)\xi}$$

Primitive Matching

$$\lim_{x \to 0} Y_0(x) = \lim_{\xi \to \infty} y_0(\xi)$$

$$\lim_{x \to 0} Be^{\int_x^1 \frac{b(s)}{a(s)} ds} = \lim_{\xi \to \infty} C_1 + (A - C_1)\underline{\underline{e^{-a(0)\xi}}}$$

The limit exists only when $a(0) \geq 0$

Assume $a(0) > 0$

Since it assumed that $a(x) \neq 0$ on $[0,1]$, $a(0) > 0$ implies that $a(x) > 0$ for all $x \in [0,1]$.

Then, $\boxed{Be^{\int_0^1 \frac{b(s)}{a(s)} ds} = C_1} = \text{Common Part}$

Composite Solution

$$Y_{comp} = Y_{OUT} + Y_{IN} - \begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}$$

$$\sim Y_0(x) + y_0(\xi) - \begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}$$

$$\sim \left[Be^{\int_x^1 \frac{b(s)}{a(s)} ds}\right] + \left[\cancel{C_1} + (A - C_1)e^{-a(0)\xi}\right] - \left[\cancel{C_1}\right]$$

$$\boxed{Y_{comp}(x) \sim Be^{\int_x^1 \frac{b(s)}{a(s)} ds} + \underbrace{\left(A - Be^{\int_0^1 \frac{b(s)}{a(s)} ds}\right)e^{-a(0)x/\varepsilon}}_{\text{TST when } x = O(1)}}$$

True when $a(x) > 0$ on $[0,1]$

TST when $x = O(1)$

(boundary layer correction to Y_{OUT})

Try a boundary layer at $x = 1$.

Outer Solution: $x_1 = 0$, $Y_0(0) = A$ $\Longrightarrow$ $\boxed{Y_0(x) = A e^{-\int_0^x \frac{b(s)}{a(s)} ds}}$

Inner Solution: Stretch near $x=1$. $\xi = \dfrac{x-1}{\varepsilon}$

$y' = \frac{1}{\varepsilon} Y_\xi$ $\quad x = 1 + \varepsilon\xi$

$y'' = \frac{1}{\varepsilon^2} Y_{\xi\xi}$ $\quad x = 1 \Rightarrow \xi = 0$

$$\varepsilon y'' + a(x) y' + b(x) y = 0$$

$$\varepsilon\left(\frac{1}{\varepsilon^2} Y_{\xi\xi}\right) + a(1+\varepsilon\xi)\left(\frac{1}{\varepsilon} Y_\xi\right) + b(1+\varepsilon\xi) Y = 0$$

$$Y_{\xi\xi} + a(1+\varepsilon\xi) Y_\xi + \varepsilon b(1+\varepsilon\xi) Y = 0$$

Taylor expand $a(1+\varepsilon\xi)$ and $b(1+\varepsilon\xi)$:

$$a(1+\varepsilon\xi) = a(1) + \varepsilon\xi a'(1) + \frac{\varepsilon^2\xi^2}{2} a''(1) + \cdots$$

$$b(1+\varepsilon\xi) = b(1) + \varepsilon\xi b'(1) + \frac{\varepsilon^2\xi^2}{2} b''(1) + \cdots$$

$$\Longrightarrow \quad Y_{\xi\xi} + \left(a(1) + \varepsilon\xi a'(1) + \cdots\right) Y_\xi + \left(b(1) + \varepsilon\xi b'(1) + \cdots\right) Y = 0$$

Expand y: $Y_{IN}(\xi) = y_0(\xi) + \varepsilon y_1(\xi) + \cdots$, $Y_{IN}(0) = B$

Leading Order: $y_{0\xi\xi} + a(1) y_{0\xi} = 0$; $y_0(0) = B$

$$\Longrightarrow y_0 = C_1 + C_2 e^{-a(1)\xi}$$

$$y_0(0) = C_1 + C_2 = B$$

$$C_2 = B - C_1$$

$$\Longrightarrow \boxed{y_0(\xi) = C_1 + (B - C_1) e^{-a(1)\xi}}$$

To leading order, we have

$$Y_{OUT} \sim Y_0(x) = A e^{-\int_0^x \frac{b(s)}{a(s)} ds}$$

$$Y_{IN} \sim y_0(\xi) = C_1 + (B - C_1) e^{-a(1)\xi}$$

Primitive Matching

$$\lim_{x\to 1} Y_0(x) = \lim_{\xi \to -\infty} y_0(\xi)$$

$$\lim_{x\to 1} A e^{-\int_0^x \frac{b(s)}{a(s)} ds} = \lim_{\xi\to -\infty} C_1 + (B - C_1)\underline{\underline{e^{-a(1)\xi}}}$$

The limit exists only when $a(1) \le 0$

Assume $a(1) < 0$

Since it is assumed that $a(x) \neq 0$ on $[0,1]$, $a(1) < 0$ implies that $a(x) < 0$ for all $x \in [0,1]$.

Then, $\boxed{A e^{-\int_0^1 \frac{b(s)}{a(s)} ds} = C_1}$ = Common Part

---

Composite Solution

$$Y_{comp} = Y_{OUT} + Y_{IN} - \begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}$$

$$\sim Y_0(x) + y_0(\xi) - \begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}$$

$$\sim \left[A e^{-\int_0^x \frac{b(s)}{a(s)} ds}\right] + \left[\cancel{C_1} + (B - C_1) e^{-a(1)\xi}\right] - \cancel{C_1}$$

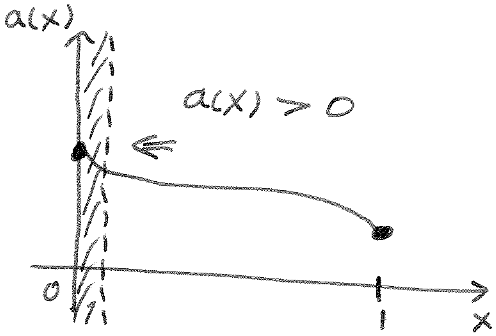
$$\boxed{Y_{comp}(x) \sim A e^{-\int_0^x \frac{b(s)}{a(s)} ds} + \underbrace{\left(B - A e^{-\int_0^1 \frac{b(s)}{a(s)} ds}\right) e^{a(1)\frac{1-x}{\varepsilon}}}_{\text{TST when } 1-x = O(1)}}$$

True when $a(x) < 0$ on $[0,1]$.

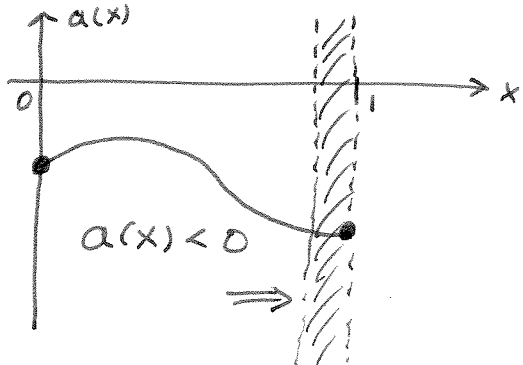
TST when $1-x = O(1)$
(boundary layer correction to Y_{OUT})

Summary: $\varepsilon y'' + a(x)y' + b(x)y = 0$; $y(0) = A$, $y(1) = B$; $0 < \varepsilon \ll 1$

$a(x) \neq 0$ on $[0,1]$



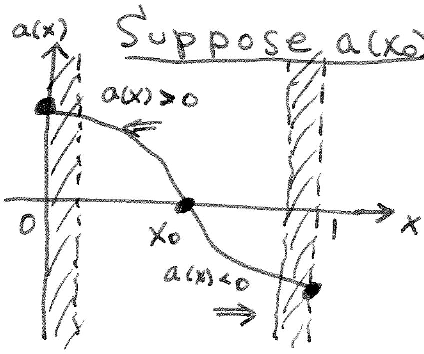
Left boundary layer of thickness $O(\varepsilon)$.



Right boundary layer of thickness $O(\varepsilon)$.

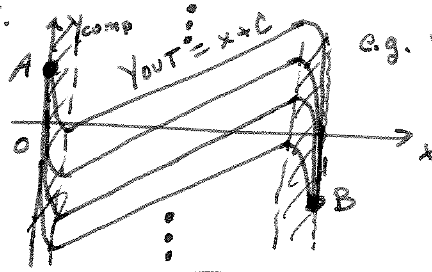
When $a(x_0) = 0$ for some $x_0 \in [0,1]$, the above conclusions can be used as a guide in determining the boundary layer location(s).

Suppose $a(x_0) = 0$ for some interior point $x_0 \in (0,1)$.

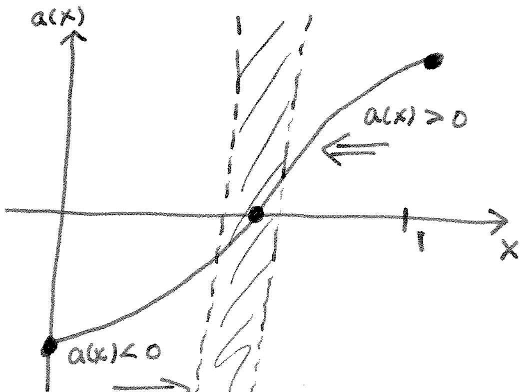


thickness $= O(\varepsilon)$

This case may lead to a one parameter family of possible composite solutions. The correct member must be determined by other means.

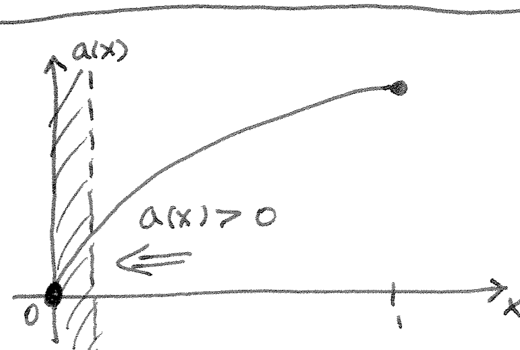


e.g. $y_{OUT} = x + C$, where C cannot be determined.



Interior Layer of thickness $O(\varepsilon^{\alpha})$, $\alpha > 0$.

Suppose $a(x) = 0$ at an endpoint $\begin{pmatrix} a(0) = 0 \\ \text{or} \\ a(1) = 0 \end{pmatrix}$.



thickness $= O(\varepsilon^{\alpha})$, $\alpha > 0$

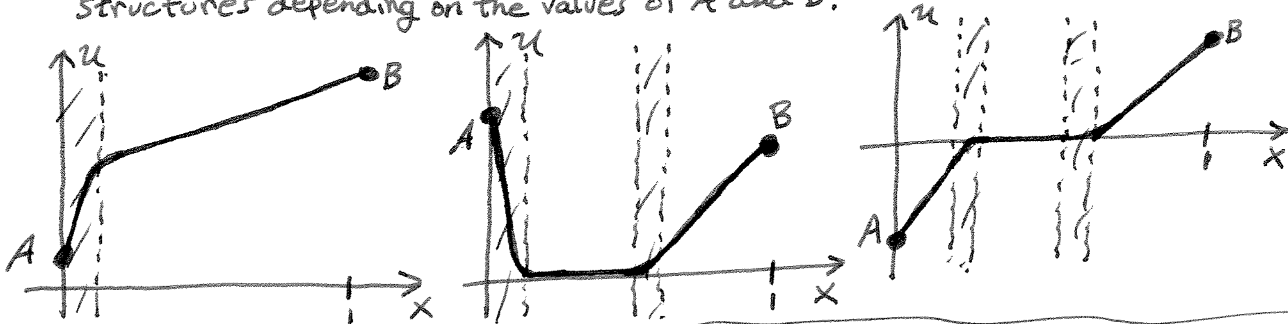
Other Boundary Layer Problems

1. Nonlinear ODEs:

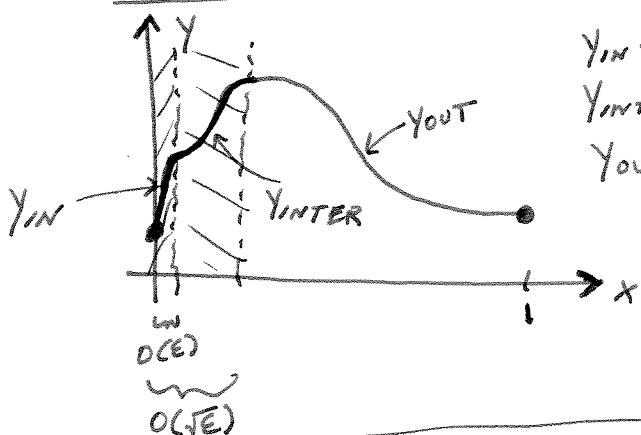
e.g. $\varepsilon u'' + uu' - u = 0$; $\begin{matrix} u(0)=A \\ u(1)=B \end{matrix}$; $0<\varepsilon<<1$

The dependent variable appears here instead of a(x).

This boundary value problem may lead to a variety of boundary layer structures depending on the values of A and B.

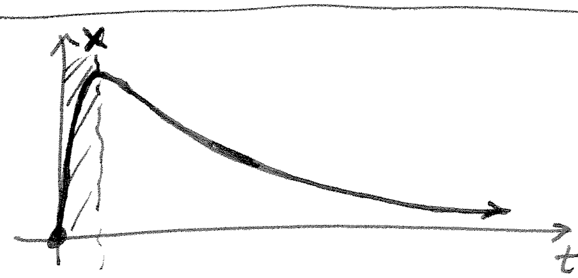


2. Two-Ply Boundary Layers (Triple-Deck)



Y_{IN} = inner solution ($O(\varepsilon)$)
Y_{INTER} = intermediate solution ($O(\sqrt{\varepsilon})$)
Y_{OUT} = outer solution ($O(1)$)

3. Initial Layers (at $t=0$)



Other Considerations

1. Systems of ODEs
2. PDEs
3. Higher Order Composite Expansions

Case: Boundary Layer at x_0, with $a(x_0) = 0$.

Consider $\varepsilon y'' + a(x)y' + b(x)y = 0$; $\begin{matrix} y(0) = A \\ y(1) = B \end{matrix}$; $0 < \varepsilon \ll 1$.

Suppose there is a boundary layer at x_0 and $a(x_0) = 0$.

Inner Solution: $\boxed{\xi = \dfrac{x - x_0}{\delta(\varepsilon)}}$, $\delta(\varepsilon) \ll 1$

$$x = x_0 + \delta\xi$$
$$y' = \frac{1}{\delta} Y_\xi$$
$$y'' = \frac{1}{\delta^2} Y_{\xi\xi}$$

$$\Rightarrow \varepsilon\left(\frac{1}{\delta^2} Y_{\xi\xi}\right) + a(x_0 + \delta\xi)\left(\frac{1}{\delta} Y_\xi\right) + b(x_0 + \delta\xi) Y = 0$$

$$\frac{\varepsilon}{\delta^2} Y_{\xi\xi} + \left[\cancel{a(x_0)}^{=0} + \delta\xi a'(x_0) + \frac{\delta^2\xi^2}{2} a''(x_0) + \cdots\right] \frac{1}{\delta} Y_\xi + \left[b(x_0) + \delta\xi b'(x_0) + \cdots\right] Y = 0$$

$$\frac{\varepsilon}{\delta^2} Y_{\xi\xi} + \left[\xi a'(x_0) + \frac{\delta\xi^2}{2} a''(x_0) + \cdots\right] Y_\xi + \left[b(x_0) + \delta\xi b'(x_0) + \cdots\right] Y = 0$$

Leading Order: $\dfrac{\varepsilon}{\delta^2} Y_{\xi\xi} + \underbrace{\xi a'(x_0) Y_\xi}_{O(1)} + \underbrace{b(x_0) Y}_{O(1)} = 0$

$$\Rightarrow \boxed{\delta = \varepsilon^{1/2}}$$

However,

i) if $a'(x_0) = a''(x_0) = \cdots = 0$ and/or $b(x_0) = b'(x_0) = \cdots = 0$, we must balance accordingly.

e.g. $\begin{matrix} a'(x_0) = a''(x_0) = 0 \\ b(x_0) = 0 \end{matrix} \Rightarrow \underline{\dfrac{\varepsilon}{\delta^2} Y_{\xi\xi}} + \dfrac{\delta^2\xi^3}{6} a'''(x_0) + \underline{\delta\xi b'(x_0)} Y = 0$

$$\frac{\varepsilon}{\delta^2} \sim \delta \Rightarrow \boxed{\delta = \varepsilon^{1/3}}$$

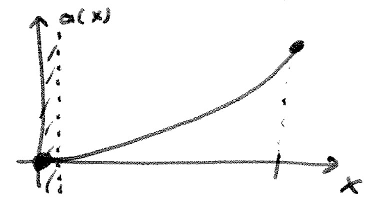
ii) the above argument doesn't apply if $a(x)$ (or $b(x)$) doesn't have a Talyor expansion at x_0.

e.g. $a(x) = \sqrt{x - x_0}$

The following example is fairly general and serves as an example of i) and ii).

Consider $\varepsilon y'' + a(x)y' + b(x)y = 0$; $y(0) = A$, $y(1) = B$; $0 < \varepsilon \ll 1$,

where $a(x) = x^\alpha$, $b(x) = x^\beta$ with $\alpha, \beta > 0$.



$a(x) > 0$ on $(0,1)$. Expect a layer at the left boundary.

Inner Solution: $\xi = \dfrac{x}{\delta(\varepsilon)}$, $\delta(\varepsilon) \ll 1$

$$\varepsilon y'' + x^\alpha y' + x^\beta y = 0$$

$$\varepsilon\left(\frac{1}{\delta^2} y_{\xi\xi}\right) + (\delta\xi)^\alpha\left(\frac{1}{\delta} y_\xi\right) + (\delta\xi)^\beta y = 0$$

$$\underbrace{\frac{\varepsilon}{\delta} y_{\xi\xi}}_{1} + \underbrace{(\delta\xi)^\alpha y_\xi}_{2} + \underbrace{\delta(\delta\xi)^\beta y}_{3} = 0$$

$1 = O(\varepsilon/\delta)$
$2 = O(\delta^\alpha)$
$3 = O(\delta^{\beta+1})$

1 should balance with the larger of 2 and 3.

Suppose 1 and 2 balance: $2 \gg 3 \Rightarrow \alpha < \beta + 1$

$$\frac{\varepsilon}{\delta} \sim \delta^\alpha \Rightarrow \boxed{\delta(\varepsilon) = \varepsilon^{\frac{1}{\alpha+1}} \text{ when } \alpha < \beta + 1}$$

Suppose 1 and 3 balance: $3 \gg 2 \Rightarrow \alpha > \beta + 1$

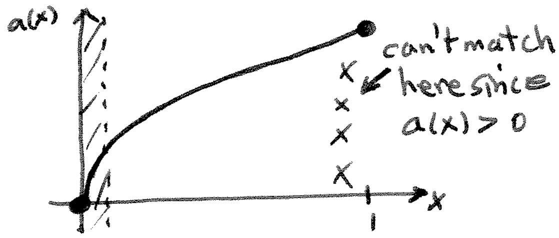
$$\frac{\varepsilon}{\delta} \sim \delta^{\beta+1} \Rightarrow \boxed{\delta(\varepsilon) = \varepsilon^{\frac{1}{\beta+2}} \text{ when } \alpha > \beta + 1}$$

Note: If $\alpha = \beta + 1$, then all three terms balance and the resulting equation $y_{\xi\xi} + \xi^\alpha y_\xi + \xi^\beta y = 0$ is as difficult to solve as the original equation. The method of matched asymptotic expansions is not helpful.

Example: $\varepsilon y'' + \sqrt{x}\, y' - y = 0$; $y(0)=0$, $y(1)=e^2$; $0 < \varepsilon << 1$

$a(x) = x^{1/2}$, $\alpha = \frac{1}{2}$

$b(x) = -1$, $\beta = 0$ $\Longrightarrow$ $\alpha < \beta + 1$ $\Longrightarrow$ $\underline{\underline{\delta(\varepsilon)}} = \varepsilon^{\frac{1}{1/2+1}} = \underline{\underline{\varepsilon^{2/3}}}$



Expect a layer of thickness $O(\varepsilon^{2/3})$ at the left boundary, with balancing occurring between the first two terms.

Outer Solution: $Y_{OUT} = Y_0(x) + \varepsilon Y_1(x) + \cdots$, $Y_{OUT}(1) = e^2$

Leading Order: $\sqrt{x}\, Y_0' - Y_0 = 0$, $Y_0(1) = e^2$

$\Rightarrow Y_0(x) = C e^{2\sqrt{x}}$

$Y_0(1) = e^2 \Rightarrow \underline{C = 1} \Longrightarrow \boxed{Y_0(x) = e^{2\sqrt{x}}}$

Inner Solution: $\boxed{\xi = \dfrac{x}{\delta(\varepsilon)}}$ $\Rightarrow \dfrac{\varepsilon}{\delta^2} Y_{\xi\xi} + (\delta\xi)^{1/2} \dfrac{1}{\delta} Y_\xi - Y = 0$

$$\underset{①}{\varepsilon Y_{\xi\xi}} + \underset{②}{\delta(\delta\xi)^{1/2} Y_\xi} - \underset{③}{\delta^2 Y} = 0$$

① and ② balance

$\varepsilon \sim \delta^{3/2}$ $\Longrightarrow$ $\boxed{Y_{\xi\xi} + \xi^{1/2} Y_\xi - \varepsilon^{1/3} Y = 0}$

Let $\boxed{\delta(\varepsilon) = \varepsilon^{2/3}}$

$$Y_{IN} = y_0(\xi) + \varepsilon y_1(\xi) + \cdots, \quad Y_{IN}(0) = 0$$

Leading Order: $y_{0\xi\xi} + \xi^{1/2} y_{0\xi} = 0$, $y_0(0) = 0$

$$\frac{y_{0\xi\xi}}{y_{0\xi}} = -\xi^{1/2}$$

$$y_{0\xi} = c_1 e^{-\frac{2}{3}\xi^{3/2}}$$

$$\int_0^\xi y_{0\xi}\, d\xi = c_1 \int_0^\xi e^{-2/3\,\xi^{3/2}}\, d\xi$$

$$\Longrightarrow \boxed{y_0(\xi) = c_1 \int_0^\xi e^{-2/3\,\xi^{3/2}}\, d\xi}$$

Primitive Matching

$$\lim_{x\to 0} Y_0(x) = \lim_{\xi\to\infty} y_0(\xi)$$

$$\lim_{x\to 0} e^{2\sqrt{x}} = \lim_{\xi\to\infty} C_1 \int_0^{\xi} e^{-\frac{2}{3}\xi^{3/2}}\, d\xi$$

$$1 = C_1 \int_0^{\infty} e^{-2/3\,\xi^{3/2}}\, d\xi = \text{Common Part}$$

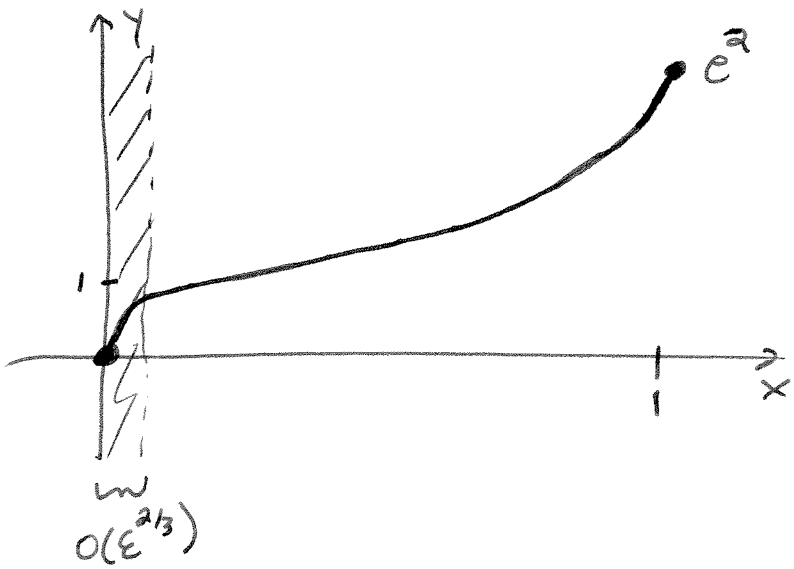
$$\boxed{C_1 = \left[\int_0^{\infty} e^{-\frac{2}{3}\xi^{3/2}}\, d\xi\right]^{-1}} \qquad \boxed{\text{Common Part} = 1}$$

Composite Solution

$$Y_{comp} = Y_{OUT} + Y_{IN} - \begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}$$

$$\sim e^{2\sqrt{x}} + C_1 \int_0^{\xi} e^{-2/3\,\xi^{3/2}}\, d\xi - 1$$

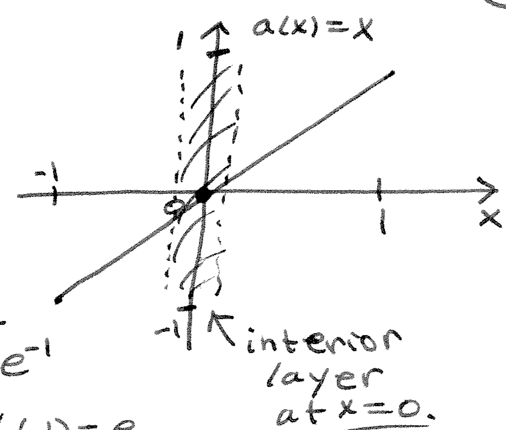
$$\boxed{Y_{comp}(x) \sim e^{2\sqrt{x}} - 1 + \frac{\int_0^{x/\varepsilon^{2/3}} e^{-2/3\,\xi^{3/2}}\, d\xi}{\int_0^{\infty} e^{-2/3\,\xi^{3/2}}\, d\xi}}$$



Interior Layers

Example: $\varepsilon y'' + x y' + x y = 0; \quad 0 < \varepsilon \ll 1$

$$y(-1) = e, \quad y(1) = 2e^{-1}$$



Outer Solution: $Y_{OUT} = Y_0(x) + \varepsilon Y_1(x) + \dots, \quad Y_{OUT}(-1) = e, \quad Y_{OUT}(1) = 2e^{-1}$

Leading Order: $x Y_0' + x Y_0 = 0, \; x \neq 0; \quad Y_0(-1) = e, \; Y_0(1) = 2e^{-1}$

$$Y_0' + Y_0 = 0$$

$$Y_0(x) = C_0 e^{-x}$$

$$Y_0(-1) = C_0^- e = e \Rightarrow C_0^- = 1$$

$$Y_0(1) = C_0^+ e^{-1} = 2e^{-1} \Rightarrow C_0^+ = 2$$

$$\Longrightarrow \boxed{Y_0(x) = \begin{cases} e^{-x}, & x < 0 \\ 2e^{-x}, & x > 0 \end{cases}}$$

Inner Solution: (at $x = 0$)

$$\boxed{\xi = \frac{x}{\delta(\varepsilon)}} \Rightarrow \frac{\varepsilon}{\delta^2} Y_{\xi\xi} + \delta \xi \frac{1}{\delta} Y_\xi + \delta \xi Y = 0$$

$$\delta(\varepsilon) = \varepsilon^{1/2} \Rightarrow Y_{\xi\xi} + \xi Y_\xi + \varepsilon^{1/2} \xi Y = 0$$

$$Y_{IN} = y_0(\xi) + \varepsilon y_1(\xi) + \dots$$

Leading Order: $y_{0\xi\xi} + \xi y_{0\xi} = 0$

$$\frac{y_{0\xi\xi}}{y_{0\xi}} = -\xi$$

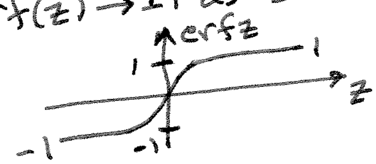
$$y_{0\xi} = C_1 e^{-\xi^2/2}$$

$$\int_0^\xi y_{0\xi} \, d\xi = C_1 \int_0^\xi e^{-s^2/2} ds$$

Recall: Error Function

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

$\operatorname{erf}(z) \to \pm 1$ as $z \to \pm\infty$



Let $t = \frac{s}{\sqrt{2}}$ $\Rightarrow y_0(\xi) - y_0(0) = C_1 \int_0^{\xi/\sqrt{2}} e^{-t^2} \cdot \sqrt{2} \, dt = C_1 \operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right)$

$$\boxed{y_0(\xi) = y_0(0) + C_1 \operatorname{erf}\left(\frac{\xi}{\sqrt{2}}\right)}$$

$y_0(0)$ and C_1 are the integration constants to be determined.

Matching to the Left ($x<0$)

$$\lim_{x\to 0^-} Y_0(x) = \lim_{\xi\to -\infty} y_0(\xi)$$

$$\lim_{x\to 0^-} e^{-x} = \lim_{\xi\to-\infty}\left[y_0(0) + c_1 \operatorname{erf}\left(\tfrac{\xi}{\sqrt{2}}\right)\right]$$

$$1 = y_0(0) + c_1(-1) = \text{CommoPart for } x<0$$

$$\boxed{y_0(0) - c_1 = 1} \qquad \boxed{\text{Common Part} = 1,\ x<0}$$

Matching to the Right ($x>0$)

$$\lim_{x\to 0^+} Y_0(x) = \lim_{\xi\to\infty} y_0(\xi)$$

$$\lim_{x\to 0^+} 2e^{-x} = \lim_{\xi\to\infty}\left[y_0(0) + c_1 \operatorname{erf}\left(\tfrac{\xi}{\sqrt{2}}\right)\right]$$

$$2 = y_0(0) + c_1(1) = \text{Common Part for } x>0$$

$$\boxed{y_0(0) + c_1 = 2} \qquad \boxed{\text{Common Part} = 2,\ x>0}$$

$$y_0(0) = \tfrac{3}{2}, \quad c_1 = \tfrac{1}{2}$$

We have $Y_{OUT} \sim Y_0(x) = \begin{cases} e^{-x}, & x<0 \\ 2e^{-x}, & x>0 \end{cases}$ $\quad$ Common Part $= \begin{cases} 1, & x<0 \\ 2, & x>0 \end{cases}$

$$Y_{IN} \sim y_0(\xi) = \tfrac{1}{2}\left[3 + \operatorname{erf}\left(\tfrac{\xi}{\sqrt{2}}\right)\right]$$

Composite Solution

$$Y_{comp} = Y_{OUT} + Y_{IN} - \begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix} \sim Y_0(x) + y_0(\xi) - \begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}$$

$$\sim \begin{cases} e^{-x}, & x<0 \\ 2e^{-x}, & x>0 \end{cases} + \tfrac{1}{2}\left[3+\operatorname{erf}\left(\tfrac{\xi}{\sqrt{2}}\right)\right] - \begin{cases} 1, & x<0 \\ 2, & x>0 \end{cases}$$

$$\sim \begin{cases} e^{-x} + \tfrac{1}{2}[3+\operatorname{erf}\left(\tfrac{\xi}{\sqrt{2}}\right) - 1, & x<0 \\ 2e^{-x} + \tfrac{1}{2}[3+\operatorname{erf}\left(\tfrac{\xi}{\sqrt{2}}\right) - 2, & x>0 \end{cases} \qquad \xi = \frac{x}{\sqrt{\varepsilon}}$$

$$\boxed{Y_{comp}(x) = \begin{cases} e^{-x} + \tfrac{1}{2}\left[1+\operatorname{erf}\left(\tfrac{x}{\sqrt{2\varepsilon}}\right)\right], & x<0 \\ 2e^{-x} - \tfrac{1}{2}\left[1-\operatorname{erf}\left(\tfrac{x}{\sqrt{2\varepsilon}}\right)\right], & x>0 \end{cases}}$$

Note: y_{comp} is continuous at $x=0$.

$$y_{comp}(0^-) = y_{comp}(0^+) = 3/2$$

y_{comp} is <u>not</u> continuously differentiable at $x=0$.

$$y'_{comp}(0^-) \neq y'_{comp}(0^+)$$

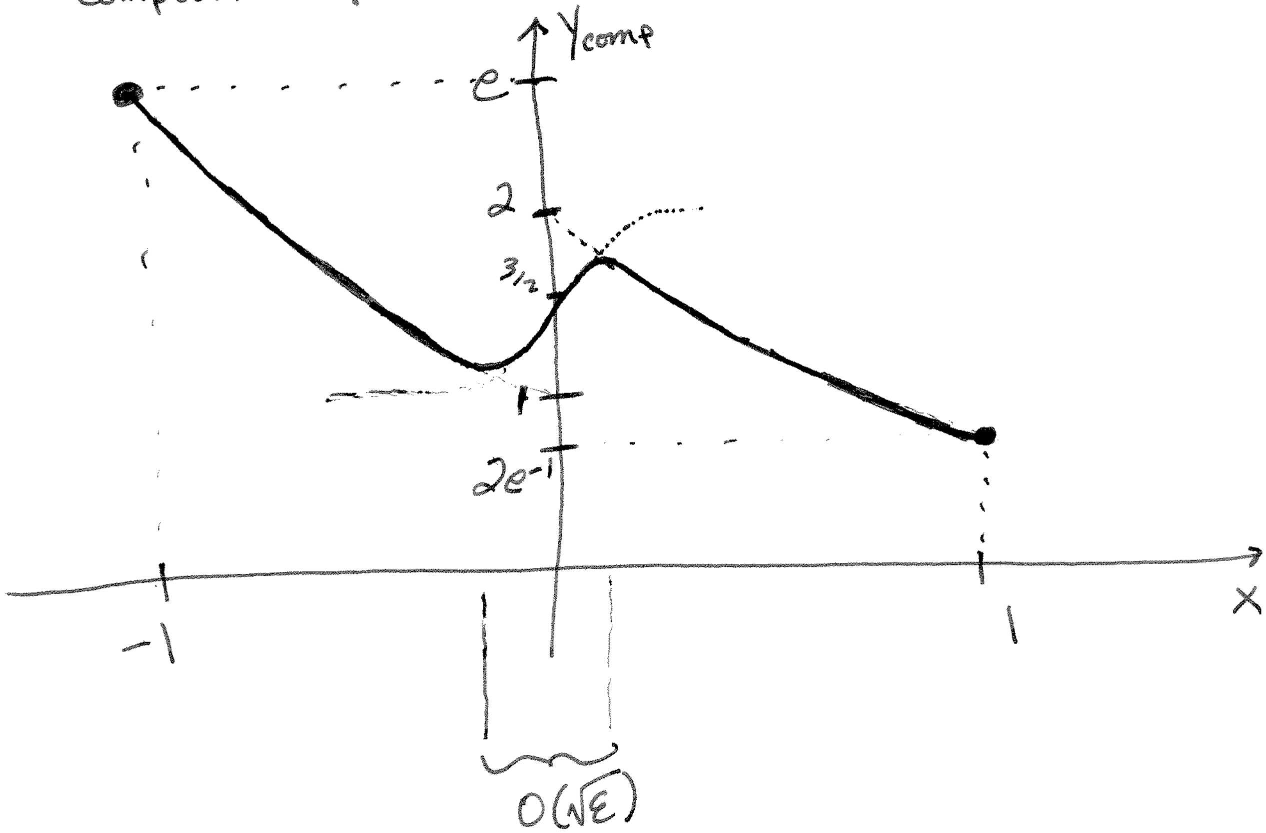
Though the left and right derivatives at $x=0$ are not equal, they <u>do</u> agree to leading order.

$$\left(\frac{d}{dx}\,\mathrm{erf}\left(\frac{x}{\sqrt{2\varepsilon}}\right) = \frac{d}{dx}\frac{2}{\sqrt{\pi}}\int_0^{x/\sqrt{2\varepsilon}} e^{-s^2}\,ds = \frac{2}{\sqrt{\pi}} e^{-x^2/2\varepsilon}\cdot\frac{1}{\sqrt{2\varepsilon}} = \sqrt{\frac{2}{\varepsilon\pi}}\, e^{-x^2/2\varepsilon} \right)$$

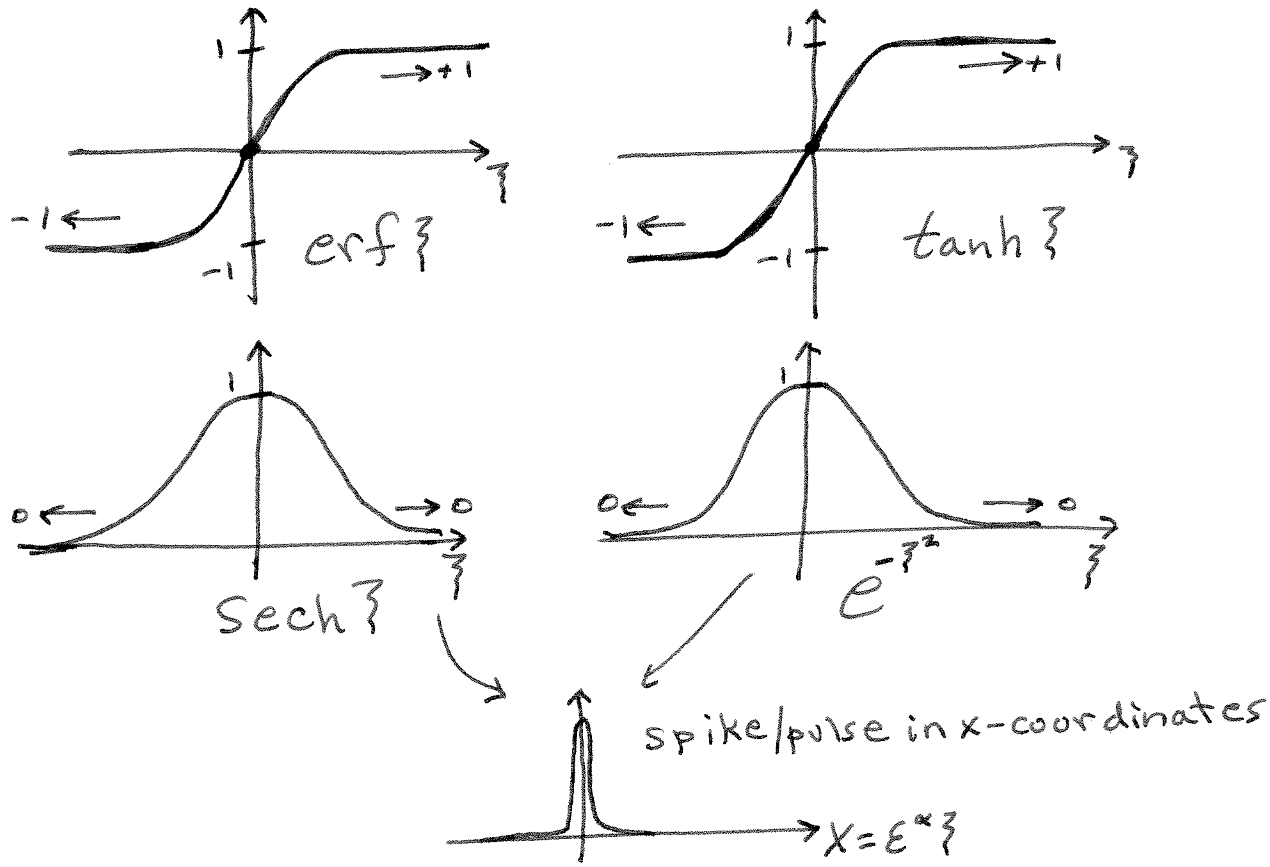
$$y'_{comp}(x) = \begin{cases} -e^{-x} + \frac{1}{2}\sqrt{\frac{2}{\varepsilon\pi}}\, e^{-x^2/2\varepsilon}, & x<0 \\ -2e^{-x} + \frac{1}{2}\sqrt{\frac{2}{\varepsilon\pi}}\, e^{-x^2/2\varepsilon}, & x>0 \end{cases}$$

$$y'_{comp}(0) = \begin{cases} -1 + \frac{1}{\sqrt{2\varepsilon\pi}}, & x<0 \\ -2 + \frac{1}{\sqrt{2\varepsilon\pi}}, & x>0 \end{cases} = \frac{1}{\sqrt{2\varepsilon\pi}} + O(1)$$

Closer agrement can be achieved by finding a higher-order composite expansion.

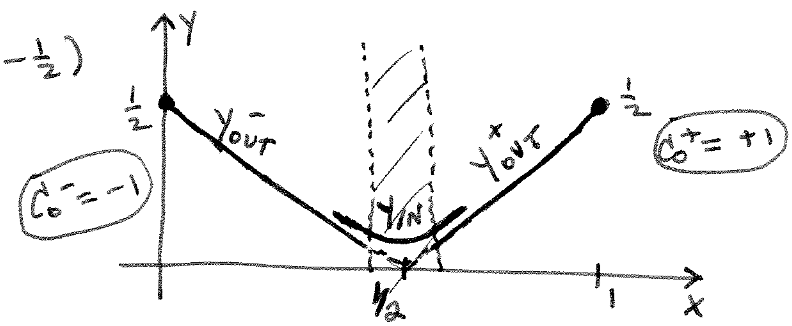


Since interior inner solutions need to match to both sides, they must exhibit limiting behavior as $\zeta \to \pm\infty$. Consequently, interior inner solutions may involve the following functions.



Interior Corner Layers

e.g. $Y_{OUT}(x) \sim C_0 \left(x - \frac{1}{2}\right)$



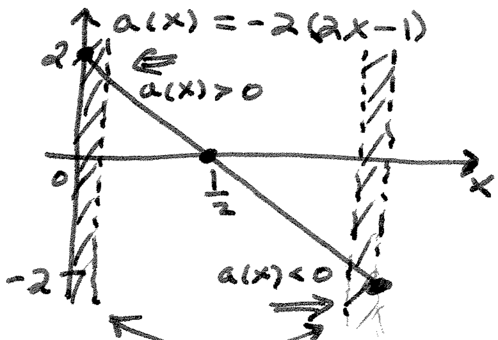
Here, $Y_{IN} \to +\infty$ as $\zeta \to \pm\infty$ and primitive matching fails. However, matching can be achieved through more sophisticated techniques, which can effectively match the derivatives of the outer and inner solutions. For the above figure,

$$\text{Common Part} = \begin{cases} -x + \frac{1}{2}, & x < \frac{1}{2} \\ x - \frac{1}{2}, & x > \frac{1}{2} \end{cases} = Y_{OUT}, \text{ and therefore, } Y_{comp} = Y_{IN}.$$

Layer at Each Boundary

Example: $\varepsilon y'' - 2(2x-1)y' + 4y = 0 \; ; \; y(0)=1 \; ; \; y(1)=2 \; ; \; 0<\varepsilon \ll 1$

Outer Solution: $Y_{OUT} = Y_0(x) + \varepsilon Y_1(x) + \cdots$



Matching is possible at both boundaries.

Leading order: $-2(2x-1)Y_0' + 4Y_0 = 0$

$$\frac{Y_0'}{Y_0} = \frac{2}{2x-1}$$

$$\ln|Y_0| = \ln|2x-1| + C$$

$$\boxed{Y_0(x) = C_0(2x-1)}$$

Try a boundary layer at $x=0$.

Inner Solution: $\xi = \frac{x}{\varepsilon}$, $x = \varepsilon\xi$ $\Rightarrow$ $\frac{\varepsilon}{\varepsilon^2} Y_{\xi\xi} - 2(2\varepsilon\xi - 1)\frac{1}{\varepsilon} Y_\xi + 4y = 0$

$$\boxed{Y_{\xi\xi} + 2(1-2\varepsilon\xi)Y_\xi + \varepsilon 4 y = 0}$$

$$Y_{IN} = y_0(\xi) + \varepsilon y_1(\xi) + \cdots \; ; \; Y_{IN}(0) = 1$$

Leading order: $y_{0\xi\xi} + 2y_{0\xi} = 0 \; ; \; y_0(0) = 1$

$$y_0 = C_1 + C_2 e^{-2\xi}$$

$y_0(0) = C_1 + C_2 = 1$

$C_2 = 1 - C_1$ $\Rightarrow$ $\boxed{y_0(\xi) = C_1 + (1-C_1)e^{-2\xi}}$

Note: A single boundary layer at $x=0$ is sufficient to determine a composite solution which satisfies the ODE and the boundary conditions to leading order.

In this case, $Y_0(1) = C_0 = 2 \implies Y_0(x) = 2(2x-1)$

Match

$$\lim_{x\to 0^+} Y_0(x) = \lim_{\xi\to+\infty} y_0(\xi)$$

$$-2 = C_1 = \text{Common Part}$$

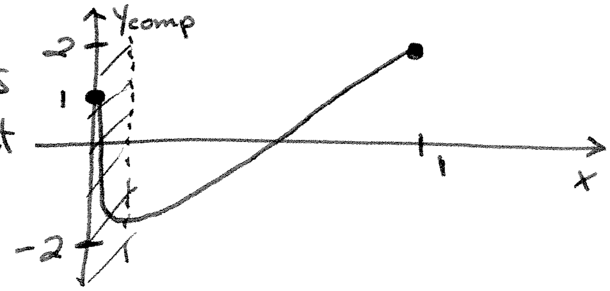
$$y_0(\xi) = -2+3e^{-2\xi}$$

$$Y_{comp} \sim Y_0(x) + y_0(\xi) - \left(\text{Common Part}\right)$$

$$\sim 2(2x-1) + \left[-2+3e^{-2\xi}\right] - (-2)$$

$$\boxed{Y_{comp}(x) = 2(2x-1) + 3e^{-2x/\varepsilon}}$$

However, this composite solution is not unique. Considering a layer at the other boundary leads to other possiblilities.



Try a boundary layer at $x=1$

Inner Solution: $\eta = \dfrac{x-1}{\varepsilon}$, $x = 1+\varepsilon\eta$

$$\implies \frac{\varepsilon}{\varepsilon^2} Y_{\eta\eta} - 2\left(2(1+\varepsilon\eta)-1\right)\frac{1}{\varepsilon} Y_\eta + 4y = 0$$

$$\boxed{Y_{\eta\eta} - 2(1+2\varepsilon\eta)Y_\eta + 4\varepsilon y = 0}$$

$$Y_{IN1} = Y_0(\eta) + \varepsilon Y_1(\eta) + \cdots \; ; \; Y_{IN}(0) = 2$$

Leading order: $Y_{0\eta\eta} - 2Y_{0\eta} = 0 \; ; \; Y_0(0) = 2$

$$\implies Y_0 = C_3 + C_4 e^{2\eta}$$

$$Y_0(0) = C_3 + C_4 = 2$$

$$C_4 = 2 - C_3 \implies \boxed{Y_0(\eta) = C_3 + (2-C_3)e^{2\eta}}$$

We have $\quad Y_{OUT} \sim Y_0(x) = C_0(2x-1)$

$Y_{IN0} \sim y_0(\xi) = C_1 + (1-C_1)e^{-2\xi}$

$Y_{IN1} \sim \mathcal{Y}_0(\eta) = C_3 + (2-C_3)e^{2\eta}$

Matching

At the left boundary ($x=0$)

$$\lim_{x\to 0^+} Y_0(x) = \lim_{\xi\to+\infty} y_0(\xi)$$

$$-C_0 = C_1 = \binom{\text{Common}}{\text{Part}}_0$$

At the right boundary ($x=1$)

$$\lim_{x\to 1^-} Y_0(x) = \lim_{\eta\to-\infty} \mathcal{Y}_0(\eta)$$

$$C_0 = C_3 = \binom{\text{Common}}{\text{Part}}_1$$

$\Rightarrow$

$$\left.\begin{aligned} Y_{OUT} &\sim Y_0(x) = C_0(2x-1) \\ Y_{IN0} &\sim y_0(\xi) = -C_0 + (1+C_0)e^{-2\xi} \\ Y_{IN1} &\sim \mathcal{Y}_0(\eta) = C_0 + (2-C_0)e^{2\eta} \end{aligned}\right\}$$

These satisfy all boundary and matching conditions for each C_0.

$$\binom{\text{Common}}{\text{Part}}_0 = -C_0, \quad \binom{\text{Common}}{\text{Part}}_1 = C_0$$

Composite Solution

$$\boxed{Y_{comp} = Y_{OUT} + \left[Y_{IN0} - \binom{\text{Common}}{\text{Part}}_0\right] + \left[Y_{IN1} - \binom{\text{Common}}{\text{Part}}_1\right]}$$

corrections at each boundary

$$Y_{comp} \sim Y_0(x) + \left[y_0(\xi) - \binom{\text{Common}}{\text{Part}}_0\right] + \left[\mathcal{Y}_0(\eta) - \binom{\text{Common}}{\text{Part}}_1\right]$$

$$\sim C_0(2x-1)\left[\cancel{-C_0} + (1+C_0)e^{-2\xi} - \cancel{(-C_0)}\right] + \left[\cancel{C_0} + (2-C_0)e^{2\eta} - \cancel{C_0}\right]$$

$$\boxed{Y_{comp}(x) \sim C_0(2x-1) + (1+C_0)e^{-2x/\varepsilon} + (2-C_0)e^{-2\frac{1-x}{\varepsilon}}}$$

$$y_{comp}(x) \sim C_0(2x-1) + (1+C_0)e^{-2x/\varepsilon} + (2-C_0)e^{-2\frac{1-x}{\varepsilon}}, \quad \underline{C_0 = ?}$$

The composite solution is uniformly valid on $[0,1]$ and it satisfies the ODE and the boundary conditions to leading order for all C_0.

The method of matched asymptotic expansions yields a one parameter (C_0) family of possible composite solutions. Only one member corresponds to the true solution. Selection of the correct member may be possible by other means.

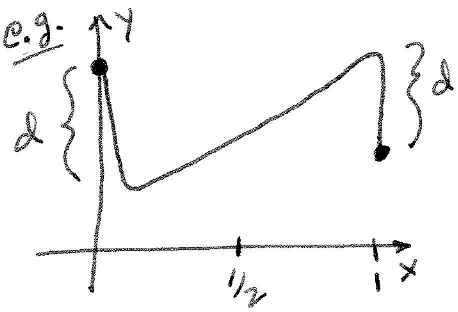
<u>e.g.</u> 1. Manipulation of the ODE may lead to an additional condition on the solution.

<u>e.g.</u> If the ODE can be written as $\frac{d}{dx}\left[f(x,y;\varepsilon)\frac{d}{dx}g(x,y;\varepsilon)\right] = 0$, then integrating from $x=0$ to $x=1$ gives an algebraic relation in terms of $y(0)$, $y'(0)$, $y(1)$, and $y'(1)$, which determines C_0.

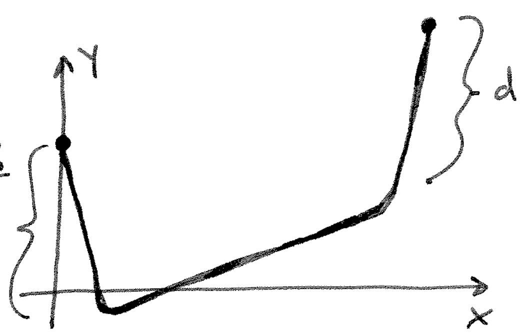
2. In a physical application, the configuration of the system may suggest a boundary layer on one side or the other.

3. It may be possible to show that the solution is symmetrical in some sense. Properties of symmetry can be exploited to determine C_0.

<u>e.g.</u>



<u>OR</u>



Determing C_0: Symmetry Considerations

$$\varepsilon y'' - 2(2x-1)y' + 4y = 0\,; \quad \begin{matrix} y(0)=1 \\ y(1)=2 \end{matrix}$$

Let $\boxed{u = y - x + \frac{1}{2}}$ $\Rightarrow$ $u(0) = y(0) - 0 + \frac{1}{2} = 3/2$

$u(1) = y(1) - 1 + \frac{1}{2} = 3/2$

$y = u + x - \frac{1}{2}$

$y' = u' + 1$

$y'' = u''$

$$\varepsilon u'' - 2(2x-1)(u'+1) + 4(u + x - \tfrac{1}{2}) = 0$$

$$\boxed{\begin{matrix} \varepsilon u'' - 2(2x-1)u' + 4u = 0 \\ u(0) = u(1) = 3/2 \end{matrix}} \quad (1)$$

Let $\boxed{t = 1 - x}$ $\Rightarrow$ $\varepsilon u_{tt} - 2(2(1-t)-1)(-u_t) + 4u = 0$

$u' = -u_t$

$u'' = u_{tt}$

$$\boxed{\begin{matrix} \varepsilon u_{tt} - 2(2t-1)u_t + 4u = 0 \\ u(0) = u(1) = 3/2 \end{matrix}} \quad (2)$$

Notice that $u(x)$ and $u(t)$ satisfy the same boundary value problem ((1)=(2)).

$\Rightarrow$ $\boxed{u(x) = u(1-x)}$, or equivalently, $u(\frac{1}{2}+x) = u(\frac{1}{2}-x)$.

Therefore, $u(x)$ is symmetric about $x = \frac{1}{2}$.

$$u(x) = u(1-x)$$

$$y(x) - x + \cancel{\tfrac{1}{2}} = y(1-x) - (1-x) + \cancel{\tfrac{1}{2}}$$

$$y(x) = y(1-x) + (2x-1)$$

Plug in the composite expansion to determine C_0.

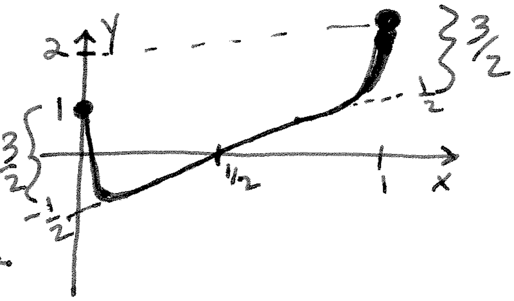
$$C_0(2x-1) + (1+C_0)e^{-2x/\varepsilon} + (2-C_0)e^{-2\frac{1-x}{\varepsilon}} = C_0\left(2(1-x)-1\right) + (1+C_0)e^{-2\frac{1-x}{\varepsilon}} + (2-C_0)e^{-2x/\varepsilon} + (2x-1)$$

$$(2x-1)(C_0 + C_0 - 1) + e^{-2x/\varepsilon}(1 + C_0 - 2 + C_0) + e^{-2\frac{1-x}{\varepsilon}}(2 - C_0 - 1 - C_0) = 0$$

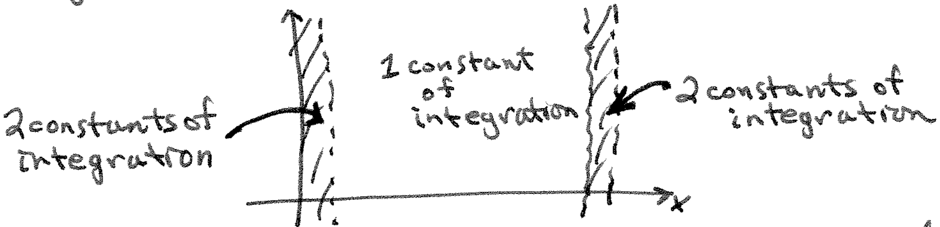
$$(2C_0 - 1)\left[(2x-1) + e^{-2x/\varepsilon} - e^{-2\frac{1-x}{\varepsilon}}\right] = 0 \quad \text{for all } x \in [0,1]$$

$$\Rightarrow \boxed{C_0 = \frac{1}{2}}$$

$$\boxed{y_{comp}(x) = \left(x - \frac{1}{2}\right) + \frac{3}{2}\left(e^{-2x/\varepsilon} + e^{-2\frac{1-x}{\varepsilon}}\right)}$$



The boundary layer problem in the above example is under-determined in that there are not enough conditions to uniquely determine all of the integration constants.



There are 5 integration constants to be determined by 4 conditions, (two boundary conditions and two matching conditions).

In the example, each matching condition determines exactly one constant.

However, it is possible for a matching condition to determine <u>two</u> constants.

e.g. Suppose $Y_{OUT} \sim C_0(2x-1)$ and $Y_{INO} \sim C_1 e^{2\xi} + (1-C_1)e^{-2\xi}$.

Match: $\lim_{x\to 0^+} C_0(2x-1) = \lim_{\xi\to+\infty}\left[C_1 e^{2\xi} + (1-C_1)e^{-2\xi}\right]$

$C_1 = 0$ for boundedness

$\Longrightarrow \quad -C_0 = 0$

The matching condition gives $\begin{cases} C_1 = 0 & \text{for boundedness of } Y_{INO} \\ C_0 = 0 & \text{for matching} \end{cases}$

and two constants are determined.

If this occurs at both boundaries, the boundary layer problem may (but not necessarily) be over-determined.

With luck, all integration constants will be uniquely determined.

This completes the analysis of the equation

$$\varepsilon y'' + a(x)y' + b(x)y = 0.$$

Example: Initial Layer / System of ODEs

$$\dot{u} = -u \,, \quad u(0)=1 \qquad ; \quad 0<\varepsilon \ll 1$$
$$\varepsilon \dot{v} = u - v \,, \quad v(0) = 0$$

Find a leading order composite expansion for $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$.

Naive Expansion: $\vec{u} = \vec{u}_0 + \varepsilon \vec{u}_1 + \cdots = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \varepsilon \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \cdots$

Leading Order: $\dot{u}_0 = -u_0 \,, \quad u_0(0) = 1$
$0 = u_0 - v_0 \,, \quad v_0(0) = 0$

$u_0 = v_0 = Ce^{-t}$ Can't satisfy both initial conditions

The leading order system is over-determined and an initial layer must be considered.

Outer Solution: $\vec{U}_{OUT} = \vec{U}_0 + \varepsilon \vec{U}_1 + \cdots = \begin{pmatrix} U_0 \\ V_0 \end{pmatrix} + \varepsilon \begin{pmatrix} U_1 \\ V_1 \end{pmatrix} + \cdots$

Leading Order: $\dot{U}_0 = -U_0$
$0 = U_0 - V_0 \implies \boxed{U_0(t) = V_0(t) = C_0 e^{-t} \text{ or } \vec{U}_0(t) = C_0 e^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

Inner Solution: $\boxed{\tau = \frac{t}{\varepsilon}} \implies u_\tau = -\varepsilon u \,, \quad u(0) = 1$
$v_\tau = u - v \,, \quad v(0) = 0$

$$\vec{u}_{IN} = \vec{u}_0 + \varepsilon \vec{u}_1 + \cdots = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + \varepsilon \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} + \cdots$$

Leading Order: $u_{0\tau} = 0 \,, \quad u_0(0) = 1$
$v_{0\tau} = u_0 - v_0 \,, \quad v_0(0) = 0$

$$\implies \boxed{u_0(\tau) = 1 \,, \quad v_0(\tau) = 1 - e^{-\tau} \quad \text{or} \quad \vec{u}_0(\tau) = \begin{pmatrix} 1 \\ 1 - e^{-\tau} \end{pmatrix}}$$

We have $\vec{U}_{OUT} \sim U_0(t) = c_0 e^{-t}\begin{pmatrix}1\\1\end{pmatrix}$

$\vec{U}_{IN} \sim u_0(\tau) = \begin{pmatrix}1\\1-e^{-\tau}\end{pmatrix}$

Primitive Matching

$$\lim_{t\to 0^+} U_0(t) = \lim_{\tau\to+\infty} \vec{u}_0(\tau)$$

$$\lim_{t\to 0^+} c_0 e^{-t}\begin{pmatrix}1\\1\end{pmatrix} = \lim_{\tau\to+\infty}\begin{pmatrix}1\\1-e^{-\tau}\end{pmatrix}$$

$$c_0\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$\Longrightarrow \quad c_0 = 1 \qquad \overrightarrow{\begin{pmatrix}\text{Common}\\\text{Part}\end{pmatrix}}_0 = \begin{pmatrix}1\\1\end{pmatrix}$$

Composite Expansion

$$\vec{U}_{comp} = \vec{U}_{OUT} + \vec{U}_{IN} - \overrightarrow{\begin{pmatrix}\text{Common}\\\text{Part}\end{pmatrix}}$$

$$\sim \vec{U}_0(t) + \vec{u}_0(\tau) - \overrightarrow{\begin{pmatrix}\text{Common}\\\text{Part}\end{pmatrix}}_0$$

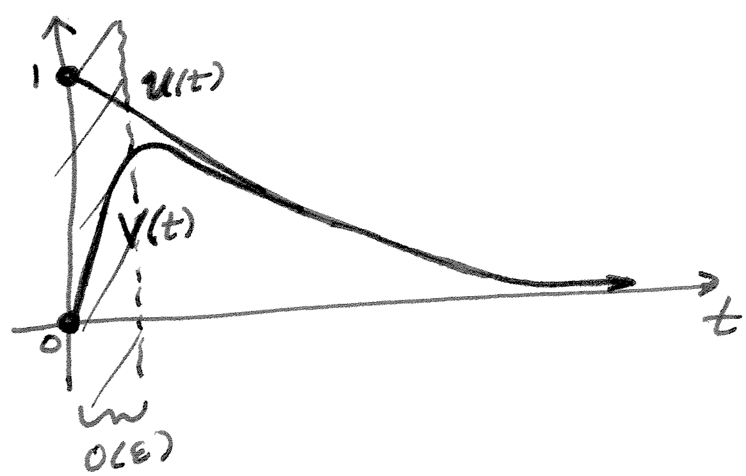
$$\sim e^{-t}\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\1-e^{-\tau}\end{pmatrix} - \begin{pmatrix}1\\1\end{pmatrix}$$

$$\sim \begin{pmatrix}e^{-t}\\e^{-t}-e^{-\tau}\end{pmatrix}$$

$$\boxed{U_{comp}(t) \sim \begin{pmatrix}e^{-t}\\e^{-t}-e^{-t/\varepsilon}\end{pmatrix}}$$

$u(t) = e^{-t}$

$v(t) = e^{-t} - e^{-t/\varepsilon}$



Example: Nonlinear System of ODEs / Implicit Composite Expansion

$$\dot{x} = -x + (x+\alpha-\beta)y, \quad x(0)=1$$
$$\varepsilon \dot{y} = x - (x+\alpha)y, \quad y(0)=0 \quad ; \quad 0<\varepsilon \ll 1 \quad \alpha,\beta \in \mathbb{R}; \ \alpha,\beta>0$$

Outer Solution: $\vec{X}_{out} = \vec{X}_0(t) + \varepsilon \vec{X}_1(t) + \cdots = \begin{pmatrix} X_0(t) \\ Y_0(t) \end{pmatrix} + \varepsilon \begin{pmatrix} X_1(t) \\ Y_1(t) \end{pmatrix} + \cdots$

Leading Order: $\dot{X}_0 = -X_0 + (X_0+\alpha-\beta)Y_0$

$0 = X_0 - (X_0+\alpha)Y_0 \implies \boxed{Y_0(t) = \dfrac{X_0(t)}{X_0(t)+\alpha}}$

$$\dot{X}_0 = -X_0 + (X_0+\alpha-\beta)\frac{X_0}{X_0+\alpha} = -\cancel{X_0} + \cancel{X_0} - \frac{\beta X_0}{X_0+\alpha}$$

$$\int\left(1+\frac{\alpha}{X_0}\right)dX_0 = -\beta\int dt$$

$$\boxed{X_0(t) + \alpha\ln|X_0(t)| = -\beta t + C_0}$$ Implicit General Solution

The initial conditions cannot be satisfied since $Y_0(0) \neq \dfrac{X_0(0)}{X_0(0)+\alpha}$.

Inner Solution: $\boxed{\tau = t/\varepsilon} \implies X_\tau = \varepsilon[-X + (X+\alpha-\beta)Y], \quad X(0)=1$

$Y_\tau = X - (X+\alpha)Y, \quad Y(0)=0$

$$\vec{X}_{in} = \vec{x}_0(\tau) + \varepsilon\vec{x}_1(\tau) + \cdots = \begin{pmatrix} x_0(\tau) \\ y_0(\tau) \end{pmatrix} + \varepsilon\begin{pmatrix} x_1(\tau) \\ y_1(\tau) \end{pmatrix} + \cdots$$

Leading order: $x_{0\tau} = 0, \quad x_0(0) = 1 \implies \boxed{x_0(\tau) = 1}$

$y_{0\tau} = x_0 - (x_0+\alpha)y_0, \quad y_0(0)=0$

$y_{0\tau} = 1 - (1+\alpha)y_0$

$y_{0\tau} + (1+\alpha)y_0 = 1$

Integrating Factor $\mu = e^{(1+\alpha)\tau}$

$$e^{(1+\alpha)\tau}y_0 = \int e^{(1+\alpha)\tau}d\tau = \frac{1}{1+\alpha}e^{(1+\alpha)\tau} + C_1$$

$$y_0(\tau) = \frac{1}{1+\alpha} + C_1 e^{-(1+\alpha)\tau}$$

$y_0(0)=0 \Rightarrow C_1 = \dfrac{-1}{1+\alpha} \implies \boxed{y_0(\tau) = \dfrac{1-e^{-(1+\alpha)\tau}}{1+\alpha}}$

We have **Outer:** $X_0(t) + \alpha \ln|X_0(t)| = C_0 - \beta t$

$$Y_0(t) = \frac{X_0(t)}{X_0(t)+\alpha}$$

Inner: $x_0(\tau) = 1$

$$y_0(\tau) = \frac{1 - e^{-(1+\alpha)\tau}}{1+\alpha}$$

Primitive Matching

$$\lim_{t \to 0^+} \begin{pmatrix} X_0(t) \\ Y_0(t) \end{pmatrix} = \lim_{\tau \to +\infty} \begin{pmatrix} x_0(\tau) \\ y_0(\tau) \end{pmatrix}$$

$$\begin{pmatrix} X_0(0) \\ Y_0(0) \end{pmatrix} = \lim_{\tau \to +\infty} \begin{pmatrix} 1 \\ \frac{1 - e^{-(1+\alpha)\tau}}{1+\alpha} \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{1+\alpha} \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{matrix} X_0(0) = 1 \\ Y_0(0) = \frac{1}{1+\alpha} \end{matrix}} \quad Y_0(0) = \frac{X_0(0)}{X_0(0)+\alpha} \checkmark \quad \boxed{\begin{pmatrix} \overrightarrow{\text{Common}} \\ \text{Part} \end{pmatrix}_0 = \begin{pmatrix} 1 \\ \frac{1}{1+\alpha} \end{pmatrix}}$$

At $t=0$, $X_0(0) + \alpha \ln|X_0(0)| = C_0 - \beta \cdot 0$

$$1 + \cancel{\alpha \ln 1} = C_0 - 0$$

$$C_0 = 1$$

Composite Expansion

$$\vec{X}_{comp} = \vec{X}_{OUT} + \vec{X}_{IN} - \begin{pmatrix} \overrightarrow{\text{Common}} \\ \text{Part} \end{pmatrix}$$

$$\sim \begin{pmatrix} X_0(t) \\ Y_0(t) \end{pmatrix} + \begin{pmatrix} x_0(\tau) \\ y_0(\tau) \end{pmatrix} - \begin{pmatrix} \overrightarrow{\text{Common}} \\ \text{Part} \end{pmatrix}_0$$

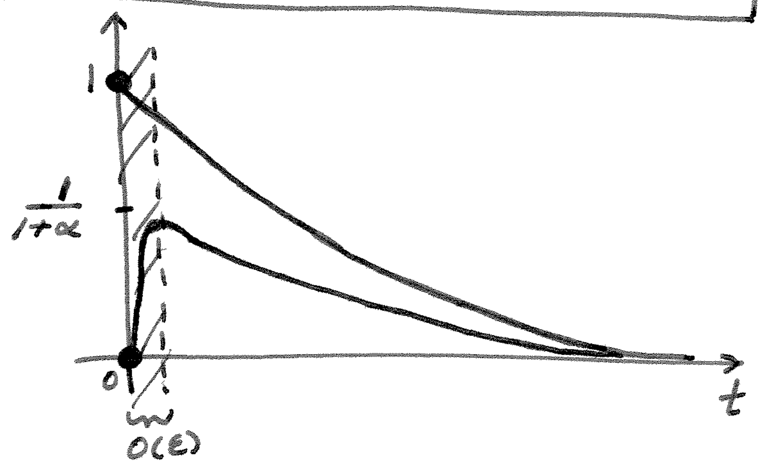
$$\sim \begin{pmatrix} X_0(t) \\ Y_0(t) \end{pmatrix} + \begin{pmatrix} 1 \\ \frac{1 - e^{-(1+\alpha)\tau}}{1+\alpha} \end{pmatrix} - \begin{pmatrix} 1 \\ \frac{1}{1+\alpha} \end{pmatrix}$$

$$\sim \begin{pmatrix} X_0(t) \\ Y_0(t) - \frac{e^{-(1+\alpha)\tau}}{1+\alpha} \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{matrix} X_{comp}(t) \sim X_0(t) \\ Y_{comp}(t) \sim Y_0(t) - \frac{e^{-(1+\alpha)t/\varepsilon}}{1+\alpha} \end{matrix}}$$

The leading order composite solution is given by

$$\boxed{\begin{aligned} &X_{comp}(t) \sim X_0(t) \\ &Y_{comp}(t) \sim Y_0(t) - \frac{e^{-(1+\alpha)t/\varepsilon}}{1+\alpha} \\ &\text{where } X_0(t) \text{ and } Y_0(t) \text{ are determined by} \\ &\qquad X_0(t) + \ln|X_0(t)| = 1 - \beta t \\ &\qquad Y_0(t) = \frac{X_0(t)}{X_0(t)+\alpha} \end{aligned}}$$



Notes: 1. $\ln|X_0(t)| \sim -\beta t$ as $t \to \infty$ $(\beta > 0)$

$X_0(t) > 0 \implies \underline{X_0(t) \sim e^{-\beta t} \text{ as } t \to \infty}$

2. If $\beta < 0$, then $X_0(t) \sim -\beta t$ as $t \to \infty$.

3. If $\alpha < 0$ or $\beta < 0$ (or both), then

$|X_0(t)|, |Y_0(t)| \to \infty$ as $t \to \infty$.

4. We haven't quite 'solved' the problem since there is still an implicit relation which needs to be solved, but we have reduced a nonlinear system of ODEs to a single transcendental equation which can be solved with Newton's method, for example.

Two-Ply Boundary Layer (Triple-Deck)

Example: $\varepsilon^3 x y'' + x^2 y' - (x^3+\varepsilon) y = 0$; $y(0)=1$, $y(1)=e^{1/2}$

Naive Expansion: $y(x) = y_0(x) + \varepsilon y_1(x) + \cdots$

$$\varepsilon^3 x(y_0'' + \varepsilon y_1'' + \cdots) + x^2(y_0' + \varepsilon y_1' + \cdots) - (x^3+\varepsilon)(y_0 + \varepsilon y_1 + \cdots) = 0$$

$O(1)$: $x^2 y_0' - x^3 y_0 = 0$; $y_0(0)=1$, $y_0(1)=e^{1/2}$

$$\Rightarrow y_0(x) = c\, e^{x^2/2}$$

$$\left.\begin{array}{l} y_0(0) = c = 1 \\ y_0(1) = c e^{1/2} = e^{1/2} \end{array}\right\} \Rightarrow c = 1 \Rightarrow \boxed{y_0(x) = e^{x^2/2}}$$

Note: $y_0(x)$ satisfies both boundary conditions. The naive expansion seems fine so far, but we haven't considered $\varepsilon > 0$ yet.

$O(\varepsilon)$: $x^2 y_1' - x^3 y_1 - y_0 = 0$; $y_1(0) = y_1(1) = 0$

$$y_1' - x y_1 = \frac{1}{x^2} e^{x^2/2} \qquad y_1(1) = (c_1 - 1) e^{1/2} = 0$$

$$\Rightarrow y_1(x) = \left(c_1 - \frac{1}{x}\right) e^{x^2/2} \qquad c_1 = 1$$

$$\Rightarrow \boxed{y_1(x) = -\frac{1-x}{x} e^{x^2/2}}$$

Note: $y_1(x) \to -\infty$ as $x \to 0^+$

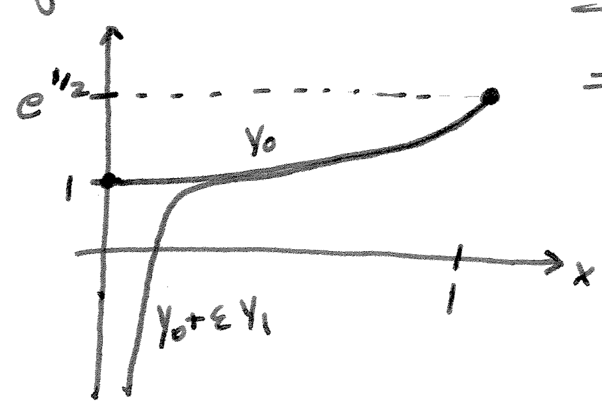
$\Rightarrow$ (i) y_1 cannot satisfy the boundary condition $y_1(0)=0$

(ii) The naive expansion

$$y(x) = e^{x^2/2} - \varepsilon \frac{1-x}{x} e^{x^2/2} + \cdots$$

is not uniformly valid on $[0,1]$.

The region of nonuniformity is $x = O(\varepsilon)$.



$\Rightarrow$ Need a boundary layer at $x=0$

$$\varepsilon^3 x y'' + x^2 y' - (x^3 + \varepsilon) y = 0 \; ; \quad \begin{array}{l} y(0) = 1 \\ y(1) = e^{1/2} \end{array}$$

Outer Solution: $Y_{OUT} = Y_0(x) + \varepsilon Y_1(x) + \cdots \; ; \; Y_{OUT}(1) = e^{1/2}$

Leading Order: $x^2 Y_0' - x^3 Y_0 = 0 \; ; \; Y_0(1) = e^{1/2}$

$$\Rightarrow \boxed{Y_0(x) = e^{x^3/2}}$$

Inner Solution: $\boxed{\xi = \frac{x}{\delta(\varepsilon)}} \Rightarrow \varepsilon^3 \delta \xi \cdot \frac{1}{\delta^2} y_{\xi\xi} + (\delta\xi)^2 \frac{1}{\delta} y_\xi - \left((\delta\xi)^3 + \varepsilon\right) y = 0$

$$\boxed{\varepsilon^3 \xi y_{\xi\xi} + \delta^2 \xi^2 y_\xi - (\delta^4 \xi^3 + \delta\varepsilon) y = 0} \quad \circledast$$

①	②	③	④
$O(\varepsilon^3)$	$O(\delta^2)$	$O(\delta^4)$	$O(\delta\varepsilon)$

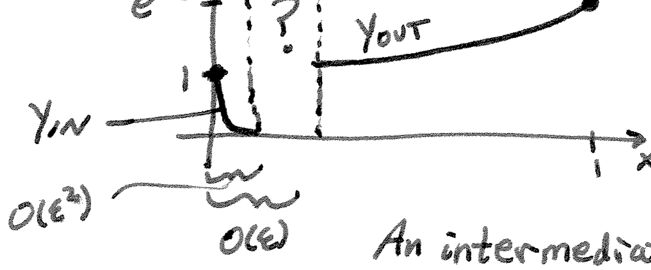
To satisfy both boundary conditions, we need to consider the second derivative term, so ① should balance with one of the other terms.

		①	②	③	④	
~~① ~ ②~~	~~$\varepsilon^3 \sim \delta^2 \Rightarrow \delta = \varepsilon^{3/2}$~~	~~ε^3~~	~~ε^3~~	~~ε^6~~	~~$\varepsilon^{5/2}$~~	X ~~④ dominates~~
~~① ~ ③~~	~~$\varepsilon^3 \sim \delta^4 \Rightarrow \delta = \varepsilon^{3/4}$~~	~~ε^3~~	~~$\varepsilon^{3/2}$~~	~~ε^3~~	~~$\varepsilon^{7/4}$~~	X ~~② dominates~~
① ~ ④	$\varepsilon^3 \sim \delta\varepsilon \Rightarrow \delta = \varepsilon^2$	ε^3	ε^4	ε^8	ε^3	✓

① and ④ balance with $\boxed{\delta(\varepsilon) = \varepsilon^2}$

Note: The boundary layer thickness is $O(\varepsilon^2)$, whereas the region of nonuniformity of the naive expansion (outer solution) is $x = O(\varepsilon)$.

What happens when $\varepsilon^2 \ll x \ll \varepsilon$?



An intermediate layer will be required to fill the gap.

$$\boxed{\varepsilon^3 \xi Y_{\xi\xi} + \delta^2 \xi^2 Y_\xi - (\delta^4 \xi^3 + \delta\varepsilon) Y = 0} \quad \circledast$$

$\underline{\delta = \varepsilon^2}$ $\Rightarrow$ $\varepsilon^3 \xi Y_{\xi\xi} + \varepsilon^4 \xi^2 Y_\xi - (\varepsilon^8 \xi^3 + \varepsilon^3) Y = 0$

$$\underline{\xi Y_{\xi\xi} + \varepsilon \xi^2 Y_\xi - (\varepsilon^5 \xi^3 + 1) Y = 0}$$

$Y_{IN} = y_0(\xi) + \varepsilon y_1(\xi) + \cdots$; $Y_{IN}(0) = 0$

Leading order: $\xi y_{0\xi\xi} - y_0 = 0$; $y_0(0) = 1$

The equation can be transformed into a

Modified Bessel Equation of Order p
$Z_{\rho\rho} + \frac{1}{\rho} Z_\rho - \left(1 + \frac{p^2}{\rho^2}\right) Z = 0$
Solutions: $Z = I_p(\rho)$ and $Z = K_p(\rho)$

where

I_p = Modified Bessel Function of the First Kind of Order p

K_p = Modified Bessel Function of the Second Kind of Order p.

Asymptotic Behavior

$$I_p(x) \sim \frac{e^x}{\sqrt{2\pi x}} + \cdots \text{ as } x \to \infty$$

$$K_p(x) \sim e^{-x} \sqrt{\frac{\pi}{2x}} + \cdots \text{ as } x \to \infty$$

Also, $K_1(x) \sim \frac{1}{x} + \cdots$ as $x \to 0$

Transform $\zeta y_{0\zeta\zeta} - y_0 = 0$ into a Modified Bessel Equation

First, let $\rho = 2\zeta^{1/2}$ $\Longrightarrow$ $\rho_\zeta = \zeta^{-1/2}$

$$\rho_{\zeta\zeta} = -\tfrac{1}{2}\zeta^{-3/2}$$

$$y_{0\zeta\zeta} = (y_{0\zeta})_\zeta = (y_{0\rho}\rho_\zeta)_\zeta = y_{0\rho\rho}\rho_\zeta^2 + y_{0\rho}\rho_{\zeta\zeta} = y_{0\rho\rho}\zeta^{-1} + y_{0\rho}(-\tfrac{1}{2}\zeta^{-3/2})$$

$$\zeta y_{0\zeta\zeta} = y_{0\rho\rho} - \frac{1}{2\zeta^{1/2}} y_{0\rho} = y_{0\rho\rho} - \frac{1}{\rho} y_{0\rho}$$

$$\zeta y_{0\zeta\zeta} = y_{0\rho\rho} - \frac{1}{\rho} y_{0\rho}$$

Then,

$$\zeta y_{0\zeta\zeta} - y_0 = \boxed{y_{0\rho\rho} - \frac{1}{\rho} y_{0\rho} - y_0 = 0}$$

Next, let $\boxed{Z = 2\rho^{-1} y_0}$

$$y_0 = \tfrac{1}{2}\rho Z$$

$$y_{0\rho} = \tfrac{1}{2}(\rho Z_\rho + Z)$$

$$y_{0\rho\rho} = \tfrac{1}{2}(\rho Z_{\rho\rho} + 2Z_\rho)$$

$$\tfrac{1}{2}(\rho Z_{\rho\rho} + 2Z_\rho) - \tfrac{1}{\rho}\cdot\tfrac{1}{2}(\rho Z_\rho + Z) - \tfrac{1}{2}\rho Z = 0$$

$$\rho Z_{\rho\rho} + 2Z_\rho - Z_\rho - \tfrac{1}{\rho} Z_\rho - \rho Z = 0$$

$$\boxed{Z_{\rho\rho} + \frac{1}{\rho} Z_\rho - (1 - 1/\rho^2) Z = 0}$$

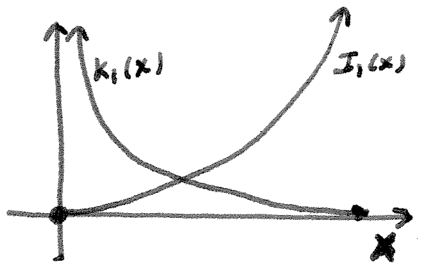
Modified Bessel Equation of Order $p = 1$.

$$\Longrightarrow Z(\rho) = C_1 I_1(\rho) + C_2 K_1(\rho)$$

$$y_0 = \tfrac{1}{2}\rho Z \Longrightarrow y_0(\rho) = \frac{C_1}{2}\rho I_1(\rho) + \frac{C_2}{2}\rho K_1(\rho)$$

$$\rho = 2\zeta^{1/2} \Longrightarrow \boxed{y_0(\zeta) = C_1 \zeta^{1/2} I_1(2\zeta^{1/2}) + C_2 \zeta^{1/2} K_1(2\zeta^{1/2})}$$

$$K_1(x) \sim \frac{1}{x} \text{ as } x \to 0^+$$



$$y_0(0) = \lim_{\zeta\to 0^+} C_2 \zeta^{1/2} K_1(2\zeta^{1/2}) = 1$$

$$\lim_{\zeta\to 0^+} C_2 \zeta^{1/2}\cdot\frac{1}{2\zeta^{1/2}} = 1$$

$$C_2/2 = 1 \quad \Longrightarrow \quad \boxed{C_2 = 2}$$

$$I_1 \sim \frac{e^x}{\sqrt{2\pi x}} \text{ as } x \to \infty$$

I_1 cannot match

$$\Longrightarrow \boxed{C_1 = 0}$$

$$\boxed{y_0(\zeta) = 2\zeta^{1/2} K_1(2\zeta^{1/2})}$$

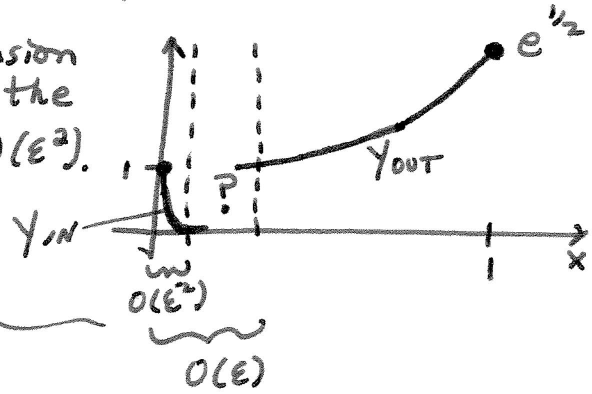
We have $Y_{OUT} \sim Y_0(x) = e^{x^3/2}$

$y_{IN} \sim y_0(\zeta) = 2\zeta^{1/2} K_1(2\zeta^{1/2})$

Try to match: $\lim_{x \to 0^+} Y_0(x) = \lim_{\zeta \to \infty} y_0(\zeta)$

$1 \neq 0$ The outer and inner expansions do not match.

The problem is that the outer expansion is not valid for $x = O(\varepsilon)$, while the boundary layer thickness is only $O(\varepsilon^2)$.



$\Rightarrow$ Need an Intermediate Layer.

Reconsider Equation ⊗

$\zeta = \dfrac{x}{\delta}$ $\quad \varepsilon^3 \zeta y_{\zeta\zeta} + \delta^2 \zeta^2 y_\zeta - (\delta^4 \zeta^3 + \delta\varepsilon) = 0$

(1) (2) (3) (4)

We have already considered Outer: (2)~(3)

Inner: $\cancel{(1)\sim(2)}$, $\cancel{(1)\sim(3)}$, (1)~(4)

Remaining Possibilities

		(1)	(2)	(3)	(4)	
(2), (4) $\delta^2 \sim \delta\varepsilon \Rightarrow \boxed{\delta = \varepsilon}$		ε^3	ε^2	ε^4	ε^2	✓
~~(3), (4) $\delta^4 \sim \delta\varepsilon \Rightarrow \delta = \varepsilon^{1/3}$~~		ε^3	$\varepsilon^{2/3}$	$\varepsilon^{4/3}$	$\varepsilon^{4/3}$	✗

(2) dominates

Let $\boxed{\eta = \dfrac{\zeta}{\varepsilon}}$ $\Rightarrow$ ⊗ becomes

$$\varepsilon^3 \eta y_{\eta\eta} + \varepsilon^2 \eta^2 y_\eta - (\varepsilon^4 \eta^3 + \varepsilon^2) y = 0$$

Intermediate Layer: $\boxed{\varepsilon \eta y_{\eta\eta} + \eta^2 y_\eta - (\varepsilon^2 \eta^3 + 1) y = 0}$

+ matching conditions

Intermediate Solution: $\varepsilon\eta Y_{\eta\eta} + \eta^2 Y_\eta - (\varepsilon^2\eta^3+1)Y = 0$

$$Y_{INT} = \mathcal{Y}_0(\eta) + \varepsilon\, \mathcal{Y}_1(\eta) + \cdots$$

Leading order: $\eta^2 \mathcal{Y}_{0\eta} + \mathcal{Y}_0 = 0$

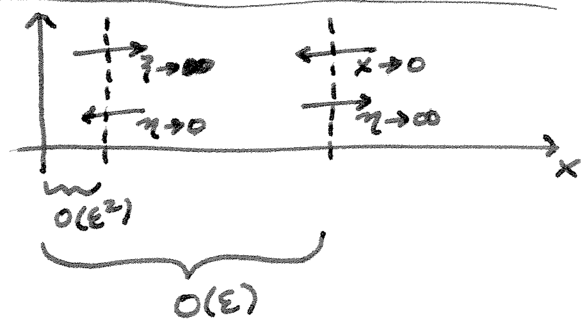
$$\Longrightarrow \boxed{\mathcal{Y}_0(\eta) = A e^{-1/\eta}}$$

We have $Y_{OUT} \sim Y_0(x) = e^{x^2/2}$, $x = O(1)$

$Y_{INT} \sim \mathcal{Y}_0(\eta) = A e^{-1/\eta}$, $x = O(\varepsilon)$

$Y_{IN} \sim y_0(\xi) = 2\xi^{1/2} K_1(2\xi^{1/2})$, $x = O(\varepsilon^2)$

Matching



Match Y_{IN} and Y_{INT}

$$\lim_{\xi\to\infty} y_0(\xi) = \lim_{\eta\to 0^+} \mathcal{Y}_0(\eta)$$

$0 = 0$ ✓ $\quad \left(\begin{smallmatrix}\text{Common}\\ \text{Part}\end{smallmatrix}\right)_0 = 0$

Match Y_{INT} and Y_{OUT}

$$\lim_{\eta\to\infty} \mathcal{Y}_0(\eta) = \lim_{x\to 0^+} Y_0(x)$$

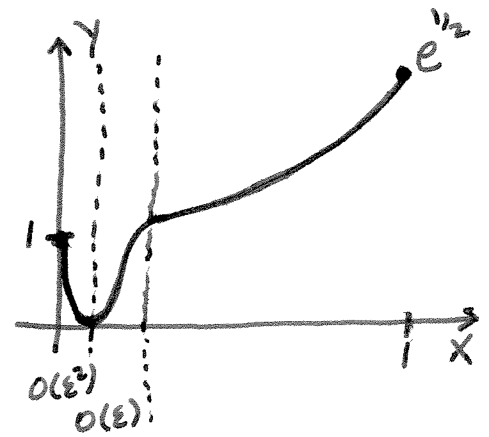
$A = 1$ $\quad \left(\begin{smallmatrix}\text{Common}\\ \text{Part}\end{smallmatrix}\right)_1 = 1$

Composite Expansion

$$Y_{comp} = \left[Y_{OUT} + \left(Y_{INT} - \left(\begin{smallmatrix}\text{Common}\\ \text{Part}\end{smallmatrix}\right)_1\right)\right] + \left(Y_{IN} - \left(\begin{smallmatrix}\text{Common}\\ \text{Part}\end{smallmatrix}\right)_0\right)$$

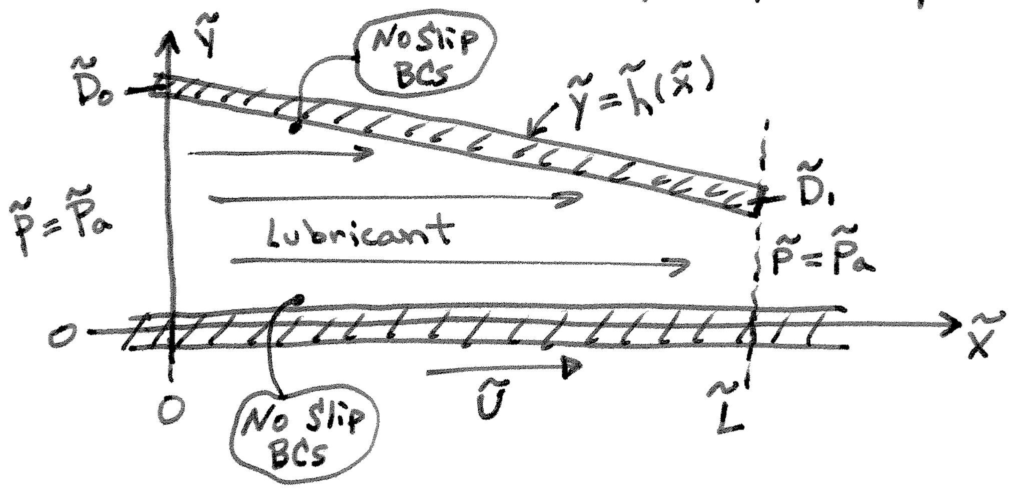
$$\sim e^{x^2/2} + (e^{-1/\eta} - 1) + (2\xi^{1/2}K_1(2\xi^{1/2}) - 0)$$

$$\boxed{Y_{comp}(x) \sim e^{x^2/2} - 1 + e^{-\varepsilon/x} + 2\frac{x^{1/2}}{\varepsilon} K_1\left(2\frac{x^{1/2}}{\varepsilon}\right)}$$



Example: Slider Bearing (Lubrication Theory) (See Holmes pg. 66, prob 3)

A thin film of viscous fluid flows between two plates. The upper plate is stationary, while the lower plate moves with velocity $\tilde{U}$, dragging the fluid as it moves. The fluid motion keeps the plates separated.



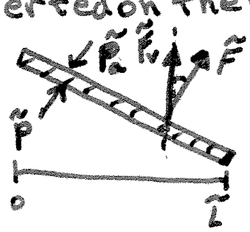
$\tilde{p}$ = pressure , $\tilde{P}_a$ = ambient pressure

$\tilde{U}$ = lower plate velocity

$\tilde{h}(\tilde{x}) = \tilde{D}_0 - \dfrac{\tilde{D}_0 - \tilde{D}_1}{\tilde{L}}\tilde{x}$ = height of the upper plate $\quad \tilde{D}_0 = \tilde{h}(0)$, $\tilde{D}_1 = \tilde{h}(\tilde{L})$

$\tilde{\mu}$ = viscosity

The bearings effectiveness is measured by the net vertical force exerted on the upper plate. The goal is to determine this quantity.



Total Vertical Force $= \boxed{\tilde{F}_T = \int_0^{\tilde{L}} (\tilde{p} - \tilde{P}_a)\, d\tilde{x}}$

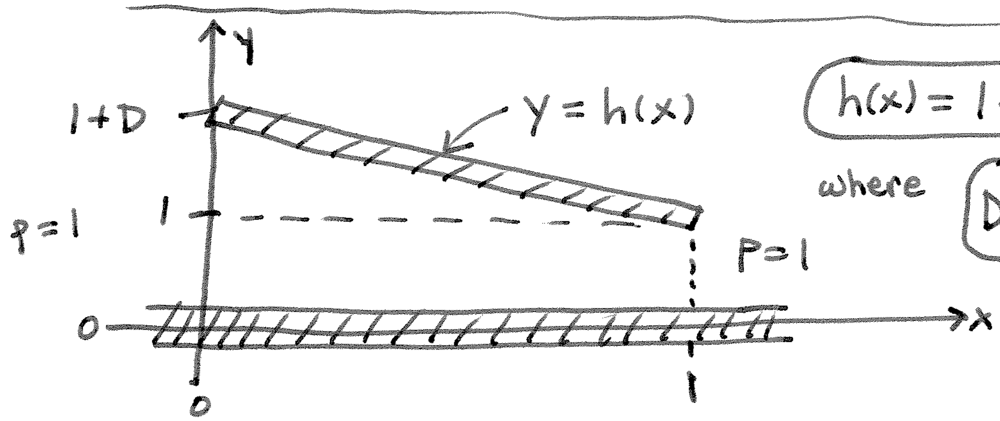
Under reasonable assumptions (e.g. constant temperature), the pressure equation decouples from the other equations of fluid dynamics. It is also assumed that pressure is a function of $\tilde{x}$ alone ($\partial/\partial\tilde{y} = 0$).

Reynold's Equation:
$$\boxed{\frac{d}{d\tilde{x}}\left(\tilde{h}^3(\tilde{x})\,\tilde{p}\,\frac{d\tilde{p}}{d\tilde{x}}\right) = 6\tilde{\mu}\tilde{U}\frac{d}{d\tilde{x}}\left(\tilde{h}(\tilde{x})\,\tilde{p}\right)} \quad 0<\tilde{x}<\tilde{L}$$

subject to $\boxed{\tilde{p}(0) = \tilde{P}_a \,;\; \tilde{P}(\tilde{L}) = \tilde{P}_a}$

Nondimensionalize

$$x = \tilde{x}/\tilde{L} \quad y = \tilde{y}/\tilde{D}_1 \quad h = \tilde{h}/\tilde{D}_1 \quad p = \tilde{p}/\tilde{p}_a \quad F_T = \frac{\tilde{F}_T}{\tilde{p}_a \tilde{L}}$$



$$\boxed{h(x) = 1 + D(1-x)}$$

where $\boxed{D = \dfrac{\tilde{D}_0 - \tilde{D}_1}{\tilde{D}_1}}$

Reynold's Equation:

$$\boxed{\frac{d}{dx}\left(h^3(x)\, p \frac{dp}{dx}\right) = \Lambda \frac{d}{dx}\left(h(x)\, p\right)} \quad \boxed{\begin{array}{c} p(0) = 1 \\ p(1) = 1 \end{array}}$$

where $\boxed{\Lambda = \dfrac{6\tilde{\mu}\tilde{L}\tilde{U}}{\tilde{p}_a \tilde{D}_1^2} = \text{bearing number}}$ (dimensionless)

Typically, lubricant is highly viscous, and $\tilde{D}_1$ is relatively small for a thin film, so it is reasonable assume that $\Lambda \gg 1$. This is a standard assumption in lubrication theory.

Let $\boxed{\varepsilon = \dfrac{1}{\Lambda} \ll 1}$ $\Longrightarrow$ $\boxed{\begin{array}{c} \varepsilon\left(h^3 p p'\right)' = (ph)' \\ p(0) = p(1) = 1 \end{array}}$

$$\boxed{h(x) = 1 + D(1-x)}$$

$$\boxed{F_T = \int_0^1 (p-1)\, dx}$$

Outer Solution: $P_{OUT} = P_0(x) + \epsilon P_1(x) + \cdots$

Leading order: $(P_0 h)' = 0$

$$P_0 h = c_0$$

$$P_0(x) = \frac{c_0}{h(x)} \Rightarrow \boxed{P_0(x) = \frac{c_0}{1+D(1-x)}}$$

Notes:

1. The problem is singular since $P_0(x)$ cannot satisfy both boundary conditions.

   $\Rightarrow$ Try a boundary layer solution.

2. The value of c_0 depends on the boundary layer location(s).

3. The thin boundary layer has little effect on the value of the integral $F_T = \int_0^1 (p-1)\,dx$.

   To leading order,

$$\boxed{F_T \sim \int_0^1 (P_0(x) - 1)\,dx}$$

The contribution from the boundary layer comes in at higher order.

Therefore, only $P_0(x)$ is needed to approximate F_T to leading order. However, we must consider the inner solution to determine c_0.

Try a Boundary Layer at $x=0$.

Outer Solution: $P_0(1)=1 \Rightarrow P_0(1)=\frac{c_0}{1+D(1-1)}=1$

$c_0 = 1$

$$\boxed{P_0(x)=\frac{1}{1+D(1-x)}}$$

Inner Solution: $\boxed{\xi=\frac{x}{\varepsilon}} \Rightarrow h = 1+D(1-x) = 1+D(1-\varepsilon\xi) = (1+D)-\varepsilon D\xi$

$h(\xi) = h_0 + O(\varepsilon)$

where $h_0 = h(0) = 1+D$

$$\frac{\varepsilon}{\varepsilon}\left(h^3 p\cdot\frac{1}{\varepsilon}P_\xi\right)_\xi = \frac{1}{\varepsilon}(ph)_\xi$$

$$(h^3 p P_\xi)_\xi = (ph)_\xi\ ;\ P(0)=1$$

Expand: $P_{IN} = g_0(\xi) + \varepsilon g_1(\xi) + \ldots$

Leading Order: $(h_0^3 g_0 g_{0\xi})_\xi = (g_0 h_0)_\xi$

$$h_0^2 (g_0 g_{0\xi})_\xi = g_{0\xi}$$

Integrate Twice $\Rightarrow$ $\boxed{h_0^2\left[g_0(\xi) - c_1 \ln|g_0(\xi)+c_1|\right] = \xi + c_2}$

Implicit Inner Solution

Matching

$$\lim_{\xi\to\infty} g_0(\xi) = \lim_{x\to 0^+} P_0(x) = \frac{1}{1+D}$$

Consider $h_0^2\left[g_0 - \underline{c_1 \ln|g_0+c_1|}\right] = \underline{\xi} + c_2$ as $\xi\to\infty$.

Dominant Terms

$\Rightarrow -h_0^2 c_1 \ln|g_0+c_1| \sim \xi$ as $\xi\to\infty$

Therefore, $\lim_{\xi\to\infty} g_0(\xi) = -c_1 = \frac{1}{1+D}$.

But then, $\underbrace{\frac{h_0^2}{1+D}\ln|g_0+c_1|}_{\to -\infty} \sim \underbrace{\xi}_{\to +\infty}$ as $\xi\to\infty$

Contradiction (LHS $\to -\infty$, RHS $\to +\infty$)

$\Rightarrow$ The inner solution at $x=0$ is not matchable.

$\Rightarrow$ No Boundary Layer at $x=0$.

Try a Boundary Layer at $x=1$.

Outer Solution: $P_0(0)=1 \Rightarrow P_0(0)=\dfrac{C_0}{1+D(1-0)}=1$ $\quad \boxed{C_0=1+D}$

$$\boxed{P_0(x)=\frac{1+D}{1+D(1-x)}}$$

Inner Solution: $\boxed{\eta=\dfrac{x-1}{\varepsilon}} \Rightarrow h=1+D(1-x)=1+D\big(1-(1+\varepsilon\eta)\big)=1-\varepsilon D\eta$

$x=1+\varepsilon\eta$ $\qquad \boxed{h(\eta)=1+O(\varepsilon)}=h(1)+O(\varepsilon)$

$$\boxed{(h^3 p p_\eta)_\eta=(ph)_\eta,\quad p(0)=1}$$

Expand: $P_{IN}=g_0(\eta)+\varepsilon g_1(\eta)+\cdots, \quad P_{IN}(0)=1$

Leading Order: $(g_0 g_{0\eta})_\eta=g_{0\eta}, \quad g_0(0)=1$

Integrate Twice $\Rightarrow \boxed{g_0(\eta)-C_1\ln|g_0(\eta)+C_1|=\eta+C_2}$

Implicit Inner Solution

Matching

$$\lim_{\eta\to-\infty}g_0(\eta)=\lim_{x\to1^-}P_0(x)=1+D=\begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}$$

Consider $g_0-\underline{C_1\ln|g_0+C_1|}=\underline{\eta}+C_2$ as $\eta\to-\infty$.

Dominant Terms

$$\Rightarrow\quad -C_1\ln|g_0+C_1|\sim\eta \text{ as } \eta\to-\infty$$

Therefore, $\lim_{\eta\to-\infty}g_0(\eta)=-C_1=1+D$

$$\boxed{C_1=-(1+D)}$$

Then, $\underbrace{(1+D)\ln|g_0-(1+D)|}_{\to-\infty}\sim\underbrace{\eta}_{\to-\infty}$ as $\eta\to\infty$.

Matching is successful.

Boundary Condition: $g_0(0)=1 \Rightarrow \boxed{1+(1+D)\ln D=C_2}$

We have
$$c_0 = 1+D$$
$$c_1 = -(1+D)$$
$$c_2 = 1+(1+D)\ln D$$

$$P_{OUT} \sim P_0(x) = \frac{1+D}{1+D(1-x)}$$

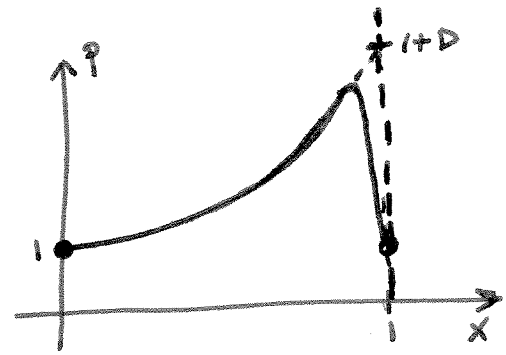
$$P_{IN} \sim g_0(\eta) \ : \quad g_0(\eta) - c_1 \ln|g_0(\eta)+c_1| = \eta + c_2$$

Plug in c_1 and c_2 $\Rightarrow$ $$g_0(\eta) + (1+D)\ln\left|\frac{g_0(\eta)-(1+D)}{D}\right| = \eta + 1$$

$$\begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix} = 1+D$$

Composite Solution

$$P_{comp} = P_{OUT} + P_{IN} - \begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}$$
$$\sim P_0(x) + g_0(\eta) - \begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}$$
$$\sim \frac{1+D}{1+D(1-x)} + g_0(\eta) - (1+D)$$

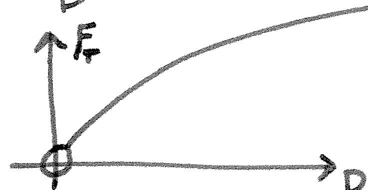


$$\boxed{P_{comp}(x) \sim -\frac{D(1+D)(1-x)}{1+D(1-x)} + g_0\left(-\frac{1-x}{\varepsilon}\right)}$$

where g_0 is determined by $\boxed{g_0(\eta) + (1+D)\ln\left|\frac{g_0(\eta)-(1+D)}{D}\right| = \eta + 1}$

$$F_T = \int_0^1 (p(x)-1)\,dx \sim \int_0^1 (P_0(x)-1)\,dx = \int_0^1 \left(\frac{1+D}{1+D(1-x)} - 1\right) dx$$
$$\sim -\frac{1+D}{D}\ln|1+D(1-x)| - x \Big|_0^1 = -1 + \frac{1+D}{D}\ln(1+D)$$

$$\boxed{F_T \sim \frac{1+D}{D}\ln(1+D) - 1}$$



Finally, convert back the ~~the~~ original dimensional quantities.