# Chapter 1: First Order ODEs

Most General Form: ~~$F(t, x, \frac{dx}{dt}) = 0$ (OR $F(x, y, dy/dx)$)~~ e.g. $\sqrt{\frac{dx}{dt}} \sin(tx^2) + e^{dx/dt} = \ln(t/x)$

~~i.e. Any expression involving $t$, $x$, and $\frac{dx}{dt}$.~~

This form is a bit too general for our purposes.

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We'll consider first order ODEs in which $\frac{dx}{dt}$ can be explicitly solved for.

$$\Rightarrow \boxed{\frac{dx}{dt} = f(t,x)} \quad \text{(Normal Form)}$$

Special Cases of the Normal Form

1. Linear: $f(t,x) = -a(t)\,x + b(t)$

(Integrating Factor Method) $\Rightarrow \frac{dx}{dt} + a(t)x = b(t)$

The equation is linear in $x$ and $dx/dt$.

e.g. The equation for a line is linear in $x$ and $y$: $y + ax = b$

Linearity depends only on how $x$ and $dx/dt$ appear in the equation. $a(t)$ and $b(t)$ may be nonlinear functions of $t$.

2. Separable: $f(t,x) = g(t)\,h(x)$ product of two functions, one is a function of $t$ alone, and the other is a function of $x$ alone.

(Method of Separable Variables) $\Rightarrow \frac{dx}{dt} = g(t)\,h(x)$ $\Rightarrow \int \frac{dx}{h(x)} = \int g(t)\,dt$

$H(x) = G(t) + C$

Solve for $x$.

All $x$'s can be put on one side of the equations and all $t$'s on the other

3. Autonomous: $f(t,x) = f(x)$ ($f$ does not depend explicitly on $t$)

However, $f(x)$ does depend on $t$ through $x$, but it does not depend explicitly on $t$. $f$ is actually a composite function, $f = f(x(t))$

Autonomous ODEs will be our main focus in this course, in 1st and higher orders.

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Let's review special cases 1. and 2., and then turn our attention to the general normal form and special case 3.

## 1. Linear ODEs and the Integrating Factor Method

General Form of a first order Linear ODE: $\boxed{1\,\dfrac{dx}{dt} + a(t)\,x = b(t)}$

Example: Find the general solution of the ODE $t(3x+x') = 3t^3 + 2x$

First recognize the ODE to be linear — we have a sum of terms, each involving at most one factor of $x$ or $x'$ raised to the first power.

To solve a linear first order ODE,

STEPS:

1) Write the ODE in the form

$$(1)\quad 1\,\frac{dx}{dt} + a(t)x = b(t)$$ , and identify $a(t)$

need a coefficient of 1 here

(Get $x'$ by itself and group all terms which have a factor of $x$)

$$t(3x+x') = 3t^3 + 2x$$
$$3x + x' = 3t^2 + \tfrac{2}{t}x$$
$$(1')\quad x' + (3 - 2/t)x = 3t^2 \Rightarrow a(t) = 3 - 2/t$$

2) Compute the Integrating Factor: $\boxed{\mu(t) = e^{\int a(t)\,dt}}$

The integration constant may be ignored here since it ends up cancelling everywhere anyway.

$$\mu(t) = e^{\int (3 - 2/t)\,dt} = e^{3t - 2\ln|t|}$$
$$= e^{3t}e^{-2\ln|t|} = e^{3t}e^{\ln(|t|^{-2})}$$
$$= e^{3t}|t|^{-2} = \frac{e^{3t}}{t^2} \Rightarrow \boxed{\mu(t) = \frac{e^{3t}}{t^2}}$$

3) Multiply equation (1) by $\mu(t)$ and write the LHS as $\frac{d}{dt}(\mu(t)x)$.

$$\mu(t) \times (1') \Rightarrow \underbrace{\frac{e^{3t}}{t^2}x' + \frac{e^{3t}}{t^2}\left(3 - \frac{2}{t}\right)x}_{\mu x' + \mu' x = (\mu x)'} = \frac{e^{3t}}{t^2}\cdot 3t^2$$

L.H.S. is equal to $\frac{d}{dt}(\mu(t)x)$

$$\frac{d}{dt}\left(\frac{e^{3t}}{t^2}x\right) = 3e^{3t}$$

4) Integrate and solve for $x$

$$\frac{e^{3t}}{t^2}x = \int 3e^{3t}\,dt = e^{3t} + C$$
$$\frac{x}{t^2} = 1 + Ce^{-3t}$$
$$\boxed{x(t) = t^2\left[1 + Ce^{-3t}\right]}$$

## 2. Separable ODEs and the Method of Separable Variables

Initial Value Problem

**Example:** Solve the IVP $\frac{dy}{dx} = \frac{x}{y+1}$, $y(0) = 2$

First recognize that the ODE is separable

To solve a separable ODE,

General Solution:

1) Separate the Variables: $(y+1)\,dy = x\,dx$

2) Integrate: $\int (y+1)\,dy = \int x\,dx$

$$\frac{y^2}{2} + y = \frac{x^2}{2} + C$$

← Both sides of equation get a constant of integration, but they can be combined into a single constant

$(2C \to C)$ $\quad y^2 + 2y = x^2 + C$

3) Solve for y (if possible)

Method 1: Complete the Square

$$y^2 + 2y + 1 = x^2 + C + 1$$

$$(y+1)^2 = x^2 + C \quad (C+1 \to C)$$

$$y + 1 = \pm\sqrt{x^2 + C}$$

$$\boxed{y = -1 \pm \sqrt{x^2 + C}}$$

Method 2: Quadratic Formula

$$y^2 + 2y - (x^2 + C) = 0$$

$$y = \frac{-2 \pm \sqrt{4 + 4(x^2 + C)}}{2}$$

$$y = -1 \pm \sqrt{1 + (x^2 + C)}$$

$(C+1 \to C)$

General Solution $\quad \boxed{y = -1 \pm \sqrt{x^2 + C}}$

This is a solution whether + or − is used. When applying the IC to find C, we must choose the appropriate sign.

4) Find C and choose + or −

$y(0) = 2$

$-1 \pm \sqrt{0^2 + C} = 2$

$\pm\sqrt{C} = 3$

$+\sqrt{C} = 3$

$\Rightarrow C = 9$

Only the + sign can satisfy the IC. i.e. $y(0)$ can be $> -1$ only if the + sign is used.

$$\Rightarrow \boxed{y = -1 \pm \sqrt{x^2 + 9}}$$

## 2. Separable ODEs and the Method of Separable Variables

**Example:** Solve the IVP (Initial Value Problem): $2y(x^2-1)\frac{dy}{dx} = x(y^2-1)$ ; $y(0) = -2$ (nonlinear)

First recognize the ODE to be separable.

To solve a separable ODE,

General Solution:

1) Separate the Variables: $\frac{2y\,dy}{y^2-1} = \frac{x\,dx}{x^2-1}$

Not valid when $y=\pm 1$. Consider this case afterwards

2) Integrate: $\int \frac{2y\,dy}{y^2-1} = \int \frac{x}{x^2-1}\,dx$

$u = x^2-1$, $du = 2x\,dx \Rightarrow \int \frac{\frac{1}{2}du}{u} = \frac{1}{2}\ln|u|+c$

$u = y^2-1$, $du = 2y\,dy \Rightarrow \int \frac{du}{u} = \ln|u|+c$

$\ln|y^2-1| = \frac{1}{2}\ln|x^2-1|+c$

Strictly speaking both sides get a constant of integration. They may combined into a single constant, usually placed on the side of the independent variable.

3) Solve for $y$: $\cancel{e}^{\ln|y^2-1|} = e^{\frac{1}{2}\ln|x^2-1|+c}$

$|y^2-1| = e^c e^{\frac{1}{2}\ln|x^2-1|}$, $e^c>0$

$y^2-1 = \pm e^c \cancel{e}^{\ln|x^2-1|^{1/2}}$, $\pm e^c \neq 0$

$y^2-1 = C|x^2-1|^{1/2}$, $C\neq 0$

$y^2 = 1+C|x^2-1|^{1/2}$, $C\neq 0$

Note: $C=0 \Rightarrow y=\pm 1$

This is the case we avoided above

What happens if $y=\pm 1$?

The ODE reduces to

$\frac{dy}{dx}=0$

$\Rightarrow$ There is no change in $y$ as $x$ varies

$\Rightarrow$ $y=\pm 1$ are constant solutions of the ODE

① $\boxed{y = \pm\left[1+C|x^2-1|^{1/2}\right]^{1/2}}$ (General Solution) for all $C$.

Two Solution 'Branches' — We must choose one or the other when applying the IC.

therefore, equation ① yields a solution when $C=0$ as well, and thus equation ① is a solution for all $C$.

Particular Solution: Apply the IC (Initial Condition) to find $C$: $y(0)=-2$

Need the absolute value

Need the − sign to satisfy the IC $\rightarrow$ $-\left[1+C|0-1|^{1/2}\right]^{1/2} = -2$

$-[1+C]^{1/2} = -2$

$1+C = 4$

$C=3$

$\boxed{y(x) = -\left[1+3|x^2-1|^{1/2}\right]^{1/2}}$

Satisfies both the ODE and the IC

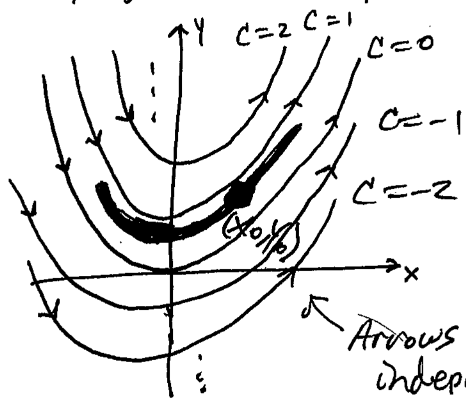
# Solution Curves

The general solution of a first order ODE involves exact one arbitrary constant. therefore, the general solution may be considered as a one-parameter family (C = parameter) of solution curves.

e.g. Consider the ODE $\frac{dy}{dx}=2x$

$$\Rightarrow \boxed{y=x^2+C} \quad \text{(General Solution)}$$

The general solution yields a distinct solution curve for each value of C.

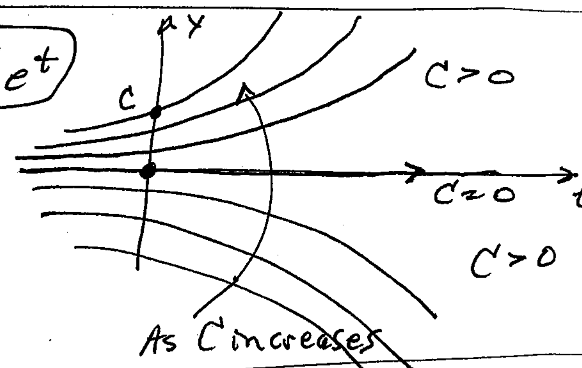



Given an IC, say $y(x_0)=y_0$, the solution of the IVP $\frac{dx}{dt}=2x$ ; $y(x_0)=y_0$ is represented by the solution curve passing through the point $(x_0,y_0)$

$y(2)=3 \Rightarrow C=-1$

Arrows indicate the direction as the independent variable increases

e.g. $\frac{dx}{dt}=x$ (separable) $\Rightarrow \boxed{x(t)=Ce^t}$

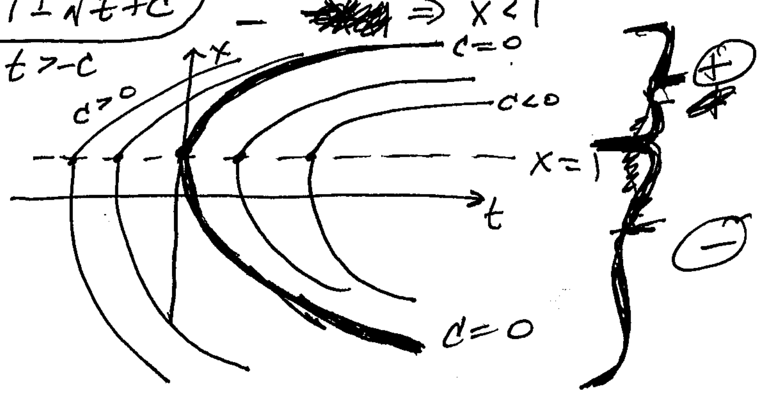



e.g. $2(x-1)\frac{dx}{dt}=1$ (separable) $\Rightarrow \boxed{x(t)=1\pm\sqrt{t+C}}$

$t>-C$

$+ \Rightarrow x>1$

$- \Rightarrow x<1$




Note: When $x=1$, the ODE reduces to $0\cdot\frac{dx}{dt}=1 \Rightarrow \frac{dx}{dt}\to\infty$ as $x\to 1$

Solutions cannot equal 1 ($x\neq 1$)

Solution curves do not intersect. Each point in the plane has exactly solution curve passing through it.

# Direction Fields

Consider the general Normal Form: $\boxed{\frac{dx}{dt} = f(t,x)}$

There are no solution methods that apply to the general normal form. Some ODEs having the normal form are solvable, but most are not. In practice, most differential equations cannot be solved exactly. However, the are several methods and techniques for describing the solution behavior without actually knowing the solution

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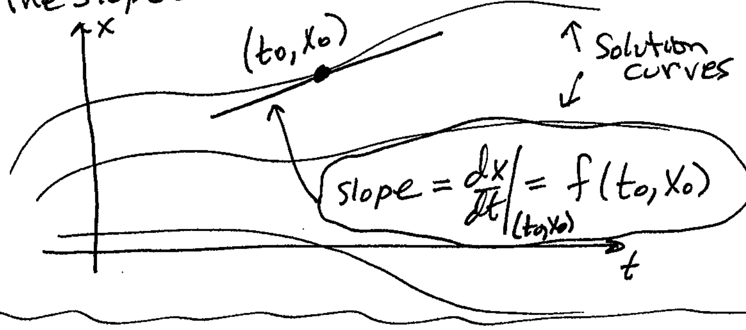
The function $f(t,x)$ is called the slope function since it gives the slope of the solution curves at each point in the tx-plane.

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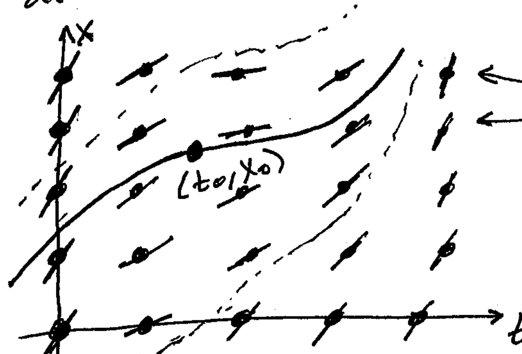
The idea of a direction field is to compute the slope $f(t,x)$ of the solution curves at several points in the tx-plane, and indicate each with a small line segment (or arrow) with the same slope.

e.g. $\frac{dx}{dt} = f(t,x)$ ; $t, x \geq 0$

It is convenient to make the points equally spaced.




Each line segment is tangent to a solution curve

Then, given an IC, say $x(t_0) = x_0$, the solution curve passing through the point $(t_0, x_0)$ may be sketch by following the path indicated by the line segments

## Show plots

dfield1.pdf
dfield2.pdf

# Autonomous ODEs

Recall the Normal Form: $\frac{dx}{dt} = f(t,x)$, $x = x(t)$.

An autonomous ODE is a special case of the normal form in which $f$ does not depend explicitly on $t$. $\Rightarrow f = f(x)$

$$\Rightarrow \boxed{\frac{dx}{dt} = f(x)} \quad (\text{autonomous})$$

Note: though not explicitly $f$ does depend on $t$ through $x$. The function $f$ is actually a composite function since $x$ is a function of $t$ $\Rightarrow f = f(x(t))$.

e.g. $\underbrace{\frac{dx}{dt} = (1-y)e^y + \sin y}_{\text{autonomous}} \cancel{+ t^2}$ ← explicit dependence on $t$

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Autonomous ODEs give rise to the notion of equilibrium solutions.

Definition: An equilibrium solution of an autonomous ODE $\frac{dx}{dt} = f(x)$ are the constant solutions of the equation $f(x) = 0$.

i.e. Equilibrium solutions are the roots of the function $f$.

i.e. $x_e$ is an equilibrium solution if $f(x_e) = 0$.

↑ constant $x$-value

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Equilibrium Solutions are also called Critical Point,
Equilibrium Points
Stationary Points
Fixed Points
⋮

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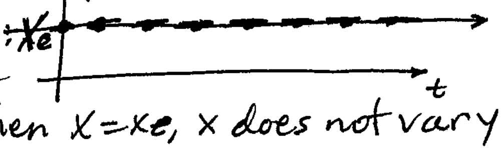
The slope of the solution curves is zero at critical points.

that is, $\left.\frac{dx}{dt}\right|_{x = x_e} = f(x_e) = 0$

Direction Field




When $x = x_e$, $x$ does not vary as $t$ varies

$\Rightarrow$ $x(t) = x_e$ is a constant solution of the ODE for all $t$

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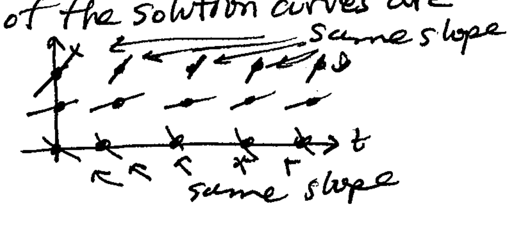
Note: For an autonomous ODE, the slopes of the solution curves are the same at each value of $x$.

i.e. The slope depends only on $x$, and not on $t$.



Extra

# Equilibrium Solutions

Eq. Sols. are the states in which a system is in equilibrium

e.g. Newton's Law of Heating/Cooling

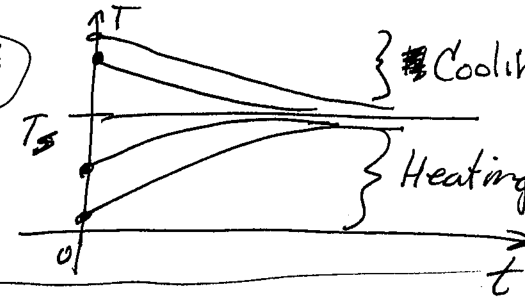
$$\frac{dT}{dt} = -k(T - T_s),\ k > 0$$

$T$ = temp of an object
$T_s$ = surrounding temp. (const.)

Eq. Sol.: $T_e = T_s$

The eq. sol. $T_e = T_s$ corresponds to the state in which the object is in thermal equilibrium with its surroundings.

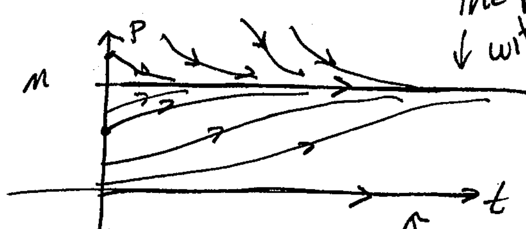
Gen. Sol. $T(t) = T_s + Ce^{-kt}$




e.g. Logistic Population Model

$$\frac{dP}{dt} = k\left(1 - \frac{P}{M}\right)P\ ;\ k, M > 0$$

Eq. Sols: $P_e = 0$, $P_e = M$




The population size is in equilibrium with the available resource
the environment can sustain at most a population size of M.

There is no population

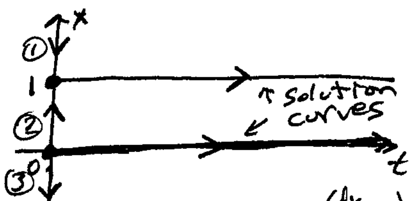
(Introductory) **Example:** Consider the autonomous ODE

$$\frac{dx}{dt} = x(1-x), \quad t \geq 0$$

Sketch some solution curves.

Equilibrium Solutions: $f(x) = x(1-x) = 0 \Rightarrow$

$x_e = 0$ $\quad$ $x_e = 1$




what happens elsewhere?

Note that $\frac{dx}{dt}$ is zero only at the equilibrium solutions, and thus, the slope ($dx/dt$) of the solution curves can change sign only at the equilibrium solutions (i.e. only at $x=0$ and $x=1$).
e.g. Between the two equilibrium solutions ($0<x<1$), the slope of the solution curves must be either positive everywhere or negative everywhere.

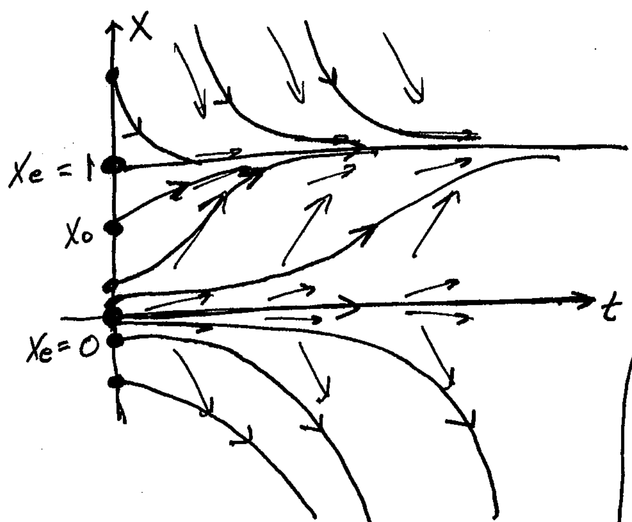
The equilibrium solutions partition the tx-plane into 3 regions i) $x>1$ ii) $0<x<1$ iii) $x<0$

① $x>1 \Rightarrow \frac{dx}{dt} = x(1-x) = (>0)(<0) < 0 \Rightarrow \frac{dx}{dt} < 0$ for $x>1$

② $0<x<1 \Rightarrow \frac{dx}{dt} = x(1-x) = (>0)(>0) > 0 \Rightarrow \frac{dx}{dt} > 0$ for $0<x<1$

③ $x<0 \Rightarrow \frac{dx}{dt} = x(1-x) = (<0)(>0) < 0 \Rightarrow \frac{dx}{dt} < 0$ for $x<0$

Also, the slope of the solution curves is near zero near the equilibrium solution, and becomes steeper as we move away from the equilibrium solutions.



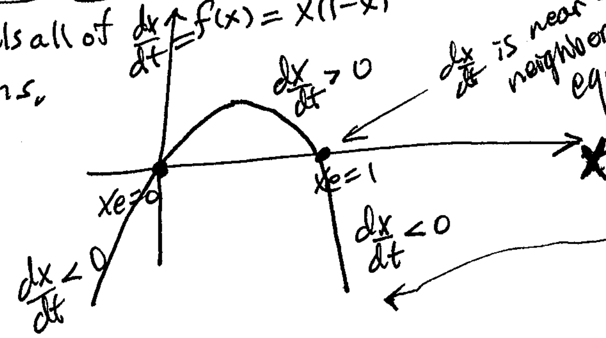

The limit $\lim_{t\to\infty} x(t)$ is an important quantity since it corresponds to the end state (ultimate fate) of the system.
Consider the IC $x(0) = x_0$.

Here,

$$\boxed{\begin{aligned} &\lim_{t\to\infty} x(t) = 1 \quad \text{for } x_0 > 0 \\ &\lim_{t\to\infty} x(t) = 0 \quad \text{for } x_0 = 0 \\ &\lim_{t\to\infty} x(t) = -\infty \quad \text{for } x_0 < 0 \end{aligned}}$$

(All solution curves converge to $x_e = 1$)

**Note:** The graph of $f(x)$ reveals all of the above observations.




$\frac{dx}{dt}$ is near zero in the neighborhood of the equilibrium solutions, and $|dx/dt|$ increase as we move away from the critical points.

Observe that as $t \to \infty$, solution converge toward the eq. sol. $X_e = 1$, and diverge away from the eq. sol. $X_e = 0$.

$X_e = 1$ is said to be asymptotically stable

$X_e = 0$ is said to be unstable.

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Definition: An equilibrium solution $X_e$ of the ODE $\frac{dx}{dt} = f(x)$ is said to be asymptotically stable (AS) if

$$\lim_{t \to \infty} X(t) = X_e$$

whenever $X$ is initially sufficiently close to $X_e$.

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All solutions curves which are sufficiently close to an asymptotically Stable eq. sol. will converge to that eq. sol. as $t \to \infty$

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If solutions diverge away from an equilibrium solution $X_e$ as $t \to \infty$, then $X_e$ is said to be unstable.

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In the above example, ~~$X_e$~~

$X_e = 1$ is an AS eq. sol.

$\lim_{t \to \infty} X(t) = 1$ if $\underbrace{0 < X(0) < \infty}_{\text{sufficiently close to } X_e}$ — may be any finite value of $t$.

$X_e = 0$ is an U eq. sol.

$\lim_{t \to \infty} X(t) = 0$ if and only if $X(0) = 0$

Finally, $\lim_{t \to \infty} X(t) = -\infty$ if $X(0) < 0$

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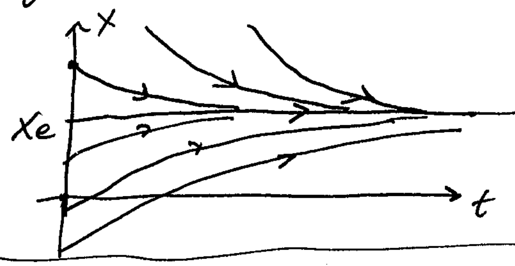
In practice, the limit $\lim_{t \to \infty} X(t)$ is an important quantity since it corresponds to the end state of a system. For example, if $X(t)$ represents the size of a population at time $t$, then $\lim_{t \to \infty} X(t)$ reveals the ultimate fate of the population, e.g. $\lim_{t \to \infty} X(t) = 0 \Rightarrow$ Extinction.

## There are four different classifications of stability.

Let $x_e$ denote an equilibrium solution.

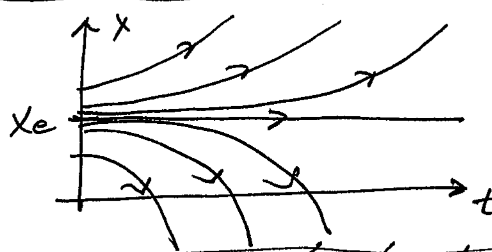
### ① Asymptotically Stable (AS)

Solutions near $x_e$ converge to $x_e$ at $t \to \infty$.



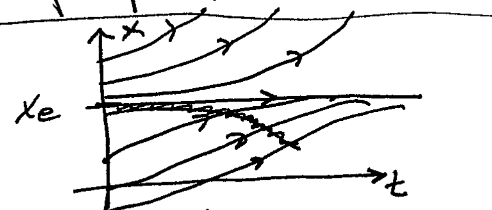

### ② Unstable (U)

Solutions near $x_e$ diverge away from $x_e$ as $t \to \infty$.

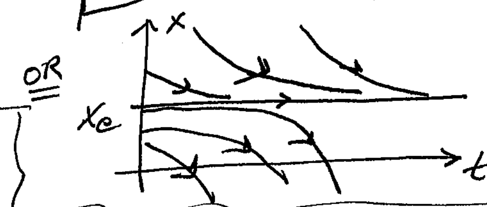



### ③ Semi-Stable (SS)

Solutions near $x_e$ converge to $x_e$ as $t \to \infty$ on one side and diverge away from $x_e$ as $t \to \infty$ on the other side.




OR




### ④ Stable (S) (Liapunov Stability)

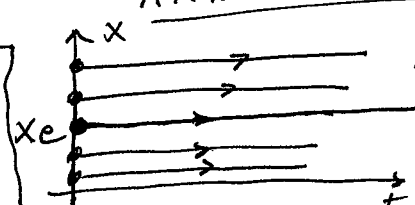
Solutions near $x_e$ stay near $x_e$ as $t \to \infty$, but they do not necessarily converge to $x_e$ as $t \to \infty$

Stable eq. sols. will arise when we consider systems of ODEs, or equivalently when we consider higher-order ODEs.

Note: Eq. sols. of first order autonomous ODEs do not fall into this category, except for the one special case in which $f(x) = 0$

$$\Rightarrow \frac{dx}{dt} = 0 \Rightarrow x(t) = C$$

All x-values are eq. sols.




Solutions near $x_e$ stay near $x_e$, but do not converge to $x_e$ as $t \to \infty$

Note: $AS \Rightarrow S$,

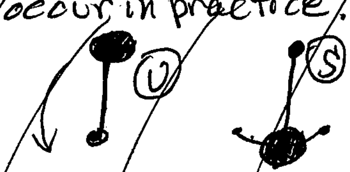
but $S \not\Rightarrow AS$.

$\Rightarrow$ AS is stronger

~~Note: Unstable solutions do not occur in practice.~~

~~e.g. pendulum~~

~~2nd order $\theta'' + \frac{g}{L}\sin\theta = 0$~~




~~Not AS (unless there is friction)~~

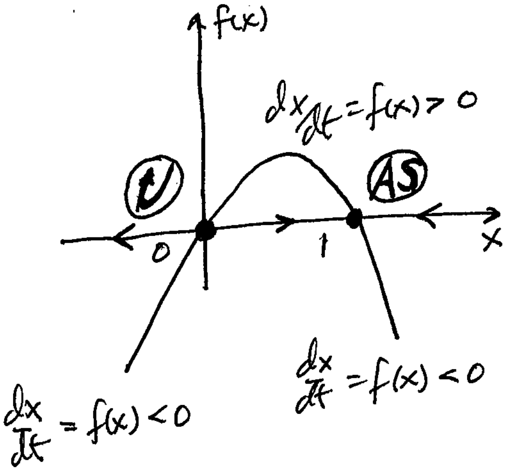
LATER

## Two Techniques for Classifying Equilibrium Solutions by Stability

**Method 0:** Plot a direction field and visually identify the eq. sol and their stability.

**Method 1:** Consider the graph of $f(x)$

e.g. $\frac{dx}{dt} = f(x) = x(1-x) \Rightarrow$ Eq. Sols: $x_e = 0, x_e = 1$




The arrows indicate the direction in which $x$ moves as $t$ increases.

$\longrightarrow \Rightarrow x$ increases as $t$ increases

$\longleftarrow \Rightarrow x$ decreases as $t$ increases.

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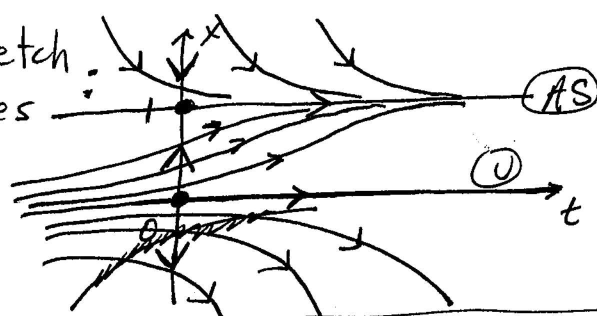
The arrows suggest that as $t \to \infty$, solutions converge toward $x_e = 1$, and diverge away from $x_e = 0$.

$\Rightarrow$ $\boxed{\begin{array}{l} x_e = 1 \text{ is asymptotically stable (AS)} \\ x_e = 0 \text{ is unstable (U)} \end{array}}$

From this, we can sketch the solution curves:

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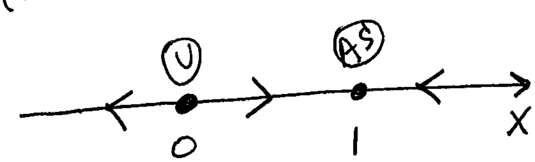
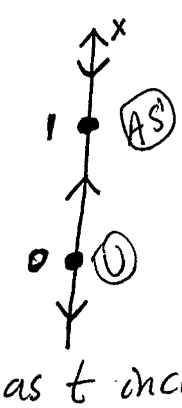
## Phase Line:

(x-axis)




OR

Often it is convenient to draw a vertical phase line.




Phase Portraits

The arrows indicate the motion of $x$ as $t$ increases.

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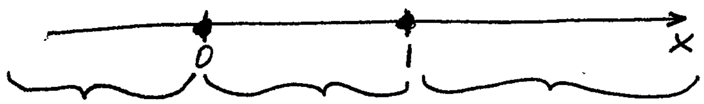
Note: In 2-D, such a plot is called the phase plane

In general n-D, such a plot is called a phase portrait.

**Shortcut for Method 1:** It is not necessary to plot $f(x)$.

e.g. $\frac{dx}{dt} = f(x) = x(1-x)$

1. Draw the phase line ($x$-axis) and label the eq. sols.

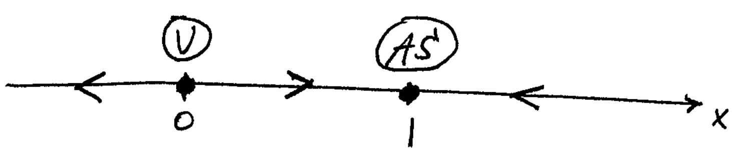



Note that $f(x)$ can change in sign only at the equilibrium solutions. Therefore, $f(x)$ must be the same sign throughout each of the subintervals resulting from the equilibrium solutions.

2. Pick a point from each subinterval and determine the sign of $f(x)$ at that point, this gives the sign of $f$ over the entire subinterval

| Interval | Pick a point | sign of $f(x)$ |
|---|---|---|
| $(-\infty, 0)$ | $x=-1$ | $f(-1) = (-1)(2) < 0$ |
| $(0, 1)$ | $x = \frac{1}{2}$ | $f(\frac{1}{2}) = (\frac{1}{2})(\frac{1}{2}) > 0$ |
| $(1, \infty)$ | $x = 2$ | $f(2) = 2(-1) < 0$ |

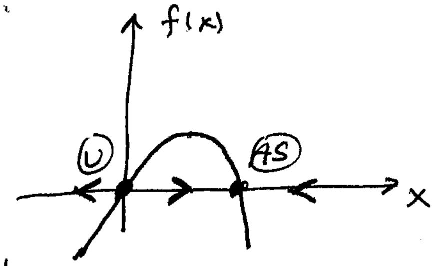
3. Complete the phase-portrait.




$\Rightarrow$ $\boxed{\begin{array}{l} x_e = 0 \text{ is Unstable} \\ x_e = 1 \text{ is Asymptotically Stable} \end{array}}$

Method 2: This method can be generalized to systems of ODEs.

e.g. $\frac{dx}{dt} = f(x) = x(1-x)$




Notice that $f'(x) > 0$ at the unstable eq. sol., and $f'(x) < 0$ at the asymptotically stable eq. sol. This is true in general.

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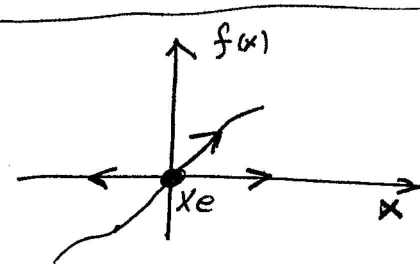


Since the arrows point away from unstable eq. sols., it must be that $f(x) < 0$ to the left and $f(x) > 0$ to the right of an unstable eq. sol. Thus, f must increase through an unstable eq. sol. $\Rightarrow f'(x_e) > 0$

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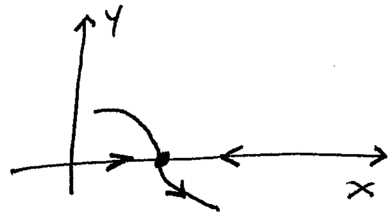



Since the arrows point toward an asymptotically stable eq. sols, it must be that $f(x) > 0$ to the left and $f(x) < 0$ to the right of an asymptotically stable eq. sol. Thus, f must decrease through an asymptotically stable eq. sol.

$\Rightarrow f'(x_e) < 0$

---

**Theorem:** Let $x_e$ be an eq. sol. of the ODE $\frac{dx}{dt} = f(x)$.

If $f'(x_e) < 0$, then $x_e$ is AS.

If $f'(x_e) > 0$, then $x_e$ is U.

If $f'(x_e) = 0$, then the test is inconclusive.

If $f'(x_e) = 0$, we must turn to method 1.

**Note:** If $f'(x_e) = 0$, then $x_e$ may be either AS, U, SS, or S.

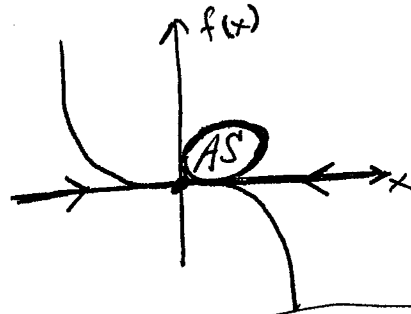
e.g. When $f'(x_e)=0$, method 2 fails and we must turn to method 1.

**AS:** $\frac{dx}{dt} = f(x) = -x^3$

$x_e = 0$

$f'(x_e) = 0$

(inconclusive)

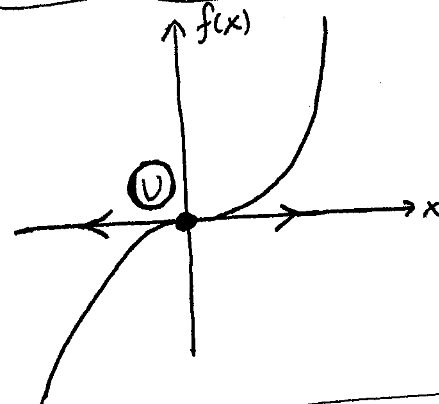



**U:** $\frac{dx}{dt} = f(x) = x^3$

$x_e = 0$

$f'(x_e) = 0$

(inconclusive)

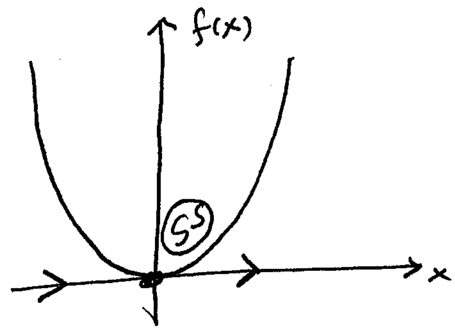



**SS:** $\frac{dx}{dt} = f(x) = x^2$

$x_e = 0$

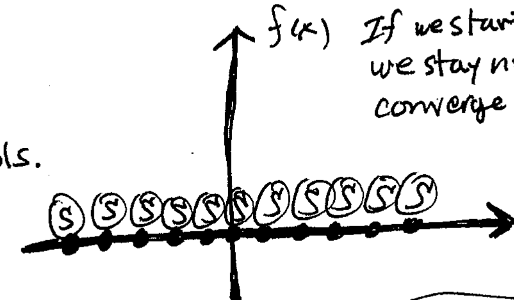
$f'(x_e) = 0$

(inconclusive)




**S:** $\frac{dx}{dt} = f(x) = 0$

All x-values are eq. sols. with $f'(x_e) = 0$




If we start near a eq. sol. $x_e$, we stay near $x_e$, but do not converge to it as $t \to \infty$

**Note:** SS and S eq. sol. may occur only when $f'(x_e) = 0$.
If $f'(x_e) \neq 0$, the the eq. sol. is necessarily AS or U.

**Note:** $f'(x_e) = 0$ at multiple roots

Back to the example: $\frac{dx}{dt} = f(x) = x(1-x)$

$x_e = 0, \quad x_e = 1$

$$f(x) = x(1-x) = x - x^2$$

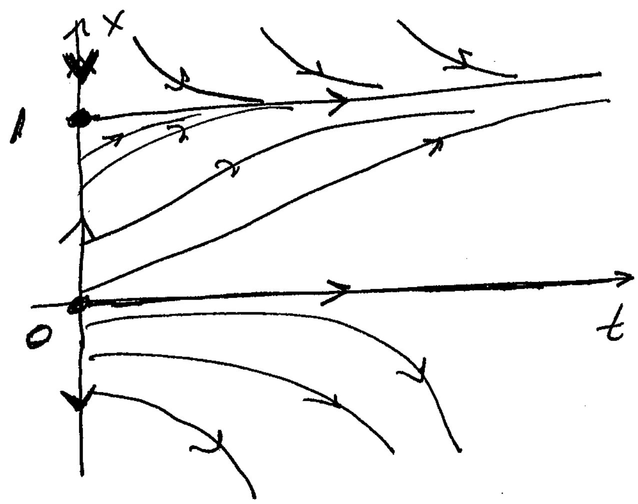
$$f'(x) = 1 - 2x$$

$$f'(0) = 1 - 0 = 1 > 0 \Rightarrow$$

$$f'(1) = 1 - 2 = -1 < 0 \Rightarrow$$

$$\boxed{\begin{array}{l} x_e = 0 \text{ is } U \\ x_e = 1 \text{ is } AS \end{array}}$$

With this, we can sketch the solution curves.




This type of analysis yields a qualitative description of the solution behaviors rather than a quantitative description

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# Preview of Systems

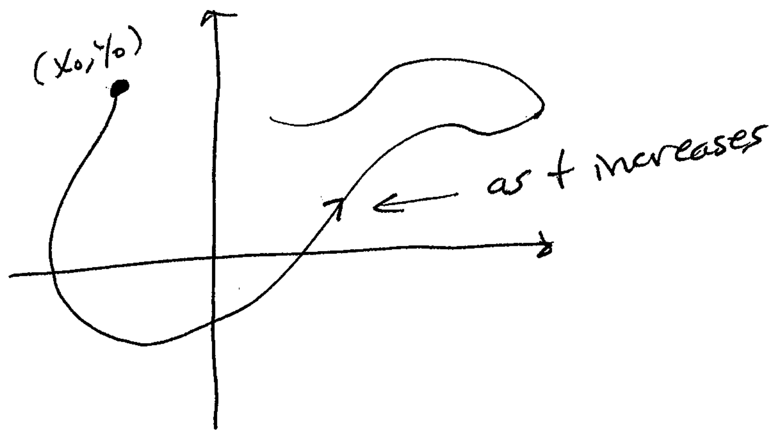
Autonomous system of 2 first order ODEs

Not $t$ dependence

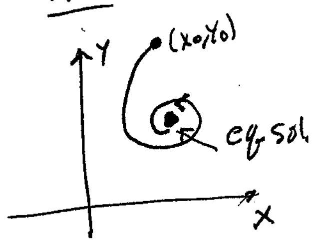
$$\frac{dx}{dt} = f(x,y), \quad x(0) = x_0$$

$$\frac{dy}{dt} = g(x,y), \quad y(0) = y_0$$

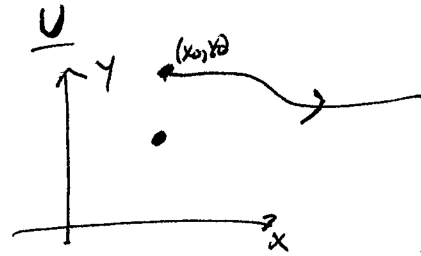
The solution is a pair of functions: $x(t), y(t)$, which can be considered as a parametric curve in the $xy$-plane.



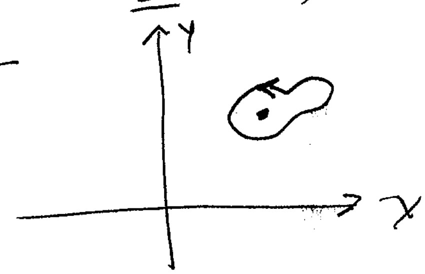

AS




U

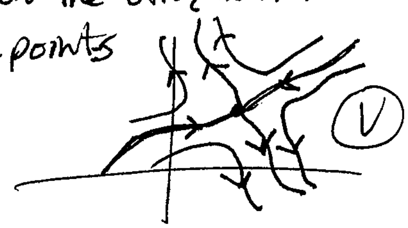



S (stable)




Semi-Stable is not relevant for systems, since the concept of AS on one side and U on the other doesn't make sense.

However, there are saddle points



**Example:** $\frac{dx}{dt} = f(x) = x(x+1)(x-2)^2$

Find all eq. sols., classify each by stability, and sketch some solution curves.

**Eq. Sols:** $f(x) = x(x+1)(x-2)^2 = 0$

$x_e = 0 \quad x_e = -1 \quad x_e = 2$

**Method 2:** Recall: $(fgh)' = f'gh + fg'h + fgh'$

$f'(x) = 1\cdot(x+1)(x-2)^2 + x\cdot 1\cdot(x-2)^2 + x(x+1)\cdot 2(x-2)$

$f'(x) = (x+1)(x-2)^2 + x(x-2)^2 + 2x(x+1)(x-2)$

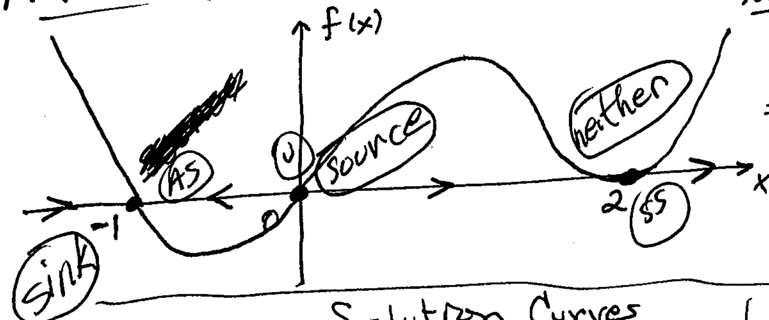
$f'(0) = (1)(-2)^2 + 0 + 0 = 4 > 0 \Rightarrow \boxed{x_e = 0 \text{ is U}}$

$f'(-1) = 0 + (-1)(-3)^2 + 0 = -9 < 0 \Rightarrow \boxed{x_e = -1 \text{ is AS}}$

$f'(2) = 0 + 0 + 0 = 0 \Rightarrow$ Inconclusive. Must use Method 1:

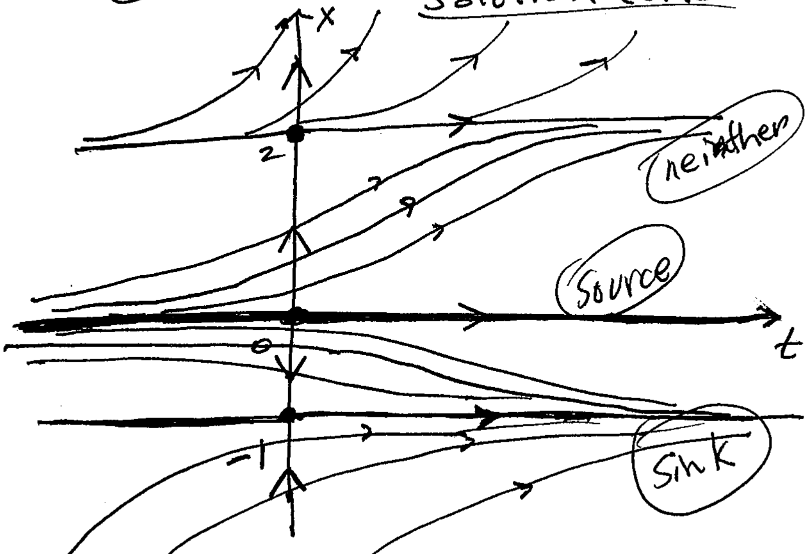
**Method 1:** $f(x) = x(x+1)(x-2)^2$

Note: $f(x) \to +\infty$ as $x \to \pm\infty$




$\Rightarrow \boxed{x_e = 2 \text{ is SS}}$

Solution Curves




Note that as $t \to -\infty$, solution curves diverge away from AS eq. sols. and converge toward U eq. sols.

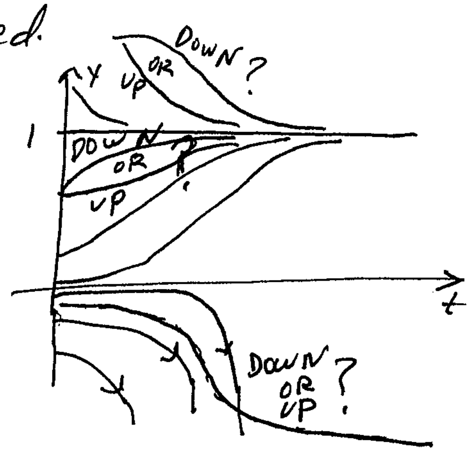
# Concavity / Inflection Points of Solutions Curves

The solution curve can be sketched a bit more accurately if concavity is considered.

**Recall:** $\frac{dx}{dt} = x(1-x)$

$x_e = 0 \quad x_e = 1$

U $\quad$ AS




$$\text{Slope} = \frac{dx}{dt} = f(x)$$

$$\text{Concavity} = \frac{d^2x}{dt^2} = ?$$

~~Common mistake~~

$$\cancel{\frac{dx}{dt} = f(x)}$$

$$\cancel{\frac{d^2x}{dt^2} \neq f'(x)}$$

Must differentiate both sides w.r.t. the same variable.

Consider f to be a composite function.

$$\frac{dx}{dt} = f(x(t))$$

$$\frac{d^2x}{dt^2} = f'(x)\frac{dx}{dt} = f'(x)f(x)$$

$$\boxed{\frac{d^2x}{dt^2} = f'(x)f(x)}$$

*Note*: Since $f(x)=0$ only at eq. sols., a change in concavity of a solution curve (i.e. an inflection point) may occur only when $f'(x)=0$.

For the above example, $f(x) = x(1-x)$ $\Rightarrow$ $\frac{d^2x}{dt^2} = f'(x)f(x) = (1-2x)\,x\,(1-x) = 0$

$f'(x) = 1-2x$

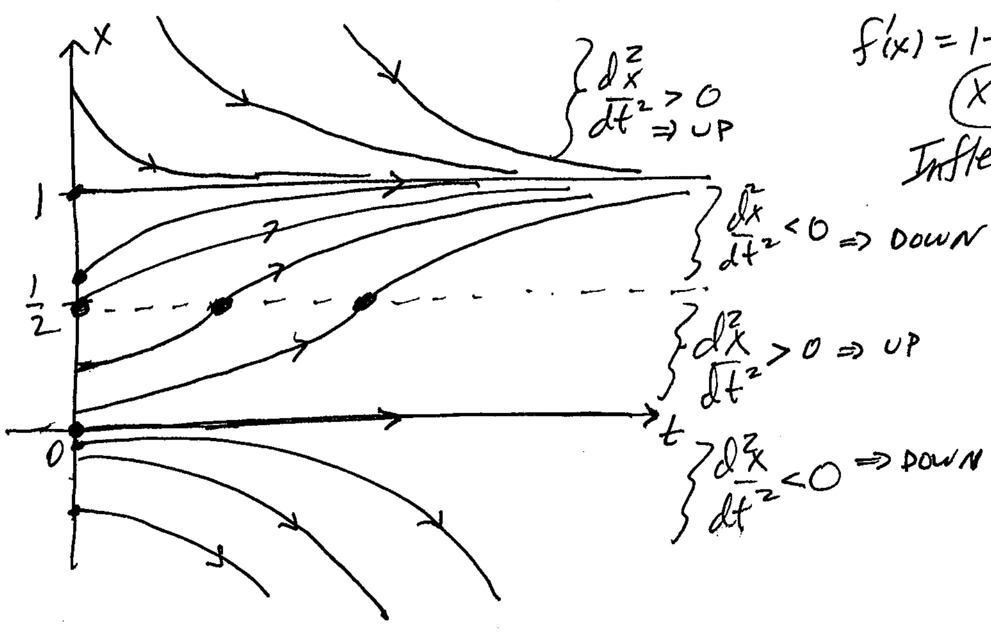
$x=\frac{1}{2} \quad x=0 \quad x=1$

eq. sol.

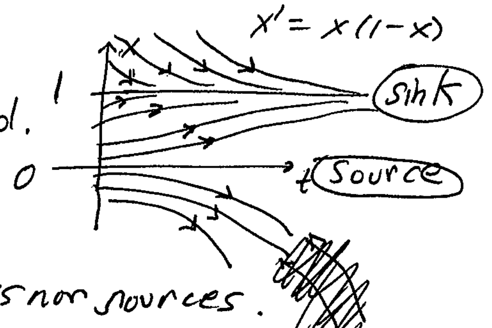
$f'(x) = 1-2x = 0$

$x = \frac{1}{2}$

Inflection Points occur at $x=\frac{1}{2}$



# Terminology




**Sink** = asymptotically stable eq. sol.

**Source** = unstable eq. sol.

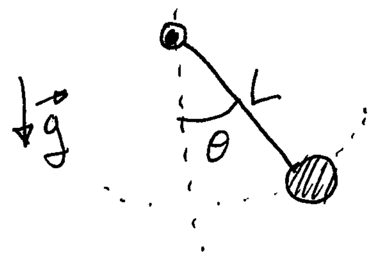
SS and S eq. sols. are neither sinks nor sources.

See "Flow" on the next page

**Note:** Autonomous 1st order ODEs have only monontonic solutions.
$\Rightarrow$ No oscillations

**Note:** Unstable eq. sols. do not occur in realistic physical systems.
When an eq. sol. is found, its stability must be checked to ensure that it is an observable possibility.

e.g. **Pendulum** (frictionless): $\theta'' + \frac{g}{L}\sin\theta = 0$ (Newton's 2nd Law)

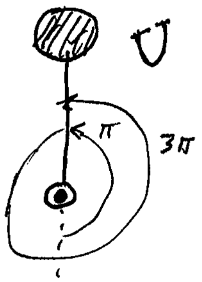



Eq. Sol. $\theta'' = 0 \Rightarrow \cancel{\frac{g}{L}} \sin\theta = 0$

$\theta = n\pi \quad n = 0, \pm 1, \pm 2, \dots$

**Odd Multiples of $\pi$**

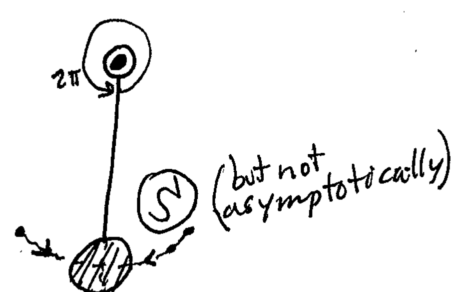
$\theta = \pm\pi, \pm 3\pi, \dots$




The eq. sol. is not physically relevant

**Even Multiples of $\pi$**

$\theta = 0, \pm 2\pi, \pm 4\pi, \dots$




If the pendulum starts near the eq. sol, it will oscillate about the eq. sol. (stay nearby), but will not converge toward it.

The eq. sols. correspond to **Static Equilibrium** (Forces are balanced)

# Flow

Consider the IVP $\boxed{\frac{dx}{dt} = f(t,x),\ x(0) = x_0}$. ①

The solution of ① depends on the value of $x_0$. We may allow $x_0$ to vary and consider the solution to be a function of two variables ($t$ and $x_0$).

Informal Description: The flow $\phi(t,x_0)$ of the ODE $\frac{dx}{dt} = f(t,x)$ represents all solutions of the IVP ① as $x_0$ varies through all possible values.
(feasible, relevant)

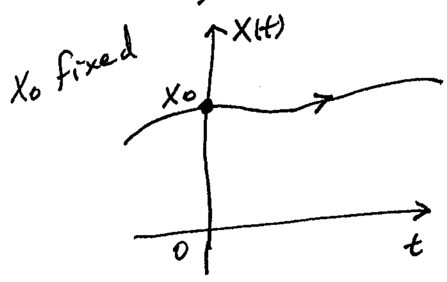
$$\Rightarrow \quad \frac{\partial \phi}{\partial t} = f(t,x), \quad \phi(0,x_0) = x_0$$

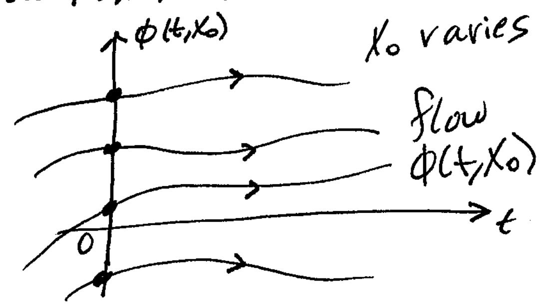
Notes: 1. The flow corresponds to the ODE, rather than the IVP. Given an ODE $\frac{dx}{dt} = f(t,x)$, the IC $x(0) = x_0$ is imposed to define the flow.

2. If $x_0$ is held fixed, $\phi$ becomes a function of a single variable, $\phi = \phi(t)$, and it is the solution of the IVP ①.

$$x_0 \text{ fixed} \Rightarrow \frac{d\phi}{dt} = f(t,x), \quad \phi(0) = x_0$$

3. A solution $x(t)$ of the IVP ① corresponds to a single solution curve, whereas the flow $\phi(t,x_0)$ represents all solution curves.




To evaluate $\phi(t,x_0)$ for some $t$ and $x_0$, go to the curve passing through the initial point $(0,x_0)$, and travel along the curve to time $t$.

**Example:** Find the flow of the ODE $\frac{dx}{dt} = tx^2$.

To find the flow of an ODE,

i) find the general solution (it will involve an arbitrary constant $C$),

and ii) impose the IC $x(0) = x_0$ and express the constant $C$ in terms of $x_0$.

---

i) General Solution

$$\frac{dx}{dt} = tx^2 \text{ (separable)}$$

$$\int \frac{dx}{x^2} = \int t\,dt$$

$$-\frac{1}{x} = \frac{t^2}{2} + c$$

$$x = \frac{1}{-c - t^2/2}$$

$$x = \frac{2}{-2c - t^2}$$

$(-2c \to c)$ $\quad \boxed{x(t) = \frac{2}{c - t^2}}$ (gen. sol.)

ii) Impose the IC $x(0) = x_0$

$$x(0) = \frac{2}{c - 0} = x_0$$

$$C = 2/x_0$$

$$\Rightarrow \quad \phi(t, x_0) = \frac{2}{2/x_0 - t^2}, \quad x_0 \neq 0$$

$$\boxed{\phi(t, x_0) = \frac{2x_0}{2 - x_0 t^2}} \quad \text{all } x_0$$

Since $x_0$ is not specified, this is essentially a general solution with $C$ expressed in terms of $x_0$.

---

Given the flow $\phi(t, x_0)$, we can find the solution satisfying any IC.

e.g. Suppose we wish to find the solution satisfying the IC $x(2) = 1$.

$$\phi(t, x_0) = \frac{2x_0}{2 - x_0 t^2}$$

$$x(2) = 1 \Rightarrow \phi(2, x_0) = 1, \quad x_0 = ?$$

$$\phi(2, x_0) = \frac{2x_0}{2 - x_0 \cdot 2^2} = 1$$

$$2x_0 = 2 - 4x_0$$

$$6x_0 = 2$$

$$x_0 = \frac{1}{3}$$

The solution passing through $(2,1)$ also passes through $(0, \frac{1}{3})$

Solution of the IVP $\frac{dx}{dt} = tx^2, \; x(2) = 1$

$$x(t) = \phi(t, \tfrac{1}{3}) = \frac{2 \cdot 1/3}{2 - \frac{1}{3}t^2}$$

$$\boxed{x(t) = \frac{2}{6 - t^2}}$$

# Bifurcations

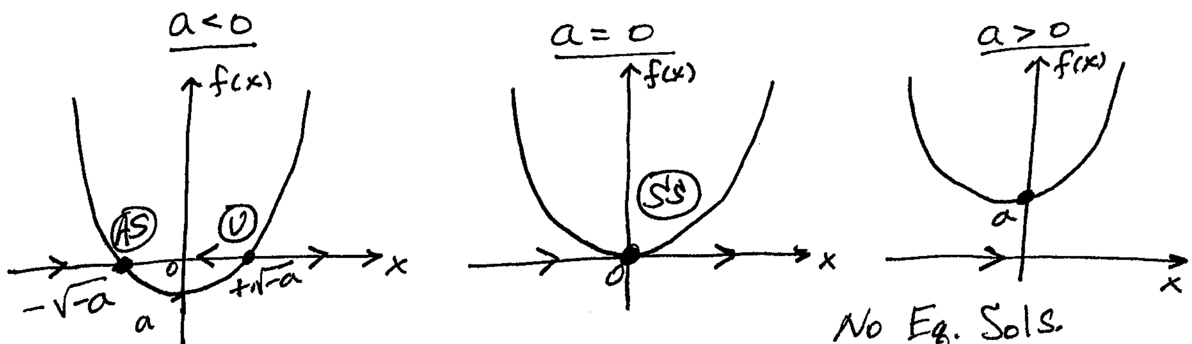
Often ODEs involve a control (adjustable) parameter.

e.g. $\boxed{\frac{dx}{dt} = x^2 + a}$, $a$ = real constant

In such a case, the qualitative structure of the phase portrait may change as the parameter is varied.

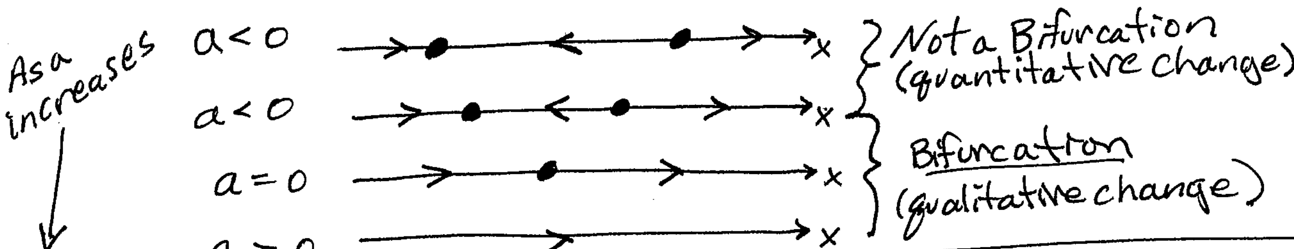
e.g. $f(x) = x^2 + a = 0$

Eq. Sols.: $\boxed{x = \pm\sqrt{-a},\ a \le 0}$




No Eq. Sols.

As a increases through 0, we see a fundamental change in the structure of the phase portrait.




A qualitative change in the dynamics (flow) of the phase portrait as a control parameter is varied is called a Bifurcation.

A Bifurcation occurs if either

i) an eq. sol(s) is created or destroyed,

or ii) an existing eq. sol(s). has a change in stability

as the control parameter is varied.

The control parameter being varied is called the Bifurcation parameter

A value of the bifurcation parameter at which a bifurcation occurs is called a Bifurcation point. (e.g. $a=0$ in the above example)

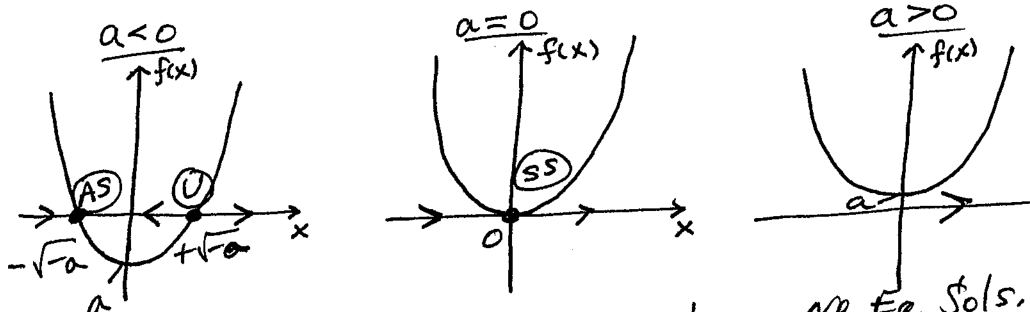
# Fundamental Bifurcations

1. Saddle-Node Bifurcation
2. Transcritical Bifurcation
3. Pitchfork Bifurcation
   (a) Supercritical (b) Subcritical

## 1. Saddle-Node Bifurcation

Classical Representative Example: $\boxed{\dfrac{dx}{dt} = x^2 + a}$ Eq. Sols. $\boxed{x_e = \pm\sqrt{-a},\ a \le 0}$

A Saddle-Node Bifurcation occurs when a local max. or min. passes through the x-axis.




No Eq. Sols.

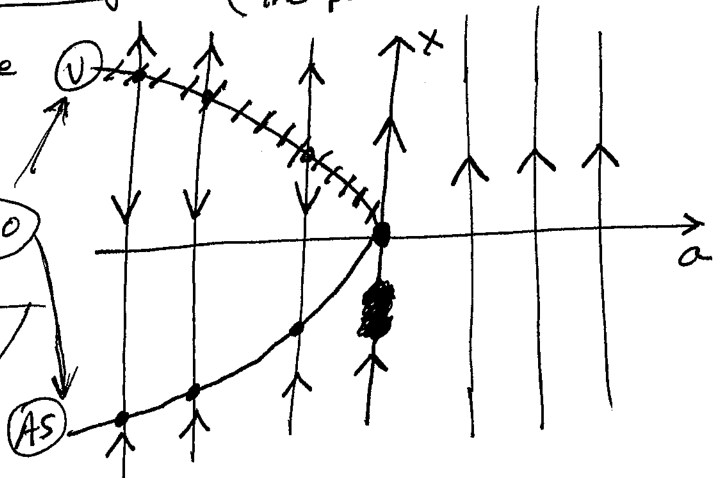
As a increases through 0, the two eq. sols. collide and are destroyed.
As a decrease through 0, two eq. sols are created.

## Bifurcation Diagram (Diagram which describes the phase line at each value of a)

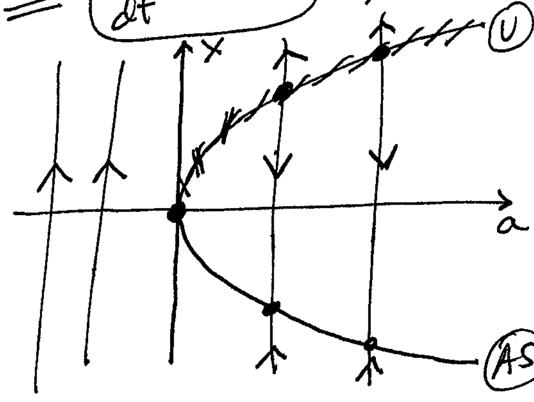
Imagine a vertical phase line located at each value of a.

Eq. Sols. $x^2 + a = 0$

$\boxed{x_e = \pm\sqrt{-a},\ a \le 0}$




OR $\boxed{\dfrac{dx}{dt} = x^2 - a}$ Eq. Sols. $x = \pm\sqrt{a}$




Solid Curves (—) correspond to Asymptotically Stable Eq. Sols.

Dashed Curves (- - - OR ++++) correspond to Unstable Eq. Sols.

# 2. Transcritical Bifurcation

Classical Representative Example: $\boxed{\dfrac{dx}{dt} = x(a-x)}$  Eq. Sols.: $\boxed{x_e = 0,\ x_e = a}$

When $a=0$, the two eq. sols. coincide.

$f(x) = x(a-x) = ax - x^2$

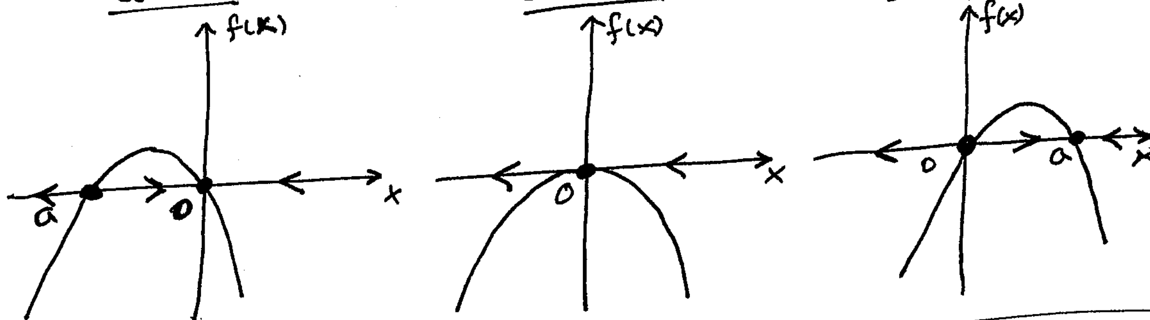
$f'(x) = a - 2x = 0$

$x = \frac{a}{2}$ (local max/min?)

$f''(x) = -2 < 0 \Rightarrow$ $\boxed{x = \frac{a}{2} \text{ is a local max}}$

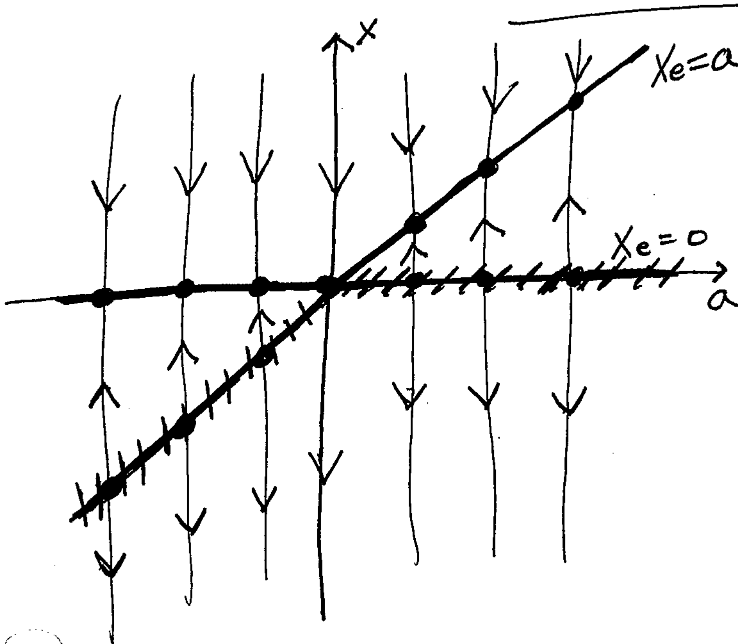
concave down

$a<0$  $a=0$  $a>0$

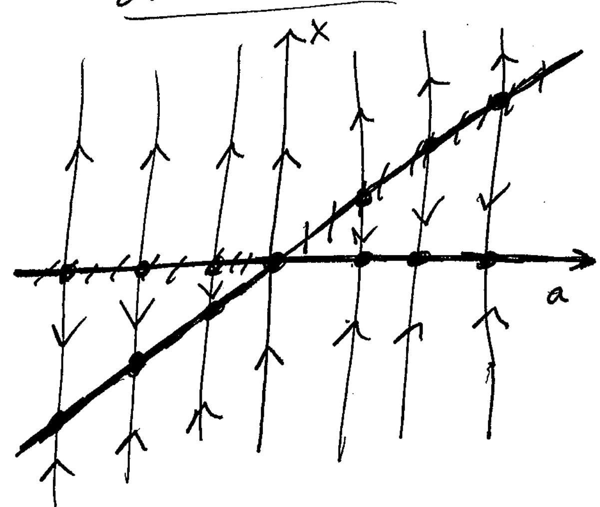



Bifurcation Diagram

As $a$ increases through 0, the two eq. sols. pass through each other and exchange stability as they do so.




OR $\dfrac{dx}{dt} = x(x-a)$




OR $\dfrac{dx}{dt} = x(x+a)$

$\dfrac{dx}{dt} = x(a+x)$

# 3. Pitchfork Bifurcation

## (a) Supercritical

Classical Representative Example: $\boxed{\frac{dx}{dt} = x(a-x^2)}$ Eq. Sols.: $\boxed{x_e = 0,\ x_e = \pm\sqrt{a},\ a \geq 0}$

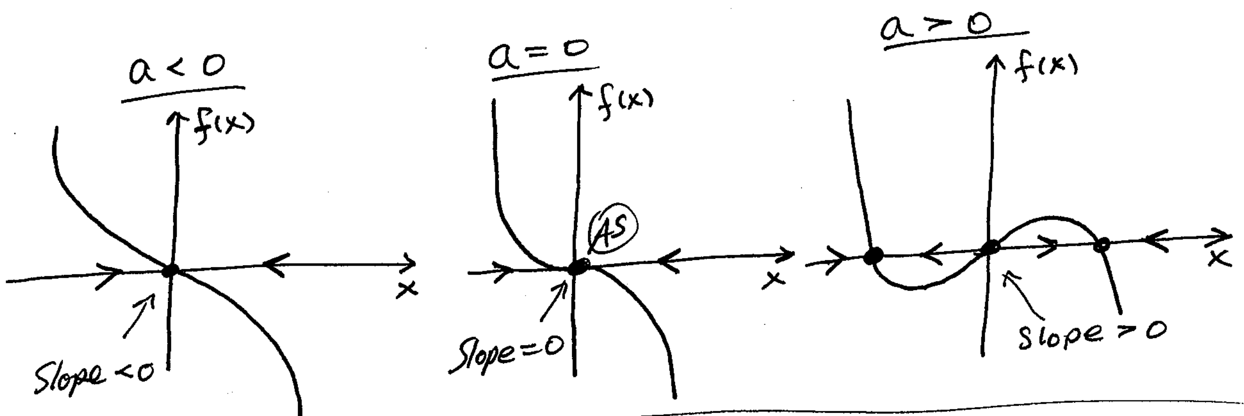
$f(x) = x(a-x^2) = ax - x^3$

$f'(x) = a - 3x^2 = 0 \implies f'(0) = a$

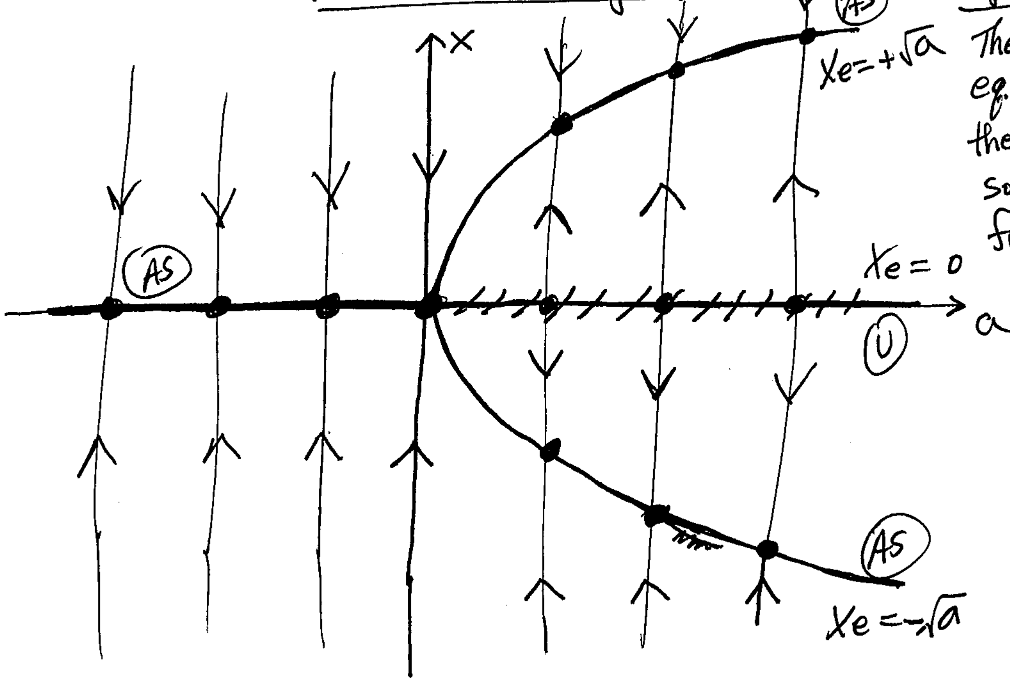
$x = \pm\sqrt{\frac{a}{3}}$ (local max/mins?) $(a \geq 0)$

$f''(x) = -6x$

$f''(\pm\sqrt{\frac{a}{3}}) = \pm 2\sqrt{a} \Rightarrow$ $x = \sqrt{\frac{a}{3}}$ is a local max, $x = -\sqrt{\frac{a}{3}}$ is a local min, $a > 0$




Bifurcation Diagram




Supercritical

The two emerging eq. sols. are AS, while the pre-existing eq. solution transits from AS to U.

## (b) Subcritical

Classical Representative Example: $\boxed{\dfrac{dx}{dt} = x(a+x^2)}$ Eq. Sols.: $\boxed{x_e = 0,\ x_e = \pm\sqrt{-a},\ a \le 0}$

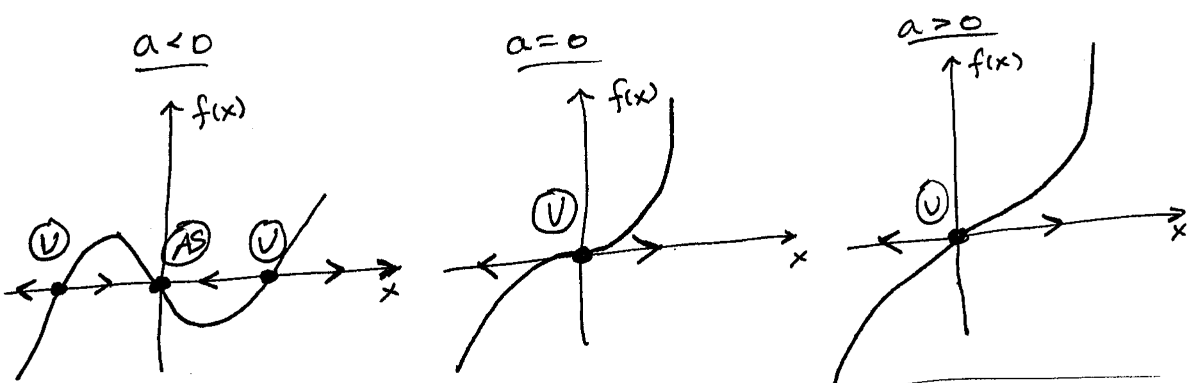
$f(x) = x(a+x^2) = ax + x^3$

$f'(x) = a + 3x^2 = 0$

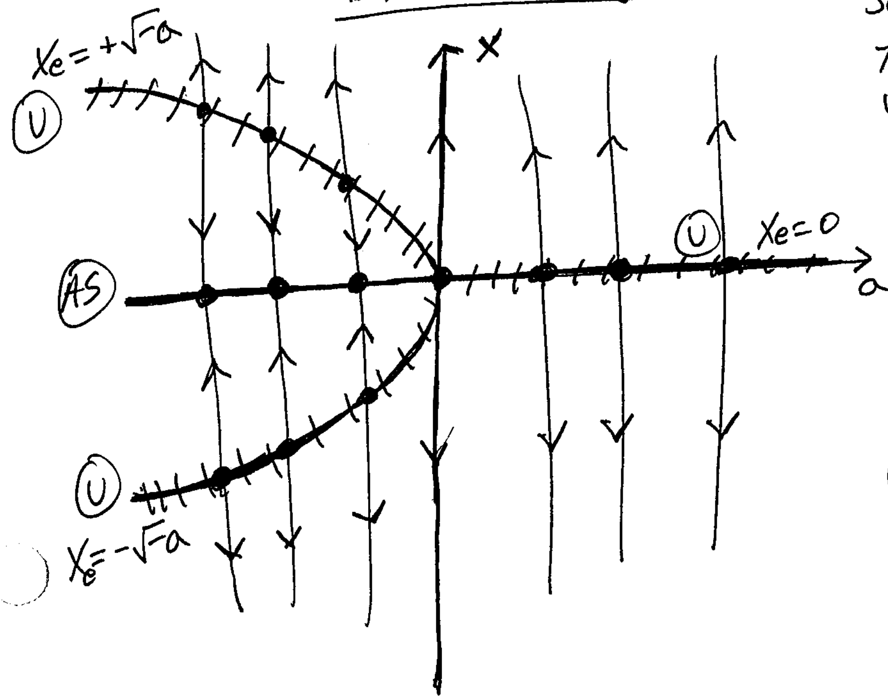
$x^* = \pm\sqrt{\frac{-a}{3}}$ (local max/min?) $(a \le 0)$

$f''(x) = 6x$

$f''\left(\pm\sqrt{\frac{-a}{3}}\right) = \pm 2\sqrt{-a} \Rightarrow$ $+\sqrt{-a/3}$ is a local min. $-\sqrt{-a/3}$ is a local max.




Bifurcation Diagram




Subcritical

The two emerging eq. sols. are U, while the pre-existing eq. sol. transits from U to AS.

Note: SS and S eq. sols can only occur at bifurcation points.

**Example:** $\frac{dx}{dt} = x(x-1) + a$

Sketch the bifurcation diagram

**Eq. Sols.:** $f(x) = x(x-1) + a = x^2 - x + a = 0$

$$X_e = \frac{1 \pm \sqrt{1-4a}}{2} = \frac{1}{2} \pm \sqrt{\tfrac{1}{4} - a}$$

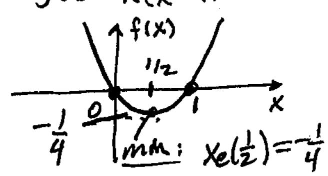
$$\boxed{X_e = \frac{1}{2} \pm \sqrt{\tfrac{1}{4} - a},\ a \le \tfrac{1}{4}} \Rightarrow X_e = X_e(a) \text{ is a square root function}$$

$$X_{max} = X_e\left(\tfrac{1}{4}\right) = \tfrac{1}{2}$$

Rather than working with this expression, we may take a graphical approach

Consider the case $a=0$.

$f(x) = x(x-1)$




The bifurcation parameter $a$ shifts this curve vertically by $a$ units. $\Rightarrow$ When $a = \frac{1}{4}$, the local min. lies on the x-axis.

$\Rightarrow$ Saddle-Node Bifurcation at $a = \frac{1}{4}$.

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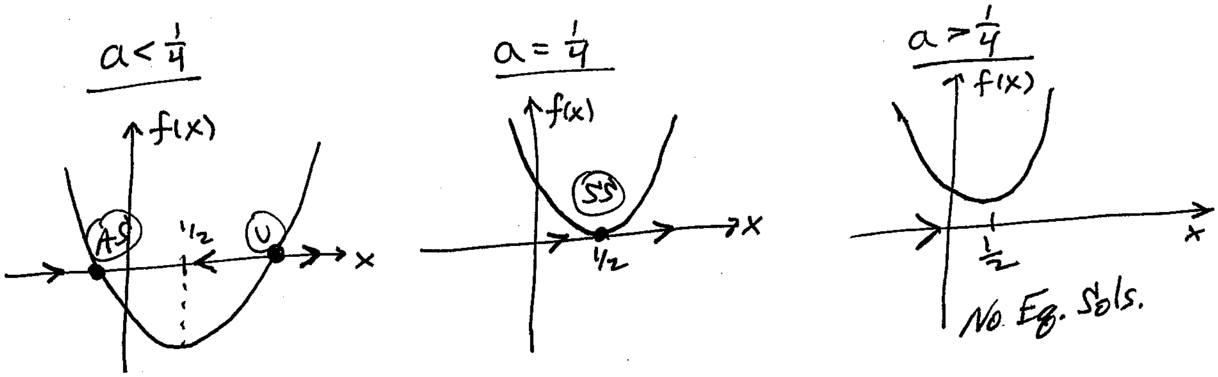
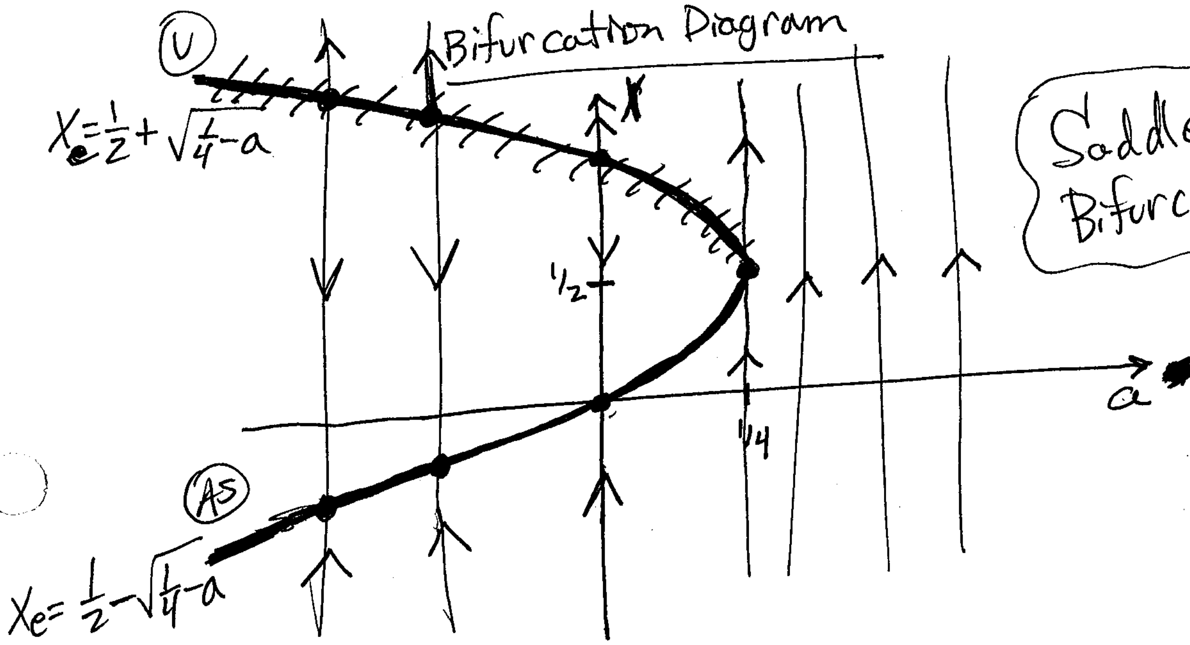


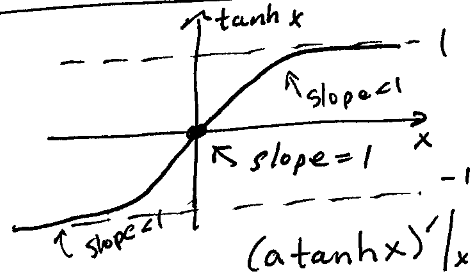




Saddle-Node Bifurcation

Example: $\frac{dx}{dt} = x - a\tanh x$

Sketch the bifurcation diagram

Recall: $\tanh = \frac{e^x - e^{-x}}{e^x + e^{-x}}$




$(a\tanh x)'|_{x=0} = a$

Eq. Sols.

$f(x) = x - a\tanh x = 0$

$x = ?$

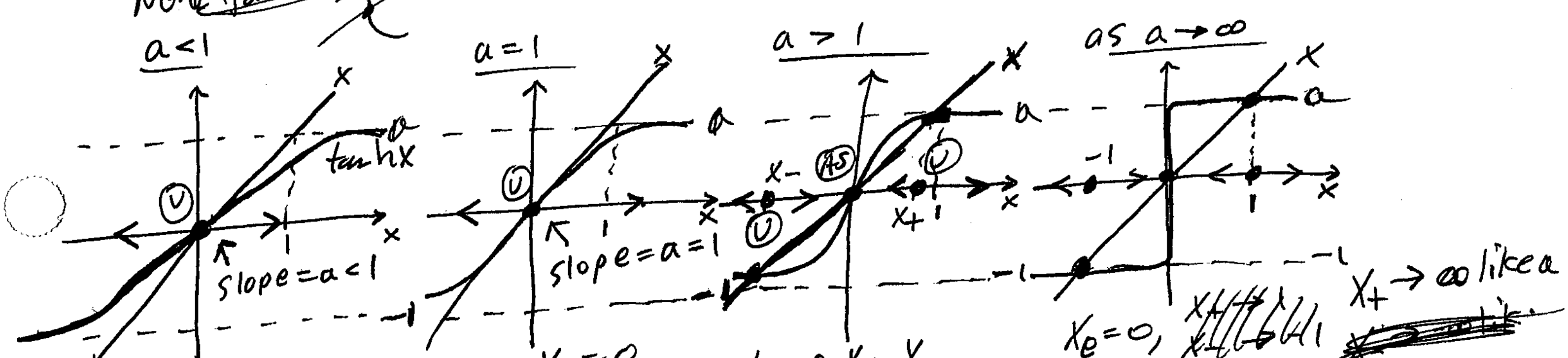
Graphical Approach

Let $g(x) = x$
$h(x) = a\tanh x$

$f(x) = 0 \iff g(x) = h(x)$

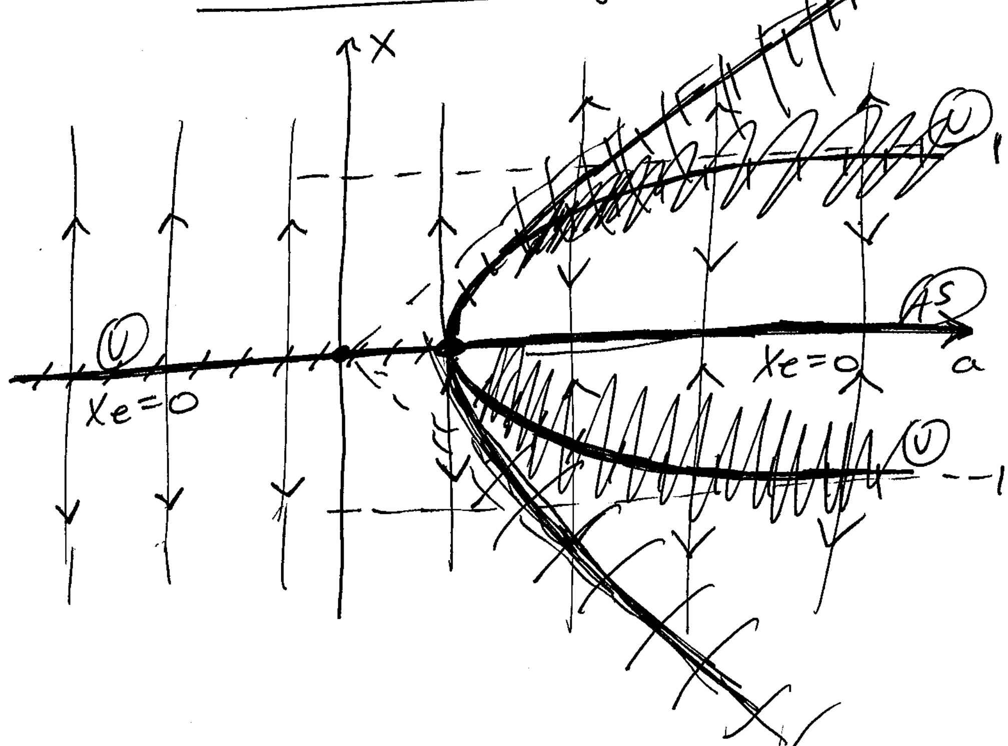
Plot $g(x) = x$ and $h(x) = a\tanh x$ on the same set of axes. The points of intersection correspond to the eq. sols.

None for $a \le 0$




$X_e = 0$    $X_e = 0$    $X_e = 0, X_+, X_-$    $X_e = 0,$    $X_+ \to \infty$ like a

## Bifurcation Diagram



# Exponential Growth/Decay

Let $x(t)$ = amount of some quantity (mass, temperature, population, ...) at time $t$.

Underlying Assumption: The rate of change of $x$ is proportional to the size of $x$

$$\frac{dx}{dt} \propto x \Rightarrow \boxed{\frac{dx}{dt} = kx}$$

$k$ = proportionality constant

If $x \geq 0$, $k>0 \Rightarrow$ growth; $k<0 \Rightarrow$ decay

Units of $k = \frac{1}{\text{time}}$

$\frac{dx}{dt}$ = growth rate

$\frac{dx/dt}{x}$ = relative growth rate

The underlying assumption is equivalent to the assumption that the relative growth rate is constant ($=k$).

Suppose $x>0$ and $k>0$. The underlying assumption suggests that it always requires the same amount of time for $x$ to double in size (regardless of the initial size).

e.g.

| $t$ | $x$ |
|---|---|
| 0 | 100 |
| 3 | 200 |
| 6 | 400 |
| 12 | 800 |
| ⋮ | ⋮ |

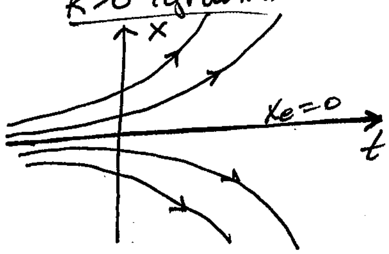
$x(9) \neq 600$
$x(9) < 600$

Same idea for decay

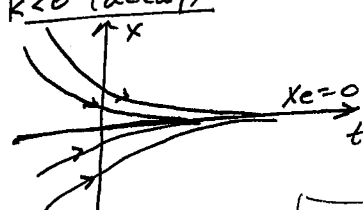
| $t$ | $x$ |
|---|---|
| 0 | 100 |
| 3 | 50 |
| 6 | 25 |
| 12 | 12.5 |
| ⋮ | ⋮ |

Gen Sol.: $x(t) = Ce^{kt}$ $\quad x_e = 0$ $\quad$ Flow: $\boxed{\phi(t, x_0) = x_0 e^{kt}}$

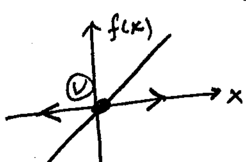
$k>0$ (growth)



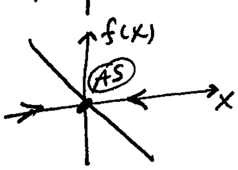

$k<0$ (decay)



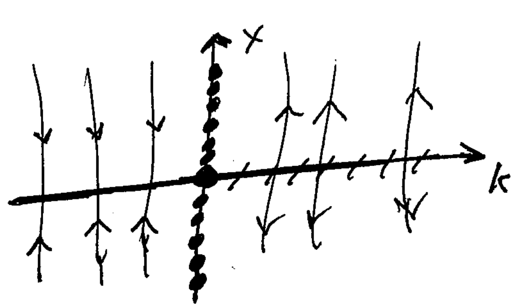

The stability of $x_e$ changes as $k$ passes through 0
$\Rightarrow$ Bifurcation
"change is stability"




$f(x) = kx$




## Bifurcation Diagram

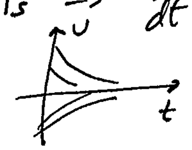


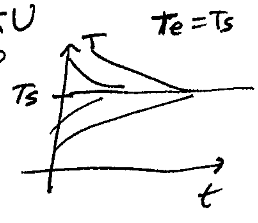

Applications of the Exponential Growth/Decay ODE.

1. Radioactive Decay
2. Carbon Dating (Carbon decay)
3. Population Growth/Decline (e.g. bacteria)
4. Newton's Law of Heating/Cooling

$$\frac{dT}{dt} = -k(T - T_s), \quad k > 0$$

$U = T - T_s \Rightarrow \frac{dU}{dt} = -kU$ $\quad U_e = 0$ $\quad T_e = T_s$



# Logistic Population Model

The exponential ODE, with $k>0$, may be used to model population growth (e.g. bacteria, mice, ...). However, it corresponds to an unlimited growth potential in that it predicts that the population size goes to infinity in time, which is not realistic. That is, the exponential model assumes that there are unlimited resources (e.g. food, water, shelter, ...) available. Such an assumption may be reasonable in the short-term, but at some point the population will outgrow its environment. For example, the size of a population of a few mice in a large field may grow exponential at first, but at some point the population size will become large enough that the mice will have to compete for the limited resources available. For such a case, rather than using an exponential model, it is more realistic to allow the relative growth rate to depend on the population size P.

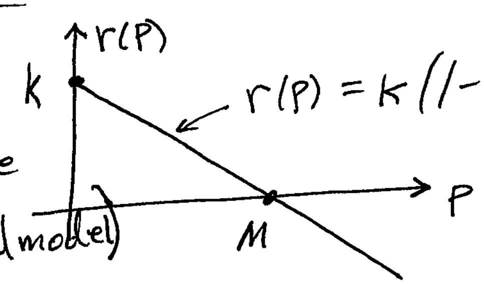
Exponential Model: $\dfrac{dP}{dt} = kP$ relative growth rate $= \dfrac{dP/dt}{P} = k = \text{constant}$

More realistic Model: $\dfrac{dP}{dt} = r(P)\cdot P$

Expect the relative growth rate $r(P)$ to decreases with P, and even be negative if P is sufficiently large.

Simplest Choice: Let r be a linear decreasing function of P.

$P \geq 0$




$k$ = intrinsic rel. growth rate (no limitations; same as for the exponential model)

When P is near 0, there is essentially no limitations $\Rightarrow$ $\boxed{\dfrac{dP}{dt} = k\left(1-\dfrac{P}{M}\right)P}$ $k, M > 0$

---

General Solution: $P(t) = \dfrac{M}{1+Ce^{-kt}}$

$C = \dfrac{M}{P_0} - 1$

Flow: $P(0) = P_0$ $\Rightarrow$ $\boxed{\phi(t,P_0) = \dfrac{M}{1+\left(\frac{M}{P_0}-1\right)e^{-kt}}}$ $P_0 \neq 0$

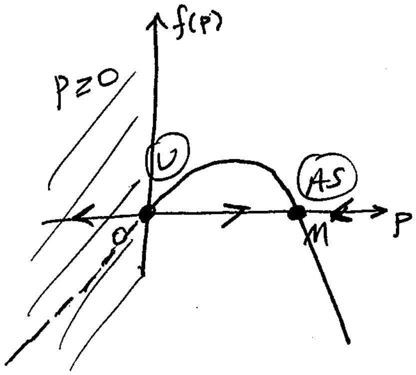
OR $\phi(t,P_0) = \dfrac{P_0 M}{P_0 + (M-P_0)e^{-kt}}$, all $P_0$

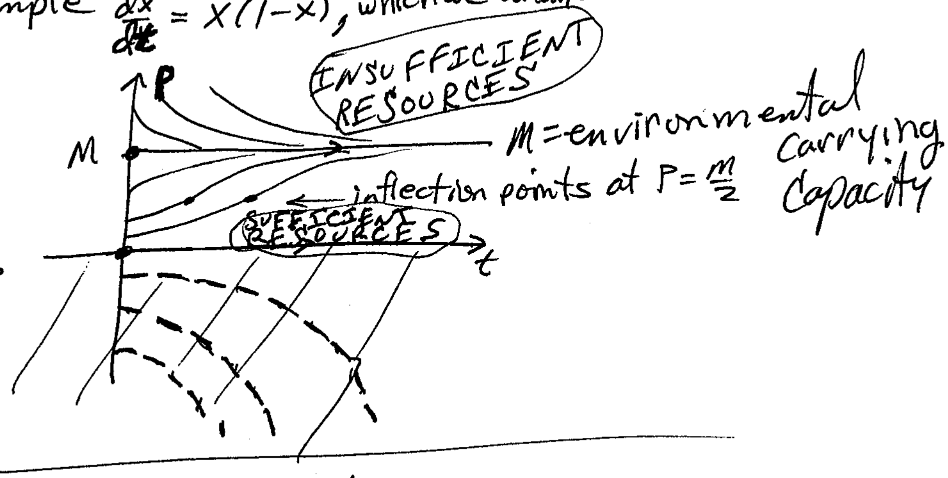
# Logistic Population Model

$$\boxed{\frac{dP}{dt} = k\left(1-\frac{P}{M}\right)P,\ P \geq 0\ ;\ k, M > 0}$$

Eq. Sols.: $P_e = 0,\ P_e = M$

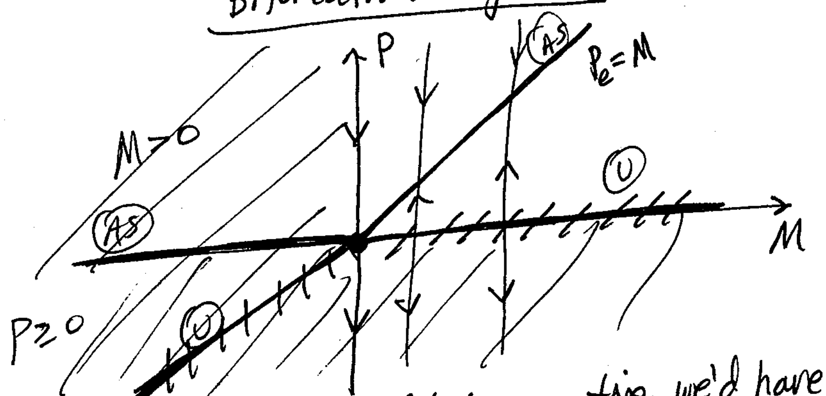
Note: When $k = M = 1$, the Logistic Population model reduces to our previous example $\frac{dx}{dt} = x(1-x)$, which we analyzed extensively.



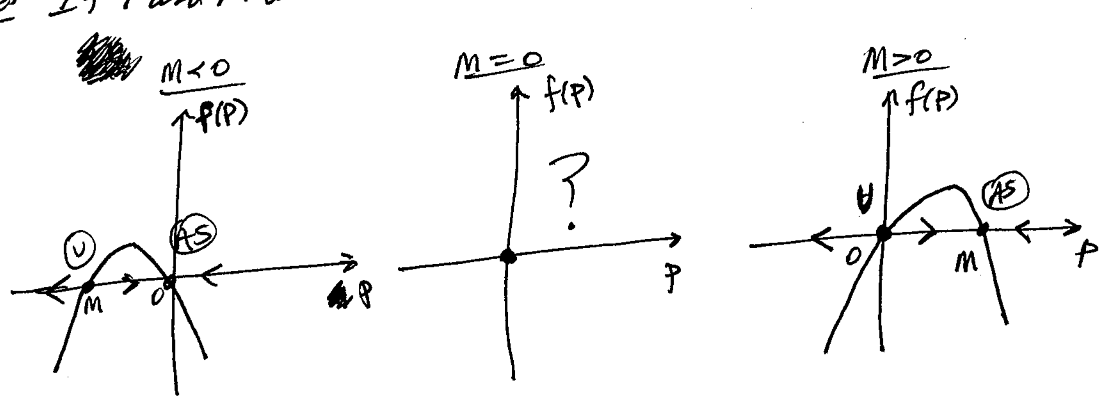

$P_e = M \Rightarrow$ Try fixing $k$ and varying $M$.

Bifurcation Diagram ($M$ = bifurcation parameter)




Transcritical Bifurcation

Note: If $P$ and $M$ were allowed to be negative, we'd have a Transcritical Bifurcation at $M = 0$.



# Logistic Population with Immigration/Emmigration

$$\frac{dP}{dt} = k\left(1-\frac{P}{M}\right)P + I(P), \quad P \ge 0, t \ge 0 \ ; \ k, M > 0$$

$I(P) > 0 \Rightarrow$ Immigration

$I(P) < 0 \Rightarrow$ Emmigration (or Harvesting a Natural Resource)

Assume $I$ is constant.

$$\Rightarrow \boxed{\frac{dP}{dt} = k\left(1-\frac{P}{M}\right)P + I}$$

Draw the bifurcation diagram with $I$ as the bifurcation parameter.

Eq. Sols: $f(P) = k\left(1-\frac{P}{M}\right)P + I = 0$

$$\cancel{-\frac{k}{M}}\left(P^2 - MP - \frac{M}{k}I\right) = 0$$

$$P_e = \frac{M \pm \sqrt{M^2 + 4\frac{M}{k}I}}{2} = \frac{M}{2}\left[1 \pm \sqrt{1 + \frac{4I}{kM}}\right]$$

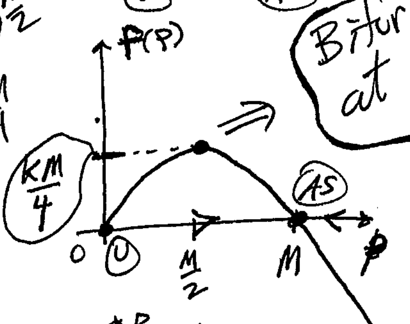
$$\boxed{P_e = \frac{M}{2}\left(1 \pm \sqrt{1 + \frac{4I}{kM}}\right)}, \quad 1 + \frac{4I}{kM} \ge 0$$

for $\boxed{I \ge -\frac{kM}{4}}$

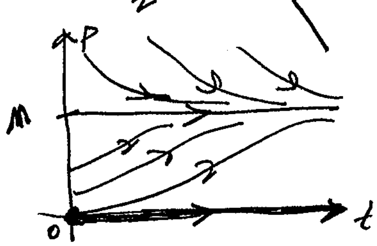
$\underline{I = 0} \Rightarrow$ Logistic Model

$P_e = 0$, $P_e = M$
(U) (AS)

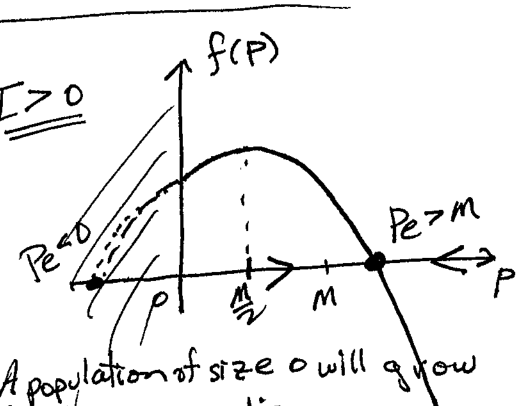
$f\left(\frac{M}{2}\right) = k\left(\frac{1}{2}\right)\frac{M}{2}$
$= +\frac{kM}{4}$



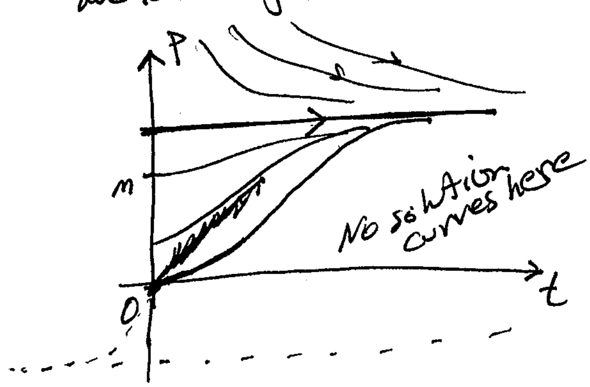

$\Rightarrow$ Bifurcation at $I = -\frac{kM}{4}$



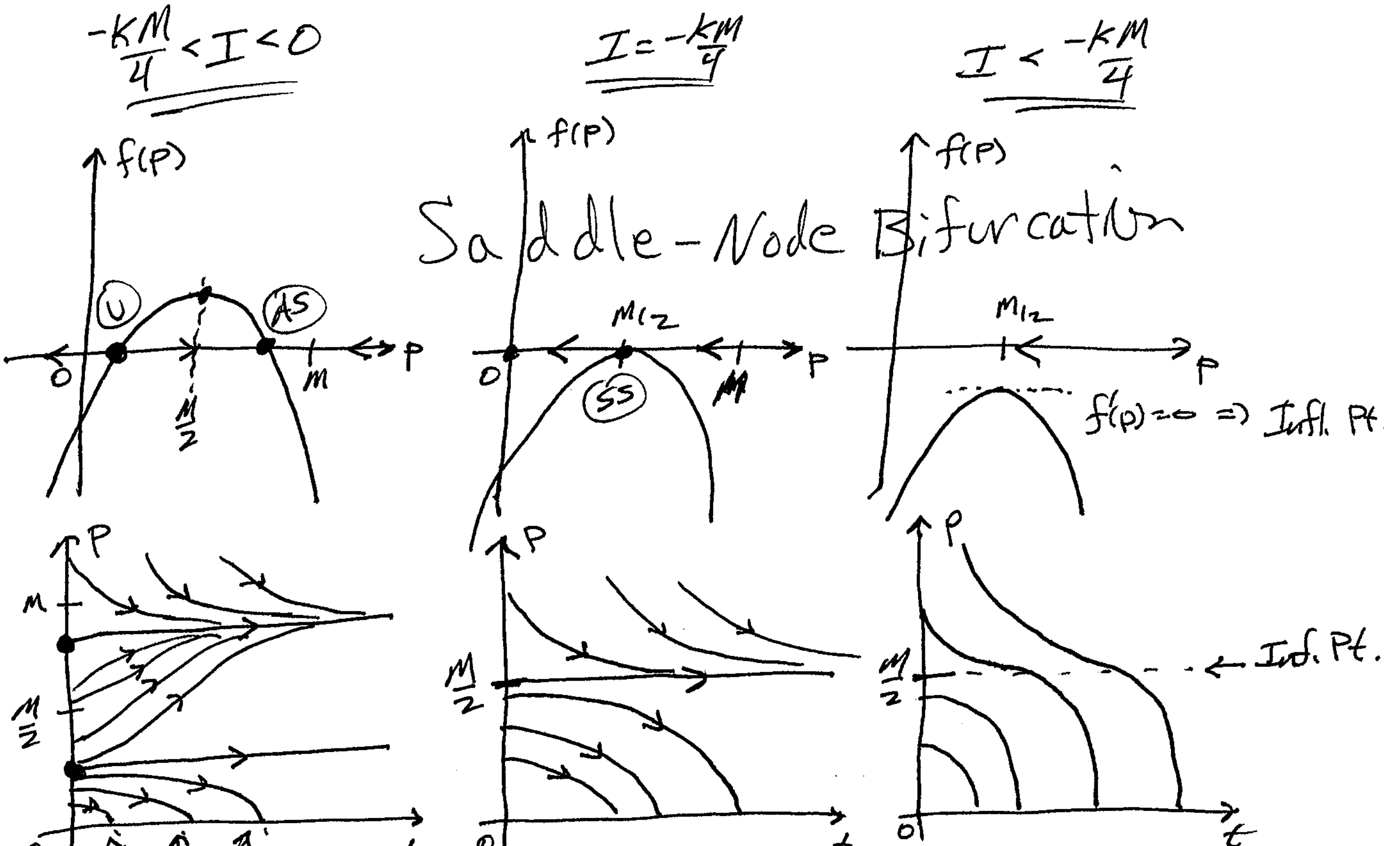

$\underline{I > 0}$




A population of size 0 will grow due to immigration



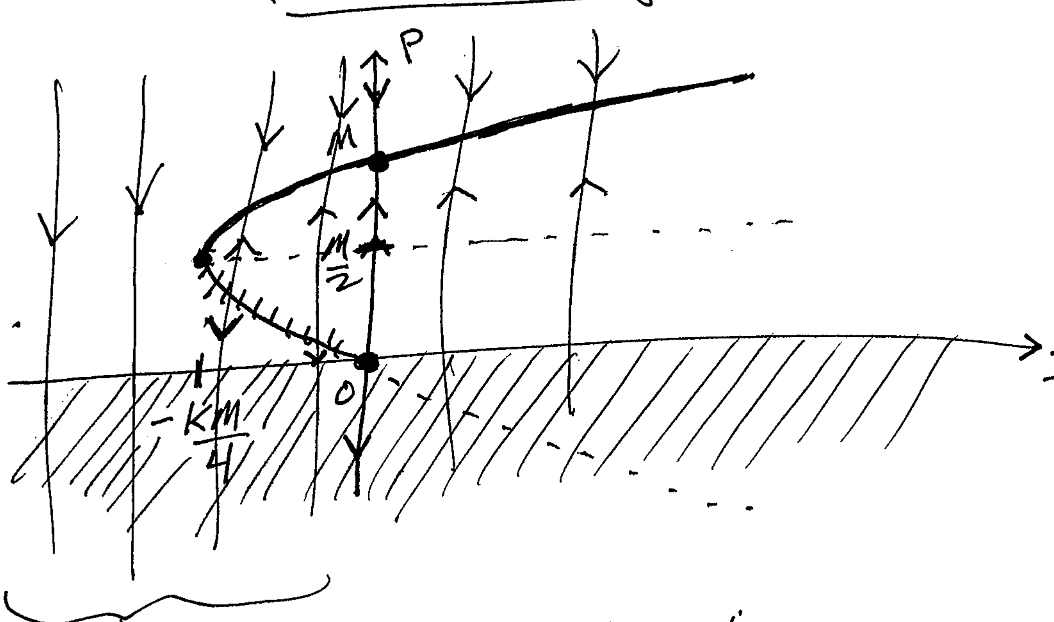
# Saddle-Node Bifurcation




Extinction in finite time

when $I=0$, the growth rate of the population is maximal at $\frac{M}{2}$

## Bifurcation Diagram




Too much Harvesting $\Rightarrow$ Extinction

# Linearization

*Idea:* Approximate a nonlinear ODE by a linear ODE.
The theory of linear ODEs is well-developed (e.g. integrating factor method).
Linear ODEs are much more reasonable to analyze than nonlinear ODEs.

Linearization can be used to determine the stability of eq. sols.

Linearization is not needed directly for $1^{st}$ order ODEs because we have the theorem ...

*Theorem:* Let $x_e$ be an eq.-sol. of $\frac{dx}{dt} = f(x)$

$f'(x_e) < 0 \Rightarrow x_e$ is AS
$f'(x_e) > 0 \Rightarrow x_e$ is U
$f'(x_e) = 0 \Rightarrow$ Inconclusive

Linear $1^{st}$ order ODE

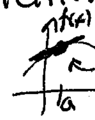
$\frac{dx}{dt} = a(t)\,x + b(t)$

a and b const. for an autonomous ODE

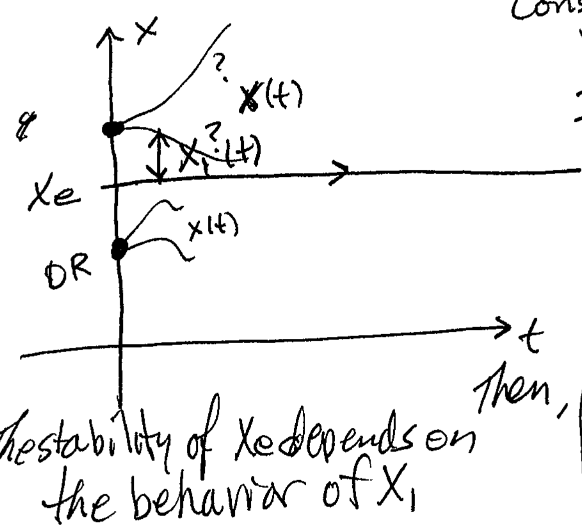
However, linearization is essential in proving this theorem.
It is important to understand linearization because it is used directly when determining the stability of eq-sols of systems of ODEs. Though the above theorem can be generalized for systems of ODEs, it is considerably more involved and it requires that we do the linearization directly. In the above theorem, the linearization was done for us once and for all, and there are the results. For systems, we must do a linearization at each eq. sol.

The ODE $\frac{dx}{dt} = f(x)$ can be linearized about $x_e$ (approximated by a linear ODE in the vicinity of $x_e$) to determine the stability of $x_e$.

e.g. tangent line




first two terms of the Taylor series $f(x) \approx f(a) + f'(a)(x-a)$




Consider a solution curve $x(t)$ which is very close to $x_e$ (as close as necessary).

If $x(t) \to x_e$, then $x_e$ is AS.
If $x(t)$ diverges from $x_e$, then $x_e$ is U

## Change of Variables

Let $\boxed{x(t) = x_e + x_1(t)}$, where $x_1(t)$ is very small in magnitude.
$x_1 = x - x_e$

Then, $x_1 \to 0 \Rightarrow x \to x_e \Rightarrow x_e$ is AS
$x_1 \to \pm\infty \Rightarrow x$ diverges from $x_e \Rightarrow x_e$ is U

The stability of $x_e$ depends on the behavior of $x_1$

## Change of Variables

$x(t) = x_e + x_1(t)$, where $x_1(t)$ is small in magnitude

Convert the nonlinear ODE for $x$ into an approximate linear ODE for $x_1$.

$\frac{dx}{dt} = f(x)$ (nonlinear) — (Expand $f$ in a Taylor Series about $x_e$)

$\frac{dx}{dt} = \frac{dx_1}{dt} \Rightarrow \frac{dx_1}{dt} = f(x_e + x_1) = \underbrace{\cancel{f(x_e)}}_{=0 \text{ since } x_e \text{ is an eq sol.}} + x_1 f'(x_e) + \frac{x_1^2}{2!} f''(x_e) + \frac{x_1^3}{3!} f'''(x_e) + \cdots$

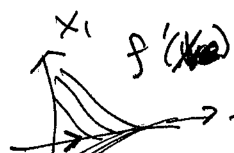
Important Idea for Systems

$\Rightarrow \frac{dx_1}{dt} = f'(x_e)x_1 + \underbrace{\cancel{\frac{x_1^2}{2!} f''(x_e)} + \cdots}_{\text{small} (\approx 0)}$ (nonlinear)

Note: If $x_1$ is small, then $x_1^2, x_1^3, \ldots$ are really small. Neglect these terms $\Rightarrow$ $\boxed{\frac{dx_1}{dt} \approx \underbrace{f'(x_e)}_{\cancel{\text{constant}}} x_1}$ (Exponential ODE (linear))

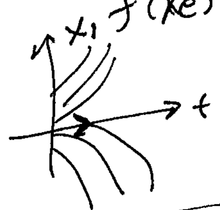
this is the linear approximation of $\frac{dx}{dt} = f(x)$ for $x$ near $x_e$, provided $f'(x_e) \neq 0$.

We have analyzed the exponential ODE. $x_{1e} = 0$ is the only eq sol. (AS if $f'(x_e) < 0$; U if $f'(x_e) > 0$)



$f'(x_e) < 0 \Rightarrow$ Exponential Decay $\Rightarrow x_1 \to 0 \Rightarrow x \to x_e \Rightarrow x_e$ is AS

$f'(x_e) < 0 \Rightarrow x_e$ is AS



$f'(x_e) > 0 \Rightarrow$ Exponential Growth $\Rightarrow x_1 \to \pm\infty \Rightarrow x$ diverges from $x_e \Rightarrow x_e$ is U

$f'(x_e) > 0 \Rightarrow x_e$ is U

---

If $f'(x_e) = 0$, we have $\frac{dx_1}{dt} = 0$ and all $x_1$ are eq sols. However, this equation is just an approximation. In exact terms, $\frac{dx_1}{dt} \neq 0$. The terms that were neglect make the right-hand side not equal to 0, and it is these terms which determine the stability of $x_e$.

---

Systems: $\frac{d\vec{x}_1}{dt} = \vec{f}'(\vec{x}_1)\, \vec{x}_1$

$\frac{d\vec{x}_1}{dt}$: vector; $\vec{f}'$: matrix (Jacobian); $\vec{x}_1$: vector

The following example illustrates the approach taken for systems.

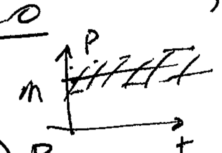
Example: Logistic Population Model

$$\frac{dP}{dt} = k\left(1-\frac{P}{M}\right)P \;;\; k, M > 0$$

Eq. Sols: $f(P) = k\left(1-\frac{P}{M}\right)P = 0$

$\cancel{P_e = M}$ $\quad P_e = 0$

$f(P) = k\left(P - \frac{P^2}{M}\right)$




$f'(P) = k\left(1 - \frac{2P}{M}\right)$

Linearized ODE for P near Pe: $\quad \frac{dP_1}{dt} = f'(P_e)\,P_1 \qquad f'(P_e) = k\left(1-\frac{2P_e}{M}\right)$

$P(t) = P_e + P_1(t) \qquad \boxed{\frac{dP_1}{dt} = k\left(1-\frac{2P_e}{M}\right)P_1}$ for any eq sol. Pe.

$P_e = 0$: $\quad \frac{dP_1}{dt} = kP_1 \;\Rightarrow$ Exponential Growth $\Rightarrow P_1 \to \pm\infty$
$k>0$
$\Rightarrow \boxed{P_e = 0 \text{ is } U}$

$P_e = M$: $\quad \frac{dP_1}{dt} = -kP_1 \;\Rightarrow$ Exponential Decay $\Rightarrow P_1 \to 0$
$\Rightarrow \boxed{P_e = M \text{ is } AS}$

---

OR Linearize directly about M ($P_e = M$)

Let $P = P_e + P_1 \qquad \frac{dP}{dt} = k\left(1-\frac{P}{M}\right)P$

$\frac{dP}{dt} = \frac{dP_1}{dt} \qquad \frac{dP_1}{dt} = k\left(1-\frac{P_e+P_1}{M}\right)(P_e+P_1)$, $P_1$ is small

e.g. $P_e = M \Rightarrow \frac{dP_1}{dt} = k\left(1-\frac{M+P_1}{M}\right)(M+P_1)$

$= k\left(\cancel{1-1}-\frac{P_1}{M}\right)(M+P_1)$

$= k\left(\frac{-P_1}{M}\right)(M+P_1)$ (nonlinear) $= -\frac{k}{M}\left(MP_1 + \cancel{P_1^2}\right)$
neglect

$P_1$ is much smaller than M

$\Rightarrow M + P_1 \approx M$ (neglect $P_1$)

$$\frac{dP_1}{dt} = -\frac{kP_1}{M}\cdot M \;\Rightarrow\; \boxed{\frac{dP_1}{dt} = -kP_1}$$

## Section 2.1: $2^{nd}$ Order ODEs (Review)

General Form: $F(t, x, x', x'') = 0$ Normal Form: $x'' = f(t, x, x')$

---

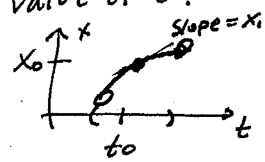
We'll consider **Linear $2^{nd}$ Order ODEs**

$1^{st}$ order
$x' + p(t)x = g(t)$

General Form: $\boxed{\underline{\underline{1}}\, x'' + p(t)x' + q(t)x = g(t)}$ ← The generalization to higher dimensions is straightforward.

IVP: $\boxed{x(t_0) = x_0 \,,\ x'(t_0) = x_1}$ ← $x$ and $x'$ are specified at a single value of $t$.

Informal theorem: If $p, q,$ and $g$ are continuous at $t_0$, then a unique solution exists in some interval containing $t_0$.

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If $g(t) \equiv 0$, the ODE is Homogeneous.

$\Rightarrow \boxed{x'' + p(t)x' + q(t)x = 0}$ (H)

Otherwise, the ODE is Nonhomogeneous ($g(t) \neq 0$ for some $t$)
$g(t)$ = nonhomogeneous term

---

For now, we'll consider Homogeneous ODEs of the form (H).

---

Principal of Superposition: If $x_1$ and $x_2$ are solutions of (H), then the linear combination $x = c_1 x_1 + c_2 x_2$ is also a solution of (H). (but not necessarily the general solution).

**Theorem:** If $x_1$ and $x_2$ are linearly independent solutions of (H), then the linear combination $x = c_1x_1 + c_2x_2$ is the general solution of (H).

**Note:** Two functions, $x_1$ and $x_2$, are linearly dependent if one is a constant multiple of the other ($x_2 = kx_1$ for some $k$ or $x_1 = kx_2$)

Otherwise, $x_1$ and $x_2$ are linearly dependent ($x_2 \neq kx_1$ for any $k$)

e.g. $x_1 = \cos^2 t$, $x_2 = 1 + \cos 2t$ $\qquad$ $\cos^2 t = \frac{1}{2}(1+\cos 2t)$, $x_1 = \frac{1}{2}x_2 \Rightarrow$ Linearly Dependent

---

Wronskian (tool for checking linear independence/dependence)

The Wronskian of two functions $x_1(t)$ and $x_2(t)$ is defined by

$$\boxed{W = \underbrace{W(x_1,x_2)(t)}_{} = \begin{vmatrix} x_1(t) & x_2(t) \\ x_1'(t) & x_2'(t) \end{vmatrix} = x_1(t)x_2'(t) - x_1'(t)x_2(t)}$$

W is a function of t, what that function is depends on $x_1$ and $x_2$.

---

**Fact:** $W(x_1,x_2)(t) \neq 0$ for all x in I ~~on I~~ $\Rightarrow$ $x_1$ and $x_2$ are linearly independent on I.

~~$W(x_1,x_2)(t) = 0$ at some point on I $\Rightarrow$ $x_1$ and $x_2$ are linearly~~ dependent on I..

Let $x_1$ and $x_2$ be solutions of (H) on an interval I.

Then, $\boxed{W \neq 0 \text{ on } I \Rightarrow x_1 \text{ and } x_2 \text{ are linearly independent on } I \Rightarrow x = c_1x_1 + c_2x_2 \text{ is the general solution of (H) on } I.}$

i.e. $\left.\begin{array}{l} x_1 \text{ and } x_2 \text{ are solutions of (H)} \\ W(x_1,x_2)(t) \neq 0 \end{array}\right\} \Rightarrow x = c_1x_1 + c_2x_2$ is the general solution of (H)

---

**Theorem:** If $x_1$ and $x_2$ are solutions of (H) on an interval I, then either $W(x_1,x_2)(t) = 0$ for all $t$ in I or $W(x_1,x_2)(t) \neq 0$ for all $t$ in I.

Examples:

1) $x'' - 2x' + x = 0$ Solutions: $x_1 = e^t$, $x_2 = te^t$

$$W(x_1, x_2)(t) = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix} = (\cancel{t}+1)e^{2t} - \cancel{te^{2t}} = e^{2t} \neq 0$$ for all $t$.

$\Rightarrow$ $x_1$ and $x_2$ are linearly independent

$\Rightarrow$ $\boxed{x = C_1 x_1 + C_2 x_2 \text{ is the general solution}}$

2) $x'' + x = 0$ Solutions: $x_1 = \cos(t + \pi/4)$, $x_2 = \sin(t - \pi/4)$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$W(x_1, x_2)(t) = \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix} = \begin{vmatrix} \cos(t+\pi/4) & \sin(t-\pi/4) \\ -\sin(t+\pi/4) & \cos(t-\pi/4) \end{vmatrix}$$

$$= \cos(t+\pi/4)\cos(t-\pi/4) + \sin(t+\pi/4)\sin(t-\pi/4)$$

$A = t + \pi/4$
$B = t - \pi/4$

$$= \cos[(t+\pi/4) - (t-\pi/4)] = \cos\frac{\pi}{2} = 0 \text{ for all } t.$$

$\Rightarrow$ $x_1$ and $x_2$ are linearly dependent

Trig. Identities $\Rightarrow$ $x_2 = -x_1$

$\Rightarrow$ $x = C_1 x_1 + C_2 x_2$ is a solution of (H), but not the general solutions

## Linear, Homogeneous, 2nd Order ODEs with Constant Coefficients

$x'' + px' + qx = 0$, where $p$ and $q$ are constants.

Let $p = \frac{b}{a}$ and $q = \frac{c}{a}$, where $a \neq 0$

$\Rightarrow \boxed{ax'' + bx' + cx = 0,\ a \neq 0}$ (HC) $a \neq 0$ ensures 2nd order

Fact: Linear, Homogeneous ODEs with Constant Coefficients of any Order have solutions of the form $x = e^{rt}$.

Plug $x = e^{rt}$ into (HC) and find $r$.

$$ax'' + bx' + cx = 0$$

$$x = e^{rt} \Rightarrow ar^2e^{rt} + bre^{rt} + ce^{rt} = 0$$

$$\cancel{e^{rt}}(ar^2 + br + c) = 0$$

$$e^{rt} \neq 0 \Rightarrow \boxed{ar^2 + br + c = 0}$$ (CE) Characteristic Equation

If $r$ is a root of (CE), then $x = e^{rt}$ is a solution of (HC).

Note: (CE) has the same coefficients as (HC). (CE) can be obtained from (HC) by replacing the $n^{th}$ derivative of $x$ by the $n^{th}$ power of $r$.

(CE) is a quadratic polynomial $\Rightarrow$ There are two roots, $r_1$ and $r_2$.

3 Cases:
1. Real Distinct Roots $(r_1 \neq r_2)$
2. Real Repeated Roots $(r_1 = r_2)$
3. Complex Conjugate Roots $(r_{1,2} = \alpha \pm i\beta,\ \beta \neq 0)$

In each we can find two linearly independent solution, and hence, the general solution

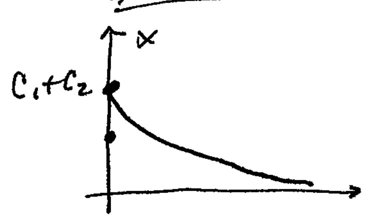
## 1. Real Distinct Roots ($r_1 \neq r_2$)

Solutions: $x_1 = e^{r_1 t}$, $x_2 = e^{r_2 t}$ $\quad W(x_1,x_2)(t) = \underbrace{(r_2 - r_1)}_{\substack{\neq 0 \\ \text{since } r_1 \neq r_2}} \underbrace{e^{(r_1+r_2)t}}_{\substack{\neq 0 \\ \text{since } e^x \neq 0}} \neq 0$

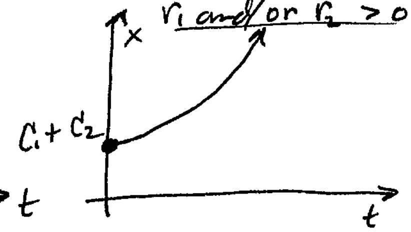
$\Rightarrow$ $x_1$ and $x_2$ are linearly independent

$\Rightarrow \boxed{x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}}$ (General Solution)

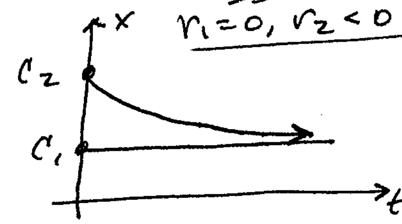
$r_1, r_2 < 0$




$r_1$ and/or $r_2 > 0$




$r_1 = 0,\ r_2 < 0$




Example: $x'' - 3x' + 2x = 0$

CE: $r^2 - 3r + 2 = 0$

$(r-1)(r-2) = 0$

$r = 1 \quad r = 2$

$\Rightarrow \boxed{x(t) = c_1 e^{t} + c_2 e^{2t}}$ (gen. sol.)

## 2. Real Repeated Roots ($r_1 = r_2$)

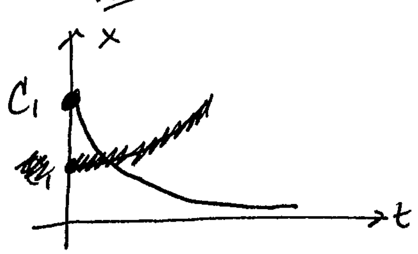
The double root yields only one solution ($x_1 = e^{r_1 t}$). The method of Reduction of Order can be used to find a second solution ($x_2 = t e^{r_1 t}$).

$x_1 = e^{r_1 t}$, $x_2 = t e^{r_1 t}$ $\quad W(x_1,x_2)(t) = e^{2r_1 t} \neq 0$ for all $t$

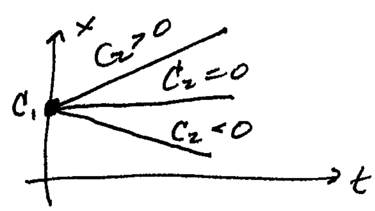
$\Rightarrow$ $x_1$ and $x_2$ are linearly independent.

$\Rightarrow \boxed{x(t) = (c_1 + c_2 t)\, e^{r_1 t}}$ (General Solution)

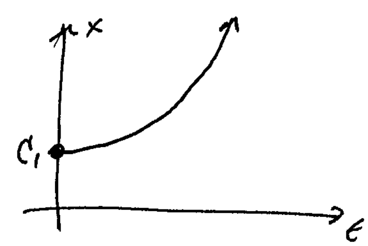
$r_1 < 0$




$r_1 = 0 \quad x = c_1 + c_2 t$




$r_1 > 0$




Example: $x'' + 2x' + x = 0$

CE: $r^2 + 2r + 1 = 0$

$(r+1)^2 = 0$

$r = -1, -1$

$\Rightarrow \boxed{x(t) = (c_1 + c_2 t)e^{-t}}$

## 3. Complex Conjugate Roots ($r_{1,2} = \alpha \pm i\beta,\ \beta \neq 0$)

the roots give two complex solutions : $V_1 = e^{(\alpha+i\beta)t} = e^{\alpha t}(\cos\beta t + i\sin\beta t)$

$V_2 = e^{(\alpha-i\beta)t} = e^{\alpha t}(\cos\beta t - i\sin\beta t)$

Real solutions are preferred.

Euler's Formula ($e^{i\theta} = \cos\theta + i\sin\theta$)

Using the Principal of Superposition, two real solutions can be found.

$x_1 = \frac{V_1+V_2}{2} = e^{\alpha t}\cos\beta t$ are solutions by the Principal of Superposition

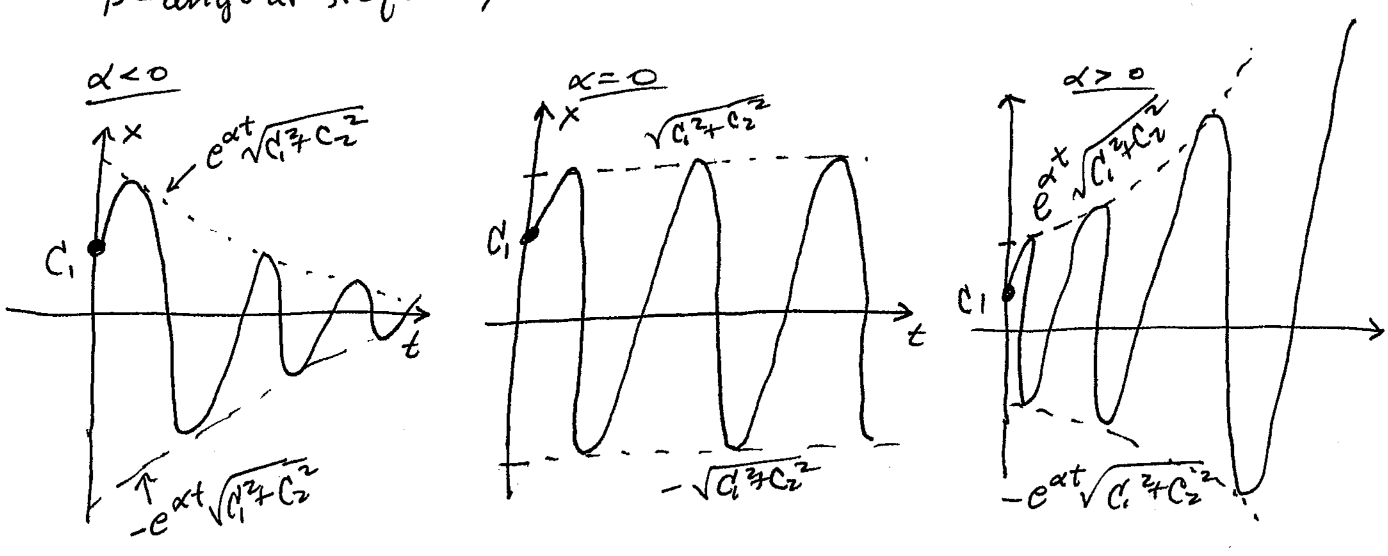
$x_2 = \frac{V_1-V_2}{2i} = e^{\alpha t}\sin\beta t$

$W(x_1,x_2)(t) = \underbrace{\beta}_{\neq 0}\, \underbrace{e^{2\alpha t}}_{\neq 0} \neq 0 \Rightarrow$ $x_1$ and $x_2$ are linearly independent

$$\Rightarrow \boxed{x(t) = e^{\alpha t}\underbrace{\left[C_1\cos\beta t + C_2\sin\beta t\right]}_{\text{periodic: Amplitude} = \sqrt{C_1^2+C_2^2}}}$$

$\alpha$ = growth rate of the amplitude

$\beta$ = angular frequency of oscillation




Example: $x''+2x'+5x=0$

CE: $r^2+2r+5=0$

$r = \frac{-2\pm\sqrt{4-20}}{2} = -1\pm 2i$

$r = -1\pm 2i \quad \alpha=-1$
$\beta = 2$

$$\Rightarrow \boxed{x(t) = e^{-t}\left[C_1\cos 2t + C_2\sin 2t\right]}$$

# Chapter 2: Planar Linear Systems of $1^{st}$ Order ODEs

2 equations/2 unknowns (x,y)

Chapter 2 bounces back and forth between systems and linear algebra (matrices). We'll discuss matrices first, and then consider systems of ODE.

## Section 2.3: Preliminary Linear Algebra

### Matrices/Vectors

Notation: Scalars: $a, b, c, \ldots$ (lower-case letters)

Vectors: $\vec{a}, \vec{b}, \vec{c}, \ldots$ (lower-case with arrows)

Matrices: $A, B, C, \ldots$ (capital letters)

Planar Systems involve $2\times2$ matrices and $2\times1$ vectors
(two dimensional) We'll focus on these for now.

Multiplication: $AB = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}$

$$A\vec{v} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} a_{11}v_1 + a_{12}v_2 \\ a_{21}v_1 + a_{22}v_2 \end{pmatrix}$$

Note: Multiplication does not necessarily commute.
i.e. $AB$ does not necessarily equal $BA$

Definition: If $AB = BA$, then $A$ and $B$ are said to commute.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, (column vector)

where $a, b, c, d, v_1,$ and $v_2$ are complex numbers in general.
(or functions)

1. Zero Matrix/vector: $\boxed{0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}}$ $\boxed{\vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}$

$0 + A = A + 0 = A$ (additive identity) $\leftrightarrow$ $\vec{0} + \vec{v} = \vec{v} + \vec{0} = \vec{v}$ (additive identity)

$0 \cdot A = A \cdot 0 = 0$ (0 commutes with all matrices) $\boxed{0\vec{v} = \vec{0}}$

2. Identity Matrix: $\boxed{I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$

$AI = IA = A$
(multiplicative identity)
($I$ commutes with all matrices)
Also, $I\vec{v} = \vec{v}$

3. Inverse Matrices:

Definition: A square matrix $A$ is invertible (or nonsingular) if there exists a matrix $A^{-1}$ such that

$$AA^{-1} = A^{-1}A = I.$$

$A^{-1}$ is called the inverse of $A$

If $A^{-1}$ does not exist, $A$ is noninvertible (or singular).

Note: If $A^{-1}$ exists, it is unique. (i.e. $A$ has at most one inverse)

4. Determinant: $\boxed{\det(A) = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}$ (scalar)

$|A|$ may be negative

Theorem: $\boxed{A \text{ is nonsingular (i.e. } A^{-1} \text{ exists) if and only if } \det(A) \neq 0}$

5. Formulas: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow \boxed{A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

trace of $A$: $\boxed{tr(A) = a + d}$

$\boxed{\det(A^{-1}) = \frac{1}{\det(A)}}$ $\boxed{\det(AB) = \det(A)\det(B)}$

6. Linear Independence: Two vectors $\vec{x}$ and $\vec{y}$ are linearly dependent if one is a constant multiple of the other. That is, if either $\vec{x} = c\vec{y}$ or $\vec{y} = c\vec{x}$ for some scalar $c$. Otherwise, $\vec{x}$ and $\vec{y}$ are linearly independent.

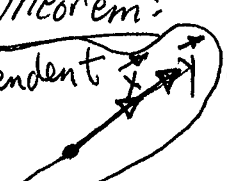
Let $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

Theorem:

$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = 0$ if and only if $\vec{x}$ and $\vec{y}$ are linearly dependent.

$\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \neq 0$ if and only if $\vec{x}$ and $\vec{y}$ are linearly independent.

Dependent $\Rightarrow$




e.g. $\vec{x} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ $\vec{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{vmatrix} -2 & 0 \\ 3 & 1 \end{vmatrix} = -2 - 0 = -2 \neq 0$

$\Rightarrow$ $\vec{x}$ and $\vec{y}$ are linearly independent

$\vec{x} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ $\vec{y} = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$ $\begin{vmatrix} -2 & 4 \\ 3 & -6 \end{vmatrix} = 12 - 12 = 0$

$\Rightarrow$ $\vec{x}$ and $\vec{y}$ are linearly dependent

$\vec{y} = -2\vec{x}$

# Linear Systems of Algebraic Equations

We'll consider 2x2 systems (2 equations/2 unknows)

General Form (2x2 system):

$$\boxed{\begin{aligned} ax+by &= b_1 \\ cx+dy &= b_2 \end{aligned}} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

Unknowns: $x$ and $y$

where $a, b, c, d, b_1$, and $b_2$ are, in general, complex constants.

---

System (1) can be written in matrix form.

Let $\boxed{A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$, $\boxed{\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}}$, and $\boxed{\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}$.

$$A\vec{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \vec{b}$$

$$\Rightarrow \boxed{A\vec{x} = \vec{b}}$$

$A$ is called the coefficient matrix.
$\vec{x}$ is the unknown to be determined.

---

System (1) represents two lines in the xy-plane.

Assume $b, d \neq 0$. (These are special cases which yield the same results)

Then,

$$(1)\quad y = -\frac{a}{b}x + \frac{b_1}{b} \qquad \text{slope} = m_1 = -\frac{a}{b}$$

$$(2)\quad y = -\frac{c}{d}x + \frac{b_2}{d} \qquad \text{slope} = m_2 = -\frac{c}{d}$$

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = bd\left(\frac{a}{b} - \frac{c}{d}\right) = bd(m_2 - m_1)$$

$$\boxed{\det(A) = bd(m_2 - m_1)}$$

**Theorem:** Consider the linear system $A\vec{x} = \vec{b}$.

1. If $A$ is nonsingular ($\det(A) \neq 0$), then $A^{-1}$ exists and $\vec{x} = A^{-1}\vec{b}$ is the unique solution.

---

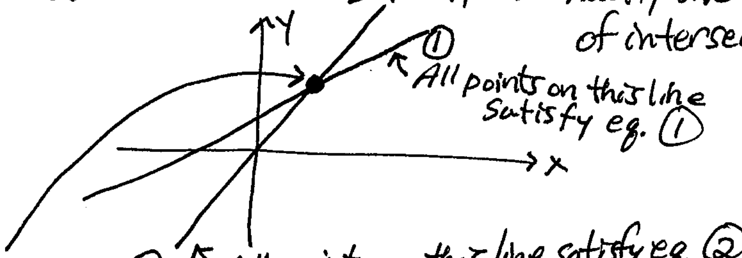
$A\vec{x} = \vec{b}$

$A^{-1}A\vec{x} = A^{-1}\vec{b}$

$I\vec{x} = A^{-1}\vec{b}$

$\vec{x} = A^{-1}\vec{b}$

$\det(A) \neq 0 \Rightarrow m_2 \neq m_1 \Rightarrow$ exactly one point of intersection



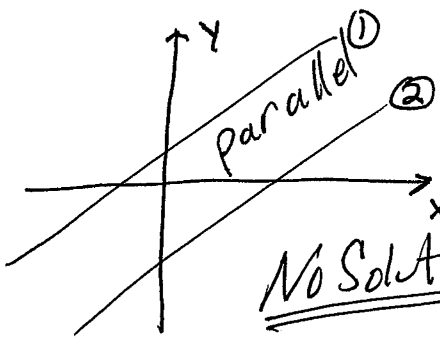

this is the unique point at which both equations are satisfied

---

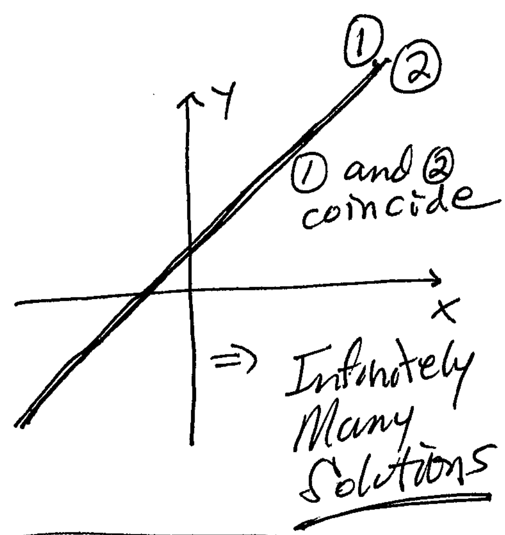
2. If $A$ is singular ($\det(A) = 0$), then $A^{-1}$ does not exist, and there are either no solutions or infinitely many solutions.

$\det(A) = bd(m_2 - m_1) = 0$

$b, d \neq 0 \Rightarrow m_1 = m_2$




No Solutions




$\Rightarrow$ Infinitely Many Solutions

---

It was assumed that $b, d \neq 0$

The above results hold true if $b=0$, $d=0$, or both.

**Example**: Solve: $\begin{array}{l} 2x+y=1 \\ -x+2y=-1 \end{array}$ (Find x and y)

**Matrix Form**: $A=\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$, $\vec{x}=\begin{pmatrix} x \\ y \end{pmatrix}$, $\vec{b}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\det(A)=\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 4-(-1) = 5 \neq 0 \Rightarrow$ $A^{-1}$ exists

$A^{-1}=\frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$

$\vec{x}=A^{-1}\vec{b}=\frac{1}{5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$\boxed{\vec{X}=\frac{1}{5}\begin{pmatrix} 3 \\ -1 \end{pmatrix}}$ OR $\boxed{\begin{array}{l} X = 3/5 \\ Y = -1/5 \end{array}}$ **Unique Solution**

---

**Example**: ① $4x+2y=2$, ② $2x+y=1$ $\Rightarrow \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

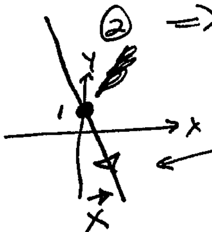
$A\vec{x}=\vec{b}$

Relates to eigenvalues/vectors

$\det(A)=\begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = 4-4=0 \Rightarrow$ $A^{-1}$ does not exist

There are either no solutions or infinitely many solutions.

① $\Rightarrow y=1-2x$
② $\Rightarrow y=1-2x$
} The two equations yield only one condition for two unknowns. $\Rightarrow$ **Infinitely Many Solutions**




$\boxed{\vec{X}=\begin{pmatrix} x \\ 1-2x \end{pmatrix} \text{ is a solution for all } x}$

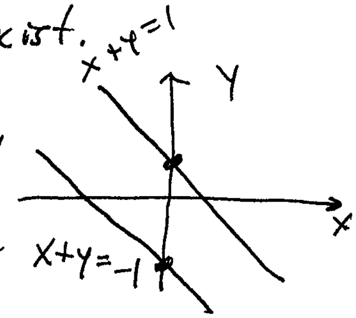
---

**Example**: $\begin{array}{l} x+y=1 \\ x+y=-1 \end{array} \Rightarrow \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$A\vec{x}=\vec{b}$

$\det(A)=\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 1-1=0 \Rightarrow$ $A^{-1}$ does not exist.

There is **No Solution** since $x+y$ cannot equal 1 and $-1$ at the same time



# Homogeneous Linear Systems of Algebraic Equations

<u>Definition</u>: The linear system $A\vec{x}=\vec{b}$ is <u>homogeneous</u> if $\vec{b}=\vec{0}$, and nonhomogeneous if $\vec{b}\neq\vec{0}$.

---

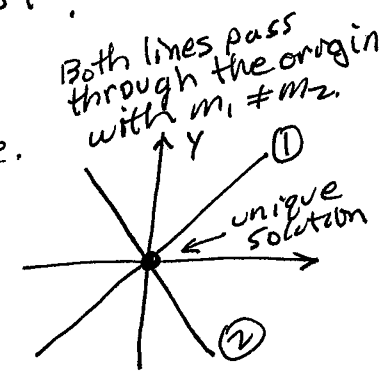
We'll be interested in homogeneous linear systems

$$\boxed{A\vec{x}=\vec{0}} \Rightarrow \begin{pmatrix} ax+by=0 \\ cx+dy=0 \end{pmatrix}$$

<u>Note</u>: The trivial solution ($\vec{x}=\vec{0}$) always exists $(A\vec{0}=\vec{0})$.
We'll be interested in finding nontrivial solutions.

<u>Question</u>: When do nontrivial solutions exist?

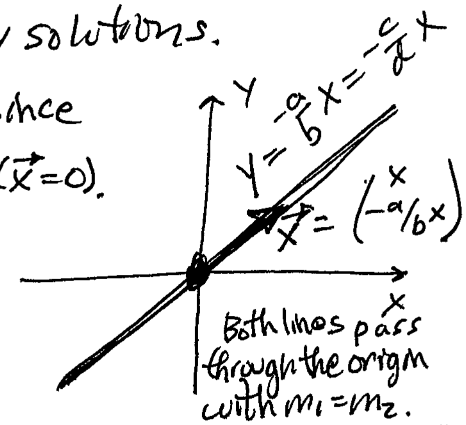
Consider the above theorem:




1. If <u>$\det(A)\neq 0$</u>, the solution is <u>unique</u>.
   $\Rightarrow$ $\vec{x}=0$ is the unique solution
   i.e. $\det(A)\neq 0 \Rightarrow$ No Nontrivial Solutions

2. If <u>$\det(A)=0$</u>, then there are either no solutions or infinitely many solutions.

   There must be infinitely many solutions since we know that there is at least one solution ($\vec{x}=0$).

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<u>Answer</u>: Nontrivial Solutions exist only when $\det(A)=0$.

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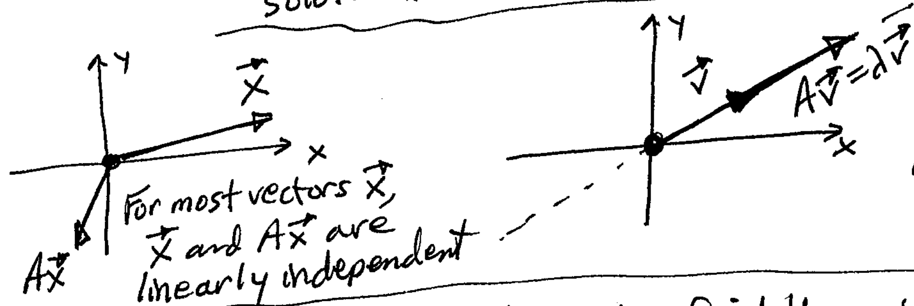
<u>Preview</u>: We'll need to find nontrivial solutions of the homogeneous linear system $(A-\lambda I)\vec{v}=0$, $\vec{v}=?$

$\Rightarrow$ <u>Need $\det(A-\lambda I)=0$</u>

Specifically, we'll need to find the values of $\lambda$ which make this true.

# Section 2.5: Eigenvalues/Eigenvectors

**Definition:** The values of $\lambda$ for which the linear system $A\vec{v} = \lambda\vec{v}$ has nontrivial solutions ($\vec{v} \neq 0$) are called the eigenvalues of $A$, and the corresponding nontrivial solutions are called the eigenvectors of $A$.




The eigenvectors of $A$ are the vectors such that $\vec{v}$ and $A\vec{v}$ are linearly dependent.

Given a matrix $A$, we'll need to find the eigenvalues/vectors.

Rewrite the equation:

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

$I\vec{v} = \vec{v}$

$$A\vec{v} - \lambda I\vec{v} = \vec{0}$$

$\vec{v} = 0$ is always a solution $\boxed{(A - \lambda I)\vec{v} = \vec{0}}$ Homogeneous Linear System

Recall: Nontrivial solutions exist only when $\det(A - \lambda I) = 0$.

$\Rightarrow$ $\boxed{\text{The eigenvalues of } A \text{ are the values of } \lambda \text{ for which } \det(A - \lambda I) = 0}$

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. To find the eigenvalues of $A$, set $\det(A - \lambda I) = 0$ and solve for $\lambda$.

$$\det(A - \lambda I) = \det\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \det\begin{pmatrix} a-\lambda & b \\ c & d-\lambda \end{pmatrix}$$

$$= (a-\lambda)(d-\lambda) - bc = ad - (a+d)\lambda + ad - bc \overset{\text{set}}{=} 0$$

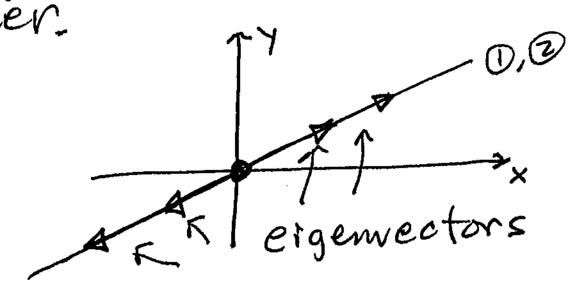
$$\boxed{\begin{aligned} \det(A - \lambda I) &= \lambda^2 - (a+d)\lambda + (ad - bc) = 0 \\ &= \lambda^2 - tr(A)\lambda + \det(A) = 0 \end{aligned}}$$

Quadratic Polynomial $\Rightarrow$ There are 2 eigenvalues ($\lambda_1$ and $\lambda_2$) (possibly repeated) $\lambda_1 = \lambda_2$

Then for each eigenvalue $\lambda_j$ ($j=1,2$), find the corresponding eigenvectors $\vec{v}_j$ by solving the homog. linear system $(A-\lambda_j I)\vec{v}_j = \vec{0}$.

Note: Since $\det(A-\lambda_j I)=0$, there are infinitely many linearly dependent solutions (eigenvectors). That is, the infinitely many eigenvectors are all constant multiples of each other.

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$




When stating the eigenvectors of a matrix $A$, it is conventional to specify only a single eigenvector. It is understood that all multiples of the specified vector are also eigenvectors.

---

Summary: To find the eigenvalues/vectors of a matrix $A$,

1) Solve $\det(A-\lambda I)=0$ to find the eigenvalues ($\lambda_1$ and $\lambda_2$)

2) For each eigenvalue, solve

$$(A-\lambda_j I)\vec{v}_j = \vec{0} \text{ to find } \vec{v}_j \ (j=1,2).$$

**Example:** Find the eigs of $A = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 1 \\ 2 & -\lambda \end{pmatrix}$$

1. $\det(A-\lambda I) = (1-\lambda)(-\lambda) - (2)(1) = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1) = 0$

$\lambda_1 = 2 \quad \lambda_2 = -1$

2. $\lambda_1 = 2$ $\quad (A-\lambda_1 I)\vec{V}_1 = \vec{0}, \quad \vec{V}_1 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$A - \lambda_1 I = \begin{pmatrix} 1-2 & 1 \\ 2 & -2 \end{pmatrix} \qquad \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \; -v_1 + v_2 = 0$

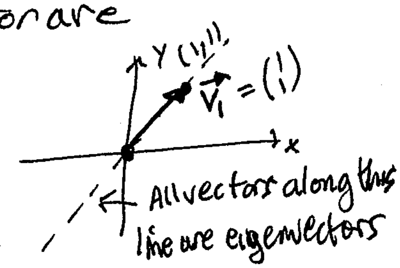
$\quad\;\; 2v_1 - 2v_2 = 0$

The two equations yield the same condition: $V_1 = V_2$ $\;\Rightarrow\; \vec{V}_1 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{V}_1 = v_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for all $v_1$

It is only necessary to indicate a single eigenvector since it is understood that all constant multiples of the specified vector are eigenvalues as well.

Pick $v_1 = 1 \Rightarrow \boxed{\lambda_1 = 2 \,;\, \vec{V}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

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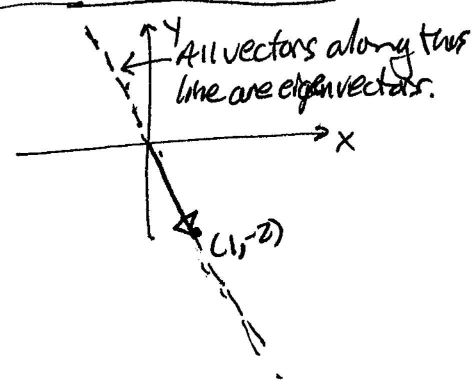
$\lambda_2 = -1$ $\quad (A-\lambda_2 I)\vec{V}_2 = \vec{0}, \quad \vec{V}_2 = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$A - \lambda_2 I = \begin{pmatrix} 1-(-1) & 1 \\ 2 & -(-1) \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \; 2v_1 + v_2 = 0$

$\quad\;\; 2v_1 + v_2 = 0$

$V_2 = -2V_1 \;\Rightarrow\; \vec{V}_2 = \begin{pmatrix} v_1 \\ -2v_1 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ for all $v_1$




Pick $v_1 = 1 \Rightarrow \boxed{\lambda_2 = -1 \,;\, \vec{V}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}}$

**Examples:** Find the eigs of $A = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$

Note: The eigenvalues of a diagonal matrix are exactly the diagonal entries of the matrix.

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 0 \\ 1 & 2-\lambda \end{pmatrix}$$

1. $\det(A - \lambda I) = (1-\lambda)(2-\lambda) - 0 = 0$

$$\lambda_1 = 1 \quad \lambda_2 = 2$$

2. $\lambda_1 = 1$: $(A - \lambda_1 I)\vec{v}_1 = \vec{0}$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $0 = 0$

$v_1 + v_2 = 0$

$v_2 = -v_1$ $\Rightarrow$ $\boxed{\lambda_1 = 1 ;\ \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

---

$\lambda_2 = 2$: $\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow$ $-v_1 = 0$

$v_1 = 0$

$v_1 = 0$ $\Rightarrow$ $\boxed{\lambda_2 = 2 ;\ \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

$i=\sqrt{-1}$
$i^2=-1$
$i^3=i^2 i=-i$
$i^4=(i^2)^2=(-1)^2=1$
$i^5=i^4 i=1\cdot i=i$
$\vdots$

# Notes on Complex Algebra

complex number: $c=a+ib$ where $i=\sqrt{-1}$ $\Rightarrow$

conjugate: $\bar{c}=a-ib$

$a=Re(c)=$ "real part of c"
$b=Im(c)=$ "imaginary part of c"

It is preferred to express complex fraction with $i$ only appearing in the numerator

$$\frac{1}{a+ib}=\frac{1}{a+ib}\cdot\frac{a-ib}{a-ib}=\frac{a-ib}{a^2+b^2} \Rightarrow \frac{1}{a+ib}=\frac{a-ib}{a^2+b^2}$$

Matrices: $\bar{A}=\begin{pmatrix}\bar{a} & \bar{b}\\ \bar{c} & \bar{d}\end{pmatrix}$

Special Case: $\underline{a=0} \Rightarrow \frac{1}{i}=-i$

OR $i^2=i\cdot i=-1$
$i=\frac{-1}{i}$
$\frac{1}{i}=-i$

**Example:** Find the eigs of $A=\begin{pmatrix}-3 & 2\\ -1 & -1\end{pmatrix}$

$$A-\lambda I=\begin{pmatrix}-3-\lambda & 2\\ -1 & -1-\lambda\end{pmatrix}$$

1. $\det(A-\lambda I)=(-3-\lambda)(-1-\lambda)-(-1)(2)=\lambda^2+4\lambda+5=0$

$$\lambda=\frac{-4\pm\sqrt{16-20}}{2}=-2\pm i$$

$$\lambda_{1,2}=-2\pm i$$

2. $\lambda_1=-2+i$ $\quad (A-\lambda_1 I)\vec{v}_1=\vec{0}$

$$\begin{pmatrix}-3-(-2+i) & 2\\ -1 & -1-(-2+i)\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}-(1+i) & 2\\ -1 & 1-i\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$\Rightarrow -(1+i)v_1+2v_2=0 \Rightarrow v_2=\frac{1+i}{2}v_1$

$-v_1+(1-i)v_2=0 \quad v_2=\frac{v_1}{1-i}\cdot\frac{1+i}{1+i}=v_1\frac{1+i}{1+1}=\frac{1+i}{2}v_1$

These are equivalent (they must be since there are infinitely many solutions)

Either $v_1$ or $v_2$ can be chosen arbitrarily, and the other is given by the relation.
For convenience (to avoid fractions)

Pick $v_1=2 \Rightarrow \boxed{\lambda_1=-2+i,\ \vec{v}_1=\begin{pmatrix}2\\ 1+i\end{pmatrix}}$

Other Possibilities

i) Pick $v_1=1 \Rightarrow \vec{v}_1=\begin{pmatrix}1\\ \frac{1+i}{2}\end{pmatrix}$

ii) Pick $v_2=1 \Rightarrow \vec{v}_1=\begin{pmatrix}1-i\\ 1\end{pmatrix}$

iii) Pick $v_1=1-i \Rightarrow \vec{v}_1=\begin{pmatrix}1-i\\ 1\end{pmatrix}$

iv) Pick $v_1=10 \Rightarrow \vec{v}_1=\begin{pmatrix}10\\ 5(1+i)\end{pmatrix}$

$\vdots$

The eigenvectors corresponding to $\lambda_1=-2+i$ are all linearly dependent.

$\underline{\underline{\lambda_2=-2-i}}$ $\quad (A-\lambda_2 I)\vec{v}_2=\vec{0}$

$$\begin{pmatrix} -3-(-2-i) & 2 \\ -1 & -1-(-2-i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1+i & 2 \\ -1 & 1+i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

equivalent

$$\Rightarrow \quad -(1-i)v_1+2v_2=0 \Rightarrow v_2=\frac{1-i}{2}v_1$$

$$-v_1+(1+i)v_2=0 \Rightarrow v_2=\frac{v_1}{1+i}\cdot\frac{1-i}{1-i}=\frac{1-i}{2}v_1$$

Pick $v_1=2 \Rightarrow \boxed{\lambda_2=-2-i\ ;\ \vec{v}_2=\begin{pmatrix} 2 \\ 1-i \end{pmatrix}}$

---

Notice that $\lambda_2=\bar{\lambda}_1$ and $\vec{v}_2=\overline{\vec{v}_1}$. This is true in general.

The eigenvalues are complex conjugates, and the conjugate of each eigenvector corresponding to $\lambda_1$ is an eigenvector corresponding to $\lambda_2$

---

Consider $(A-\lambda I)\vec{v}=\vec{0} \Rightarrow$ $\lambda$ is an eigenvalue of $A$ with corresponding eigen vector $\vec{v}$

Take the conjugate

$$\overline{(A-\lambda I)\vec{v}}=\overline{\vec{0}}$$

$$(\bar{A}-\bar{\lambda}\bar{I})\overline{\vec{v}}=\overline{\vec{0}}$$

Since $A$ is real $A=\bar{A}$. The same is true of $I$ and $\vec{0}$

$$(A-\bar{\lambda}I)\overline{\vec{v}}=\vec{0}$$

If $\lambda$ and $\vec{v}$ satisfy the equation, then so do $\bar{\lambda}$ and $\overline{\vec{v}}$

That is, if $\lambda$ and $\vec{v}$ ~~are an~~ is an eigenvalue/vector pair, then so is $\bar{\lambda}$ and $\overline{\vec{v}}$.

$\Rightarrow$ Once $\vec{v}_1$ is known, $\vec{v}_2$ follows immediately,

$$\boxed{\vec{v}_2=\overline{\vec{v}_1}}$$

There is no need to explicitly calculate $\vec{v}_2$, as was done in the above example.

real or complex

**Fact:** If $\lambda_1 \neq \lambda_2$, then the eigenvectors corresponding to $\lambda_1$ are <u>linearly independent</u> of those corresponding to $\lambda_2$.

This is an important fact in regard to solving linear systems of ODEs. Two linearly independent solution vectors will be needed.

---

If the eigenvalues are complex, then are conjugates, and thus $\lambda_1 \neq \lambda_2$. Repeated eigenvalues must be real.

## Real Repeated Eigenvalues

~~Example~~ If the eigenvalues are equal, ~~we may or may not be able~~ there may or may not be two linearly independent eigenvectors.

**Example:**

$A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$\lambda_1 = \lambda_2 = 2$

$(A - \lambda I)\vec{v} = \vec{0}$

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{matrix} 0 = 0 \\ 0 = 0 \end{matrix}$

There is no restriction on $v_1$ and $v_2$.

$\Rightarrow$ $v_1$ <u>and</u> $v_2$ are arbitrary

<u>All</u> vectors are eigenvectors

Pick any two linearly independent ones

$\Rightarrow \boxed{\lambda_1 = 2;\ v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$ $\boxed{\lambda_2 = 2;\ v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$

$\lambda_1 = \lambda_2 = 2$

$(A - \lambda I)\vec{v} = 0$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{matrix} v_2 = 0 \\ 0 = 0 \end{matrix} \Rightarrow \begin{matrix} v_2 = 0 \\ v_1 \text{ is arbitrary} \end{matrix}$

There is only one linearly independent eigenvector.

Pick $v_1 = 1$

$\Rightarrow \boxed{\lambda_1 = \lambda_2 = 2;\ \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

This case requires special treatment.

# Planar Systems of ODEs (2 equations/2 unknowns)

For now, we'll consider IVPs which consist of 2 first order ODEs, with 2 unknowns, $x(t)$ and $y(t)$.

Normal Form:

$$\boxed{\begin{aligned} x' &= F(t,x,y)\ ; \ x(0) = x_0 \\ y' &= G(t,x,y)\ ; \ y(0) = y_0 \end{aligned}} \quad (1)$$

Solution $x(t)$, $y(t)$

The two equations must be solved simultaneously since the equation for $x$ depends on $y$, and vice versa. The equations are said to be coupled. In the special case in which the equations can be solved independently, they are said to be decoupled.

A solution $x(t), y(t)$ satisfies both equations and the IC's.

Autonomous: $F$ and $G$ do not depend explicitly on $t$.

$$\boxed{\begin{aligned} x' &= F(x,y)\ ; \ x(0) = x_0 \\ y' &= G(x,y)\ ; \ y(0) = y_0 \end{aligned}}$$

## Linear Planar Systems

Definition: System (1) is linear if $F$ and $G$ have a linear dependence on $x$ and $y$ (but not necessarily on $t$). Otherwise, (1) is nonlinear.

Linear System:

$$\boxed{\begin{aligned} x' &= a(t)x + b(t)y + f(t)\ ; \ x(0) = x_0 \\ y' &= c(t)x + d(t)y + g(t)\ ; \ y(0) = y_0 \end{aligned}} \quad (2)$$

Vector Form:

$$\boxed{\vec{x}\,' = A(t)\vec{x} + \vec{f}(t)\ ; \ \vec{x}(0) = \vec{x}_0}$$

$$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}, \quad \vec{f}(t) = \begin{pmatrix} f(t) \\ g(t) \end{pmatrix}, \quad \vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

~~Definition: System (2) is homogeneous if $\vec{f} \equiv \vec{0}$, and nonhomogeneous otherwise.~~

~~Defn~~ If $A$ is a constant matrix, system (2) is said to have constant coefficients.

Linear + Autonomous $\Rightarrow$ Constant Coefficients

**Note:** The concepts, definitions, and theory associated with planar linear systems parallels that of $2^{\text{nd}}$ order linear ODEs.

We'll see that $\vec{x}' = A(t)\vec{x} + \vec{f}(t)$, $\vec{x}(0) = \vec{x}_0$ and $x'' + p(t)x' + q(t)x = g(t)$, $x(0) = x_0$, $x'(0) = y_0$ are equivalent.

---

This statement can be generalized to higher dimension/order.

---

## Homogeneous Linear Systems

**Definition:** System ② is homogeneous if $\vec{f}(t) \equiv \vec{0}$, and nonhomogeneous otherwise.

Homogeneous $\Rightarrow$ $\boxed{\vec{x}' = A(t)\vec{x}}$ (H)

**Principal of Superposition:** If $\vec{x}_1$ and $\vec{x}_2$ are solutions of (H), then the linear combination $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$ is also a solution of (H) (but not nec. the gen. sol.)

**Theorem:** If $\vec{x}_1$ and $\vec{x}_2$ are linearly independent solutions of (H), then $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$ is the general solution of (H).

Recall: Two vectors $\vec{x}_1 = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$ and $\vec{x}_2 = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$ are linearly independent on an interval $I$ if and only if $\begin{vmatrix} x_{11} & x_{21} \\ x_{12} & x_{22} \end{vmatrix} \neq 0$ for each $t$ in $I$.

---

Recall: Let $\lambda_1, \vec{v}_1$ and $\lambda_2 \vec{v}_2$ be the eigs of a constant matrix $A$.

If $\lambda_1 \neq \lambda_2$, then $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

real or complex

Note: Any linear $2^{nd}$ order ODE can be written as a linear planar system, and vice versa.

Consider: $\boxed{\begin{array}{c} x''+p(t)x'+q(t)x = g(t) \\ x(0)=x_0,\ x'(0)=y_0 \end{array}}$ (A)

Let $y=x'$ $\Rightarrow$ $y'=x''=-p(t)x-q(t)y+g(t)$ $\quad y(0)=x'(0)=y_0$ $\quad y(0)=y_0$

$$\Rightarrow \boxed{\begin{array}{ll} x'=y & ,\ x(0)=x_0 \\ y'=-p(t)x-q(t)y+g(t), & y(0)=y_0 \end{array}} \quad (B)$$

OR $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -p(t) & -q(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ g(t) \end{pmatrix}, \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$$\Rightarrow \vec{x}' = A(t)\vec{x} + \vec{f}(t), \quad \vec{x}(0) = \vec{x}_0$$

(A) homogeneous $(g(t)\equiv 0)$ $\Rightarrow$ (B) homogeneous $(\vec{f}(t)\equiv\vec{0})$

(A) has constant coefficients $(p,q)$ $\Rightarrow$ (B) has constant coefficients $(A)$

Consider: $\boxed{\begin{array}{l} x'=a(t)x+b(t)y+f(t),\ x(0)=x_0 \\ y'=c(t)x+d(t)y+g(t),\ y(0)=y_0 \end{array}}$ (B') $\rightarrow$ $x'(0)=a(0)x(0)+b(0)y(0)+f(0)$

$x'(0)=a(0)x_0+b(0)y_0+f(0)$

Solve for y: $\quad y=\frac{1}{b(t)}\left[x'-a(t)x-f(t)\right]$

Plug in y: $\quad \dfrac{b(t)\left[x''-a(t)x'-a'(t)x-f'(t)\right]-\left[x'-a(t)x-f(t)\right]b'(t)}{(b(t))^2} = c(t)x+\frac{d(t)}{b(t)}\left[x'-a(t)x-f(t)\right]+g(t)$

$$\left[x''-a(t)x'-a'(t)x-f'(t)\right]-\frac{b'(t)}{b(t)}\left[x'-a(t)x-f(t)\right]=b(t)c(t)x+d(t)\left[x'-a(t)x-f(t)\right]+b(t)g(t)$$

$$\boxed{\begin{array}{l} x''-\left[a(t)+d(t)+\frac{b'(t)}{b(t)}\right]x'+\left[a(t)d(t)-b(t)c(t)-a'(t)+\frac{b'(t)a(t)}{b(t)}\right]x \\ x(0)=x_0,\ x'(0)=a(0)x_0+b(0)y_0+f(0) \quad = f'(t)-\frac{b'(t)}{b(t)}f(t)-d(t)f(t)+b(t)g(t) \end{array}} \quad (A')$$

(B') homogeneous $\Rightarrow$ (A') homogeneous

(B') has constant coefficients $(a,b,c,d)$ $\Rightarrow$ (A') has constant coefficients

The roles of x and y can be reversed. That is, we may eliminate x to obtain ~~any~~ a $2^{nd}$ order linear ODE for y.

For now, we'll consider

Linear Homogeneous Planar Systems with Constant Coefficients.

$$x' = ax + by \qquad \underline{\underline{OR}} \qquad \vec{x}' = A\vec{x}$$
$$y' = cx + dy \qquad \qquad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \; A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Eigenvalues of A: $\det(A - \lambda I) = \lambda^2 - (a+d)\lambda + (ad - bc) = 0$

OR $\lambda^2 - tr(A)\lambda + \det(A) = 0$

Equivalent 2nd Order ODE: $x'' - (a+d)x' + (ad-bc)x = 0$ (linear homog. const. coefs.)

Characteristic Equation: $r^2 - (a+d)r + (ad-bc) = 0$

Observe that the eigenvalues of the coefficient matrix A of the planar system are the same as the roots of the characteristic equation of the equivalent 2nd order ODE.

Solutions: $x_1 = e^{r_1 t} = e^{\lambda_1 t}$
$x_2 = e^{r_2 t} = e^{\lambda_2 t}$

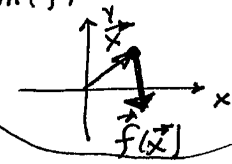
# Autonomous Planar Systems

**General Form:** $\boxed{\begin{aligned} x' &= f(x,y) \\ y' &= g(x,y) \end{aligned}}$ OR $\boxed{\vec{x}\,' = \vec{f}(\vec{x})}$ $\qquad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad \vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$$\vec{f}(\vec{x}) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$$

Vector Field

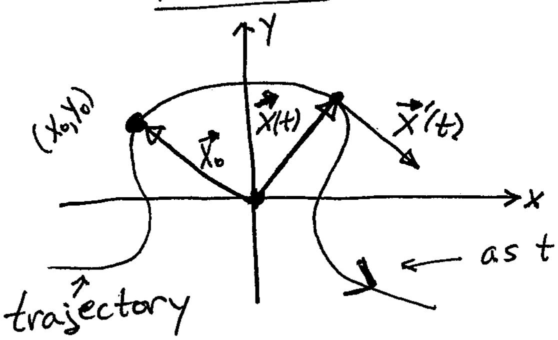
$\dim(\vec{f}) = \dim(\vec{x})$




**Initial Conditions:** $\boxed{\begin{aligned} x(0) &= x_0 \\ y(0) &= y_0 \end{aligned}}$ $\qquad \boxed{\vec{x}(0) = \vec{x}_0}$

---

For a given initial condition ($\vec{x}(0) = \vec{x}_0$), the solution, $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, yields a parametric solution curve (or <u>trajectory</u>) in the xy-plane (or <u>phase-plane</u>).




The solution vector sweeps out the solution curve as $t$ varies. The vector $\vec{x}\,'(t)$ is tangent to the solution curve at each point on the curve, and it points in the direction of increasing $t$.

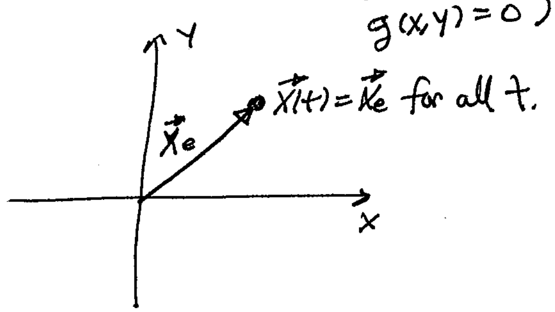
---

**Equilibrium Solutions:** (time-independent solutions)

$\vec{x}\,' = \vec{f}(\vec{x})$

Set $\vec{x}\,' = 0 \implies \boxed{\vec{f}(\vec{x}_e) = \vec{0}}$

$$\left.\begin{aligned} f(x,y) &= 0 \\ g(x,y) &= 0 \end{aligned}\right\}$$ 2 equations / 2 unknowns (x and y)

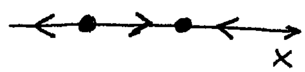



May yield multiple equilibrium solutions.

# Phase Portraits / Direction Fields

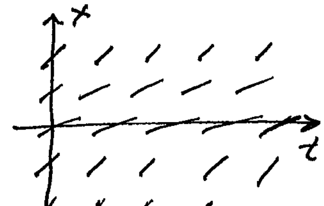
**Recall: First Order ODEs:** $x' = f(x)$ = slope function (slope of the solution curve in the tx-plane)

**Phase Portrait (Phase Line):**




$x = x(t)$ is a parametric equation for linear motion

**Direction Field:**




This works for autonomous equations also. ($x' = f(t,x)$)

For autonomous equations, the slope depends only on $x$.

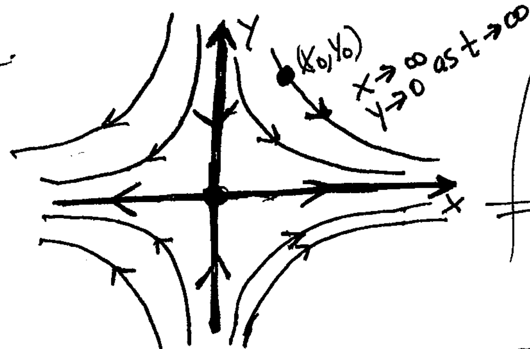
---

**Planar Systems:** $\vec{x}' = \vec{f}(\vec{x})$ = slope function (slope of the solution curves in the xy-plane)

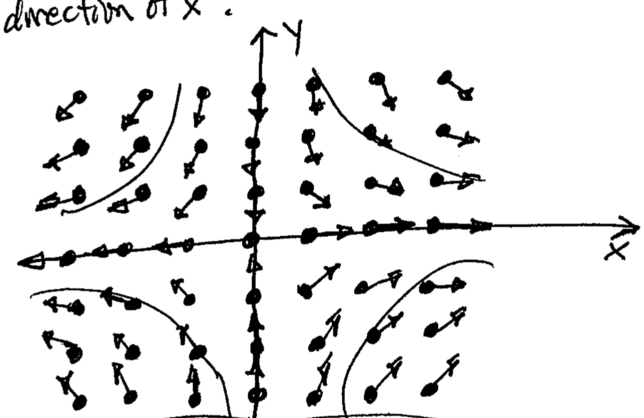
**Phase Portrait (Phase Plane):** The phase portrait is a plot of several representative trajectories (corresponding to various initial conditions)

**Direction Field:** Pick several points in the xy-plane. At each point $\vec{x}$, compute $\vec{x}' = \vec{f}(\vec{x})$. $\vec{x}'$ is tangent to the solution curve and it points in the direction of increasing $t$. Draw an arrow at $\vec{x}$ which points in the direction of $\vec{x}'$.

e.g.




$\Rightarrow$




**Summary:** At each point $\vec{x}$ in the xy-plane $\vec{f}(\vec{x})$ gives the slope of the solution curve and the direction of increasing $t$.

$\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ $\quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)} \Rightarrow \boxed{\frac{dy}{dx} = \frac{y'(t)}{x'(t)}}$

$\vec{x}'(t) = \vec{f}(\vec{x}(t))$

$\vec{x}'(t)$ points in the direction of increasing $t$.

## Classifications of Planar Systems (Planar $\Rightarrow \vec{x}$ and $\vec{f}$ have 2 components)

General Form: $\vec{f}(t, \vec{x}, \vec{x}\,') = 0$

1. Normal Form: $\vec{x}\,' = \vec{f}(t, \vec{x})$
2. Autonomous: $\vec{x}\,' = \vec{f}(\vec{x})$ $\leftarrow$ primary interest
3. Linear: $\vec{x}\,' = A(t)\vec{x} + \vec{f}(t)$
   - (a) Homogeneous: $\vec{x}\,' = A(t)\vec{x}$ $(\vec{f} \equiv 0)$
   - (b) Constant Coefficients: $\vec{x}\,' = A\vec{x} + \vec{f}(t)$ ($A$ constant)

---

## Autonomous, Linear, Homogeneous, Planar Systems $\Rightarrow (\vec{f}(\vec{x}) = A\vec{x})$

General Form: $\boxed{\begin{array}{l} x' = ax + by, \; x(0) = x_0 \\ y' = cx + dy, \; y(0) = y_0 \end{array}}$

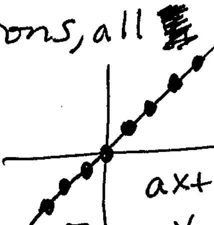
OR $\boxed{\vec{x}\,' = A\vec{x}, \; \vec{x}(0) = \vec{x}_0}$

Equilibrium Solutions: Set $\vec{x}\,' = 0 \Rightarrow A\vec{x} = 0$

If $\det(A) \neq 0$, then $\vec{x}_e = 0$ is the unique solution.

$\det(A) \neq 0 \Rightarrow \vec{x}_e = 0$ is the only eq. sol.

If $\det(A) = 0$, then there are infinitely many solutions, all lying along a lie that passes through the origin.



$ax + by = 0$

$y = -\frac{ax}{b}$

assuming $b \neq 0$ ($b = 0 \Rightarrow x = 0$)

Eigenvalues of $A$: $\det(A - \lambda I) = \lambda^2 - tr(A)\lambda + \det(A) = 0$

$$\lambda = \frac{tr(A) \pm \sqrt{(tr(A))^2 - 4\det(A)}}{2}$$

$\lambda = 0$ is a borderline case

$\det(A) \neq 0 \Leftrightarrow \lambda_1, \lambda_2 \neq 0$

$\det(A) = 0 \Leftrightarrow \lambda_1 = 0, \lambda_2 = tr(A)$

$tr(A) = 0 \Rightarrow \lambda_2 = 0$ also

# Eigenvalue Solution Method

The planar system $\vec{x}' = A\vec{x}$ is equivalent to a linear homogeneous 2nd order ODE with constant coefficients ($ax''+bx'+cx=0$). Since the latter has solutions of the form $x=e^{\lambda t}$, the former must have vector solutions of the form $\vec{x}=\vec{v}e^{\lambda t}$.

Try $\boxed{\vec{x}=\vec{v}e^{\lambda t}}$ as a solution.

$\lambda$ = some constant
$\vec{v}$ = some constant vector $\Big\}$ to be determined. (want $\vec{v}\neq 0$)

---

$\vec{x}=\vec{v}e^{\lambda t}$ Plug into $\vec{x}'=A\vec{x}$

$\vec{x}'=\vec{v}\lambda e^{\lambda t}$

$$\vec{v}\lambda \cancel{e^{\lambda t}} = A\vec{v}\cancel{e^{\lambda t}}$$

$$A\vec{v}=\lambda\vec{v}$$

OR $\boxed{(A-\lambda I)\vec{v}=\vec{0}}$ This is the equation that determines the eigs of $A$.

$\Rightarrow$ Nontrivial solutions ($\vec{x}=\vec{v}e^{\lambda t}\neq 0$) exist only when $\lambda$ is an eigenvalue of $A$, and $\vec{v}$ is the corresponding eigenvector.

Since $A$ has 2 eigenvalues ($\lambda_1$ and $\lambda_2$) with corresponding eigenvectors ($\vec{v}_1$ and $\vec{v}_2$), we have 2 solutions

$$\boxed{\begin{aligned}\vec{x}_1 &= \vec{v}_1 e^{\lambda_1 t}\\ \vec{x}_2 &= \vec{v}_2 e^{\lambda_2 t}\end{aligned}}$$

$\leftarrow$ $\vec{x}_1$ and $\vec{x}_2$ are eigenvectors for all $t$.

There are the 3 usual cases

1) Real Distinct Eigenvalues ($\lambda_1\neq\lambda_2$)
2) Real Repeated Eigenvalues ($\lambda_1=\lambda_2$)
3) Complex Conjugate Eigenvalues ($\lambda_1,\lambda_2=\alpha\pm i\beta$, $\beta\neq 0$)

---

Real Repeated Eigenvalues and Zero Eigenvalues are cases which require special treatment.

End Chap. 2

## Section 3.1: Real Distinct Eigenvalues ($\lambda_1 \neq \lambda_2$)

$\vec{X}_1 = \vec{V}_1 e^{\lambda_1 t}$
$\vec{X}_2 = \vec{V}_2 e^{\lambda_2 t}$

Since $\lambda_1 \neq \lambda_2$, $\vec{V}_1$ and $\vec{V}_2$ are linearly independent $\Rightarrow \begin{vmatrix} V_{11} & V_{21} \\ V_{12} & V_{22} \end{vmatrix} \neq 0$.

Then,

$$\begin{vmatrix} X_{11} & X_{21} \\ X_{12} & X_{22} \end{vmatrix} = \begin{vmatrix} V_{11}e^{\lambda_1 t} & V_{21}e^{\lambda_2 t} \\ V_{12}e^{\lambda_1 t} & V_{22}e^{\lambda_2 t} \end{vmatrix} = \underbrace{e^{(\lambda_1+\lambda_2)t}}_{\neq 0} \underbrace{\begin{vmatrix} V_{11} & V_{21} \\ V_{12} & V_{22} \end{vmatrix}}_{\neq 0} \neq 0$$

$\Rightarrow \vec{X}_1$ and $\vec{X}_2$ are linearly independent solutions

$\Rightarrow$ The general solution is $\vec{X} = C_1\vec{X}_1 + C_2\vec{X}_2$.

$$\Rightarrow \boxed{\vec{X}(t) = C_1\vec{V}_1 e^{\lambda_1 t} + C_2\vec{V}_2 e^{\lambda_2 t}}$$

---

**Example:** Solve $\vec{X}' = A\vec{X}$, where $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$

Eigenvalues

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda)-3 = \lambda^2-4=0$$

$\lambda = \pm 2$

Eigenvectors

$\lambda_1 = +2$: $(A-\lambda_1 I)\vec{V}_1 = \vec{0}$

$$\begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow V_1 = 3V_2 \Rightarrow \lambda_1 = 2,\ \vec{V}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$\lambda_2 = -2$: $(A-\lambda_2 I)\vec{V}_2 = \vec{0}$

$$\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow V_2 = -V_1 \Rightarrow \lambda_2 = -2,\ \vec{V}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Then, the general solution is $\vec{X} = C_1\vec{V}_1 e^{\lambda_1 t} + C_2\vec{V}_2 e^{\lambda_2 t}$

$$\Rightarrow \boxed{\vec{X}(t) = C_1\begin{pmatrix} 3 \\ 1 \end{pmatrix}e^{2t} + C_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-2t}} \quad \text{OR} \quad \boxed{\begin{aligned} X(t) &= 3C_1 e^{2t} + C_2 e^{-2t} \\ Y(t) &= C_1 e^{2t} - C_2 e^{-2t} \end{aligned}}$$

Initial Condition: $\vec{X}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \Rightarrow \begin{matrix} X(0) = 1 \\ Y(0) = 3 \end{matrix}$

$$\vec{X}(t) = \begin{pmatrix} 3C_1 e^{2t} + C_2 e^{-2t} \\ C_1 e^{2t} - C_2 e^{-2t} \end{pmatrix}$$

$$\vec{X}(0) = \begin{pmatrix} 3C_1 + C_2 \\ C_1 - C_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

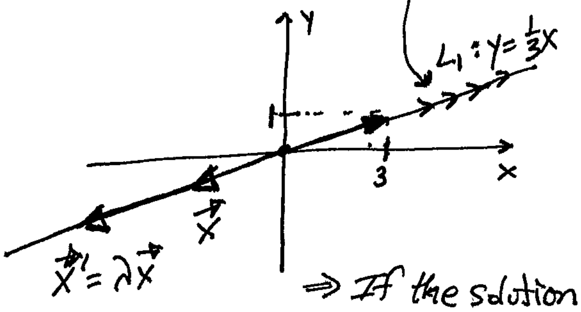
$$\begin{aligned} 3C_1 + C_2 &= 1 \\ C_1 - C_2 &= 3 \\ \hline 4C_1 \quad &= 4 \end{aligned}$$

$C_1 = 1$
$C_2 = -2$

$$\Rightarrow \boxed{\vec{X}(t) = \begin{pmatrix} 3e^{2t} - 2e^{-2t} \\ e^{2t} + 2e^{-2t} \end{pmatrix}}$$

**Phase Portrait:** The eigs of $A$ are helpful in sketching the phase portrait

Consider the direction field along the line $L_1$ containing the eigenvector $\vec{v}_1$.




$\vec{x}' = A\vec{x}$ ← slope function

Suppose $\vec{x}$ is an eigenvector.

Then, $\vec{x}' = A\vec{x} = \lambda \vec{x}$

$\Rightarrow \underline{\vec{x}' = \lambda \vec{x}} \Rightarrow \vec{x}'$ lies on the line $L_1$

$\Rightarrow$ If the solution is on the eigenline, it stays on the eigenline.

---

Consider the two solutions $c_1\vec{X}_1$ and $c_2\vec{X}_2$ ("eigenvectors")

$\binom{x_1}{y_1} = c_1\vec{X}_1 = c_1\vec{v}_1 e^{\lambda_1 t} = c_1\binom{3}{1}e^{2t}$

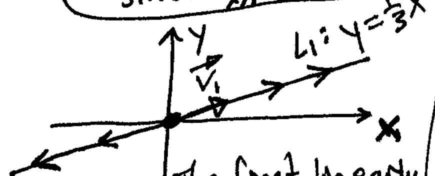
$\Rightarrow x_1 = 3c_1 e^{2t}$

$y_1 = c_1 e^{2t}$

$c_2 = 0$

$\Rightarrow$ $y_1 = \frac{1}{3}x_1$ for all $t$

$x_1, y_1 \to \pm\infty$ as $t \to \infty$ since $\lambda_1 = 2 > 0$




The first linearly independent solution stays on $L_1$

$\binom{x_2}{y_2} = c_2\vec{X}_2 = c_2\vec{v}_2 e^{\lambda_2 t} = c_2\binom{1}{-1}e^{-2t}$

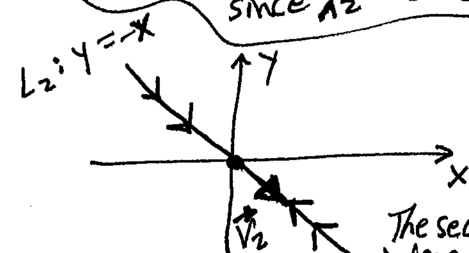
$\Rightarrow x_2 = c_2 e^{-2t}$

$y_2 = -c_2 e^{-2t}$

$c_1 = 0$

$\Rightarrow$ $y_2 = -x_2$ for all $t$

$x_2, y_2 \to 0$ as $t \to \infty$ since $\lambda_2 = -2 < 0$




The second linearly independent solution stays on $L_2$

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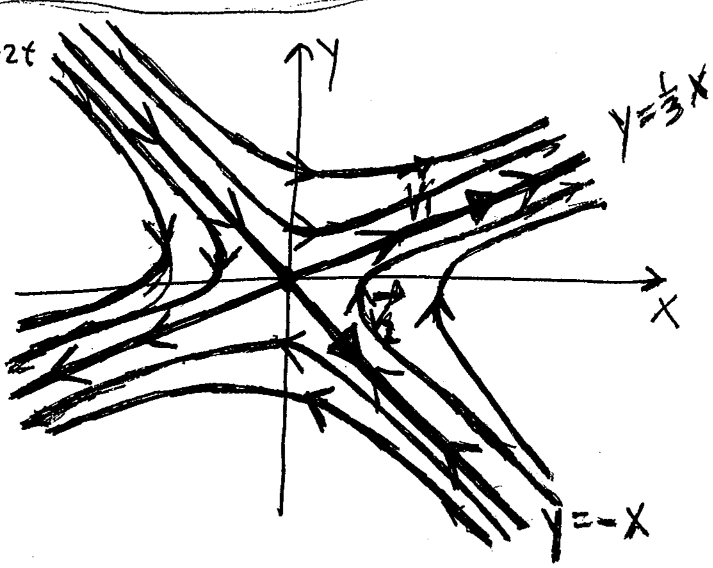
**Elsewhere:** $\vec{x} = c_1\binom{3}{1}e^{2t} + c_2\binom{1}{-1}e^{-2t}$

As $t \to +\infty$, $\vec{x} \to c_1\binom{3}{1}e^{2t}$

$\Rightarrow$ $y \to \frac{1}{3}x$

As $t \to -\infty$, $\vec{x} \to c_2\binom{1}{-1}e^{-2t}$

$\Rightarrow$ $y \to -x$



## Cases for Real Distinct Eigenvalues

1. $\lambda_1 < 0,\ \lambda_2 > 0$ (opposite sign) $(\lambda_1 \cdot \lambda_2 < 0)$
2. $\lambda_1, \lambda_2 < 0$ or $\lambda_1, \lambda_2 > 0$ (same sign) $(\lambda_1 \cdot \lambda_2 > 0)$
3. $\lambda_1 = 0,\ \lambda_2 < 0$ or $\lambda_2 > 0$ (zero eigenvalue) $(\lambda_1 = 0, \lambda_2 \neq 0)$

---

**Case 1:** $\lambda_1 < 0,\ \lambda_2 > 0$ (opposite sign) $\det(A) \neq 0 \Rightarrow \vec{x}_e = \vec{0}$ is the only eq. sol.

Simple Example: $\vec{x}\,' = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \vec{x} \Rightarrow \begin{array}{l} \lambda_1 = -1,\ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \lambda_2 = 2,\ \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array}$

$x' = -x \Rightarrow x = c_1 e^{-t}$

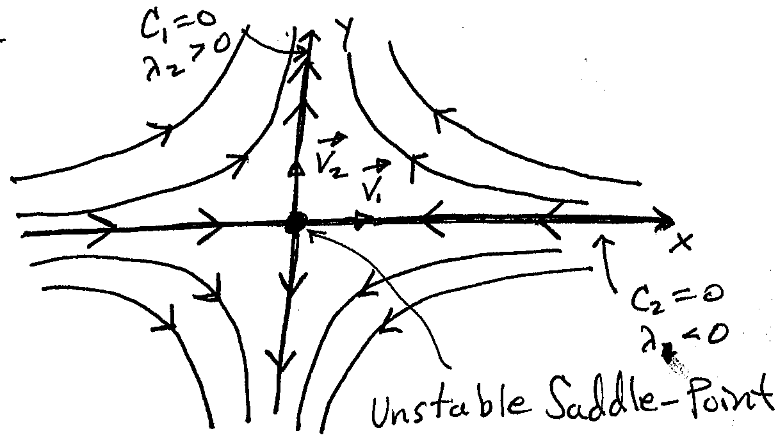
$y' = 2y \Rightarrow y = c_2 e^{2t}$

Eliminate $t$

$e^{2t} = \left(\frac{c_1}{x}\right)^2 = \frac{y}{c_2}$

$y = \frac{c_1^2 c_2}{x^2}$

$k = c_1^2 c_2 \Rightarrow \boxed{y = k/x^2}$

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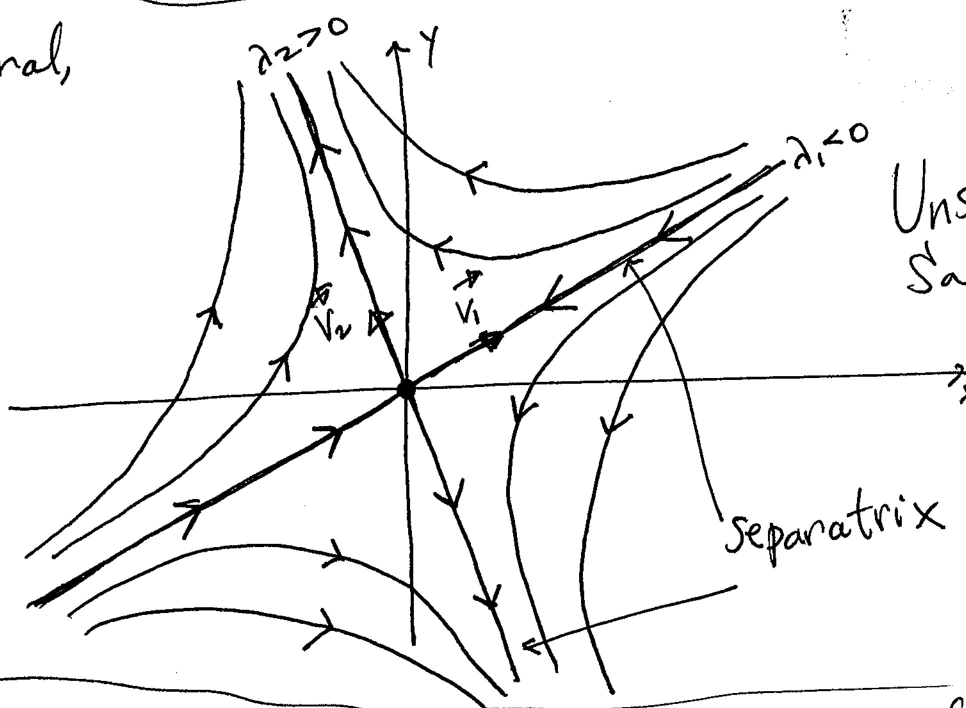
In general,




Unstable Saddle-Point

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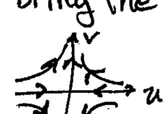
Note: A change of variable can be done to bring the general case into the form of the simple case: $\begin{array}{l} u = ax + by \\ v = cx + dy \end{array} \Rightarrow$



Case 2: $\lambda_1, \lambda_2 < 0$ or $\lambda_1, \lambda_2 > 0$ (same sign)

$\det(A) \neq 0 \Rightarrow \vec{X}_e = \vec{0}$ is the only eq. sol.

Simple Example: $\vec{X}' = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} \vec{X} \Rightarrow \begin{array}{l} \lambda_1 = -1, \ \vec{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \lambda_2 = -2, \ \vec{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{array}$

$\dot{x} = -x \Rightarrow x = c_1 e^{-t}$

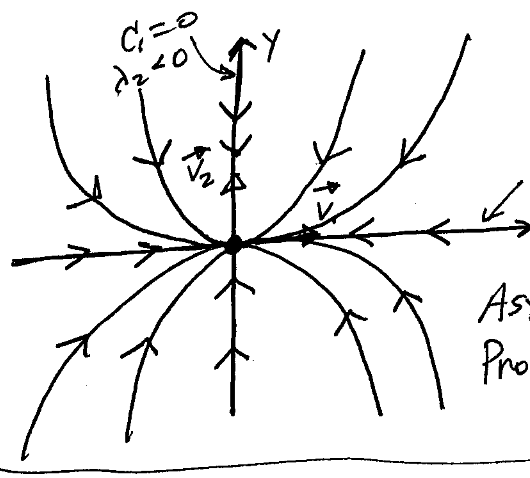
$\dot{y} = -2y \Rightarrow y = c_2 e^{-2t}$

Eliminate $t$

$e^{-2t} = \left(\frac{x}{c_1}\right)^2 = \frac{y}{c_2}$

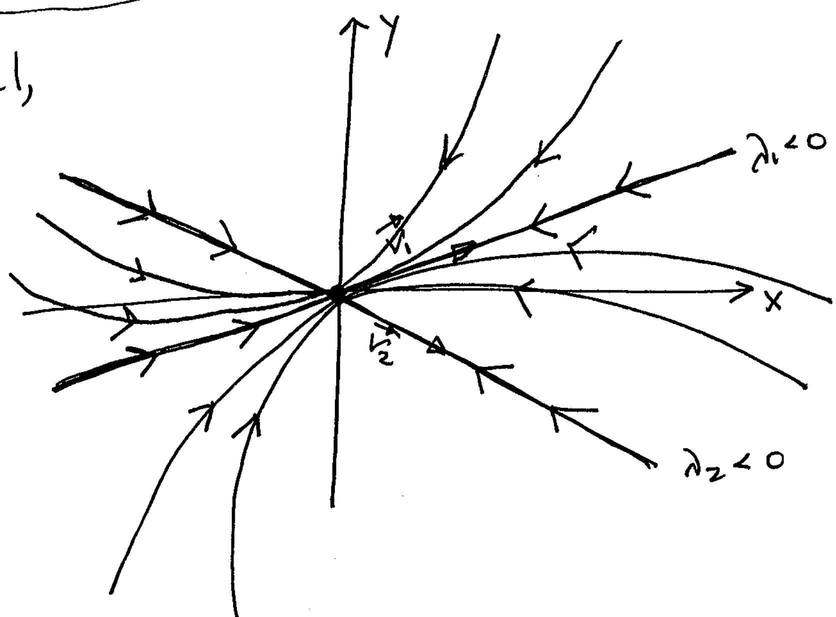
$y = \frac{c_2}{c_1^2} x^2$

$k = \frac{c_2}{c_1^2} \Rightarrow \boxed{y = kx^2}$




Asymptotically Stable Proper Node

In general,




Asymptotically Stable Node (Sink)

If $\lambda_1, \lambda_2 > 0$, the phase portraits are the same as those above, except all arrows point in the opposite direction (outward, away from $\vec{X}_e = \vec{0}$).

Unstable Node (Source)

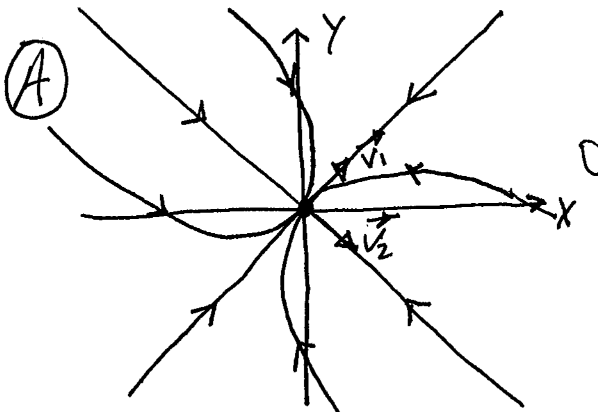
## One more feature to consider.

Suppose we have $\vec{x}' = A\vec{x}$ where $\lambda_1 = -1$, $\vec{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$

$\Rightarrow \boxed{\vec{x}(t) = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-2t}}$ $\quad \lambda_2 = -2$, $\vec{v}_2 = \begin{pmatrix}1\\-1\end{pmatrix}$

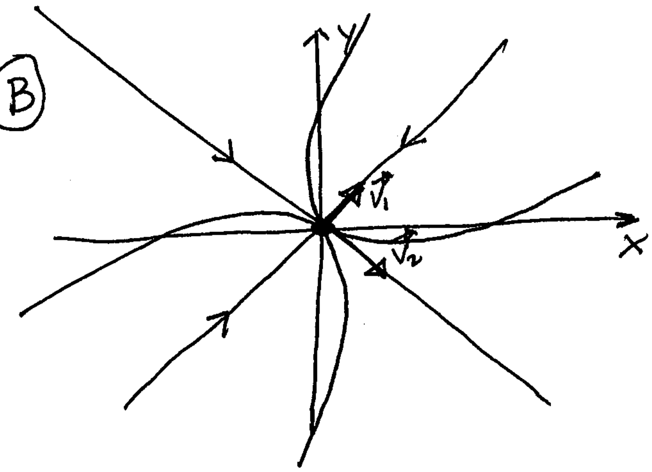
According to the above discussion, there are two possible phase portraits.

Ⓐ




OR

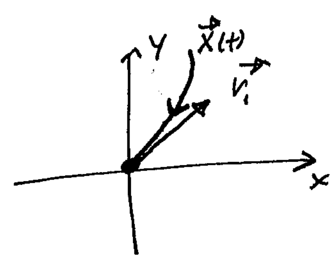
Ⓑ




## Which is correct?

As $t \to \infty$, $\vec{x} \sim c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-t}$ + much smaller term ($e^{-2t}$) $\longrightarrow \vec{0}$

Therefore, $\vec{x} \to \vec{0}$ "along" the eigenvector $\vec{v} = \begin{pmatrix}1\\1\end{pmatrix}$

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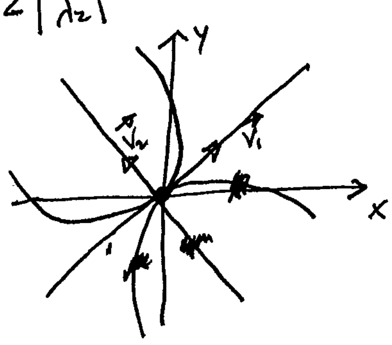
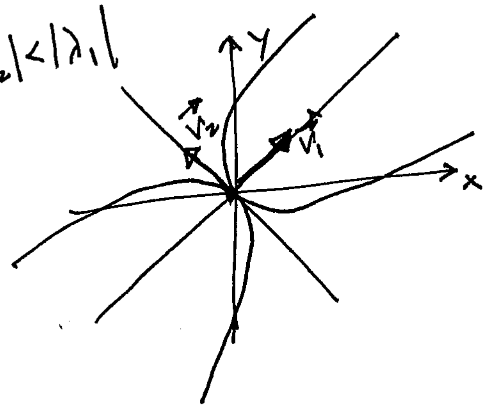
Rule of Thumb: (works for both $\lambda_1, \lambda_2 < 0$ and $\lambda_1, \lambda_2 > 0$)

At $\vec{x}_e = \vec{0}$, the trajectories are tangent to the eigenvector corresponding to the eigenvalue of smaller magnitude.

$|\lambda_1| < |\lambda_2|$




$|\lambda_2| < |\lambda_1|$




Simple Example: $\vec{x}' = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\vec{x}$

$\vec{v}_1 = \begin{pmatrix}1\\0\end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix}0\\1\end{pmatrix}$

$y = kx^{\lambda_2/\lambda_1}$

$\frac{\lambda_2}{\lambda_1} > 0$

$\lambda_2 > \lambda_1 \Rightarrow$ parabola-like

$\lambda_2 < \lambda_1 \Rightarrow$ square root-like

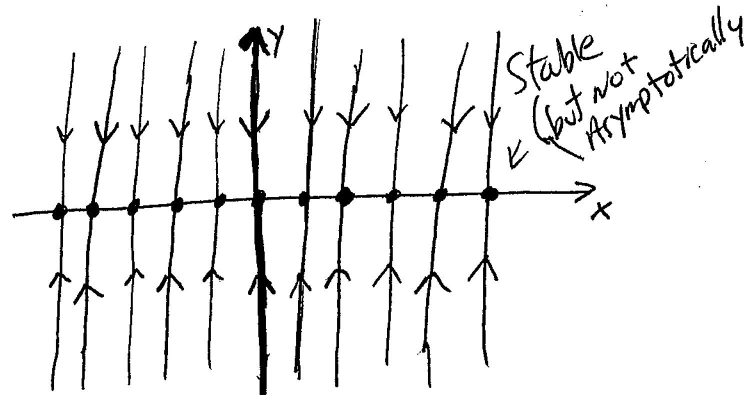
## Case 3: $\lambda_1 = 0$, $\lambda_2 \neq 0$ (zero eigenvalue) det(A) = 0 ⇒ Infinitely many eq. sol.

Simple Example: $\vec{x}' = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \vec{x}$ ⇒ $\lambda_1 = 0$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = -1$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$x' = 0 \Rightarrow x = c_1$

$y' = -y \Rightarrow y = c_2 e^{-t}$



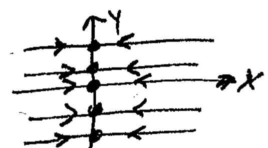

Eq. Sols.: $x' = 0$

$y' = -y = 0$

$\Rightarrow \vec{x}_e = \begin{pmatrix} x \\ 0 \end{pmatrix}$ for all $x$

---

OR $\vec{x}' = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \vec{x}$ ⇒

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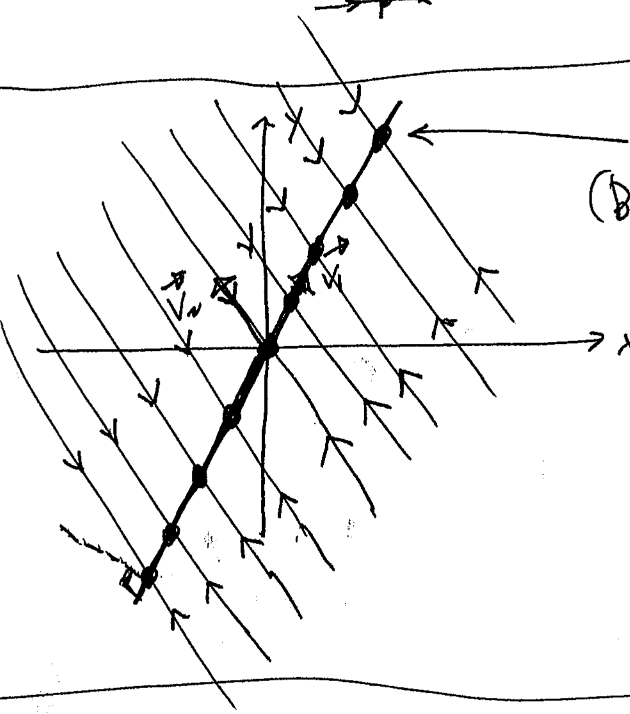
In general,

$\lambda_1 = 0$

$\lambda_2 < 0$




Stable (But Not Asymptotically)

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If $\lambda_1 = 0$ and $\lambda_2 > 0$, the phase portraits are the same as those above, except all arrows point in the opposite direction.

Unstable

## Section 3.2: Complex Conjugate Eigenvalues/vectors

Consider $\vec{x}' = A\vec{x}$, where $A$ has complex conjugate eigenvalues and eigenvectors, $\lambda, \vec{v}$ and $\bar{\lambda}, \bar{\vec{v}}$. $\Rightarrow$ $\det(A) \neq 0$ since the eigenvalues are nonzero.

Then, $\vec{z} = \vec{v}e^{\lambda t}$ and $\bar{\vec{z}} = \bar{\vec{v}}e^{\bar{\lambda} t}$ are solutions

$$\bar{\vec{z}} = \overline{\vec{v}e^{\lambda t}} = \bar{\vec{v}}e^{\bar{\lambda} t}$$

$z$ and $\bar{z}$ are conjugates

(These two solutions are complex-valued. It is preferrable to express the general solution as a linear combination of two real-valued solutions.)

*Note*: A single complex-valued solution yields two real-valued solutions

By the principal of superposition,

$$\vec{x}_1 = \frac{\vec{z} + \bar{\vec{z}}}{2} = Re(\vec{z}) = Re(\vec{v}e^{\lambda t})$$

and

$$\vec{x}_2 = \frac{\vec{z} - \bar{\vec{z}}}{2i} = Im(\vec{z}) = Im(\vec{v}e^{\lambda t})$$

are solutions as well.

It can be shown that $\vec{x}_1$ and $\vec{x}_2$ are linearly independent.
(We'll skip the proof since it is fairly long and not worth the time)

Therefore, the general solution is

$$\boxed{\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 = c_1 Re(\vec{v}e^{\lambda t}) + c_2 Im(\vec{v}e^{\lambda t})}.$$

Alternative Derivation: Let $\vec{z} = \vec{u} + i\vec{v}$, where $\vec{u} = Re(\vec{z})$ and $\vec{v} = Im(\vec{z})$ $\Rightarrow$ $\vec{u}$ and $\vec{v}$ are real vectors

Since $\vec{z}$ is a solution, we have $\vec{z}' = A\vec{z}$.

$$(\vec{u} + i\vec{v})' = A(\vec{u} + i\vec{v})$$

$$\vec{u}' + i\vec{v}' = A\vec{u} + iA\vec{v}$$

The equality holds only if the real parts and the imaginary parts are equal $\Rightarrow$ $\vec{u}' = A\vec{u}$ and $\vec{v}' = A\vec{v}$

So, $\vec{u} = Re(\vec{z}) = Re(\vec{v}e^{\lambda t})$ and $\vec{v} = Im(\vec{z}) = Im(\vec{v}e^{\lambda t})$ are solutions also.

**Example:** Find the real-valued general solution of $\vec{x}' = A\vec{x}$, where $A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$

**Eigenvalues:** $\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix} = (3-\lambda)(1-\lambda)+2 = \lambda^2 - 4\lambda + 5 = 0$

$$\lambda = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$$

$$\lambda = 2 \pm i$$

**Eigenvector:** $\lambda = 2+i$

$(A-\lambda I)\vec{v} = 0$

$$\begin{pmatrix} 1-i & 2 \\ -1 & -1-i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Rightarrow \underline{v_1 = -(1+i)v_2} \quad \text{Pick } v_2 = -1 \Rightarrow \boxed{\vec{v} = \begin{pmatrix} 1+i \\ -1 \end{pmatrix}}$$

Therefore, a complex-valued solution is

$$\vec{z} = \vec{v}e^{\lambda t} = \begin{pmatrix} 1+i \\ -1 \end{pmatrix} e^{(2+i)t}$$

The real and imaginary parts of $\vec{z}$ are real-valued solutions.
The real and imaginary parts of $\vec{z}$ are determined as follows.

Recall: Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\vec{z} = \begin{pmatrix} 1+i \\ -1 \end{pmatrix} e^{(2+i)t} = \begin{pmatrix} 1+i \\ -1 \end{pmatrix} e^{2t} e^{it} = e^{2t}\begin{pmatrix} 1+i \\ -1 \end{pmatrix}(\cos t + i\sin t)$$

$$= e^{2t}\begin{pmatrix} (1+i)(\cos t + i\sin t) \\ (-1)(\cos t + i \sin t) \end{pmatrix} = e^{2t}\begin{pmatrix} \cos t + i\sin t + i\cos t - \sin t \\ -\cos t - i\sin t \end{pmatrix}$$

$$\vec{z} = \begin{pmatrix} 1+i \\ -1 \end{pmatrix} e^{(2+i)t} = e^{2t}\left[\begin{pmatrix} \cos t - \sin t \\ -\cos t \end{pmatrix} + i\begin{pmatrix} \sin t + \cos t \\ -\sin t \end{pmatrix}\right]$$

Then, $\boxed{\vec{x}_1 = \mathrm{Re}(\vec{z}) = e^{2t}\begin{pmatrix} \cos t - \sin t \\ -\cos t \end{pmatrix}}$ $\boxed{\vec{x}_2 = \mathrm{Im}(\vec{z}) = e^{2t}\begin{pmatrix} \sin t + \cos t \\ -\sin t \end{pmatrix}}$

the real-valued general solution is $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$

$$\boxed{\vec{x}(t) = e^{2t}\left[c_1\begin{pmatrix} \cos t - \sin t \\ -\cos t \end{pmatrix} + c_2\begin{pmatrix} \sin t + \cos t \\ -\sin t \end{pmatrix}\right]}$$

OR

$$\boxed{\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{2t}[c_1(\cos t - \sin t) + c_2(\sin t + \cos t)] \\ e^{2t}[-c_1\cos t - c_2 \sin t] \end{pmatrix}}$$

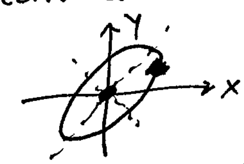
# Phase Portrait

We have
$$x(t) = e^{2t}\left[(C_1+C_2)\cos t + (-C_1+C_2)\sin t\right]$$
$$y(t) = e^{2t}\left[-C_1\cos t - C_2\sin t\right]$$

---

The parametric representation of an ellipse centered at the origin is

$$X(t) = a\cos t + b\sin t$$
$$Y(t) = c\cos t + d\sin t$$

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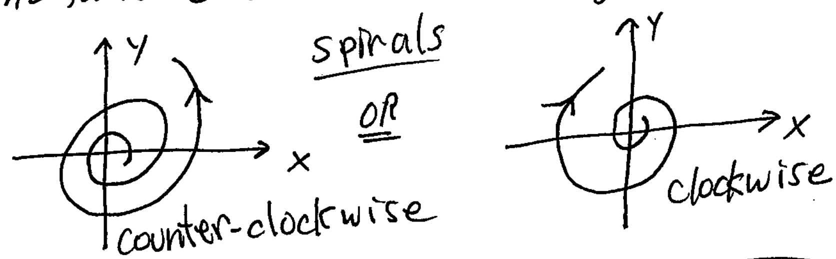
Our solution is identical to the parametric representation of an ellipse, except with an additional factor of $e^{2t}$.

The factor $e^{2t}$ acts as an increasing "radius".

$\Rightarrow$

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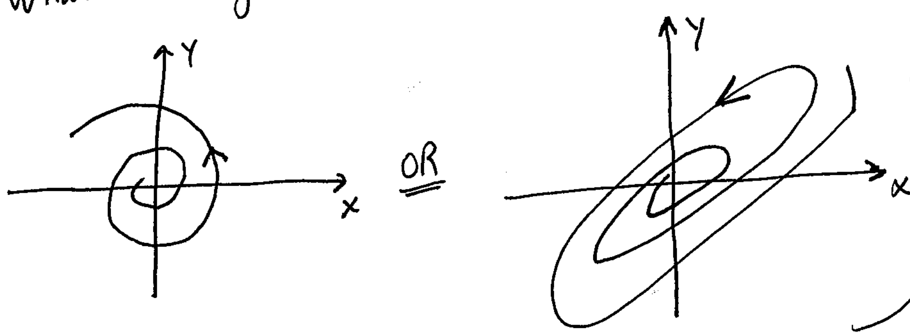
Questions:

1) Are the trajectories directed clockwise or counter-clockwise?
2) Do the trajectories spiral inward or outward?
3) What is the general shape of the trajectories?




We won't worry about this question. It is not so important.

## Clockwise or Counter-Clockwise?

We have $\vec{x}' = A\vec{x}$, where $A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$.

The direction of the spiral can be determined by the direction of the tangent vector $\vec{x}'$ at any point.

For example, consider the point $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

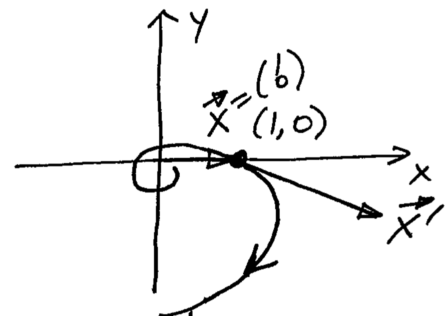
The tangent vector $\vec{x}'$ to the trajectory at $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is

$$\vec{x}' = A\vec{x} = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\vec{x}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$x' = 3$

$y' = -1 < 0$




$y' < 0 \Rightarrow$ clockwise

Since the tangent vector $\vec{x}'$ is directed downward at $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the spiral must be directed clockwise.

---

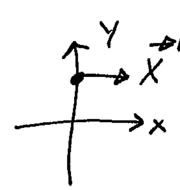
In general, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then at $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ the tangent vector $\vec{x}'$ is $\vec{x}' = A\vec{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$

$\Rightarrow$ $y' = c$ at $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Therefore,

$$\boxed{\begin{array}{ll} c < 0 & \Rightarrow \text{clockwise} \\ c > 0 & \Rightarrow \text{counter-clockwise} \end{array}}$$

Equivalently, if our test point is $\vec{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,




$$\vec{x}' = A\vec{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

$x' = b$ at $\vec{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$b > 0 \Rightarrow$ clockwise

$b < 0 \Rightarrow$ counterclockwise

# Inward or Outward?

The trajectories may spiral inward, outward, or neither, as $t$ increases.

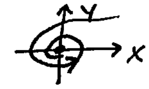
If $\lambda = \alpha \pm i\beta$, then solutions are of the form

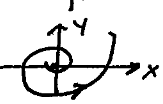
$$x(t) = e^{\alpha t}[a\cos t + b\sin t]$$
$$y(t) = e^{\alpha t}[c\cos t + d\sin t].$$

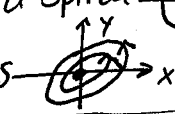
If $\alpha = Re(\lambda) < 0$, then $x, y \to 0$ as $t \to \infty$.

If $\alpha = Re(\lambda) > 0$, then $\sqrt{x^2+y^2} \to \infty$ as $t \to \infty$.

If $\alpha = Re(\lambda) = 0$, then $x$ and $y$ oscillate harmonically.

$Re(\lambda) < 0 \Rightarrow$ Inward Spiral 

$Re(\lambda) > 0 \Rightarrow$ Outward Spiral 

$Re(\lambda) = 0 \Rightarrow$ Ellipses 

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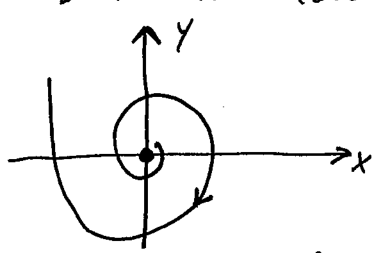
There are 6 (qualitative) types of phase portraits depending on whether $Re(\lambda) \lesseqgtr 0$ and $c \lessgtr 0$.

1) $Re(\lambda) > 0,\ c < 0$

$$\vec{x}' = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \vec{x}$$

$\lambda = 1 + i,\ \vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\vec{x}(t) = e^t\left[C_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right]$$



$Re(\lambda) > 0 \Rightarrow$ Outward

$c = -1 < 0 \Rightarrow$ clockwise

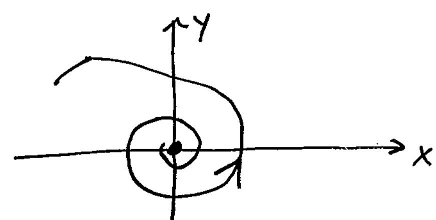
$\vec{x}_e = \vec{0}$ is an Unstable Spiral Point.

2) $Re(\lambda) > 0,\ c > 0$

$$\vec{x}' = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \vec{x}$$

$\lambda = 1 + i,\ \vec{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$$\vec{x}(t) = e^t\left[C_1\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + C_2\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}\right]$$



$Re(\lambda) > 0 \Rightarrow$ Outward

$c = 1 > 0 \Rightarrow$ Counter-clockwise

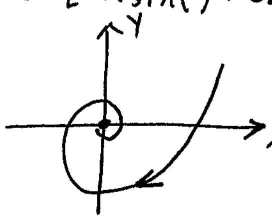
$\vec{x}_e = \vec{0}$ is an Unstable Spiral Point.

3) $Re(\lambda) < 0, \ c < 0$

$$\vec{x}' = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix} \vec{x}$$

$\lambda = -1 + i, \ \vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\vec{x}(t) = e^{-t}\left[c_1\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right]$$




$Re(\lambda) < 0 \Rightarrow$ Inward

$c < 0 \Rightarrow$ Clockwise

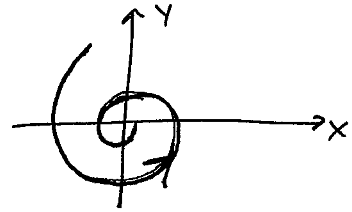
$\vec{x}_e = \vec{0}$ is an Asymptotically Stable Spiral Point

4) $Re(\lambda) < 0, \ c > 0$

$$\vec{x}' = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \vec{x}$$

$\lambda = -1 + i, \ \vec{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$$\vec{x}(t) = e^{-t}\left[c_1\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}\right]$$




$Re(\lambda) < 0 \Rightarrow$ Inward

$c > 0 \Rightarrow$ Counter-Clockwise

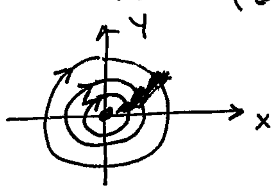
$\vec{x}_e = \vec{0}$ is an Asymptotically Stable Spiral Point

5) $Re(\lambda) = 0, \ c < 0$

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{x}$$

$\lambda = i, \ \vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$$\vec{x}(t) = c_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$




$Re(\lambda) = 0 \Rightarrow$ Ellipse (circle for this example)

$c < 0 \Rightarrow$ Clockwise

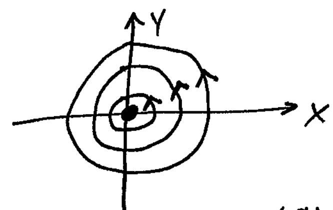
$\vec{x}_e = \vec{0}$ is a Stable Center. but not asymptotically

6) $Re\,\lambda = 0, \ c > 0$

$$\vec{x}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \vec{x}$$

$\lambda = i, \ \vec{v} = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$$\vec{x}(t) = c_1\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + c_2\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$




$Re(\lambda) = 0 \Rightarrow$ Ellipse (circle for this example)

$c > 0 \Rightarrow$ Counter-clockwise

$\vec{x}_e = \vec{0}$ is a Stable Center but not asymptotically

When sketching the phase portraits, two features should be indicated,

1) Clockwise / Counter-Clockwise
2) Inward / Outward (or neither)

## Section 3.3: Repeated Eigenvalues ($m=2$)

Consider $\vec{x}' = A\vec{x}$ where $A$ has real repeated eigenvalues, $\lambda_1 = \lambda_2$ ($=\lambda$)
The lone eigenvalue $\lambda$ may yield either one or two linearly independent eigenvectors.

Definitions: Let $A$ be an $n \times n$ matrix. Then, the eigenvalues of $A$ are the roots of an $n^{th}$ order polynomial $P_n(\lambda)$.

The algebraic multiplicity ($m$) of $\lambda$ is the multiplicity of $\lambda$ as a root of the polynomial $P_n(\lambda)$.

The geometric multiplicity ($k$) of $\lambda$ is the number of linearly independent eigenvectors associated with $\lambda$.

---

Notes: 1. $\boxed{1 \le k \le m \le n}$ A root of multiplicity $m$ of $P_n$ has at least one, but no more than $m$, linearly independent eigenvectors.

2. If $A$ is a $2 \times 2$, matrix, then $1 \le k \le m \le 2$ for each eigenvalue

3. If $\lambda$ is a distinct eigenvalue, then $k = m = 1$.

---

Let $A$ be a $2 \times 2$ matrix with a repeated eigenvalue $\lambda$ ($m=2$).

Then, either

i) $k=2 \Rightarrow \lambda$ has to linearly independent eigenvectors

or ii) $k=1 \Rightarrow \lambda$ has only one linearly independent eigenvector.

---

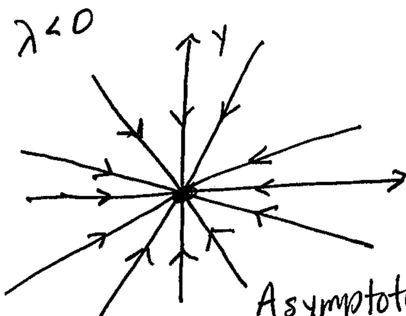
Case 1: $m=2,\ k=2 \Rightarrow \lambda$ yields 2 linearly independent eigenvectors.

Simple Example: $\vec{x}' = A\vec{x}$ where $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$, $\lambda \neq 0$

$$\vec{x}' = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \vec{x} \Rightarrow \begin{array}{l} x' = \lambda x \Rightarrow x = c_1 e^{\lambda t} \\ y' = \lambda y \Rightarrow y = c_2 e^{\lambda t} \end{array} \Rightarrow \begin{array}{l} y = \frac{c_2}{c_1} x \\ y = kx \end{array}$$

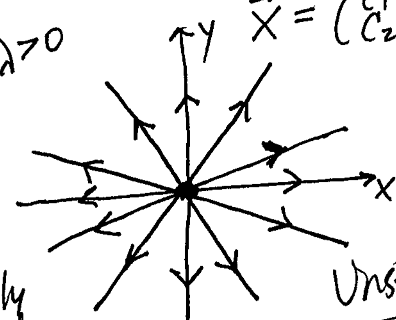
$$\vec{x} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t}$$

$\lambda < 0$




Asymptotically Stable (Star) Node

$\lambda > 0$




Unstable (Star) Node

The solution curves are straight lines passing through the origin.

**Eigenvalue Solution Method:** $\lambda$ is a repeated eigenvalue ($\lambda \neq 0$)

$A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$

**Eigenvectors:** $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$0v_1 + 0v_2 = 0$$

$\Rightarrow$ All vectors are eigenvectors

Pick any two linearly independent eigenvectors, say $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$\Rightarrow \vec{x}_1 = \vec{v}_1 e^{\lambda t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda t}$, $\vec{x}_2 = \vec{v}_2 e^{\lambda t} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\lambda t}$

Then, $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{\lambda t}$

$$\boxed{\vec{x}(t) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^{\lambda t}}$$ (general solution)

equivalent

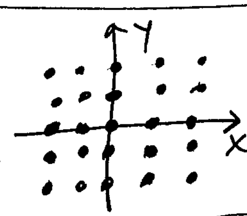
OR Pick $\vec{v}_1 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 7 \\ -1 \end{pmatrix}$ $\Rightarrow$ $\vec{x}(t) = \begin{pmatrix} -2c_1 + 7c_2 \\ 5c_1 - c_2 \end{pmatrix} e^{\lambda t}$

---

$\lambda = 0$: $\vec{x}' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\vec{x}$

$\Rightarrow x' = 0 \Rightarrow x = c_1$

$y' = 0 \Rightarrow y = c_2$




All points are Stable (but not asymptotically) equilibrium solutions.

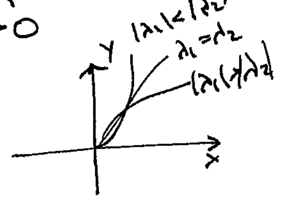
---

If a repeated eigenvalue has two linearly independent eigenvectors, it resembles the the case of real distinct eigenvalues of the same sign, and it represents the borderline case between $|\lambda_1| < |\lambda_2|$ and $|\lambda_1| > |\lambda_2|$.

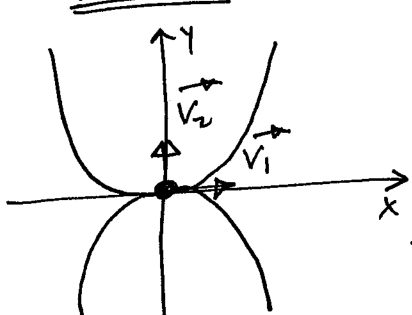
$\Rightarrow$ same sign

e.g. Consider $\vec{x}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\vec{x}$, where $\lambda_1, \lambda_2 \neq 0$ and $\lambda_1 \cdot \lambda_2 > 0$

$\Rightarrow y = k|x|^{\lambda_2/\lambda_1}$, $\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

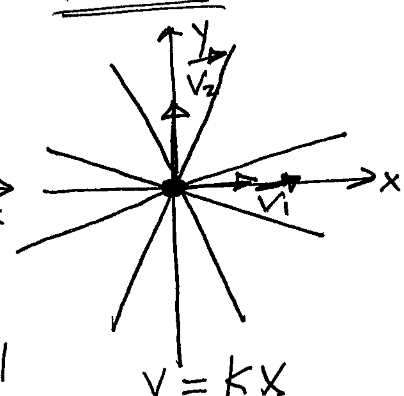



$|\lambda_1| < |\lambda_2|$



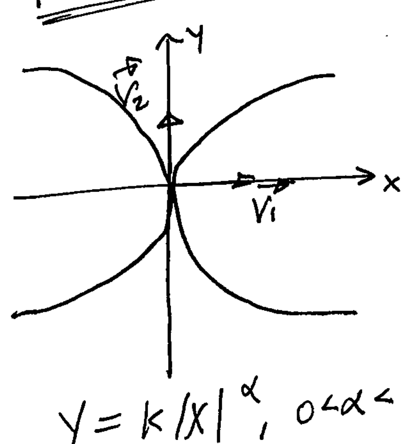

$y = kx^{\alpha}$, $\alpha > 1$

$\lambda_1 = \lambda_2$




$y = kx$

$|\lambda_1| > |\lambda_2|$




$y = k|x|^{\alpha}$, $0 < \alpha < 1$

**Case 2:** $m=2,\ k=1 \Rightarrow \lambda$ yields only one linearly independent eigenvector

Simple Example: $\vec{x}' = A\vec{x}$, where $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\lambda \neq 0$

$\lambda$ is a repeated eigenvalue

Eigenvector: $(A - \lambda I)\vec{v} = \vec{0}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$v_2 = 0$

Pick $v_1 = 1$

$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\Rightarrow \boxed{\vec{x}_1 = \vec{v}e^{\lambda t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda t}}$$

A second linearly independent solution is needed to find the general solution.

---

Componentwise Solution (Textbook):

$\vec{x}' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\vec{x}$

$x' = \lambda x + y$

$y' = \lambda y \Rightarrow y = c_2 e^{\lambda t}$

$x' - \lambda x = y = c_2 e^{\lambda t}$

Integrating Factor: $x = (c_1 + c_2 t)e^{\lambda t}$

$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix} e^{\lambda t}$ (general solution)

$$\vec{x} = \left[c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 \end{pmatrix}\right] e^{\lambda t}$$

$\uparrow \vec{x}_1$ $\qquad$ $\uparrow$ ? How do we find this vector using the eigenvalue solution method?

It is useful to find the general solution using the Eigenvalue Solution Method, which may be used for more general problems, and more importantly, for higher dimensions.

---

Eigenvalue Solution Method:

$\vec{x}' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\vec{x}$

$x' = \lambda x + y \longrightarrow$

$y' = \lambda y \longrightarrow$

Convert to a 2nd order linear ODE

$y = x' - \lambda x$

$(x' - \lambda x)' = \lambda(x' - \lambda x)$

$x'' - 2\lambda x + \lambda^2 x = 0$

$r^2 - 2\lambda r + \lambda^2 = 0$

$r = \lambda, \lambda \Rightarrow x_1 = e^{\lambda t}$

Reduction of order $\Rightarrow x_2 = t x_1 = t e^{\lambda t}$

$\Rightarrow$ Try $\vec{x}_2 = t\vec{x}_1$

We have $\vec{x}_1 = \vec{v}e^{\lambda t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda t}$

Try $\vec{x}_2 = t\vec{x}_1 = \vec{v}te^{\lambda t}$

Plug into $\vec{x}_2' = A\vec{x}_2$

$\vec{v}(\lambda t + 1)e^{\lambda t} = \underbrace{A\vec{v}}_{=\lambda\vec{v}} t e^{\lambda t}$

$\cancel{\vec{v}\lambda t e^{\lambda t}} + \vec{v}e^{\lambda t} = \cancel{\lambda\vec{v}te^{\lambda t}}$

$\vec{v}e^{\lambda t} \neq 0 \Rightarrow$ There are no solutions of the form $\vec{v}te^{\lambda t}$

Try $\vec{x}_2 = \vec{v}te^{\lambda t} + \vec{w}e^{\lambda t} \Rightarrow \boxed{\vec{x}_2 = (\vec{v}t + \vec{w})e^{\lambda t}}$

Plug into $\vec{x}_2{}' = A\vec{x}_2$

$$(\lambda\vec{v}t + \lambda\vec{w} + \vec{v})\cancel{e^{\lambda t}} = A(\underbrace{\vec{v}t}_{=\lambda\vec{v}t} + \vec{w})\cancel{e^{\lambda t}}$$

$$\cancel{\lambda\vec{v}t} + \lambda\vec{w} + \vec{v} = \cancel{\lambda\vec{v}t} + A\vec{w}$$

$$A\vec{w} - \lambda I\vec{w} = \vec{v}$$

$$\boxed{(A-\lambda I)\vec{w} = \vec{v}} \text{ (nonhomogeneous)}$$

$\det(A-\lambda I) = 0 \Rightarrow$ There are either no solutions or infinitely many solutions (linearly dependent).

It can be shown that there are always infinitely many nontrivial solutions.

<u>Definition</u>: Let $A$ be a $2\times 2$ matrix and $\lambda$ be a repeated eigenvalue of $A$, with algebraic multiplicity $m=2$ and geometric multiplicity $k=1$. Let $\vec{v}$ be an eigenvector corresponding to $\lambda$. The infinitely many linearly dependent nontrivial solutions $\vec{w}$ of

$$\boxed{(A-\lambda I)\vec{w} = \vec{v}}$$

are the <u>generalized eigenvectors</u> of $A$.

This definition can be generalized to higher dimensions.

$$\boxed{\vec{X}_2 = (\vec{v}t + \vec{w})e^{\lambda t}}$$

<u>General Solution</u>: $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2$

$$\boxed{\vec{x} = c_1\vec{v}e^{\lambda t} + c_2(\vec{v}t + \vec{w})e^{\lambda t}}$$

(Repeated Eigenvalue with $m=2, k=1$)

Back to the Simple Example; $\vec{X}' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \vec{X}$ $\quad \lambda \neq 0$

$\lambda$ is a repeated eigenvalue

$\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the only linearly independent eigenvector

$$\vec{X}_1 = \vec{v} e^{\lambda t} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda t}$$

Generalized Eigenvector: $(A - \lambda I)\vec{w} = \vec{v}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{Pick } w_1 = 0$$

$$w_2 = 1 \implies \vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\cancel{0w_1 + 0w_2 = 0}$

$$\Rightarrow \vec{X}_2 = (\vec{v}t + \vec{w})e^{\lambda t} = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] e^{\lambda t}$$

$$\vec{X}_2 = \begin{pmatrix} t \\ 1 \end{pmatrix} e^{\lambda t}$$

$$\Rightarrow \vec{X} = c_1 \vec{X}_1 + c_2 \vec{X}_2 = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\lambda t} + c_2 \begin{pmatrix} t \\ 1 \end{pmatrix} e^{\lambda t}$$

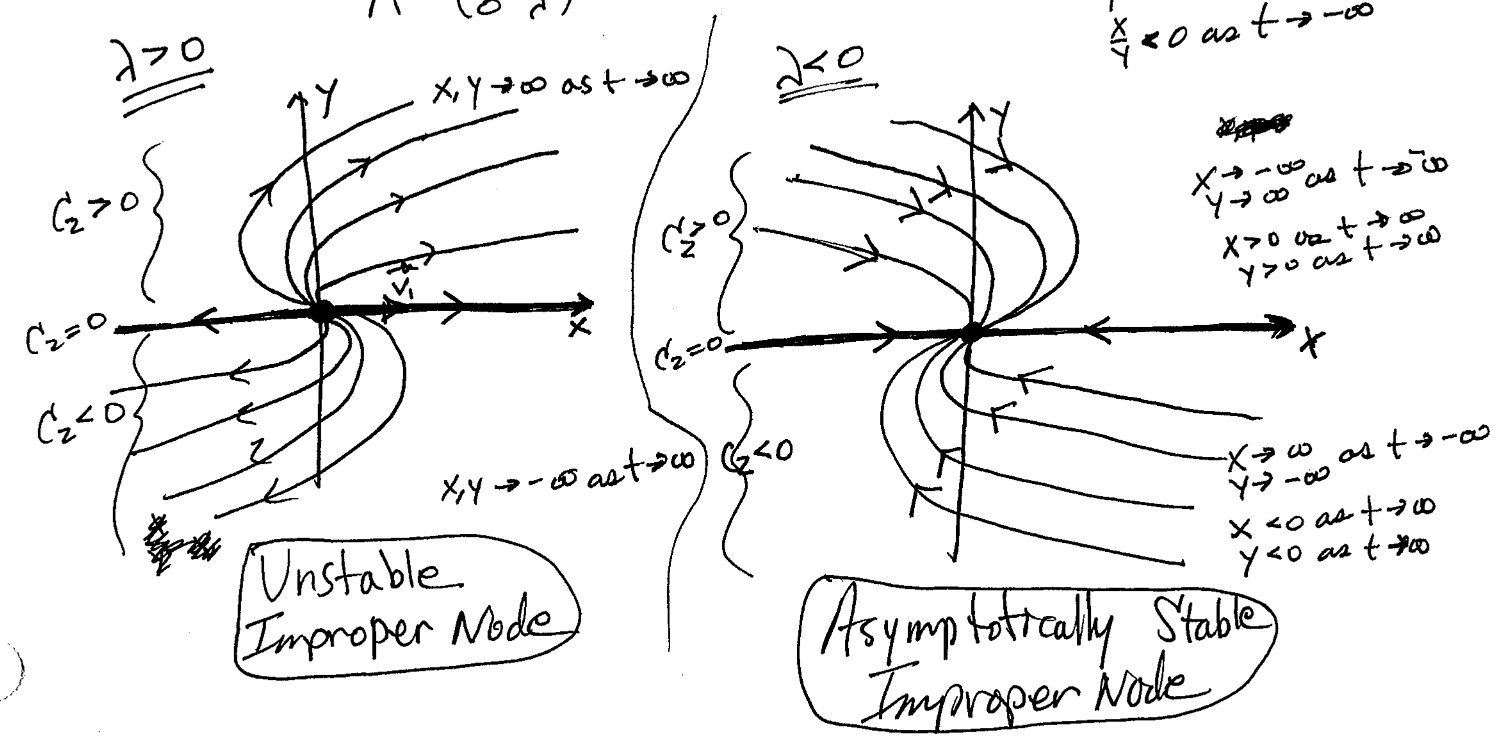
$$\boxed{\vec{X} = \begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix} e^{\lambda t}} \text{ (general solution)}$$

$$\Rightarrow \begin{aligned} x &= (c_1 + c_2 t) e^{\lambda t} \\ y &= c_2 e^{\lambda t} \end{aligned} \qquad \frac{x}{y} = \frac{c_1}{c_2} + t$$

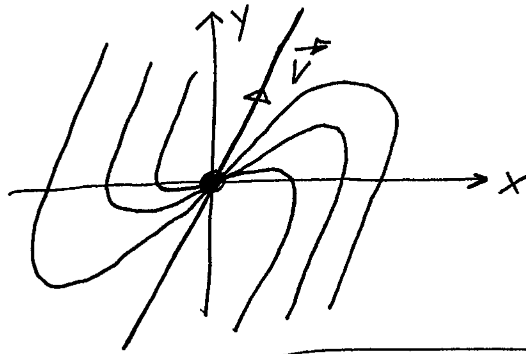
$\frac{x}{y} > 0$ as $t \to \infty$

$\frac{x}{y} < 0$ as $t \to -\infty$

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$



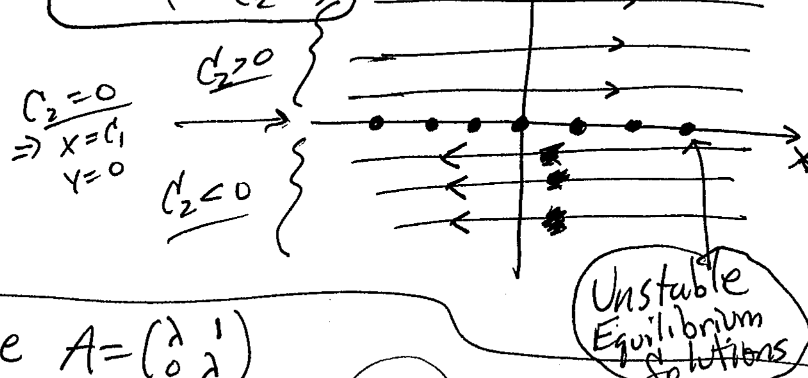
In general, we have



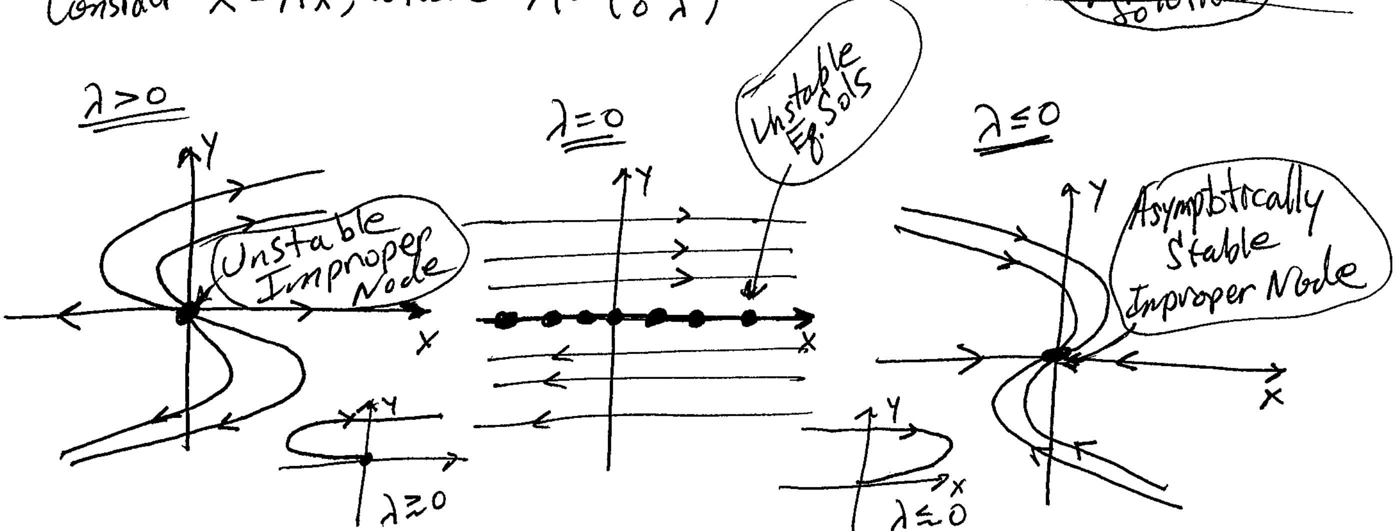

Special Case: $\lambda = 0 \Rightarrow \vec{x}' = A\vec{x}$, where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{matrix} x' = y \\ y' = 0 \end{matrix} \Rightarrow y = c_2 \Rightarrow \begin{matrix} x' = c_2 \\ x = c_2 t + c_1 \end{matrix}$

$\vec{x} = \begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix} e^{\lambda t} \Rightarrow \boxed{\vec{x} = \begin{pmatrix} c_1 + c_2 t \\ c_2 \end{pmatrix}}$




Consider $\vec{x}' = A\vec{x}$, where $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$




$\lambda$ may be considered as a bifurcation parameter, with a bifurcation occurring at $\lambda = 0$.

$\vec{x}_e = \vec{0}$ has a change in stability as $\lambda$ passes through 0.

Extra

Between 74 and 75

# Notation / Linear Algebra

Consider $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$

Let $\boxed{(\vec{v}, \vec{w}) = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}}$ matrix with $\vec{v}$ and $\vec{w}$ as its first and second columns, respectively.

Also, $\boxed{\underset{2\times 2}{A}\ \underset{2\times 2}{(\vec{v}, \vec{w})} = (A\vec{v}, A\vec{w})}$

## Section 3.4: Changing Coordinates

We'll consider a change of coordinates by a linear transformation $L$, $\vec{u} = L\vec{x}$.

Definition: A linear transformation (or mapping) $L$ is a function $L: \mathbb{R}^2 \to \mathbb{R}^2$ of the form $L\vec{x} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$. $\quad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$L$ maps one vector $\vec{x}$ into another $\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$, where the components of the new vector have a linear dependence on the components of the old vector.

Note: $L\vec{x} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\vec{x}$.

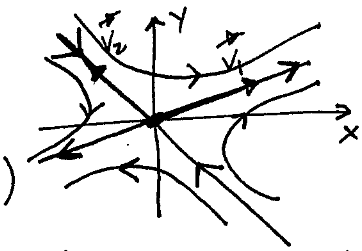
Apply the linear mapping $L$ to a vector $\vec{x}$ is equivalent to multiplying $\vec{x}$ by a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The linear mapping $L$ and its associated matrix are interchangable. It is convenient (but not quite correct) to write

$$\boxed{L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}}$$

Recall the Example: $\vec{x}' = A\vec{x}$, where $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \end{pmatrix}$

$\lambda_1 = 2, \ \vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\lambda_2 = -2, \ \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



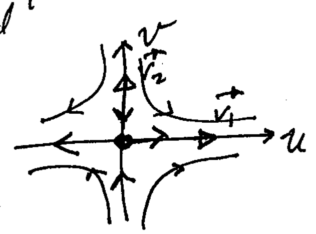

By making an appropriate change of variables $\vec{u} = L\vec{x}$, where $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$, the system can be ~~transformed~~ into the simpler system

$\vec{u}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\vec{u}$ u and v are decoupled

$\vec{u}' = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}\vec{u}$ $\Rightarrow$ $\lambda_1 = 2, \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \Rightarrow$

$\lambda_2 = -2, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The eigenvectors are parallel to the coordinate axes.




This can be solved without much difficulty, $\vec{u} = c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{2t} + c_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}e^{-2t} = \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-2t} \end{pmatrix}$.

If $\det(L) \neq 0$, then $L$ is invertible and $\vec{x} = L^{-1}\vec{u}$

$$L^{-1} = \frac{1}{\det(L)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \Rightarrow \vec{x} = \frac{1}{\underset{\uparrow\ ad-bc}{\det(L)}}\begin{pmatrix} dc_1e^{2t} - bc_2e^{-2t} \\ -cc_1e^{2t} + ac_2e^{-2t} \end{pmatrix}$$

By making an appropriate change of variable $\vec{u} = L\vec{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\vec{x}$, any planar system can be transformed into one of the following three forms.

These are called Jordan forms or canonical forms.

---

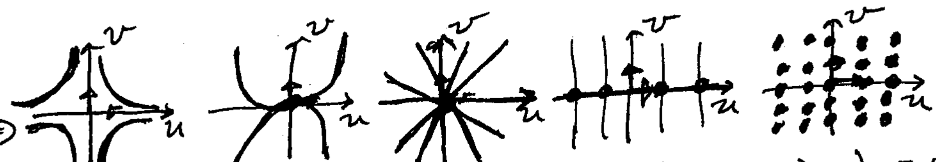
1. $\vec{u}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\vec{u}$ : Real (distinct or repeated) eigenvalues for which $k=2$.

$k=2 \Rightarrow$ Two linearly independent eigenvectors.

$\lambda_1, \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2, \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\left.\begin{array}{l} \vec{u}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\vec{u} \\ \vec{u} = \begin{pmatrix} u \\ v \end{pmatrix} \end{array}\right\} \Rightarrow$




$\lambda_1, \lambda_2$ are opposite in sign | $\lambda_1, \lambda_2$ are the same in sign $(\lambda_1 \neq \lambda_2)$ | $\lambda_1 = \lambda_2$ | $\lambda_1 = 0, \lambda_2 \neq 0$ (or vice versa) | $\lambda_1 = \lambda_2 = 0$

with 2 linearly independent eigenvectors

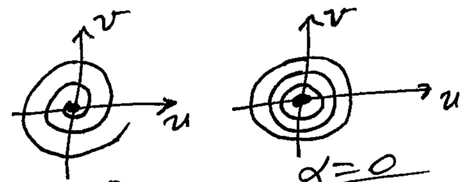
---

2. $\vec{u}' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}\vec{u}$ : Complex Conjugate eigenvalues

$\beta \neq 0$

$\lambda = \alpha \pm i\beta, \ \vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\vec{u}' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}\vec{u} \Rightarrow$




$\alpha \neq 0$ (Circular Spirals) | $\alpha = 0$ (Circles)

---

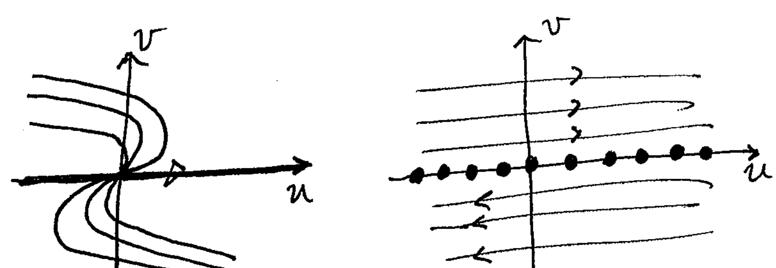
3. $\vec{u}' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\vec{u}$ : Repeated eigenvalues for which $k=1$.

$k=1 \Rightarrow$ Only one linearly independent eigenvector

$\lambda, \vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

generalized eigenvector : $\vec{w} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

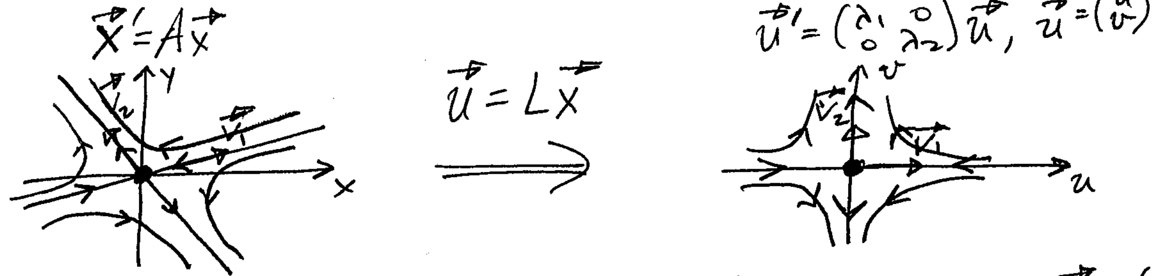
$\vec{u}' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\vec{u} \Rightarrow$




$\lambda \neq 0$ | $\lambda = 0$

**Case 1: $\vec{u}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \vec{u}$** : Real (distinct or repeated) Eigenvalues with two linearly independent eigenvectors, $\lambda_1, \vec{v}_1$; $\lambda_2, \vec{v}_2$

Suppose we have a planar system $\vec{x}' = A\vec{x}$ with a saddle point at the origin.




The appropriate transformation $L$ must map $\vec{v}_1$ onto the x-axis, $L\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, and map $\vec{v}_2$ onto the y-axis, $L\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Satisfying these properties will lead to the correct mapping $L$.

---

Consider $\vec{x}' = A\vec{x}$, where $A$ has real eigenvalues (distinct or repeated) and two linearly independent eigenvectors, $\vec{v}_1 = \begin{pmatrix} v_{11} \\ v_{12} \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix}$. Form the matrix $T$ which has the components of $\vec{v}_1$ in the first column and the components of $\vec{v}_2$ in the second column.

$$\boxed{T = (\vec{v}_1, \vec{v}_2) = \begin{pmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{pmatrix}}$$

Note: $T$ is invertible since $\det(T) = \begin{vmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{vmatrix} \neq 0$ since $\vec{v}_1$ and $\vec{v}_2$ are linearly independent.

Then, $$\boxed{T^{-1} = \frac{1}{\det(T)} \begin{pmatrix} v_{22} & -v_{21} \\ -v_{12} & v_{11} \end{pmatrix}}$$

Note: $\boxed{L = T^{-1}}$ is the desired linear mapping

To see this, we have $T^{-1}T = I$

$$T^{-1}(\vec{v}_1, \vec{v}_2) \longrightarrow T^{-1}\begin{pmatrix} v_{11} & v_{21} \\ v_{12} & v_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$(T^{-1}\vec{v}_1, T^{-1}\vec{v}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \boxed{T^{-1}\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \quad \boxed{T^{-1}\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

$L = T^{-1}$ maps $\vec{v}_1$ into $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2$ into $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Let $\vec{u} = L\vec{x} = T^{-1}\vec{x}$, where $\vec{u} = \binom{u}{v}$

$$\boxed{\vec{u} = T^{-1}\vec{x}, \text{ where } T = (\vec{v}_1, \vec{v}_2)}$$ (Change of Coordinates)

What equation does $\vec{u}$ satisfy?

$\vec{u} = T^{-1}\vec{x} \Rightarrow \vec{x} = T\vec{u}$

$$\vec{x}' = A\vec{x}$$

$$(T\vec{u})' = A(T\vec{u})$$

$$T\vec{u}' = AT\vec{u}$$

$$\boxed{\vec{u}' = (T^{-1}AT)\vec{u}}, \quad T^{-1}AT = ?$$

$$\begin{aligned} T^{-1}AT &= T^{-1}A(\vec{v}_1, \vec{v}_2) \\ &= T^{-1}(A\vec{v}_1, A\vec{v}_2) \\ &= T^{-1}(\lambda_1\vec{v}_1, \lambda_2\vec{v}_2) \\ &= (\lambda_1 T^{-1}\vec{v}_1, \lambda_2 T^{-1}\vec{v}_2) \\ &= \left(\lambda_1\binom{1}{0}, \lambda_2\binom{0}{1}\right) \end{aligned}$$

$A\vec{v}_1 = \lambda_1\vec{v}_1$
$A\vec{v}_2 = \lambda_2\vec{v}_2$

$T^{-1}\vec{v}_1 = \binom{1}{0}$
$T^{-1}\vec{v}_2 = \binom{0}{1}$

$$T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\Rightarrow \boxed{\vec{u}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \vec{u}}$$

$\lambda_1, \vec{v}_1 = \binom{1}{0}$
$\lambda_2, \vec{v}_2 = \binom{0}{1}$

$\vec{u}_1 = \binom{1}{0} e^{\lambda_1 t}$
$\vec{u}_2 = \binom{0}{1} e^{\lambda_2 t}$

General Solution: $\vec{u} = c_1\vec{u}_1 + c_2\vec{u}_2 = c_1\binom{1}{0}e^{\lambda_1 t} + c_2\binom{0}{1}e^{\lambda_2 t}$

$$\vec{u} = \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{pmatrix}$$

Then, $\boxed{\vec{x} = T\vec{u}}$

Previous Example: $\vec{x}\,' = A\vec{x}$, where $A = \begin{pmatrix}1 & 3\\ 1 & -1\end{pmatrix}$

Recall $\lambda_1 = 2,\ \vec{v}_1 = \begin{pmatrix}3\\1\end{pmatrix}$

$\lambda_2 = -2,\ \vec{v}_2 = \begin{pmatrix}1\\-1\end{pmatrix}$

$T = (\vec{v}_1, \vec{v}_2) = \begin{pmatrix}3 & 1\\ 1 & -1\end{pmatrix} \Rightarrow \det(T) = -4$

$L = T^{-1} = \frac{1}{-4}\begin{pmatrix}-1 & -1\\ -1 & 3\end{pmatrix}$

Check: $T^{-1}\vec{v}_1 = \frac{1}{4}\begin{pmatrix}1 & 1\\ 1 & -3\end{pmatrix}\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$ ✓

$T^{-1}\vec{v}_2 = \frac{1}{4}\begin{pmatrix}1 & 1\\ 1 & -3\end{pmatrix}\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$ ✓

$$\boxed{L = T^{-1} = \frac{1}{4}\begin{pmatrix}1 & 1\\ 1 & -3\end{pmatrix}}$$

Let $\vec{u} = T^{-1}\vec{x} = \frac{1}{4}\begin{pmatrix}x+y\\ x-3y\end{pmatrix}$

$\Rightarrow \vec{u}\,' = (T^{-1}AT)\vec{u} = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}\vec{u}$

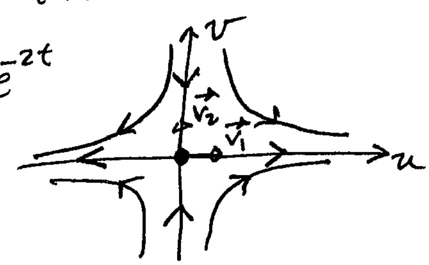
$\vec{u}\,' = \begin{pmatrix}2 & 0\\ 0 & -2\end{pmatrix}\vec{u}$

$\lambda_1 = 2,\ \vec{v}_1 = \begin{pmatrix}1\\0\end{pmatrix} \Rightarrow \vec{u}_1 = \begin{pmatrix}1\\0\end{pmatrix}e^{2t}$

$\lambda_2 = -2,\ \vec{v}_2 = \begin{pmatrix}0\\1\end{pmatrix} \Rightarrow \vec{u}_2 = \begin{pmatrix}0\\1\end{pmatrix}e^{-2t}$

$\vec{u} = c_1\vec{u}_1 + c_2\vec{u}_2 = c_1\begin{pmatrix}1\\0\end{pmatrix}e^{2t} + c_2\begin{pmatrix}0\\1\end{pmatrix}e^{-2t}$

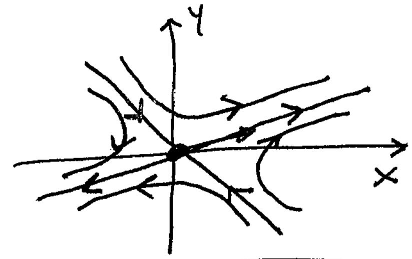
$$\boxed{\vec{u} = \begin{pmatrix}c_1e^{2t}\\ c_2e^{-2t}\end{pmatrix}}$$




Then, $\vec{x} = T\vec{u} = \begin{pmatrix}3 & 1\\ 1 & -1\end{pmatrix}\begin{pmatrix}c_1e^{2t}\\ c_2e^{-2t}\end{pmatrix}$

$$\boxed{\vec{x} = \begin{pmatrix}3c_1e^{2t} + c_2e^{-2t}\\ c_1e^{2t} - c_2e^{-2t}\end{pmatrix}}$$

Agrees with our previous solution

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It seems that we did a lot of extra work. The eigenvalues/vectors given in the first place are sufficient to write down the general solution and sketch the phase portrait. So what do we gain by considering the canonical form?

1. It is convenient in theory since we need only consider the canonical forms in proving theorems which apply to general matrices. At most, three cases need to be considered.
2. $e^A = \sum_{n=0}^{\infty}\frac{A^n}{n!}$ : $T^{-1}AT = \begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}$ $\Rightarrow e^A = e^{T\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}T^{-1}} = Te^{\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}}T^{-1}$ $= T\sum_{n=0}^{\infty}\frac{\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}^n}{n!}T^{-1}$

   $A = T\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}T^{-1}$

   $$e^A = T\sum_{n=0}^{\infty}\frac{\begin{pmatrix}\lambda_1^n & 0\\ 0 & \lambda_2^n\end{pmatrix}}{n!}T^{-1}$$

## Case 2: $\vec{u}' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \vec{u}$: Complex Conjugate Eigenvalues

Consider $\vec{x}' = A\vec{x}$, where $A$ has complex conjugate eigenvalues, $\lambda = \alpha \pm i\beta$.

Suppose one pair is $\lambda = \alpha + i\beta$ and $\vec{v} = Re(\vec{v}) + i\, Im(\vec{v})$.

Then, the appropriate choice for $T$ is

$$\boxed{T = (Re(\vec{v}), Im(\vec{v}))}$$

and $\boxed{L = T^{-1}}$.

Change of coordinate: $\boxed{\vec{u} = L\vec{x} = T^{-1}\vec{x}}$

$$\Rightarrow \boxed{\begin{array}{l} \vec{u}' = (T^{-1}AT)\vec{u}, \\ \text{where } T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \end{array}}$$

Previous Example: $\vec{x}' = A\vec{x}$, where $A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$ pick the sign of $\beta$ so that it agrees with this entry.

Recall: $\lambda = 2 + i$, $\vec{v} = \begin{pmatrix} 1+i \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + i \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\alpha = 2$, $\beta = 1$ $\quad Re(\vec{v}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $Im(\vec{v}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Then, $T = (Re(\vec{v}), Im(\vec{v})) = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \Rightarrow \det(T) = 1$

$$T = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$T^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

Check: $T^{-1}AT = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$

$$= \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \checkmark$$

Then, $\vec{u}' = (T^{-1}AT)\vec{u}$

$$\vec{u}' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}\vec{u} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\vec{u}$$

$$\underline{\lambda = \alpha + i\beta = 2+i}, \quad \underline{\vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix}} \qquad \begin{matrix}\alpha = 2 \\ \beta = 1\end{matrix}$$

<u>Complex Solution</u>: $\vec{v}e^{\lambda t} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{(2+i)t} = e^{2t}\begin{pmatrix} 1 \\ i \end{pmatrix}(\cos t + i\sin t)$

$$= e^{2t}\left[\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right]$$

$$\Rightarrow \vec{u}_1 = e^{2t}\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \quad \vec{u}_2 = e^{2t}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

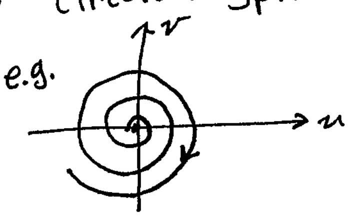
$$\vec{u} = c_1\vec{u}_1 + c_2\vec{u}_2 = e^{2t}\left[c_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right]$$

$$\boxed{\vec{u} = e^{2t}\begin{pmatrix} c_1\cos t + c_2\sin t \\ -c_1\sin t + c_2\cos t \end{pmatrix}} = \begin{pmatrix} u \\ v \end{pmatrix}$$

$$u^2 + v^2 = e^{4t}\left[(c_1^2\cos^2 t + \cancel{2c_1c_2\sin t\cos t} + c_2^2\sin^2 t) + (c_1^2\sin^2 t - \cancel{2c_1c_2\sin t\cos t} + c_2^2\cos^2 t)\right]$$

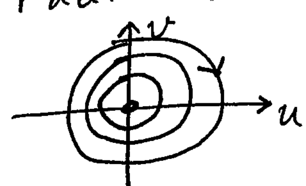
$$u^2 + v^2 = e^{4t}(c_1^2 + c_2^2) \Rightarrow \text{"Circular" Spirals}$$

e.g.




$\underline{\alpha = 0} \Rightarrow u^2 + v^2 = c_1^2 + c_2^2$

<u>Circles</u>: radius $= \sqrt{c_1^2 + c_2^2}$




$c = \beta = -1 < 0$
$\Rightarrow$ <u>clockwise</u>
$\alpha = 2 > 0 \Rightarrow$ <u>outward</u>

---

Finally, $\vec{x} = T\vec{u} = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} e^{2t}\begin{pmatrix} c_1\cos t + c_2\sin t \\ -c_1\sin t + c_2\cos t \end{pmatrix}$

$$\Rightarrow \boxed{\vec{x} = e^{2t}\begin{pmatrix} (c_1+c_2)\cos t + (c_2-c_1)\sin t \\ -c_1\cos t - c_2\sin t \end{pmatrix}} \quad \text{"Elliptical" Spiral}$$

Agrees with our previous solution

Case 3: $\vec{u}' = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \vec{u}$ : Repeated Eigenvalues with only one linearly independent eigenvector.

Consider $\vec{x}' = A\vec{x}$, where $A$ has a repeated eigenvalue $\lambda$ with only one linearly independent eigenvector $\vec{v}$.

Let $\vec{w}$ be a generalized eigenvector;

$$\boxed{(A - \lambda I)\vec{w} = \vec{v}}.$$

The appropriate choice for $T$ is $\boxed{T = (\vec{v}, \vec{w})}$

Example: $\vec{x}' = A\vec{x}$, where $A = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$

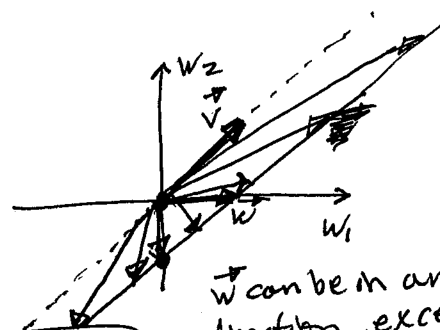
$\lambda = 2, 2$, $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

generalized eigenvector: $(A - \lambda I)\vec{w} = \vec{v}$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$w_1 - w_2 = 1 \Rightarrow w_2 = w_1 - 1$

pick $w_1 = 1 \Rightarrow w_2 = 0 \Rightarrow \vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$




$\vec{w}$ can be in any direction, except that of $\vec{v}$.

Then, $T = (\vec{v}, \vec{w}) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \det(T) = -1$ $\quad T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$T^{-1} = \frac{1}{-1}\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad T^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$

Check: $T^{-1}AT = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ ✓

$\Rightarrow \vec{u}' = (T^{-1}AT)\vec{u} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\vec{u}$

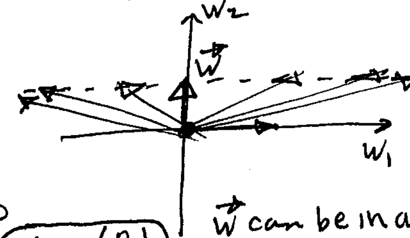
$$\boxed{\vec{u}' = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\vec{u}}$$

We have $\vec{u}' = \begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}\vec{u}$, $\Rightarrow \lambda = 2, 2;\ \vec{v} = \begin{pmatrix}1\\0\end{pmatrix}$

generalized eigenvector: $(A - \lambda I)\vec{w} = \vec{v}$

$$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}w_1\\ w_2\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$$

$w_2 = 1$ Pick $w_1 = 0$ $\Rightarrow \vec{w} = \begin{pmatrix}0\\1\end{pmatrix}$



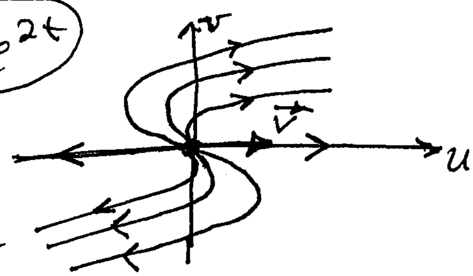

$\vec{w}$ can be in any direction, except that of $\vec{v}$.

$$\vec{u}_1 = \vec{v}e^{\lambda t} = \begin{pmatrix}1\\0\end{pmatrix}e^{2t}$$

$$\vec{u}_2 = (\vec{v}t + \vec{w})e^{2t} = \left(\begin{pmatrix}1\\0\end{pmatrix}t + \begin{pmatrix}0\\1\end{pmatrix}\right)e^{2t} = \begin{pmatrix}t\\1\end{pmatrix}e^{2t}$$

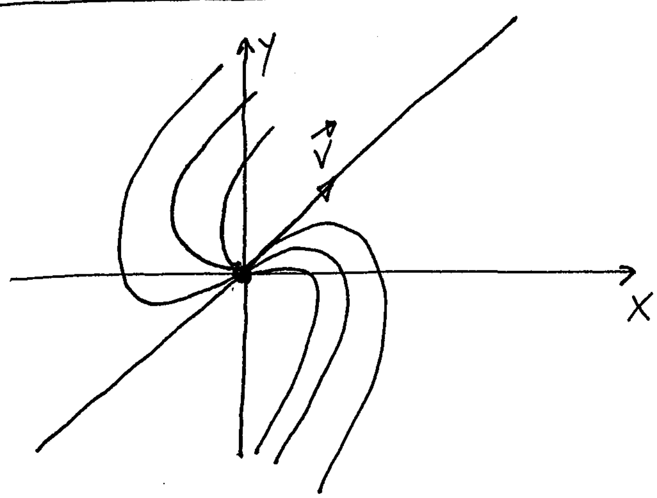
$$\Rightarrow \vec{u} = c_1\vec{u}_1 + c_2\vec{u}_2 = c_1\begin{pmatrix}1\\0\end{pmatrix}e^{2t} + c_2\begin{pmatrix}t\\1\end{pmatrix}e^{2t} = \begin{pmatrix}c_1 + c_2 t\\ c_2\end{pmatrix}e^{2t}$$

$$\vec{u} = \begin{pmatrix}c_1 + c_2 t\\ c_2\end{pmatrix}e^{2t}$$




Then, $\vec{x} = T\vec{u} = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}c_1 + c_2 t\\ c_2\end{pmatrix}e^{2t}$

$$\boxed{\vec{x} = \begin{pmatrix}c_1 + c_2 + c_2 t\\ c_1 + c_2 t\end{pmatrix}e^{2t}}$$



# Summary

$$\vec{x}' = A\vec{x}$$

| | $T$ | $T^{-1}AT$ |
|---|---|---|
| Real Eigenvalues ($\lambda_1$ and $\lambda_2$) with two lin. indep. eigenvectors ($\vec{v}_1$ and $\vec{v}_2$) | $(\vec{v}_1, \vec{v}_2)$ | $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ |
| Real Repeated Eigenvalues ($\lambda$) with one lin. indep. eigenvector $\vec{v}$, and one generalized eigenvector $\vec{w}$ | $(\vec{v}, \vec{w})$ | $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ |
| Complex Conjugate Eigenvalues $\lambda = \alpha + i\beta$, $\vec{v}$ | $(Re(\vec{v}), Im(\vec{v}))$ | $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ |

Then, let $\vec{u} = T^{-1}\vec{x} \Rightarrow \vec{x} = T\vec{u}$

$$\vec{u}' = (T^{-1}AT)\vec{u}$$

# Section 6.4: The Exponential of a Matrix

**Definition:** The exponential of an $n\times n$ matrix $A$ is defined by

$$\boxed{e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$$

$e^A$ is an $n\times n$ matrix

If $t$ is a scalar, we have $\boxed{e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$

## Properties

1) If $A$ and $B$ commute ($AB=BA$), then $e^{A+B} = e^A e^B = e^B e^A$

2) $(e^A)^{-1} = e^{-A}$ (i.e. the inverse matrix of $e^A$ is $e^{-A}$) $\Rightarrow$ $e^A e^{-A} = e^{-A} e^A = I$

3) $e^{T^{-1}AT} = T^{-1} e^A T$ <u>OR</u> $\boxed{e^{(T^{-1}AT)t} = T^{-1} e^{(At)} T}$

Proof: $e^{T^{-1}AT} = \sum_{k=0}^{\infty} \frac{(T^{-1}AT)^k}{k!} = \sum_{k=0}^{\infty} \overbrace{T^{-1}A\cancel{T}\cdot\cancel{T^{-1}}A\cancel{T}\cdot\ldots\cdot\cancel{T^{-1}}A\cancel{T}\cdot\cancel{T^{-1}}AT}^{k\text{-fold}}$

$$= \sum_{k=0}^{\infty} \frac{T^{-1}A^kT}{k!} = T^{-1}\sum_{k=0}^{\infty}\frac{A^k}{k!}\cdot T = T^{-1}e^A T \quad \text{Q.E.D.}$$

4) $\boxed{\frac{d}{dt}(e^{At}) = Ae^{At}}$

Proof: $\frac{d}{dt}(e^{At}) = \frac{d}{dt}\sum_{k=0}^{\infty}\frac{A^kt^k}{k!} = \sum_{k=0}^{\infty}\frac{A^k}{k!}\frac{d}{dt}(t^k) = \sum_{k=0}^{\infty}\frac{A^k}{k!}kt^{k-1}$

$$= \sum_{k=1}^{\infty}\frac{A^k}{k!}kt^{k-1} = \sum_{k=1}^{\infty}\frac{A^k}{(k-1)!}t^{k-1} \underset{\ell=k-1}{=} \sum_{\ell=0}^{\infty}\frac{A^{\ell+1}}{\ell!}t^{\ell}$$

$$= A\sum_{\ell=0}^{\infty}\frac{A^{\ell}t^{\ell}}{\ell!} = Ae^{At} \quad \text{Q.E.D.}$$

zero matrix

5) $e^{0} = I$

6) $A$, $e^A$, and $e^{-A}$ commute pairwise

i.e. $Ae^A = e^A A$
$Ae^{-A} = e^{-A}A$
$e^Ae^{-A} = e^{-A}e^A = I$

7) $e^I = e\cdot I$ <u>OR</u> $e^{It} = e^t I$

## Fundamental Theorem of Linear Systems

Let $A$ be an $n \times n$ matrix. For each $\vec{x}_0 \in \mathbb{R}^n$, the initial value problem $\vec{x}' = A\vec{x},\ \vec{x}(0) = \vec{x}_0$

has a unique solution for all $t \in \mathbb{R}$ given by $\boxed{\vec{x}(t) = e^{At}\vec{x}_0}$

---

The matrix $e^{At}$ can be difficult to calculate for a general matrix $A$.
The calculation is simplified by transforming the system to canonical form.

### Planar Systems ($n=2$)

With the appropriate linear transformation $T$, the matrix $B = T^{-1}AT$ will have one of the following three forms

(1) $B = T^{-1}AT = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ $\quad T = (\vec{v}_1, \vec{v}_2)$

(2) $B = T^{-1}AT = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ $\quad T = (\vec{v}, \vec{w})$

(3) $B = T^{-1}AT = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$ $\quad T = (\mathrm{Re}(\vec{v}), \mathrm{Im}(\vec{v}))$

---

Given the IVP $\vec{x}' = A\vec{x}, \vec{x}(0) = \vec{x}_0$, let $B = T^{-1}AT$.

Then, $A = TBT^{-1}$ and

$$e^{At} = e^{(TBT^{-1})t} = e^{T(Bt)T^{-1}} = Te^{Bt}T^{-1}$$

$$e^{At} = Te^{Bt}T^{-1}$$

The unique solution of the IVP is

$$\boxed{\vec{x} = e^{At}\vec{x}_0 = Te^{Bt}T^{-1}\vec{x}_0}$$

---

OR Let $\vec{u} = T^{-1}\vec{x}$ $\qquad \vec{u}_0 = T^{-1}\vec{x}_0 \Rightarrow \vec{u}(0) = \vec{u}_0$

$\Rightarrow \vec{u}' = (T^{-1}AT)\vec{u} = B\vec{u} \longrightarrow \vec{u}' = B\vec{u},\ \vec{u}(0) = \vec{u}_0$

$\Rightarrow \vec{u} = e^{Bt}\vec{u}_0$

Then, $\vec{x} = T\vec{u} = Te^{Bt}\vec{u}_0 = Te^{Bt}T^{-1}\vec{x}_0$ ✓

To complete the solution we need to compute $e^{Bt}$ for each of the three cases.

Case (1): $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

$$e^{Bt} = \sum_{k=0}^{\infty} \frac{B^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{\begin{pmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{pmatrix} t^k}{k!} = \sum_{k=0}^{\infty} \frac{\begin{pmatrix} (\lambda_1 t)^k & 0 \\ 0 & (\lambda_2 t)^k \end{pmatrix}}{k!}$$

$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(\lambda_1 t)^k}{k!} & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(\lambda_2 t)^k}{k!} \end{pmatrix} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

$$\boxed{e^{Bt} = e^{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} t} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}}$$

---

Case(2): $B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \Rightarrow B = \lambda I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$\lambda I$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ commute

$$e^{Bt} = e^{(\lambda I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})t} = e^{\lambda I t} \cdot e^{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t}$$

$$= e^{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} t} \cdot \left[ I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t + \frac{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 t^2}{2!} + \frac{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^3 t^3}{3!} + \cdots \right]$$

by the above result

$$= \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{pmatrix} \left[ I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t \right]$$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^n = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for $n = 2, 3, \ldots$

$$= e^{\lambda t} I \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\boxed{e^{Bt} = e^{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} t} = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}$$

Case (3): $B = \begin{pmatrix}\alpha & \beta \\ -\beta & \alpha\end{pmatrix}$, $\lambda = \alpha + i\beta$

It can be shown by induction that

$$B^k = \begin{pmatrix}\alpha & \beta \\ -\beta & \alpha\end{pmatrix} = \begin{pmatrix} Re(\lambda^k) & Im(\lambda^k) \\ -Im(\lambda^k) & Re(\lambda^k)\end{pmatrix}$$

Note: $Re(\lambda^k) \neq (Re\,\lambda)^k = \alpha^k$

$$e^{Bt} = \sum_{k=0}^{\infty} \frac{B^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{\begin{pmatrix} Re(\lambda^k) & Im(\lambda^k) \\ -Im(\lambda^k) & Re(\lambda^k)\end{pmatrix} t^k}{k!} = \begin{pmatrix} Re\left(\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}\right) & Im\left(\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}\right) \\ -Im\left(\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}\right) & Re\left(\sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!}\right)\end{pmatrix}$$

$$= \begin{pmatrix} Re(e^{\lambda t}) & Im(e^{\lambda t}) \\ -Im(e^{\lambda t}) & Re(e^{\lambda t})\end{pmatrix}$$

$$e^{\lambda t} = e^{(\alpha + i\beta)t} = e^{\alpha t}(\cos(\beta t) + i \sin(\beta t))$$

$$\Rightarrow Re(e^{\lambda t}) = e^{\alpha t}\cos(\beta t)$$
$$Im(e^{\lambda t}) = e^{\alpha t}\sin(\beta t)$$

$$\boxed{e^{Bt} = e^{\begin{pmatrix}\alpha & \beta \\ -\beta & \alpha\end{pmatrix}t} = e^{\alpha t}\begin{pmatrix}\cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t)\end{pmatrix}}$$

---

Example: $\vec{x}' = A\vec{x}$, $A = \begin{pmatrix} 3 & 2 \\ -1 & 1\end{pmatrix}$, $\vec{x}(0) = \vec{x}_0 = \begin{pmatrix} x_0 \\ y_0\end{pmatrix}$ (Same as a previous example)

Recall: $\lambda = 2 + i$ and $\vec{v} = \begin{pmatrix} 1+i \\ -1\end{pmatrix} = \begin{pmatrix} 1 \\ -1\end{pmatrix} + i\begin{pmatrix} 1 \\ 0\end{pmatrix}$

$\alpha = 2 \quad \beta = 1$ Complex conjugate eigenvalues $\Rightarrow B = \begin{pmatrix}\alpha & \beta \\ -\beta & \alpha\end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 2\end{pmatrix}$

Let $B = T^{-1}AT = \begin{pmatrix}\alpha & \beta \\ -\beta & \alpha\end{pmatrix} \Rightarrow A = TBT^{-1}$

$$\vec{x} = e^{At}\vec{x}_0 = e^{(TBT^{-1})t}\vec{x}_0 = Te^{Bt}T^{-1}\vec{x}_0 \Rightarrow \vec{x} = e^{At}\vec{x}_0 = Te^{Bt}T^{-1}\vec{x}_0$$

$$T = (Re(\vec{v}), Im(\vec{v})) = \begin{pmatrix} 1 & 1 \\ -1 & 0\end{pmatrix} \Rightarrow T = \begin{pmatrix} 1 & 1 \\ -1 & 0\end{pmatrix} \Rightarrow T^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 1\end{pmatrix}$$

$$e^{Bt} = e^{\alpha t}\begin{pmatrix}\cos(\beta t) & \sin(\beta t) \\ -\sin(\beta t) & \cos(\beta t)\end{pmatrix} = e^{2t}\begin{pmatrix}\cos t & \sin t \\ -\sin t & \cos t\end{pmatrix} \Rightarrow e^{Bt} = e^{2t}\begin{pmatrix}\cos t & \sin t \\ -\sin t & \cos t\end{pmatrix}$$

Then, $\vec{x} = Te^{Bt}T^{-1}\vec{x}_0 = \begin{pmatrix} 1 & 1 \\ -1 & 0\end{pmatrix} e^{2t}\begin{pmatrix}\cos t & \sin t \\ -\sin t & \cos t\end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1\end{pmatrix}\begin{pmatrix} x_0 \\ y_0\end{pmatrix} = e^{2t}\begin{pmatrix}\cos t - \sin t & \cos t + \sin t \\ -\cos t & -\sin t\end{pmatrix}\begin{pmatrix} -y_0 \\ x_0 + y_0\end{pmatrix}$

$$= e^{2t}\begin{pmatrix} -y_0(\cos t - \sin t) + (x_0 + y_0)(\cos t + \sin t) \\ y_0 \cos t - (x_0 + y_0)\sin t\end{pmatrix} = e^{2t}\begin{pmatrix} x_0 \cos t + (x_0 + 2y_0)\sin t \\ y_0 \cos t - (x_0 + y_0)\sin t\end{pmatrix}$$

$$\boxed{\vec{x} = e^{2t}\begin{pmatrix} x_0 \cos t + (x_0 + 2y_0)\sin t \\ y_0 \cos t - (x_0 + y_0)\sin t\end{pmatrix}}$$

(these results are equivalent to those of a previous example)

## Section 6.5: Nonhomogeneous Linear Systems with Constant Coefficients

**Scalar (1-D):** general form: $x' = f(t,x)$

nonhomogeneous/constant coefficients: $x' = ax + g(t)$ ← nonhomogeneous/nonautonomous

Assume $a$ is constant, though the integrating factor method works for $a(t)$ as well.

**Integrating Factor Method:** $x' - ax = g(t)$

$$\mu = e^{-at}$$

$$\Rightarrow x = e^{at}\left[x(0) + \int_0^t e^{-as} g(s)\,ds\right]$$

**Systems (n-D):** general form: $\vec{x}' = \vec{f}(t,\vec{x})$

nonhomogeneous/constant coefficients: $\boxed{\vec{x}' = A\vec{x} + \vec{g}(t)}$ ← nonhomogeneous/nonautonomous

**Integrating Factor Method:**

$$\vec{x}' - A\vec{x} = \vec{g}(t) \Rightarrow \mu = e^{-At}$$

$$e^{-At}\vec{x}' - Ae^{-At}\vec{x} = e^{-At}\vec{g}(t)$$

$$\frac{d}{dt}\left(e^{-At}\vec{x}\right) = e^{-At}\vec{g}(t)$$

$$\int_0^t \frac{d}{ds}\left(e^{-As}\vec{x}\right)ds = \int_0^t e^{-As}\vec{g}(s)\,ds$$

$$e^{-As}\vec{x}\Big|_0^t = \int_0^t e^{-As}\vec{g}(s)\,ds$$

$$e^{-At}\vec{x}(t) - \vec{x}(0) = \int_0^t e^{-As}\vec{g}(s)\,ds$$

$$\boxed{\vec{x}(t) = e^{At}\left[\vec{x}(0) + \int_0^t e^{-As}\vec{g}(s)\,ds\right]}$$

**Example:** $x'' - 3x' + 2x = e^t \; ; \; x(0) = x_0, \; x'(0) = y_0$

Method of Undetermined Coefficients $\Rightarrow$ $x = (2x_0 - y_0 - 1)e^t + (y_0 - x_0 + 1)e^{2t} - te^t$

$e^t$ is a homogeneous solution $\Rightarrow x_p = Ate^t$, $A = -1$

---

Convert to a Planar System:

$$x' = y$$
$$y' = -2x + 3y + e^t$$

$$\Rightarrow \vec{x}' = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}\vec{x} + \begin{pmatrix} 0 \\ e^t \end{pmatrix}$$

$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$ $\quad \vec{x}(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$

$\vec{g}(t) = \begin{pmatrix} 0 \\ e^t \end{pmatrix}$

eigs of A: $\lambda_1 = 1, \; \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 2, \; \vec{v}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

transformation: $T = (\vec{v}_1, \vec{v}_2) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $\Rightarrow \det(T) = 1$

$$T^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$T^{-1}AT = B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\Rightarrow A = TBT^{-1} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Then, $e^{At} = e^{(TBT^{-1})t} = e^{T(Bt)T^{-1}} = Te^{Bt}T^{-1}$

$$= Te^{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}t}T^{-1} = T\begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}T^{-1}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} e^t & 0 \\ 0 & e^{2t} \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} e^t & e^{2t} \\ e^t & 2e^{2t} \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 2e^t - e^{2t} & -e^t + e^{2t} \\ 2e^t - 2e^{2t} & -e^t + 2e^{2t} \end{pmatrix}$$

$$\Rightarrow e^{-At} = e^{A(-t)} = \begin{pmatrix} 2e^{-t} - e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{pmatrix}$$

$$e^{-At}\vec{g}(t) = \begin{pmatrix} 2e^{-t}-e^{-2t} & -e^{-t}+e^{-2t} \\ 2e^{-t}-2e^{-2t} & -e^{-t}+2e^{-2t} \end{pmatrix}\begin{pmatrix} 0 \\ e^{t} \end{pmatrix}$$

$$\boxed{e^{-At}\vec{g}(t) = \begin{pmatrix} -1+e^{-t} \\ -1+2e^{-t} \end{pmatrix}}$$

$$\int_0^t e^{-As}\vec{g}(s)\,ds = \int_0^t \begin{pmatrix} -1+e^{-s} \\ -1+2e^{-s} \end{pmatrix} ds = \begin{pmatrix} -s-e^{-s} \\ -s-2e^{-s} \end{pmatrix}\Bigg|_0^t = -\begin{pmatrix} s+e^{-s} \\ s+2e^{-s} \end{pmatrix}\Bigg|_0^t$$

$$= -\begin{pmatrix} (t+e^{-t})-(0+1) \\ (t+2e^{-t})-(0+2) \end{pmatrix} = -\begin{pmatrix} t+e^{-t}-1 \\ t+2e^{-t}-2 \end{pmatrix}$$

$$\boxed{\int_0^t e^{-As}\vec{g}(s)\,ds = -\begin{pmatrix} t+e^{-t}-1 \\ t+2e^{-t}-2 \end{pmatrix}}$$

$$\vec{X} = e^{At}\left[\vec{X}(0) + \int_0^t e^{-As}\vec{g}(s)\,ds\right]$$

$$= \begin{pmatrix} 2e^{t}-e^{2t} & -e^{t}+e^{2t} \\ 2e^{t}-2e^{2t} & -e^{t}+2e^{2t} \end{pmatrix}\left[\begin{pmatrix} X_0 \\ Y_0 \end{pmatrix} - \begin{pmatrix} t+e^{-t}-1 \\ t+2e^{-t}-2 \end{pmatrix}\right]$$

$$\begin{pmatrix} X_0+1-t-e^{-t} \\ Y_0+2-t-2e^{-t} \end{pmatrix}$$

$$= \begin{pmatrix} (2e^{t}-e^{2t})(X_0+1-t-e^{-t}) + (-e^{t}+e^{2t})(Y_0+2-t-2e^{-t}) \\ (2e^{t}-2e^{2t})(X_0+1-t-e^{-t}) + (-e^{t}+2e^{2t})(Y_0+2-t-2e^{-t}) \end{pmatrix}$$

$$\boxed{\vec{X} = \begin{pmatrix} e^{t}(2X_0-Y_0-1-t) + e^{2t}(Y_0-X_0+1) \\ e^{t}(2X_0-Y_0-2-t) + 2e^{2t}(Y_0-X_0+1) \end{pmatrix}}$$

$$\Rightarrow \boxed{X = (2X_0-Y_0-1)e^{t} + (Y_0-X_0+1)e^{2t} - te^{t}}$$

Method of Undetermined Coefficients: homogeneous $X_h = c_1e^{t}+c_2e^{2t}$

particular: $X_p = \alpha te^{t}$, $\alpha = -1$

$\Rightarrow X = c_1e^{t}+c_2e^{2t} - te^{t}$ Initial Conditions $\Rightarrow$ $c_1 = 2X_0-Y_0-1$, $c_2 = Y_0-X_0+1$

## Section 4.1: The Trace-Determinant Plane

Consider $\vec{x}' = A\vec{x}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The eigenvalues of $A$ are determined by

$$\lambda^2 - tr(A)\lambda + det(A) = 0,$$

where $tr(A) = a + d$

and $det(A) = ad - bc$.

Quadratic Formula $\Rightarrow$ $$\boxed{\lambda = \frac{tr(A) \pm \sqrt{(tr(A))^2 - 4\,det(A)}}{2}}$$

Since the eigenvalues of $A$ depend only on $tr(A)$ and $det(A)$, we can make some conclusions about the equilibrium solution $\vec{x} = \vec{0}$ based on these quantities. The results can be illustrated in the Trace-Determinant Plane.

For convenience, let $\boxed{\tau = tr(A)} = a + d$

and $\boxed{\delta = det(A)} = ad - bc$

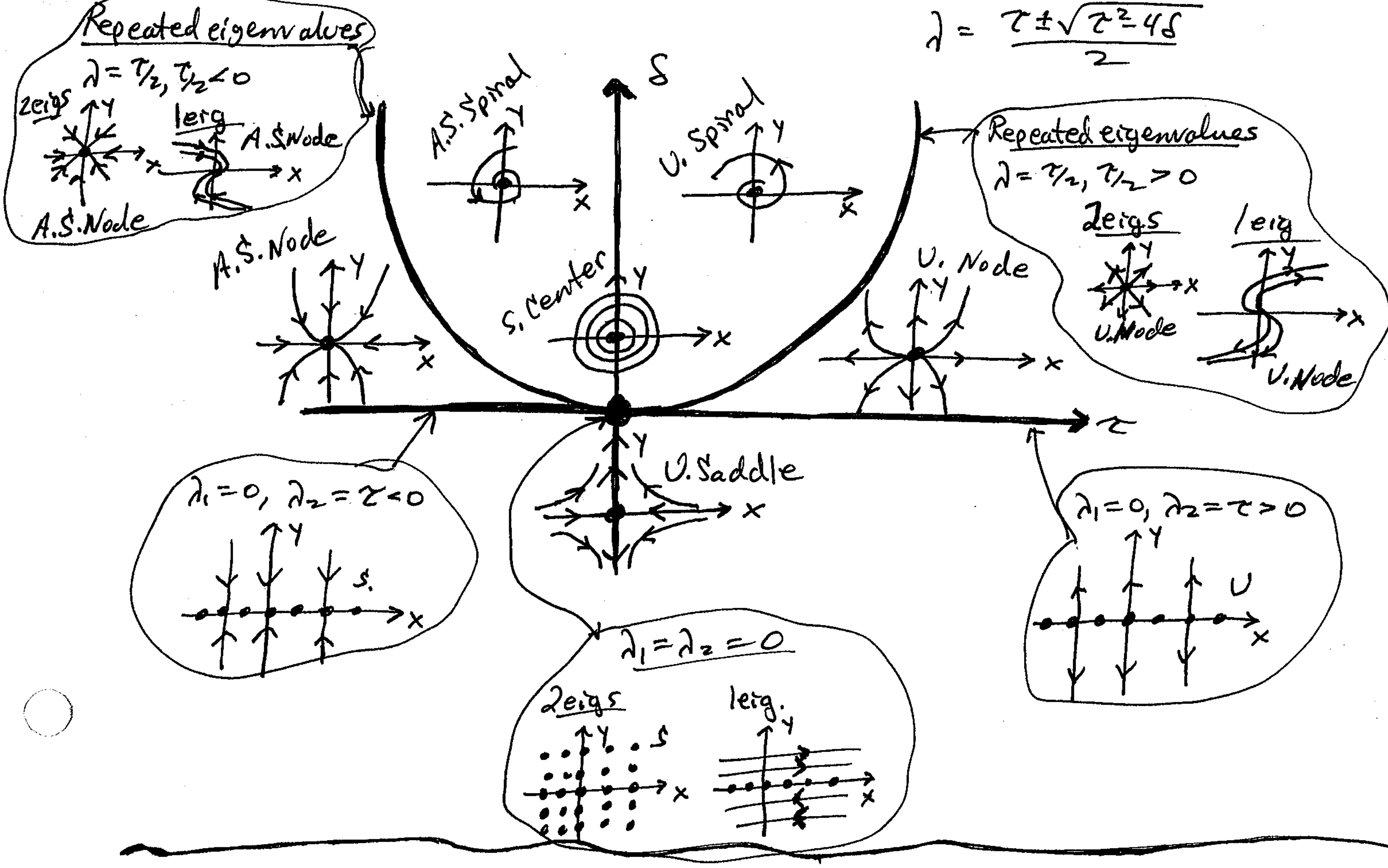
$$\Rightarrow \quad \boxed{\lambda = \frac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}}$$

# Trace-Determinant Plane

$\tau = tr(A) = a + d$

$\delta = det(A) = ad - bc$

$\lambda = \dfrac{\tau \pm \sqrt{\tau^2 - 4\delta}}{2}$




Complex Eigenvalues: $\tau^2 - 4\delta < 0$

$$\delta > \tau^2/4$$

$Re(\lambda) = \tau/2$
- $\tau < 0 \Rightarrow$ Asymptotically Stable Spiral
- $\tau = 0 \Rightarrow$ Stable Center
- $\tau > 0 \Rightarrow$ Unstable Spiral

} clockwise or counter-clockwise?

The direction of the spiral (clockwise or counter-clockwise) cannot be determined from $\tau$ and $\delta$.

e.g. $\vec{x}' = A\vec{x}$ $\quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\tau = 0$

$\delta = 1$

clockwise counter-clockwise

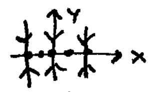
**Real Distinct Eigenvalues:** $\tau^2 - 4\delta > 0$

$\delta < \tau^2/4$

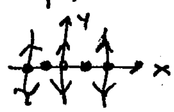
opposite sign: $\delta < 0$ $\Rightarrow$ Unstable Saddle

same sign: $\delta > 0$ $\rightarrow$ $\tau < 0 \Rightarrow$ Asymptotically Stable Node

$\tau > 0 \Rightarrow$ Unstable Node

$\lambda_1 = 0,\ \lambda_2 = \tau$: $\delta = 0$ $\rightarrow$ $\tau < 0 \Rightarrow$  Stable (but not asymptotically)



$\tau > 0 \Rightarrow$ 



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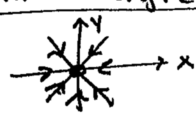
**Repeated Eigenvalues** $\tau^2 - 4\delta = 0$

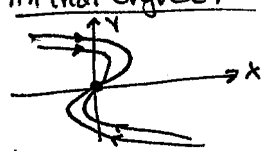
$\delta = \tau^2/4$ $\Rightarrow$ $\lambda = \tau/2, \tau/2$

$\lambda = \tau/2 < 0$ $\Rightarrow$ 2 lin. ind. eigvecs. | 1 lin ind. eigvec.



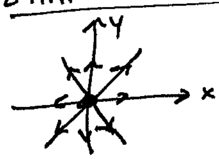


A.S.





$\lambda = \tau/2 > 0$ $\Rightarrow$ 2 lin. ind. eigvecs. | 1 lin ind. eigvec.





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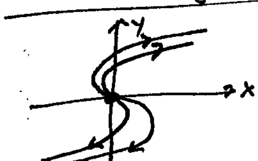
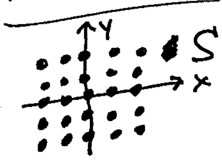




$\lambda = \tau/2 = 0$ $\Rightarrow$ 2 lin. ind. eigvecs. | 1 lin ind. eigvec.

$\Rightarrow \delta = 0$





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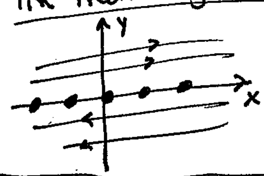



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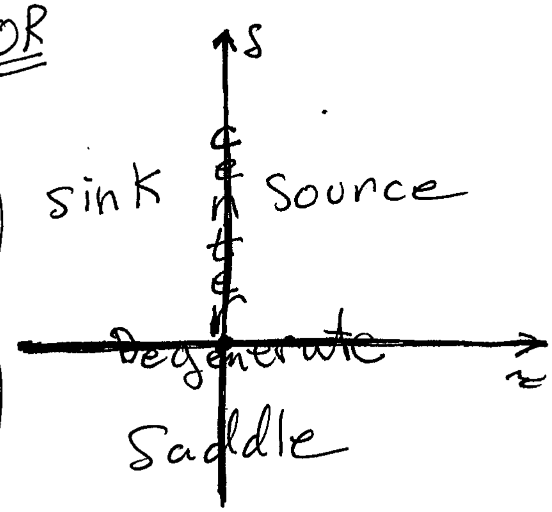
*Note:* $\tau$ and $\delta$ reveal the type and stability of the equilibrium solution $\vec{x} = \vec{0}$, but they do not reveal and info about the eigenvectors, clockwise/counter-clockwise, etc.

The matrix $A$ has four independent parameters, whereas $\tau$ and $\delta$ represent only two of these parameters, and hence, they cannot contain all of the information.

OR



# Chapter 7: Nonlinear Systems

Nonlinear ODEs admit much more complicated and rich solution behavior. The theory of linear ODEs is well-developed, unique solutions exist, and they are often solvable. On the other hand, nonlinear systems zero, multiple, or infinitely many solutions. Nonlinear ODEs are usually not solvable.

## Section 7.1: Dynamical Systems

Recall: $x' = f(x)$, $x(0) = x_0$
flow $= \phi(t, x_0)$
$x_0$ fixed $\Rightarrow x(t) = \phi(t, x_0)$

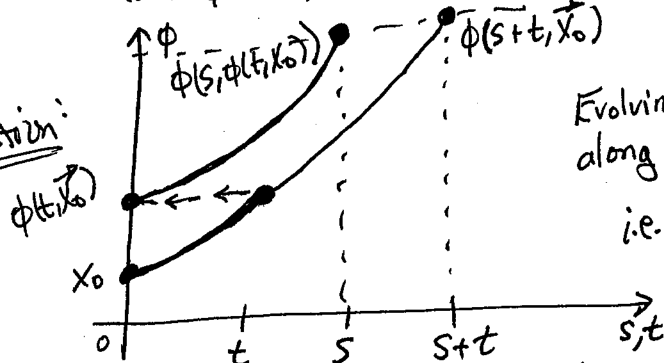
n-dim: flow $= \vec{\phi}(t, \vec{x_0})$, $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

Informal Definition: $\vec{\phi}(t, \vec{x_0})$ is a dynamical system if

(i) $\vec{\phi}(0, \vec{x_0}) = \vec{x_0}$

(ii) $\vec{\phi}(s+t, \vec{x_0}) = \vec{\phi}(s, \vec{\phi}(t, \vec{x_0}))$ for each $s, t, \vec{x_0}$.

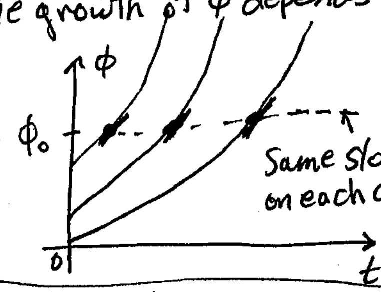
1-dim illustration:



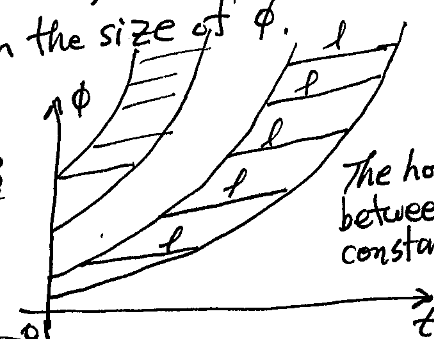

Evolving a solution over a finite time T along any curve yields the same change in $\phi$.
i.e. The growth of $\phi$ at $\phi = \phi_0$ is the same along any feasible solution curve

$\Rightarrow$ The growth of $\phi$ depends only on the size of $\phi$.

other ways to think about it:




OR




The horizontal distance between curves is constant as $\phi$ varies.

Since $\frac{d\phi}{dt}$ depends only on the value of $\phi$, it must be that $\phi$ is the flow of an autonomous differential equation,

$$x' = f(x)$$

change in $x$ depends only on $x$ through some function $f$

The same ideas can be generalized to higher dimensions.

In summary, a dynamical system can be considered as the flow of an autonomous system of first order ODEs, $\vec{x}' = \vec{f}(\vec{x})$

# Autonomous Nonlinear Systems

Normal form of $1^{st}$ order 2x2 systems : $x' = f(t,x,y)$, $y' = g(t,x,y)$ OR $\vec{x}' = \vec{f}(t,\vec{x})$, $\vec{x} = \binom{x}{y}$, $\vec{f} = \binom{f}{g}$

The system is autonomous if $f$ and $g$ do not depend on $t$.

Autonomous : $\boxed{\begin{array}{l} x' = f(x,y) \\ y' = g(x,y) \end{array} \quad \text{OR} \quad \vec{x}' = \vec{f}(\vec{x})}$ $\vec{f}$ is $n \times 1$, $\vec{x}$ is $n \times 1$

Note: $\vec{f}: \mathbb{R}^n \to \mathbb{R}^n$ is a vector field (i.e. $\vec{f}$ maps an n-dim vector into an n-dim vector)

Note: $\vec{f}(\vec{x}) = A\vec{x} \Rightarrow \vec{x}' = A\vec{x}$ (Linear with Constant Coefficients)

---

Population models often involve autonomous ODEs.

We considered a single species (e.g. exponential and logistic population models) in which the ODE was of the form $x' = f(x)$ (autonomous).

Likewise, autonomous systems of ODEs may be used to model populations of two or more interacting species.

e.g. (i) predator/prey populations (fox/rabbits, snakes/mice)

(ii) competing species (snakes and owls compete for mice) (two or more species compete for the same resources)

(iii) big fish/little fish/fisherman

(iv) etc.

# Section 7.2: Existence and Uniqueness

Recall: The linear IVP $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \vec{x}_0$ has a unique solution ($\vec{x} = e^{At}\vec{x}_0$), which is valid for all $t \in \mathbb{R}$.

Nonlinear IVPs may have any number of solutions, (0, 1, 2, ..., or even $\infty$) which may exists only for $t$ in a restricted interval.

---

Classical Example of Nonuniqueness: $x' = 2x^{1/2}$, $x(0) = 0$

$x = 0$ is clearly a solution

A second solution can be found by the method of Separable Variables.

$$\int \frac{dx}{x^{1/2}} = 2\int dt \qquad x(0) = 0 \Rightarrow c = 0$$

$$2x^{1/2} = 2t + c$$

$$x = t^2$$

2 solutions (not unique)

---

Consider the general form of an autonomous, nonlinear IVP,

$$\vec{x}' = \vec{f}(\vec{x}), \quad \vec{x}(t_0) = \vec{x}_0.$$

If $\vec{f}$ is a smooth function, a unique solution exists on some interval containing $t_0$.

---

Existence and Uniqueness Theorem: Consider the IVP

$$\vec{x}' = \vec{f}(\vec{x}), \quad \vec{x}(t_0) = \vec{x}_0.$$

$\vec{x}_0 \in \mathbb{R}^n$
$\vec{f}: \mathbb{R}^n \to \mathbb{R}^n$

If $\vec{f}$ is continuously differentiable at $t_0$, then a unique solution exists on an interval $(t_0 - a, t_0 + a)$ for some $a > 0$.

---

The above example fails to have a unique solution since $f(x) = 2x^{1/2}$ is not continuously differentiable at $t_0 = 0$.

# Equilibrium Solutions (Critical Points)

The equilibrium solutions (or critical points) of the system $\vec{x}' = \vec{f}(\vec{x})$ are the points $\vec{x}_e$ at which $\vec{x}' = \vec{0}$, or equivalently, where $\boxed{\vec{f}(\vec{x}_e) = \vec{0}}$.

i.e. $\left.\begin{aligned} f(x_e, y_e) &= 0 \\ g(x_e, y_e) &= 0 \end{aligned}\right\}$ 2 equations / 2 unknowns.

Note: Unlike $\vec{x}' = A\vec{x}$, $\vec{x}' = \vec{f}(\vec{x})$ may have multiple (isolated) equilibrium solution.

---

Example: Consider $\left.\begin{aligned} x' &= x(1-2x-y) \\ y' &= y(2+x+y) \end{aligned}\right\}$ This form is typical of population models. Expect 3 or 4 eq. sols.

Find all equilibrium solutions

Set $x' = y' = 0 \Rightarrow \boxed{\begin{aligned} x(1-2x-y) &= 0 \\ y(2+x+y) &= 0 \end{aligned}}$ $\begin{aligned} &\to \text{either } x=0 \text{ or } 1-2x-y=0 \\ &\to \text{either } y=0 \text{ or } 2+x+y=0 \end{aligned}$

Consider the four combinations

1) $\begin{aligned} x&=0 \\ y&=0 \end{aligned} \Rightarrow \boxed{(0,0)}$

2) $\begin{aligned} x&=0 \\ 2+x+y&=0 \end{aligned} \Rightarrow \begin{aligned} x&=0 \\ y&=-2 \end{aligned} \Rightarrow \boxed{(0,-2)}$

3) $\begin{aligned} 1-2x-y&=0 \\ y&=0 \end{aligned} \Rightarrow \begin{aligned} x&=\tfrac{1}{2} \\ y&=0 \end{aligned} \Rightarrow \boxed{(\tfrac{1}{2},0)}$

4) $\begin{aligned} 1-2x-y&=0 \\ 2+x+y&=0 \end{aligned} \Rightarrow \begin{aligned} 2x+y&=1 \\ x+y&=-2 \\ \hline x&=3 \Rightarrow y=-5 \end{aligned} \Rightarrow \boxed{(3,-5)}$

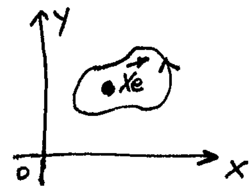
---

Except for special cases, we can classify the equilibrium solutions of autonomous nonlinear systems by type (e.g. spiral, node, saddle) and by stability.

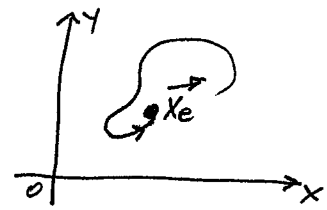
# Stability of Equilibrium Solutions

**Informal Definitions**: An equilibrium solution $\vec{X}_e$ is stable if all solutions (trajectories) near $\vec{X}_e$ stay near $\vec{X}_e$ as $t \to \infty$.

e.g.




OR

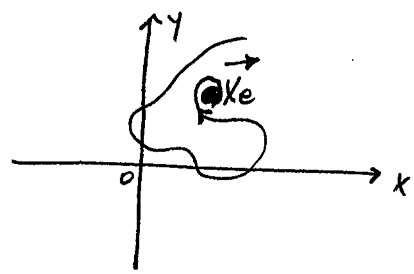



Otherwise, $\vec{X}_e$ is unstable.

$\Rightarrow$ Only one divergent trajectory is enough to render an equilibrium solution as unstable.

$\vec{X}_e$ is asymptotically stable if all solutions (trajectories) near $\vec{X}_e$ approach $\vec{X}_e$ as $t \to \infty$.

i.e. $\lim_{t \to \infty} \vec{X}(t) = \vec{X}_e$ if $\vec{X}$ is sufficiently close to $\vec{X}_e$.




**Note:** Asymptotically Stable $\Longrightarrow$ Stable

$\not\Longleftarrow$

# Linearization About Equilibrium Solutions

Example: $x' = x + y^2, \ x(0) = x_0$ (nonlinear)
$y' = -y, \ y(0) = y_0$

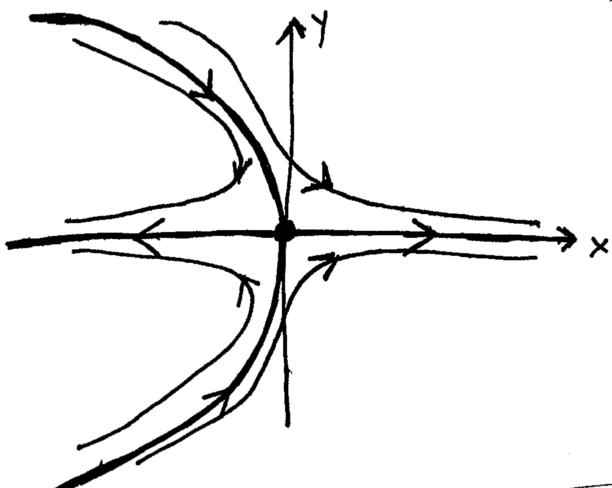
$\vec{x}_e = \vec{0}$ is the only equilibrium solution

$y' = -y, y(0) = y_0 \implies \boxed{y = y_0 e^{-t}}$

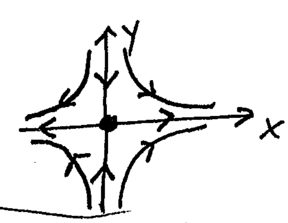
$x' = x + y^2, x(0) = x_0 \implies \boxed{x = (x_0 + \frac{1}{3}y_0^2)e^t - \frac{1}{3}y_0^2 e^{-2t}}$
(integrating factor method)

Note: $y_0 = 0 \implies x = x_0 e^t, \ y = 0$ for all $t$
$x \to \pm\infty$ as $t \to \infty$

$x_0 = \frac{-1}{3}y_0^2 \implies x = \frac{-1}{3}y_0^2 e^{-2t}, \ y = y_0 e^{-t}$
$\implies x = -\frac{1}{3}y^2$ for all $t$
and $x, y \to 0$ as $t \to \infty$.




Near the equilibrium solution $\vec{x}_e = \vec{0}$, $y^2 \sim 0$.

$\implies \begin{matrix} x' \approx x \\ y' = -y \end{matrix} \implies \begin{matrix} \lambda_1 = 1, \ \vec{v}_1 = \binom{1}{0} \\ \lambda_2 = -1, \ \vec{v}_2 = \binom{0}{1} \end{matrix} \implies$ 


The nonlinear term $(y^2)$ bends the vertical separatrix. Otherwise, the equilibrium solution $\vec{x}_e = \vec{0}$ is qualitatively equivalent to a saddle point. As we zoom in closer and closer to the equilibrium solution $\vec{x}_e = \vec{0}$ of the nonlinear system, the phase portrait approaches that of the linear system.
(The nonlinear term $(y^2)$ is most significant near the $y$-axis where $x=0$, and it has no effect along the $x$-axis, where $y=0$.)

Consider the general case: $\vec{x}\,' = \vec{f}(\vec{x})$.

<u>Idea</u>: Near each equilibrium solution $\vec{x}_e$, the nonlinear system $\vec{x}\,'=\vec{f}(\vec{x})$ can be approximated by a linear system $\vec{u}\,' = A\vec{u}$ in attempt to classify the equilibrium solution $\vec{x}_e$ by type and stability.

---

## Jacobian Matrix $\quad \vec{f}(\vec{x}) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}$

$$\boxed{\text{Jacobian of } \vec{f}(\vec{x}) = \vec{f}\,'(\vec{x}) = \frac{d\vec{f}}{d\vec{x}}(\vec{x}) = \begin{pmatrix} f_x(x,y) & f_y(x,y) \\ g_x(x,y) & g_y(x,y) \end{pmatrix}}$$

e.g. $\vec{f}(\vec{x}) = \begin{pmatrix} x^2 \\ ye^x \end{pmatrix} \Rightarrow \vec{f}\,'(\vec{x}) = \begin{pmatrix} (x^2)_x & (x^2)_y \\ (ye^x)_x & (ye^x)_y \end{pmatrix} = \begin{pmatrix} 2x & 0 \\ ye^x & e^x \end{pmatrix}$

though differentiation with respect to a vector is not defined, the Jacobian matrix can be thought of as the derivative of a vector with respect to a vector.

---

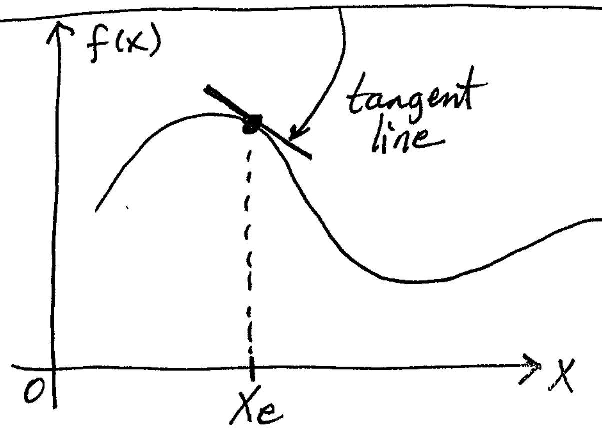
## Linear Approximation of $\vec{f}(\vec{x})$.

Recall the Taylor series of a scalar function $f(x)$ about a point $x = x_e$.

$$f(x) \approx \underbrace{f(x_e) + (x-x_e)f'(x_e)}_{\text{linear part}} + \frac{(x-x_e)^2}{2!} f''(x_e) + \underbrace{\cdots}_{\text{(smaller terms)}}$$

linear approximation of $f(x)$ near $x = x_e$ :

$$f(x) \approx \overbrace{f(x_e)}^{=0 \text{ if } x_e \text{ is an eq. sol.}} + f'(x_e)(x-x_e) \quad \text{for } x \text{ near } x_e$$



**Vector Form**: Taylor series of $\vec{f}(\vec{x})$ near $\vec{x} = \vec{x}_e$.

$$\vec{f}(\vec{x}) = \underbrace{\vec{f}(\vec{x}_e) + (\vec{x} - \vec{x}_e)\vec{f}\,'(\vec{x}_e)}_{\text{linear part}} + \cdots \text{ (smaller terms)}$$

**Linear Approximation of $\vec{f}\,'(\vec{x})$ near $\vec{x} = \vec{x}_e$**: $\boxed{\vec{f}(\vec{x}) \approx \vec{f}(\vec{x}_e) + \vec{f}\,'(\vec{x}_e)(\vec{x} - \vec{x}_e) \text{ for } \vec{x} \text{ near } \vec{x}_e}$

---

Let $\vec{x}_e$ be an equilibrium solution of the system $\vec{x}\,' = \vec{f}(\vec{x})$.

Then,

$$\vec{x}\,' = \vec{f}(\vec{x}) = \underbrace{\cancel{\vec{f}(\vec{x}_e)}}_{\substack{= 0 \text{ since } \vec{x}_e \text{ is} \\ \text{an eq. sol.}}} + \vec{f}\,'(\vec{x}_e)(\vec{x} - \vec{x}_e) + \cdots$$

$$\Rightarrow \boxed{\vec{x}\,' \approx \vec{f}\,'(\vec{x}_e)(\vec{x} - \vec{x}_e)}$$

Linear Approximation of $\vec{x}\,' = \vec{f}(\vec{x})$ for $x$ near $\vec{x}_e$.

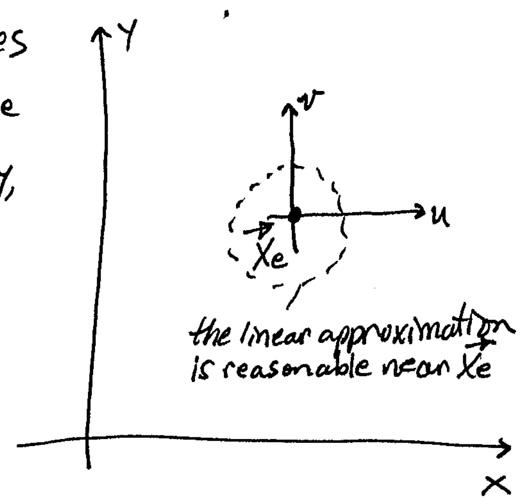
Finally, make the substitution $\vec{u} = \vec{x} - \vec{x}_e$ to shift the equilibrium solution to the origin ($\vec{u}_e = \vec{0}$).

$$\begin{aligned} \vec{u} &= \vec{x} - \vec{x}_e \\ \vec{u}\,' &= \vec{x}\,' \end{aligned} \quad \Longrightarrow \quad \vec{u}\,' = \vec{f}\,'(\vec{x}_e)\,\vec{u}$$

constant Jacobian matrix of $\vec{f}(\vec{x})$ at $\vec{x} = \vec{x}_e$.

**OR** $\boxed{\vec{u}\,' = A\vec{u}, \text{ where } A = \vec{f}\,'(\vec{x}_e)}$ $\quad \vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$

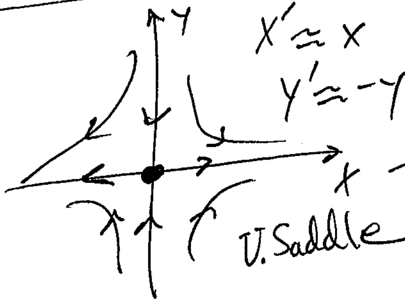
The linear system $\vec{u}\,' = A\vec{u}$ approximates the nonlinear system $\vec{x}\,' = \vec{f}(\vec{x})$ near the equilibrium solution $\vec{x}_e = \vec{x}_e$, or equivalently, near $\vec{u}_e = \vec{0}$.



Recall: $x' = x + y^2, \; x(0) = x_0$
$y' = -y, \; y(0) = y_0$

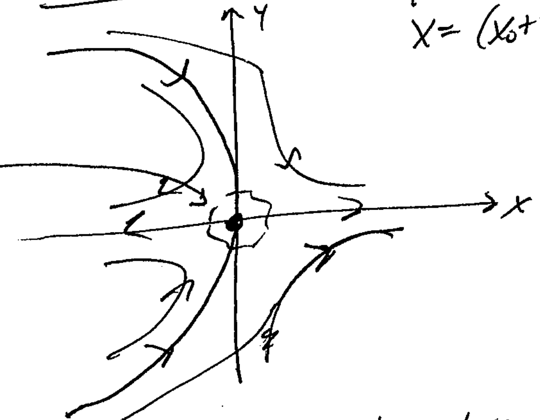
Local Behavior near $\vec{x}_e = \vec{0}$

$x' \simeq x$
$y' \simeq -y$




Global Behavior

$y = y_0 e^{-t}$
$x = (x_0 + \frac{1}{3}y_0^2)e^t - \frac{1}{3}y_0^2 e^{-2t}$

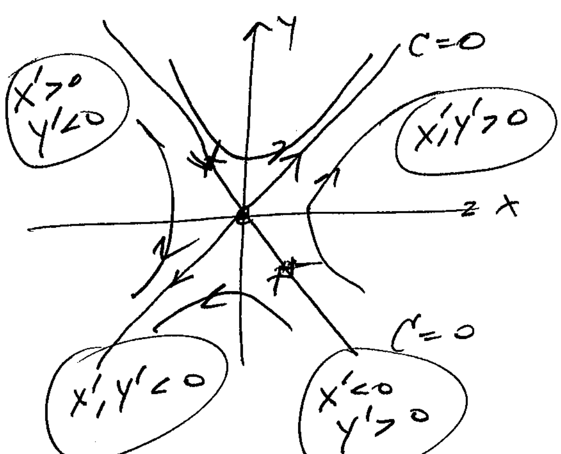



For simpler problems, we can solve for x and y to get the global phase portrait.

Alternative Approach $x' = y$
$y' = x$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'}{x'} = \frac{x}{y}$$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$




$$\frac{dy}{dx} = \frac{x}{y}$$

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + c$$

$$y^2 = x^2 + C$$

---

Book Problems 1 (ii, iii, iv)

Use the approaches described above to find equations for the trajectories of the nonlinear system and to plot the global phase portrait.

# Introduction

Nonlinear Autonomous Planar System:

$x' = f(x,y)$
$y' = g(x,y)$ OR $\vec{x}' = \vec{f}(\vec{x})$

Eq. Sols: $f(x,y)=0$, $g(x,y)=0$ OR $\vec{f}(\vec{x}) = \vec{0}$

---

Linearize about each $\vec{x}_e$ to classify it by type and stability

Let $\vec{u} = \vec{x} - \vec{x}_e$, $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ $\Rightarrow$ $\boxed{\begin{array}{c} \vec{u}\,' \approx A\vec{u} \\ A = \vec{f}\,'(\vec{x}_e) \\ \text{Jacobian} \end{array}}$

$\vec{x} = \vec{x}_e$
$\Rightarrow \vec{u} = 0$
$\Rightarrow \vec{u}_e = 0$

$\vec{x}$ near $\vec{x}_e$ $\Rightarrow$ $|\vec{u}|$ is small

The approximate linear system describes the local behavior of the nonlinear system near $\vec{x}_e$

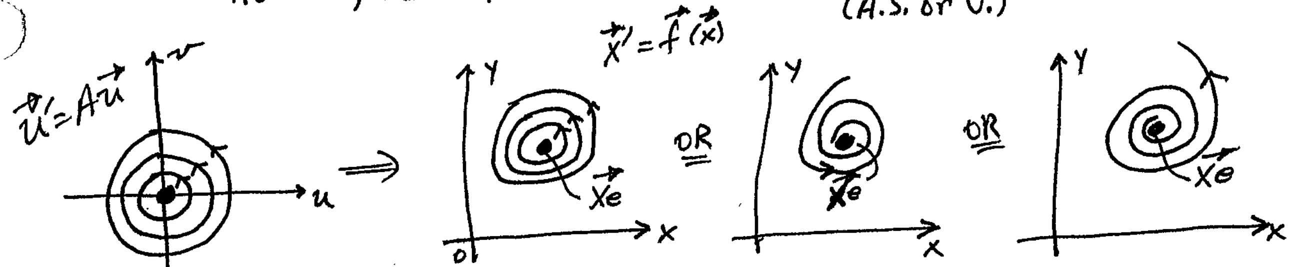
**Theorem:** Simplified version of the Hartman – Grobman Theorem

Let $\vec{x}_e$ be an equilibrium solution of the system $\vec{x}' = \vec{f}(\vec{x})$, and let $\lambda_1$ and $\lambda_2$ be the eigenvalues of the matrix $A = \vec{f}'(\vec{x}_e)$.

If $Re(\lambda_{1,2}) \neq 0$, then the equilibrium solution $\vec{u}_e = \vec{0}$ of the linear system $\vec{u}' = A\vec{u}$ is of the same type (saddle, spiral, ...) and stability as the equilibrium solution $\vec{x}_e$ of the nonlinear system $\vec{x}' = \vec{f}(\vec{x})$. Furthermore, the trajectories of the two systems have the same qualitative behavior near their respective equilibrium solutions.

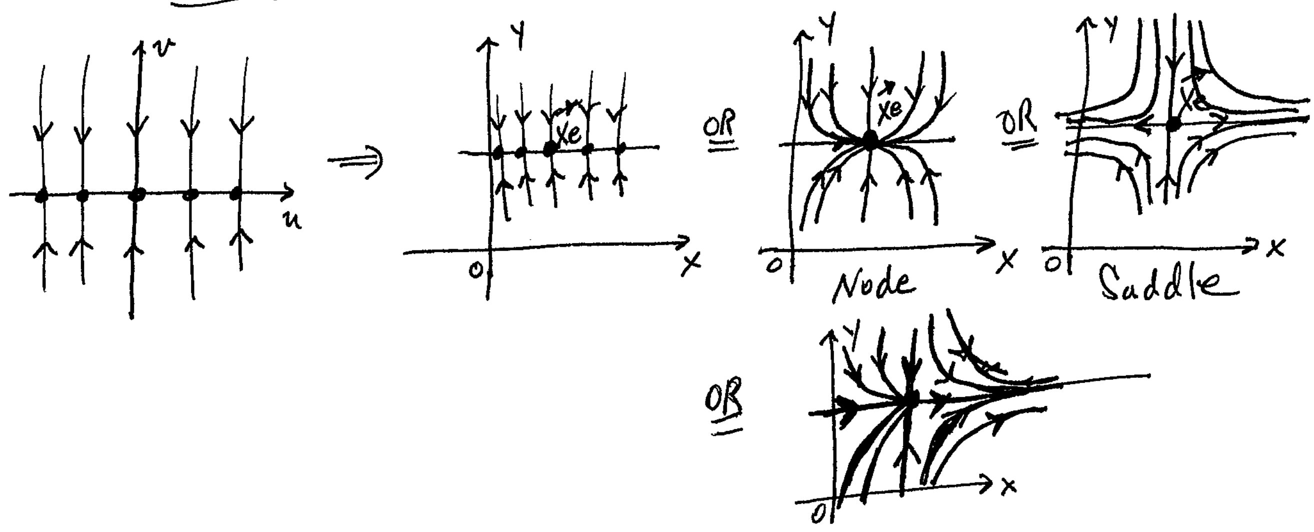
**Note:** If $Re(\lambda_1) = 0$ and/or $Re(\lambda_2) = 0$, then the linear analysis is inconclusive.

e.g. 1. Suppose $\lambda = \pm i\beta$. $\Rightarrow$ $\vec{u}_e = \vec{0}$ is a center of the linear system. However, $\vec{x}_e$ may be either a center or a spiral of the nonlinear system.
(A.S. or U.)




A center is a borderline case. The nonlinear terms may be enough to make the center either a A.S. spiral or a U. spiral.

2. Suppose $\lambda_1 = 0$ and $\lambda_2 \neq 0$.



Example: $x' = x(1-2x-y) = f(x,y)$
$y' = y(2+x+y) = g(x,y)$
Determine the type and stability of all equilibrium solutions.

In a previous example, we found the equilibrium solutions to be

$(0,0)$, $(0,-2)$, $(1/2, 0)$, $(3,-5)$.

Jacobian: $\vec{f}'(\vec{x}) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 1-4x-y & -x \\ y & 2+x+2y \end{pmatrix}$

$\vec{f}'(0,0) = \overset{A}{\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}} \Rightarrow \begin{matrix} \lambda_1 = 1 \\ \lambda_2 = 2 \end{matrix} \Rightarrow$ $\boxed{\vec{x}_e = (0,0) \text{ is an unstable node}}$ $\quad \vec{u}' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\vec{u}$

$\vec{f}'(0,-2) = \overset{A}{\begin{pmatrix} 3 & 0 \\ -2 & -2 \end{pmatrix}} \Rightarrow \begin{matrix} \lambda_1 = 3 \\ \lambda_2 = -2 \end{matrix} \Rightarrow$ $\boxed{\vec{x}_e = (0,-2) \text{ is an unstable saddle}}$ $\quad \vec{u}' = \begin{pmatrix} 3 & 0 \\ -2 & -2 \end{pmatrix}\vec{u}$

$\vec{f}'(\tfrac{1}{2},0) = \overset{A}{\begin{pmatrix} -1 & -1/2 \\ 0 & 5/2 \end{pmatrix}} \Rightarrow \begin{matrix} \lambda_1 = -1 \\ \lambda_2 = 5/2 \end{matrix} \Rightarrow$ $\boxed{\vec{x}_e = (\tfrac{1}{2},0) \text{ is an unstable saddle}}$ $\quad \vec{u}' = \begin{pmatrix} -1 & -1/2 \\ 0 & 5/2 \end{pmatrix}\vec{u}$

$\vec{f}'(3,-5) = \overset{A}{\begin{pmatrix} -6 & -3 \\ -5 & -5 \end{pmatrix}} \Rightarrow \lambda_{1,2} = \dfrac{-11 \pm \sqrt{61}}{2} < 0 \Rightarrow$ $\boxed{\vec{x}_e = (3,-5) \text{ is an asymptotically stable node}}$ $\quad \vec{u}' = \begin{pmatrix} -6 & -3 \\ -5 & -5 \end{pmatrix}\vec{u}$

Notes: 1) $\vec{f}'(\vec{0})$ is the linear part of $\vec{f}(\vec{x})$.
The nonlinear terms are relatively small near $\vec{x}_e = \vec{0}$.

2) We can find the eigenvectors of the linear system near each equilibrium solution, and plot the phase portrait provided $Re(\lambda_{1,2}) \neq 0$.
Often the information can be merged together to sketch the phase portrait over the entire phase plane.

3) If $Re(\lambda_1) = 0$ and/or $Re(\lambda_2) = 0$, the behavior of the linear system may not accurately predict the behavior of the nonlinear system near the equilibrium solution.

To analyze $\vec{x}' = \vec{f}(\vec{x})$,

1. Find all equilibrium solutions $\vec{x}_e$ such that $\vec{f}(\vec{x}_e) = \vec{0}$.
2. Find an expression for the Jacobian of $\vec{f}(\vec{x})$, $\vec{f}'(\vec{x})$.
3. For each equilibrium solution $\vec{x}_e$, analyze the linear system $\vec{u}' = A\vec{u}$, where $A = \vec{f}'(\vec{x}_e)$ as before.
   (a) find the eigenvalues/vectors of $A$
   (b) sketch the phase portrait

   If $Re(\lambda_{1,2}) \neq 0$, then the phase portrait of the nonlinear system resembles that of the linear system near $\vec{x}_e$.
4. Merge the results near each equilibrium solution $\vec{x}_e$ from step 3 to sketch the phase portrait of the nonlinear system over the entire phase plane, if possible.

---

Example: $x' = 1 - y^2 = f(x,y)$ Sketch the phase portrait.
$y' = x + y = g(x,y)$

1. Set $x' = 1 - y^2 = 0 \Rightarrow y = \pm 1$
$y' = x + y = 0 \Rightarrow x = -y$ $\Rightarrow$ Equilibrium solutions $\boxed{(1,-1) \\ (-1,1)}$

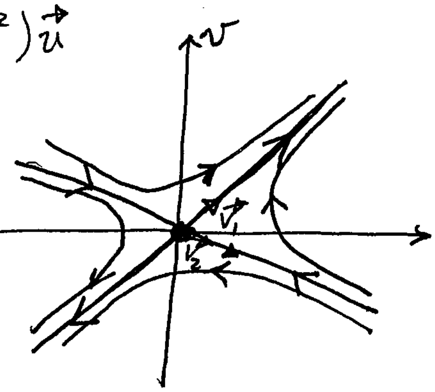
2. $\vec{f}'(\vec{x}) = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} = \begin{pmatrix} 0 & -2y \\ 1 & 1 \end{pmatrix}$

3. $\vec{x}_e = (1,-1) \Rightarrow \vec{f}'(1,-1) = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \Rightarrow$ $\lambda_1 = 2,\ \vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; $\lambda_2 = -1,\ \vec{v}_2 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\Rightarrow$ unstable saddle

$\vec{u}' = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}\vec{u}$

$\vec{u}_e = \vec{0}$




These trajectories approximate those of the nonlinear system near $\vec{x}_e = (1,-1)$

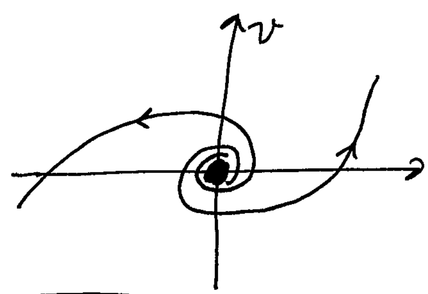
$\underline{\underline{\vec{X}_e = (-1,1)}} \Rightarrow \vec{f}'(-1,1) = \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix} \Rightarrow \boxed{\lambda_1, \lambda_2 = \frac{1}{2} + i\frac{\sqrt{7}}{2}}$

Unstable Spiral

$\vec{u}' = \begin{pmatrix} 0 & -2 \\ 1 & 1 \end{pmatrix} \vec{u}$

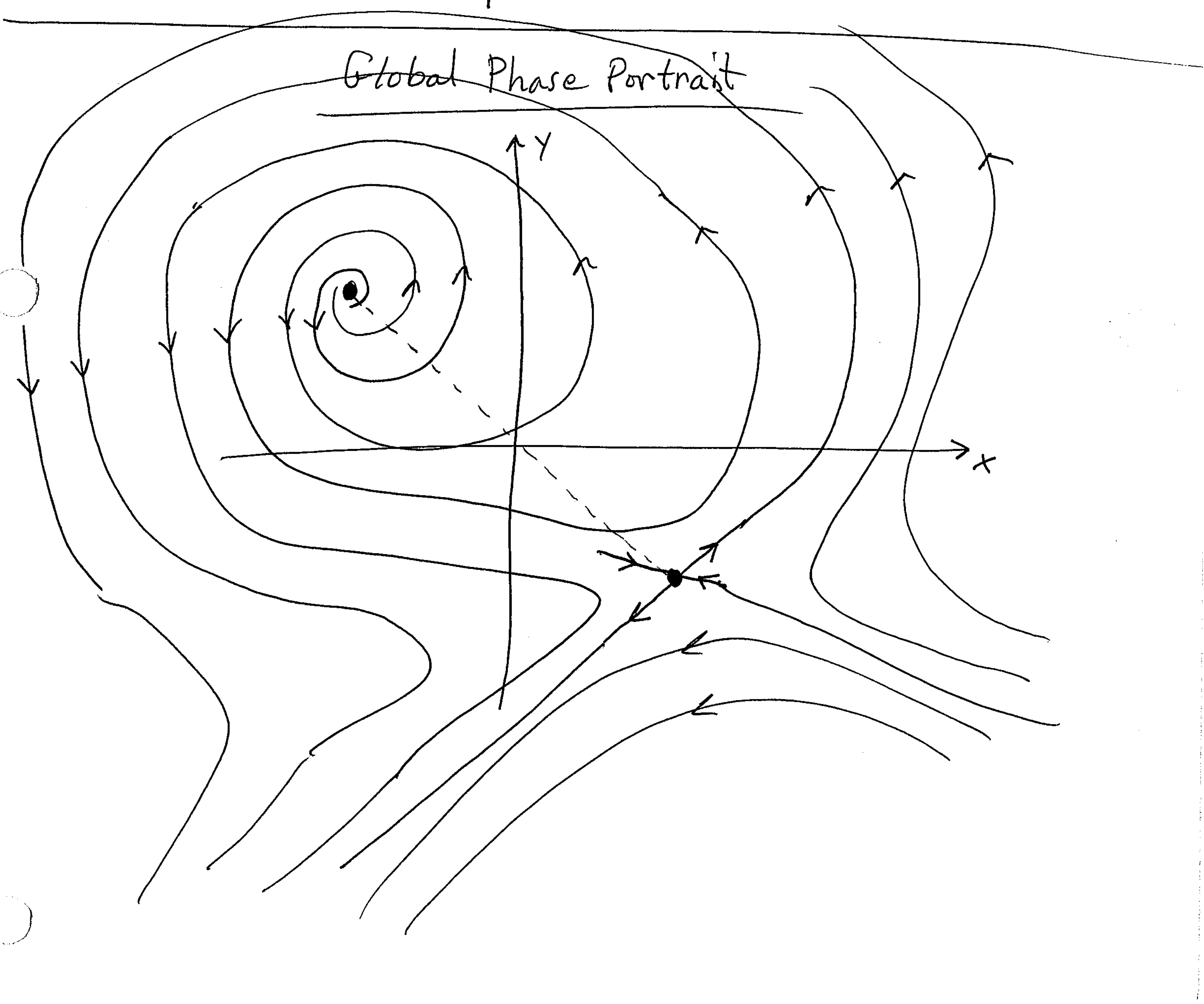
$c = 1 > 0 \Rightarrow$ Counter-Clockwise

$\vec{u}_e = \vec{0}$




These trajectories approximate those of the nonlinear system near $\vec{X}_e = (-1,1)$.

## Global Phase Portrait



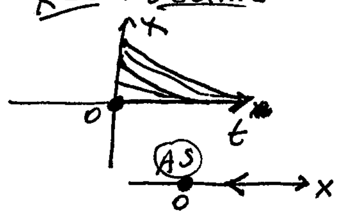
## Section 11.2 and 11.3: Population Models

Recall: Population models for a single species are often of the form

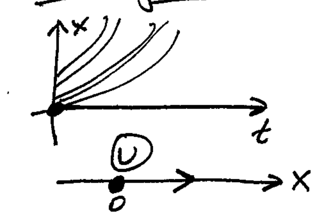
$\boxed{x' = R(x)\cdot x}$, where $R(x) = \frac{x'}{x}$ = relative growth rate

Exponential Population Model: $R(x) = k$ = constant $\Rightarrow \boxed{x' = kx}$

$k<0$: decline



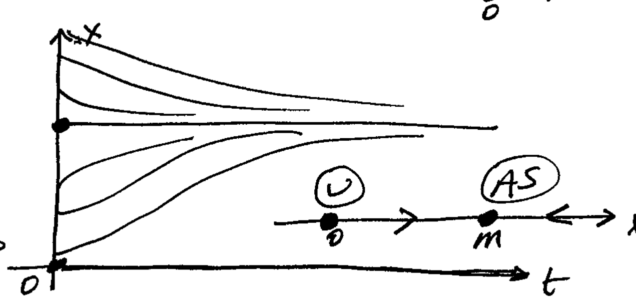

$k>0$: growth




Logistic Population Model: $R(x) = k(1-\frac{x}{m})$

$\Rightarrow \boxed{x' = k(1-\frac{x}{m})x}$

OR $\boxed{x' = x(a-bx)}$ $a = k > 0$, $b = \frac{k}{m} > 0$

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Now consider two interacting populations, $x$ and $y$. Each species, in the absence of the other, is assumed to obey some single species population model, such as the exponential or logistic population model.

e.g. If $y=0$, $x$ may be described by the logistic model.

When both species are present, additional terms must be included to describe the influence each species has on the other as they interact. The simplest meaningful term is of the form $\pm cxy$.

e.g. $x' = \underbrace{x(a-bx)}_{x \text{ is logistic when } y=0} - cxy$

The negative sign suggests that population y has a negative effect on population x. i.e. The presence of y hinders the growth of x.

$\Rightarrow \boxed{x' = x\underbrace{(a-bx-cy)}_{\text{relative growth rate} = R(x,y)}}$, $a,b,c>0$

# Section 11.2: Predator-Prey Models

$x$ = prey (e.g. field mice, rabbits, little fish)
$y$ = predator (e.g. snakes, fox, big fish)

**Model 1:** Assume that $x$ grows exponentially when $y=0$ ($x$ has unlimited resources) and $y$ declines exponentially when $x=0$ ($y$ has no food without $x$).

exponential growth when $y=0$

$$\Rightarrow \quad x' = ax - bxy = x(a-by)$$
$$y' = -cy + dxy = y(-c+dx)$$

exponential decline when $x=0$

The interaction hinders the growth of $x$ and enhances the growth of $y$.

$$\Rightarrow \quad \boxed{\begin{aligned} x' &= x(a-by) \\ y' &= y(-c+dx) \end{aligned}} \qquad a,b,c,d>0$$

---

**Example:** $x' = x(2-y)$ $\quad a=2 \quad b=1$
$y' = 2y(-3+x)$ $\quad c=6 \quad d=2$

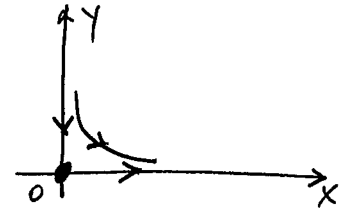
Eq. Sols: $x' = x(2-y) = 0$
$y' = 2y(-3+x) = 0$ $\Rightarrow$ $(0,0)$, $(3,2)$

## Linearize about each equilibrium solution

Jacobian: $\vec{f}'(\vec{x}) = \begin{pmatrix} 2-y & -x \\ 2y & 2(x-3) \end{pmatrix}$

$\underline{(0,0)}$: $\vec{f}'(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -6 \end{pmatrix} \Rightarrow \boxed{\begin{matrix} \lambda_1 = 2, \vec{v}_1 = \binom{1}{0} \\ \lambda_2 = -6, \vec{v}_2 = \binom{0}{1} \end{matrix}}$

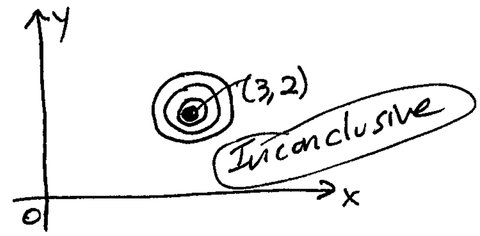
Unstable Saddle




$\underline{(3,2)}$: $\vec{f}'(3,2) = \begin{pmatrix} 0 & -3 \\ 4 & 0 \end{pmatrix} \Rightarrow \boxed{\lambda = \pm 2\sqrt{3}\, i}$

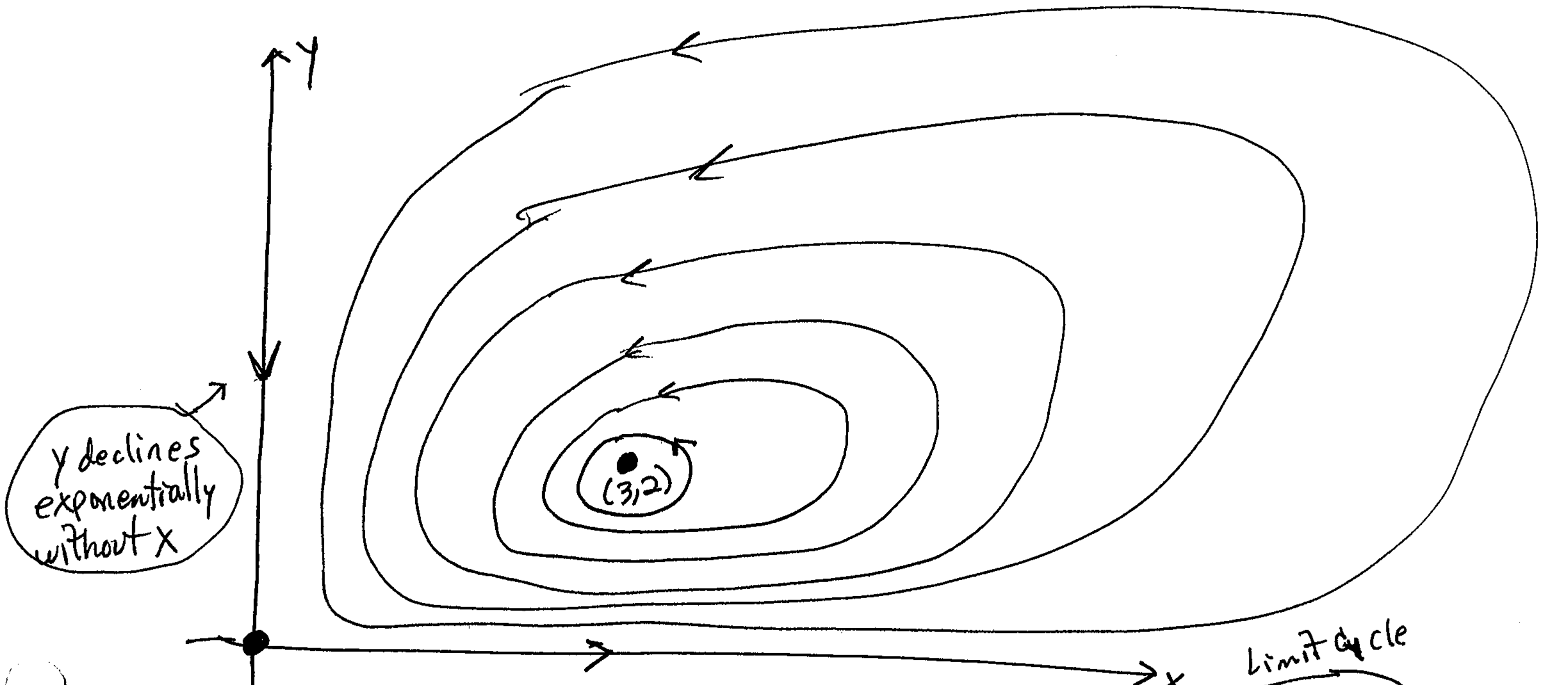
Stable Center

$c = 4 > 0 \Rightarrow$ Counter-Clockwise

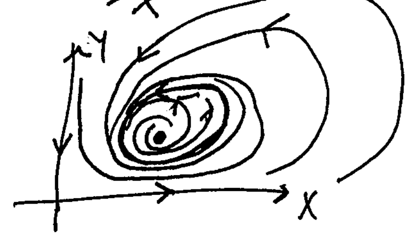



The linear analysis predicts that the equilibrium solution (3,2) is a stable center. However, since Re($\lambda$) the linear analysis is inconclusive for the nonlinear system, in which the equilibrium solution ~~[illegible]~~ (3,2) may be either a stable center or a spiral (either asymptotically stable or unstable).

## According to the Linear Analysis




OR



**Model 2:** (more realistic)

Assume that $x$ is logistic when $y=0$ ($x$ has limited resources)

and $y$ declines exponentially when $x=0$ ($y$ has no food without $x$).

$$\Rightarrow \boxed{\begin{aligned} x' &= x(a-bx-cy) \\ y' &= y(-d+ex) \end{aligned}} \quad a,b,c,d,e>0$$

---

**Example:** $x' = x(2-\frac{1}{2}x-y)$

$y' = 2y(-3+x)$

Equilibrium Solutions: $x' = x(2-\frac{1}{2}x-y) = 0$, $y' = 2y(-3+x) = 0$ $\Rightarrow$ $(0,0)$, $(4,0)$, $(3,\frac{1}{2})$

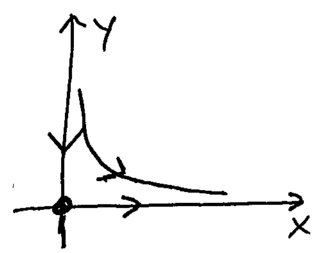
---

Linearize about each equilibrium solution

Jacobian: $\vec{f}'(\vec{x}) = \begin{pmatrix} 2-x-y & -x \\ 2y & 2(x-3) \end{pmatrix}$

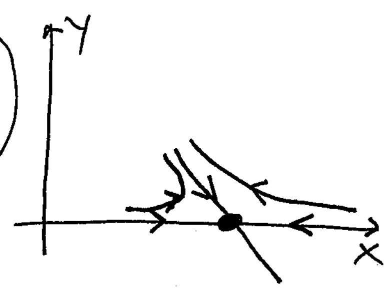
$(0,0)$: $\vec{f}'(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -6 \end{pmatrix} \Rightarrow \lambda_1 = 2,\ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda_2 = -6,\ \vec{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Unstable Saddle




$(4,0)$: $\vec{f}'(4,0) = \begin{pmatrix} -2 & -4 \\ 0 & 2 \end{pmatrix} \Rightarrow \lambda_1 = -2,\ \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda_2 = 2,\ \vec{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Unstable Sadde

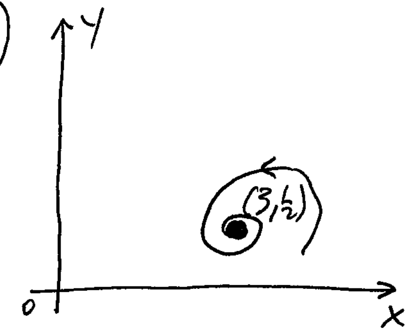


$\underline{(3,\tfrac{1}{2}):}$ $\vec{f}'(3,\tfrac{1}{2}) = \begin{pmatrix} -3/2 & -3 \\ 1 & 0 \end{pmatrix} \Rightarrow \lambda = \frac{-3}{4} \pm i\frac{\sqrt{39}}{4}$

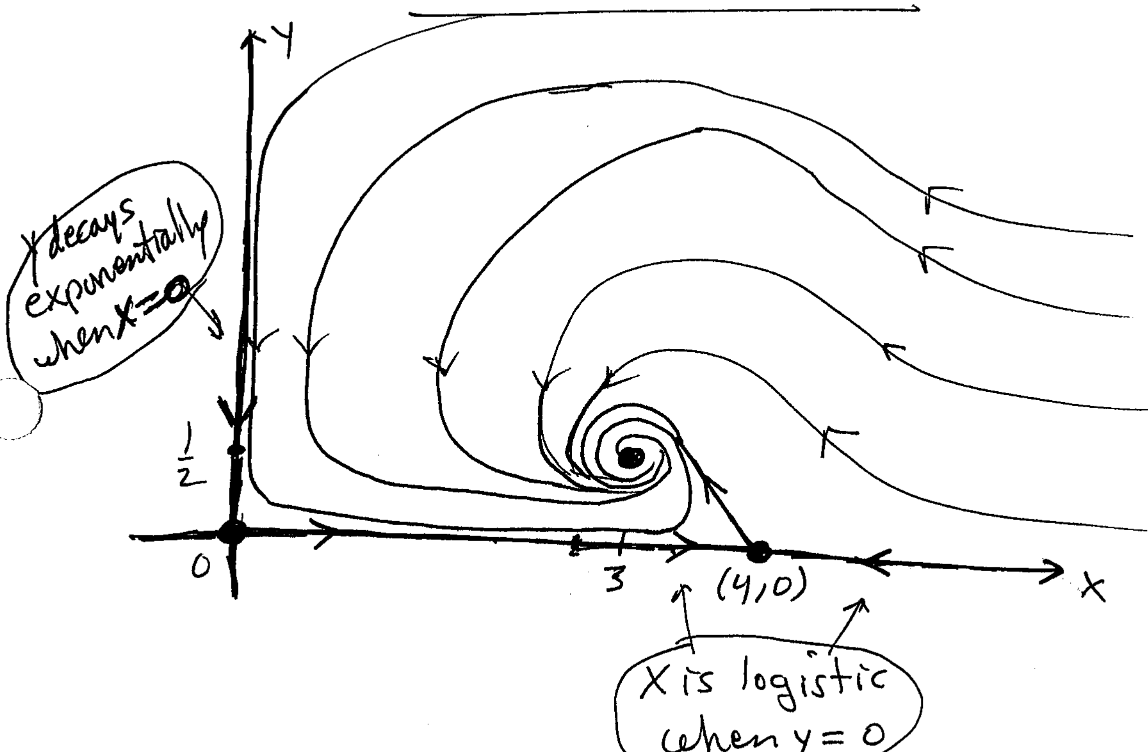
Asymptotically Stable Spiral

$Re(\lambda) < 0 \Rightarrow$ Inward

$c = 1 > 0 \Rightarrow$ Counter-clockwise




## Global Phase Portrait




The behavior along the **axes** agrees with the assumptions about each population in the absence of the other.

If both species are present, their population sizes will ultimately reach a balance and the two species will coexist thereafter at the equilibrium solution $(3, \frac{1}{2})$.

## Section 11.3: Competing Species

Two species compete for the same resources.

e.g. $x$ = owls } owls and snakes compete for field mice.
$y$ = snakes } (It is assumed that the field mice are limited)

It is assumed that each population is logistic in the absence of the other.
Since the two species compete for the same resources, the interaction decreases the growth rates of both populations.

$$\Rightarrow \boxed{\begin{aligned} x' &= x(a-bx-cy) \\ y' &= y(d-ey-fx) \end{aligned}} \quad \underline{a,b,c,d,e,f > 0}$$

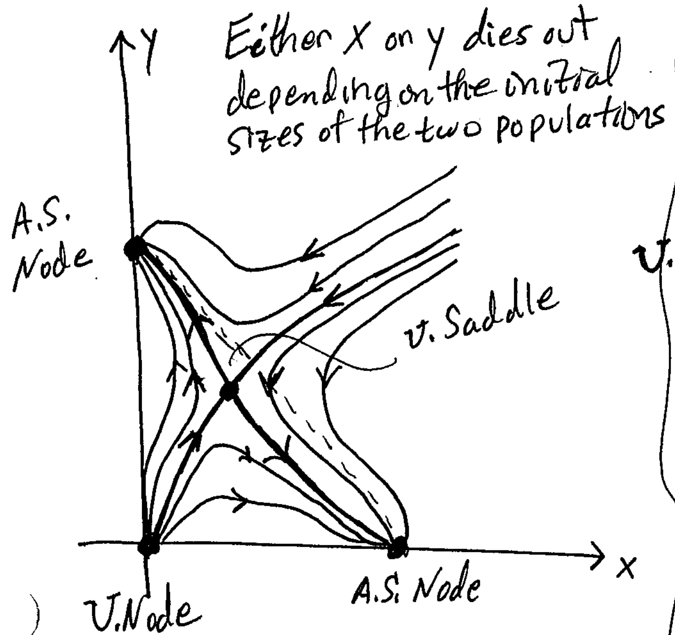
---

There are various possible outcomes depending on the values of $a,b,c,d,e,f$, and the initial population sizes.

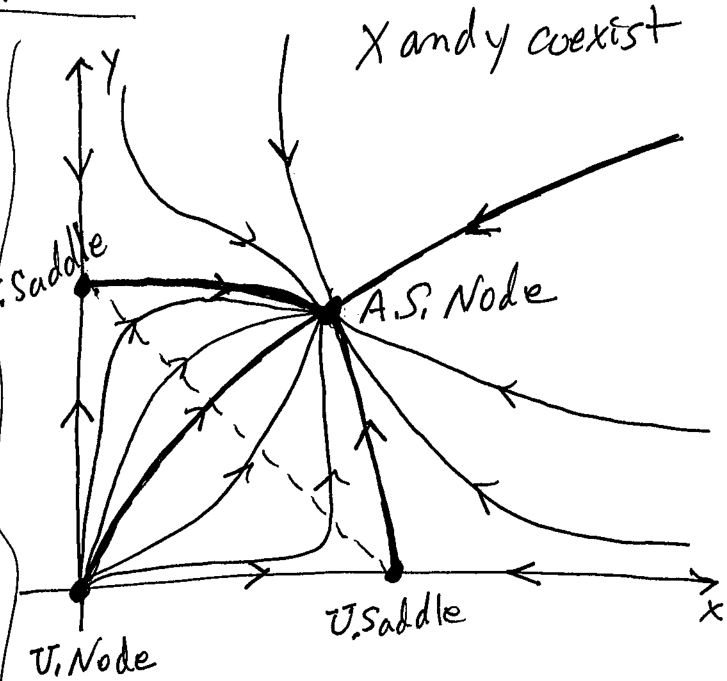
i) $x$ may become extinct
ii) $y$ may become extinct
iii) $x$ and $y$ may coexist.

---

Two Classical Possibilities




X and y coexist



# Chapter 9: Global Techniques



## Section 9.1: Nullclines

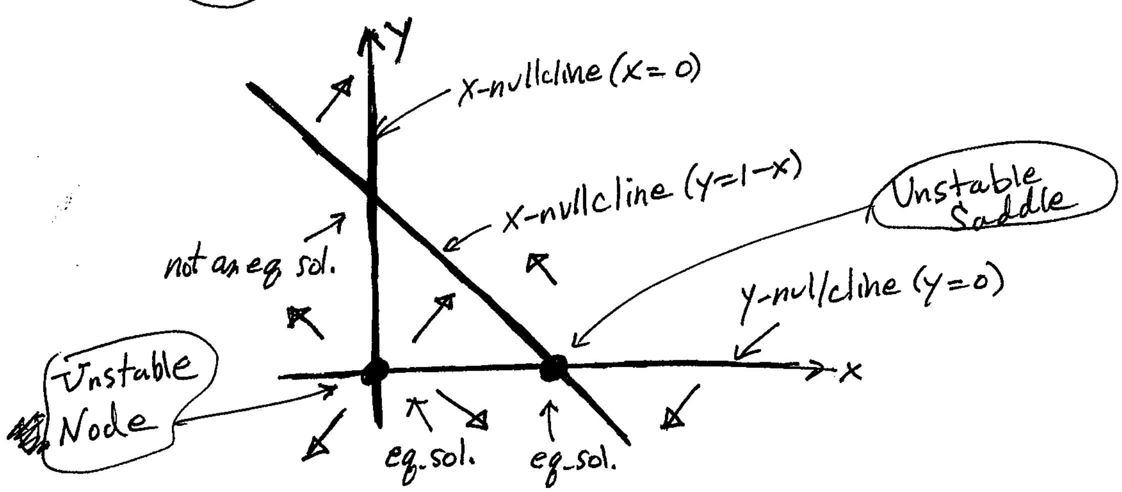
Consider $x' = f(x,y)$
$y' = g(x,y)$

The x-nullclines are curves along which $x' = 0 \Rightarrow$ Vertical Tangent Vector.
" y-nullclines " " " " $y' = 0 \Rightarrow$ Horizontal " ".

$$\boxed{\begin{array}{ll} x\text{-nullclines}: & f(x,y)=0 \ \text{(Vertical Tangent Vector)} \\ y\text{-nullclines}: & g(x,y)=0 \ \text{(Horizontal Tangent Vector)} \end{array}}$$

Equilibrium solutions occur at points where the x-nullclines intersect the y-nullclines.

e.g. $x' = x(1-x-y) \Rightarrow$ x-nullclines: $x=0$, $y=1-x$
$y' = y \Rightarrow$ y-nullclines: $y=0$




The nullclines divide the xy-plane into regions where the tangent vectors to the solution curves are either NE, NW, SE, SW in direction

e.g. x-nullcline: east, west  y-nullcline: north, south

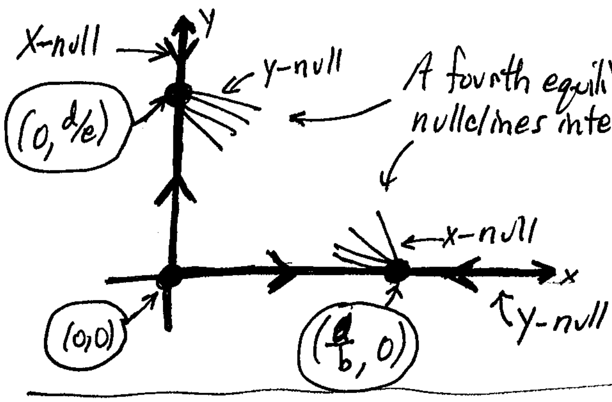
---

Nullclines are a powerful tool for studying bifurcations.

Example: Competing Species

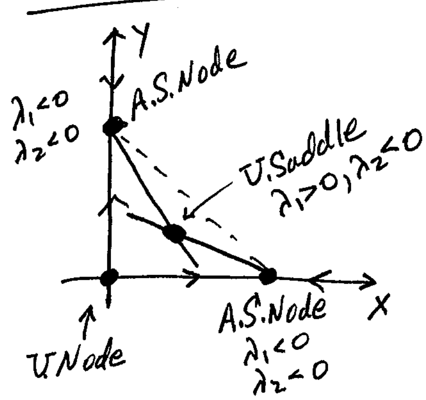
nullclines

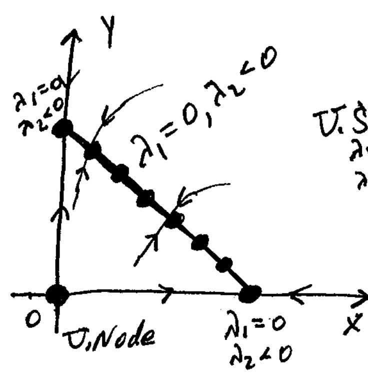
$$x' = x(a-bx-cy) \Rightarrow x=0,\ y=\tfrac{1}{c}(a-bx)$$
$$y' = y(d-ex-fy) \Rightarrow y=0,\ y=\tfrac{1}{e}(d-fx)$$

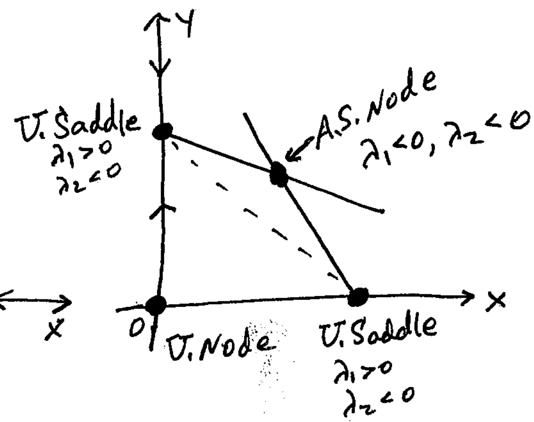



A fourth equilibrium solution exists if these nullclines intersect in the first quadrant

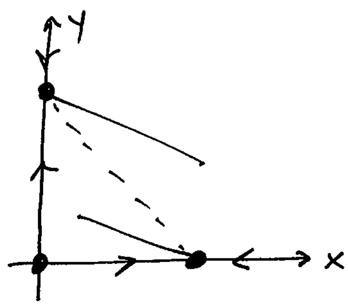
Cases:

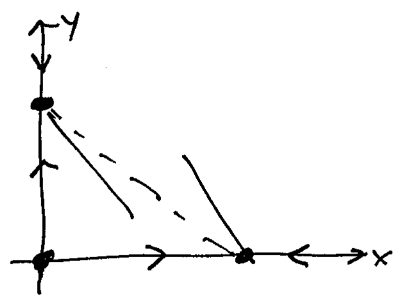



No Intersections



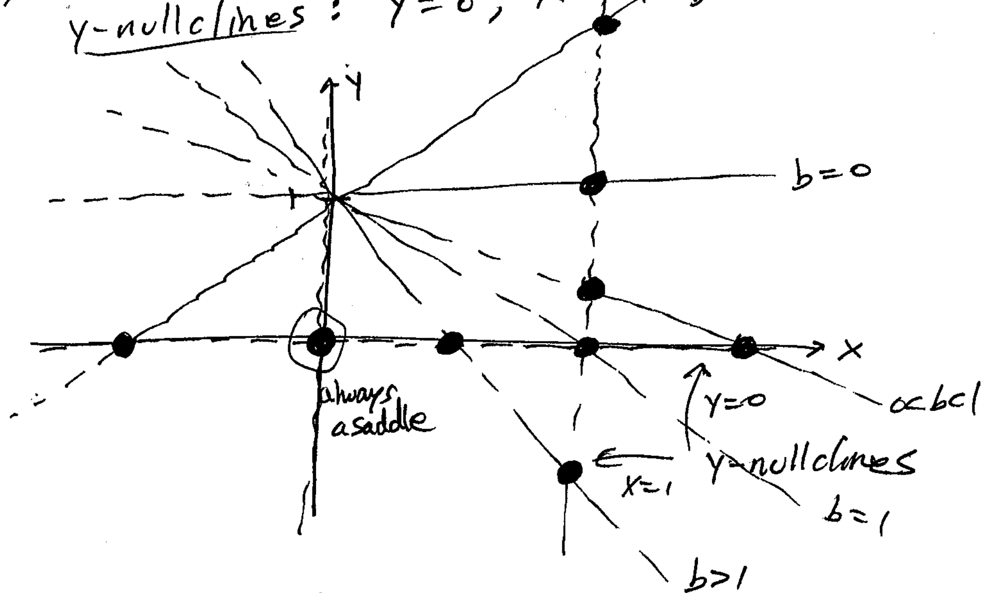
# Bifurcations

**Example:** $x' = x(1-bx-y)$
$y' = y(-1+x)$

*Note:* $x, y \geq 0$, $b \geq 0$ $\Rightarrow$ Predator-Prey Model
$b=0 \Rightarrow x$ is exponential
$b>0 \Rightarrow x$ is logarithmic

**Eq Sols:** $\boxed{(0,0),\ (1/b, 0),\ (1, 1-b)}$

x-nullclines : $x=0$, $y = 1-bx$
y-nullclines : $y=0$, $x=1$




$$\vec{f}'(\vec{x}) = \begin{pmatrix} 1-2bx-y & -x \\ y & -1+x \end{pmatrix}$$

$$\vec{f}'(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \begin{matrix} \lambda_1 = 1 > 0 \\ \lambda_2 = -1 < 0 \end{matrix} \Rightarrow \text{unstable saddle}$$

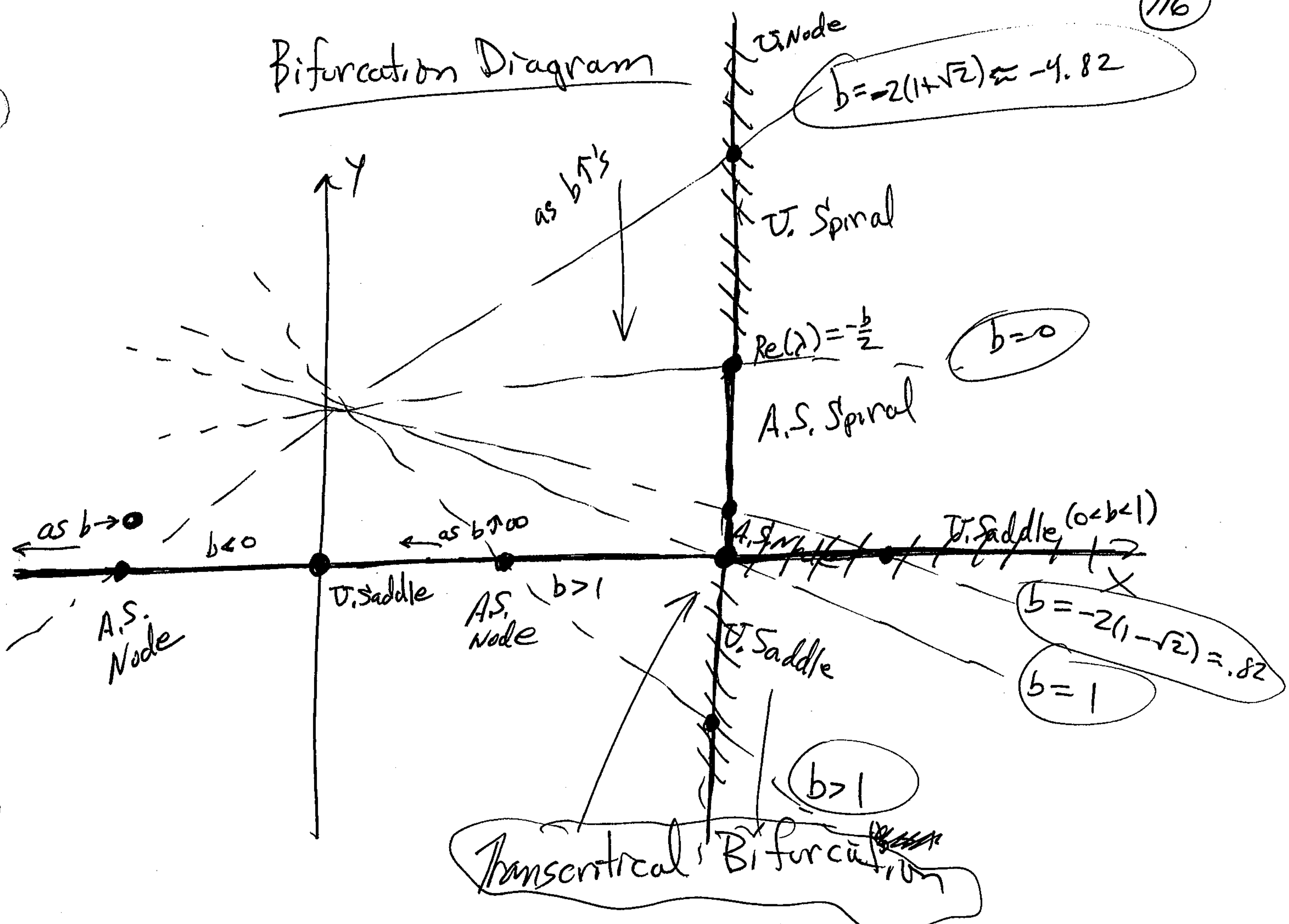
$$\vec{f}'(1/b, 0) = \begin{pmatrix} -1 & -1/b \\ 0 & -1+\frac{1}{b} \end{pmatrix} \quad \begin{matrix} \lambda_1 = -1 < 0 \\ \lambda_2 = -1 + \frac{1}{b} \end{matrix} \quad \begin{matrix} < \text{ for } b<0, b>1 \\ > \text{ for } 0<b<1 \end{matrix}$$

$$\vec{f}'(1, 1-b) = \begin{pmatrix} -b & -1 \\ 1-b & 0 \end{pmatrix} \quad \lambda = \frac{-b \pm \sqrt{b^2+4b-4}}{2}$$

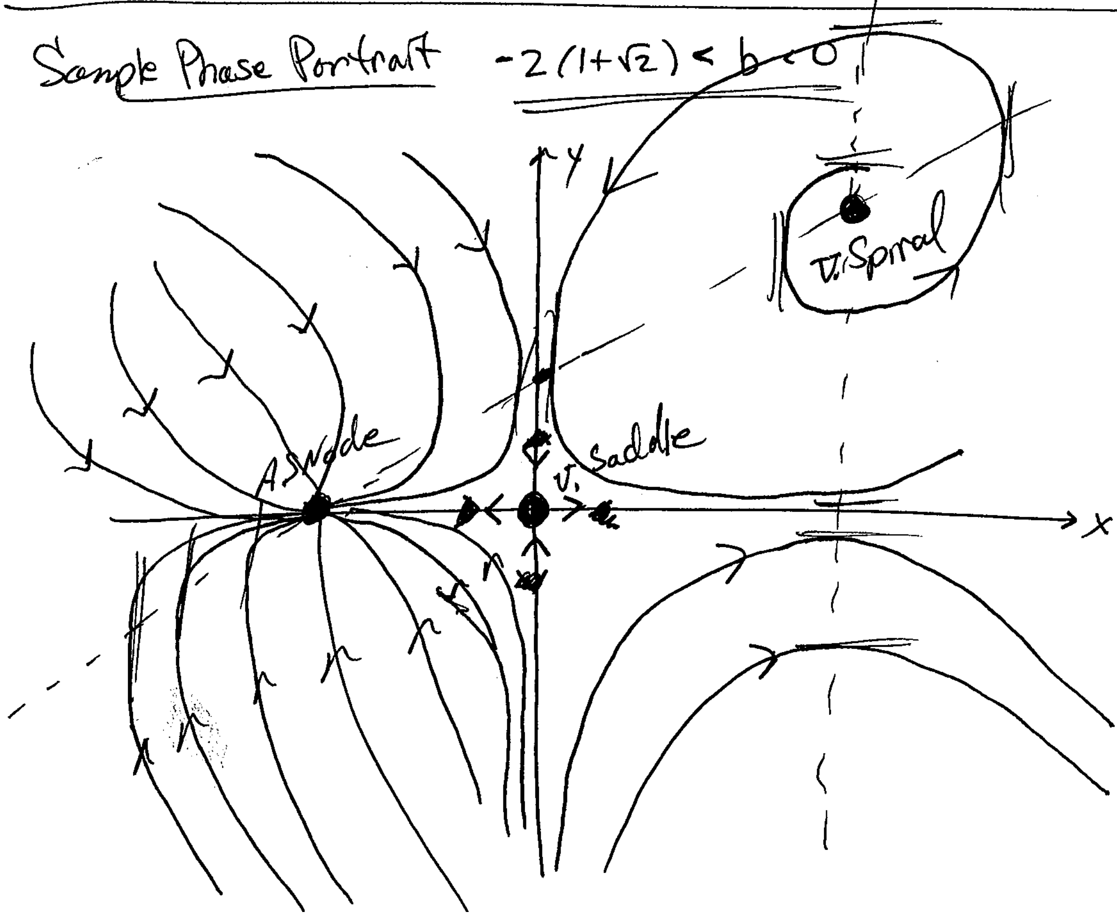
complex for
$-2(1+\sqrt{2}) < b < -2(1-\sqrt{2})$

$Re(\lambda) = -\frac{b}{2}$

# Bifurcation Diagram




## Sample Phase Portrait $-2(1+\sqrt{2}) < b < 0$



# Linearization / Eigenvalues : Scalar vs. System

**System:** $\vec{x}' = \vec{f}(\vec{x})$

Linearize about an equilibrium solution $\vec{x}_e$

$$\vec{u} = \vec{x} - \vec{x}_e$$

$$\Rightarrow \vec{u}' = \vec{f}'(\vec{x}_e)\,\vec{u}$$

Eigenvalues: $\det(\vec{f}'(\vec{x}_e) - \lambda I) = 0$

$\lambda_i < 0$ for each $i \Rightarrow \vec{x}_e$ is Asymptotically Stable

$\lambda_i > 0$ for some $i \Rightarrow \vec{x}_e$ is Unstable

$Re(\lambda_i) = 0$ is an Inconclusive borderline case.

---

**Scalar:** $x' = f(x)$

Linearize about an equilibrium solution $x_e$

$$u = x - x_e$$

$$\Rightarrow u' = f'(x_e)\,u$$

Recall:
$f'(x_e) < 0 \Rightarrow x_e$ is A.S
$f'(x_e) > 0 \Rightarrow X$ is U
$f'(x_e) = 0 \Rightarrow$ Inconclusive

Eigenvalues: $\det(\underbrace{f'(x_e)}_{\text{scalar}} - \lambda \overset{=1}{I}) = 0$

$$f'(x_e) - \lambda = 0$$

$$\lambda = f'(x_e)$$

$\Rightarrow \lambda = f'(x_e) < 0 \Rightarrow x_e$ is Asymptotically Stable

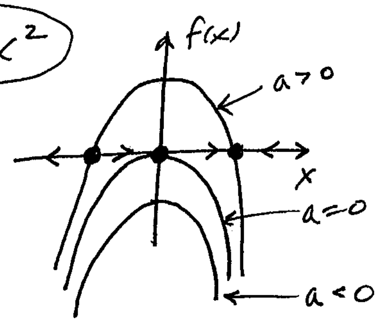
$\lambda = f'(x_e) > 0 \Rightarrow x_e$ is Unstable

$\lambda = f'(x_e) = 0 \Rightarrow$ Inconclusive borderline case

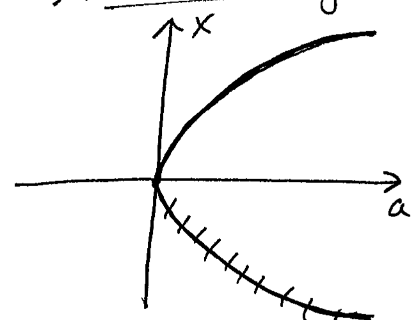
# Bifurcations

## Saddle Node

**Scalar:** $x' = a - x^2$




Bifurcation Diagram




Eq. Sols. $f(x) = a - x^2 = 0$

$x = \pm\sqrt{a},\ a>0$

---

### Analogous Examples for Systems

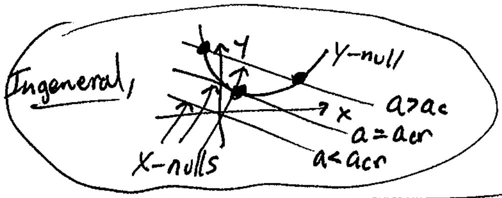
Since $Y_e = 0$, we may draw a bifurcation diagram x vs. a.

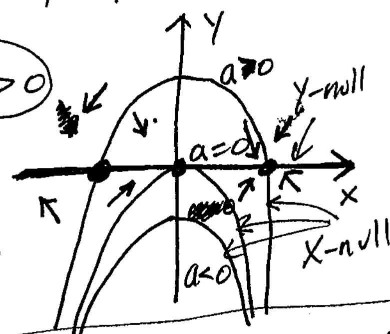
nullclines

1. $x' = (a - x^2) - y \Rightarrow x:\ y = a - x^2$

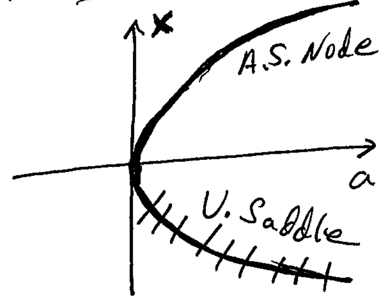
$y' = -y \Rightarrow y:\ y = 0$

Eq. Sols. $(\pm\sqrt{a}, 0),\ a>0$




Bifurcation Diagram

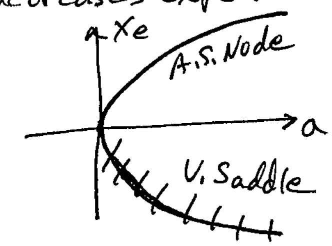



(*) (2) $x' = a - x^2$

$y' = -y$

The equations are decoupled. x undergoes a saddlenode bifurcation, while y decreases exponentially.

Bifurcation Diagram:

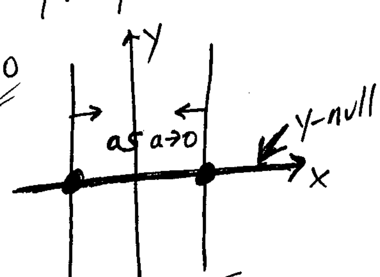



Eq. Sols: $(\pm\sqrt{a}, 0),\ a>0$

nullclines

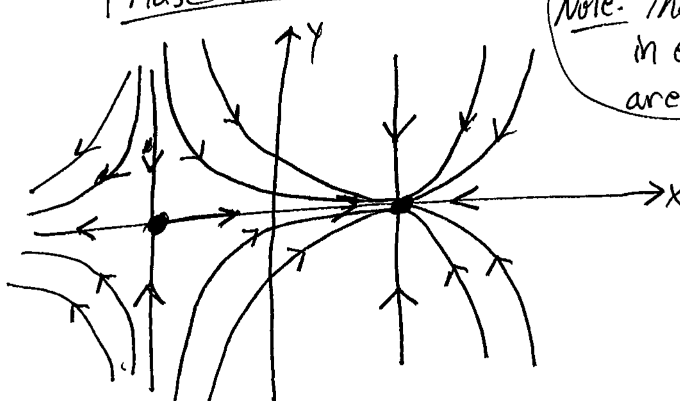
x: $x = \pm\sqrt{a}$

y: $y = 0$

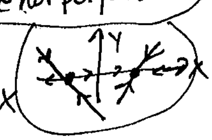



no x-nullclines for $a<0$

Phase Portrait $(a>0)$




Note: The eigenvectors in example 1 are not perpendicular



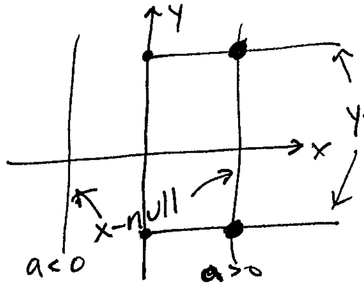
3. $\begin{cases} x' = a - x \\ y' = a - y^2 \end{cases}$ $\Longrightarrow$ **nullclines** $x$: $x = a$; $y$: $y = \pm\sqrt{a}$, $a > 0$

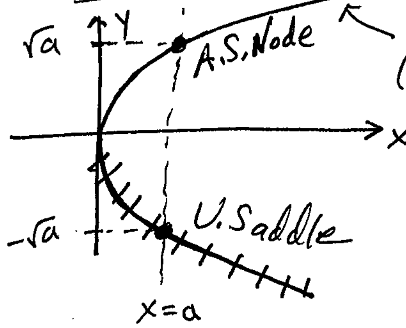
These may be considered as a set of parametric equations with parameter $a$.

$$x = a,\quad y = \pm\sqrt{x},\quad a > 0$$

Eq. Sols: $(a, \pm\sqrt{a}),\ a > 0$

Bifurcation Diagram?



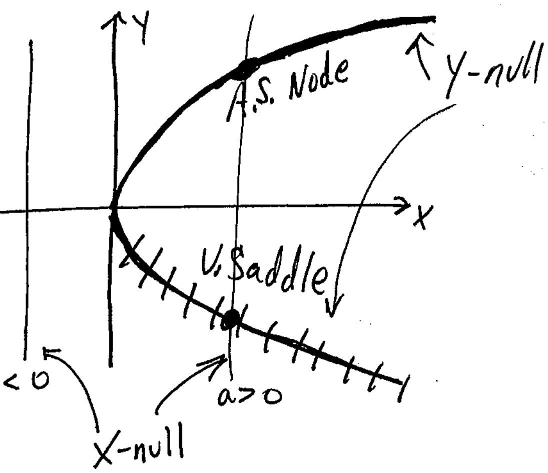

Parametric Curve: $(x(a), y(a))$
(Eq. Sols. as $a$ varies)

---

4. $\begin{cases} x' = a - x \\ y' = x - y^2 \end{cases}$ $\Longrightarrow$ **nullclines** $x$: $x = a$; $y$: $y = \pm\sqrt{x}$

Eq. Sols: $(a, \pm\sqrt{a}),\ a > 0$

Bifurcation Diagram?




---

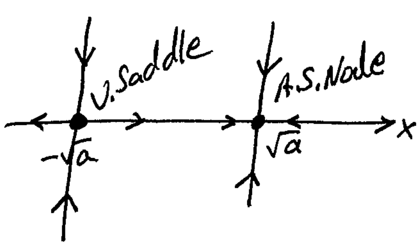
Note! For scalar equations, one eq. sol. is A.S., while the other is U.

For systems, both eq. sols. may be U.

In general, there is one saddle (U) and one Node (either A.S. or U).

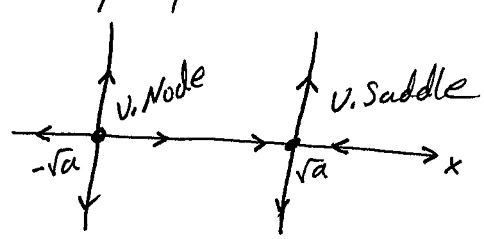
e.g. In example 2, $y$ decreases exponentially.

$$x' = a - x^2$$
$$y' = -y$$




If $y$ increases exponentially, we have...

$$x' = a - x^2$$
$$y' = y$$



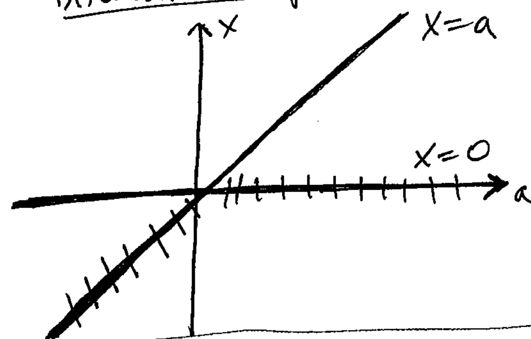
# Transcritical

**Scalar:** $x' = x(a-x)$

**Eq. Sols:** $f(x) = x(a-x) = 0$

$\boxed{x=0,\ x=a}$

Bifurcation Diagram

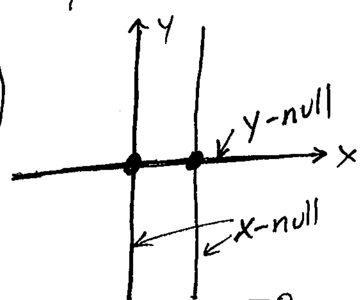



## Analogous Examples for Systems

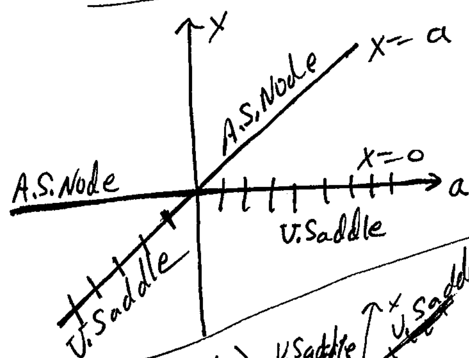
nullclines

1. $\begin{cases} x' = x(a-x) \\ y' = -y \end{cases}$ $\Rightarrow$ $x$: $x=0,\ x=a$ ; $\Rightarrow$ $y$: $y=0$

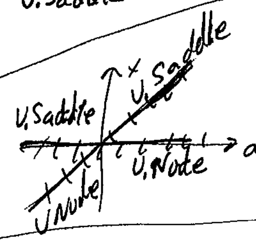
**Eq. Sols:** $(0,0)$, $(a,0)$




Bifurcation Diagram ($y_e = 0$)




**Note:** $y' = y \Rightarrow$ Both eq. sols. are U (one saddle and one node)



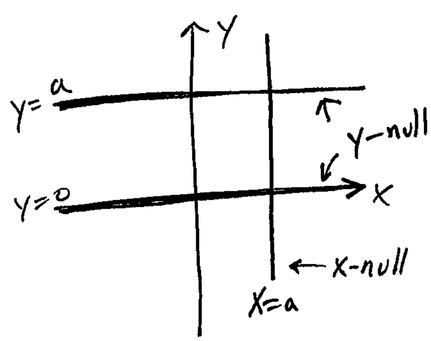

nullclines

2. $\begin{cases} x' = a-x \\ y' = y(a-y) \end{cases}$ $\Rightarrow$ $x$: $x=a$ ; $\Rightarrow$ $y$: $y=0,\ y=a$

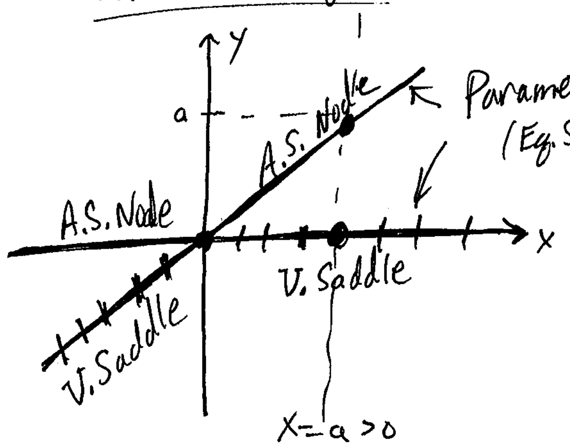
Parametric Curves: $y=x$

$y=0$

**Eq. Sols:** $(a,0)$, $(a,a)$




Bifurcation Diagram?



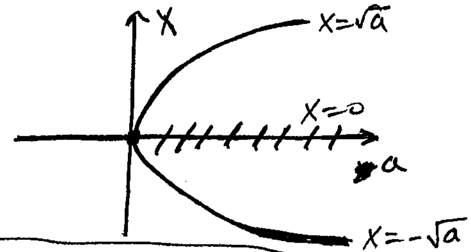

Parametric Curves $(x(a), y(a))$
(Eq. Sols. as $a$ varies)

## Supercritical Pitchfork

Scalar: $x' = x(a - x^2)$

Eq. Sols: $x = 0$, $x = \pm\sqrt{a},\ a > 0$

Bifurcation Diagram

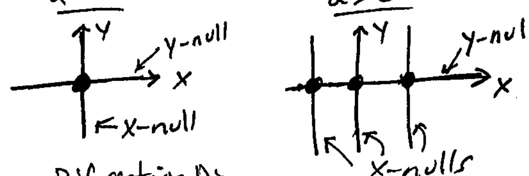



### Analogous Examples for Systems

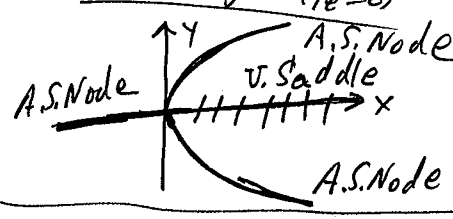
1. $x' = x(a - x^2)$, $y' = -y$

nullclines
$\Rightarrow$ $x$: $x = 0$ $x = \pm\sqrt{a}$
$\Rightarrow$ $y$: $y = 0$

Eq. Sols.: $(0,0)$, $(\pm\sqrt{a}, 0),\ a > 0$




Bifurcation Diagram ($y_e = 0$)



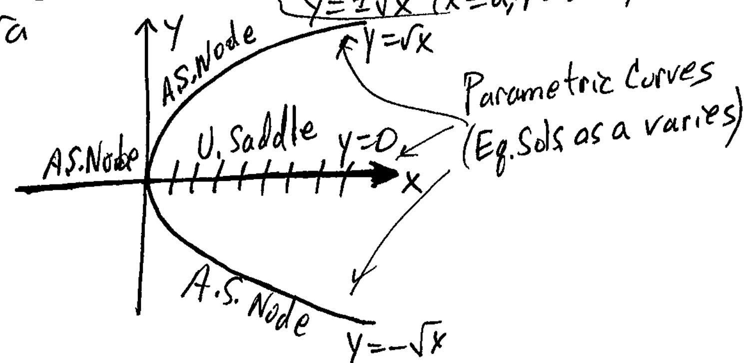

2. $x' = a - x$, $y' = y(a - y^2)$

nullclines
$\Rightarrow$ $x$: $x = a$
$\Rightarrow$ $y$: $y = 0,\ y = \pm\sqrt{a}$

Eq. Sols.: $(a, 0)$, $(a, \pm\sqrt{a}),\ a > 0$

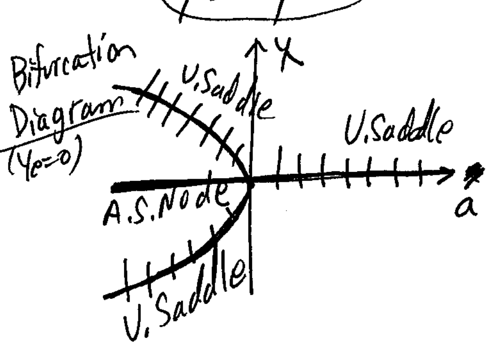
Bifurcation Diagram?

Parametric Curves: $y = 0$ $(x = a, y = 0)$
$y = \pm\sqrt{x}$ $(x = a, y = \pm\sqrt{a})$

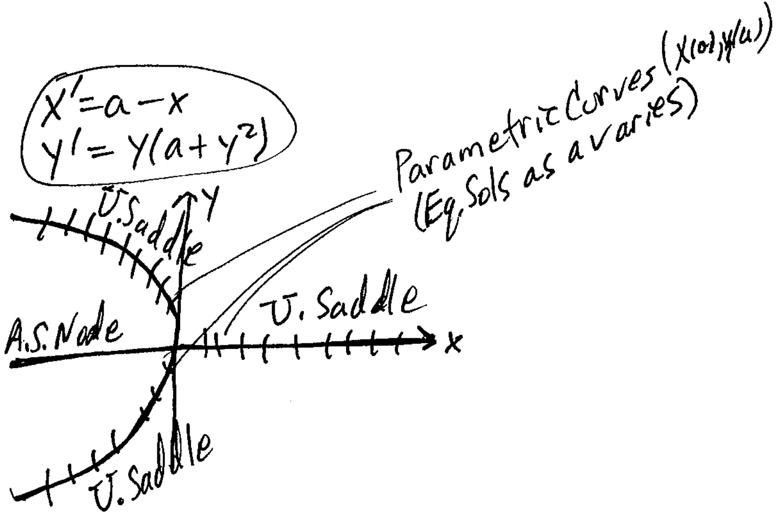



## Subcritical Pitchfork

1. $x' = x(a + x^2)$, $y' = -y$



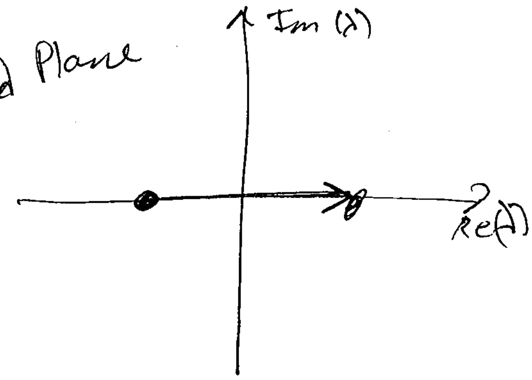

2. $x' = a - x$, $y' = y(a + y^2)$



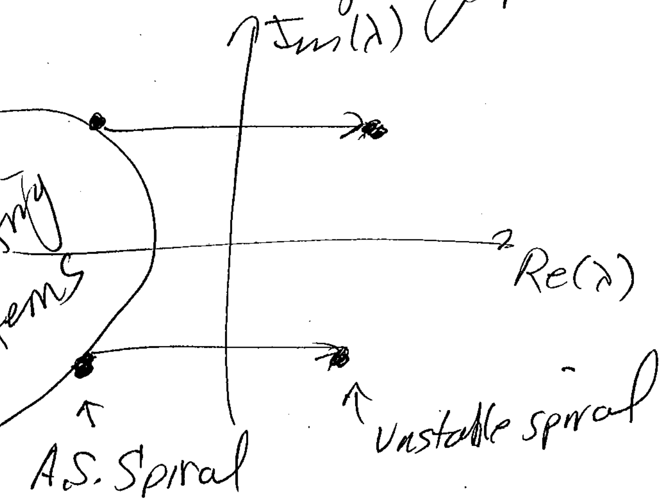
Note:

The above examples involve Node/Saddle bifurcations, which involve only real eigenvalues. An eq. sol. changes stability as an eigenvalue crosses the Imag. axis ~~with Im(λ)=0~~ along the real axis (i.e. with $Im(\lambda) = 0$)

Complex $\lambda$ Plane



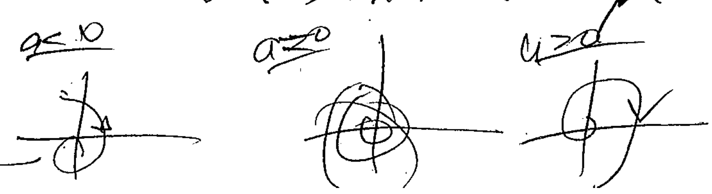

Alternatively, a bifurcation occurs as a pair of complex conjugate eigenvalues cross the Imag axis with nonzero imaginary part.




The bifurcation can be much more interesting in nonlinear systems

We have seen one such linear example $\vec{x}' = \begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix} \vec{x}$

$\lambda = a \pm i$



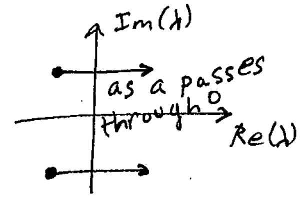
# Hopf Bifurcations

Consider $\begin{cases} x' = y - x(x^2+y^2-a) \\ y' = -x - y(x^2+y^2-a) \end{cases}$ Eq. Sol.: $\vec{X} = \vec{0}$

Linear Analysis: $\vec{f}'(\vec{x}) = \begin{pmatrix} -3x^2-y^2+a & 1-2xy \\ -1-2xy & -x^2-3y^2+a \end{pmatrix}$

$\vec{f}'(\vec{0}) = \begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix} \Rightarrow \lambda = a \pm i$

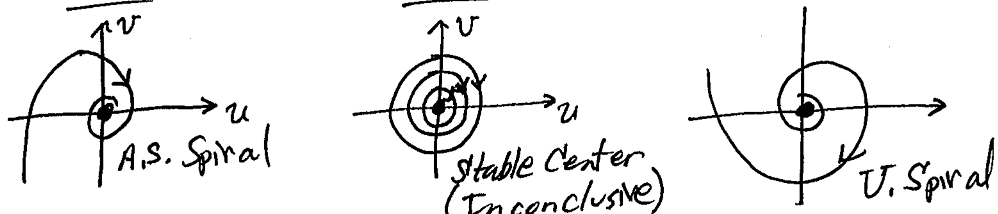
clockwise




$\vec{u}' = \begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix} \vec{u}$

## Local Behavior (of the linear system)

$a<0$ $\quad$ $a=0$ $\quad$ $a>0$




Global Behavior? Do the trajectories spiral inward ($a<0$) or outward ($a>0$) from 0 to infinity?

## Global Behavior

Convert to Polar Coordinates: $r^2 = x^2+y^2, \; r \geq 0$

$\tan\theta = \frac{y}{x}, \; -\pi < \theta \leq \pi$ (OR $0 \leq \theta < 2\pi$)

Want $r' = f(r,\theta)$
$\theta' = g(r,\theta)$

$r^2 = x^2+y^2$

$\frac{d}{dt}$: $2rr' = 2xx' + 2yy'$

$rr' = xx' + yy'$

$rr' = x[\cancel{y} - x(x^2+y^2-a)] + y[\cancel{-x} - y(x^2+y^2-a)]$

$= -x^2(x^2+y^2-a) - y^2(x^2+y^2-a)$

$rr' = -(x^2+y^2)(x^2+y^2-a)$

$\cancel{r}r' = -r^{\cancel{2}}(r^2-a)$

$\boxed{r' = r(a-r^2)}$

$\tan\theta = \frac{y}{x}$

$\frac{d}{dt}$: $\sec^2\theta\,\theta' = \frac{xy' - yx'}{x^2}$

$\frac{r^2}{\cancel{x^2}}\theta' = \frac{xy'-yx'}{\cancel{x^2}}$

$r^2\theta' = xy' - yx'$

$r^2\theta' = x[-x - \cancel{y(x^2+y^2-a)}] - y[y - \cancel{x(x^2+y^2-a)}]$

$= -x^2 - y^2 = -r^2$

$r^2\theta' = -r^2$

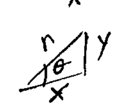
$\boxed{\theta' = -1}$ (clockwise)

$\sec^2\theta = 1 + \tan^2\theta$

$= 1 + \frac{y^2}{x^2}$

$= \frac{x^2+y^2}{x^2}$

$\sec^2\theta = r^2/x^2$



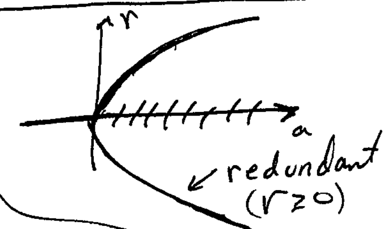
We have $\boxed{\begin{aligned} r' &= r(a-r^2) \\ \theta' &= -1 \end{aligned}}$ ← The ODEs are decoupled.

Ignoring $\theta$, $r$ undergoes a supercritical pitchfork bifurcation.

$r' = r(a-r^2)$

Eq. Sol.: $r=0$

$r = \pm\sqrt{a},\ a>0$




Eq Sol: $\boxed{r=0}$ ($\theta$ is irrelevant at the origin)

$a<0 \Rightarrow r=0$ is an A.S. Spiral point (as seen above)

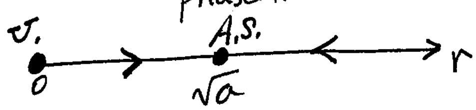
$a>0 \Rightarrow r=0$ is an U Spiral point (as seen above)

Notice that $r=\sqrt{a}$ is an "equilibrium radius". ($r=-\sqrt{a}$ is redundant)

Suppose $\begin{aligned} r(0) &= \sqrt{a} \\ \theta(0) &= 0 \end{aligned} \Rightarrow \begin{aligned} r' &= 0 \quad \leftarrow \text{radius never changes} \\ \theta' &= -1 \quad \leftarrow \theta \text{ decreases linearly} \end{aligned}$

---

Global Analysis: $r' = r(a-r^2)$

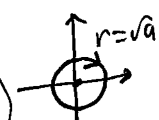



$0<r<\sqrt{a}$: $r' = \underbrace{r}_{>0}\underbrace{(a-r^2)}_{>0} > 0$

$r>\sqrt{a}$: $r' = \underbrace{r}_{>0}\underbrace{(a-r^2)}_{<0} < 0$

$\Rightarrow \boxed{r \to \sqrt{a} \text{ as } t\to\infty}$

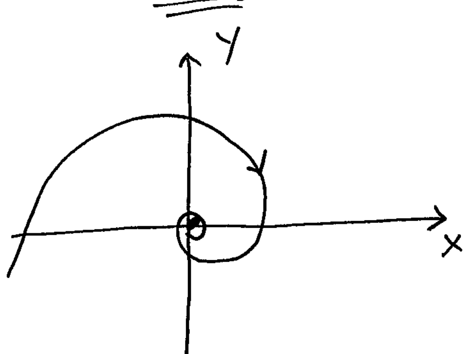
$\theta = -1$



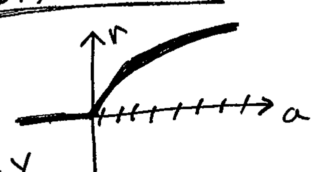

$\Rightarrow$ All nonzero Initial Conditions will yield trajectories that converge to the limit cycle $r=\sqrt{a},\ a>0$.

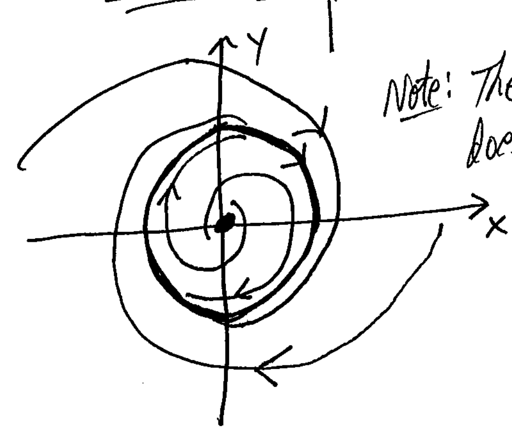
Such a Bifurcation is called a Hopf Bifurcation.

$a<0$




$a>0$

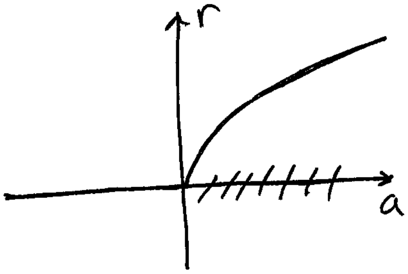


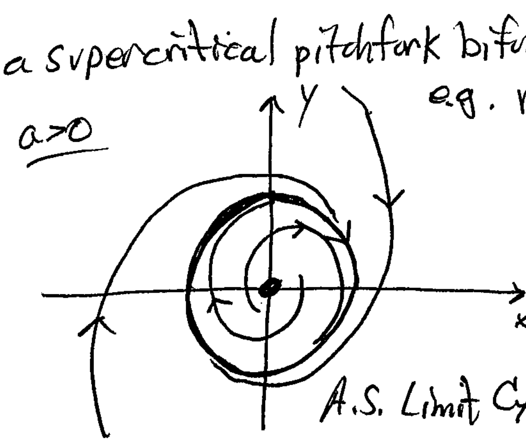

Note: The linear analysis does not reveal this behavior.

Supercritical Hopf: $r$ undergoes a supercritical pitchfork bifurcation.

e.g. $r' = r(a - r^2)$



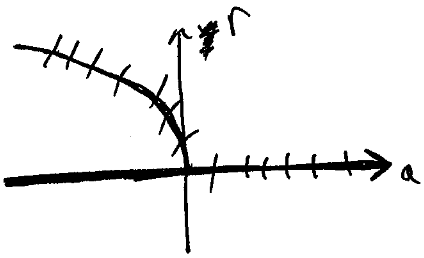

$\Rightarrow$



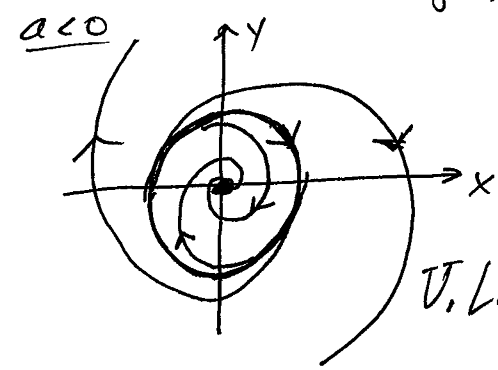

A.S. Limit Cycle

Subcritical Hopf: $r$ undergoes a subcritical pitchfork bifurcation

e.g. $r' = r(a + r^2)$




$\Rightarrow$

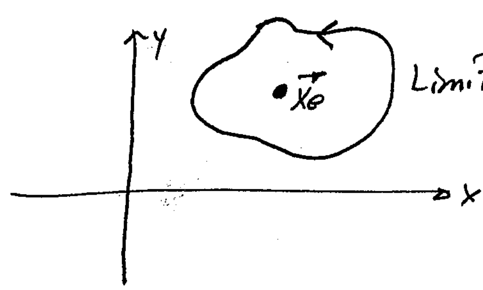



U. Limit Cycle.

---

Note: Hopf Bifurcations are typically much more difficult to analyze than seen in the above examples. The above examples are convenient since the limit cycle is a circle of radius $\sqrt{a}$, and hence, can be described using polar coordinates. The limit cycle need not be a circle, which significantly complicates the analysis.

e.g.




Limit Cycle

# Hysteresis

Consider $\boxed{\begin{aligned} r' &= r(a+r^2-r^4) \\ \theta' &= -1 \end{aligned}}$

Eq. Radii: $r(a+r^2-r^4)=0$

$(r_0, r_1, r_2)$ $\quad r_0 = 0 \quad$ $r^4 - r^2 - a = 0$

$$r^2 = \frac{1 \pm \sqrt{1+4a}}{2} \Rightarrow \boxed{r = \sqrt{\frac{1 \pm \sqrt{1+4a}}{2}}}$$

$$\boxed{r_1 = \sqrt{\frac{1+\sqrt{1+4a}}{2}}, \quad a \geq -\frac{1}{4}} \quad (1+4a \geq 0)$$

$$\boxed{r_2 = \sqrt{\frac{1-\sqrt{1+4a}}{2}}, \quad -\frac{1}{4} \leq a \leq 0}$$

$1+4a \geq 0$ and $1-\sqrt{1+4a} \geq 0$

$a \geq -\frac{1}{4}$ $\quad$ $1+4a \leq 1$

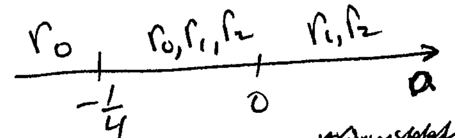
$a \leq 0$

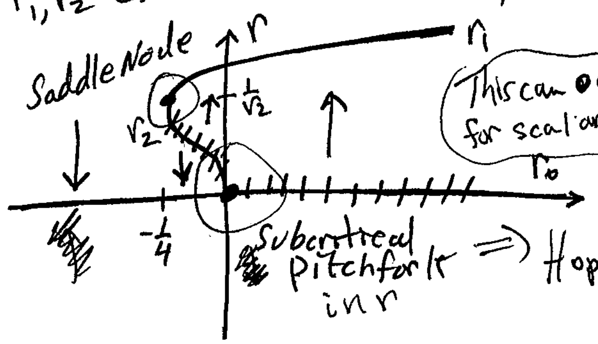
$\Rightarrow -\frac{1}{4} \leq a \leq 0$

$a < -\frac{1}{4} \Rightarrow r_0$ exists

$-\frac{1}{4} < a < 0 \Rightarrow r_0, r_1, r_2$ exist

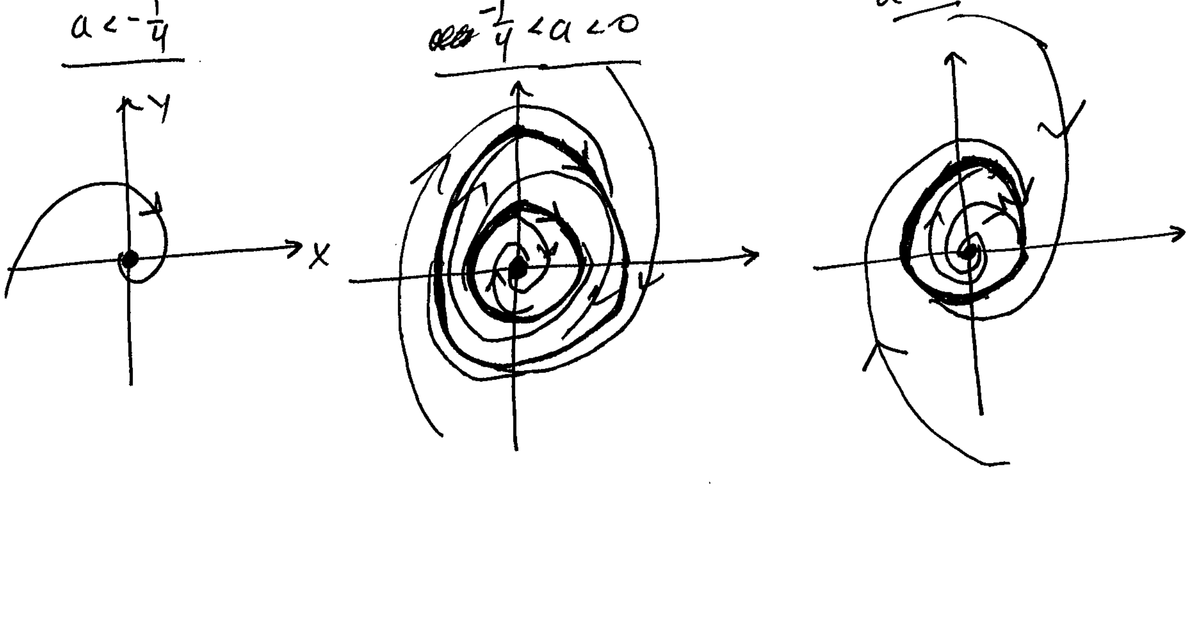
$a > 0 \Rightarrow r_1, r_2$ exist



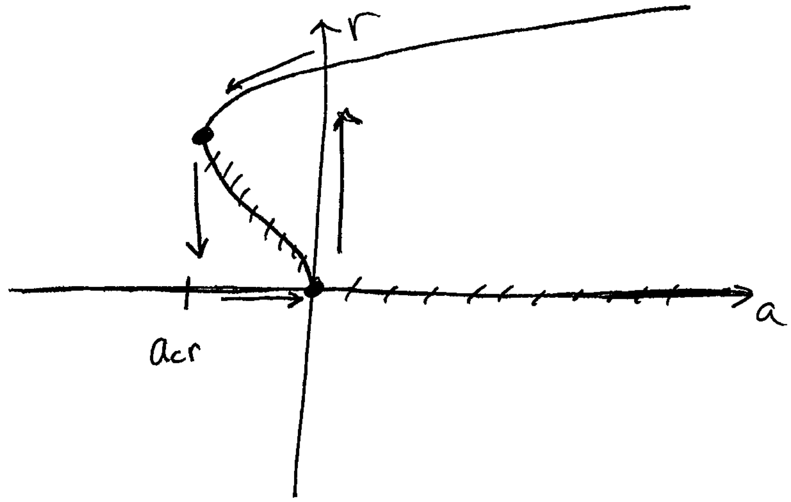




Subcritical pitchfork in r $\Rightarrow$ Hopf Bifurcation (Subcritical)

This can occur for scalar ODEs also.



Hysteresis:




Two asymptotically stable eq sols. exist for $-\frac{1}{4} < a < 0$.
The limiting behavior of a solution depends on the initial conditions,

If we start on the lower eq sol. $(r=0)$, we will stay there until $a$ is increased passed $0$, at which point, the solution jumps to the upper eq. sol.

If we then decrease $a$, the solution stays on the upper branch until $a$ is decreased passed $a_{cr}$, at which point, the solution jumps to the lower eq. sol.

~~e.g. fluttering blinds~~

# Higher-Dimensional Systems

An $n^{th}$ order linear ODE can be converted to an equivalent $n\times n$ linear system of ODEs, and vice versa.

i.e. $y^{(n)}+a_{n-1}(t)y^{(n-1)}+a_{n-2}(t)y^{(n-2)}+\dots+a_2y''+a_1y'+a_0y=f(t)$

$$\Longleftrightarrow \quad \vec{X}'=A(t)\vec{X}+\vec{f}(t)$$

Example: $x^{iv}+2tx'''-x''+3x=f(t)$

Let $x_1=x$ $\qquad \vec{X}=\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}$

$x_2=x'=x_1'$

$x_3=x''=x_2'$

$x_4=x'''=x_3'$

$$\Rightarrow \quad \vec{X}'=\begin{pmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-3&0&1&-2t\end{pmatrix}\vec{X}+\begin{pmatrix}0\\0\\0\\f(t)\end{pmatrix}$$

$x_4'=x^{iv}=-3x+x''-2tx'''+f(t)$

$x_4'=-3x_1+x_3-2tx_4+f(t)$

ODE has const. coefs. $\Longleftrightarrow$ System has const. coefs.

ODE is homog. $\Longleftrightarrow$ System is homog.

---

## Linear Homog. ~~Eqs.~~ Systems $\quad \vec{X}'=A(t)\vec{X}$ (1)

$A$ is an $n\times n$ matrix $\quad \vec{X}=\begin{pmatrix}x_1\\\vdots\\x_n\end{pmatrix}$

Gen. Sol: $\vec{X}=c_1\vec{X}_1+c_2\vec{X}_2+\dots+c_n\vec{X}_n$

where $\vec{X}_1,\dots,\vec{X}_n$ are linearly indep. sols. of (1).

Linear Indep. $\vec{X}_1,\dots,\vec{X}_n$ are linearly indep. on an interval $I$ iff

$$\det(\vec{X}_1,\dots,\vec{X}_n)=\begin{vmatrix}x_{11}&\cdots&x_{n1}\\x_{12}&&\vdots\\\vdots&&\\x_{1n}&\cdots&x_{nn}\end{vmatrix}\neq 0 \text{ for all } t \text{ in } I.$$

## Linear/Autonomous/Homog. ODEs $\vec{x}' = A\vec{x}$ A is a const. $n \times n$ matrix

$\vec{x}' = A\vec{x}$ Eq. Sol: $\vec{x}_e = \vec{0}$ $\det(A) \neq 0 \Rightarrow \vec{x}_e$ is the only eq. sol

Eigenvalue Solution Method: Eigenvalues: $\det(A - \lambda I) = 0$ ($n^{th}$ order polynomial in $\lambda$)
$\Rightarrow$ get $n$ (possibly repeated) eigenvalues.

In higher dimensions, there are 4 Cases.

1) Real Distinct
2) Real Repeated
3) Complex Conjugate Distinct
4) Complex Conjugate Repeated ($n \geq 4$)

To find the gen. sol, we need $n$ linearly indep. sols.
Each of the $n$ eigenvalues/vectors leads to one of the $n$ linearly indep. sols.

- Each real distinct eigenvalue/vector $(\lambda, \vec{v})$ yields a solution of the form
$$\vec{v}e^{\lambda t}$$

- Each distinct complex conjugate pair ($\lambda = \alpha \pm i\beta$) yields two sols. of the form
$$Re(\vec{v}e^{\lambda t})$$
$$Im(\vec{v}e^{\lambda t})$$

- Repeated (Real or Complex) eigenvalues, together with generalized eigenvectors, yield the appropriate number of linear indep. sols as well.

**Example:** $\vec{x}' = A\vec{x}$, where $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (decoupled)

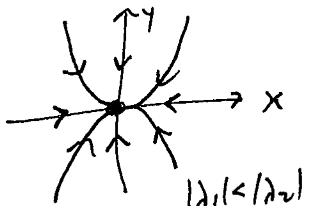
$$\Rightarrow \quad \begin{aligned} &\lambda_1 = -1, \ \vec{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow \vec{x}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}e^{-t} \\ &\lambda_2 = -2, \ \vec{v}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix} \Rightarrow \vec{x}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}e^{-2t} \\ &\lambda_3 = 1, \ \vec{v}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix} \Rightarrow \vec{x}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}e^{t} \end{aligned}$$

Eq. Sol. $\vec{x}_e = \vec{0}$

gen sol: $\boxed{\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \begin{pmatrix} c_1 e^{-t} \\ c_2 e^{-2t} \\ c_3 e^{t} \end{pmatrix}}$

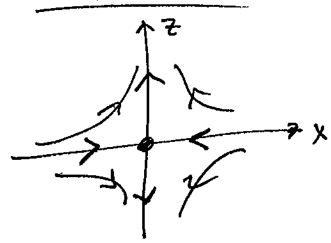
$x, y \to 0$, $z \to \pm\infty$ as $t \to \infty$

xy-plane ($c_3 = 0$)

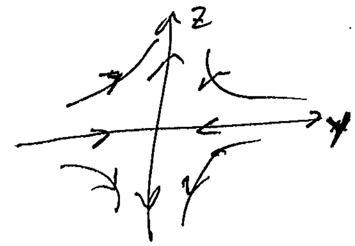


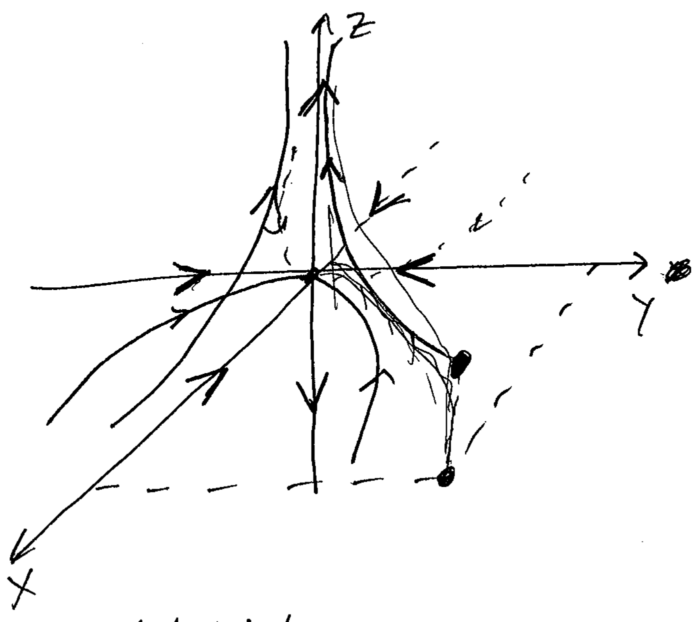

$|\lambda_1| < |\lambda_2|$

xz-plane ($c_2 = 0$)



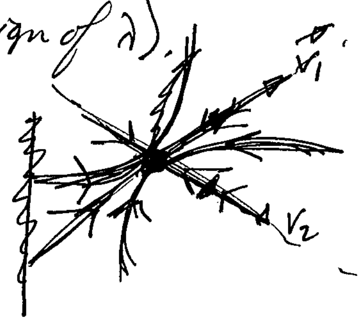

yz-plane ($c_1 = 0$)




The general real distinct eigenvalue case is essentially the same, except for the direction of the eigenvectors and the direction of the arrows (which depend on the sign of $\lambda$).

e.g. span($\vec{v}_1, \vec{v}_2$) plane



(Canonical Form)

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**Example:** $\vec{x}' = A\vec{x}$, where $A = \begin{pmatrix} -2 & 1 & 0 \\ -1 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ (x and y decouple from z)

$$\Rightarrow \left.\begin{array}{l} \lambda_1 = -2+i,\ \vec{v}_1 = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \\ \lambda_2 = -2-i,\ \vec{v}_2 = \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \end{array}\right\} \Rightarrow \begin{array}{l} \vec{x}_1 = \mathrm{Re}(\vec{v}_1 e^{\lambda_1 t}) \\ \vec{x}_2 = \mathrm{Im}(\vec{v}_1 e^{\lambda_1 t}) \end{array}$$

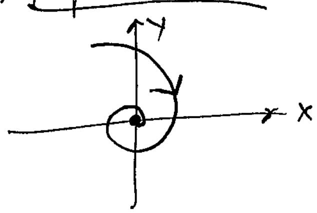
Eq. Sol: $\vec{x}_e = \vec{0}$

$$\lambda_3 = -1,\ \vec{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \vec{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t}$$

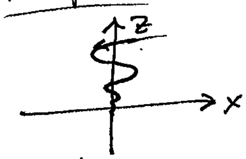
gen. sol.
$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$$
$$= e^{-2t}\left[c_1\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + c_2\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right] + c_3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-t}$$

$$\boxed{\vec{x} = \begin{pmatrix} e^{-2t}(c_1\cos t + c_2 \sin t) \\ e^{-2t}(-c_1 \sin t + c_2 \cos t) \\ c_3 e^{-t} \end{pmatrix}}$$

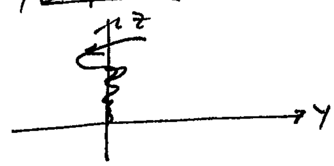
xy-plane ($c_3=0$)

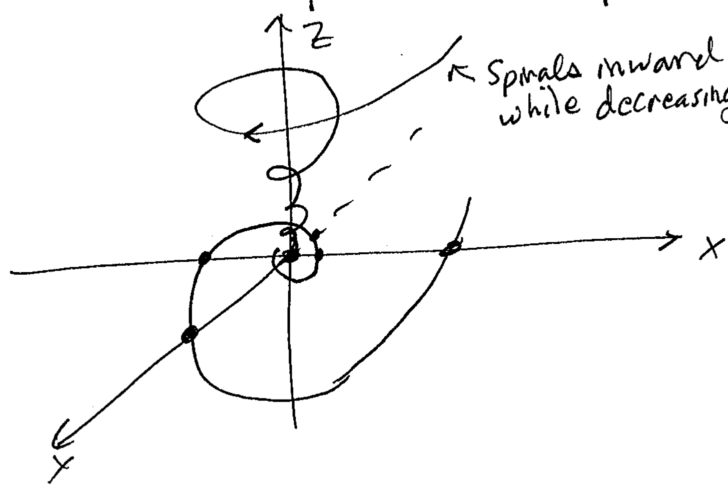



xz-plane




yz-plane




Spirals inward toward the z-axis, while decreasing to 0 in the z-direction

Show Plots

# Canonical Forms

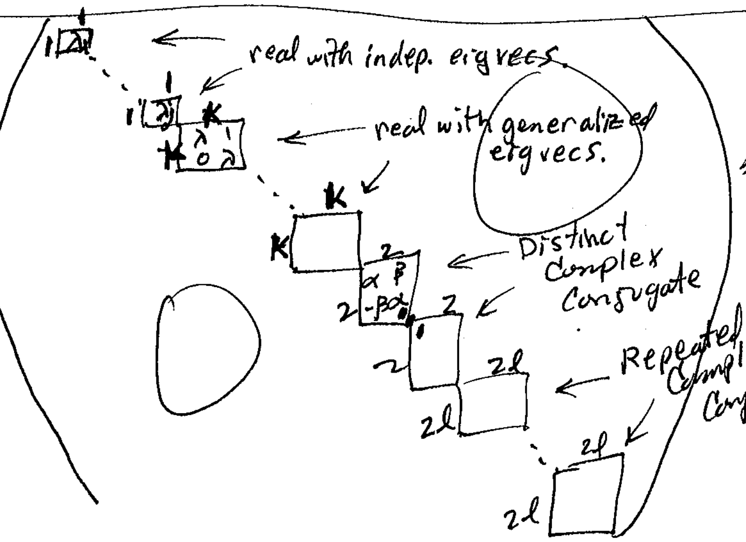
## 3x3 Systems

Real Eigs

1. 3 linearly indep. eigenvectors (real distinct or repeated eigenvalues) $\Rightarrow \begin{pmatrix} \boxed{\lambda_1} & 0 & 0 \\ 0 & \boxed{\lambda_2} & 0 \\ 0 & 0 & \boxed{\lambda_3} \end{pmatrix}$
2. 2 linearly indep. eigenvectors (at least two real repeated eigenvalues) $\Rightarrow \begin{pmatrix} \boxed{\lambda_1} & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$
3. 1 linearly indep. eigenvectors $(\lambda_1 = \lambda_2 = \lambda_3)$ $\Rightarrow \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}$
4. Complex conjugate pair of eigenvalues + one real eigenvalue $(\lambda_1$ and $\lambda_{2,3} = \alpha \pm i\beta)$ $\Rightarrow \begin{pmatrix} \boxed{\lambda_1} & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{pmatrix}$

Block Diagonal Matrices

## Higher Dimensions:




~~The computations~~

Linear Transformation: $T = (\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$

As for planar Systems:
- real with linear indepen eigvects. $\Rightarrow$ $\vec{v}_i$ = eigenvector
- repeated real without a full set of linear indep. eigvecs $\Rightarrow$ $\vec{v}_i$ = generalized eigenvectors
- complex conjugate $\Rightarrow$ $\vec{v}_i = Re(\vec{v}e^{\lambda t}), Im(\vec{v}e^{\lambda t})$

$e^{At}$ is computed as before.

# Real Repeated Eigenvalues

Double eigenvalues are treated in the same way as for planar systems

## 3x3 Systems

A 3x3 system may have a triple eigenvalue which may have either one, two, or three linearly independent eigenvectors.

If there are 3 linearly indep. eigenvectors, then
$\vec{X}_1 = \vec{V}_1 e^{\lambda t}$, $\vec{X}_2 = \vec{V}_2 e^{\lambda t}$, $\vec{X}_3 = \vec{V}_3 e^{\lambda t}$, and there is no problem.

The following examples demonstrate the technique for treating the cases of only 1 or 2 linearly indep. eigvecs.

**Example:** $\vec{X}' = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \vec{X}$ (canonical form)

Eigs: $\lambda_1 = \lambda_2 = \lambda_3 = 2$

$(A - \lambda I)\vec{V} = \vec{0}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \\ V_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} V_2 = V_3 = 0 \\ V_1 \text{ is arbitrary} \end{matrix} \Rightarrow \vec{V} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \vec{X}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^{2t}$$

Need 2 more linearly indep. solutions.

n Generalized Eigenvectors:
$$(A - \lambda I)\vec{W}_1 = \vec{V}$$
$$(A - \lambda I)\vec{W}_j = \vec{W}_{j-1} \quad j = 2, \ldots, n$$

$\Rightarrow \quad (A - \lambda I)\vec{W}_1 = \vec{V}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} W_2 = 1 \\ W_3 = 0 \\ W_1 \text{ is arbitrary} \end{matrix} \Rightarrow \vec{W}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\Rightarrow \vec{X}_2 = (\vec{V}t + \vec{W}_1)e^{\lambda t}$

$$\vec{X}_2 = \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} e^{2t}$$

$(A - \lambda I)\vec{W}_2 = \vec{W}_1$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} W_1 \\ W_2 \\ W_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} W_2 = 0 \\ W_3 = 1 \\ W_1 \text{ is arbitrary} \end{matrix} \Rightarrow \vec{W}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Then,
$$\vec{X}_3 = \Big(\underset{k=2}{\frac{t^2}{2}\vec{V}} + \underset{k=1}{t\vec{W}_1} + \underset{k=0}{\vec{W}_2}\Big) e^{\lambda t}$$

Taylor coefficients $t^k/k!$

$$\vec{X}_3 = \begin{pmatrix} t^2/2 \\ t \\ 1 \end{pmatrix} e^{2t} \Rightarrow \vec{X} = \begin{pmatrix} (C_1 + C_2 t + C_3 t^2/2)e^{2t} \\ (C_2 + C_3 t)e^{2t} \\ C_3 e^{2t} \end{pmatrix} = \begin{pmatrix} C_1 + C_2 t + C_3 \frac{t^2}{2} \\ C_2 + C_3 t \\ C_3 \end{pmatrix} e^{2t}$$

Example: $\vec{X}' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \vec{X}$ (x and z decouple)

Eigs: $\lambda_1 = \lambda_2 = \lambda_3 = 3$

$(A - \lambda I)\vec{v} = \vec{0}$

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$ $v_1, v_2$ are arbitrary, $v_3 = 0$

We can find 2 linearly indep eigvecs by choosing $v_1$ and $v_2$ appropriately

$\vec{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{v_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Two linearly indep. eigvecs $\Rightarrow$ $\vec{X_1} = \vec{v_1} e^{\lambda t}$, $\vec{X_2} = \vec{v_2} e^{\lambda t}$ Need 1 more sol.

---

Method: Guess a solution of the form

$\vec{X_3} = (\vec{v} t + \vec{w}) e^{\lambda t}$

$\Rightarrow (A - \lambda I)\vec{w} = \vec{v}$ where $\vec{v} = c_1 \vec{v_1} + c_2 \vec{v_2} = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}$

($c_1$ and $c_2$ are arbitrary at this point)

$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}$

$\Rightarrow$ $0 = c_1$

$w_3 = c_2 \Rightarrow c_2 \neq 0$

$w_1$ and $w_2$ are arbitrary

$c_1 = 0$
$w_1, w_2$ are arbitrary and $w_3 = c_2 \neq 0$ $\Rightarrow$ Pick $w_1 = w_2 = 0$, $w_3 = c_2 = 1$ $\Rightarrow \vec{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\Rightarrow \vec{X_3} = (\vec{v} t + \vec{w}) e^{\lambda t} = \left( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) e^{2t}$

$\vec{X_3} = \begin{pmatrix} 0 \\ t \\ 1 \end{pmatrix} e^{2t}$

Then $\vec{X} = c_1 \vec{X_1} + c_2 \vec{X_2} + c_3 \vec{X_3}$ ← not the same c's as above

$\boxed{\vec{X} = \begin{pmatrix} c_1 \\ c_2 + c_3 t \\ c_3 \end{pmatrix} e^{2t}}$

# Nonlinear Systems

$$\vec{x}\,' = \vec{f}(\vec{x})$$

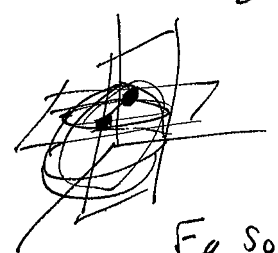
Eq. Sols: $\vec{f}(\vec{x_e}) = \vec{0}$ (n equations/n unknowns)

Direction Field: $\vec{x}\,' = \vec{f}\,'(\vec{x})$ is tangent to the trajectories and points in the direction of increasing $t$.

Nullclines: 3-D:

$$\begin{aligned} x' &= f(x,y,z) \\ y' &= g(x,y,z) \\ z' &= h(x,y,z) \end{aligned} \Rightarrow \left.\begin{aligned} f(x,y,z) &= 0 \\ g(x,y,z) &= 0 \\ h(x,y,z) &= 0 \end{aligned}\right\} \text{surfaces}$$

e.g.



2-D linear ⇒ lines (intersect at the origin)

3-D linear ⇒ planes

Eq Sols occur at the points of intersection of the 3 surfaces

---

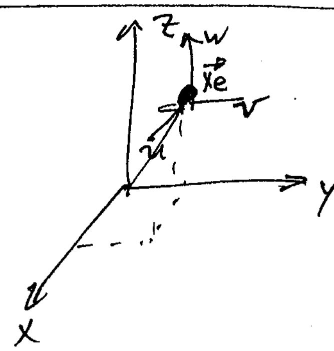
## Linearization

$$\vec{u} = \vec{x} - \vec{x_e}$$

$$\vec{u}\,' = \vec{f}\,'(x_e)\,\vec{u}$$

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \vec{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\vec{f} = \begin{pmatrix} f \\ g \\ h \end{pmatrix} \quad \vec{f}\,'(\vec{x}) = \begin{pmatrix} f_x & f_y & f_z \\ g_x & g_y & g_z \\ h_x & h_y & h_z \end{pmatrix}$$

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## Linear Stability Analysis

Let $\lambda_1, \ldots, \lambda_n$ be the $n$ eigenvalues of $\vec{f}\,'(\vec{x_e})$.

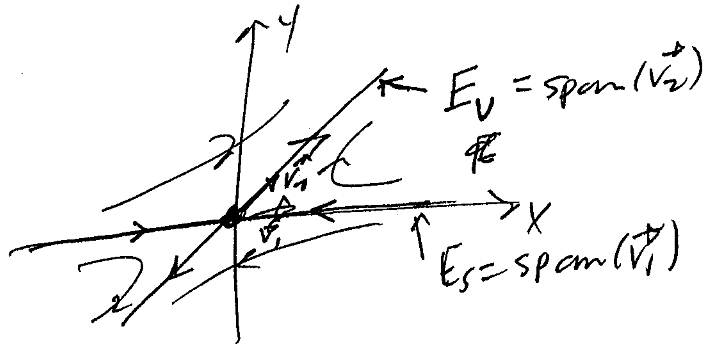
$\lambda_i < 0$ for each $i$ ⇒ $\vec{x_e}$ is A.S.

$\lambda_i > 0$ for some $i$ ⇒ $\vec{x_e}$ is U

$\lambda_i \leq 0$ for each $i$ and $\lambda_j = 0$ for some $j$ ⇒ Inconclusive

# Manifolds (E)

e.g.




e.g.