1016-808: Partial Differential Equations

$$\vec{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} \qquad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Jacobian Matrix

Consider a vector function (field) $\vec{f}: \mathbb{R}^n \to \mathbb{R}^n$. ($\vec{f}$ is a vector field)

$$\Rightarrow \vec{f}(\vec{x}) = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{pmatrix} = \vec{f}(x_1, \dots, x_n) = \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{pmatrix}$$

The Jacobian Matrix of $\vec{f}(\vec{x})$ is defined to be

$$\underbrace{J = \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} = \frac{\partial \vec{f}}{\partial \vec{x}} = \vec{f}'(\vec{x})}_{\text{notations}} \overset{\text{def.}}{=} \begin{pmatrix} \partial f_1/\partial x_1 & \cdots & \partial f_1/\partial x_n \\ \vdots & & \vdots \\ \partial f_n/\partial x_1 & & \partial f_n/\partial x_n \end{pmatrix}$$

$$\Rightarrow J_{ij} = \frac{\partial f_i}{\partial x_j}$$

"Chain Rule" for Jacobian Matrices

Consider the composite vector function $\vec{F}(\vec{t}) = \vec{f}(\vec{x}(\vec{t}))$.
$\vec{F}: \mathbb{R}^n \to \mathbb{R}^n$

k^{th} row $\Rightarrow F_k(t_1, \dots, t_n) = f_k(x_1(t_1, \dots, t_n), \dots, x_n(t_1, \dots, t_n))$ for each $k = 1, \dots, n$

Then,

$$\frac{\partial(F_1, \dots, F_n)}{\partial(t_1, \dots, t_n)} = \frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} \cdot \frac{\partial(x_1, \dots, x_n)}{\partial(t_1, \dots, t_n)}$$

OR $$\frac{\partial \vec{F}}{\partial \vec{t}} = \frac{\partial \vec{f}}{\partial \vec{x}} \cdot \frac{\partial \vec{x}}{\partial t}$$

OR $$\vec{F}'(\vec{t}) = f'(\vec{x}) \cdot \vec{x}'(\vec{t})$$

Chain Rule: Composite vector-valued functions of a vector

Let $\vec{f}(\vec{x}) = \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_k(\vec{x}) \end{pmatrix}$, $\vec{x}(\vec{t}) = \begin{pmatrix} x_1(\vec{t}) \\ \vdots \\ x_n(\vec{t}) \end{pmatrix}$, $\vec{t} = \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix}$, and

$$\boxed{\vec{F}(\vec{t}) = \vec{f}(\vec{x}(\vec{t}))}. \qquad \begin{array}{l} \vec{F}(\vec{t}) : \mathbb{R}^m \to \mathbb{R}^k \\ \vec{f}(\vec{x}(\vec{t})) : \mathbb{R}^m \to \mathbb{R}^n \to \mathbb{R}^k \end{array}$$

$$\Rightarrow \begin{pmatrix} F_1(t_1,\dots,t_m) \\ \vdots \\ F_k(t_1,\dots,t_m) \end{pmatrix} = \begin{pmatrix} f_1(x_1(t_1,\dots,t_m), \dots, x_n(t_1,\dots,t_m)) \\ \vdots \\ f_k(x_1(t_1,\dots,t_m),\dots,x_n(t_1,\dots,t_m)) \end{pmatrix}$$

$$\Rightarrow F_i(t_1,\dots,t_m) = f_i(x_1(t_1,\dots,t_m),\dots,x_n(t_1,\dots,t_m)) \text{ for each } i=1,\dots,k$$

Then for each $i=1,\dots,k$ and $j=1,\dots,m$,

$$\boxed{\frac{\partial F_i}{\partial t_j} = \sum_{l=1}^{n} \frac{\partial f_i}{\partial x_l}\cdot\frac{\partial x_l}{\partial t_j} = \frac{\partial f_i}{\partial x_1}\cdot\frac{\partial x_1}{\partial t_j} + \dots + \frac{\partial f_i}{\partial x_n}\cdot\frac{\partial x_n}{\partial t_j} = \nabla f_i \bullet \frac{\partial \vec{x}}{\partial t_j}} \sim \frac{\partial f_i}{\partial \vec{x}} \bullet \frac{\partial \vec{x}}{\partial t_j}$$

$$\nabla f_i = \langle \partial f_i/\partial x_1, \dots, \partial f_i/\partial x_n \rangle$$

Example: $\begin{pmatrix} F_1(s,t) \\ F_2(s,t) \end{pmatrix} = \begin{pmatrix} xyz^2 \\ 3x^2+2xy-5z \end{pmatrix}$ where $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} s-t^2 \\ st+2 \\ s^2t \end{pmatrix}$. $\frac{\partial F_2}{\partial s} = ?$

$$\frac{\partial F_2}{\partial s} = \frac{\partial F_2}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial F_2}{\partial y}\cdot\frac{\partial y}{\partial s} + \frac{\partial F_2}{\partial z}\cdot\frac{\partial z}{\partial s}$$

$m=2$: $t_1 = s$, $t_2 = t$ $\quad$ $n=3$: $x_1 = x$, $x_2 = y$, $x_3 = z$ $\quad$ $k=2$: $f_1(x,y,z) = xyz^2$, $f_2(x,y,z) = 3x^2+2xy-5z$

$$= (6x+2y)\cdot(1) + (2x)(t) + (-5)(2st)$$

$$= 2x(3+t) + 2y - 10st = 2(s-t^2)(3+t) + 2(st+2) - 10st$$

$$= 2(3s-3t^2+st-t^3+st+2-5st) = 2(3s-3t^2-t^3-3st+2)$$

$$\boxed{\frac{\partial F_2}{\partial s} = 2(3s-3t^2-t^3-3st+2)}$$

Implicit Function Theorem

Vector Form

Consider a system of n equations with $m+n$ variables,

$$F_K(x_1,\dots,x_m,y_1,\dots,y_n)=0 \quad \text{for } K=1,\dots,n,$$

$$\vec{F}(\vec{x},\vec{y})=\vec{0}$$

$$\underset{n\times 1}{\vec{F}}\;\underset{m\times 1}{\vec{x}}\;\underset{n\times 1}{\vec{y}}\;\underset{n\times 1}{\vec{0}}$$

where each F_K is continuously differentiable with respect to each variable. Suppose the system is satisfied at a point $P_0:(x_1^0,\dots,x_m^0,y_1^0,\dots,y_n^0)$

$$P_0:(\vec{x}_0,\vec{y}_0)$$

with

$$\left|\frac{\partial(F_1,\dots,F_n)}{\partial(y_1,\dots,y_n)}\right|_{P_0}\neq 0.$$

← determinant of the Jacobian evaluated at P_0.

$$\frac{\partial\vec{F}}{\partial\vec{y}}(\vec{x}_0,\vec{y}_0)\neq 0$$

Then, the y_k's can be expressed as functions of the x_j's in some neighborhood N of P_0.

$$\Rightarrow \quad y_K=\phi_K(x_1,\dots,x_m) \text{ in } N, \quad K=1,\dots,n$$

$$\vec{y}=\vec{\phi}(\vec{x}) \text{ in } N$$

Furthermore, each ϕ_K is continuously differentiable with respect to each x_j at all points in N

Example:
$$F_1(u,v,x,y) = y - u = 0$$
$$F_2(u,v,x,y) = v - y - x^2 = 0 \quad (1)$$

Where can x and y be expressed as functions of u and v?

Let $P_0 : (u_0, v_0, x_0, y_0)$ be a point at which (1) is satisfied.

$$\Rightarrow \; y_0 - u_0 = 0 \;\Rightarrow\; y_0 = u_0 \qquad\qquad u_0 = v_0 - x_0^2$$
$$v_0 - y_0 - x_0^2 = 0 \;\Rightarrow\; y_0 = v_0 - x_0^2 \;\Rightarrow\; v_0 = u_0 + x_0^2$$
$$x_0^2 \geq 0 \;\Rightarrow\; v_0 \geq u_0$$

Implicit Function Theorem:
$$\left|\frac{\partial(F_1,F_2)}{\partial(x,y)}\right|_{P_0} = \begin{vmatrix} F_{1x} & F_{1y} \\ F_{2x} & F_{2y} \end{vmatrix}_{P_0} = \begin{vmatrix} 0 & 1 \\ -2x & -1 \end{vmatrix}_{P_0} = 2x_0 \quad \begin{matrix} = 0 \text{ for } x_0 = 0 \\ \neq 0 \text{ otherwise} \end{matrix}$$

$\Rightarrow$ There exists continuously differentiable functions

$$\begin{matrix} x = \phi_1(u,v) \\ y = \phi_2(u,v) \end{matrix} \quad \text{in some neighborhood of } P_0 \text{ provided } x_0 \neq 0.$$

Solve for x and y:
$$y - u = 0 \;\Rightarrow\; y = u$$
$$v - y - x^2 = 0 \;\Rightarrow\; x^2 = v - y = v - u$$
$$x = \pm\sqrt{v - u}$$

$$\Rightarrow \; y = \phi_1(u,v) = u$$

$x \geq 0$: $\quad x = \phi_2^+(u,v) = +\sqrt{v-u}$ $\qquad$ for $v \geq u$

$x < 0$: $\quad x = \phi_2^-(u,v) = -\sqrt{v-u}$

Therefore, x and y can be expressed as continuously differentiable functions of u and v in any region in which $x \neq 0$.

There's no neighborhood about $x = 0$ in which a single function $x = \phi_2(u,v)$ exists.

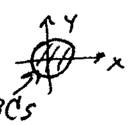
Notation: $x = \phi(u,v)$

dependent variable (x) — function (ϕ) — independent variables (u,v)

It is often convenient to assign the same label to both the dependent variable and the function. $\Rightarrow$ $x = x(u,v)$.
Whether x denotes the dependent variable or the function is usually clear from the context. If not, the function is written as $x(u,v)$.

Invertible Coordinate Transformations

Many problems can be simplified by introducing a new coordinate system.

e.g. $u_{xx} + u_{yy} = 0$ on a unit disk 

$\Rightarrow$ Convert to polar and use separation of variables.

It is often desirable that the coordinate transformation be invertible (i.e. one-to-one) so that we may convert back to the original coordinate system

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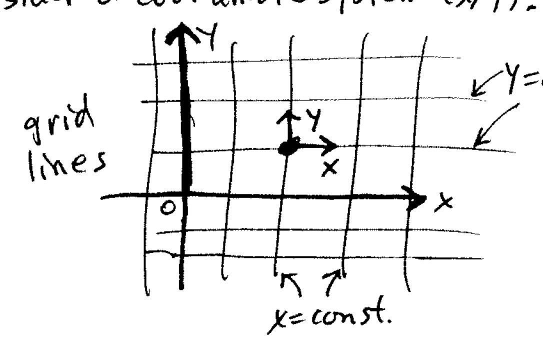
Consider a coordinate system (x,y).



((x,y) need not be a rectangular coordinate system)

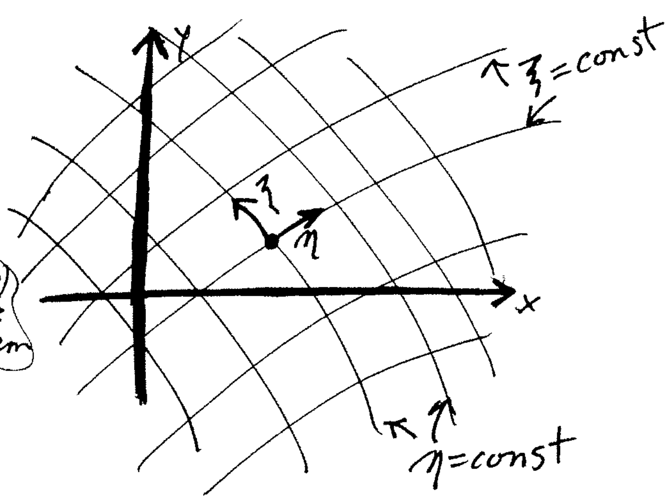
Make a transformation of coordinates.

Given: $\xi = \xi(x,y)$
$\eta = \eta(x,y)$

The grid lines are the level curves of the (ξ,η) coordinate system

$\xi(x,y) = C_1$

$\eta(x,y) = C_2$



Single variable case

If $y = f(x)$ is one-to-one, then $x = g(y)$ ← inverse function

OR $\xi = \xi(x)$, $x = x(\xi)$

If $\xi = \xi(x,y)$, $\eta = \eta(x,y)$ is one-to-one, then

$x = x(\xi,\eta)$

$y = y(\xi,\eta)$

inverse transformation

The transformation is <u>invertible</u> on a region R if x and y can be expressed as functions of ξ and η in R. $\Rightarrow$ $\begin{array}{l} x = x(\xi,\eta) \\ y = y(\xi,\eta) \end{array}$.

The Implicit Function theorem reveals where this is possible.

Given $\begin{array}{l} \xi = \xi(x,y) \\ \eta = \eta(x,y) \end{array}$, let $\begin{array}{l} F_1(\xi,\eta,x,y) = \xi(x,y) - \xi = 0 \\ F_2(\xi,\eta,x,y) = \eta(x,y) - \eta = 0 \end{array}$.

Then,

$$\left|\frac{\partial(F_1,F_2)}{\partial(x,y)}\right| = \begin{vmatrix} F_{1x} & F_{1y} \\ F_{2x} & F_{2y} \end{vmatrix} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix}$$

$\Rightarrow$ the transformation is invertible on any region R in which $\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} \neq 0$.

<u>Brief Summary:</u> Given $\begin{array}{l} \xi = \xi(x,y) \\ \eta = \eta(x,y) \end{array}$. (coordinate transformation).

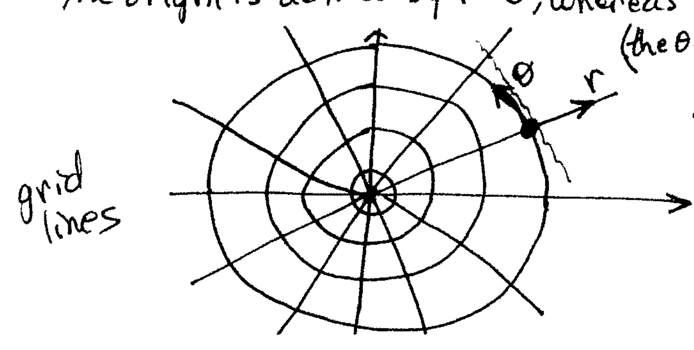
We can write $\begin{array}{l} x = x(\xi,\eta) \\ y = y(\xi,\eta) \end{array}$ wherever $\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x\eta_y - \xi_y\eta_x \neq 0$.
(i.e. we can invert the functions)

<u>Example:</u> Start with polar coordinates and define $\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}$.
Where is the coordinate transformation invertible?
i.e. Given x and y, where can we uniquely determine r and θ?

$$\begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \quad (=0 \text{ only at the origin})$$

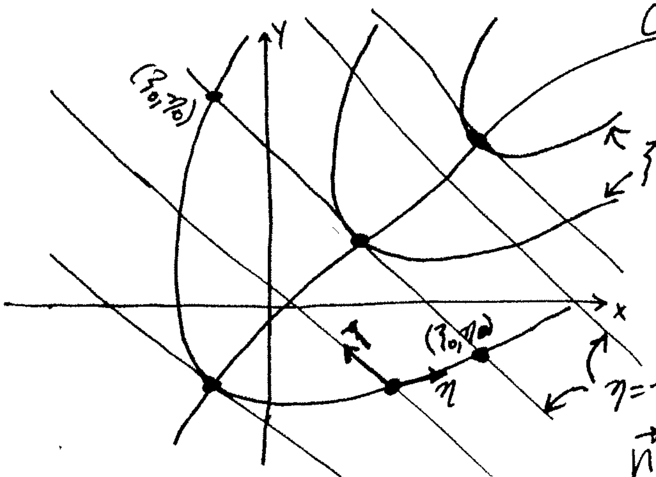
$\Rightarrow$ the transformation is invertible everywhere $\begin{pmatrix} r = \sqrt{x^2+y^2} \\ \theta = \tan^{-1}\frac{y}{x} \end{pmatrix}$, except at the origin $(r=0)$, at which point $\theta = \tan^{-1}\frac{0}{0}$ $\leftarrow$ indeterminate form

The origin is defined by $r=0$, whereas θ is not defined at the origin.
(the θ grid lines intersect at the origin)



At the origin, r increases in all directions

Example: Given cartesian coordinates (x,y), let $\begin{array}{l}\xi=\xi(x,y)\\ \eta=\eta(x,y)\end{array}$ and suppose that ξ and η have the level curves shown in the figure.



Along the curve C, the ξ-level curves and the η-level curves are tangent to each other

$\xi=\xi(x,y)=\text{const}$

$\vec{n}=\nabla\xi=\langle \xi_x,\xi_y\rangle$

$\Rightarrow \vec{t}=\langle -\xi_y,\xi_x\rangle$

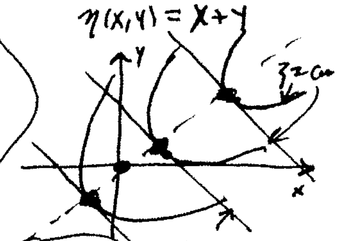
$\eta=\eta(x,y)=\text{const}.$

$\vec{n}=\nabla\eta=\langle \eta_x,\eta_y\rangle$

$\Rightarrow \vec{t}=\langle -\eta_y,\eta_x\rangle$

e.g. $\xi(x,y)=(x+y)-(x-y)^2$

$\eta(x,y)=x+y$



$C: y=x$ $\quad$ $\eta=x+y=c$

$y=c-x$

$\begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix}=-4(x-y)$

The normal vectors, $\nabla\xi$ and $\nabla\eta$, are parallel at points where $\nabla\xi\times\nabla\eta=\vec{0}$.

$$\nabla\xi\times\nabla\eta=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \xi_x & \xi_y & 0\\ \eta_x & \eta_y & 0\end{vmatrix}=\vec{k}\underbrace{\begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix}}_{\text{Implicit Function Theorem}}=\vec{0}\ \text{when}\ \nabla\xi\parallel\nabla\eta.$$

$$\Rightarrow \begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix}=0$$

Therefore, the coordinate transformation is not invertible on regions which contain points at which the normal vectors are parallel.

$\Rightarrow$ the tangent vectors are parallel $\Rightarrow$ the level curves are tangent

$\Rightarrow$ The coordinate transformation is not invertible on regions which contain points at which the ξ-level curve and the η-level curves are tangent to each other. (along curve C in the above figure)

ξ,η fixed
vary elsewhere

At such points, ξ and η are not independent variables since we cannot vary one while holding the other fixed. Both ξ and η don't change in the tangent direction, and both vary in all other directions.

The transformation is invertible on any region which lies entirely on one side of the curve C.

Furthermore, notice that for each point (ξ_0,η_0) on one side of C, there is another point with these same coordinates on the other side of C. $\Rightarrow$ The transformation is not one-to-one, unless we limit its domain to one side of C or the other.

i.e. the ξ-level curves and the η-level curves intersect at two points.

Linear Equations

An equation is <u>linear</u> in the quantities $A_1, \dots, A_n$ if it has the form $c_1 A_1 + \cdots + c_n A_n = c_0$, where the coefficients c_i's are independent of the A_j's.

e.g. line: $ax + by = c$ is linear in x and y

1st order ODE: $y' + p(x)y = g(x)$ is linear in y and y'

A system of equations is said to be linear in $A_1, \dots, A_n$ if each equation has the above form.

e.g. $\begin{matrix} ax + by = c \\ dx + ey = f \end{matrix}$ is linear in x and y

$\begin{matrix} u_x + x v_y = u \\ v_x + y^2 u_y = v \end{matrix}$ is linear in u, u_x, u_y, v, v_x, v_y

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A differential equation is said to be <u>linear</u> if the equation is linear in the dependent variable and its derivatives.

e.g. Linear: $x^2 u'' + e^x u' + 2u = \frac{1}{x}$

$$u_t = \kappa u_{xx} + \lambda u_{xt} - xt u_x + 3\alpha u - 5t^3$$

$$\begin{cases} u_x + x v_y = u \\ v_x + y^2 u_y = v \end{cases}$$

Nonlinear: $u_t + \underline{u u_x} = x + t$

$$u_t = \kappa u_{xx} + \underline{\underline{u^2}}$$

$$u' = \underline{\underline{\sqrt{u}}} + \underline{\underline{e^u}}$$

I. First Order PDEs (with two independent variables)

First Order $\Rightarrow$ The highest-ordered derivatives appearing in the PDE are first order partial derivatives (e.g. $u_t - x^2 u_x = tu$)

We'll mainly focus on PDEs involving two independent variables (e.g. x and y or x and t).
Much of the theory and many of the techniques can be generalized to more independent variables.

Scalar First Order PDEs $\Rightarrow$ One equation with one unknown dependent variable

General Form : $F(x,t,u,u_t,u_x) = 0 \qquad u(x,t) = ?$

Classifications of special cases

nonlinear

Quasi-linear: $\boxed{a(x,t,u)\,u_t + b(x,t,u)\,u_x = c(x,t,u)}$

The PDE is linear in the first order partial derivatives, u_t and u_x, but not necessarily in u.

Semi-linear (or Almost-linear): $\boxed{a(x,t)\,u_t + b(x,t)\,u_x = \underbrace{c(x,t,u)}_{\text{nonlinear (in general)}}}$

The PDE is linear in u_t and u_x, with coefficients that are independent of u, but the PDE is not necessarily linear in u.

Linear: $\boxed{a(x,t)\,u_t + b(x,t)\,u_x = c(x,t)u + d(x,t)}$

Note: Linear $\Rightarrow$ Semilinear $\Rightarrow$ Quasilinear

Systems of First Order PDEs (with 2 independent variables)

n equations with n unknown dependent variables $u_1, \dots, u_n$

General Form: $F_i(x, t, u_1, \dots, u_n, u_{1t}, \dots, u_{nt}, u_{1x}, \dots, u_{nx}) = 0$ for $i = 1, \dots, n$

Vector Form: $\vec{F}(x, t, \vec{u}, \vec{u}_t, \vec{u}_x) = \vec{0}$

Consider the special case in which

$$\boxed{F_i = \sum_{j=1}^{n} a_{ij} \frac{\partial u_j}{\partial t} + \sum_{j=1}^{n} b_{ij} \frac{\partial u_j}{\partial x} = c_i \quad \text{for } i = 1, \dots, n} \quad (S)$$

$$\Rightarrow F_i = (a_{i1} u_{1t} + \dots + a_{in} u_{nt}) + (b_{i1} u_{1x} + \dots + b_{in} u_{nx}) = c_i$$

Quasilinear:
$$a_{ij} = a_{ij}(x, t, \vec{u})$$
$$b_{ij} = b_{ij}(x, t, \vec{u})$$
$$c_i = c_i(x, t, \vec{u})$$

Semilinear:
$$a_{ij} = a_{ij}(x, t)$$
$$b_{ij} = b_{ij}(x, t)$$
$$c_i = c_i(x, t, \vec{u})$$

Linear:
$$a_{ij} = a_{ij}(x, t)$$
$$b_{ij} = a_{ij}(x, t)$$
$$c_i = \vec{d}_i(x, t) \cdot \vec{u} + e_i(x, t)$$
$$= d_{i1} u_1 + \dots + d_{in} u_n + e_i$$

Matrix Notation: Let $A = (a_{ij})$ and $B = (b_{ij})$ and $\vec{c} = (c_i)$

$$(S) \Rightarrow \boxed{A \vec{u}_t + B \vec{u}_x = \vec{c}}$$

Quasilinear:
$$A = A(x, t, \vec{u})$$
$$B = B(x, t, \vec{u})$$
$$\vec{c} = \vec{c}(x, t, \vec{u})$$

Semilinear:
$$A = A(x, t)$$
$$B = B(x, t)$$
$$\vec{c} = \vec{c}(x, t, \vec{u})$$

Linear:
$$A = A(t, x)$$
$$B = B(t, x)$$
$$\vec{c} = D(x, t) \cdot \vec{u} + \vec{e}$$

Classifications of First Order PDEs (with 2 independent variables)

Consider the nxn system $\boxed{A\vec{u}_t + B\vec{u}_x = \vec{c}}$ (quasilinear in general)

We may be able to classify the system as either Elliptic, Hyperbolic, or Parabolic, if either A or B is nonsingular ($\det \neq 0$), or both.

Eigenvalue Problem:

	eigenvalues		eigenvectors	
	$\boxed{\det(B-\lambda A)=0}$	$\longleftrightarrow$	$\boxed{(B-\lambda A)\vec{v}=\vec{0}}$	if A is nonsingular
OR	$\boxed{\det(A-\lambda B)=0}$	$\longleftrightarrow$	$\boxed{(A-\lambda B)\vec{v}=\vec{0}}$	if B is nonsingular

nonsingular

Notes:
1) Either problem may be used if both A and B are nonsingular. The two problems yield different eigenvalues, but they lead to the same conclusions.
2) If A is [redacted] singular in the first eigenvalue problem (or B in the second), the coefficient of λ^n is $\det A = 0$, and therefore, there will not be a full set of n eigenvalues.
3) It is recommended to use the first eigenvalue problem when possible (i.e. when A is nonsingular). If t represents time, A is often the identity matrix, in which case the first eigenvalue problem reduces to the usual eigenvalue problem for the matrix B. In this case, the first eigenvalue problem will involve less algebra than the second.

Classifications

1) Elliptic: No Real Eigenvalues (n must be even since complex eigenvalues come in conjugate pairs)
2) Hyperbolic: n Real Eigenvalues with a full set of n linearly independent eigenvectors.
   a) n Real Distinct Eigenvalues $\Longrightarrow$ n linearly independent eigenvectors
   b) n Real Eigenvalues with Repeats, but still yielding n linearly independent eigenvectors
3) Parabolic: n Real Eigenvalues that do not yield a full set of n linearly independent eigenvectors
   This case occurs only when there is at least one repeated eigenvalue.

Notes:
1) Since A and B depend on x and t (and u), the classification of a PDE be different in various regions of the xt-plane.
2) No such classifications exist for systems that have a mix of real and complex eigenvalues. Most physical problems have either all real or all complex eigenvalues.
3) Scalar Case: $a(x,t,u)\,u_t + b(x,t,u)\,u_x = c(x,t,u)$

$$\lambda = \frac{b(x,t,u)}{a(x,t,u)} \text{ if } a(x,t,u)\neq 0 \quad \underline{OR} \quad \lambda = \frac{a(x,t,u)}{b(x,t,u)} \text{ if } b(x,t,u)\neq 0$$

There is one real eigenvalue $\Rightarrow$ Hyperbolic
(if $a(x,t,u)\neq 0$ or $b(x,t,u)\neq 0$)

Examples: Classify the following systems of PDEs

1) Cauchy - Riemann Equations: $u_x = v_y$, $v_x = -u_y$ (linear)

Observe that u and v satisfy the 2-D Laplace Equation.

$$u_{xx} = (u_x)_x = (v_y)_x = (v_x)_y = (-u_y)_y = -u_{yy}$$

$\Rightarrow$ $u_{xx} + u_{yy} = 0$ (2-D Laplace Equation)

Similarly, $v_{xx} + v_{yy} = 0$ $\Rightarrow$ Expect the system to be *Elliptic*.

$A\vec{u}_x + B\vec{u}_y = \vec{c}$, where $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$

$$\begin{matrix} u_x - v_y = 0 \\ v_x + u_y = 0 \end{matrix} \Rightarrow \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{A = I} \begin{pmatrix} u \\ v \end{pmatrix}_x + \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{B} \begin{pmatrix} u \\ v \end{pmatrix}_y = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\vec{c}}$$

Eigenvalue problem: $\det(B - \lambda A) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1 = 0$

$\lambda = \pm i \Rightarrow$ $\boxed{\text{Elliptic}}$

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2) $u_t = v_x$, $\chi u_x = v$ (linear)

Observe that u satisfies the 1-D Heat Equation

$$u_t = v_x = (\chi u_x)_x = \chi u_{xx}$$

$u_t = \chi u_{xx}$ (1-D Heat Equation)

$\Rightarrow$ Expect the system to be *Parabolic*

$A\vec{u}_t + B\vec{u}_x = \vec{c}$, $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$

$$\begin{matrix} u_t - v_x = 0 \\ \chi u_x = v \end{matrix} \Rightarrow \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}_{A} \begin{pmatrix} u \\ v \end{pmatrix}_t + \underbrace{\begin{pmatrix} 0 & -1 \\ \chi & 0 \end{pmatrix}}_{B} \begin{pmatrix} u \\ v \end{pmatrix}_x = \underbrace{\begin{pmatrix} 0 \\ v \end{pmatrix}}_{\vec{c}}$$

$\det(A) = 0$

Eigenvalue problem: $\det(A - \lambda B) = \det\begin{pmatrix} 1 & \lambda \\ -\chi\lambda & 0 \end{pmatrix} = \chi\lambda^2 = 0$

$\lambda = 0, 0$ hyperbolic or parabolic ?

$\lambda = 0$

eigenvectors: $(A - \lambda B)\vec{V} = \vec{0}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$V_1 = 0$, $0 = 0$ V_2 is arbitrary

$\Rightarrow$ $\boxed{\vec{V} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$ The double eigenvalue $\lambda = 0$ yields only one linearly independent eigenvector $\Rightarrow$ $\boxed{\text{Parabolic}}$

3) $u_t - c v_x = 0$
$v_t - c u_x = 0$ (linear)

observe that u and v satisfy the 1-D Wave Equation

$$u_{tt} - c v_{xt} = 0 \qquad \Rightarrow \quad u_{tt} = c\, v_{xt} = c(c u_{xx}) = c^2 u_{xx}$$
$$v_{tx} - c u_{xx} = 0$$

$u_{tt} = c^2 u_{xx}$ (1-D wave Equation) $\Rightarrow$ Expect the system to be Hyperbolic.

Similarly, $v_{tt} = c^2 v_{xx}$

$A\vec{u}_t + B\vec{u}_x = \vec{c}$, $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$

$$\begin{matrix} u_t - c v_x = 0 \\ v_t - c u_x = 0 \end{matrix} \Rightarrow \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{A = I} \begin{pmatrix} u \\ v \end{pmatrix}_t + \underbrace{\begin{pmatrix} 0 & -c \\ -c & 0 \end{pmatrix}}_{B} \begin{pmatrix} u \\ v \end{pmatrix}_x = \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\vec{c}}$$

Eigenvalue Problem: $\det(B - \lambda A) = \begin{vmatrix} -\lambda & -c \\ -c & -\lambda \end{vmatrix} = \lambda^2 - c^2 = 0$

$\lambda = \pm c$

$c \neq 0 \Rightarrow$ 2 Distinct Eigenvalues $\Rightarrow$ 2 linearly independent eigenvectors

$\Rightarrow$ $\boxed{\text{Hyperbolic for } c \neq 0}$

$c = 0 \Rightarrow \lambda = 0$ is a Repeated Eigenvalue $\Rightarrow$ hyperbolic or parabolic ?

$B - \lambda A = \begin{pmatrix} -\lambda & 0 \\ 0 & -\lambda \end{pmatrix}$

$\lambda = 0 \Rightarrow \quad (B - \lambda A)\vec{v} = \vec{0}$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ $0 = 0$ No restrictions on v_1 and v_2
$0 = 0$ $\Rightarrow$ v_1 and v_2 are arbitrary

$\Rightarrow \vec{V}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{V}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The double eigenvalue $\lambda = 0$ yields a full set of 2 linearly independent eigenvectors

$\Rightarrow$ $\boxed{\text{Hyperbolic for } c = 0}$

4) Euler Equations of 1-D High-Speed Flow of an Ideal Gas

ρ = density
v = velocity
P = pressure

$\gamma = \dfrac{\text{specific heat at constant pressure}}{\text{specific heat at constant volume}} > 1$

For an Ideal Gas, $\gamma \approx 1.4$

and sound speed $= c = \sqrt{\dfrac{\gamma P}{\rho}}$

$$\boxed{\begin{aligned} &\rho_t + (\rho v)_x = 0 \\ &v_t + v v_x + \tfrac{1}{\rho} P_x = 0 \\ &P_t + \gamma P v_x + v P_x = 0 \end{aligned}}$$

(conservation of mass)
(derived from conservation of momentum)
(derived from conservation of energy)

(quasilinear)

$$A\vec{u}_t + B\vec{u}_x = \vec{c}, \quad \vec{u} = \begin{pmatrix} \rho \\ v \\ P \end{pmatrix}$$

$$\begin{aligned} &\rho_t + v\rho_x + \rho v_x = 0 \\ &v_t + v v_x + \tfrac{1}{\rho} P_x = 0 \\ &P_t + \gamma P v_x + v P_x = 0 \end{aligned} \Rightarrow \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{A = I} \begin{pmatrix} \rho \\ v \\ P \end{pmatrix}_t + \underbrace{\begin{pmatrix} v & \rho & 0 \\ 0 & v & 1/\rho \\ 0 & \gamma P & v \end{pmatrix}}_{B} \begin{pmatrix} \rho \\ v \\ P \end{pmatrix}_x = \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}}_{\vec{c}}$$

Eigenvalue Problem: $\det(B - \lambda A) = \begin{pmatrix} v-\lambda & \rho & 0 \\ 0 & v-\lambda & 1/\rho \\ 0 & \gamma P & v-\lambda \end{pmatrix} = 0$

$$(v-\lambda)\left[(v-\lambda)^2 - \gamma P/\rho\right] = 0$$

$$\lambda = v, \quad \lambda = v \pm \sqrt{\frac{\gamma P}{\rho}}$$

$$\lambda = v \pm c$$

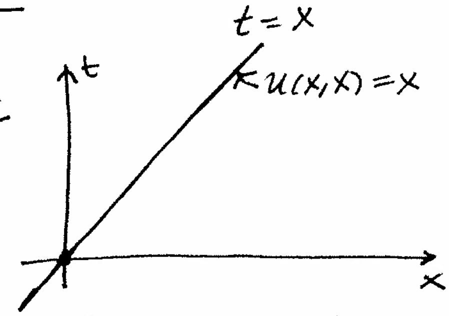
$c = 0$ only at absolute zero temperature

Real Distinct Eigenvalues $\Rightarrow$ $\boxed{\text{Hyperbolic}}$

Method of Characteristics

Introductory Example: Hyperbolic ($\lambda = 1$)

$$u_t - u_x = u, \quad u(x,x) = x$$



C: Initial Data Curve

Let $\tau = \frac{x+t}{2}$ and $s = \frac{x-t}{2}$

Transform Coordinates: $(x,t) \to (\tau, s)$

$$u(x,t) \to u(\tau,s) = u(x(\tau,s), t(\tau,s))$$

PDE:

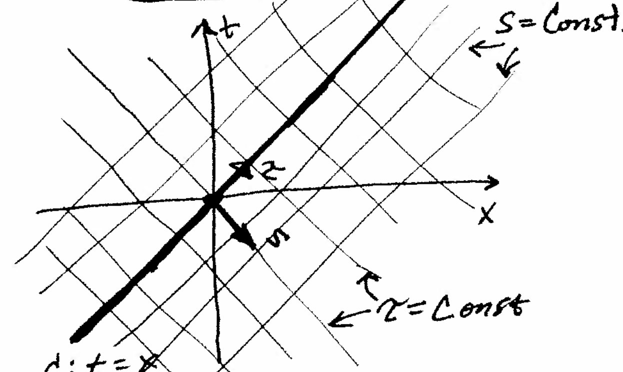
$$u_t = u_\tau \tau_t + u_s s_t = \tfrac{1}{2}(u_\tau - u_s)$$
$$u_x = u_\tau \tau_x + u_s s_x = \tfrac{1}{2}(u_\tau + u_s)$$

$\Rightarrow$ $u_t - u_x = u$

$$\tfrac{1}{2}(\cancel{u_\tau} - u_s) - \tfrac{1}{2}(\cancel{u_\tau} + u_s) = u$$

$u_s + u = 0$

C: $s = 0$



level curves

$\tau(x,t) = \frac{x+t}{2} = c$, $s(x,t) = \frac{x-t}{2} = c$

$\underline{t = -x + 2c}$, $\underline{t = x - 2c}$

IC: On C, $t = x \Rightarrow \tau = \frac{x+x}{2} = x$

$$s = \frac{x-x}{2} = 0$$

$u(x,x) = x$

$$u(\tau,s)\Big|_C = u(\tau,0) = \tau$$

$$\Rightarrow \boxed{\begin{aligned} &\frac{\partial u}{\partial s} + u = 0 \\ &u(\tau,0) = \tau \end{aligned}}$$

separable and linear

Integrating Factor

$\mu = e^s$

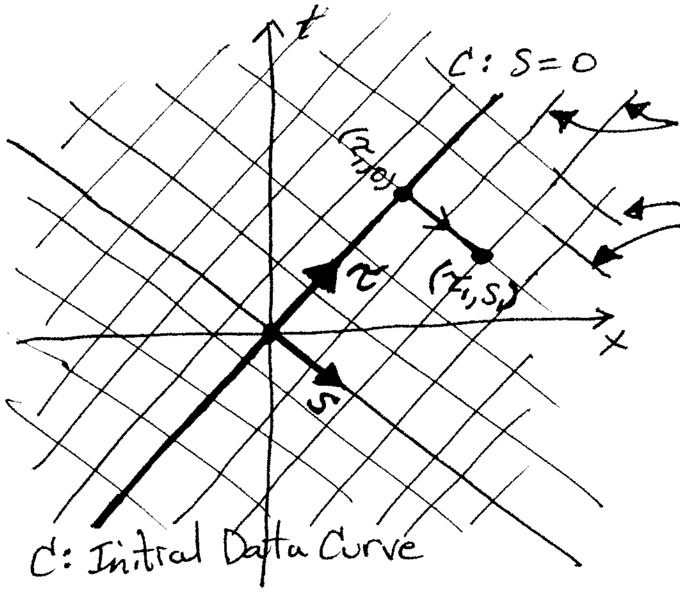
$$\Rightarrow e^s \frac{\partial u}{\partial s} + e^s u = 0$$
$$\frac{d}{ds}(e^s u) = 0$$
$$e^s u(\tau,s) = A(\tau)$$
$$u(\tau,s) = A(\tau) e^{-s}$$

IC: $u(\tau,0) = A(\tau) e^0 = \tau$

$A(\tau) = \tau$ $\Rightarrow$ $u(\tau,s) = \tau e^{-s}$

$\tau = \frac{x+t}{2}$, $s = \frac{x-t}{2}$ $\Rightarrow$

$$\boxed{u(x,t) = \frac{x+t}{2} e^{-\left(\frac{x-t}{2}\right)}}$$



Along the levelcurves $\tau(x,t)=c$ the PDE reduces to an ODE

The characteristics of a PDE are curves along which the PDE reduces to an ODE.

In the above example, the characteristics of the PDE $u_t - u_x = u$ are the level curves $\tau(x,t) = \frac{x+t}{2} = c$, along which $u_s + u = 0$.

Notice that the solution at a point (τ_1, s_1) depends only on the value of u at a single point $(\tau_1, 0)$ on the Initial Data Curve.

i.e. The evolution of u along a characteristic is independent of the evolution of u along any other characteristic.

A key feature of Hyperbolic PDEs is that the solution at any point depends only on the initial values over a finite interval of the Initial Data Curve, whereas the solution of a Parabolic PDE depends on the initial values over the entire Initial Data Curve.

i.e. For a hyperbolic PDE, an initial disturbance at a single point influences the solution over a limited domain.

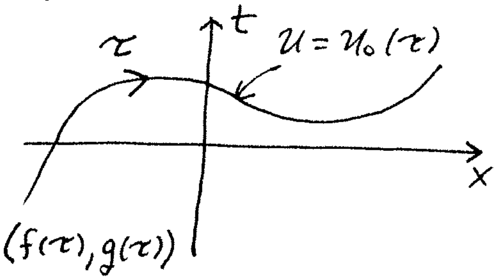
For a parabolic PDE, an inital disturbance at a single point influences the solution over the entire domain.

Quasilinear Scalar First Order PDEs

PDE: $\boxed{a(x,t,u)\dfrac{\partial u}{\partial t} + b(x,t,u)\dfrac{\partial u}{\partial x} = c(x,t,u)}$ Hyperbolic (wherever a and b are not both zero)

Initial Data: Suppose that u is prescribed along some parametric curve C.

$$\boxed{\begin{aligned} &C:\ x = f(\tau),\ t = g(\tau) \\ &u\big|_C = u(f(\tau), g(\tau)) = u_0(\tau) \end{aligned}}$$



u_0 is the given initial data on C.

The Cauchy Problem of a PDE is the initial value problem consisting of the PDE subject to initial data prescribed along some Initial Data Curve C.

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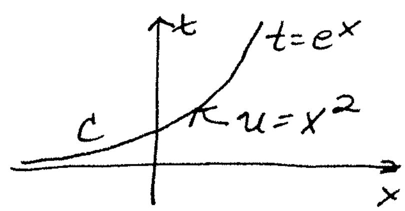
e.g. $\left.\begin{aligned} &a\dfrac{\partial u}{\partial t} + b\dfrac{\partial u}{\partial x} = c \\ &u(x, e^x) = \underbrace{x^2}_{u_0(x)} \end{aligned}\right\}$ Cauchy Problem



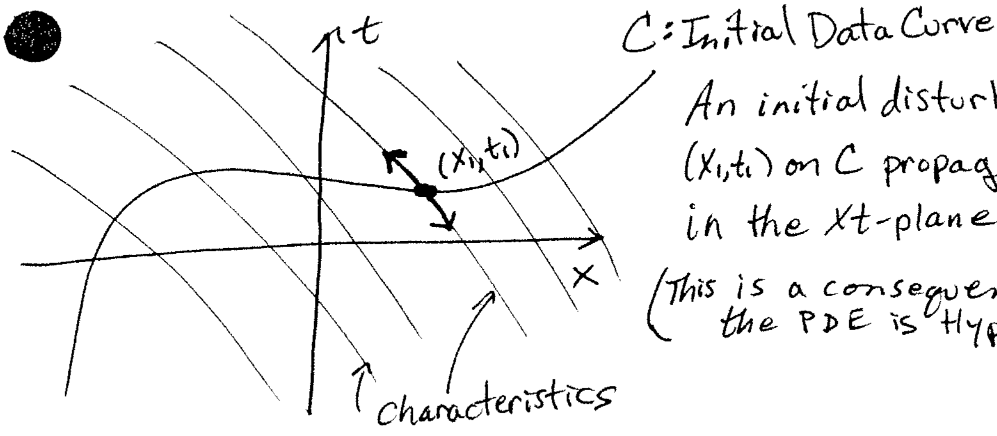
Let Parametrize C: $x = \tau$ and $t = e^\tau$

$\Rightarrow$ **Initial Data:** $x = \tau$, $t = e^\tau$, $u = \tau^2$, $-\infty < \tau < \infty$

Note: A theorem exist which states the conditions under which the Cauchy Problem of a quasilinear scalar first order PDE has a unique solution in some neighborhood of the Initial Data Curve.

In short, if $a(x,t,u)$, $b(x,t,u)$, and $c(x,t,u)$ are sufficiently smooth, then the Cauchy Problem has a unique solution if the Initial Data Curve is nowhere tangent to the characteristics of the PDE.

Method of Characteristics



An initial disturbance at some point (x_1, t_1) on C propagates along a curve in the xt-plane called a <u>characteristic</u>.

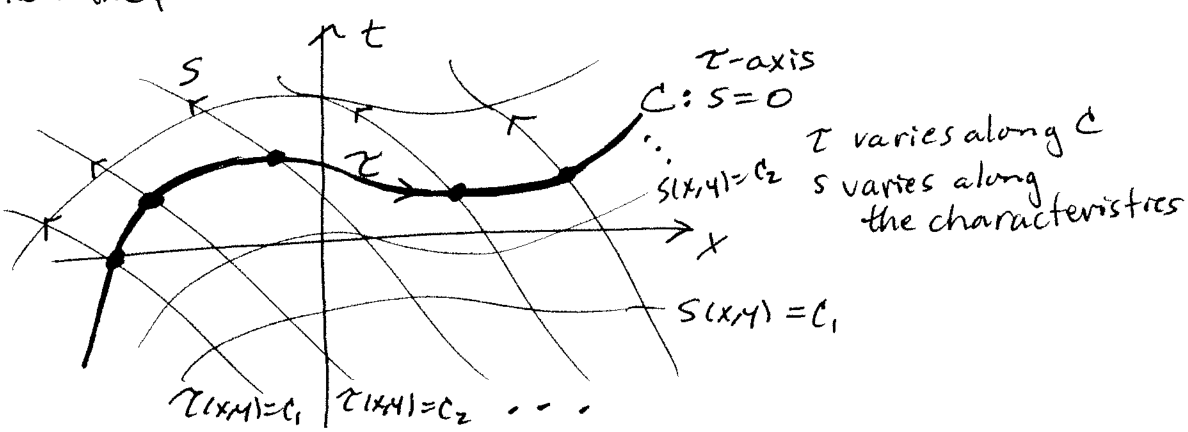
(This is a consequence of the fact that the PDE is Hyperbolic.)

<u>Idea:</u> Make a coordinate transformation, $(x,t) \to (\tau, s)$, so that

(i) the Initial Data Curve C corresponds to the τ-axis.
(i.e. to the level curve $s(x,y) = 0$)

and (ii) the characteristics of the PDE are the grid lines of the τ-coordinate (i.e. the level curves $\tau(x,y) = c$), so that only s varies along the characteristics, while τ is constant along each.

Since an initial disturbance propagates only along the characteristics, where s varies and τ is constant, the PDE reduces to an ODE along each of the characteristics, with s being the independent variable. Thus, we have a one-parameter family of ODEs, where the parameter τ selects a specific characteristic.



Developing the Method of Characteristics

Consider the following Cauchy problem for $u(x,y)$.

PDE: $a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$

Initial Data: u is prescribed on a parametric curve $C: x=f(\tau),\ y=g(\tau)$.

$$\Rightarrow u(\tau) = u(f(\tau), g(\tau)) = u_0(\tau)$$ ← given function

---

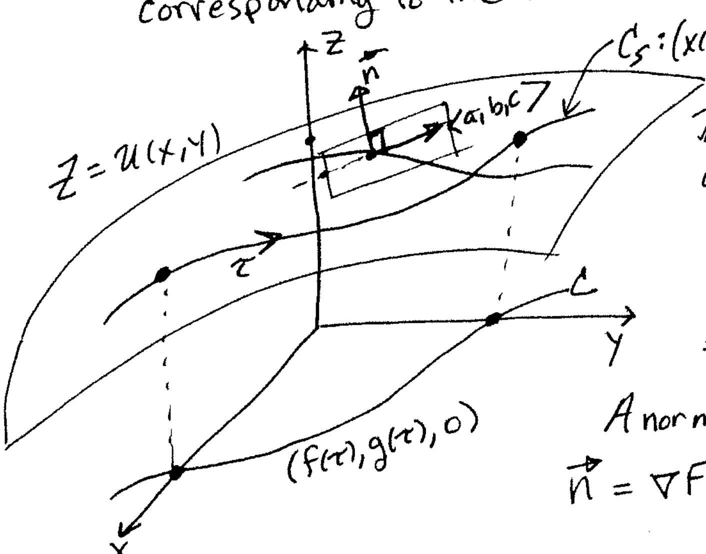
Transform the coordinates, $(x,y)\to(\tau,s)$, so that the Initial Data Curve C corresponds to the τ-axis. (i.e. to the level curve $s(x,y)=0$).

$x = x(\tau,s)$ $\quad \tau=\tau(x,y)$
$y = y(\tau,s)$ $\quad s=s(x,y)$

$\Rightarrow$ Require that $x(\tau,0)=f(\tau)$
$y(\tau,0)=g(\tau)$

$$\Rightarrow u(\tau,0) = u(x(\tau,0), y(\tau,0)) = u_0(\tau)$$

The solution of the Cauchy problem is a surface $z=u(x,y)$ in xyz-space, which contains the parametric curve $C_s: (x(\tau,0), y(\tau,0), u(\tau,0))$ corresponding to the initial data.



The solution surface is the level surface of the function
$W = F(x,y,z) = u(x,y) - z$
corresponding to $w=0$.

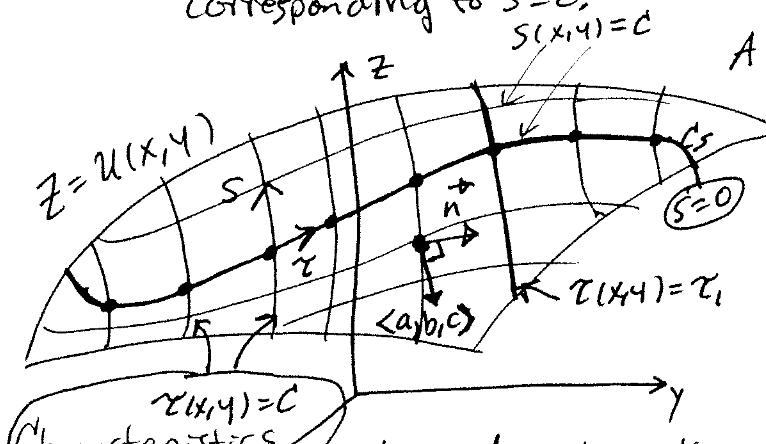
$$\Rightarrow F(x,y,z) = u(x,y) - z = 0$$

A normal vector to the level surface is
$\vec{n} = \nabla F = \langle u_x, u_y, -1\rangle$ at all points on the surface

Then, $\langle a,b,c\rangle \cdot \vec{n} = \langle a,b,c\rangle\cdot\langle u_x,u_y,-1\rangle = a u_x + b u_y - c = 0$. ← PDE

Since the vector $\langle a,b,c\rangle$ is perpendicular to $\vec{n}$, it lies in the tangent plane to the solution surface at all points on the surface. Therefore, as we move in the direction of $\langle a,b,c\rangle$, a curve is traced out on the solution surface, which will have $\langle a,b,c\rangle$ as a tangent vector at all points on the curve.

Suppose we generate a family of such curves, each of which passes through a distinct point on C_s. The family of curves is parametrized by the value of τ at the point of intersection on C_s. The τ-coordinate is defined by requiring τ to be constant along each of the curves. That is, τ is defined so that the curves are the gridlines (level curves, $\tau(x,y)=c$) of the τ-coordinate. Only s varies along each curve, with the points on C_s corresponding to $s=0$.



A curve corresponding to a fixed τ, say $\tau=\tau_1$, is a parametric curve with parameter s.

$$\left.\begin{aligned} x &= x(\tau_1,s) \\ y &= y(\tau_1,s) \\ u &= u(\tau_1,s) \end{aligned}\right\}$$ parametric representation with parameter s of the level curve $\tau(x,y)=\tau_1$.

A tangent vector to the parametric curve is $\left\langle \frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial u}{\partial s} \right\rangle$.

As seen above, the vector $\langle a,b,c\rangle$ is also tangent to the curve.

$$\Rightarrow \left\langle \frac{\partial x}{\partial s}, \frac{\partial y}{\partial s}, \frac{\partial u}{\partial s} \right\rangle = k\,\langle a,b,c\rangle$$

Scale s in such a way that $k=1$.

$$\Rightarrow \boxed{\begin{aligned} \frac{\partial x}{\partial s} &= a(x,y,u) \\ \frac{\partial y}{\partial s} &= b(x,y,u) \\ \frac{\partial u}{\partial s} &= c(x,y,u) \end{aligned}}$$

We obtain a system of ODEs for each fixed τ.

<u>Note</u>: With (τ,s) defined as described above, we have

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s} = a u_x + b u_y = c.$$

Therefore, the PDE reduces to an ODE, $\frac{\partial u}{\partial s} = c(x(\tau,s), y(\tau,s), u(\tau,s))$, along each of the level curves $\tau(x,y)=c$. Therefore,

the level curves <u>$\tau(x,y)=c$ are the characteristics</u> of the PDE.

The above system of ODEs, together with the initial data on C_s ($s=0$), yields an initial value problem along each characteristic (i.e. for each fixed τ). We have a one-parameter family of initial value problems, with τ being the parameter and with s being the independent variable. The system can be solved to yield

$$x = x(\tau,s)$$
$$y = y(\tau,s)$$
$$u = u(\tau,s).$$

If the coordinate transformation is invertible, we can solve for τ and s in terms of x and y, $\Rightarrow$ $\begin{array}{l}\tau = \tau(x,y)\\ s = s(x,y)\end{array}$, and the solution in the (x,y) coordinate system is given by

$$u(x,y) = u(\tau(x,y), s(x,y)).$$

Note: The coordinate transformation (τ, s) is not defined beforehand, but rather determined as part of the solution process.

Steps in Applying the Method of Characteristics

Given: PDE: $a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u)$

Initial Data: $u(x,y)\big|_C = u_0$ (given function)

Step 1: Parametrize the initial data, with parameter τ, and with $s=0$.

$$C: \begin{cases} x = x(\tau,0) \\ y = y(\tau,0) \end{cases}$$
$$\Rightarrow u(\tau,0) = u(x(\tau,0), y(\tau,0)) = u_0(\tau)$$

Step 2: Solve the system of ODEs

Characteristic Equations:
$$\frac{\partial x}{\partial s} = a(x,y,u)$$
$$\frac{\partial y}{\partial s} = b(x,y,u)$$
$$\frac{\partial u}{\partial s} = c(x,y,u)$$

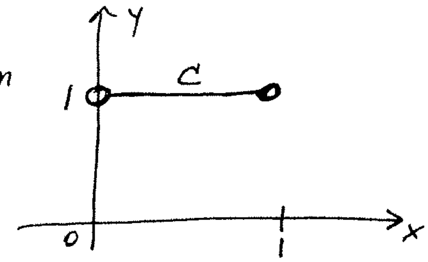
subject to the initial data in step 1.

This yields
$$x = x(\tau,s)$$
$$y = y(\tau,s)$$
$$u = u(\tau,s)$$

Step 3: Solve for τ and s in terms of x and y, if possible, to obtain the solution $u(x,y) = u(\tau(x,y), s(x,y))$.

Example: Solve the following Cauchy problem

$$x u_x - y u_y = 2xy^2$$
$$u(x,1) = x^2+1, \quad 0<x<1$$



Step 1: Parametrize C $(s=0)$

C: $x(\tau,0) = \tau \quad , 0<\tau<1$
$y(\tau,0)=1$
$u(\tau,0)=\tau^2+1$

Step 2: Solve the system of ODEs

$x_s = a(x,y,u) = x$
$y_s = b(x,y,u) = -y$ subject to the initial conditions in Step 1.
$u_s = c(x,y,u) = 2xy^2$

$\frac{\partial x}{\partial s} = x, \; x(\tau,0)=\tau$
$x(\tau,s) = A(\tau)e^{s}$
$x(\tau,0) = A(\tau) = \tau$
$x(\tau,s) = \tau e^{s}$

$\frac{\partial y}{\partial s} = -y, \; y(\tau,0)=1$
$y(\tau,s) = B(\tau)e^{-s}$
$y(\tau,0) = B(\tau) = 1$
$y(\tau,s) = e^{-s}$

$\frac{\partial u}{\partial s} = 2xy^2 = 2(\tau e^{s})(e^{-s})^2, \; u(\tau,0) = \tau^2+1$
$\frac{\partial u}{\partial s} = 2\tau e^{-s}$
$u(\tau,s) = -2\tau e^{-s} + C(\tau)$
$u(\tau,0) = -2\tau + C(\tau) = \tau^2+1$
$C(\tau) = (\tau+1)^2$
$u(\tau,s) = -2\tau e^{-s} + (\tau+1)^2$

Step 3: Find $u(x,y) = u(\tau(x,y), s(x,y))$

$y = e^{-s} \Rightarrow s = -\ln y, \; y>0$

$x = \tau e^{s}$
$\tau = xe^{-s} = xy$
$\tau = xy$

$\Rightarrow u(x,y) = -2(xy)\cdot y + (xy+1)^2$

$$\boxed{u(x,y) = (xy+1)^2 - 2xy^2}$$

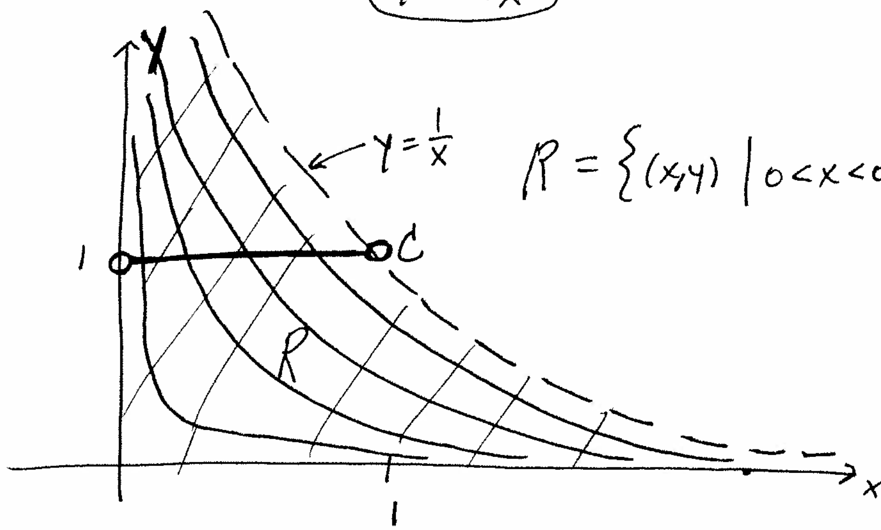
Check: $xu_x - yu_y = x\left[\cancel{2y(xy+1)} - 2y^2\right] - y\left[\cancel{2x(xy+1)} - 4xy\right] = 2xy^2$ ✓
$u(x,1) = (x+1)^2 - 2x = x^2+2x+1-2x = x^2+1$ ✓

We have $u(x,y) = (xy+1)^2 - 2xy^2$

Characteristics : $\tau(x,y) = k$

$$xy = k$$

$$y = k/x$$



$R = \{(x,y) \mid 0 < x < \infty,\ 0 < y < \frac{1}{x}\}$

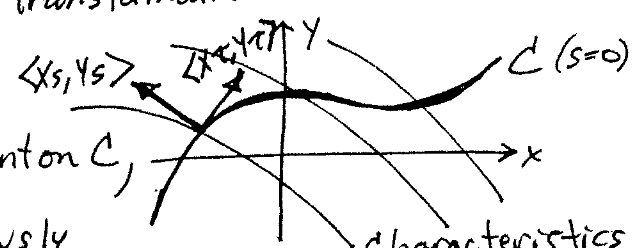
We solved an ODE along each characteristic subject to the initial condition specified on C.

though $u(x,y) = (xy+1)^2 - 2xy^2$ is defined for all (x,y), the solution is not fully determined outside of R. A general solution solution exists outside of R, but there are no initial conditions specified for the characteristics lying outside of R. The function $u(x,y) = (xy+1)^2 - 2xy^2$ is only one of the infinitely many solutions outside of R.

In short, although $u(x,y) = (xy+1)^2 - 2xy^2$ is defined outside of R and it satisfies the PDE, it is not more important than the other possible solutions.

Invertibility of the Coordinate Transformation

Given $\begin{matrix} x = x(\tau, s) \\ y = y(\tau, s) \end{matrix}$, the Method of Characteristics requires that we solve for τ and s in terms of x and y. To achieve a solution by the Method of Characteristics, the coordinate transformation must be invertible in some region of the Initial Data Curve C. The Implicit Function theorem guarantees that the coordinate transformation is invertible in a neighborhood of C if



$$\begin{vmatrix} x_\tau & x_s \\ y_\tau & y_s \end{vmatrix} \neq 0 \text{ at each point on } C,$$

provided that x and y are continuously differentiable with respect to τ and s.

This condition can be written is an alternative form.

$a u_x + b u_y = c$
$x_s = a$
$y_s = b$

Using the relations $x_s = a(x,y,u)$ and $y_s = b(x,y,u)$, we have

$$\begin{vmatrix} x_\tau & x_s \\ y_\tau & y_s \end{vmatrix} = \begin{vmatrix} x_\tau & a \\ y_\tau & b \end{vmatrix} = -\begin{vmatrix} a & b \\ x_\tau & y_\tau \end{vmatrix},$$ and therefore

$$\begin{vmatrix} x_\tau & x_s \\ y_\tau & y_s \end{vmatrix} \neq 0 \text{ if and only if } \begin{vmatrix} a & b \\ x_\tau & y_\tau \end{vmatrix} \neq 0$$

<u>Note</u>: $\langle x_\tau, y_\tau \rangle$ is tangent to C at each point on C

$\langle x_s, y_s \rangle$ is tangent to the characteristics at each point.

The tangent vectors are <u>not parallel</u> on C if and only if

$$|\langle x_\tau, y_\tau \rangle \times \langle x_s, y_s \rangle| = \left| \begin{vmatrix} x_\tau & y_\tau \\ x_s & y_s \end{vmatrix} \right| = \left| \begin{vmatrix} x_\tau & x_s \\ y_\tau & y_s \end{vmatrix} \right| \neq 0 \text{ at each point on } C$$

$\Rightarrow$ The coordinate transformation is invertible in a neighborhood of the Initial Data Curve C if C is nowhere tangent to the characteristics.

In this case, the Initial Data Curve C is said to be <u>noncharacteristic</u>.

Theorem: Cauchy–Kowaleski

Consider the Cauchy problem

$$a(x,y,u)\,u_x + b(x,y,u)\,u_y = c(x,y,u),$$

where u is prescribed on a parametric curve C.

$$C: \begin{array}{l} x = f(\tau) \\ y = g(\tau) \end{array},\ \alpha < \tau < \beta \quad \Rightarrow \quad u = u(f(\tau), g(\tau)) = u_0(\tau).$$

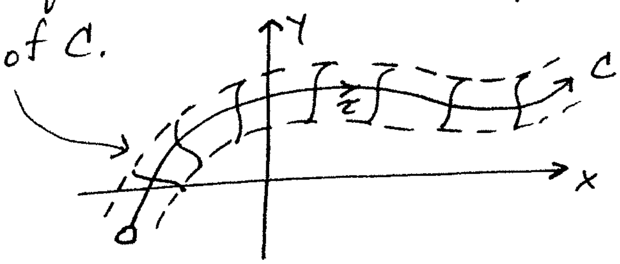
Suppose that $a(x,y,u)$, $b(x,y,u)$, and $c(x,y,u)$ are continuously differentiable in some neighborhood of the initial data, that is, in some neighborhood of the space curve

$$(f(\tau), g(\tau), u_0(\tau)),\ \alpha < \tau < \beta.$$

If $\begin{vmatrix} a & b \\ x_\tau & y_\tau \end{vmatrix}_C = \begin{vmatrix} a & b \\ f'(\tau) & g'(\tau) \end{vmatrix}_C \neq 0$ at each point on C,

$\begin{vmatrix} x_s & y_s \\ x_\tau & y_\tau \end{vmatrix}$

then there exists a unique solution to the Cauchy problem in a neighborhood of C.



Furthermore, if $\begin{vmatrix} a & b \\ f'(\tau) & g'(\tau) \end{vmatrix} = 0$ at all points on C, then the Initial Data Curve C is a characteristic, and the Cauchy problem has either infinitely many solutions or no solutions in a neighborhood of C.

Example: $u_x + u_y = u^2$ $\quad C: \; x(\tau,0) = \tau$

$u(x,0) = x$ $\quad\quad y(\tau,0) = 0$

$\quad\quad u(\tau,0) = \tau$

$x_s = 1,\; x(\tau,0) = \tau$

$x = s + A(\tau)$

$x(\tau,0) = 0 + A(\tau) = \tau$

$x(\tau,s) = s + \tau$

$y_s = 1,\; y(\tau,0) = 0$

$y = s + B(\tau)$

$y(\tau,0) = 0 + B(\tau) = 0$

$y(\tau,s) = s$

$u_s = u^2,\; u(\tau,0) = \tau$

$\int \frac{du}{u^2} = \int ds$

$-\frac{1}{u} = s - C(\tau)$

$u = \frac{1}{C(\tau) - s}$

$u(\tau,0) = \frac{1}{C(\tau) - 0} = \tau$

$C(\tau) = 1/\tau$

$u(\tau,s) = \frac{\tau}{1 - \tau s}$

Invert: $s = y$

$\tau = x - \tau = x - y$

$\tau = x - y$

$$\Rightarrow \boxed{u(x,y) = \frac{x-y}{1 - y(x-y)}} \text{ for } (x,y) \in R$$

$R = ?$

Invertibility: $\begin{vmatrix} a & b \\ x_\tau & y_\tau \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$

$\Rightarrow$ The coordinate transformation is invertible for all (x,y).

Continuity: u is not defined for (x,y) such that

$1 - y(x-y) = 0$

$\Rightarrow x = y + \frac{1}{y}$

$\left(\begin{array}{l} x > y \\ x \to y \text{ as } y \to \pm\infty \\ x \to \pm\infty \text{ as } y \to 0^{\pm} \end{array} \right)$

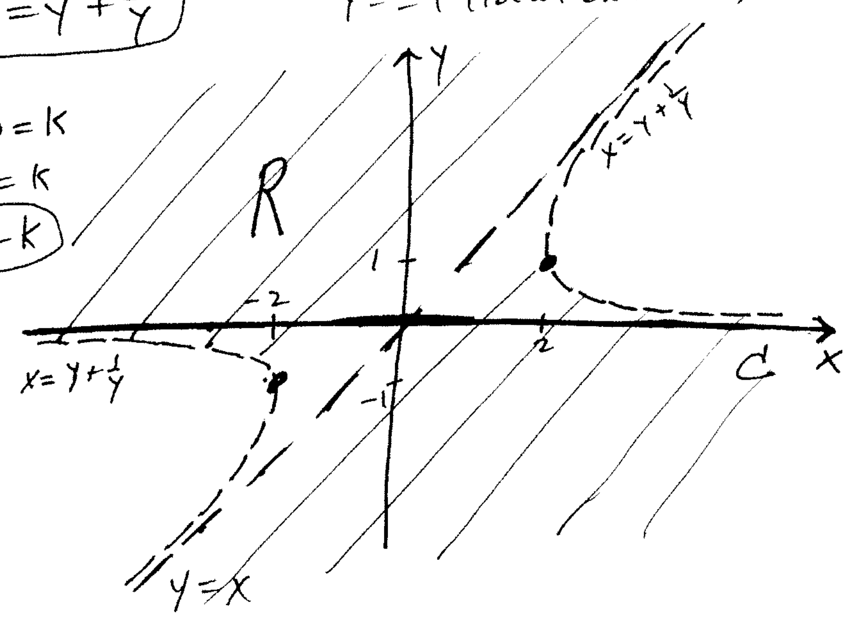
$\frac{dx}{dy} = 1 - 1/y^2 = 0$

$y = \pm 1$ (local extrema)

Characteristics: $\tau(x,y) = k$

$x - y = k$

$y = x - k$



R = domain of the solution

Example: $u_x + 3u_y = u$ $\quad$ C: $x(\tau,0) = \tau$
$u(x, 3x) = \cos x$ $\quad$ $y(\tau, 0) = 3\tau$
$u(\tau, 0) = \cos\tau$

$x_s = 1,\ x(\tau,0) = \tau$	$y_s = 3,\ y(\tau,0) = 3\tau$	$u_s = u,\ u(\tau,0) = \cos\tau$
$x = s + A(\tau)$	$y = 3s + B(\tau)$	$u = C(\tau)e^s$
$x(\tau,0) = 0 + A(\tau) = \tau$	$y(\tau,0) = 0 + B(\tau) = 3\tau$	$u(\tau,0) = C(\tau) = \cos\tau$
$x(\tau,s) = s + \tau$	$y(\tau,s) = 3(s+\tau)$	$u(\tau,s) = e^s\cos\tau$

$= -\begin{vmatrix} x_\tau & x_s \\ y_\tau & y_s \end{vmatrix}$

Invert: $\begin{vmatrix} a & b \\ x_\tau & y_\tau \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$ for all (x,y) Invertible

The Cauchy-Kowaleski Theorem does not guarantee a unique solution.
In fact, it states that there are either infinitely many solutions or no solutions.

We have $\begin{array}{l} s+\tau = x \\ s+\tau = y/3 \end{array} \Rightarrow \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{\det A \neq 0}\begin{pmatrix} s \\ \tau \end{pmatrix} = \begin{pmatrix} x \\ y/3 \end{pmatrix}$

$\Rightarrow$ Given a point (x,y), there are either infinitely many (when $y = 3x$) solutions (τ, s) or no (when $y \neq 3x$) solutions (τ,s).

Therefore, the transformation is not invertible over any region.

$\Rightarrow$ The Method of Characteristics Fails

The fact that $\begin{vmatrix} a & b \\ x_\tau & y_\tau \end{vmatrix} = 0$ for all points on C suggests that the Initial Data Curve C is everywhere tangent to characteristic

$\Rightarrow$ C is a characteristic
i.e. C is not noncharacteristic

Though we can find a solution $u(\tau,s)$ in the (τ,s) coordinate system, it cannot be converted back to the (x,y) coordinate system.

Since the coordinate transformation in the above example is not invertible, we are not able to solve for $\tau(x,y)$, and therefore, we do not obtain the characteristics of the PDE.

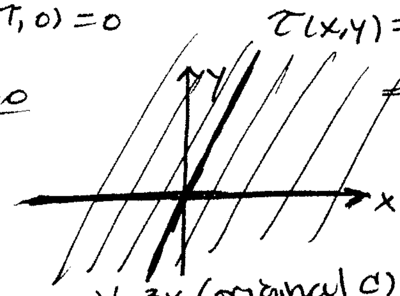
The characteristics are a feature of the PDE alone — they are independent of the initial data. The Method of Characteristics failed, not because of the PDE, but because of the Initial Data Curve C. We could find the characteristics by using different initial data. Any choice will do provided the Initial Data Curve C is noncharacteristic.

e.g. Solve $u_x + 3u_y = u$, $u(x,0) = 0$ $\quad$ C: $x(\tau,0) = \tau$, $y(\tau,0) = 0$, $u(\tau,0) = 0$ $\quad$ Characteristics

$x_s = 1 \Rightarrow x = s + \tau$ $\qquad$ $y_s = 3 \Rightarrow y = 3s$ $\qquad$ $u_s = u \Rightarrow u = 0$

$\tau = x - \frac{y}{3}$ $\qquad$ $s = y/3$

$\tau(x,y) = x - \frac{y}{3} = k \Rightarrow \boxed{y = 3x + k}$



Alternatively, rather than using the Method of Characteristics, the characteristics can be found by finding the general solution of a single ODE for $\frac{dy}{dx}$, so no initial data is need, as should be the case since the characteristics of a PDE are independent of the initial data.

Recall: If x and y are functions of a single variable, say $x = x(t)$, $y = y(t)$,

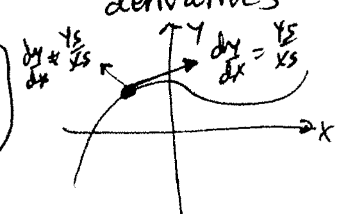
then $dx = x'(t)\,dt$, $dy = y'(t)\,dt \Rightarrow \frac{dy}{dx} = \frac{y'(t)\,dt}{x'(t)\,dt} = \frac{dy/dt}{dx/dt}$ $\qquad \boxed{\frac{dy}{dx} = \frac{dy/dt}{dx/dt}}$

ordinary derivatives

The same idea does not apply to partial derivatives

$x = x(\tau,s)$, $y = y(\tau,s)$ $\quad$ i.e. $\frac{dy}{dx} \neq \frac{\partial y/\partial s}{\partial x/\partial s}$

Instead, $dx = X_\tau d\tau + X_s ds$, $dy = Y_\tau d\tau + X_s ds$ $\Rightarrow$ $\boxed{\frac{dy}{dx} = \frac{Y_\tau d\tau + Y_s ds}{X_\tau d\tau + X_s ds}}$



However, on C, where $\tau = \text{const}$, we have $d\tau = 0$, in which case

$\Rightarrow \frac{dy}{dx} = \frac{\cancel{Y_\tau}\cdot 0 + Y_s ds}{\cancel{X_\tau}\cdot 0 + X_s ds} = \frac{Y_s}{X_s} = \frac{b(x,y,u)}{a(x,y,u)} \Rightarrow \boxed{\frac{dy}{dx} = \frac{b(x,y,u)}{a(x,y,u)}}$ The solution curves are the characteristics

$\frac{dy}{dx} = 3 \Rightarrow y = 3x + k$

Example: $uu_t + u_x = 0$, $u(x,1) = 1/x,\ x \ge 1$

C: $x(\tau,0) = \tau,\ \tau \ge 1$
$t(\tau,0) = 1$
$u(\tau,0) = 1/\tau$

$t_s = u,\ t(\tau,0) = 1$
$t_s = \frac{1}{\tau}$
$t = \frac{s}{\tau} + C(\tau)$
$t(\tau,0) = 0 + C(\tau) = 1$
$\Rightarrow t(\tau,s) = \frac{s}{\tau} + 1$

$x_s = 1,\ x(\tau,0) = \tau$
$x = s + A(\tau)$
$x(\tau,0) = 0 + A(\tau) = \tau$
$x(\tau,s) = s + \tau$

$u_s = 0$
$u = B(\tau)$
$u(\tau,0) = B(\tau) = 1/\tau$
$u(\tau,s) = 1/\tau$

Invertible? Homework Problem

Note: $\begin{vmatrix} x_\tau & x_s \\ t_\tau & t_s \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ -s/\tau^2 & 1/\tau \end{vmatrix} = \frac{1}{\tau} + \frac{s}{\tau^2} = \frac{1}{\tau^2}(\tau + s) = 0$ when $s = -\tau$

$s = -\tau \Rightarrow (x,t) = (0,0)$

$\Rightarrow$ the coordinate transformation may not be invertible in a region containing the point $(x,t) = (0,0)$.

This does not reveal the whole story.

Note: the coordinate transformation is invertible in a neighborhood of C.

$s + \tau = x$
$s + \tau = \tau t$ $\Rightarrow$ $x = \tau t$, $\tau = x/t$

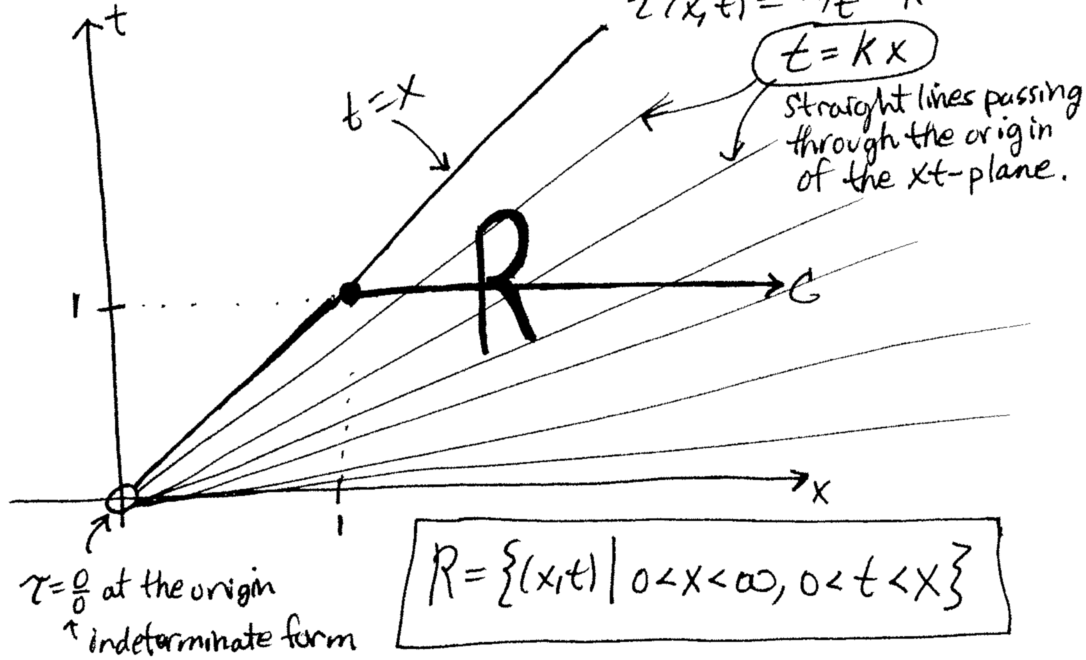
$s = x - \tau = x - x/t = x(1 - 1/t)$
$s = x(1 - 1/t)$

$\Rightarrow \boxed{u(x,t) = t/x}$

Characteristics

$\tau(x,t) = x/t = k$
$t = kx$

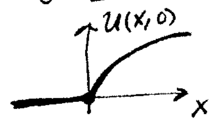
$\frac{dx}{dt} = \frac{b}{a} = \frac{1}{u} = \frac{x}{t}$
$\int \frac{dx}{x} = \int \frac{dt}{t}$
$\ln|x| = \ln|t| + \cancel{k}$
$x = kt$
$\frac{x}{t} = k$



$\boxed{R = \{(x,t) \mid 0 < x < \infty,\ 0 < t < x\}}$

Example: $u_t + u_x = -\dfrac{\alpha u}{L-t}, \quad 0<t<L$

$$u(x,0) = \begin{cases} 0, & x \le 0 \\ \sqrt{x}, & x>0 \end{cases}$$



C: $x(\tau,0)=\tau$

$t(\tau,0)=0$

$$u(\tau,0) = \begin{cases} 0, & \tau \le 0 \\ \sqrt{\tau}, & \tau>0 \end{cases}$$

$t_s = 1, \; t(\tau,0)=0$

$t = s + A(\tau)$

$t(\tau,0) = 0 + A(\tau) = 0$

$t(\tau,s) = s$

$x_s = 1, \; x(\tau,0)=\tau$

$x = s + B(\tau)$

$x(\tau,0) = 0 + B(\tau) = \tau$

$x(\tau,s) = s+\tau$

$u_s = \dfrac{-\alpha u}{L-t} = -\dfrac{\alpha u}{L-s}, \quad u(\tau,0) = \begin{cases} 0, & \tau\le 0 \\ \sqrt{\tau}, & \tau>0 \end{cases}$

$$\int \frac{du}{u} = \int \frac{-\alpha\, ds}{L-s}$$

$$\ln|u| = \alpha \ln|L-s| + C(\tau)$$

$$u = C_1(\tau)\,|L-s|^\alpha$$

Invert: $\begin{vmatrix} x_\tau & x_s \\ t_\tau & t_s \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

$\Rightarrow$ Invertible for all (x,t)

$$u(\tau,0) = C_1(\tau)\, L^\alpha = \begin{cases} 0, & \tau \le 0 \\ \sqrt{\tau}, & \tau>0 \end{cases}$$

$$C_1(\tau) = \begin{cases} 0, & \tau\le 0 \\ \sqrt{\tau}/L^\alpha, & \tau>0 \end{cases}$$

$s = t$

$\tau = x - t$

$\tau \le 0 \Rightarrow x \le t$ Also, $L-t>0$

$\tau > 0 \Rightarrow x > t$

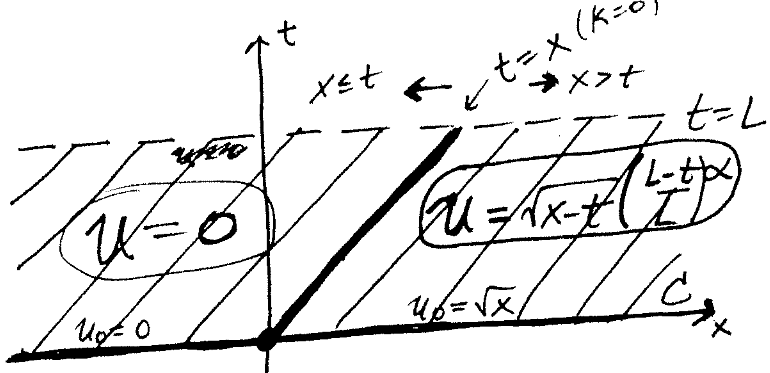
$$\Rightarrow u(\tau,s) = \begin{cases} 0, & \tau \le 0 \\ \sqrt{\tau}\left|\dfrac{L-s}{L}\right|^\alpha, & \tau > 0 \end{cases}$$

$$\Rightarrow \boxed{u(x,t) = \begin{cases} 0, & x \le t \\ \sqrt{x-t}\left(\dfrac{L-t}{L}\right)^\alpha, & x>t \end{cases}}$$

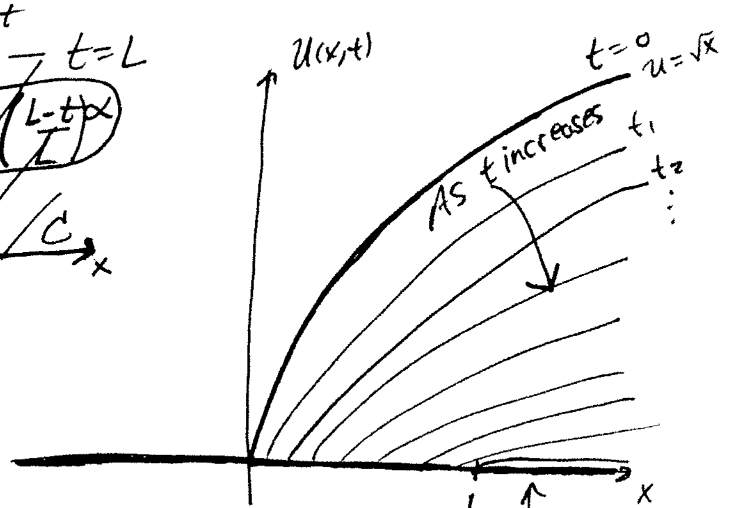
Characteristics

$\tau(x,t) = x-t = k$

$t = x-k$



$u \to 0$ as $t \to L$



$u \to 0$ as $t \to L$

$n+1$ Independent Variables $u(x_1,\dots,x_n,t) = u(\vec{x},t)$

$\vec{x} = \langle x_1, \dots, x_n \rangle$

Cauchy Problem: Quasilinear

PDE: $\boxed{a_0(\vec{x},t,u)u_t + a_1(\vec{x},t,u)u_{x_1} + \cdots + a_n(\vec{x},t,u)u_{x_n} = c(\vec{x},t,u)}$

Initial Data: the initial data is prescribed on an n-dimensional surface S defined parametrically by $t = \bar{t}(\vec{\tau})$ $\quad \vec{\tau} = \langle \tau_1, \dots, \tau_n \rangle$

$$x_i = \bar{x}_i(\vec{\tau}), \; i = 1, \dots, n$$

$$\Rightarrow \quad u|_S = u(\bar{x}_1(\vec{\tau}), \dots, \bar{x}_n(\vec{\tau}), \bar{t}(\vec{\tau})) = u_0(\vec{\tau}).$$

Method of Characteristics: ~~Use the following~~ $(\vec{x},t) \to (\vec{\tau},s)$

$n+2$ nonlinear ODEs with $n+2$ unknowns $(\vec{x},t,u)$

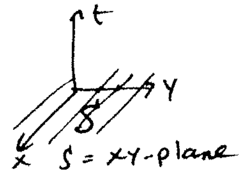
$$\begin{cases} \dfrac{\partial t}{\partial s} = a_0(\vec{x},t,u), & t(\vec{\tau},0) = \bar{t}(\vec{\tau}) \\ \dfrac{\partial x_i}{\partial s} = a_i(\vec{x},t,u), & x_i(\vec{\tau},0) = \bar{x}_i(\vec{\tau}), \; i = 1,\dots,n \\ \dfrac{\partial u}{\partial s} = c(\vec{x},t,u), & u(\vec{\tau},0) = u_0(\vec{\tau}) \end{cases}$$

A unique solution exists in a neighborhood of the initial data if

$$\left| \frac{\partial(x_1,\dots,x_n,t)}{\partial(\tau_1,\dots,\tau_n,s)} \right| \neq 0 \text{ at all points on } S$$

$\Rightarrow$ S is noncharacteristic

Example: $u_t + u_x + u_y = u$
$u(x,y,0) = x + y$



$S:$ $x = \bar{x}(\tau_1,\tau_2) = \tau_1$
$y = \bar{y}(\tau_1,\tau_2) = \tau_2$
$t = \bar{t}(\tau_1,\tau_2) = 0$
$u = u_0(\tau_1,\tau_2) = \tau_1 + \tau_2$

$t_s = 1, \; t(\vec{\tau},0) = 0$
$t = s + A(\vec{\tau}), \; A(\vec{\tau}) = 0$
$t = s$

$x_s = 1, \; x(\vec{\tau},0) = \tau_1$
$x = s + B(\vec{\tau}), \; B(\vec{\tau}) = \tau_1$
$x = s + \tau_1$

$(x,y,t) \to (\tau_1,\tau_2,s)$

$y_s = 1, \; y(\vec{\tau},0) = \tau_2$
$y = s + C(\vec{\tau}), \; C(\vec{\tau}) = \tau_2$
$y = s + \tau_2$

$u_s = u, \; u(\vec{\tau},0) = \tau_1 + \tau_2$
$u = D(\vec{\tau})e^s, \; D(\vec{\tau}) = \tau_1 + \tau_2$
$u = (\tau_1 + \tau_2)e^s$

Invert:
$$\begin{vmatrix} x_{\tau_1} & x_{\tau_2} & x_s \\ y_{\tau_1} & y_{\tau_2} & y_s \\ t_{\tau_1} & t_{\tau_2} & t_s \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0 \checkmark$$

$s = t$
$\tau_1 = x - t$
$\tau_2 = y - t$

$\Rightarrow \boxed{u(x,y,t) = (x + y - 2t)e^t}$

Check

General First Order PDEs (with two independent variables)

Cauchy Problem:

PDE: $F(x, y, u, u_x, u_y) = 0$

parametrization of the initial function on C

Initial Data: $C: \; x = \bar{x}(\tau), \; y = \bar{y}(\tau)$ $\quad u(\bar{x}(\tau), \bar{y}(\tau)) = \bar{u}(\tau)$

Method of Characteristics

This condition guarantees that the equation is a PDE at all points.

Let $p = u_x$ and $q = u_y$, and suppose that $p^2 + q^2 \neq 0$. $\;(u_x^2 + u_y^2 \neq 0) \Rightarrow$ $z = u(x,y)$ has no local extrema or inflection points.

$\Rightarrow F(x, y, u, p, q) = 0$

Solve

$$\boxed{\begin{array}{ll} x_s = F_p & x(\tau,0) = \bar{x}(\tau) \\ y_s = F_q & y(\tau,0) = \bar{y}(\tau) \\ u_s = pF_p + qF_q & u(\tau,0) = \bar{u}(\tau) \\ p_s = -(F_x + pF_u) & p(\tau,0) = \bar{p}(\tau) \\ q_s = -(F_y + qF_u) & q(\tau,0) = \bar{q}(\tau) \end{array}},$$

where $\bar{p}$ and $\bar{q}$ are determined from the following two conditions

1) $F = 0$ on $C \Rightarrow \boxed{F(\bar{x}, \bar{y}, \bar{u}, \bar{p}, \bar{q}) = 0}$

2) $u_\tau = u_x x_\tau + u_y y_\tau = p x_\tau + q y_\tau$

On $C \Rightarrow \boxed{\bar{u}'(\tau) = \bar{p}(\tau)\bar{x}'(\tau) + \bar{q}(\tau)\bar{y}'(\tau)}$

Notes: 1) The roles of τ and s are the same as before.
$\Rightarrow$ The characteristics of the PDE are the level curves $\tau(x,y) = k$.

2) $\begin{vmatrix} x_\tau & x_s \\ y_\tau & y_s \end{vmatrix} = \begin{vmatrix} x_\tau & F_p \\ y_\tau & F_q \end{vmatrix} = -\begin{vmatrix} F_p & F_q \\ x_\tau & y_\tau \end{vmatrix}$

If F is continuously differentiable with respect to $x, y, u, p,$ and q, then there exists a unique solution of the Cauchy problem in a neighborhood of C if

$$\begin{vmatrix} F_p & F_q \\ x_\tau & y_\tau \end{vmatrix} \neq 0 \text{ on } C \Rightarrow C \text{ is noncharacteristic}$$

3) The above general method of characteristics reduces appropriately if the PDE is quasilinear.

Example: $u_x u_y = 1$, $u(x,1) = 2x^{1/2}, x>0$

$p = u_x$, $q = u_y$ $\Rightarrow$ $F(x,y,u,p,q) = pq - 1$

$F_x = F_y = F_u = 0$

$F_p = q \quad F_q = p$

$x_s = F_p = q$

$y_s = F_q = p =$

$u_s = pF_p + qF_q = pq + qp = 2pq$

$p_s = -(F_x + pF_u) = 0$

$q_s = -(F_y + pF_u) = 0$

C: $x = \bar{x}(\tau) = \tau$, $y = \bar{y}(\tau) = 1$, $u = \bar{u}(\tau) = 2\tau^{1/2}$, $p = \bar{p}(\tau) = \tau^{-1/2}$, $q = \bar{q}(\tau) = \tau^{1/2}$

1) $F(\bar{x},\bar{y},\bar{u},\bar{p},\bar{q}) = \bar{p}\bar{q} - 1 = 0$

$\bar{q} = \dfrac{1}{\bar{p}}$

2) $\bar{u}' = \bar{p}\bar{x}' + \bar{q}\bar{y}'$

$\tau^{-1/2} = \bar{p}\cdot 1 + \bar{q}\cdot 0$

$\bar{p} = \tau^{-1/2} \Rightarrow \bar{q} = \tau^{1/2}$

$p_s = 0,\ p(\tau,0) = \bar{p}(\tau) = \tau^{-1/2}$

$\Rightarrow p = \tau^{-1/2}$

$q_s = 0,\ q(\tau,0) = \bar{q}(\tau) = \tau^{1/2}$

$\Rightarrow q = \tau^{1/2}$

$x_s = q = \tau^{1/2},\ x(\tau,0) = \bar{x}(\tau) = \tau$

$\Rightarrow x = \tau^{1/2}s + \tau$

$y_s = p = \tau^{-1/2},\ y(\tau,0) = \bar{y}(\tau) = 1$

$\Rightarrow y = \tau^{-1/2}s + 1$

$u_s = 2pq = 2,\ u(\tau,0) = \bar{u}(\tau) = 2\tau^{1/2}$

$\Rightarrow u = 2s + 2\tau^{1/2}$

Invert: $\begin{vmatrix} F_p & F_q \\ x_\tau & y_\tau \end{vmatrix} = \begin{vmatrix} q & p \\ 1 & 0 \end{vmatrix} = -p = -\tau^{-1/2}$ $\quad \tau = 0?$ $\Rightarrow x = 0$

$$\begin{aligned} &x = \tau^{1/2}s + \tau \\ -&(\tau y = \tau^{1/2}s + \tau) \\ \hline &x - \tau y = 0 \end{aligned}$$

$\tau = \dfrac{x}{y}$

$s = \tau^{1/2}(y-1) = \left(\dfrac{x}{y}\right)^{1/2}(y-1)$

$s = \left(\dfrac{x}{y}\right)^{1/2}(y-1)$

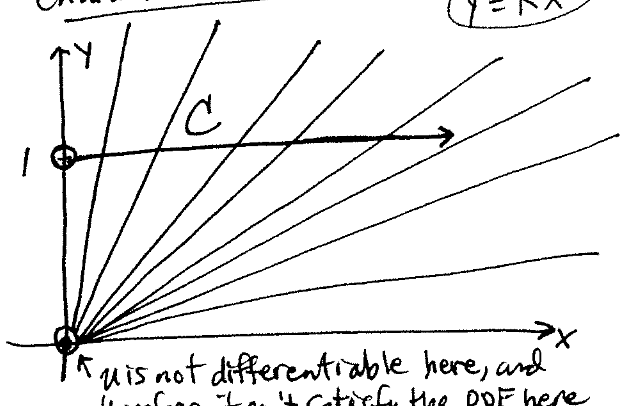
$\Rightarrow u = 2(s + \tau^{1/2}) = 2\left[\left(\dfrac{x}{y}\right)^{1/2}(y \cancel{-1}) + \cancel{\left(\dfrac{x}{y}\right)^{1/2}}\right]$

$= 2\left(\dfrac{x}{y}\right)^{1/2} y = 2(xy)^{1/2}$

$$\boxed{u(x,y) = 2(xy)^{1/2}}$$

Characteristics: $\tau(x,y) = \dfrac{x}{y} = k$

$y = kx$



Similarity Solutions

There are many technique for reducing a PDE to an ODE, or to another PDE with fewer independent variables.

e.g. (i) Integral Transform Methods (e.g. Laplace Transform Method)
(ii) Eigenfunction Expansions
(iii) Similarity Transformations

PDEs may yield solutions in which some or all of the independent variables appear only through a certain grouping, rather than appearing independently.

Example: Consider the 2-D heat equation, $u_t = \kappa(u_{xx}+u_{yy})$.
We can search for solutions in which x and y appear only through the grouping $r(x,y)=\sqrt{x^2+y^2}$

Transform : $(x,y,t) \to (r(x,y),t) = (\sqrt{x^2+y^2},t)$

$$\Rightarrow u(x,y,t) = u(r(x,y),t)$$

With this transformation, the number of independent variables is reduced from 3 to 2.

Then, write the PDE in terms of r and t, rather than in terms of x, y, and t.

$$u_x = u_r r_x$$

$$u_{xx} = u_{rr}r_x^2 + u_r r_{xx} = u_{rr}\frac{x^2}{r^2} + u_r\frac{y^2}{r^3}$$

$$+\quad u_{yy} = u_{rr}r_y^2 + u_r r_{yy} = u_{rr}\frac{y^2}{r^2} + u_r\frac{x^2}{r^3}$$

$$u_{xx}+u_{yy} = u_{rr}\frac{x^2+y^2}{r^2} + u_r\frac{x^2+y^2}{r^3}$$

$$u_{xx}+u_{yy} = u_{rr} + \frac{1}{r}u_r$$

$$\Rightarrow \boxed{u_t = \kappa\left(u_{rr}+\frac{1}{r}u_r\right)}$$

(radially symmetric heat equation.
$\Rightarrow$ Polar form with θ-derivatives set equal to 0)

The PDE now yields only radially symmetric (similarity) solutions.

Notes: 1) Solutions may be lost through a similarity transformation. Only solutions of the assumed form, with the independent variables grouped in a specified way, can be found. In the above example, the similarity transformation yields only radially symmetric solutions. The similarity transformation works well for those solutions, but no others.

2) A similarity transformation may limit the variety of ICs and BCs that can be satisfied

Example: $u_t + a u_x = 0$, $a =$ real

Search for solutions of the form $u(x,t) = u(\xi(x,t))$, where $\xi(x,t) = x - at$

$$u_t = u_\xi \xi_t = -a u_\xi \qquad u_t + a u_x = 0$$
$$u_x = u_\xi \xi_x = u_\xi \qquad -a u_\xi + a u_\xi = 0$$
$$0 = 0 \Rightarrow$$ All differentiable functions of ξ satisfy the PDE.

$$\Rightarrow u = f(\xi) = f(x-at)$$
$$u(x,t) = f(x-at)$$
for any differentiable function f.

Check: $u_t + a u_x = 0$
$$-af' + af' = 0 \checkmark$$

Note: No solutions are lost here.

To see this, we can do a full coordinate transformation, $(x,y) \to (\xi, \eta)$.

Let $\xi = \xi(x,y) = x - at$

$\eta = \eta(x,y) =$ any function such that

$$\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_t \end{vmatrix} = \begin{vmatrix} 1 & -a \\ \eta_x & \eta_t \end{vmatrix} = \eta_t + a\eta_x \neq 0$$ for all x and t.

$\Rightarrow$ The coordinate transformation is invertible for all x and t.

then,
$$u_t = u_\xi \xi_t + u_\eta \eta_t = -a u_\xi + u_\eta \eta_t$$
$$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta \eta_x$$

$$u_t + a u_x = 0 \Rightarrow (\cancel{-a u_\xi} + u_\eta \eta_t) + a(\cancel{u_\xi} + u_\eta \eta_x) = 0$$
$$u_\eta (\underbrace{\eta_t + a\eta_x}_{\neq 0}) = 0$$
$$u_\eta = 0 \Rightarrow$$ u does not depend on any variable, other than ξ.

Integrate: $u = A(\xi)$

$$\Rightarrow \boxed{u(x,t) = f(x-ct)}$$ is the general solution

Note: The solution $u(x,t)$ of $u_t + a u_x = 0$ depends only on the variable $\xi = x - ct$, and no others. Furthermore, u's dependence on $\xi = x - ct$ is arbitray.

The function f is determined by given initial conditions.

e.g. $u(x,0) = x^2$

$u(x,0) = f(x) = x^2$

$\Rightarrow$

$$\boxed{u(x,t) = f(x-ct) = (x-ct)^2}$$

Similarity variables often have the form $\xi(x_1, \dots, x_n) = x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$.

Classical Example: Consider the 1-D heat equation, $u_t = \kappa u_{xx}$

Search for solutions of the form $u(x,t) = u(\xi(x,t))$, where $\boxed{\xi = x^\alpha t^\beta;\ \alpha, \beta \neq 0}$

$$u_t = u_\xi \xi_t = \beta x^\alpha t^{\beta-1} u_\xi = \beta \frac{\xi}{t} u_\xi$$

$$u_x = u_\xi \xi_x$$

$$u_{xx} = u_{\xi\xi}\xi_x^2 + u_\xi \xi_{xx} = u_{\xi\xi}\alpha^2 x^{2\alpha-2}t^{2\beta} + u_\xi \alpha(\alpha-1)x^{\alpha-2}t^\beta$$

$$= u_{\xi\xi}\alpha^2\frac{\xi^2}{x^2} + u_\xi \alpha(\alpha-1)\frac{\xi}{x^2}$$

$$u_t = \kappa u_{xx} \Rightarrow \beta \frac{\xi}{t} u_\xi = \kappa\left[\alpha^2 \frac{\xi^2}{x^2}u_{\xi\xi} + \alpha(\alpha-1)\frac{\xi}{x^2}u_\xi\right]$$

$$\boxed{\beta \xi \frac{x^2}{t} u_\xi = \kappa\left[\alpha^2\xi^2 u_{\xi\xi} + \alpha\underline{(\alpha-1)}\xi u_\xi\right]}$$

This is the only piece that is not in terms of ξ. It must be a power of ξ if the similarity transformation is to be successful.

$$\frac{x^2}{t} = \xi^p = (x^\alpha t^\beta)^p$$

x: $2 = \alpha p$ $\Rightarrow$

t: $-1 = \beta p$

divide $-2 = \dfrac{\alpha}{\beta}$

$$\boxed{\beta = -\frac{\alpha}{2}}$$

The PDE can be written in terms of ξ alone if $\boxed{\beta = -\frac{\alpha}{2}}$.

Since there is only one constraint on α and β, either α or β can be chosen, while the other follows from the constraint.

Pick $\underline{\alpha = 1}$ to kill this term.

$$\boxed{\alpha = 1} \Rightarrow \boxed{\beta = -\frac{1}{2}} \Rightarrow \boxed{\xi = \frac{x}{\sqrt{t}}}$$

$$-\frac{1}{2}\xi \cdot \cancel{\xi^2} u_\xi = \kappa\left[\cancel{\xi^2} u_{\xi\xi} + 0\right]$$

$$\boxed{u_{\xi\xi} + \frac{\xi}{2\kappa}u_\xi = 0}, \text{ where } \boxed{\xi = \frac{x}{\sqrt{t}}}$$

Linear second order ODE, but with variable coefficients.

We have $u_{\xi\xi} + \frac{\xi}{2\kappa} u_\xi = 0$. (separable 1st order ODE for u_ξ)

$$\frac{d(u_\xi)}{d\xi} = -\frac{\xi}{2\kappa} u_\xi$$

$$\int \frac{d(u_\xi)}{u_\xi} = \frac{-1}{2\kappa}\int \xi \, d\xi$$

$$\ln|u_\xi| = \frac{1}{2\kappa}\frac{\xi^2}{2} + c \Rightarrow u_\xi = Ce^{-\xi^2/4\kappa}$$

Integrate from 0 to ξ.

$$\int_0^\xi u_s(s)\,ds = C\int_0^\xi e^{-s^2/4\kappa}\,ds \qquad t = \frac{s}{2\sqrt{\kappa}}, \quad ds = 2\sqrt{\kappa}\,dt$$

$$u(\xi) - u(0) = C\int_0^{\xi/2\sqrt{\kappa}} e^{-t^2}\cdot 2\sqrt{\kappa}\,dt$$

$$u(\xi) = u(0) + C_1 \cdot \frac{2}{\sqrt{\pi}}\int_0^{\xi/2\sqrt{\kappa}} e^{-t^2}\,dt$$

$$u(\xi) = C_0 + C_1 \operatorname{erf}\left(\xi/2\sqrt{\kappa}\right)$$

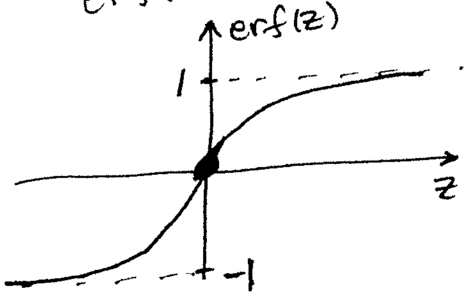
$$\Rightarrow \boxed{u(x,t) = C_0 + C_1 \operatorname{erf}\left(\frac{x}{2\sqrt{\kappa t}}\right)}$$

Error Function

$$\boxed{\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt}$$

$\operatorname{erf}(\pm\infty) = \pm 1$

$\operatorname{erf}(z)$ is odd



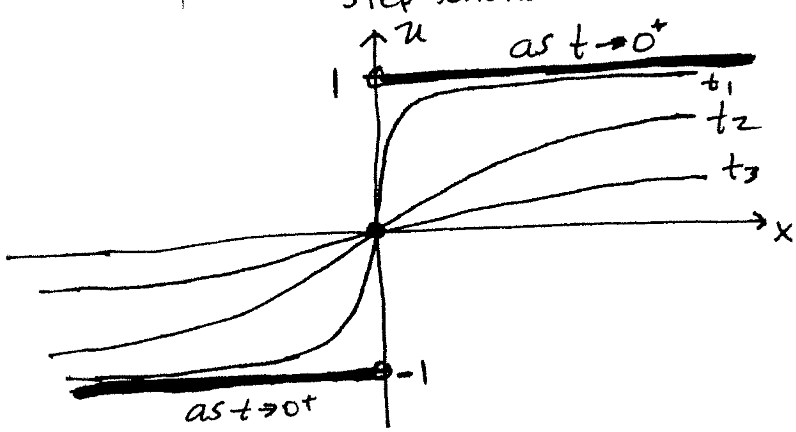
Let $C_0 = 0$ and $C_1 = 1$. $\Rightarrow u(x,t) = \operatorname{erf}\left(\frac{x}{2\sqrt{\kappa t}}\right)$

Note: As $t \to 0^+$, $\frac{x}{2\sqrt{\kappa t}} \to$ $+\infty$ for $x>0$, $-\infty$ for $x<0$.

$\Rightarrow u(x,t) \to \operatorname{erf}(\infty) = 1$ for $x>0$, $\to \operatorname{erf}(-\infty) = -1$ for $x<0$ as $t \to 0^+$

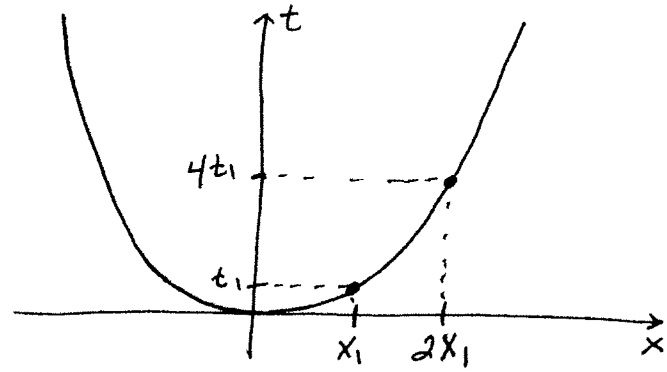
$\Rightarrow u(x,t) \to$ a step function as $t \to 0^+$

$\Rightarrow$ The similarity solution corresponds to a step function initial condition.

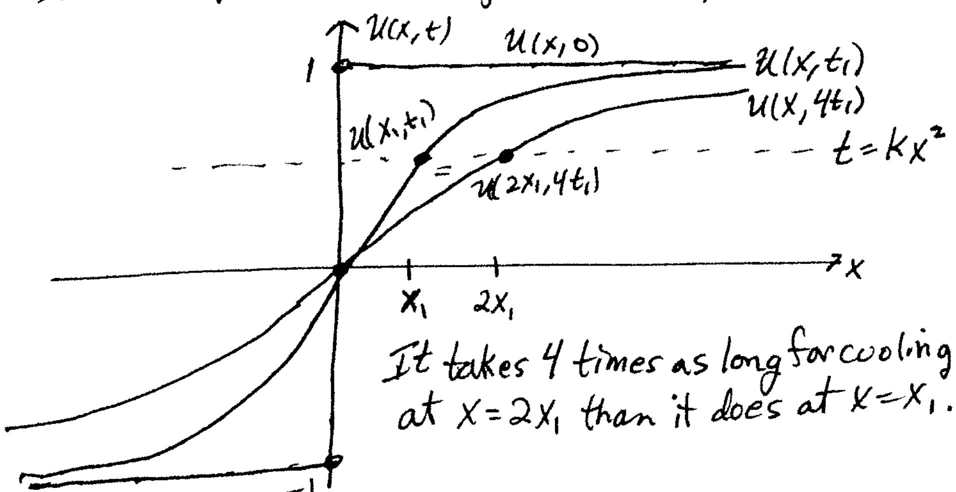


The similarity variable $\xi = x/\sqrt{t}$ reveals the relationship between the space scale and the time scale associated with (heat) diffusion.

Observe that $u(x,t) = \operatorname{erf}\left(\frac{x}{2\sqrt{\kappa t}}\right)$ is constant along the level curves $\xi(x,y) = x/\sqrt{t} = C$. $\Rightarrow$ $t = kx^2,\ k>0$



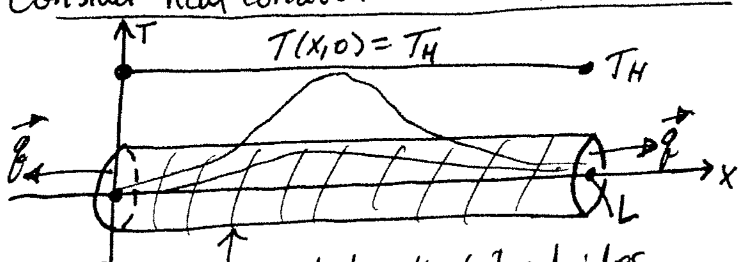
~~If the space variable x gets stretched by a~~



It takes 4 times as long for cooling to occur at $x = 2x_1$ than it does at $x = x_1$.

$$\Rightarrow \boxed{u(x,t) = u(\lambda x, \lambda^2 t)}$$

Consider heat conduction in a metal rod



Insulated along the lateral sides, but not at the ends.

If the length of the rod is doubled to $2L$, it will take 4 times as long to cool. i.e. the peak temperature at the center of the rod will decay 4 times as slowly.

Conservation Laws

Scalar: one equation/one unknown

1-D: $\boxed{u_t + (f(u))_x = 0}$, $u = u(x,t)$

u is the conserved quantity: $\int_{-\infty}^{\infty} u(x,t)\,dx$ is constant with respect to t

$f(u)$ is the flux function

Chain Rule: $u_t + f'(u)\,u_x = 0$ (quasilinear)

2-D $u_t + (f(u))_x + (g(u))_y = 0$, $u = u(x,y,t)$

n-D: $u_t + \sum_{i=1}^{n} (f_i(u))_{x_i} = 0$, $u = u(\vec{x},t) : \mathbb{R}^{n+1} \to \mathbb{R}$
$\vec{x} = \langle x_1, \dots, x_n \rangle$

u conserved $\Rightarrow \int_{\mathbb{R}^n} u(\vec{x},t)\,d\vec{x}^n$ is constant with respect to t.

Systems: m equations/m unknowns

Same as above, except that u, 0, and the flux functions are m-dimensional vectors: $\vec{u} = \langle u_1, \dots, u_m \rangle$ $\vec{u}(\vec{x},t) : \mathbb{R}^{n+1} \to \mathbb{R}^m$

$\vec{f} = \langle f_1, \dots, f_m \rangle$ $\vec{f}(\vec{u}) : \mathbb{R}^m \to \mathbb{R}^m$

e.g. **1-D** $\boxed{\vec{u}_t + (\vec{f}(\vec{u}))_x = \vec{0}}$

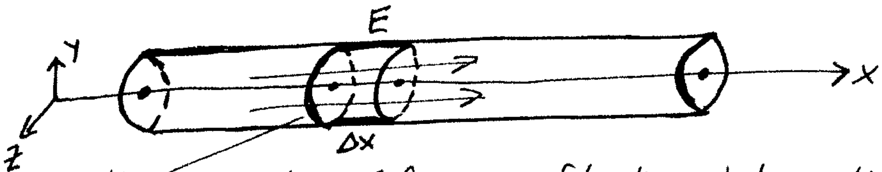
Chain Rule: $\vec{u}_t + \underbrace{\vec{f}'(\vec{u})}_{} \vec{u}_x = \vec{0}$

Jacobian: $\vec{f}'(\vec{u}) = \frac{\partial \vec{f}}{\partial \vec{u}} = \frac{\partial(f_1, \dots, f_m)}{\partial(u_1, \dots, u_m)}$

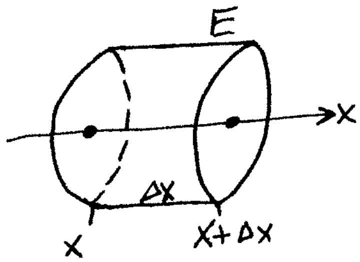
2-D
n-D Same as above, but vector equations.

Conservation of Mass

- Consider a fluid flowing through a pipe lying along the x-axis. Assume that the fluid properties are uniform throughout any cross-section of the pipe. That is, assume that the fluid density ρ and velocity v vary only in the x-direction (and in time).



Consider the flow through a section E of the pipe of length Δx between times t and $t+\Delta t$



$\rho(x,t) =$ density (kg/m) $\quad \rho v =$ flux (rate of flow) (kg/s)

$v(x,t) =$ Velocity (m/s)

(net change in mass of E from time t to $t+\Delta t$) = (net inflow of mass through the boundary of E between times t and $t+\Delta t$)

$m(t+\Delta t) - m(t) =$ (net inflow of mass at x) $-$ (net outflow of mass at $x+\Delta x$) during the time interval $(t, t+\Delta t)$

- $$\int_x^{x+\Delta x} \rho(x,t+\Delta t)\,dx - \int_x^{x+\Delta x} \rho(x,t)\,dx = \int_t^{t+\Delta t} \rho(x,t)v(x,t)\,dt - \int_t^{t+\Delta t} \rho(x+\Delta x,t)v(x+\Delta x,t)\,dt$$

$$\times \frac{1}{\Delta x \Delta t} \Rightarrow \quad \frac{\frac{1}{\Delta x}\int_x^{x+\Delta x} \rho(x,t+\Delta t)\,dx - \frac{1}{\Delta x}\int_x^{x+\Delta x} \rho(x,t)\,dx}{\Delta t} = \frac{\frac{1}{\Delta t}\int_t^{t+\Delta t} \rho(x,t)v(x,t)\,dt - \frac{1}{\Delta t}\int_t^{t+\Delta t} \rho(x+\Delta x,t)v(x+\Delta x,t)\,dt}{\Delta x}$$

Let $\Delta t, \Delta x \to 0 \Rightarrow$

$$\lim_{\Delta t\to 0} \frac{\rho(x,t+\Delta t) - \rho(x,t)}{\Delta t} = \lim_{\Delta x \to 0} \frac{\rho(x,t)v(x,t) - \rho(x+\Delta x,t)v(x+\Delta x,t)}{\Delta x}$$

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho v)$$

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0}$$

Recall:

$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$

$f_{ave} \to f(a)$ as $b \to a$

a ← b

- $\rho_{ave} = \frac{1}{\Delta x}\int_x^{x+\Delta x} \rho(x,t)\,dx$ (time $= t$) $\longrightarrow \rho(x,t)$ as $\Delta x \to 0$

Note: There are two quantities (ρ and v) to be determined here. A single equation is not sufficient for doing so. To obtain a closed system of PDEs, we must also consider conservation of momentum and energy.

Euler Equations of Gas Dynamics

The Euler equations are a system of three PDEs which determine the density, velocity, and pressure of a gas. The Euler equations are a reduction of the full set of Navier-Stokes equations of fluid dynamics. For a gas, it is reasonable to neglect the effects of visocity and diffusion (thermal and mass). The equations of momentum and energy conservation are ? derived in a similar way as the equation of mass conservation.

$\rho(x,t)$ = density
$v(x,t)$ = velocity
$P(x,t)$ = pressure

$$\boxed{\begin{aligned} &\rho_t + (\rho v)_x = 0 && \text{conservation of mass} \\ &(\rho v)_t + (\rho v^2 + P)_x = 0 && \text{conservation of momentum} \\ &(\rho E)_t + (v(\rho E + P))_x = 0 && \text{conservation of energy} \end{aligned}}$$

where $E(\rho, v, P) = \underset{\substack{\uparrow \\ \text{internal energy}}}{e(\rho, P)} + \underset{\substack{\uparrow \\ \text{kinetic energy}}}{v^2/2}$ is the gas energy

For an ideal gas,

$$e(\rho, P) = \frac{P}{\rho(\gamma - 1)}, \quad \gamma \approx 1.4 \rightarrow \underline{\text{Equation of State}}$$

$$\Rightarrow \boxed{E = \frac{P}{\rho(\gamma-1)} + \frac{v^2}{2}}$$

$$e = \frac{P}{\rho(\gamma-1)}, \quad \rho = \frac{m}{V}$$

$$e = \frac{P}{m/V(\gamma-1)}$$

$$PV = \underbrace{(\gamma-1)me}_{nRT}$$

$$PV = nRT$$

The Euler equations can be written in the form of a conservation law, $\vec{u}_t + (\vec{f}(\vec{u}))_x = \vec{0}$,

where $\vec{u} = \begin{pmatrix} \rho \\ \rho v \\ \rho E \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ and $\vec{f}(\vec{u}) = \vec{f}(u_1, u_2, u_3) = ?$

<u>homework</u>

Conservation of Mass in 3-D

Recall from multivariable calculus

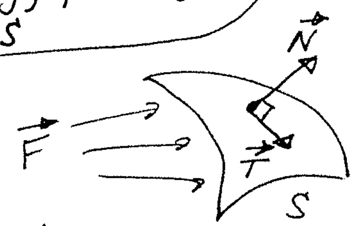
Given a vector field $\vec{F}$ and a surface S,

$$\text{Flux of } \vec{F} \text{ across } S = \iint_S \vec{F}\cdot\vec{N}\,dS$$

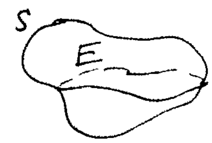
$\vec{N}$ = unit normal to S

(positively oriented
open surface ⇒ upward
closed surface ⇒ outward)

(rate of flow through S)



Gauss' Divergence Theorem



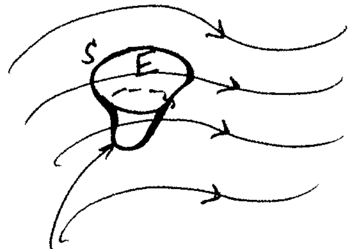
Let E be a region with boundary surface S.

$$\iint_S \vec{F}\cdot\vec{N}\,dS = \iiint_E \nabla\cdot\vec{F}\,dV$$

$\vec{N}$ = outward unit normal to S

Conservation of Mass:

Mass flow: $\vec{F} = \rho\vec{V}$ $\left(\frac{kg}{m^2 s}\right)$ = rate of flow per unit area



E = arbitrary region
S = boundary of E

(rate of change of mass of E) = (rate of mass inflow through the boundary of E)
= − (outward flux of $\vec{F} = \rho\vec{V}$ across S)

$$\iiint_E \frac{\partial\rho}{\partial t}\,dV = -\oint\!\!\oint_S \vec{F}\cdot\vec{N}\,dS \quad (kg/s)$$

$$= -\iiint_E \nabla\cdot\vec{F}\,dV$$

$$= -\iiint_E \nabla\cdot(\rho\vec{V})\,dV$$

$$\Rightarrow \iiint_E \left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V})\right)dV = 0$$

The fact that the definite integral is equal to zero does not imply that the integrand is zero. However, E is an arbitrary region, so the integral is zero over any region, which then does imply that the integrand is equal to zero ⇒

$$\boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\vec{V}) = 0}$$

Scalar Conservation Laws (1-D)

$$\boxed{u_t + (f(u))_x = 0}$$

Method of Characteristics: $u_t + f'(u)\,u_x = 0$

$t_s = 1 \quad x_s = f'(u) \quad u_s = 0 \quad \Rightarrow$ u is constant along the characteristics

---

Linear Case: The coefficient $f'(u)$ of u_x does not depend on u.

$$\Rightarrow f'(u) = a \quad \Rightarrow f(u) = au + b$$

$$\Rightarrow \boxed{u_t + a\,u_x = 0}$$

$$t_s = 1 \quad x_s = a \quad u_s = 0$$

Along the characteristics ($d\tau = 0$),

$$\frac{dx}{dt} = \frac{\cancel{x_\tau d\tau} + x_s\,ds}{\cancel{t_\tau d\tau} + t_s\,ds} = \frac{x_s}{t_s} = \frac{a}{1} = a.$$

The characteristics are the solution curves of the ODE $\frac{dx}{dt} = a$.

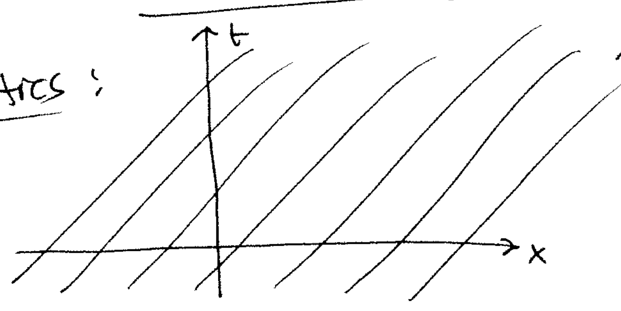
$$\Rightarrow \text{Characteristics: } x - at = x_0, \quad x_0 = \text{constant}$$

On the characteristics, $x = x(t) = at + x_0$ and $u = u(x(t), t)$.

$$\Rightarrow \frac{d}{dt} u(x(t),t) = u_x x'(t) + u_t = u_t + a u_x = 0$$

$\Rightarrow$ u is constant along the characteristics.

Characteristics:



$x - at = x_0 \Rightarrow t = \frac{x - x_0}{a}$

$\text{Slope} = \frac{dt}{dx} = \frac{1}{a}$

Consider the Cauchy problem

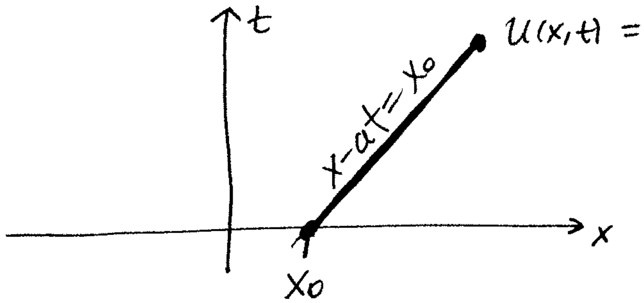
$$\begin{cases} u_t + a\,u_x = 0 \\ u(x,0) = u_0(x) \end{cases}$$

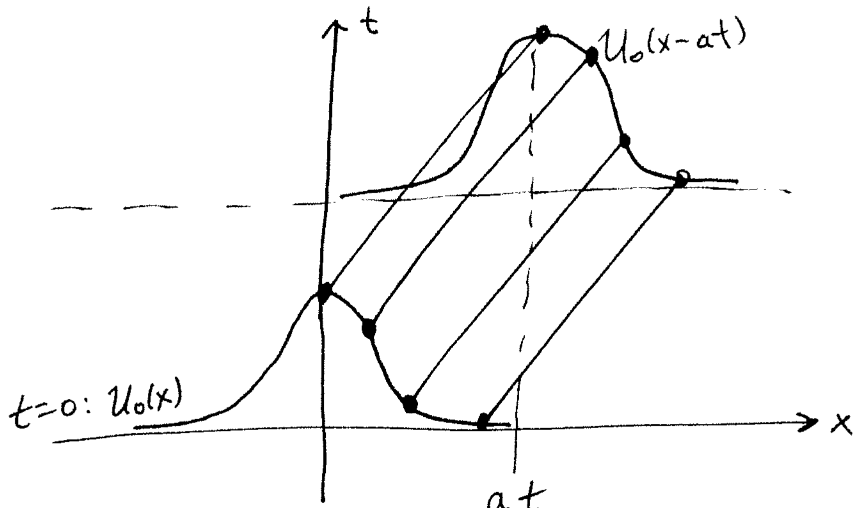
Solution: $\boxed{u(x,t) = u_0(x-at)}$

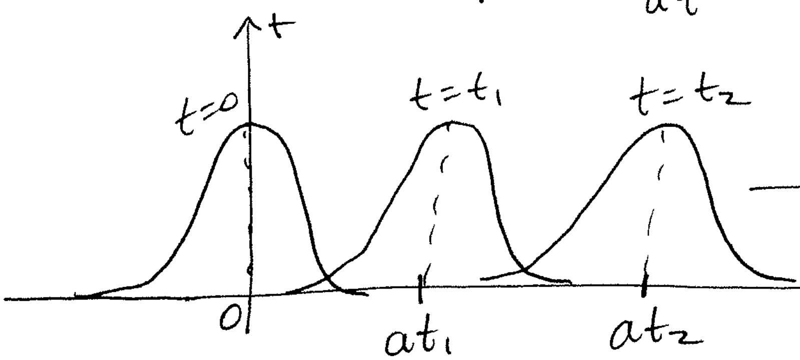
The solution is constant along the characteristics $x - at = x_0$

$$u(x,t) = u_0(x-at) = u_0(x_0)$$

Along the characteristic $x-at=x_0$, u is equal to the initial function u_0 evaluated at x_0.

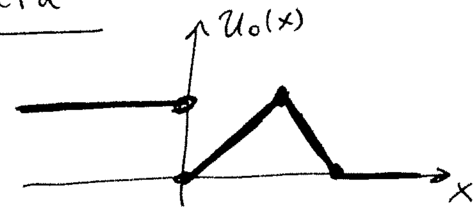






$\longrightarrow$ speed $= \dfrac{dx}{dt} = a$

Nonsmooth Initial Data



Consider $\quad u_t + a u_x = 0$

$\qquad\qquad\quad u(x,0) = u_0(x)$

The solution u at a point (x,t) depends on only one value of the initial data $u_0(x_0)$. Consequently, smoothness of $u_0(x)$ is not required to construct a solution. However, the solution does not satisfy the "differential form" of the conservation law at points where u or u_x are discontinuous.
(is not differentiable)

Note: u is smooth along the characteristics which pass through the smooth portions of the initial data. A fundamental property of linear hyperbolic equations is that singularities propagate only along the characteristics.

If $u_0(x)$ is not differentiable at some point, then $u(x,t)$ is not a solution of the initial value problem in the classical sense.

Solutions of the conservation law $\vec{u}_t + (\vec{f}(\vec{u}))_x = \vec{0}$ are redefined as follows. "differential form"

A function $u(x,t)$ is said to be a solution of the "differential form" if it satisfies the "integral form" of the conservation law,

$$\int_{x_1}^{x_2}\int_{t_1}^{t_2} (\vec{u}_t + \vec{f}(\vec{u})_x)\, dt\, dx = 0 \quad \text{for all } x_1, x_2, t_1, \text{and } t_2 \text{ in the domain}$$

$$\Rightarrow \boxed{\int_{x_1}^{x_2} \left(\vec{u}(t_2,x) - \vec{u}(t_1,x)\right) dx + \int_{t_1}^{t_2} \left(\vec{f}(\vec{u}(x_2,t)) - \vec{f}(\vec{u}(x_1,t))\right) dt = 0} \quad \text{for all } x_1, x_2, t_1, \text{and } t_2 \text{ in the domain.}$$

Notes:
1) Unlike derivatives, integrals are well-defined for nondifferentiable functions.
2) If $u(x,t)$ satisfies the integral form, then $u(x,t)$ satisfies the differentiable form at all points where u is smooth.
3) A nonsmooth solution is called a weak solution.

Review

1-D Conservation Laws: $\vec{u}_t + (\vec{f}(\vec{u}))_x = \vec{0}$, typically $t > 0$

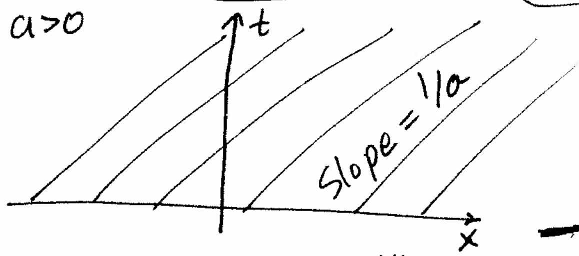
$f(\vec{u})$ = flux function

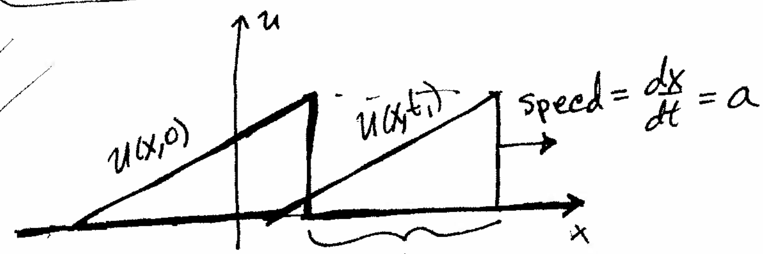
Typical IC: $\vec{u}(x,0) = \vec{u}_0(x)$

Linear Scalar Case: $f(u) = au \Rightarrow (f(u))_x = au_x$

$$\Rightarrow \boxed{u_t + au_x = 0}$$

Characteristics: $x - at = x_0$ OR $t = \dfrac{x - x_0}{a}$





a = "wave speed" (velocity)

$a > 0 \Rightarrow$ right traveling

$a < 0 \Rightarrow$ left traveling

u is constant along the characteristics.

Nonsmooth function are considered to be solutions if they satisfy the integral form of the conservation law. $\int_{x_1}^{x_2}\int_{t_1}^{t_2} \left(\vec{u}_t + \vec{f}(\vec{u})_x\right) dt\, dx$ for all x_1, x_2, t_1, and t_2 in the domain.

$$\Rightarrow \int_{x_1}^{x_2} \left(\vec{u}(x,t_2) - \vec{u}(x,t_1)\right) dx + \int_{t_1}^{t_2} \left(f(\vec{u}(x_2,t)) - f(\vec{u}(x_1,t))\right) dt = 0.$$

Such a solution satisfies the PDE at all points at which it is smooth. Nonsmooth solutions that satisfy the integral form, but not the differential form at all points, are called <u>weak solutions</u>. (Informal Definition)

<u>Note</u>: Given a linear conservation law with nonsmooth initial data, it is straightforward to construct a unique weak solution since the initial data simply travels with velocity a. Smooth initial data remains smooth as time evolves, and discontinuities in u or its derivatives propagate along characteristics.

On the other hand, non-smooth solutions of nonlinear conservation laws are <u>not</u> unique. Furthermore, smooth initial data may lead to non-smooth (non-unique) weak solutions, and continuous solutions may exists for discontinuous initial data.

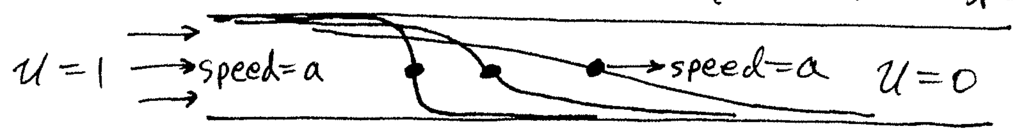
Convection-Diffusion Equation

$$\boxed{u_t + a u_x = \kappa u_{xx}}$$

convection term — diffusion term

The convection-diffusion equation describes diffusion within a media which travels with speed a.

e.g. Hot gas is blown into a stream-tube initially filled with cool gas. (which travels at speed a)

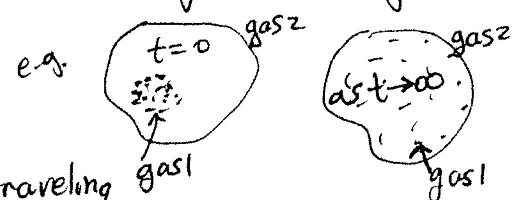


u = gas velocity

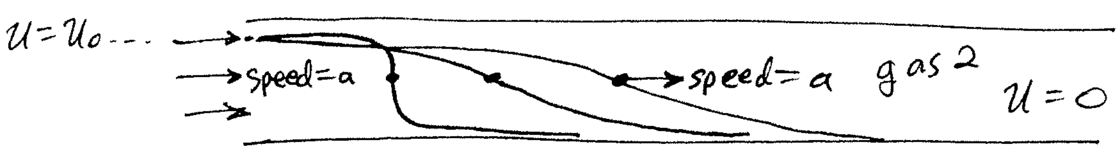
Heat diffuses as usual, but in a reference that travels with the gas at a speed a.

e.g. Two-component mixture of gas with mass diffusion of gas 1 within gas 2.

u = concentration of gas 1
(mass fraction)
(proportion)

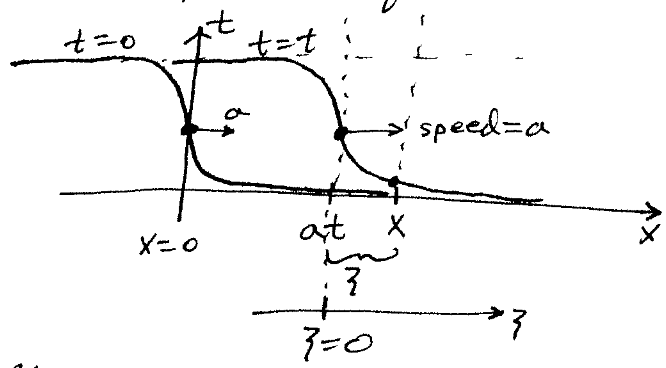


The stream-tube is initially filled with gas 2, traveling at speed a. For $t > 0$, a mixture of gas 1 and gas 2, with $u = u_0$, $0 < u_0 \le 1$ is blown into the tube.



The convection-diffusion equation is simply the heat equation in a moving coordinate system.

- The convection-diffusion equation is simply the heat equation in a moving coordinate system.



Coordinate Transformation: $(x,t) \to (\xi, \tau)$

$$\boxed{\begin{aligned}\tau &= t \\ \xi &= x - at\end{aligned}}$$

$u_t = u_\tau \tau_t + u_\xi \xi_t = u_\tau - a u_\xi$

$u_x = \cancel{u_\tau \tau_x} + u_\xi \xi_x = u_\xi$

$u_{xx} = u_{\xi\xi}$

$$u_t + a u_x = \kappa u_{xx} \quad \text{(C-D)}$$

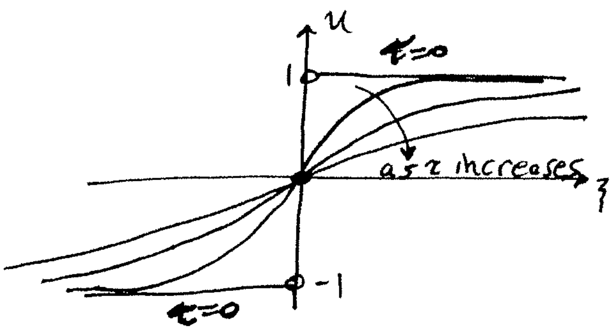
$$(u_\tau - \cancel{a u_\xi}) + \cancel{a u_\xi} = \kappa u_{\xi\xi}$$

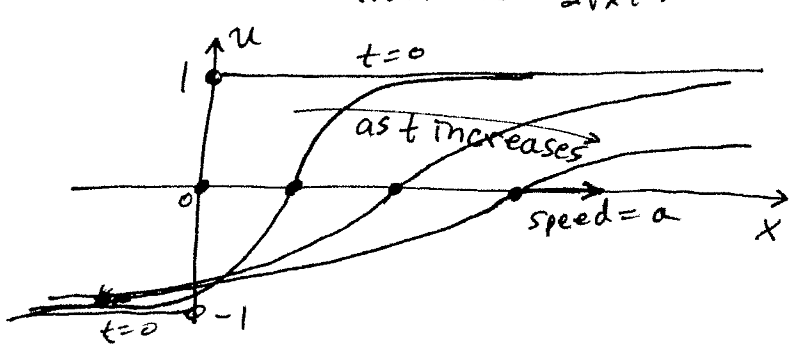
$$\boxed{u_\tau = \kappa u_{\xi\xi}} \quad \text{(D)}$$

- If $u(\xi,\tau)$ is a solution of (D), then $u(x-at,t)$ is a solution of (C-D).

Example: It was shown that $u_1(\xi,\tau) = \mathrm{erf}\left(\frac{\xi}{2\sqrt{\kappa\tau}}\right)$ is a similarity solution of $u_\tau = \kappa u_{\xi\xi}$. Therefore,

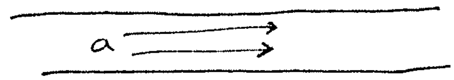
$u(x,t) = u_1(x-at,t) = \mathrm{erf}\left(\frac{x-at}{2\sqrt{\kappa t}}\right)$ is a similarity solution of $u_t + a u_x = \kappa u_{xx}$. $\left(\text{similarity variable} = \frac{x-at}{2\sqrt{\kappa t}}\right)$





- The same ideas can be applied to other solutions of the heat equation, as well as to other evolution equations that have a convective term of the form $a u_x$.

Consider a media (e.g. fluid) that travels with a speed a.



Material Derivative (or convective derivative): $\boxed{\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + a\dfrac{\partial}{\partial x}}$ Time derivative in the coordinate system attached to the motion of the fluid

e.g. The convection-diffusion equation becomes $\dfrac{Du}{Dt} = \varkappa u_{xx}$.

More generally, if the fluid velocity is given by

$$\vec{v} = \vec{v}(x,y,z) = \langle v_1, v_2, v_3 \rangle,$$

then

$$\boxed{\frac{D}{Dt} = \frac{\partial}{\partial t} + v_1\frac{\partial}{\partial x} + v_2\frac{\partial}{\partial y} + v_3\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla}$$

Example: Conservation of Mass

$$\rho_t + \nabla\cdot(\rho\vec{v}) = 0$$

$$\rho_t + \rho\nabla\cdot\vec{v} + \nabla\rho\cdot\vec{v} = 0$$

$$(\rho_t + \vec{v}\cdot\nabla\rho) + \rho\nabla\cdot\vec{v} = 0$$

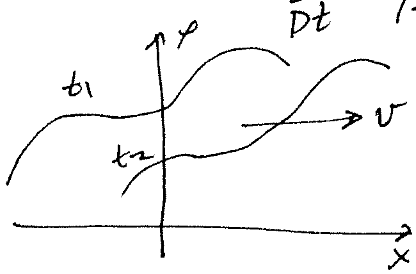
$$\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{v} = 0$$

Incompressible fluid: $\nabla\cdot\vec{v} = 0$

$$\Rightarrow \frac{D\rho}{Dt} = 0$$

1-D case: Incompressible $\Rightarrow v_x = 0 \Rightarrow v$ is constant in space (All points of the solution move at the same speed for a fixed time)

$$\frac{D\rho}{Dt} = \rho_t + v\rho_x = 0 \text{ (linear conservation law)}$$



The density profile retains its shape as it travels.

Nonlinear Scalar Conservation Laws

Solutions of the linear transport equation ($u_t + a u_x = 0$) are relatively uninteresting. Initial Data merely propagates with speed a. Discontinuities in the solution exist only if the initial data has discontinuities. Such discontinuities propagate along characteristics.

Nonlinear conservation laws exhibit much richer and more interesting behavior. For example, discontinuities (shocks) may form from smooth initial data. Nonuniqueness of solutions becomes an issue as well.

General Form: $\boxed{u_t + (f(u))_x = 0}$ Typically, $t > 0$.

OR $u_t + f'(u)\, u_x = 0$

We'll consider initial data of the form $\boxed{u(x,0) = u_0(x)}$

Method of Characteristics: C: $x(\tau,0) = \tau$
$t(\tau,0) = 0$
$u(\tau,0) = u_0(\tau)$

$t_s = 1$ $\quad$ $x_s = f'(u)$ $\quad$ $u_s = 0$

$t = s$ $\quad$ $x = f'(u)s + \tau$ $\quad$ $u = u_0(\tau)$

$\Rightarrow \tau = x - f'(u)t \Rightarrow$ Implicit Solution $\boxed{u = u_0(x - f'(u)t)}$

Implicit Function Theorem: $F(x,t,u) = u - u_0(x - f'(u)t)$
We can solve for u in regions where $\frac{\partial F}{\partial u} \neq 0$

Characteristics: $\tau = \text{constant} = x_0$
$\Rightarrow x = x_0 + f'(u)t$

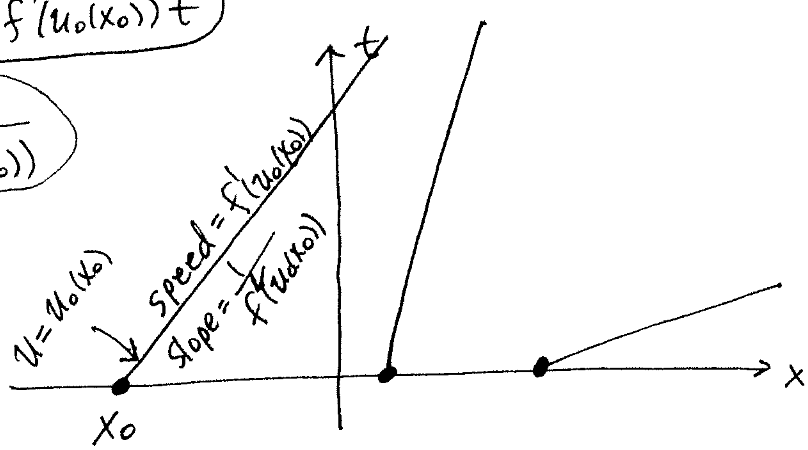
u is constant along the characteristics
$u = u_0(\tau) = u_0(x - f'(u)t) = u_0(x_0)$

$x = x_0 + f'(u_0(x_0))\, t$

OR $t = \dfrac{x - x_0}{f'(u_0(x_0))}$

$\text{speed} = \dfrac{dx}{dt} = f'(u_0(x_0))$

The speed (slope) of the characteristics varies from point to point.



Burger's Equation

- Burger's equation is the simplest nonlinear conservation law. It is convenient to consider Burger's equation in investigating the fundamental aspects of the solution behavior of nonlinear conservation laws.

"Inviscid" Burger's Equation: $f(u) = u^2/2 \Rightarrow (f(u))_x = u u_x$

$$\Rightarrow \boxed{u_t + u u_x = 0}$$

(quasilinear, but still nonlinear)

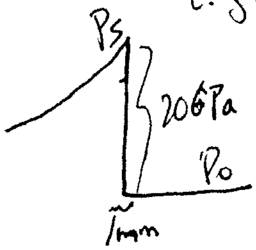
The conservation laws of gas dynamics are based on a reduction of the Navier-Stokes equations in the limit of vanishing viscosity.
Burger actually studied the equation with a viscosity term included.

Viscous Burger Equation: $$\boxed{u_t + u u_x = \varepsilon u_{xx}}$$

viscosity term (diffusive)
(This term has a smoothing effect. It smooths discontinuities)

- The Viscous Burger equation unique smooth solutions, whereas the Inviscid Burger equation may have discontinuous and nonunique solutions. The fact that the solutions of the Inviscid Burger equation may be nonsmooth and nonunique is a consequence of neglecting viscosity. Physically, solutions are smooth, though they may involve very large gradients. With viscosity neglected, these large gradients are modeled by discontinuities. This modeling choice leads to nonunique solutions. One way to select the correct physically relevant solution of Inviscid Burger equation is to consider the Viscous Burger equation in the limit of vanishing viscosity ($\varepsilon \to 0$). That is, find a unique smooth solution of the Viscous Burger equation and let $\varepsilon \to 0$. This approach yields the physically relevant solution of the Inviscid Burger equation.

- e.g. Explosion involve a shock wave. The pressure may increase greatly across the shock. It is feasible to see a 20 GPa increase in pressure over the distance of 1 mm.



Consider $u_t + u u_x = 0,\ t > 0$ $\quad f(u) = u^2/2$

$u(x,0) = u_0(x)$ $\quad f'(u) = u$

Implicit Solution: $u = u_0(x - f'(u)t)$

$$\boxed{u = u_0(x - ut)}$$

$\boxed{\text{speed} = \frac{dx}{dt} = u_0(x_0) = u}$

The speed of the characteristic passing through a point (x,t) is equal to the value of u at that point.

Characteristics: $x = x_0 + f'(u_0(x_0))t$

$$\boxed{\begin{array}{c} x = x_0 + u_0(x_0)t \\ \text{OR} \\ x = x_0 + ut \end{array}} \quad \text{OR} \quad \boxed{t = \frac{x - x_0}{u_0(x_0)} = \frac{x - x_0}{u}}$$

Along the characteristics, $\boxed{u = u_0(x - ut) = u_0(x_0)}$

Example: $u_t + u u_x = 0$

$u(x,0) = u_0(x) = 2x$

Solution: $u = u_0(x - ut)$

$u = 2(x - ut)$

$u(1 + 2t) = 2x$

$$\boxed{u = \frac{2x}{1 + 2t}}$$

Characteristics:

$\boxed{\text{Speed} = \frac{dx}{dt} = u_0(x_0) = 2x_0}$

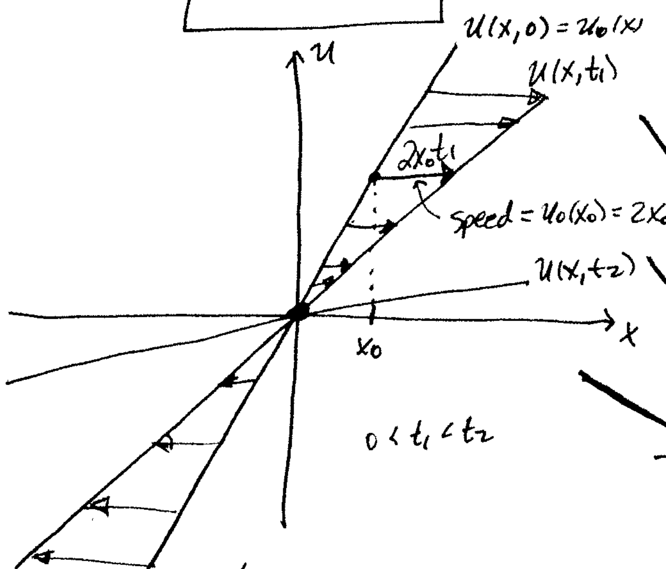
$x = x_0 + u_0(x_0)t$

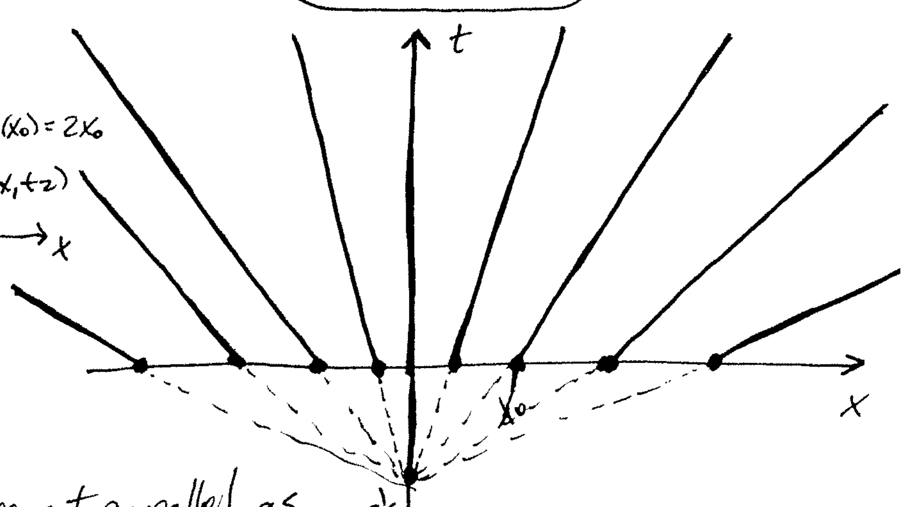
$x = x_0 + 2x_0 t$

$\boxed{x = x_0(1 + 2t)}$ OR $\boxed{t = \frac{1}{2}\left(\frac{x}{x_0} - 1\right)}$

Along the characteristic with x intercept x_0,

$\boxed{u = u_0(x_0) = 2x_0}$

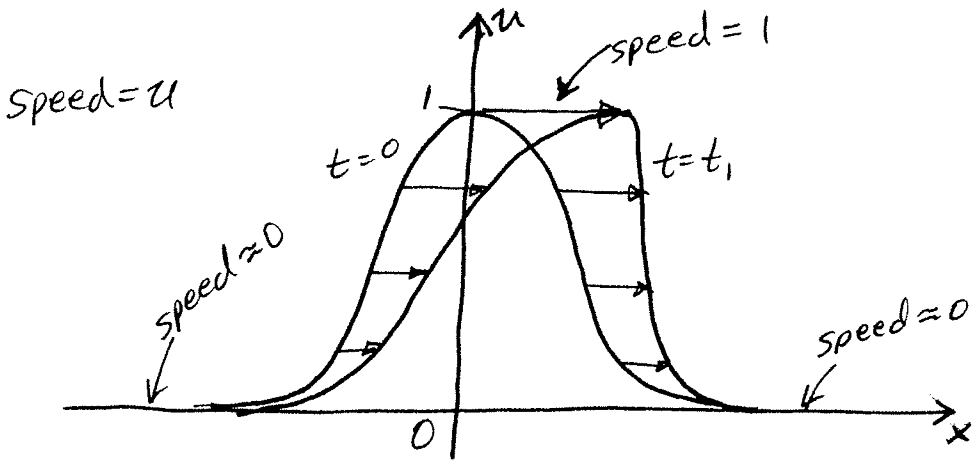




The characteristics are not parallel as they are in the linear case. Each characteristic travels with its own speed, $u_0(x_0) = 2x_0$

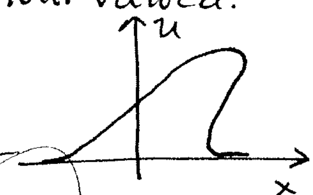
Example: $u_t + u u_x = 0,\ t > 0$

$u(x,0) = u_0(x) = e^{-x^2}$



Each point on the curve travels with its own speed u, so the top of the curve travels faster than the bottom of the curve.

This trend cannot continue indefinitely since at some time the curve would fold over itself and u would become multi-valued.



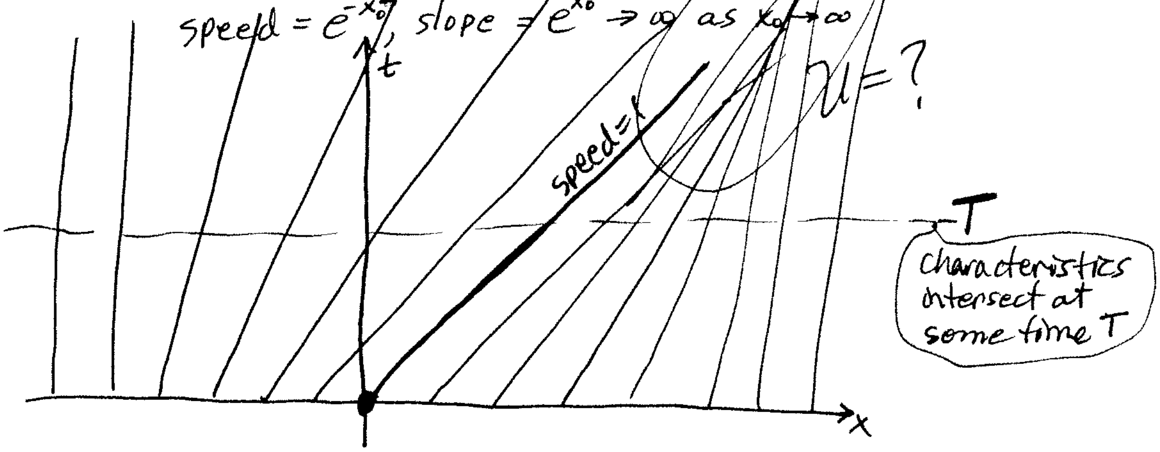
So what happens instead?

Characteristics: $x = x_0 + e^{-x_0^2} t$

speed $= e^{-x_0^2}$, slope $= e^{x_0^2} \to \infty$ as $x_0 \to \infty$

An expression for T can be derived. (homework)

For this problem, $T = \sqrt{\frac{e}{2}}$.

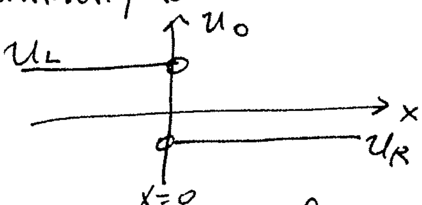


At some time T, the characteristics will intersect. Intersecting characteristics are not admissible since it suggests that u is multivalued at points of intersection. The method of characteristics can be used to find the solution in regions in which the characteristics do not intersect. To investigate this further, we'll consider a simpler problem, the so-called Riemann problem.

Riemann Problem

- The Riemann problem corresponds to initial data which consists of two constant states separated by a discontinuity. Without loss of generality, we may consider the discontinuity to be located at $x=0$.

$$\boxed{u_0(x) = \begin{cases} u_L, & x<0 \\ u_R, & x>0 \end{cases}}$$

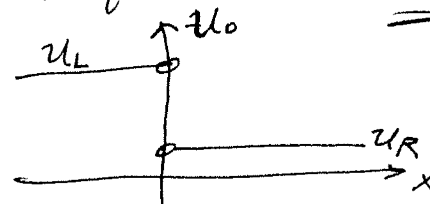


The Riemann problem is useful in studying the behavior of conservation laws. The Riemann problem also plays a role in numerical algorithms for computing solutions of conservation laws.

Consider the Riemann problem for Burger's equation with $u_L > u_R$.

$$u_t + u u_x = 0, \quad t>0$$

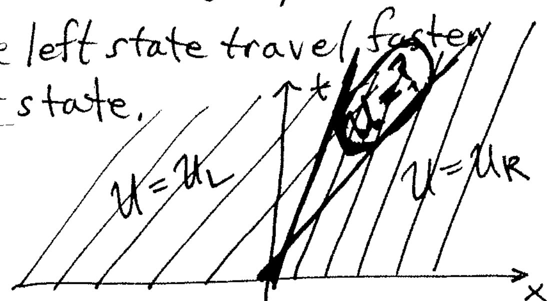
$$u(x,0) = u_0(x) = \begin{cases} u_L, & x<0 \\ u_R, & x>0 \end{cases}$$



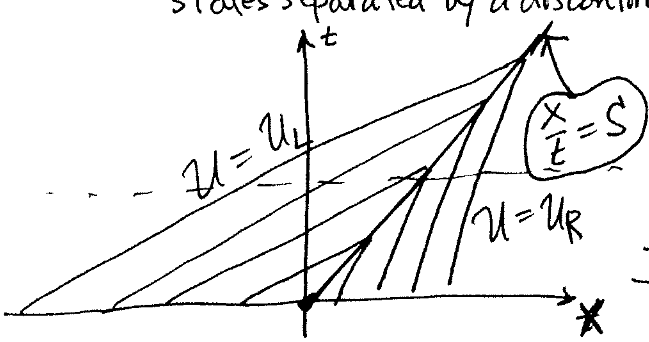
The characteristics travel with speed $u_0(x_0) = \begin{cases} u_L, & x_0<0 \\ u_R, & x_0>0 \end{cases}$,

so the characteristics associated with the left state travel faster than those associated with the right state.

The method of characteristics fails in region where the characteristics intersect. To determine the behavior of the solution in this region, we may consider the Viscous Burger equation ($u_t + u u_x = \epsilon u_{xx}$) in the limit of vanishing viscocity ($\epsilon \to 0$).

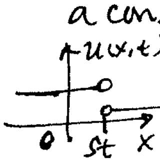


Numerical figures: The plots show that the solution consists of two constant states separated by a discontinuity (shock) which travels with constant speed S.



Rather than intersecting, the characteristics impinge upon a shock which separates the two constant states. The shock travels with a constant speed S.

$$\boxed{u(x,t) = \begin{cases} u_L, & x/t < S \\ u_R, & x/t > S \end{cases}}$$



To complete the solution, we must compute S.

Figures

1. <u>Inviscid Burger Equation</u>: Each point on the curve travels with its own speed u, so the top of the curve, where u is larger, travels faster than the bottom of the curve. This trend cannot continue indefinitely since at some time the curve would fold over itself and u would become multivalued, which is inadmissible.

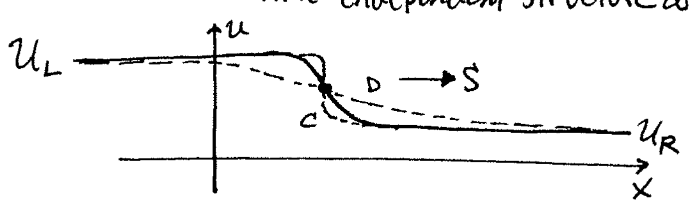
2. <u>Shock</u>:

   A. <u>Diffusion vs. Convection-Diffusion</u>: $u_t = u_{xx}$ vs. $u_t + a u_x = u_{xx}$

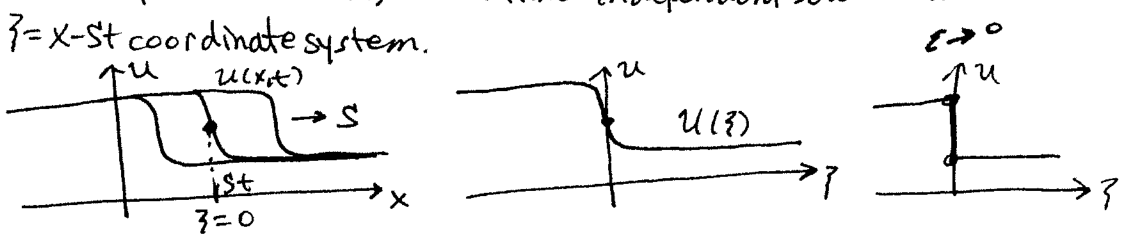
   The curves in the convection-diffusion plot are precisely the same shape as those in the diffusion plot, but they travel with speed $a=1$.

   B. <u>Viscous Burger Equation</u>: $u_t + u u_x = \varepsilon u_{xx}$

   This is not a convection-diffusion equation. The effect of each point on the curve traveling with its own speed u balances with the diffusion effect. The curve travels with a time-independent structure after the transients die out.



<u>Homework</u>: Show that $u(x,t) = U(x-St) = A + B\tanh(k(x-St))$ is a similarity solution. Here, U is a time-independent solution in the $\xi = x - St$ coordinate system.



Alternatively, we may introduce a change of coordinates, $(x,t)\to(\xi,\tau)$,

$$\begin{aligned}\tau &= t\\ \xi &= x - St\end{aligned} \implies \cancel{U_\tau} + (U-S)U_\xi = \varepsilon U_{\xi\xi}.$$

Set $U_\tau = 0$ to find time-independent solutions. The similarity solution corresponds to the steady solution that travels with speed S.

<u>Homework Hint</u>: Write $(U-S)U_\xi$ as $\left(\frac{(U-S)^2}{2}\right)_\xi$ and integrate both sides of the equation from $-\infty$ to ξ (or from ξ to ∞).

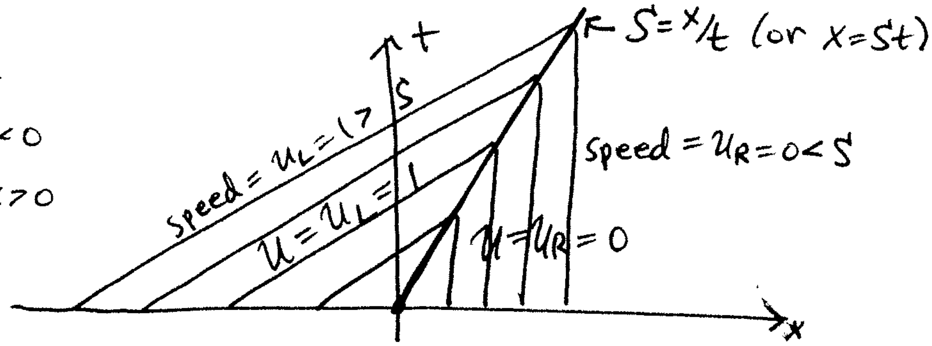
Shock = discontinuous jump in u

As $\varepsilon \to 0$, we see a discontinuity (shock), with u being constant on either side of the shock. The shock travels with a constant speed S.

Characteristics

$$\text{speed} = \begin{cases} u_L = 1, & x < 0 \\ u_R = 0, & x > 0 \end{cases}$$

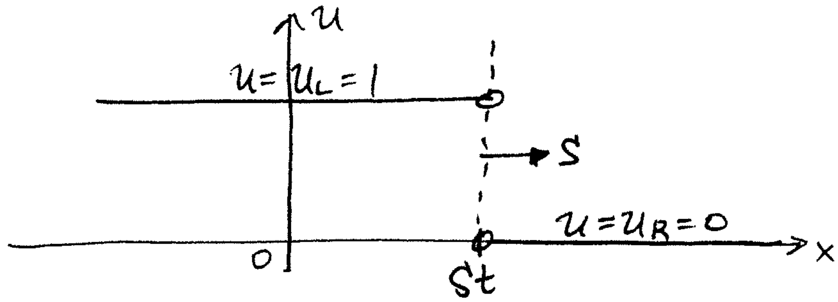
$u_R < S < u_L$



Rather than intersecting, the characteristics impinge upon a shock, which travels with a speed S.

Intersecting characteristics suggest the formation of a shock. Here, the shock forms (is already formed) at time $t = 0$.

At time t, we have



Therefore, the solution is

$$\boxed{u(x,t) = \begin{cases} u_L = 1, & x/t < S \quad (x < St) \\ u_R = 0, & x/t > S \quad (x > St) \end{cases}}$$

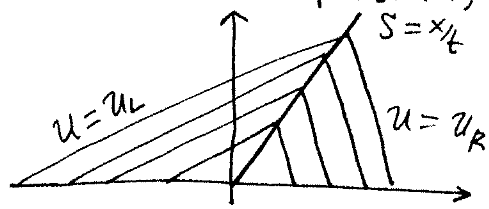
To complete the solution, we must compute S.

Computation of $\dot{S}$

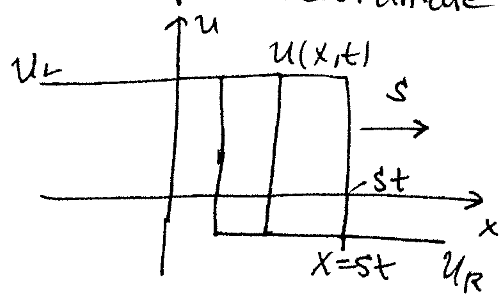
To be more general, consider a general Riemann problem,

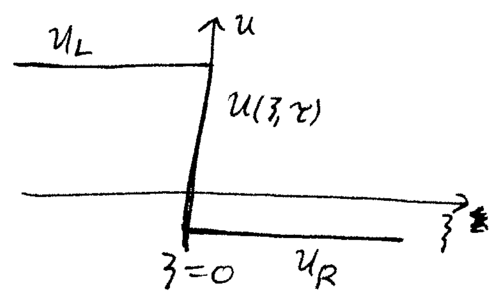
$$u_t + (f(u))_x = 0$$

$$u(x,0) = \begin{cases} u_L, & x<0 \\ u_R, & x>0 \end{cases}$$



Introduce a spatial coordinate that is attached to the shock.





Let $\tau = t$, $\xi = x - \dot{S}t$. With this, the position of the shock is fixed at $\xi = 0$

$$\Rightarrow \quad (u_\tau - \dot{S}u_\xi) + (f(u))_\xi = 0$$

The solution is time-independent in the (ξ,τ) coordinate system $\Rightarrow$ $u_\tau = 0$.
Then, we have $\quad -\dot{S}u_\xi + (f(u))_\xi = 0$

Integrate across the shock from $\xi = 0^-$ to $\xi = 0^+$ (e.g. $\xi = -\varepsilon$ to $\xi = \varepsilon$):

$$\int_{0^-}^{0^+} \left(-\dot{S}u_\xi + (f(u))_\xi\right) d\xi = 0$$

$$-\dot{S}u\Big|_{0^-}^{0^+} + f(u)\Big|_{0^-}^{0^+} = 0$$

$$-\dot{S}(u_R - u_L) + (f(u_R) - f(u_L)) = 0$$

$$\boxed{\dot{S} = \frac{f(u_R) - f(u_L)}{u_R - u_L}}$$

For Burger's equation ($f(u) = u^2/2$), we have

$$S = \frac{u_R^2/2 - u_L^2/2}{u_R - u_L} = \frac{1}{2}\frac{(u_R + u_L)\cancel{(u_R - u_L)}}{\cancel{u_R - u_L}} = \frac{u_R + u_L}{2}.$$

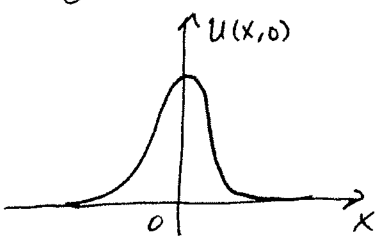
Then, the complete solution is

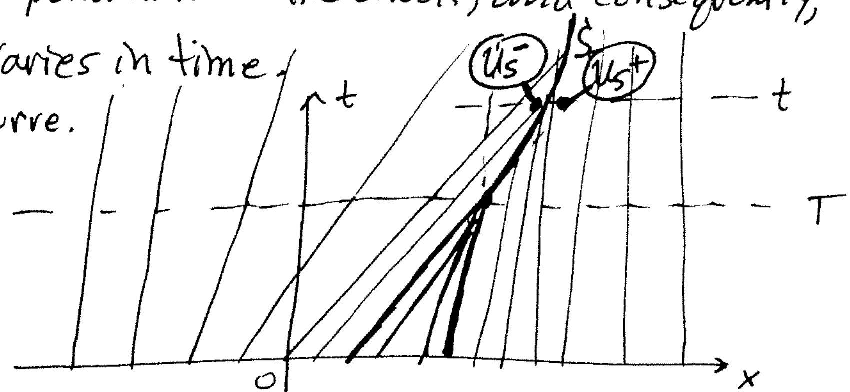
$$\boxed{u(x,t) = \begin{cases} u_L, & x/t < S \\ u_R, & x/t > S \end{cases} \quad \text{where } S = \frac{u_L + u_R}{2}}$$

Shock for $u_L > u_R$

The shock speed is constant for the Riemann problem since the solution is constant on either side of the shock. In general, the solution is time-dependent near the shock, and consequently, the shock speed S varies in time.

Consider an initial bell curve.





$$\boxed{S = \frac{f(u_s^+) - f(u_s^-)}{u_s^+ - u_s^-}}$$

$u_s^+ = u_s^+(t)$ = value of u to the immediate right of the shock

$u_s^- = u_s^-(t)$ = value of u to the immediate left of the shock.

For systems of conservation laws,

$$\boxed{S(\vec{u}_s^+ - \vec{u}_s^-) = \vec{f}(\vec{u}_s^+) - \vec{f}(\vec{u}_s^-)}$$

(Rankine–Hugoniot Relation)

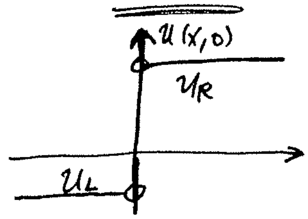
can't divide by a vector. This factor must be written on the left-hand side.

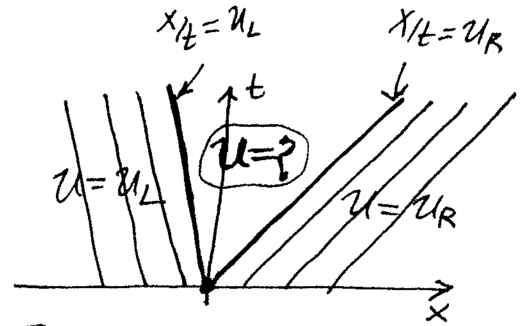
Relates the shock speed, the jump in u, and the jump in the flux function

Now consider the Riemann problem with $u_L < u_R$.

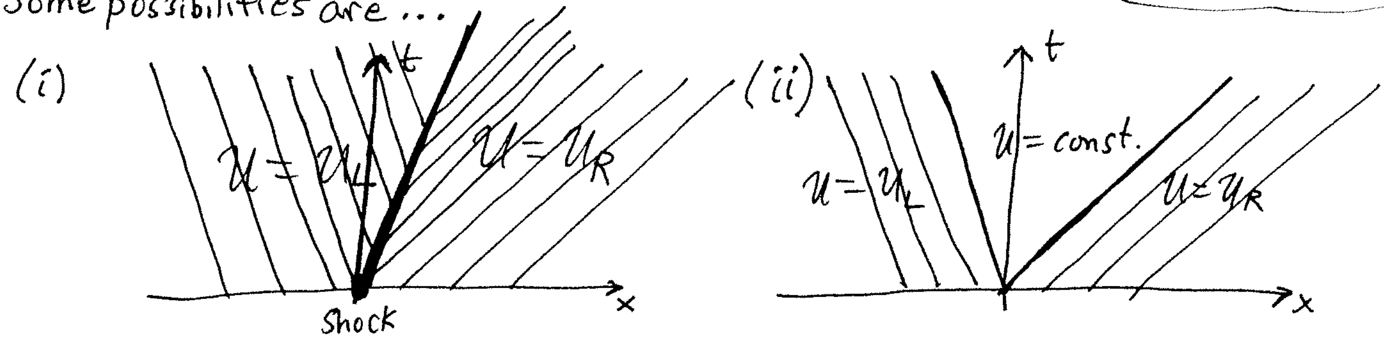
$$u_t + u u_x = 0, \quad t > 0$$

$$u(x,0) = u_0(x) = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0 \end{cases}$$



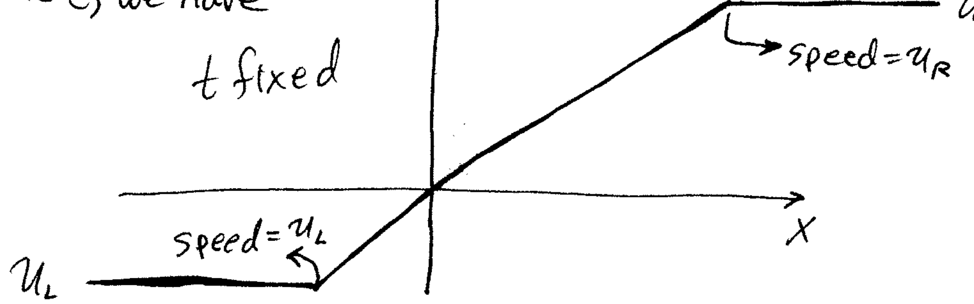


Some possibilities are ...



Though these possibilities are weak solutions of Burger's equation with the given initial data, neither is physically admissible. Both violate entropy considerations of thermodynamics. For instance, the laws of thermodynamics suggest that characteristics can only impinge on a shock, and not emerge from a shock. To find the physically admissible solution we may consider the viscous Burger equation in the limit as $\varepsilon \to 0$. It can be seen that the solution is a continuous transition between u_L and u_R.

At time t, we have



- The solution is a similarity solution with similarity variable $\xi = x/t$.

Let $\boxed{\xi = x/t}$

$$u_t = u_\xi \xi_t = \frac{-x}{t^2} u_\xi = -\frac{\xi}{t} u_\xi \qquad\qquad u_t + u u_x = 0$$

$$u_x = u_\xi \xi_x = \frac{1}{t} u_\xi \qquad \Rightarrow \qquad -\frac{\xi}{\cancel{t}} u_\xi + \frac{1}{\cancel{t}} u u_\xi = 0$$

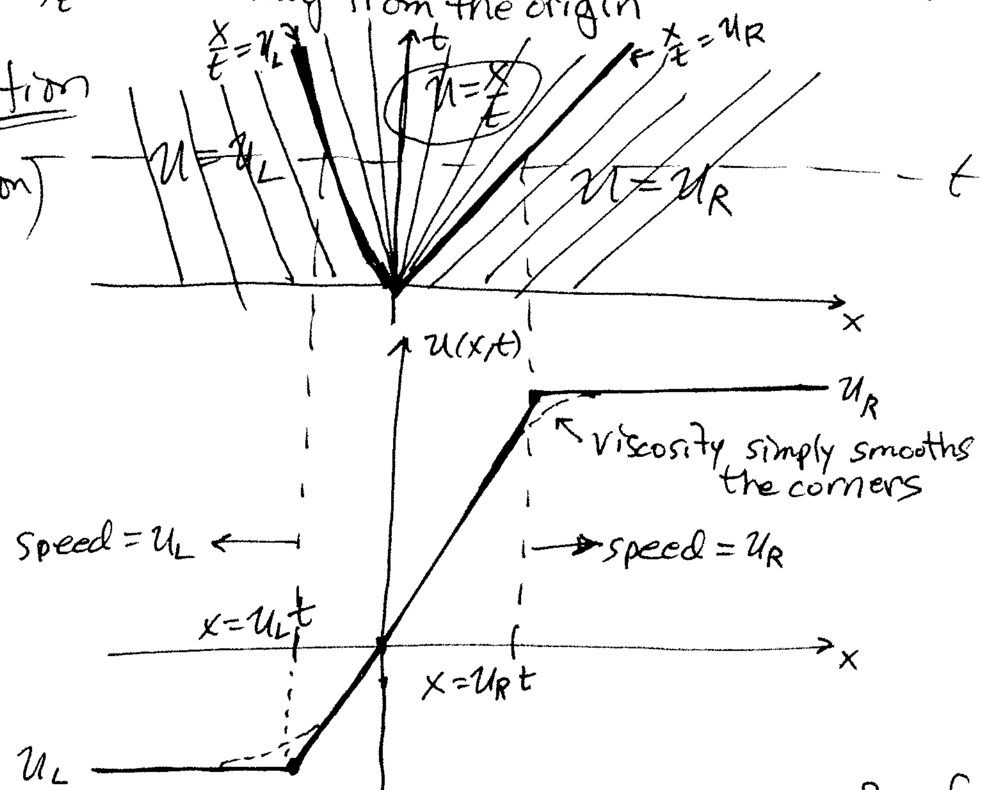
$$(u - \xi) u_\xi = 0$$

$$u_\xi \neq 0 \Rightarrow \boxed{u = \xi = x/t}$$

Between the left and right states u is constant along the rays $u = x/t$ emanating from the origin

rarefaction

- (or expansion fan)



The solution is

$$u(x,t) = \begin{cases} u_L, & x/t < u_L \\ x/t, & u_L < x/t < u_R \\ u_R, & x/t > u_R \end{cases}$$

Rarefaction (Expansion Fan) for $\underline{u_L < u_R}$

-

- Recall that a weak solution is defined to be a solution that satisfies the differential form of a conservation law at points of smoothness, ~~but fails to satisfy the integral form of the conservation law.~~

Example: Alternative weak solution of Burgers equation with Riemann initial conditions to demonstrate the nonuniqueness of weak solutions.

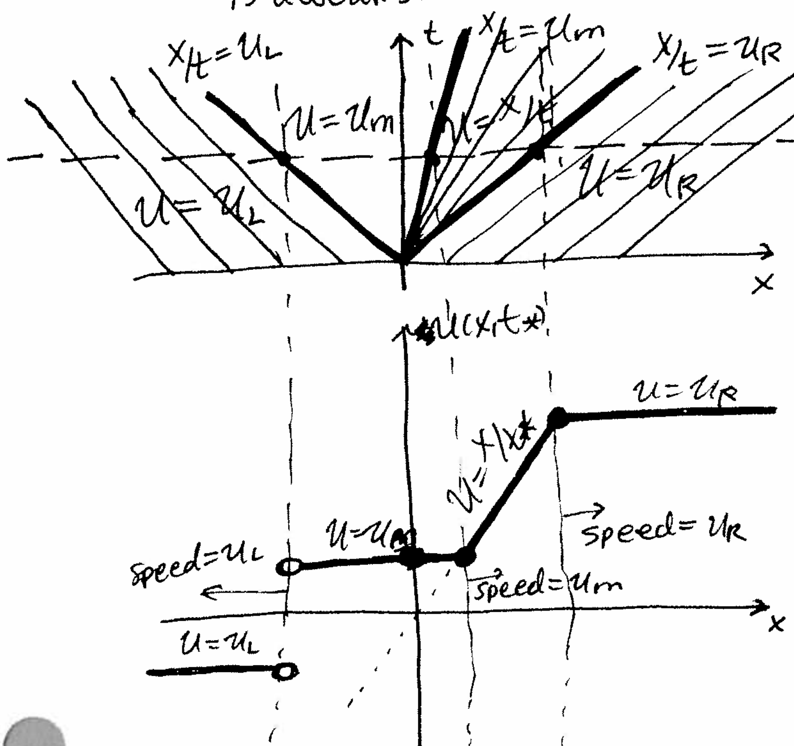
Consider the Riemann problem for Burger's equation with $u_L < u_R$.

$$u_t + u u_x = 0, \quad t > 0 \qquad \text{where } u_L < u_R$$
$$u(x,0) = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0 \end{cases}$$

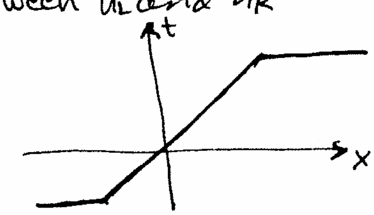
The function

$$u(x,t) = \begin{cases} u_L, & x/t < u_L \\ u_m, & u_L < x/t < u_m \\ x/t, & u_m < x/t < u_R \\ u_R & x/t > u_R \end{cases}$$

is a weak solution for all u_m such that $u_L \le u_m \le u_R$.



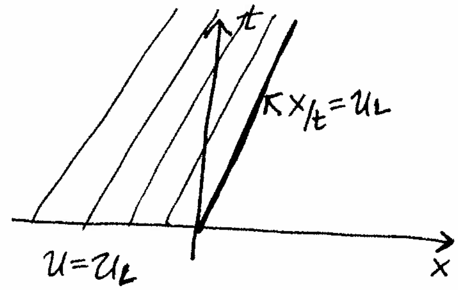
This solution does not satisfy the Rankine-Hugoniot relation which implies that $s = \frac{u_L + u_m}{2}$, whereas the discontinuity propagates with speed u_L. Furthermore, this solution does not agree with the viscous Burger equation in the limit $\varepsilon \to 0$, which yields a single rarefaction between u_L and u_R



For Burger's equation,

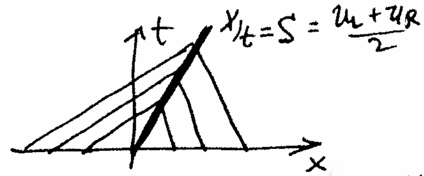
the characteristics are given by $x = x_0 + ut$,

and the characteristic speed is $s = u$.

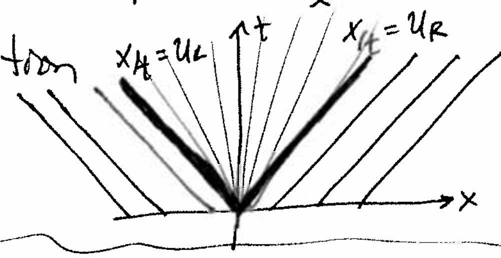


Whether a shock or a rarefaction occurs depends on whether the left Characteristic speed ($s_L = u_L$) is faster (shock) or slower (rarefaction) than the right characteristic speed ($s_R = u_R$).

$\underline{s_L > s_R} \Rightarrow \underline{u_L > u_R} \Rightarrow$ Shock



$\underline{s_L < s_R} \Rightarrow \underline{u_L < u_R} \Rightarrow$ rarefaction



For a general conservation law, $u_t + (f(u))_x = 0$,

the characteristics are given by $x = x_0 + f'(u)t$,

and the characteristic speed is $s = f'(u)$.

Therefore,

$\underline{s_L > s_R} \Rightarrow$ $f'(u_L) > f'(u_R) \Rightarrow$ Shock

$\underline{s_L < s_R} \Rightarrow$ $f'(u_L) < f'(u_R) \Rightarrow$ Rarefaction

Summary of Nonlinear Scalar Conservation Laws

	General	Burger
Conservation Law:	$u_t + (f(u))_x = 0,\ t>0$ OR $u_t + f'(u)u_x = 0,\ t>0$	$u_t + (u^2/2)_x = 0,\ t>0$ OR $u_t + uu_x = 0,\ t>0$ $\quad f(u) = u^2/2$, $f'(u) = u$
Implicit Solution:	$u = u_0(x - f'(u)t)$	$u = u_0(x - ut)$
Characteristics:	$x = x_0 + f'(u_0(x_0))t$ OR $x = x_0 + f'(u)t$	$x = x_0 + u_0(x_0)t$ OR $x = x_0 + ut$

Riemann Problem $\quad u(x,0) = \begin{cases} u_L, & x<0 \\ u_R, & x>0 \end{cases}$

	General	Burger
Shock: s = characteristic speed, $s_L > s_R$	$f'(u_L) > f'(u_R)$ $\quad (s = f'(u))$	$u_L > u_R$ $\quad (s = u)$
Rarefaction: $s_L < s_R$	$f'(u_L) < f'(u_R)$	$u_L < u_R$

Traffic Flow

Traffic Flow models are good examples of applying 1-D conservation laws since the solutions have intuitive appeal and the analytical solutions of simpler models can be found and described in full detail.

We'll consider the simplest such model. The idea is to fully understand the simplest model, after which additional features (e.g. on/off ramps, potholes, etc.) that affect traffic flow can be included.

Consider an infinite, single-lane, one-way highway with no entry or exit ramps. Since the number of vehicles is then a conserved quantity it is reasonable to model the traffic flow with the continuity equation,

$$\tilde{\rho}_{\tilde{t}} + (\tilde{\rho}\tilde{v})_{\tilde{x}} = 0 \ ; \quad \tilde{t} > 0 \ , \ -\infty < \tilde{x} < \infty ,$$

subject to some initial conditions, where

$\tilde{\rho}(\tilde{x},\tilde{t})$ = traffic density (e.g. $\frac{\text{vehicles}}{\text{meter}}$)

and $\tilde{v}(\tilde{x},\tilde{t})$ = traffic velocity (e.g. $\frac{\text{meters}}{\text{sec}}$).

Assume that vehicles travel in the $+\tilde{x}$ direction. $\Rightarrow$ $\boxed{\tilde{v} \geq 0}$

$\underline{\tilde{v} = ?}$ In gas dynamics, this equation must be supplemented with the equations of conservation of momentum and energy, thus leading to a system of 3 equations/3 unknowns $(\tilde{\rho}, \tilde{v}, \tilde{P})$.
(Euler equations)

For traffic flow, it is reasonable to assume that the traffic velocity $\tilde{v}$ at location $\tilde{x}$ and time $\tilde{t}$ depends only on the traffic density $\tilde{\rho}$ at that point. For simplicity, we'll assume that the relationship between $\tilde{v}$ and $\tilde{\rho}$ is <u>linear</u>:

$$\Rightarrow \ \tilde{v}(\tilde{\rho}) = m\tilde{\rho} + b$$

Assumptions (to determine the linear relationship between $\tilde{\rho}$ and $\tilde{v}$)

1) Assume that the vehicles are of equal length.

vehicle length = L (L is actually the mean vehicle length)

The traffic density is maximized when the vehicles are jammed bumper-to-bumper, in which case,

$$\tilde{\rho}(\tilde{x},\tilde{t}) = \rho_{max} = \frac{1 \text{ vehicle}}{L \text{ meters}} = \frac{1}{L} \qquad 0 \le \tilde{\rho} \le \rho_{max}$$

$\rho_{max} = \frac{1}{L}$ ρ_{max} can be measured, or estimated, and it is assumed to be a known quantity

3) Assume that all drivers would travel at the same maximum velocity V_{max} on an empty highway. $\Rightarrow$ $0 \le \tilde{v} \le V_{max}$

e.g. V_{max} may be the speed limit,

$$\Rightarrow \tilde{v}(0) = V_{max}$$

V_{max} is assumed to be known.

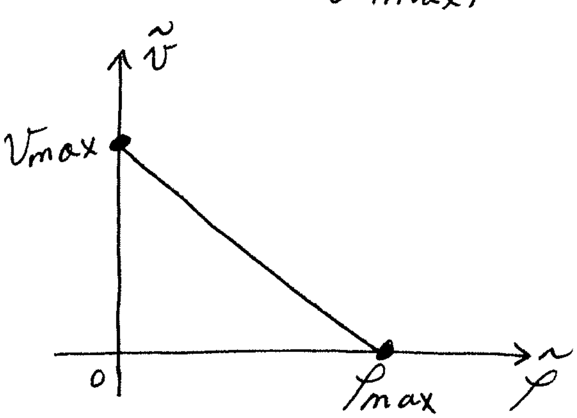
2) Assume that the velocity of bumper-to-bumper traffic is zero.

$$\Rightarrow \tilde{v}(\rho_{max}) = 0$$

We have $\tilde{v}(\tilde{\rho}) = m\tilde{\rho} + b$

$\tilde{v}(0) = V_{max}$ $\Longrightarrow$ $b = V_{max}$

$\tilde{v}(\rho_{max}) = 0$ $\quad m = -\frac{V_{max}}{\rho_{max}}$



$$\Rightarrow \boxed{\tilde{v}(\tilde{\rho}) = V_{max}\left(1 - \frac{\tilde{\rho}}{\rho_{max}}\right)}$$

- More realistic expressions for $\tilde{v}$ include other factors that affect traffic velocity, such as
  1) features of the road (curves hills potholes, tollbooths, ...)
  2) time of day (day/night, school hours, ...)
  3) density at a distance d in front of a vehicle $\Rightarrow$ $x \to x+d$
     (Drivers can see the traffic conditions ahead of their vehicle and make decisions based on the traffic/road conditions at a distance d in front of their vehicle)
  4) driver reaction time T $\Rightarrow$ $t \to t-T$ (all events are delayed by a time T)

  e.g. $\boxed{\tilde{v} = \tilde{v}(x+d,\, t-T,\, \rho(x+d, t-T))}$

Scalings

We have $\tilde{\rho}_{\tilde{t}} + (\tilde{\rho}\tilde{v})_{\tilde{x}} = 0$, $\tilde{t} > 0$, $-\infty < \tilde{x} < \infty$,

where $\tilde{v} = V_{max}(1 - \tilde{\rho}/\rho_{max})$. $0 \le \tilde{\rho} \le \rho_{max}$, $0 \le \tilde{v} \le V_{max}$

- For convenience, we may scale the variables to simplify the notation.

Let $\rho = \tilde{\rho}/\rho_{max}$ $\Rightarrow$ $(0 \le \rho \le 1)$

$v = \tilde{v}/V_{max}$ $\Rightarrow$ $(0 \le v \le 1)$

$x = \tilde{x}/L$ $\Rightarrow$ By this choice, unit length = vehicle length

$t = \dfrac{\tilde{t}}{L/V_{max}} = \dfrac{V_{max}\tilde{t}}{L}$ $\Rightarrow$ unit time = time it takes for a vehicle to travel one vehicle length at a velocity of V_{max}.

Then, $\tilde{\rho} = \rho_{max}\rho$

$\tilde{v} = V_{max} v$

$$\frac{\partial}{\partial \tilde{t}} = \frac{dt}{d\tilde{t}}\cdot\frac{\partial}{\partial t} = \frac{V_{max}}{L}\frac{\partial}{\partial t}$$

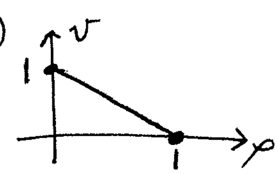
- $$\frac{\partial}{\partial \tilde{x}} = \frac{dx}{d\tilde{x}}\cdot\frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x}$$

$$\cancel{\frac{V_{max}}{L}}(\cancel{\rho_{max}}\rho)_t + \frac{1}{L}(\cancel{\rho_{max}}\rho\cdot \cancel{V_{max}}v)_x = 0$$

$$\boxed{\rho_t + (\rho v)_x = 0}$$

$\tilde{v} = V_{max}(1 - \tilde{\rho}/\rho_{max})$

$\Rightarrow$ $\boxed{v = 1 - \rho}$



We have $\boxed{\rho_t + (\rho(1-\rho))_x = 0,\ t>0,\ -\infty<x<\infty}$

$0 \le \rho \le 1$

$\boxed{v(\rho) = 1-\rho}$ $\quad 0 \le v \le 1$ $\qquad f(\rho) = \rho(1-\rho)$

$f'(\rho) = 1-2\rho$

Initial Data: $\boxed{\rho(x,0) = \rho_0(x),\ -\infty<x<\infty}$

Implicit Solution: $\rho = \rho_0(x - f'(\rho)t)$

$\rho = \rho_0(x-(1-2\rho)t)$

Characteristics: $x = x_0 + f'(\rho_0(x_0))t$

$x = x_0 + (1-2\rho_0(x_0))t$ OR $x = x_0 + (1-2\rho)t$

Characteristic Speed $= S = \frac{dx}{dt} = 1-2\rho_0(x_0) = 1-2\rho = f'(\rho)$

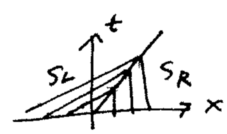
Consider the Riemann problem: $\rho(x,0) = \begin{cases} \rho_L, & x<0 \\ \rho_R, & x>0 \end{cases}$

S = characteristic speed

The left characteristic speed is $S_L = f'(u_L) = 1-2\rho_L$.

The right characteristic speed is $S_R = f'(u_R) = 1-2\rho_R$.

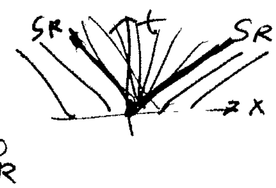
A shock occurs when $S_L > S_R$.



$\Rightarrow 1-2\rho_L > 1-2\rho_R$

$\Rightarrow \rho_L < \rho_R$

A rarefaction occurs when $S_L < S_R$.



$\Rightarrow 1-2\rho_L < 1-2\rho_R$

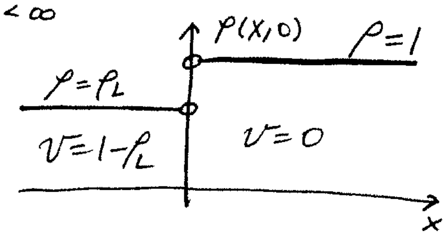
$\Rightarrow \rho_L > \rho_R$

$\boxed{\begin{array}{l} \rho_L < \rho_R \Rightarrow \text{Shock} \\ \rho_L > \rho_R \Rightarrow \text{Rarefaction} \end{array}}$

Example: $\rho_t + (\rho(1-\rho))_x = 0 \; ; \; t>0, \; -\infty < x < \infty$

$$\rho(x,0) = \begin{cases} \rho_L, & x<0 \\ 1, & x>0 \end{cases} \quad \text{where } 0<\rho_L<1$$

$$v = 1-\rho \implies v = 1-\rho_L$$



The initial data corresponds to the case in which vehicles traveling with velocity $v = 1-\rho_L$ suddenly encounter a bumper-to-bumper traffic jam and slam on their brakes. The model assumes that vehicles stop instantaneously as they reach the traffic jam.

Characteristics: $x = x_0 + (1-2\rho)t \implies t = \dfrac{x-x_0}{1-2\rho} \qquad -1 < \dfrac{1}{1-2\rho} < 1$

Characteristic speed $= s = 1-2\rho$

$$\left.\begin{aligned} s_L &= 1-2\rho_L > -1 \\ s_R &= 1-2\rho_R = -1 \end{aligned}\right\} \implies \underline{\underline{s_L > s_R}} \implies \underline{\underline{\text{Shock}}}$$

Shock Speed: Rankine-Hugoniot Relation

$$\dot{S} = \frac{f(\rho_R)-f(\rho_L)}{\rho_R-\rho_L} \quad \text{where } f(\rho) = \rho(1-\rho)$$

$$\dot{S} = \frac{\rho_R(1-\rho_R)-\rho_L(1-\rho_L)}{\rho_R-\rho_L} = \frac{\rho_R-\rho_L-(\rho_R^2-\rho_L^2)}{\rho_R-\rho_L} = 1-(\rho_R+\rho_L)$$

$$\boxed{\dot{S} = 1-(\rho_L+\rho_R)} \quad \text{(general case)}$$

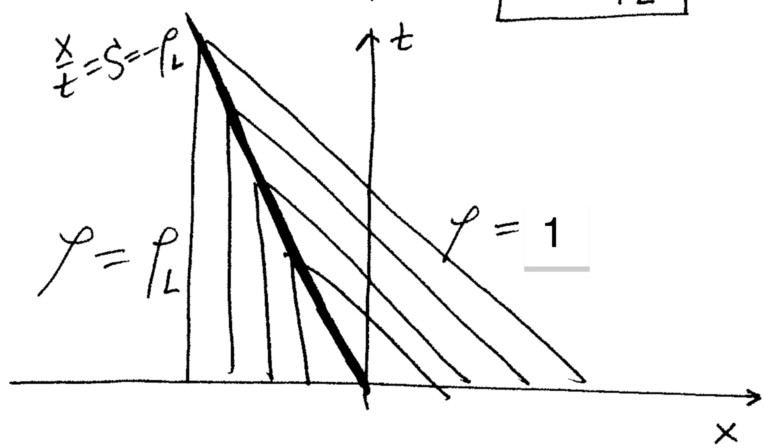
$$\dot{S} = 1-(\rho_L+1) = -\rho_L \qquad \boxed{\dot{S} = -\rho_L} \implies -1 < \dot{S} < 0$$

Also, $\rho_L < 1$

$-\rho_L > -1$

$-2\rho_L > -\rho_L - 1$

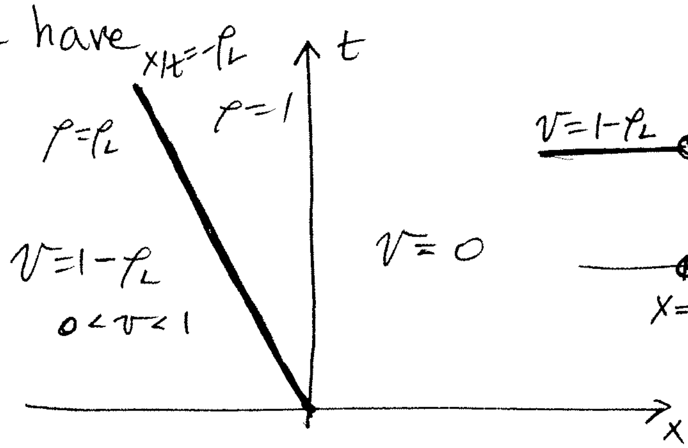
$1-2\rho_L > -\rho_L$

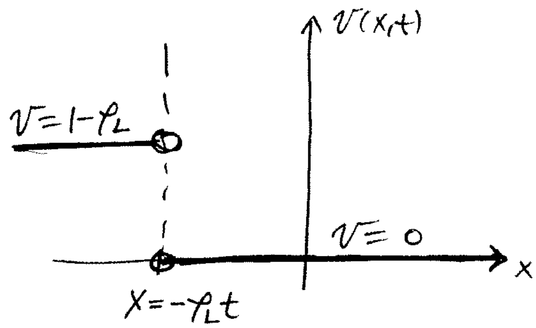


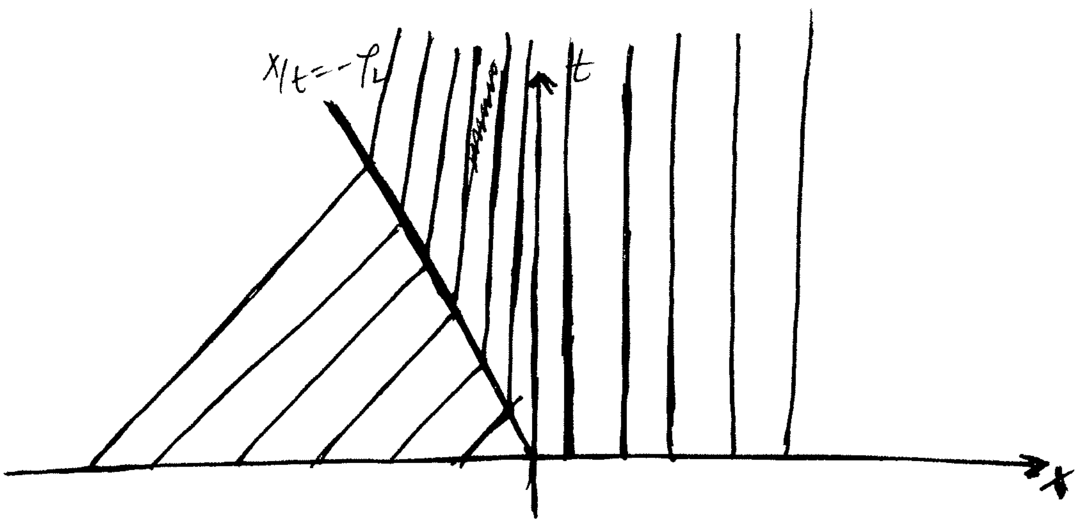
$$\boxed{\rho(x,t) = \begin{cases} \rho_L, & x/t < -\rho_L \\ 1, & x/t > -\rho_L \end{cases}}$$

Vehicle Trajectories

We have





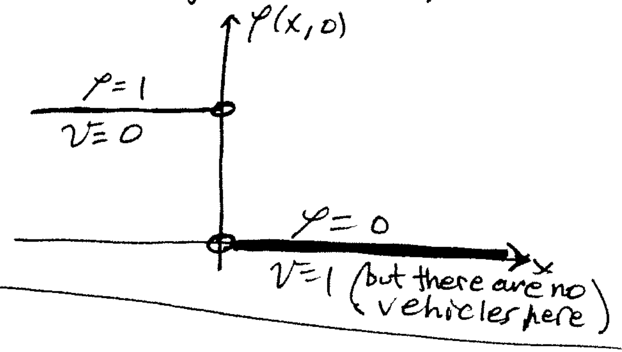


Note: These are vehicle trajectories – not characteristics.

Example: Suppose that vehicles are initially bumper-to-bumper waiting at a traffic light (located at $x=0$), which turns green at $t=0$.

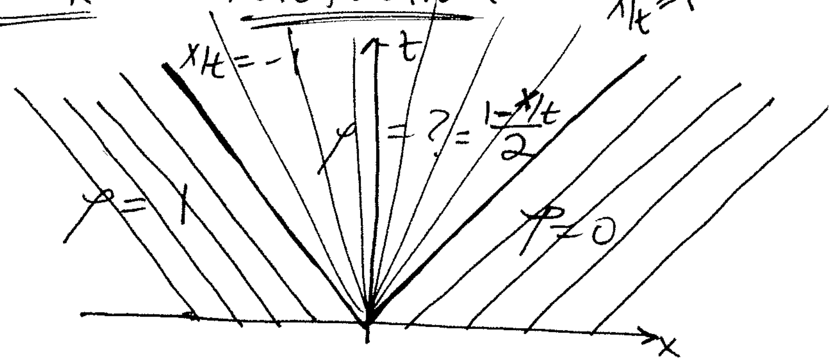
$$\Rightarrow \quad \rho_t + (\rho(1-\rho))_x = 0, \quad t>0$$

$$\rho(x,0) = \begin{cases} \rho_L = 1, & x<0 \\ \rho_R = 0, & x>0 \end{cases}$$



Characteristic speed $= s = 1-2\rho$

$$S_L = 1-2 = -1$$
$$S_R = 1-0 = 1 \quad \Rightarrow \quad \underline{S_L < S_R} \Rightarrow \underline{\text{Rarefaction}}$$



Similarity solution inside the fan

$$\xi = x/t$$

$$\Rightarrow \quad -\frac{\xi}{t}\rho_\xi + \frac{1}{t}(\rho(1-\rho))_\xi = 0$$

$$-\xi\rho_\xi + (1-2\rho)\rho_\xi = 0$$

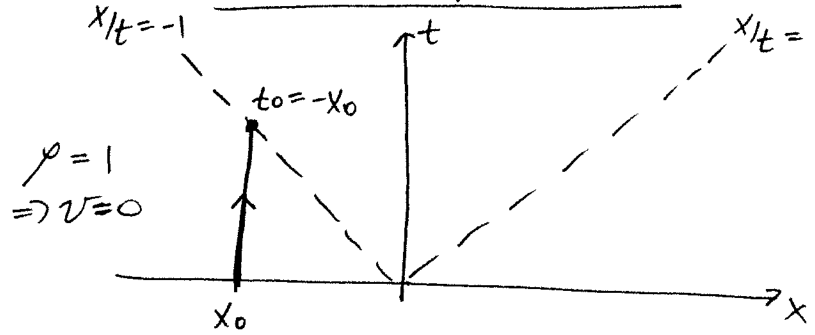
$$(1-2\rho-\xi)\rho_\xi = 0$$

$$\rho_\xi \neq 0 \Rightarrow 1-2\rho-\xi = 0$$

$$\rho = \frac{1-\xi}{2} = \frac{1-x/t}{2}$$

$$\boxed{\rho(x,t) = \begin{cases} 1, & x/t < -1 \\ \dfrac{1-x/t}{2}, & -1 < x/t < 1 \\ 0, & x/t > 1 \end{cases}}$$

Vehicle Trajectories



Inside the rarefaction, $(t > t_0 = -x_0)$, $\rho = \frac{1 - x/t}{2}$,

and $v(\rho) = 1 - \rho = \frac{1}{2}(1 + x/t)$.

Vehicle velocity $= v = \frac{dx}{dt} = \frac{1}{2}(1 + x/t)$, $t > t_0 = -x_0$

The vehicle trajectory is found by solving the initial value problem

$$\boxed{\begin{array}{l} \frac{dx}{dt} = \frac{1}{2}\left(1 + \frac{x}{t}\right), \; t > -x_0 \\ \text{subject to } \; x(-x_0) = x_0 \end{array}}$$

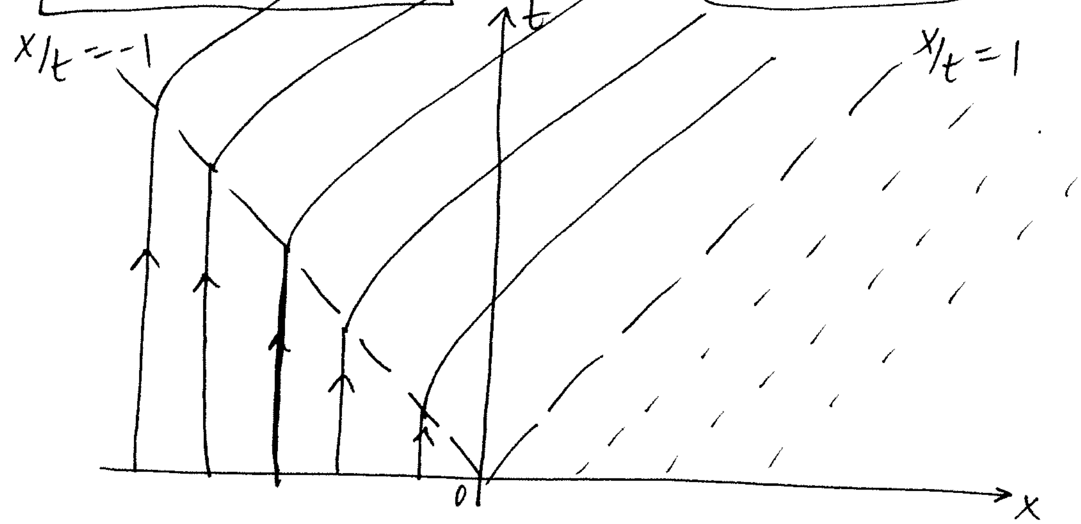
Integrating Factor

$\frac{dx}{dt} - \frac{x}{2t} = \frac{1}{2}$ $\quad \mu(t) = e^{\int -\frac{dt}{2t}} = e^{-\frac{1}{2}\ln t} = t^{-1/2}$

$t^{-1/2}x = \frac{1}{2}\int t^{-1/2}dt = t^{1/2} + C$ $\qquad x(-x_0) = -x_0 + C(-x_0)^{1/2} = x_0$

$x = t + C\,t^{1/2}$ $\qquad C = \frac{2x_0}{(-x_0)^{1/2}} = 2 \cdot \frac{-(-x_0)^{1/2}\cancel{(-x_0)^{1/2}}}{\cancel{(-x_0)^{1/2}}}$

$\Rightarrow \boxed{x(t) = t - 2\sqrt{-x_0 t}}$ $\qquad C = -2(-x_0)^{1/2}$

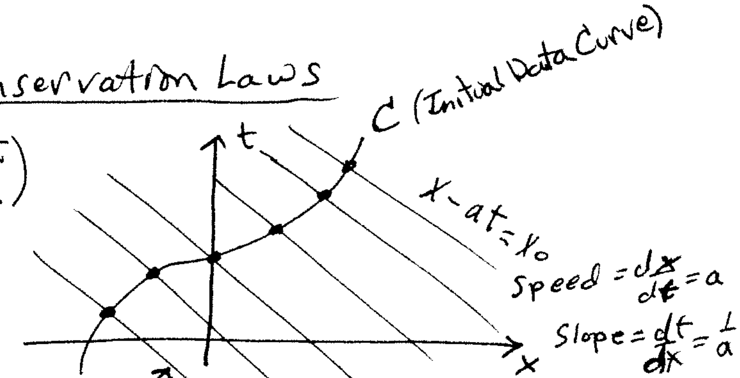


Recall: Linear Scalar ($n=1$) Conservation Laws

$$u_t + a u_x = 0 \quad \begin{pmatrix}\text{transport}\\ \text{equation}\end{pmatrix}$$

Characteristics: $x - at = x_0$

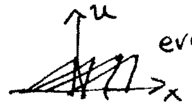
Characteristic speed: $\frac{dx}{dt} = a$



u is constant along the characteristics at the value prescribed on C.

general solution: $\boxed{u(x,t) = \phi(x-at)}$ (f satisfies the PDE at all points at which it is smooth)

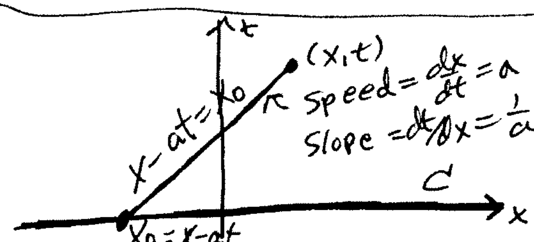
Note: If the initial data is not smooth, it will yield a unique weak solution. (unique: contrary to the nonlinear case)

e.g. 

every thing travels with speed a

For the initial data $u(x,0) = u_0(x)$,

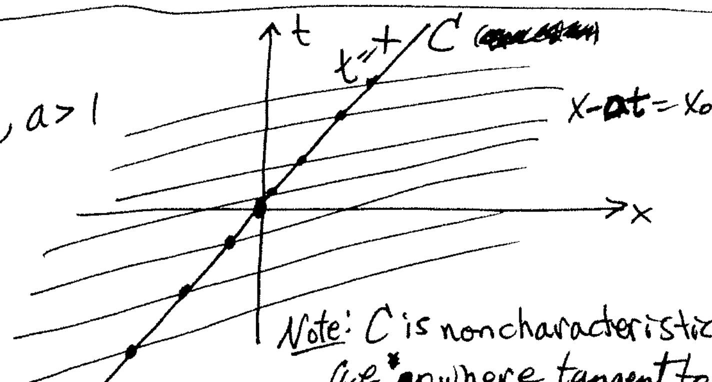
we have
$$u(x,t) = \phi(x-at)$$
$$u(x,0) = \phi(x)$$
$$u_0(x) = \phi(x)$$
$$u_0(x-at) = \phi(x-at)$$



$$\Rightarrow \boxed{u(x,t) = u_0(x-at)} = u_0(x_0)$$

Example: Solve $u_t + a u_x = 0$, $a > 1$

$u(x,x) = x^2$



Characteristics: $x - at = x_0$

Characteristic Speed: $\frac{dx}{dt} = a$

Note: C is noncharacteristic (i.e. nowhere tangent to the characteristics)

We could solve by the method of characteristics, but we'll use the general solution instead.

$$u(x,t) = \phi(x-at)$$
$$u(x,x) = \phi((1-a)x) = x^2$$
$$x \to \frac{x}{1-a} \Rightarrow \phi(x) = \left(\frac{x}{1-a}\right)^2$$

$$\boxed{u(x,t) = \phi(x-at) = \left(\frac{x-at}{1-a}\right)^2} = \left(\frac{x-at}{a-1}\right)^2 = \left(\frac{x_0}{a-1}\right)^2$$

Linear Hyperbolic Systems of Conservation Laws

Nonlinear: $\vec{u}_t + (\vec{f}(\vec{u}))_x = \vec{0}$ $\vec{u}$ is $n\times 1$, $\vec{f}$ is $n\times 1$

OR $\vec{u}_t + \vec{f}'(\vec{u})\,\vec{u}_x = \vec{0}$ where $\vec{f}'(\vec{u}) = \dfrac{\partial(f_1,\dots,f_n)}{\partial(u_1,\dots,u_n)}$ (Jacobian $n\times n$)

Linear: $\vec{f}(\vec{u}) = A\vec{u}$

$\vec{f}'(\vec{u}) = A \quad \Rightarrow \quad \boxed{\vec{u}_t + A\vec{u}_x = \vec{0}}$ A is $n\times n$ and constant

$(\vec{f}(\vec{u}))_x = A\vec{u}_x$

Recall that a system of PDEs, $B\vec{u}_t + A\vec{u}_x = \vec{0}$, is classified according to the eigenvalues defined by $\det(A-\lambda B)=0$.

i) no real eigenvalues $\Rightarrow$ Elliptic

ii) n real eigenvalues with n lin. indep. eigenvectors $\Rightarrow$ Hyperbolic

iii) n real eigenvectors without n lin. indep. eigenvectors $\Rightarrow$ Parabolic

---

Scalar conservation laws are necessarily hyperbolic.

$$u_t + f'(u)\,u_x = 0 \qquad A = f'(u),\ B = 1$$

$$\det(A-\lambda B) = \det(f'(u)-\lambda) = f'(u)-\lambda = 0$$

$\underline{\lambda = f'(u)}$ one real eigenvector

A system of conservation laws need not be hyperbolic.

$$\vec{u}_t + \vec{f}'(\vec{u})\,\vec{u}_x = \vec{0} \qquad A = \vec{f}'(\vec{u}),\ B = I$$

$\det(A-\lambda B) = \det(\vec{f}'(\vec{u}) - \lambda I) = 0$ The classification depends on the "classical" eigenvalues of $\vec{f}'(\vec{u})$.

We'll assume that our linear system ($\vec{f}'(\vec{u}) = A$) is <u>hyperbolic</u>.

$\Rightarrow$ A has n real eigenvalues with n linearly independent eigenvectors.

If A has n real eigenvalues $(\lambda_1, \ldots, \lambda_n)$ with n linearly independent corresponding eigenvectors $(\vec{r}_1, \ldots, \vec{r}_n)$, then A is <u>diagonalizable</u>. That is, A can be written as $\boxed{A = RDR^{-1}}$, where D is a diagonal matrix with the eigenvalues of A along its diagonal, $\boxed{D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}}$, and R is a matrix whose columns consist of the corresponding eigenvectors, $\boxed{R = (\vec{r}_1 | \vec{r}_2 | \cdots | \vec{r}_n)}$.

<u>Note</u>: The letter r is used to denote the eigenvectors of A since the $\vec{r}_j$'s are the <u>right</u> eigenvectors of A $(A\vec{r}_j = \lambda_j \vec{r}_j)$, rather than the left eigenvectors of A $(\vec{l}_j^{\,T} A = \lambda_j \vec{l}_j^{\,T})$

Furthermore, we'll assume that our linear system of conservation laws is <u>strictly hyperbolic</u>, meaning that the n real eigenvalues of A are distinct.

Solution and Characteristics

Consider the following strictly hyperbolic linear system.

$$\vec{u}_t + A\vec{u}_x = \vec{0}, \quad t>0 \quad (n \times n \text{ system})$$

Let λ_j and $\vec{r}_j$ $(j=1,\dots,n)$ be the eigenvalue/vector pairs of A. $\lambda_i \neq \lambda_j$ for $i \neq j$

Let $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ and $R = (\vec{r}_1 | \vec{r}_2 | \cdots | \vec{r}_n)$.

$$\Rightarrow \quad A = RDR^{-1}$$

$$\vec{u}_t + A\vec{u}_x = \vec{0}$$

Multiply by $R^{-1} \Rightarrow \quad R^{-1}\vec{u}_t + R^{-1}A\vec{u}_x = \vec{0}$

$A = RDR^{-1} \Rightarrow \quad R^{-1}\vec{u}_t + \cancel{R^{-1}R}DR^{-1}\vec{u}_x = \vec{0}$

R^{-1} is constant $\Rightarrow \quad (R^{-1}\vec{u})_t + D(R^{-1}\vec{u})_x = \vec{0}$

Let $\vec{v} = R^{-1}\vec{u} \Rightarrow \boxed{\vec{v}_t + D\vec{v}_x = 0}$

$\Rightarrow \vec{u} = R\vec{v}$

$$\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_t + \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_2 \end{pmatrix}\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}_x = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$v_{1t} + \lambda_1 v_{1x} = 0$$
$$\vdots$$
$$v_{nt} + \lambda_1 v_{nx} = 0$$

Since D is a diagonal matrix the system decouples into n independent linear scalar conservation laws,

$$\boxed{v_{jt} + \lambda_j v_{jx} = 0; \quad j=1,\dots,n}$$

↑ transport equation

$(u_t + a u_x = 0)$

$u \to v_j$

$a \to \lambda_j$

- We have $v_{jt} + \lambda_j v_{jx} = 0$ $\begin{pmatrix}\text{transport}\\ \text{equation}\end{pmatrix}$ $j = 1, \dots, n$

  <u>Characteristics</u>: $x - \lambda_j t = x_0$

  <u>Characteristic Speed</u>: $\frac{dx}{dt} = \lambda_j$ (eigenvalues of A)

  <u>general solution</u>: $\boxed{v_j(x,t) = \phi_j(x - \lambda_j t)}$

<u>Note</u>: If the initial data $u(x,0) = u_0(x)$ is given, then since $\vec{v} = R^{-1}\vec{u}$

$$v_j(x,0) = (R^{-1}\vec{u}(x,0))_j = (R^{-1}\vec{u}_0(x))_j = \phi(x)$$

$$\Rightarrow \boxed{v_j(x,t) = (R^{-1}\vec{u}_0(x - \lambda_j t))_j}$$

However, we'll keep the general solutions for now, convert back to $\vec{u}$, after which initial conditions can be imposed.

- $\boxed{v_j(x,t) = \phi_j(x - \lambda_j t)}$ <u>Note</u>: $v_j(x,0) = \phi_j(x)$
  $= v_j(x - \lambda_j t, 0)$ $\Rightarrow \phi(x - \lambda_j t) = v_j(x - \lambda_j t, 0)$

Then, $\vec{u} = R\vec{v} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & & \vdots \\ r_{n1} & r_{n2} & \cdots & r_{nn} \end{pmatrix} \begin{pmatrix} \phi_1(x - \lambda_1 t) \\ \phi_2(x - \lambda_2 t) \\ \vdots \\ \phi_n(x - \lambda_n t) \end{pmatrix}$

Let $\vec{r}_j = \begin{pmatrix} r_{1j} \\ r_{2j} \\ \vdots \\ r_{nj} \end{pmatrix}$

$$= \begin{pmatrix} r_{11}\phi_1 + r_{12}\phi_2 + \cdots + r_{1n}\phi_n \\ r_{21}\phi_1 + r_{22}\phi_2 + \cdots + r_{2n}\phi_n \\ \vdots \qquad \vdots \qquad \vdots \\ r_{n1}\phi_1 + r_{n2}\phi_2 \cdots r_{nn}\phi_n \end{pmatrix}$$

$$= \phi_1(x - \lambda_1 t)\vec{r}_1 + \phi_2(x - \lambda_2 t)\vec{r}_2 + \cdots + \phi_n(x - \lambda_n t)\vec{r}_n$$

$$\Rightarrow \boxed{\vec{u}(x,t) = \sum_{j=1}^{n} \phi_j(x - \lambda_j t)\vec{r}_j} \begin{pmatrix}\text{general solution}\\ \text{of } \vec{u}_t + A\vec{u}_x = \vec{0}\end{pmatrix}$$

- Initial Data determines the ϕ_j's.

$$\boxed{\vec{u}(x,t) = \sum_{n=1}^{n} \alpha_j(x-\lambda_j t)\, \vec{r}_j}$$

Note: $\vec{u}(x,t)$ depends on the initial data only at n points: $x-\lambda_j t$, $j=1,\dots,n$.
The j^{th} term of the sum, $\alpha_j(x-\lambda_j t)\vec{r}_j$, is constant along the lines $x-\lambda_j t = x_0$.

e.g. Consider the case of $n=3$ with our usual initial data,

$$\vec{u}_t + A\vec{u}_x = \vec{0}, \quad t>0 \qquad A \text{ is a } 3\times3 \text{ constant matrix}$$
$$\vec{u}(x,0) = \vec{u}_0(x),$$

and suppose the eigenvalues of A are $\lambda_1=-1$, $\lambda_2=0$, $\lambda_3=1$.

$$\Rightarrow \quad \vec{u}(x,t) = \alpha_1(x+t)\vec{r}_1 + \alpha_2(x)\vec{r}_2 + \alpha_3(x-t)\vec{r}_3$$

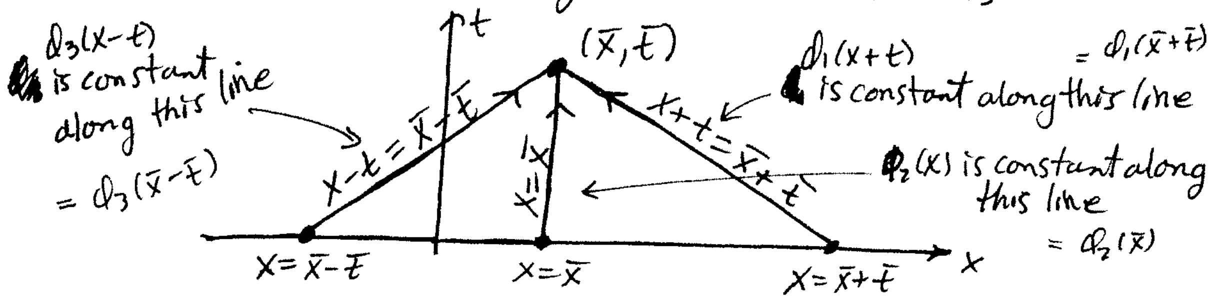
$$\vec{u}(x,0) = \alpha_1(x)\vec{r}_1 + \alpha_2(x)\vec{r}_2 + \alpha_3(x)\vec{r}_3 = \vec{u}_0(x)$$

This vector relation yields 3 scalar equations which determine the 3 unknown functions α_1, α_2, and α_3

Consider the solution $\vec{u}$ at a fixed point $(\bar{x},\bar{t})$.

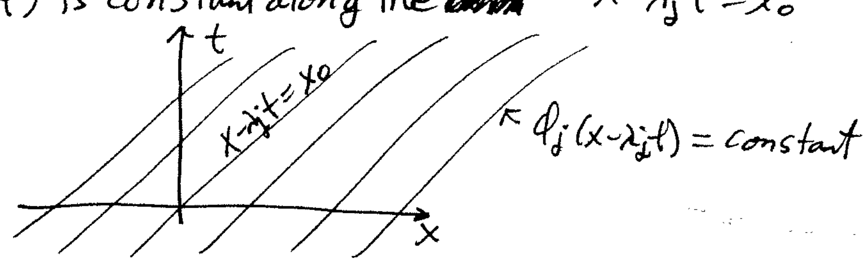
$$\vec{u}(\bar{x},\bar{t}) = \alpha_1(\bar{x}+\bar{t})\vec{r}_1 + \alpha_2(\bar{x})\vec{r}_2 + \alpha_3(\bar{x}-\bar{t})\vec{r}_3$$

$\alpha_1(\bar{x}+\bar{t})$ is constant along the line $x+t=\bar{x}+\bar{t}$: $\alpha_1(\bar{x}+\bar{t})$
$\alpha_2(\bar{x})$ is constant along the line $x=\bar{x}$: $\alpha_2(\bar{x})$
$\alpha_3(\bar{x}-\bar{t})$ is constant along the line $x-t=\bar{x}-\bar{t}$: $\alpha_3(\bar{x}-\bar{t})$



$\vec{u}(x,t)$ is the sum of 3 terms, each of which can be evaluated at some point on the initial data curve.

For a fixed j, $\alpha_j(x-\lambda_j t)$ is constant along the ~~curve~~ lines $x-\lambda_j t = x_0$

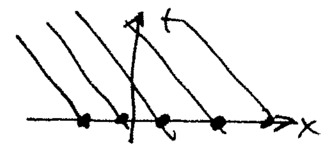


- A strictly hyperbolic system of conservation laws has n families of characteristics.

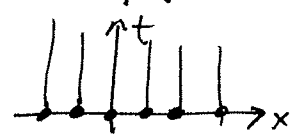
$$\boxed{\text{The lines } x-\lambda_j t = x_0 \text{ are the \underline{characteristics of the } j^{\text{th}} \text{ family}}, \quad j=1,\dots,n}$$

In the above example, there are $n=3$ families of characteristics.

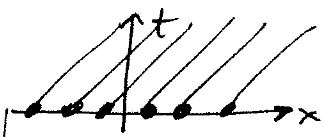
$\underline{j=1} \Rightarrow \lambda_1 = -1 \Rightarrow x - \lambda_1 t = x_0$
$x + t = x_0$


family 1

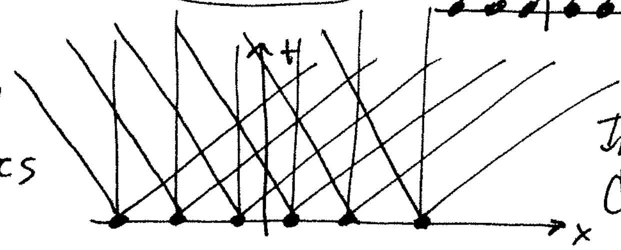
$\underline{j=2} \Rightarrow \lambda_2 = 0 \Rightarrow x - \lambda_2 t = x_0$
$x = x_0$


family 2

$\underline{j=3} \Rightarrow \lambda_3 = 1 \Rightarrow x - \lambda_3 t = x_0$
$x - t = x_0$


family 3

- Superimpose all 3 families of characteristics


Intersecting Characteristics?

Note: Shocks form only when characteristics from the same family intersect. However, shocks do not form for linear systems since the characteristics of any single family is a set of non-intersecting parallel lines, with the same characteristic speed λ_j. If the initial data has discontinuities, the solution will have discontinuities, but discontinuities cannot form from smooth initial data if the system is linear.

- Note: If the linear system is strictly hyperbolic ($\lambda_i \neq \lambda_j$ for $i \neq j$), then each family of characteristics has a distinct speed λ_j, so no two families of characteristics coincide

Recall: $\vec{v} = R^{-1}\vec{u} \Rightarrow v_j = (R^{-1}\vec{u})_j$

where $v_j(x,t) = d_j(x - \lambda_j t)$.

$\Rightarrow v_j(x,t)$ is constant along the characteristics of family j

$(x - \lambda_j t = x_0)$

The v_j's are called the characteristic variables of the system.

To express the characteristic variables in terms of $\vec{u}$, observe that v_j is a linear combination of the components u_K of the vector $\vec{u}$.

Let $R^{-1} = (s_{ij})$. $\Rightarrow R^{-1} = \begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & & \vdots \\ s_{n1} & \cdots & s_{nn} \end{pmatrix}$

Then, $\boxed{v_j = (R^{-1}\vec{u})_j = \sum_{k=1}^{n} s_{jk} u_k \quad j=1,\dots,n}$ Characteristic Variables

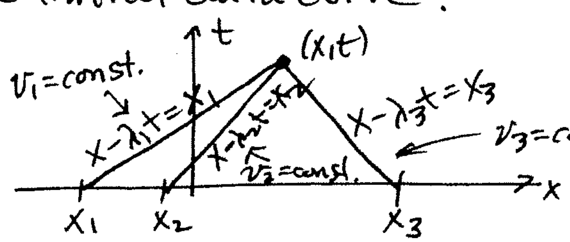
It is this linear combination of the u_k's that is constant along the characteristics of the jth family.

Each point in the xt-plane receives information from exactly n points on the initial data curve.

e.g. $n=3$

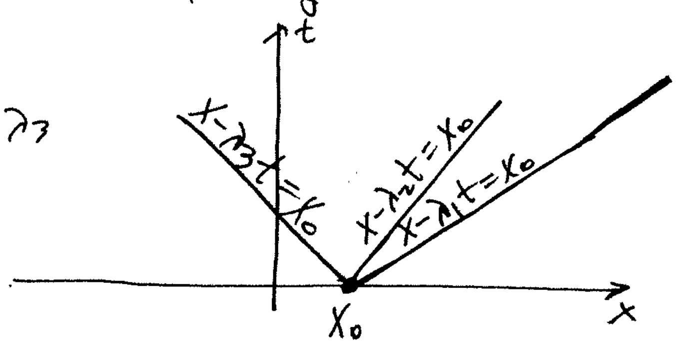
$C: t=0$

Assume $\lambda_1 > \lambda_2 > \lambda_3$



Likewise, the initial data at each point on the initial data curve affects the solution only along the n characteristics emerging from it.

$\lambda_1 > \lambda_2 > \lambda_3$



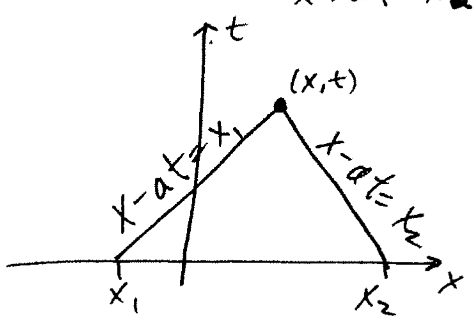
The initial data at x_0 propagates only along the 3 characteristics that emerge from the point $(x_0, 0)$ in the xt-plane.

Note: In the above example, the value of $\vec{u}$ at any point in the xt-plane depends on the initial data only at 3 points. This illustrates a key difference between hyperbolic and parabolic PDEs. The solutions of hyperbolic PDEs at any point (x,t) depends on the initial data over at most a finite interval, whereas the solutions of parabolic PDEs depends on the initial data everywhere.

e.g.

Hyperbolic — Wave Equation $\quad u(x,0) = u_0(x)$, $\quad u_{tt} = c^2 u_{xx}$

Characteristics: $x - at = x_1$, $\quad x + at = x_2$

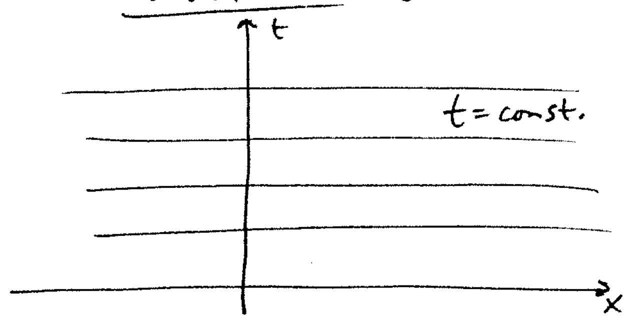


$u(x,t)$ depends only on $u_0(x_1)$ and $u_0(x_2)$.

Initial data travels along the characteristics as t increases

Parabolic — Heat Equation $\quad u_t = k\, u_{xx}$

Characteristics: $t = t_0$



The entire solution moves from characteristic to characteristic as t increases.

$u(x,t)$ depends on the initial data at all x.

Wave Equation

Example: $u_{tt} = c^2 u_{xx}$; $u(x,0) = f(x)$, $u_t(x,0) = 0$

This is a special case. More generally, $u_t(x,0) = g(x)$. We'll solve the more general case at a later time. The solution of the more general probem is called the d'Alembert solution.

Consider $\boxed{\begin{aligned} u_t + c v_x &= 0 \\ v_t + c u_x &= 0 \end{aligned}}$ $\Rightarrow u_{tt} = -c v_{xt} = -c(-c u_{xx}) = c^2 u_{xx}$

$u_{tt} = c^2 u_{xx}$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Let $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ $\Rightarrow$ $\vec{u}_t + A\vec{u}_x = \vec{0}$, where $A = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$

Eigenvalues/vectors: $\det(A - \lambda I) = \begin{vmatrix} -\lambda & c \\ c & -\lambda \end{vmatrix} = \lambda^2 - c^2 = 0$

$\lambda = \pm c$

$\lambda_1 = +c$ $\quad (A - \lambda_1 I)\vec{r}_1 = \vec{0}$

$\begin{pmatrix} -c & c \\ c & -c \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow$ $\boxed{\lambda_1 = c, \ \vec{r}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$r_1 = r_2$

$\lambda_2 = -c$ $\quad (A - \lambda_2 I)\vec{r}_2 = \vec{0}$

$\begin{pmatrix} c & c \\ c & c \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow$ $\boxed{\lambda_2 = -c; \ \vec{r}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

$r_1 = -r_2$

Characteristics

$x - \lambda_1 t = x_0$

$x - ct = x_0$

$x - \lambda_2 t = x_0$

$x + ct = x_0$

Then, $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \sum_{j=1}^{2} \phi_j(x - \lambda_j t)\,\vec{r}_j = \phi_1(x - ct)\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \phi_2(x + ct)\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow \boxed{u(x,t) = \phi_1(x - ct) + \phi_2(x + ct)}$ General Solution of the Wave Equation (Compare to Advection $u = \phi(x - ct)$)

$u_t(x,t) = -c\phi_1'(x - ct) + c\phi_2'(x + ct)$

Initial Data:

$u(x,0) = \phi_1(x) + \phi_2(x) = f(x)$ $\longrightarrow$ $\phi_1(x) + (\phi_1(x) - k) = f(x)$

$u_t(x,0) = -c(\phi_1'(x) - \phi_2'(x)) = 0$

$\phi_1(x) - \phi_2(x) = k$

$\phi_2(x) = \phi_1(x) - k$

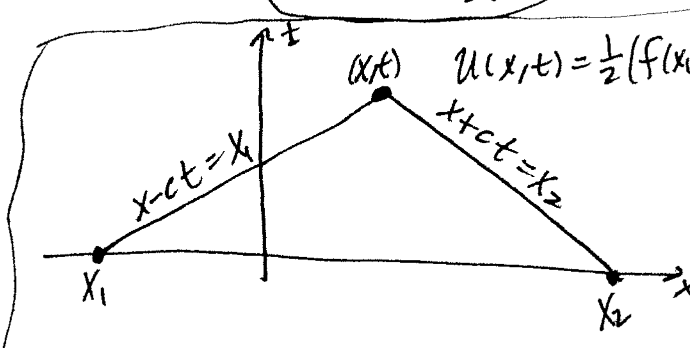
$\phi_1(x) = \dfrac{f(x) + k}{2}$

$\phi_2(x) = \dfrac{f(x) - k}{2}$

$u(x,t) = \phi_1(x - ct) + \phi_2(x + ct)$

$u(x,t) = \dfrac{f(x - ct) + \cancel{k}}{2} + \dfrac{f(x + ct) - \cancel{k}}{2}$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x - ct) + f(x + ct)\right]}$$



$u(x,t) = \frac{1}{2}(f(x_1) + f(x_2))$

Example: Solve the following Riemann problem.

$$u_t + 4u_x + 3v_x = 0, \quad u(x,0) = \begin{cases} 3, & x<0 \\ 0, & x>0 \end{cases}$$

$$v_t + u_x + 2v_x = 0, \quad v(x,0) = \begin{cases} -1, & x<0 \\ 3, & x>0 \end{cases}$$

Eigenvalues/vectors: $A = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{vmatrix} = (\lambda-4)(\lambda-2) - 3 = \lambda^2 - 6\lambda + 5 = 0$$

$$(\lambda-1)(\lambda-5) = 0$$

$$\lambda_1 = 1, \quad \lambda_2 = 5$$

$\lambda_1 = 1$: $(A - \lambda_1 I)\vec{r}_1 = \vec{0}$

$$\begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \lambda_1 = 1; \ \vec{V}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$r_2 = -r_1$

$\lambda_2 = 5$: $(A - \lambda_2 I)\vec{V}_2 = \vec{0}$

$$\begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \lambda_2 = 5; \ \vec{V}_2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$r_1 = 3r_2$

Characteristics: $x - \lambda_j t = x_j$

$$x - t = x_1$$
$$x - 5t = x_2$$

Characteristic Variables: $D = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ $R = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}$

$$R^{-1} = \frac{1}{4}\begin{pmatrix} 1 & -3 \\ 1 & 1 \end{pmatrix}$$

$$\Rightarrow \vec{V} = R^{-1}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{4}\begin{pmatrix} u - 3v \\ u + v \end{pmatrix}$$

Then,
$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum_{j=1}^{2} \phi_j(x - \lambda_j t)\vec{r}_j = \phi_1(x-t)\vec{r}_1 + \phi_2(x-5t)\vec{r}_2$$

$$= \phi_1(x-t)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \phi_2(x-5t)\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{aligned} u(x,t) &= \phi_1(x-t) + 3\phi_2(x-5t) \\ v(x,t) &= -\phi_1(x-t) + \phi_2(x-5t) \end{aligned}$$

We have
$$u(x,t) = \phi_1(x-t) + 3\phi_2(x-5t)$$
$$v(x,t) = -\phi_1(x-t) + \phi_2(x-5t)$$

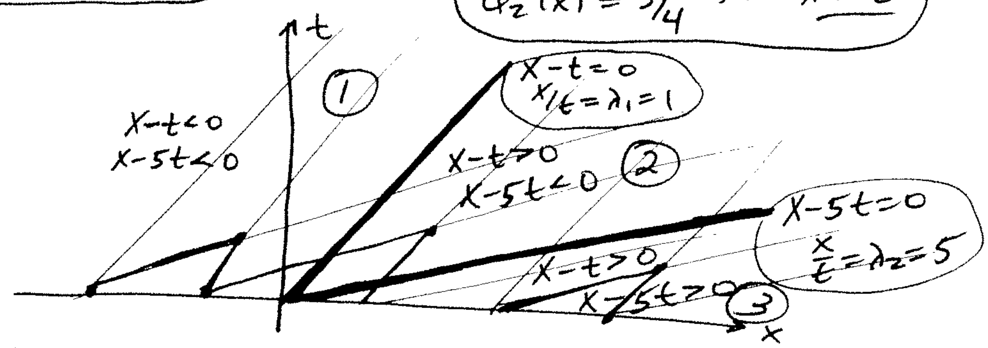
Initial Data:

$x<0$: $u(x,0) = \phi_1(x) + 3\phi_2(x) = 3$
$\quad v(x,0) = -\phi_1(x) + \phi_2(x) = -1$

$$\Rightarrow \boxed{\begin{matrix}\phi_1(x) = 3/2 \\ \phi_2(x) = \frac{1}{2}\end{matrix} \text{ for } x<0}$$

$x>0$: $u(x,0) = \phi_1(x) + 3\phi_2(x) = 0$
$\quad v(x,0) = -\phi_1(x) + \phi_2(x) = 3$

$$\Rightarrow \boxed{\begin{matrix}\phi_1(x) = -9/4 \\ \phi_2(x) = 3/4\end{matrix} \text{ for } x>0}$$



$$u(x,t) = \begin{cases} \frac{3}{2} + 3\cdot\frac{1}{2}, & x/t<1 \quad (1) \\ -9/4 + 3\cdot 1/2, & 1<x/t<5 \quad (2) \\ -9/4 + 3\cdot 3/4, & x/t>5 \quad (3)\end{cases}$$

$$v(x,t) = \begin{cases} -3/2 + 1/2, & x/t<1 \quad (1) \\ +9/4 + 1/2, & 1<\frac{x}{t}<5 \quad (2) \\ +9/4 + 3/4, & x/t>5 \quad (3)\end{cases}$$

$$\boxed{u(x,t) = \begin{cases} 3, & x/t<1 \\ -3/4, & 1<x/t<5 \\ 0, & x/t>5\end{cases}}$$

3 constant states

$$\boxed{v(x,t) = \begin{cases} -1, & x/t<1 \\ 11/4, & 1<x/t<5 \\ 3, & x/t>5\end{cases}}$$

$u_L = 3,\ u_R = 0$ ✓ $\qquad v_L = -1,\ v_R = 3$ ✓

$$\begin{pmatrix}u\\v\end{pmatrix} = \begin{cases}\begin{pmatrix}3\\-1\end{pmatrix}, & x/t<1 \\ \begin{pmatrix}-3/4\\11/4\end{pmatrix}, & 1<x/t<5 \\ \begin{pmatrix}0\\3\end{pmatrix}, & x/t>5\end{cases}$$

Note: Jump in $\begin{pmatrix}u\\v\end{pmatrix}$ across $\frac{x}{t}=\lambda_j$ is proportional to $\vec{r}_j$.

$$\Rightarrow \boxed{\begin{pmatrix}u\\v\end{pmatrix}\Big|_{\frac{x}{t}=\lambda_j^+} - \begin{pmatrix}u\\v\end{pmatrix}\Big|_{\frac{x}{t}=\lambda_j^-} = k\vec{r}_j}$$

e.g. Jump across $\frac{x}{t}=\lambda_1=1$ $= \begin{pmatrix}u\\v\end{pmatrix}\Big|_{\frac{x}{t}=1^+} - \begin{pmatrix}u\\v\end{pmatrix}\Big|_{\frac{x}{t}=1^-} = \begin{pmatrix}-3/4\\11/4\end{pmatrix} - \begin{pmatrix}3\\-1\end{pmatrix} = \begin{pmatrix}-15/4\\15/4\end{pmatrix} = -\frac{15}{4}\vec{r}_1$

Jump across $\frac{x}{t}=\lambda_2=5$ $= \begin{pmatrix}u\\v\end{pmatrix}\Big|_{\frac{x}{t}=5^+} - \begin{pmatrix}u\\v\end{pmatrix}\Big|_{\frac{x}{t}=5^-} = \begin{pmatrix}0\\3\end{pmatrix} - \begin{pmatrix}-3/4\\11/4\end{pmatrix} = \begin{pmatrix}3/4\\1/4\end{pmatrix} = \frac{1}{4}\vec{r}_2$

Linearization of Nonlinear Systems (of Conservation Laws)

Consider a general nonlinear system: $\vec{u}_t + (\vec{f}(\vec{u}))_x = \vec{0}$ (conservative form)

$\vec{u}_t + \vec{f}'(\vec{u})\,\vec{u}_x = \vec{0}$

$\Rightarrow \boxed{\vec{u}_t + A(\vec{u})\,\vec{u}_x = \vec{0}}$ where $A(\vec{u}) = \vec{f}'(\vec{u})$

(quasilinear form)

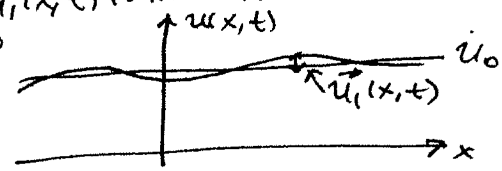
eigs: $\lambda = \lambda(\vec{u})$ wave speeds depend on $\vec{u}$

Suppose we wish to consider small disturbances from some constant state $\vec{u}_0$.

Assuming that $\vec{u}(x,t)$ remains near $\vec{u}_0$ for all (x,t), we may write

$\vec{u}(x,t) = \vec{u}_0 + \underbrace{\vec{u}_1(x,t)}_{\text{small disturbance}}$ where $\vec{u}_1(x,t)$ is much less than $\vec{u}_0$ (small compared to) for all (x,t).

1-D



then, $\vec{u}_t + A(\vec{u})\,\vec{u}_x = \vec{0}$

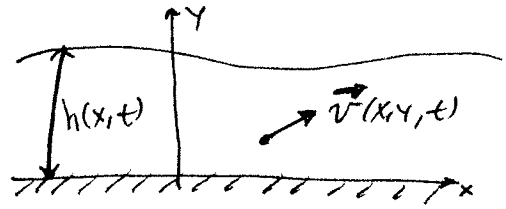
$\vec{u}_0 = \text{const} \Rightarrow (\cancel{\vec{u}_0} + \vec{u}_1)_t + A(\vec{u}_0 + \vec{u}_1)(\cancel{\vec{u}_0} + \vec{u}_1)_x = \vec{0}$

$\vec{u}_{1t} + A(\vec{u}_0 + \vec{u}_1)\,\vec{u}_{1x} = \vec{0}$

fix the Jacobian $A(\vec{u}_0 + \vec{u}_1) \approx A(\vec{u}_0) \Rightarrow \boxed{\vec{u}_{1t} + A(\vec{u}_0)\,\vec{u}_{1x} = \vec{0}}$ Approximate Linear System

constant matrix

The linearization yields a solution which approximates the exact solution $\vec{u}(x,t)$ when $\vec{u}$ is near $\vec{u}_0$. $\vec{u}(x,t) \approx \vec{u}_0 + \vec{u}_1(x,t)$



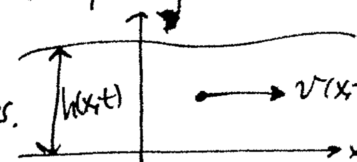
$\vec{v} = \langle v, \omega \rangle$

Example: Shallow Water Waves

Under reasonable assumptions, the 2-D problem can be reduced to a 1-D problem.

Assumptions: i) The x-component of the velocity is assumed to be constant throughout any vertical cross-section, whereas the y-component of the velocity is assumed to be negligible.

$|w| \ll |v| \Rightarrow$



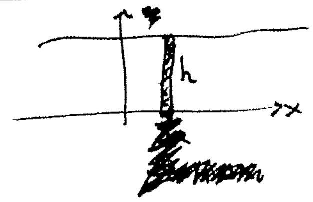
This is the case for small amplitude, shallow water waves.

"Shallow" means that h(x,t) is much smaller than wave lengths.

ii) It is assumed that the water is incompressible.

$\Rightarrow \rho_{water}(x,y,t) = \rho_0 = \text{constant}$ $\left(\frac{\text{mass}}{\text{area}}\right)$

Let $\rho = \dfrac{\text{mass of column of water } dx}{\text{length}}$ $\left(\frac{\text{mass}}{\text{length}}\right)$



Clearly, $\rho(x,t) = \rho_0 h(x,t)$

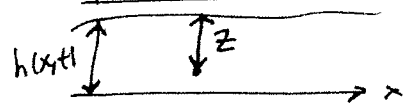
Conservation Laws (1-D)

mass: $\rho_t + (\rho v)_x = 0$

$(\cancel{\rho_0} h)_t + (\cancel{\rho_0} h v)_x = 0$

$\boxed{h_t + (hv)_x = 0}$

Hydrostatic Law



pressure at a depth z $= \rho_0 g z$

momentum: $(\rho v)_t + (\rho v^2 + P)_x = 0$

$P = $ Total pressure felt at x $= \int_0^h \rho_0 g z \, dz$

$P = \rho_0 g \frac{h^2}{2}$

$(\cancel{\rho_0} hv)_t + (\cancel{\rho_0} hv^2 + \cancel{\rho_0} g \frac{h^2}{2})_x = 0$

$hv^2 = (hv)v$

$h_t v + h v_t + (hv)_x v + hv v_x + g h h_x = 0$

$v\cancel{(h_t + (hv)_x)} + \cancel{h} v_t + \cancel{h} v v_x + g \cancel{h} h_x = 0$

$v_t + v v_x + g h_x = 0$

$v_t + \left(\frac{v^2}{2} + gh\right)_x = 0$

Conservation Form

$$\boxed{\begin{aligned} h_t + (hv)_x &= 0 \\ v_t + \left(\tfrac{v^2}{2} + gh\right)_x &= 0 \end{aligned}}$$

Nonlinear

Unknowns: h and v

We have $h_t + (hv)_x = 0$
$v_t + (v^2/2 + gh)_x = 0$

Let $u = gh$ $\Rightarrow$ $u_t + (uv)_x = 0$, $v_t + (v^2/2 + u)_x = 0$ conservation form $\Rightarrow$ $u_t + vu_x + uv_x = 0$, $v_t + vv_x + u_x = 0$

Let $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ $\Rightarrow \begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} v & u \\ 1 & v \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \vec{u}_t + \begin{pmatrix} v & u \\ 1 & v \end{pmatrix}\vec{u}_x = \vec{0}$

OR $\boxed{\vec{u}_t + A(\vec{u})\vec{u}_x = \vec{0}, \text{ where } A = \begin{pmatrix} v & u \\ 1 & v \end{pmatrix}}$ quasilinear form
Nonlinear
where $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} gh \\ v \end{pmatrix}$

Shallow Water Waves Equation

eigvals of A: $\lambda = v \pm \sqrt{u}$
(not constant)

Example: Linearization of the Shallow Water Wave Equations

Consider the initial data $u(x,0) = 1 + \varepsilon \sin(\varepsilon x)$, where ε is small and constant
$v(x,0) = 1$

Small amplitude disturbance

long waves
$\ell = 2\pi/\varepsilon$
$h = \frac{u}{g} \approx \frac{1}{9.8} \approx .1$
h is much less than ℓ
$\Rightarrow$ shallow

Linearize the system about $\vec{u}_0 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let $\vec{u}(x,t) = \vec{u}_0 + \vec{u}_1(x,t)$ where $\vec{u}_1$ is assumed to be much less than $\vec{u}_0$

Linearize System: $\vec{u}_{1t} + A(\vec{u}_0)\vec{u}_{1x} = \vec{0}$ $\quad A(\vec{u}_0) = A(1,1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$\boxed{\vec{u}_{1t} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\vec{u}_{1x} = \vec{0}}$$

Initial Data

$u(x,t) = u_0 + u_1(x,t) = 1 + u_1(x,t)$ $\qquad v(x,t) = v_0 + v_1(x,t) = 1 + v_1(x,t)$

$u_1(x,0) = u(x,0) - 1 = \varepsilon \sin(\varepsilon x)$ $\qquad v_1(x,0) = v(x,0) - 1 = 0$

$\boxed{u_1(x,0) = \varepsilon \sin(\varepsilon x)}$ $\qquad \boxed{v_1(x,0) = 0}$

- We have $\vec{U}_{1t} + \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}}_{A(\vec{u}_0)} \vec{U}_{1x}$, $\vec{u}_1 = \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}$; $\quad u_1(x,0) = \varepsilon \sin(\varepsilon x)$, $v_1(x,0) = 0$

Solution: $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 - 1 = \lambda^2 - 2\lambda + \cancel{1} - \cancel{1} = 0$

$\lambda(\lambda-2) = 0$

$\lambda_1 = 0 \quad \lambda_2 = 2$

Eigs:

$\lambda_1 = 0$: $(A - \lambda_1 I)\vec{r}_1 = \vec{0}$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \lambda_1 = 0;\ \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$r_1 = -r_2$

Characteristics: $x - \lambda_1 t = x_1$, $x = x_1$

$\lambda_2 = 2$: $(A - \lambda_2 I)\vec{r}_2 = \vec{0}$

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \lambda_2 = 2;\ \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$r_1 = r_2$

$x - \lambda_2 t = x_2$, $x - 2t = x_2$

Then, $\vec{u}_1(x,t) = \sum_{j=1}^{2} \phi_j(x - \lambda_j t)\vec{r}_j = \phi_1(x)\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \phi_2(x-2t)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow \begin{cases} u_1(x,t) = \phi_1(x) + \phi_2(x-2t) \\ v_1(x,t) = -\phi_1(x) + \phi_2(x-2t) \end{cases}$

- Initial Data:

$u_1(x,0) = \phi_1(x) + \phi_2(x) = \varepsilon \sin(\varepsilon x)$

$v_1(x,0) = -\phi_1(x) + \phi_2(x) = 0$

$\Rightarrow \phi_1(x) = \phi_2(x) = \frac{\varepsilon}{2}\sin(\varepsilon x)$

$\Rightarrow \begin{cases} u_1(x,t) = \frac{\varepsilon}{2}\sin(\varepsilon x) + \frac{\varepsilon}{2}\sin(\varepsilon(x-2t)) \\ v_1(x,t) = -\frac{\varepsilon}{2}\sin(\varepsilon x) + \frac{\varepsilon}{2}\sin(\varepsilon(x-2t)) \end{cases}$ Solution of the linear initial value problem

Recall: $\vec{u}(x,t) = \vec{u}_0 + \vec{u}_1(x,t)$ where $\vec{u}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\Rightarrow \boxed{\begin{aligned} u(x,t) &= 1 + u_1(x,t) = 1 + \tfrac{\varepsilon}{2}\left[\sin(\varepsilon x) + \sin(\varepsilon(x-2t))\right] \\ v(x,t) &= 1 + v_1(x,t) = 1 - \tfrac{\varepsilon}{2}\left[\sin(\varepsilon x) - \sin(\varepsilon(x-2t))\right] \end{aligned}}$$

and $h(x,t) = \frac{1}{g}u(x,t)$

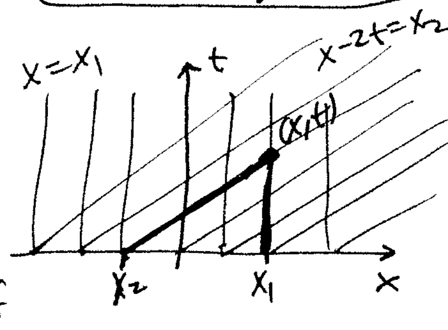
Characteristics

$x = x_1$

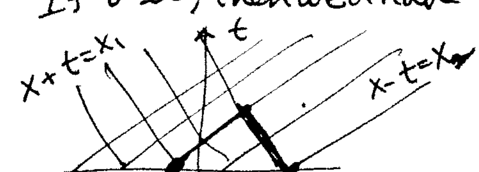
$x - 2t = x_2$

- The characteristics are not symmetric.

Since $v = 1$, the speed of both characteristics is increased by one



If $v \approx 0$, then we'd have



Alternative Solution of the linear IVP: $\vec{U}_{1t} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{U}_{1x} = \vec{0}$, $\vec{U}_1(x,0) = \begin{pmatrix} \varepsilon \sin(\varepsilon x) \\ 0 \end{pmatrix}$

$$\Rightarrow \quad U_{1t} + U_{1x} + V_{1x} = 0, \quad U_1(x,0) = \varepsilon \sin(\varepsilon x)$$
$$- \left(V_{1t} + U_{1x} + V_{1x} = 0 \right), \quad V_1(x,0) = 0$$

$$(U_1 - V_1)_t = 0$$
$$U_1(x,t) - V_1(x,t) = \phi(x)$$

$$U_1(x,0) - V_1(x,0) = \varepsilon \sin(\varepsilon x) - 0 = \phi(x) \Rightarrow \phi(x) = \varepsilon \sin(\varepsilon x)$$

$$\Rightarrow \quad V_1(x,t) = U_1(x,t) - \varepsilon \sin(\varepsilon x)$$

Then, $U_{1t} + U_{1x} + (U_1 - \varepsilon \sin(\varepsilon x))_x = 0$

$$U_{1t} + 2U_{1x} = \varepsilon^2 \cos(\varepsilon x)$$
$$U_1(x,0) = \varepsilon \sin(\varepsilon x)$$

Method of Characteristics

C: $x(\tau,0) = \tau$, $t(\tau,0) = 0$, $U_1(\tau,0) = \varepsilon \sin(\varepsilon \tau)$

$t_s = 1$ $\quad x_s = 2$ $\quad U_s = \varepsilon^2 \cos(\varepsilon x)$

$t = s$ $\quad x = 2s + \tau$ $\quad U_s = \varepsilon^2 \cos(\varepsilon(2s+\tau))$

$$\begin{vmatrix} x_\tau & x_s \\ t_\tau & t_s \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \neq 0 \ \checkmark$$

$$U_1 = \frac{\varepsilon^{\cancel{2}}}{2\cancel{\varepsilon}} \sin(\varepsilon(2s+\tau)) + C(\tau)$$

$$U_1(\tau,0) = \frac{\varepsilon}{2}\sin(\varepsilon\tau) + C(\tau) = \varepsilon \sin(\varepsilon\tau)$$
$$C(\tau) = \frac{\varepsilon}{2}\sin(\varepsilon\tau)$$

$$U_1(\tau,s) = \frac{\varepsilon}{2}\left[\sin(\varepsilon(2s+\tau)) + \sin(\varepsilon\tau)\right]$$

$2s + \tau = x \qquad \tau = x - 2t$

$$\boxed{U_1(x,t) = \frac{\varepsilon}{2}\left[\sin(\varepsilon x) + \sin(\varepsilon(x-2t))\right]}$$

and $V_1(x,t) = U_1(x,t) - \varepsilon \sin(\varepsilon x)$

$$\Rightarrow \boxed{V_1(x,t) = -\frac{\varepsilon}{2}\left[\sin(\varepsilon x) - \sin(\varepsilon(x-2t))\right]}$$

Second Order (Semilinear) PDEs (with two independent variables)

Semilinear $\Rightarrow$ Linear in the second derivatives, with coefficients depending only on the independent variables

General Form:

for convenience

$$\boxed{\underbrace{a(x,y)\,u_{xx} + 2b(x,y)\,u_{xy} + c(x,y)\,u_{yy}}_{\text{principal part}} = \underbrace{f(x,y,u,u_x,u_y)}_{\text{nonlinear}}} \quad \circledast$$

The classification of the PDE depends only on the principal part (i.e. on a, b, and c)

Classifications: Let $v = u_x$, $w = u_y$ $\qquad u_{xy} = v_y = w_x$

$$\Rightarrow \boxed{v_y - w_x = 0}$$

$$\circledast \Rightarrow \boxed{a v_x + 2b v_y + c w_y = f(x,y,u,v,w)}$$

v_y OR w_x OR $\frac{1}{2}(v_y + w_x)$ $\qquad$ u inconsequential: $u = \int v\,dx = \int w\,dy$

$$\Rightarrow \underbrace{\begin{pmatrix} 0 & -1 \\ a & 0 \end{pmatrix}}_{A} \begin{pmatrix} v \\ w \end{pmatrix}_x + \underbrace{\begin{pmatrix} 1 & 0 \\ 2b & c \end{pmatrix}}_{B} \begin{pmatrix} v \\ w \end{pmatrix}_y = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

$$\det(B - \lambda A) = \begin{vmatrix} 1 & \lambda \\ 2b - a\lambda & c \end{vmatrix} = c - \lambda(2b - a\lambda) = a\lambda^2 - 2b\lambda + c = 0$$

$(\det A \neq 0)$

$$\lambda = \frac{2b \pm \sqrt{4b^2 - 4ac}}{2a}$$

$$\lambda = \frac{b \pm \sqrt{b^2 - ac}}{a}$$

Then,

① $\boxed{b^2 - ac < 0 \Rightarrow \text{Elliptic}}$

$\lambda_1, \lambda_2 = \alpha \pm i\beta$

② $\boxed{b^2 - ac > 0 \Rightarrow \text{Hyperbolic}}$

$\lambda_1 \neq \lambda_2$ are real

$$b^2 - ac = 0 \Rightarrow \lambda_1 = \lambda_2 = \frac{b}{a} \Rightarrow$$

Parabolic (1 linearly independent eigenvecto) or Hyperbolic (2 linearly independent eigenvectors)

Eigenvectors: $(B - \lambda A)\vec{v} = \vec{0}$

$$\begin{pmatrix} 1 & b/a \\ 2b - a\frac{b}{a} & c \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$c = b^2/a \Rightarrow \begin{pmatrix} 1 & b/a \\ b & b^2/a \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$v_1 = -\frac{b}{a} v_2 \Rightarrow \vec{v} = \begin{pmatrix} b \\ -a \end{pmatrix}$$

$\Rightarrow$ Only one linearly independent eigenvector $\Rightarrow$ Parabolic

③ $\boxed{b^2 - ac = 0 \Rightarrow \text{Parabolic}}$

Note: Since the coefficients a, b, and c are functions of x and y, the classification of $\circledast$ may vary from point to point in the xy-plane

Classical Examples of Each Type

1. Wave Equation: $\boxed{u_{tt} = k^2 u_{xx}}$ $k^2 u_{xx} - u_{tt} = 0$
   $a = k^2 \quad b = 0 \quad c = -1$

   $b^2 - ac = 0 + k^2 = k^2 > 0 \Rightarrow$ $\boxed{\text{Hyperbolic}}$ (Wave propagation)

2. Diffusion (Heat) Equation: $\boxed{u_t = \kappa u_{xx},\ \kappa > 0}$ $\kappa u_{xx} = u_t$
   $a = \kappa \quad b = 0 \quad c = 0$

   $b^2 - ac = 0 - 0 = 0 \Rightarrow$ $\boxed{\text{Parabolic}}$ (Diffusion)

3. Laplace Equation: $\boxed{u_{xx} + u_{yy} = 0}$
   $a = 1 \quad b = 0 \quad c = 1$

   $b^2 - ac = 0 - 1 = -1 < 0 \Rightarrow$ $\boxed{\text{Elliptic}}$ (Steady-State) time-independent

   e.g. The Laplace equation yields steady-state solutions of the 2-D heat equation, $u_t = \kappa(u_{xx} + u_{yy})$.

These are the classical representative examples of each classification. The behavior of the solutions of these PDEs characterizes many of the general features of the solutions of the PDEs from each classification.

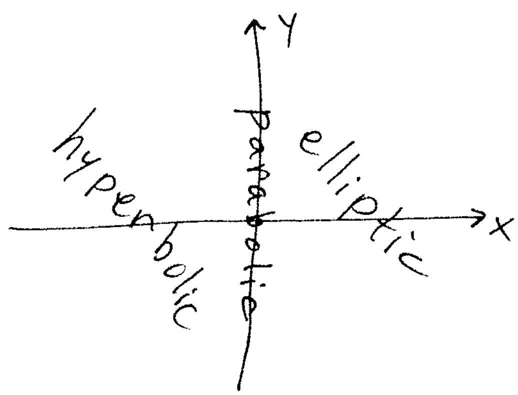
Examples: $x u_{xx} + u_{yy} = \sin x$

$a = x \quad b = 0 \quad c = 1$

$b^2 - ac = 0 - x = -x$

$b^2 - ac = -x$

$\Rightarrow$ $\boxed{\begin{array}{l}\text{Elliptic for } x > 0 \\ \text{Parabolic for } x = 0 \\ \text{Hyperbolic for } x > 0\end{array}}$



Canonical Forms

With an appropriate coordinate transformation, the general PDE, $a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = f(x,y,u,u_x,u_y)$, ⊗ can be cast into one of the following 3 canonical forms, depending on its classification.

Elliptic: ($b^2 - ac < 0$) $\quad u_{\xi\xi} + u_{\eta\eta} = F(\xi,\eta,u,u_\xi,u_\eta)$

Parabolic: ($b^2 - ac = 0$) $\quad u_{\xi\xi} = F(\xi,\eta,u,u_\xi,u_\eta)$

Hyperbolic: ($b^2 - ac > 0$) $\quad u_{\xi\xi} - u_{\eta\eta} = F(\xi,\eta,u,u_\xi,u_\eta)$

OR

$u_{\xi\eta} = F(\xi,\eta,u,u_\xi,u_\eta)$

The hyperbolic case has two equivalent canonical forms.

OR $\quad u_{\xi\xi} + k\,u_{\eta\eta} = F(\xi,\eta,u,u_\xi,u_\eta)$

$a=1 \quad b=0 \quad c=k$

$b^2 - ac = -k$

$k>0 \Rightarrow$ Elliptic

$k=0 \Rightarrow$ Parabolic

$k<0 \Rightarrow$ Hyperbolic

Note: Canonical Forms are useful because ...

i) Much of the theory of the 3 classifications assumes the corresponding canonical form. In this way, equation ⊗ can be cast into canonical form, and the theory applied.

ii) Computational methods and software assume that the PDE ⊗ is written in canonical form.

- The two canonical forms for hyperbolic PDEs are equivalent in that we can easily convert either into the other.

Example: Consider the wave equation

$$\boxed{u_{tt} = c^2 u_{xx}} \quad \left(c=1 \Rightarrow \text{first canonical form } u_{tt} - u_{xx} = 0 \right)$$

Let $\boxed{\begin{array}{l} \xi = x - ct \\ \eta = x + ct \end{array}}$

The grid lines of the (ξ,η) coordinate system are the characteristics of the wave equation

$$u_x = u_\xi + u_\eta$$
$$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$
$$u_t = -c(u_\xi - u_\eta)$$
$$u_{tt} = c^2(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})$$

$$\cancel{c^2}(\cancel{u_{\xi\xi}} - 2u_{\xi\eta} + \cancel{u_{\eta\eta}}) = \cancel{c^2}(\cancel{u_{\xi\xi}} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}})$$
$$-4u_{\xi\eta} = 0$$

$$\boxed{u_{\xi\eta} = 0} \quad \text{(second canonical form)}$$

$$u_{xx} - u_{yy} = f(x,y,u,u_x,u_y) \xrightarrow{\substack{\xi = x-y \\ \eta = x+y}} u_{\xi\eta} = F(\xi,\eta,u,u_\xi,u_\eta)$$
$$u_{\xi\eta} = F(\xi,\eta,u,u_\xi,u_\eta) \xrightarrow{\substack{x = \xi-\eta \\ y = \xi+\eta}} u_{xx} - u_{yy} = f(x,y,u,u_x,u_y)$$

Solve: $u_{\xi\eta} = 0$

$\int \cdot \, d\eta \Rightarrow u_\xi = \psi_1(\xi)$

$\int \cdot \, d\xi \Rightarrow u = \underbrace{\int \psi_1(\xi)\, d\xi}_{\phi_1(\xi)} + \phi_2(\eta)$

$$\boxed{u = \phi_1(\xi) + \phi_2(\eta) = \phi_1(x-ct) + \phi_2(x+ct)}$$

This is the same general solution found by converting the wave equation into a linear system of two first order PDEs

Note: The characteristics of the wave equation are the level curves of the coordinates, $\xi(x,y) = \xi_0$ and $\eta(x,y) = \eta_0$, that bring the wave equation into canonical form ($u_{\xi\eta} = 0$).

Wave Equation + L.O.T. $\longrightarrow$ $\begin{array}{l} \xi = x-ct \\ \eta = x+ct \end{array}$ $\longrightarrow$ Canonical Form

$$u_{tt} = c^2 u_{xx} + f(x,t,u,u_x,u_t) \qquad u_{\xi\eta} = F(\xi,\eta,u,u_\xi,u_\eta)$$

Characteristics: $x - ct = \xi_0$, $x + ct = \eta_0$

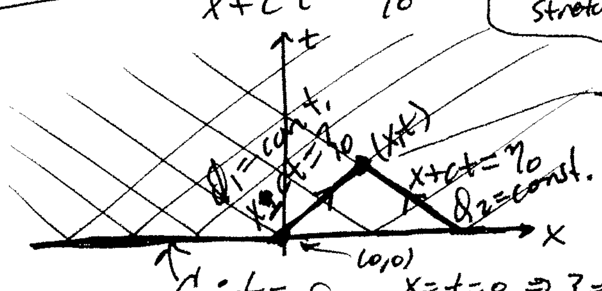
$c=1$: $\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \sqrt{2}\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}\begin{pmatrix}x\\ t\end{pmatrix}$

$\sqrt{2}$ factor stretch, $45°$ rotation

$\xi = \xi_0$, $\eta = \eta_0$

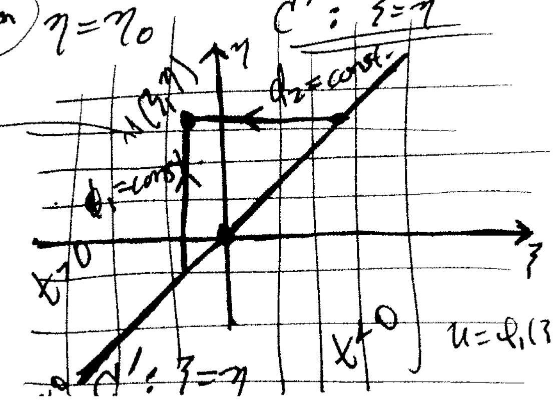
C': $t = 0 \Rightarrow \xi = \eta = x$

C': $\xi = \eta$



$C: t = 0$ $\quad x = t = 0 \Rightarrow \xi = \eta = 0$

$u = \phi_1(x-ct) + \phi_2(x+ct)$



$u = \phi_1(\xi) + \phi_2(\eta)$

Coordinate Transformation

$$a(x,y)\,u_{xx} + 2b(x,y)\,u_{xy} + c(x,y)\,u_{yy} = L.O.T.$$

Lower Order Terms (i.e. $f(\xi,\eta,u,u_\xi,u_\eta)$)

Let $\xi=\xi(x,y)$, $\eta=\eta(x,y)$ where $\begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix} = \xi_x\eta_y - \xi_y\eta_x \neq 0$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$

$$u_{xx} = (u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x)\xi_x + \underbrace{u_\xi \xi_{xx}}_{L.O.T.} + (u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x)\eta_x + \underbrace{u_\eta \eta_{xx}}_{L.O.T.}$$

$$u_{xx} = \xi_x^2 u_{\xi\xi} + 2\xi_x\eta_x u_{\xi\eta} + \eta_x^2 u_{\eta\eta} + L.O.T.$$

Similarly $$u_{yy} = \xi_y^2 u_{\xi\xi} + 2\xi_y\eta_y u_{\xi\eta} + \eta_y^2 u_{\eta\eta} + L.O.T.$$

$$u_{xy} = (u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y)\xi_x + u_\xi \xi_{xy} + (u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y)\eta_x + u_\eta \eta_{xy}$$

$$u_{xy} = \xi_x\xi_y u_{\xi\xi} + (\xi_x\eta_y + \xi_y\eta_x)u_{\xi\eta} + \eta_x\eta_y u_{\eta\eta} + L.O.T.$$

$$a(\xi_x^2 u_{\xi\xi} + 2\xi_x\eta_x u_{\xi\eta} + \eta_x^2 u_{\eta\eta}) + 2b(\xi_x\xi_y u_{\xi\xi} + (\xi_x\eta_y + \xi_y\eta_x)u_{\xi\eta} + \eta_x\eta_y u_{\eta\eta}) + c(\xi_y^2 u_{\xi\xi} + 2\xi_y\eta_y u_{\xi\eta} + \eta_y^2 u_{\eta\eta}) = L.O.T.$$

$$\underbrace{(a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2)}_{A} u_{\xi\xi} + 2\underbrace{(a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y)}_{B} u_{\xi\eta} + \underbrace{(a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2)}_{C} u_{\eta\eta} = L.O.T.$$

$$\Rightarrow \boxed{\begin{aligned} &A u_{\xi\xi} + 2B u_{\xi\eta} + C u_{\eta\eta} = f(\xi,\eta,u,u_\xi,u_\eta)\\ &\text{where } A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2\\ &\qquad\;\; B = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y\\ &\qquad\;\; C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2\end{aligned}}$$

$$B^2 - AC = \underbrace{\begin{pmatrix} \cancel{a^2\xi_x^2\eta_x^2} + b^2(\xi_x^2\eta_y^2 + \cancel{2\xi_x\xi_y\eta_x\eta_y} + \xi_y^2\eta_x^2) + \cancel{c^2\xi_y^2\eta_y^2}\\ + \cancel{2ab\xi_x\eta_x(\xi_x\eta_y + \xi_y\eta_x)} + 2ac\xi_x\xi_y\eta_x\eta_y + \cancel{2bc\xi_y\eta_y(\xi_x\eta_y + \xi_y\eta_x)}\end{pmatrix}}_{B^2} - \underbrace{\begin{pmatrix} \cancel{a^2\xi_x^2\eta_x^2} + \cancel{2ab\xi_x^2\eta_x\eta_y} + ac\xi_x^2\eta_y^2 + \cancel{2ab\xi_x\xi_y\eta_x^2} + 4b^2\xi_x\xi_y\eta_x\eta_y\\ + \cancel{2bc\xi_x\xi_y\eta_y^2} + ac\xi_y^2\eta_x^2 + \cancel{2bc\xi_y^2\eta_x\eta_y} + \cancel{c^2\xi_y^2\eta_y^2}\end{pmatrix}}_{AC}$$

$$= b^2(\xi_x^2\eta_y^2 + \xi_y^2\eta_x^2 - 2\xi_x\xi_y\eta_x\eta_y) - ac(\xi_x^2\eta_y^2 + \xi_y^2\eta_x^2 - 2\xi_x\xi_y\eta_x\eta_y)$$

$$= (b^2 - ac)(\xi_x\eta_y - \xi_y\eta_x)^2 \qquad \begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix} = \xi_x\eta_y - \xi_y\eta_x \neq 0 \Rightarrow (\xi_x\eta_y - \xi_y\eta_x)^2 > 0$$

$$\boxed{B^2 - AC = (b^2 - ac)(\xi_x\eta_y - \xi_y\eta_x)^2}$$

$\Rightarrow$ the classification of a PDE is <u>Invariant</u> under an invertible transformation.

i.e. An invertible coordinate tranformation does not change the classification of a PDE.

Converting to Canonical Form

Hyperbolic PDEs ($b^2-ac>0$): To convert to canonical form ($u_{\xi\eta}=F$), choose $\xi(x,y)$ and $\eta(x,y)$ so that $A=0$ and $C=0$.

Set $A = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2 = 0$ and $C = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2$

$$\Rightarrow \quad a\left(\frac{\xi_x}{\xi_y}\right)^2 + 2b\left(\frac{\xi_x}{\xi_y}\right) + c = 0 \qquad\qquad a\left(\frac{\eta_x}{\eta_y}\right)^2 + 2b\left(\frac{\eta_x}{\eta_y}\right) + c = 0$$

$$\frac{\xi_x}{\xi_y} = \frac{-\cancel{2}b \pm \sqrt{\cancel{4}b^2-\cancel{4}ac}}{\cancel{2}a} \qquad\qquad \frac{\eta_x}{\eta_y} = \frac{-\cancel{2}b \pm \sqrt{\cancel{4}b^2-\cancel{4}ac}}{\cancel{2}a}$$

$$\frac{\xi_x}{\xi_y} = \frac{-b \pm \sqrt{b^2-ac}}{a} \qquad\qquad \frac{\eta_x}{\eta_y} = \frac{-b \pm \sqrt{b^2-ac}}{a}$$

double families of curves

Let '−' correspond to ξ and '+' correspond to η.

$$\Rightarrow \quad \frac{\xi_x}{\xi_y} = \frac{-b-\sqrt{b^2-ac}}{a} \qquad\qquad \frac{\eta_x}{\eta_y} = \frac{-b+\sqrt{b^2-ac}}{a}$$

The grid lines of the ξ coordinate are the level curves $\xi(x,y)=\xi_0$

Along the level curves, we have

$$\xi(x,y) = \xi_0$$

$$d\xi = \xi_x\,dx + \xi_y\,dy = 0$$

$$\boxed{\frac{dy}{dx} = \frac{-\xi_x}{\xi_y} = \frac{b+\sqrt{b^2-ac}}{a}}$$

Along the second set of level curves ($\eta(x,y)=\eta_0$), we have

$$\boxed{\frac{dy}{dx} = \frac{-\eta_x}{\eta_y} = \frac{b-\sqrt{b^2-ac}}{a}}$$

Solve these two ODEs to obtain the implicit solutions

$$\xi(x,y) = \xi_0 \quad \text{and} \quad \eta(x,y) = \eta_0.$$

Example: Convert the PDE $u_{xx} - 4u_{yy} + u_x = 0$ to canonical form.

$$a = 1,\ b = 0,\ c = -4 \Rightarrow b^2 - ac = 4 > 0$$
$$\Rightarrow \text{Hyperbolic}$$

ξ-coordinate

$$\frac{dy}{dx} = \frac{\cancel{b} + \sqrt{b^2 - ac}}{a} = 2$$
$$y = 2x + \xi_0$$

η-coordinate

$$\frac{dy}{dx} = \frac{\cancel{b} - \sqrt{b^2 - ac}}{a} = -2$$
$$y = -2x + \eta_0$$

Implicit Solutions

$$\boxed{\xi(x,y) = y - 2x = \xi_0} \qquad \boxed{\eta(x,y) = y + 2x = \eta_0}$$

then,
$$u_{xx} - 4u_{yy} = \overset{A=0}{\cancel{A}} u_{\xi\xi} + 2B u_{\xi\eta} + \overset{C=0}{\cancel{C}} u_{\eta\eta}$$
$$= 2\left[a\xi_x\eta_x + \overset{b=0}{\cancel{b(\xi_x\eta_y + \xi_y\eta_x)}} + c\xi_y\eta_y\right] u_{\xi\eta}$$
$$= 2\left[(1)(-2)(2) + (-4)(1)(1)\right] u_{\xi\eta}$$
$$u_{xx} - 4u_{yy} = -16\, u_{\xi\eta}$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x = -2u_\xi + 2u_\eta$$
$$u_x = 2(u_\eta - u_\xi)$$

$$\Rightarrow \quad u_{xx} - 4u_{yy} + u_x = 0$$
$$-16\, u_{\xi\eta} + 2(u_\eta - u_\xi) = 0$$
$$\boxed{u_{\xi\eta} = \frac{1}{8}(u_\eta - u_\xi)}$$

Note: The level curves $\xi(x,y)=\xi_0$ and $\eta(x,y)=\eta_0$ are the characteristics of the PDE. To see this, write the PDE as

$$\underbrace{u_{xx} = 4u_{yy}}_{\text{wave equation with } c=2} - u_x$$

Recall: $u_{tt} = c^2 u_{xx}$ (wave equation)

Characteristics: $x - ct = x_0$
$x + ct = x_0$

$\Rightarrow$ Characteristics: $y - 2x = \xi_0$, $y + 2x = \eta_0$ These are the level curves $\xi(x,y)=\xi_0$ and $\eta(x,y)=\eta_0$

$$\Rightarrow \boxed{\begin{aligned} \xi(x,y) &= y - 2x = \xi_0 \\ \eta(x,y) &= y + 2x = \eta_0 \end{aligned}} \quad \text{(Characteristics)}$$

In general, consider $u_{\xi\eta} = F(\xi,\eta,u,u_\xi,u_\eta)$ (1)

Let $x = \xi - \eta$ and $t = \xi + \eta$ $\Rightarrow$ $\boxed{\begin{aligned} \xi(x,t) &= \frac{x+t}{2} \\ \eta(x,t) &= -\frac{x-t}{2} \end{aligned}}$ $\quad x+t = 2\xi(x,t)$, $\quad x-t = -2\eta(x,t)$

(1) $\Rightarrow$ $u_{tt} - u_{xx} = L.O.T.$ (alternative canonical form)
wave equation with $c=1$.

$\Rightarrow$ Characteristics: $x + t = C_1 \qquad x - t = C_2$

$2\xi(x,t) = C_1 \qquad -2\eta(x,t) = C_2$

$\xi(x,t) = \xi_0 = \frac{C_1}{2} \qquad \eta(x,t) = \eta_0 = -\frac{C_2}{2}$

The characteristics of $u_{\xi\eta} = F$ are given by $\xi = \xi_0$ and $\eta = \eta_0$.

Conclusion: The characteristics of a hyperbolic PDE are the level curves, $\xi(x,y)=\xi_0$ and $\eta(x,y)=\eta_0$, of the coordinates, ξ and η, which bring the PDE to the canonical form $u_{\xi\eta} = F(\xi,\eta,u,u_\xi,u_\eta)$.

$$\boxed{\text{Characteristics: } \begin{aligned} \xi(x,y) &= \xi_0 \\ \eta(x,y) &= \eta_0 \end{aligned}} \quad \text{(Hyperbolic PDEs)}$$

<u>Parabolic PDEs ($b^2-ac=0$)</u>: To convert to canonical form ($u_{\xi\xi}=F$), choose $\xi(x,y)$ and $\eta(x,y)$ so that $B=0$ and $C=0$.

<u>First set $C=0$</u>: $\Rightarrow$ $C=a\eta_x^2+2b\eta_x\eta_y+c\eta_y^2=0$

As before, $a\left(\frac{\eta_x}{\eta_y}\right)^2+2b\left(\frac{\eta_x}{\eta_y}\right)+c=0$

$$\frac{\eta_x}{\eta_y}=\frac{-2b\pm\cancel{\sqrt{4b^2-4ac}}}{2a} \quad \to b^2-ac=0$$

$$\frac{\eta_x}{\eta_y}=\frac{-b}{a}$$

Along the level curves,

$\eta(x,y)=\eta_0$

$d\eta=\eta_x dx+\eta_y dy \Rightarrow \boxed{\frac{dy}{dx}=\frac{-\eta_x}{\eta_y}=\frac{b}{a}}$ Solve to get $\eta(x,y)=\eta_0$. (implicit solution)

Then,

$$B=a\xi_x\eta_x+b(\xi_x\eta_y+\xi_y\eta_x)+c\xi_y\eta_y$$

$$=\eta_y\left[a\xi_x\frac{\eta_x}{\eta_y}+b\left(\xi_x+\xi_y\frac{\eta_x}{\eta_y}\right)+c\xi_y\right]$$

$\frac{\eta_x}{\eta_y}=\frac{-b}{a}\Rightarrow$ $$=\eta_y\left[a\xi_x\left(\frac{-b}{a}\right)+b\left(\xi_x+\xi_y\left(\frac{-b}{a}\right)\right)+c\xi_y\right]$$

$$=\eta_y\left[\cancel{-b\xi_x+b\xi_x}-\frac{\xi_y}{a}\underbrace{(b^2-ac)}_{=0}\right]=0$$

<u>$B=0$</u> $\Rightarrow$ Choosing $\eta(x,y)$ so that $C=0$ results in $B=0$ as well.

Therefore, there are no additional constraints on $\xi(x,y)$.

$\xi(x,y)$ can be any coordinate such that
$$\begin{vmatrix}\xi_x & \xi_y\\ \eta_x & \eta_y\end{vmatrix}=\xi_x\eta_y-\xi_y\eta_x\neq 0$$
(to ensure invertibility)

Possible Convenient Choices: $\xi(x,y)=x$ OR $\xi(x,y)=y$

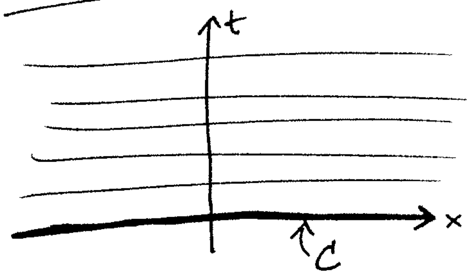
Characteristics

Parabolic PDEs have only one family of characteristics, $\eta(x,y)=\eta_0$

Example: $u_t = \kappa u_{xx}$, $\kappa > 0$ (Heat Equation)
$u(x,0) = u_0(x)$

$\kappa u_{xx} = u_t$ $\quad$ $b^2 - ac = 0 \Rightarrow$ Parabolic
$a = \kappa \quad b = 0 \quad c = 0$ $\quad$ Canonical Form: $u_{xx} = u_t/\kappa$

Characteristics: Solve $-\dfrac{\eta_x}{\eta_t} = \dfrac{dt}{dx} = \dfrac{b}{a} = 0$
$\Rightarrow t = \text{const.}$
$$\boxed{\eta(x,t) = t = \eta_0}$$



Information does not propagate along the characteristics of parabolic PDEs.

Change of Coordinates: $\eta(x,t) = t$ $\quad \Rightarrow \quad u_{\xi\xi} = u_\eta/\kappa$
Pick $\xi(x,t) = x$ $\quad$ (canonical form)

Other choices of $\xi(x,t)$ will affect only the lower order terms.
We'd still have $u_{\xi\xi} = F(\xi, \eta, u, u_\xi, u_\eta)$.

Example: Convert $u_{xx} + 2u_{xy} + u_{yy} = u_x + u_y$ to canonical form.

$a=1 \quad b=1 \quad c=1 \qquad b^2 - ac = 0 \Rightarrow$ Parabolic

Pick $\eta(x,y)$ so that $\dfrac{-\eta_x}{\eta_y} = \dfrac{dy}{dx} = \dfrac{b}{a} = 1$

$\Rightarrow \; y = x + \eta_0$

Implicit Solution: $\boxed{\eta(x,y) = y - x = \eta_0}$ (characteristics)

Pick $\xi(x,y)$ so that $\begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \begin{vmatrix} \xi_x & \xi_y \\ -1 & 1 \end{vmatrix} = \xi_x + \xi_y \neq 0.$

Let $\boxed{\xi(x,y) = x}$

Then, $u_{xx} + 2u_{xy} + u_{yy} = A u_{\xi\xi} + \cancel{2B u_{\xi\eta}} + \cancel{C u_{\eta\eta}}$ $\quad B=0 \quad C=0 \quad$ ← by the choice of η.

$= (a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2)\, u_{\xi\xi}$

$a=b=c=1$ $\qquad = ((1)(1)^2 + \cancel{2(1)(1)(0)} + \cancel{(1)(0)^2})\, u_{\xi\xi} = u_{\xi\xi}$

$u_{xx} + 2u_{xy} + u_{yy} = u_{\xi\xi}$

$u_x + u_y = (u_\xi \xi_x + u_\eta \eta_x) + (u_\xi \xi_y + u_\eta \eta_y)$

$= u_\xi - \cancel{u_\eta} + 0 + \cancel{u_\eta} = u_\xi$

$u_x + u_y = u_\xi$

The PDE becomes $u_{xx} + 2u_{xy} + u_{yy} = u_x + u_y$

$\boxed{u_{\xi\xi} = u_\xi}$ (Canonical Form)

Characteristics: $\boxed{\eta(x,y) = y - x = \eta_0}$

Note: The choice of $\xi(x,y)$ affects only the L.O.T.
The principal part is the same regardless of $\xi(x,y)$.

Solve: $u_{\xi\xi} = u_\xi$

$\int \cdot\, d\xi \Rightarrow u_\xi = u + \phi(\eta)$

$u_\xi - u = \phi(\eta)$

Integrating Factor $= e^{-\xi} \Rightarrow e^{-\xi}u = \int \phi(\eta) e^{-\xi} d\xi$

$e^{-\xi}u = -\phi(\eta)e^{\xi} + \psi(\eta)$

$-\phi(\eta) \to \phi(\eta) \quad \Longrightarrow \quad u(\xi,\eta) = \phi(\eta) + \psi(\eta)e^{\xi}$

$$\boxed{u(x,y) = \phi(y-x) + \psi(y-x)e^{x}} \quad \text{(general solution)}$$

OR $\quad u_{\xi\xi} = u_\xi$

$\dfrac{u_{\xi\xi}}{u_\xi} = 1$

$\int \cdot\, d\xi \Rightarrow \ln|u_\xi| = \xi + \psi(\eta)$

$|u_\xi| = e^{\xi + \psi(\eta)} = e^{\psi(\eta)}e^{\xi}$

$e^{\psi(\eta)} \to \psi(\eta) > 0 \Rightarrow |u_\xi| = \psi(\eta)e^{\xi}$

$u_\xi = \pm\psi(\eta)e^{\xi}$

$\pm\psi(\eta) \to \psi(\eta) \Rightarrow u_\xi = \psi(\eta)e^{\xi}$

$\int \cdot\, d\xi \Rightarrow u(\xi,\eta) = \psi(\eta)e^{\xi} + \phi(\eta)$ Same as above

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Note: The orginal PDE is symmetric with respect to x and y.

The general solution should be symmetric as well. Is it?

Yes it is. To see this, let $\psi(y-x) = e^{y-x}\sigma(y-x)$ (i.e. $\sigma(y-x) = e^{-(y-x)}\psi(y-x)$)

$\Rightarrow u(x,y) = \phi(x,y) + \sigma(y-x)e^{y}$

OR let $\psi(y-x) = e^{\frac{1}{2}(y-x)}\sigma(y-x)$

$$\Rightarrow \boxed{u(x,y) = \phi(y-x) + \sigma(y-x)e^{\frac{x+y}{2}}} \quad \text{(general solution)}$$

can replace y−x with x−y

Elliptic PDEs ($b^2-ac<0$): To convert to canonical form ($u_{\xi\xi}+u_{\eta\eta} = LOT$), find $\xi(x,y)$ and $\eta(x,y)$ so that $B=0$ and $A=C$.

The equations $B=0$ and $A=C$ are not readily solvable.
A slightly different approach must be taken for the elliptic case.

Consider $u_{\xi\xi}+u_{\eta\eta} = F(\xi,\eta,u,u_\xi,u_\eta)$

Let $\beta = i\eta$ $\Rightarrow$ $\underbrace{u_{\xi\xi}-u_{\beta\beta}} = F(\xi,-i\beta,u,u_\xi, iu_\beta)$

$\Rightarrow u_\eta = i u_\beta$

$u_{\eta\eta} = -u_{\beta\beta}$

Resembles the hyperbolic canonical form, but with a complex independent variable ($\beta = i\eta$) and LOTs!

Idea: Given $au_{xx}+2bu_{xy}+cu_{yy} = LOT$ ⊛, the elliptic canonical form can be realized by making two coordinate transformats: the first brings us to the land of complex variables, whereas the second returns us to the realm of real variables.

1) If the technique used for the hyperbolic case is applied, i.e. solve the two ODEs $\frac{dy}{dx} = \frac{b\pm\sqrt{b^2-ac}}{a}$, (complex RHS) it leads to a complex coordinate transformation in which the new independent variables are complex conjugates, say $s(x,y)$ and $\bar{s}(x,y)$, the resulting PDE is the canonical form of the hyperbolic case, but with complex conjugate independent variables, and complex LOTs.

$\Rightarrow$ $u_{s\bar{s}} = LOTs$

2) Let $\xi = s+\bar{s}$ and $\eta = i(s-\bar{s})$. Then, ξ and η are real, as is the PDE, which takes the form $u_{\xi\xi}+u_{\eta\eta} = LOTs$ (elliptic canonical form)

$$au_{xx}+2bu_{xy}+cu_{yy} = LOTs \xrightarrow[\bar{s}=\bar{s}(x,y)]{s=s(x,y)} u_{s\bar{s}} = LOTs \xrightarrow[\eta=i(s-\bar{s})]{\xi=s+\bar{s}} u_{\xi\xi}+u_{\eta\eta} = LOTs.$$

Given $au_{xx} + 2bu_{xy} + cu_{yy} = LOT.$

1) Use the technique of the hyperbolic case to find $s(x,y)$ and $\bar{s}(x,y)$.

Solve $\dfrac{dy}{dx} = \dfrac{b \pm \sqrt{b^2 - ac}}{a} = \dfrac{b \pm i\sqrt{ac-b^2}}{a}$ $(ac-b^2>0)$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{b+i\sqrt{ac-b^2}}{a}$ and $\dfrac{dy}{dx} = \dfrac{b-i\sqrt{ac-b^2}}{a}$

Implicit Solutions: $\boxed{s(x,y) = C_1}$ and $\boxed{\bar{s}(x,y) = C_2}$

$\Longrightarrow \boxed{u_{s\bar{s}} = LOT}$ (Complex PDE)

2) Let $\boxed{\xi = s + \bar{s}}$ and $\boxed{\eta = i(s - \bar{s})}$.

Then, $\xi = 2Re(s)$ and $\eta = -2Im(s)$ $\Rightarrow$ ξ and η are real

$u_s = u_\xi \xi_s + u_\eta \eta_s = u_\xi + i u_\eta$

$u_{s\bar{s}} = (u_{\xi\xi}\xi_{\bar{s}} + u_{\xi\eta}\eta_{\bar{s}}) + i(u_{\xi\eta}\xi_{\bar{s}} + u_{\eta\eta}\eta_{\bar{s}})$

$= u_{\xi\xi}(1) + \cancel{u_{\xi\eta}(-i)} + \cancel{i u_{\xi\eta}(1)} + i u_{\eta\eta}(-i)$

$= u_{\xi\xi} + u_{\eta\eta}$

$\Longrightarrow \boxed{u_{\xi\xi} + u_{\eta\eta} = LOT}$ (elliptic canonical form)

(real PDE)

Example: Convert $u_{xx} + 4x^2 u_{yy} = y + u_x$ to canonical form.

$a = 1 \quad b = 0 \quad c = 4x^2 \Rightarrow b^2 - ac = -4x^2 < 0 \Rightarrow$ Elliptic

1) Solve $\dfrac{dy}{dx} = \dfrac{b \pm i\sqrt{ac - b^2}}{a} = \pm i\sqrt{c} = \pm i2x$

$\Rightarrow y = \pm i x^2 + C$

Implicit solutions: $y \mp i x^2 = C \Rightarrow \boxed{\begin{array}{l} s(x,y) = y - ix^2 \\ \bar{s}(x,y) = y + ix^2 \end{array}}$

$\Rightarrow \boxed{u_{s\bar{s}} = LOT}$ (complex PDE)

2) Let $\boxed{\xi = s + \bar{s} = 2y}$ and $\boxed{\eta = i(s - \bar{s}) = 2x^2 > 0}$

Then, $u_x = \cancel{u_\xi \xi_x} + u_\eta \eta_x = 4x u_\eta$

$u_{xx} = 4x(\cancel{u_{\eta\xi}\xi_x} + u_{\eta\eta}\eta_x) \overset{+4u_\eta}{=} 16x^2 u_{\eta\eta} + 4u_\eta$

$u_y = u_\xi \xi_y + \cancel{u_\eta \eta_y} = 2u_\xi$ $\qquad y = \frac{\xi}{2}$

$u_{yy} = 2(u_{\xi\xi}\xi_y + \cancel{u_{\xi\eta}\eta_y}) = 4u_{\xi\xi}$ $\qquad x^2 = \eta/2$

$x = \sqrt{\eta/2}$

$\Rightarrow \quad u_{xx} + 4x^2 u_{yy} = y + u_x$

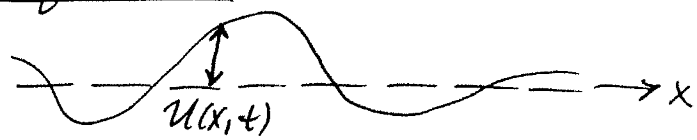
$(16x^2 u_{\eta\eta} + 4u_\eta) + 4x^2 \cdot 4u_{\xi\xi} = \frac{\xi}{2} + 4x u_\eta$

$16x^2(u_{\xi\xi} + u_{\eta\eta}) = \frac{\xi}{2} + 4u_\eta(x - 1)$

$$\boxed{u_{\xi\xi} + u_{\eta\eta} = \frac{1}{8\eta}\left[\frac{\xi}{2} + 4\left(\sqrt{\eta/2} - 1\right)u_\eta\right]}$$

Wave Equation

$$\boxed{u_{tt} = c^2 u_{xx}, \quad c>0}$$



d'Alembert Solution

The d'Alembert solution is the solution of the wave equation subject to a specified initial position and velocity.

$$\boxed{\begin{aligned} &u_{tt} = c^2 u_{xx}, \quad c>0 \\ &u(x,0) = f(x) \\ &u_t(x,0) = g(x) \end{aligned}} \quad \begin{aligned} &\\ &\text{(initial position)} \\ &\text{(initial velocity)} \end{aligned}$$

The general solution of the wave equation is

$$u(x,t) = \phi_1(x+ct) + \phi_2(x-ct).$$

$\phi_1(x+ct)$ = left moving wave
$\phi_2(x-ct)$ = right moving wave

Apply the initial data to determine the functions ϕ_1 and ϕ_2.

$$u(x,0) = \phi_1(x) + \phi_2(x) = f(x) \quad (1)$$

$$u_t(x,t) = c\phi_1'(x+ct) - c\phi_2'(x-ct)$$

$$u_t(x,0) = c\phi_1'(x) - c\phi_2'(x) = g(x)$$

$$\phi_1'(x) - \phi_2'(x) = \frac{1}{c} g(x) \quad (2)$$

Integrate (2) from 0 to x.

$$\Rightarrow \left(\phi_1(x) - \phi_1(0)\right) - \left(\phi_2(x) - \phi_2(0)\right) = \frac{1}{c}\int_0^x g(s)\,ds$$

$$\phi_1(x) - \phi_2(x) = \phi_1(0) - \phi_2(0) + \frac{1}{c}\int_0^x g(s)\,ds$$

$$(1) \quad \phi_1(x) + \phi_2(x) = f(x)$$

Add $\Rightarrow$ $\phi_1(x) = \frac{1}{2}\left[f(x) + \phi_1(0) - \phi_2(0) + \frac{1}{c}\int_0^x g(s)\,ds\right]$

Subtract $\Rightarrow$ $\phi_2(x) = \frac{1}{2}\left[f(x) - \phi_1(0) + \phi_2(0) - \frac{1}{c}\int_0^x g(s)\,ds\right]$

Then, $u(x,t) = \phi_1(x+ct) + \phi_2(x-ct)$

$$= \frac{1}{2}\left[f(x+ct) + \cancel{\phi_1(0)} - \cancel{\phi_2(0)} + \frac{1}{c}\int_0^{x+ct} g(s)\,ds\right]$$

$$+ \frac{1}{2}\left[f(x-ct) - \cancel{\phi_1(0)} + \cancel{\phi_2(0)} - \frac{1}{c}\int_0^{x-ct} g(s)\,ds\right]$$

$$= \frac{1}{2}\left[f(x+ct) + f(x-ct) + \frac{1}{c}\int_0^{x+ct} g(s)\,ds + \int_{x-ct}^{0} g(s)\,ds\right]$$

$$\boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds}$$

Two Special Cases:

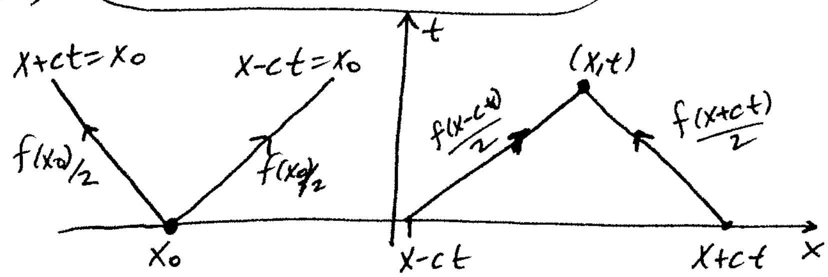
1) $u_t(x,0) = g(x) = 0$

2) $u(x,0) = f(x) = 0$

1) $u_{tt} = c^2 u_{xx}$ $\Longrightarrow$ $u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right]$

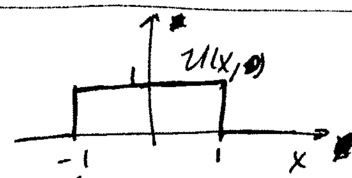
$u(x,0) = f(x)$

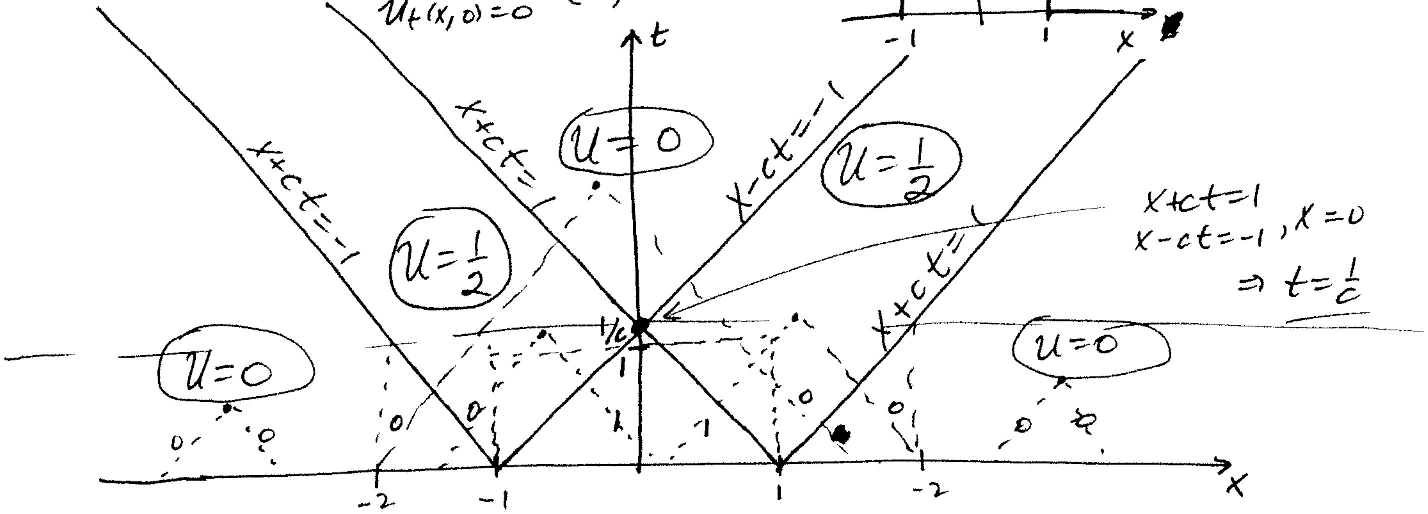
$u_t(x,0) = 0$



Example: $u(x,0) = f(x) = \begin{cases} 1, & -1<x<1 \\ 0, & \text{otherwise} \end{cases}$

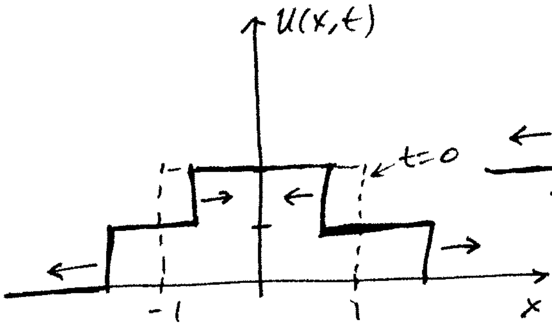
$u_t(x,0) = 0$



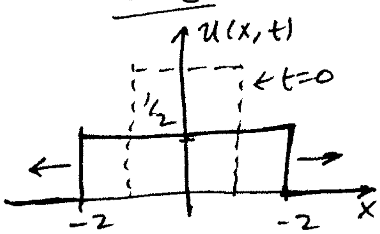


$x+ct=1$, $x-ct=-1$, $x=0$ $\Rightarrow t = \frac{1}{c}$

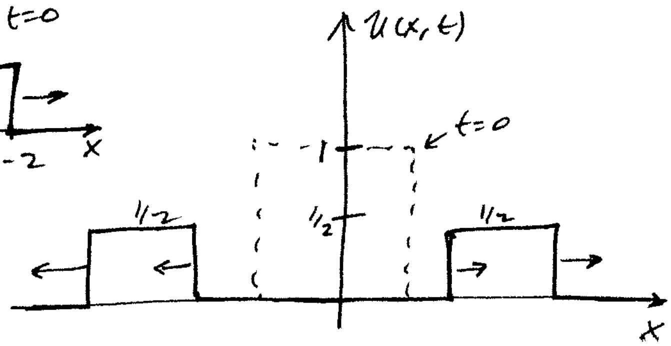
$0 < t < \frac{1}{c}$



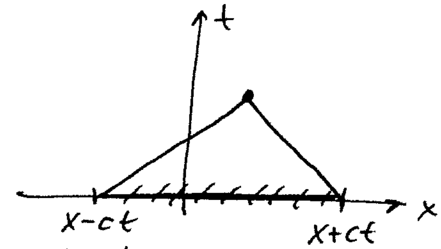
$t = \frac{1}{c}$



$t > \frac{1}{c}$

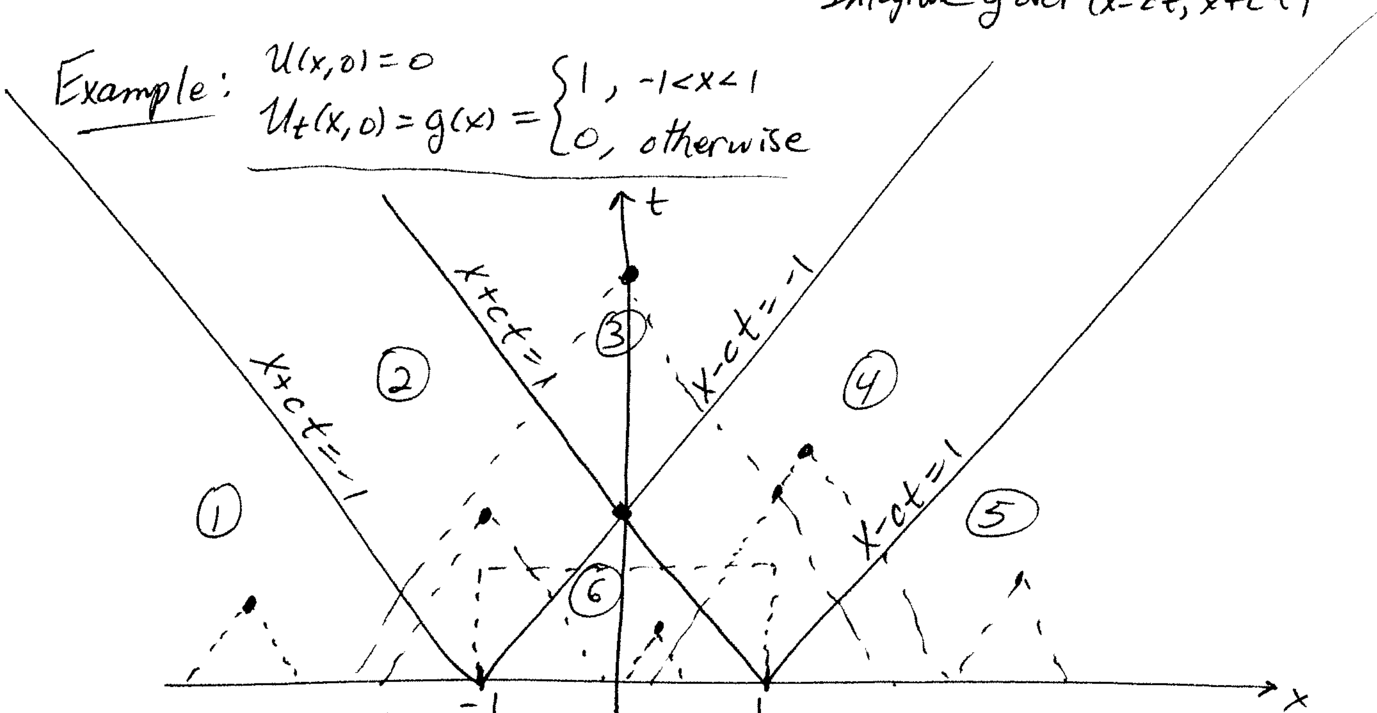


2) $u(x,0)=0$
$\underline{u_t(x,0)=g(x)}$ $\Rightarrow$ $u(x,t)=\dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(s)\,ds$



Integrate g over $(x-ct, x+ct)$

$\underline{\text{Example:}}$ $u(x,0)=0$
$\underline{u_t(x,0)=g(x)=\begin{cases}1, & -1<x<1\\ 0, & \text{otherwise}\end{cases}}$



① $u(x,t)=\dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} \underset{g=0}{g(s)\,ds} = \dfrac{1}{2c}\int_{x-ct}^{x+ct} 0\,ds = 0$ $\quad u(x,t)=0$

② $u(x,t)=\dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(s)\,ds = \dfrac{1}{2c}\left[\int_{x-ct}^{-1} 0\,ds + \int_{-1}^{x+ct} 1\,ds\right] = \dfrac{1}{2c}\left[0+(x+ct+1)\right]$

$u(x,t)=\dfrac{1}{2c}(x+ct+1)$

③ $u(x,t)=\dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(s)\,ds = \dfrac{1}{2c}\left[\int_{x-ct}^{-1} 0\,ds + \int_{-1}^{1} 1\,ds + \int_{1}^{x+ct} 0\,ds\right] = \dfrac{1}{2c}\left[0+2+0\right]$

$u(x,t)=\dfrac{1}{c}$

④ $u(x,t)=\dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} = \dfrac{1}{2c}\left[\int_{x-ct}^{1} 1\,ds + \int_{1}^{x+ct} 0\,ds\right] = \dfrac{1}{2c}\left[1-(x-ct)+0\right]$

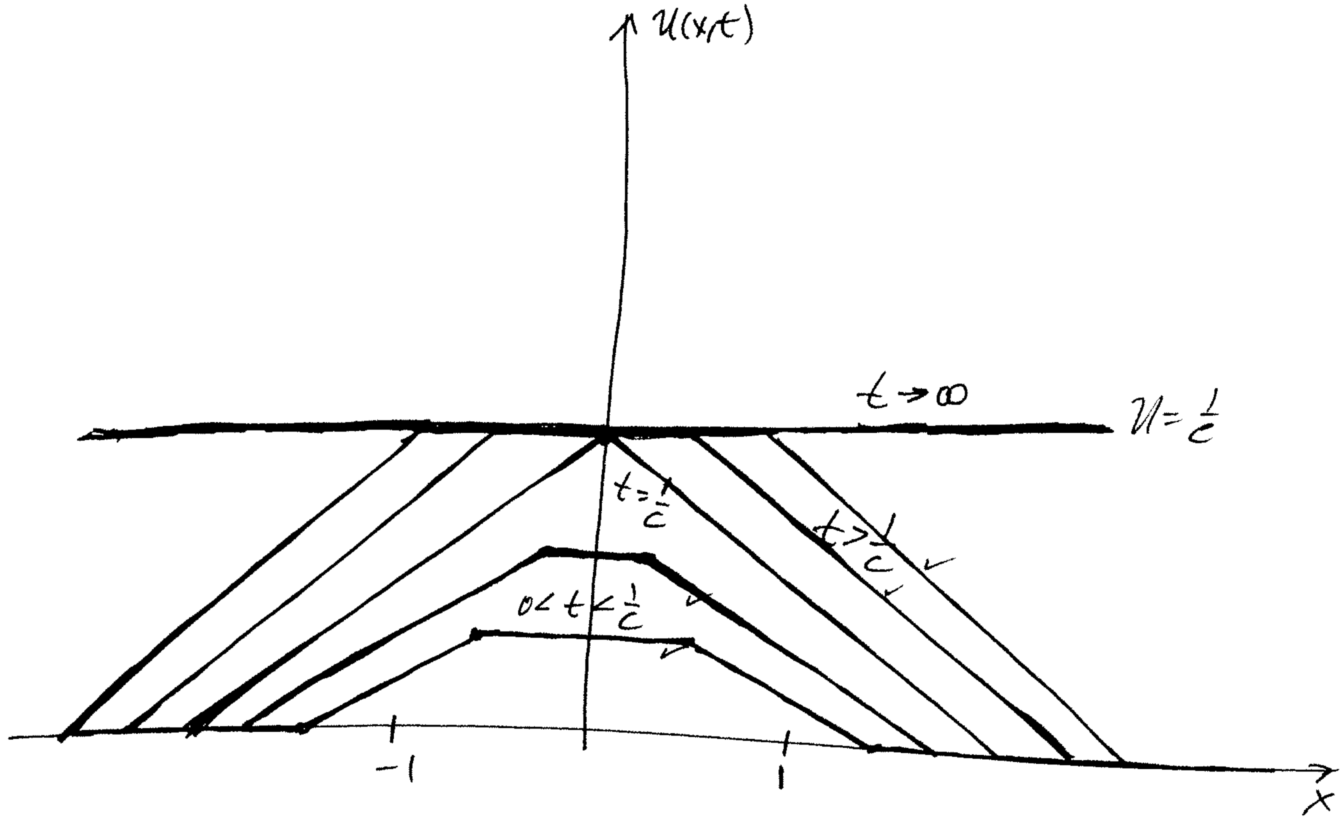
$u(x,t)=\dfrac{1}{2c}(1-x+ct)$

⑤ $u(x,t)=\dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(s)\,ds = \dfrac{1}{2c}\int_{x-ct}^{x+ct} 0\,ds$

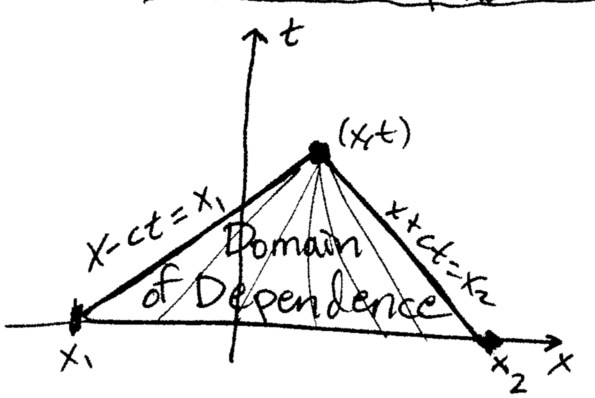
$u(x,t)=0$

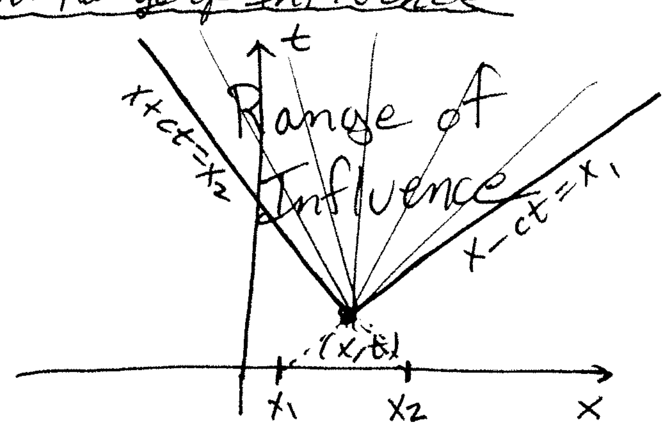
⑥ $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds = \frac{1}{2c}\int_{x-ct}^{x+ct} 1\,ds = \frac{1}{2c}\left[(x+ct)-(x-ct)\right] = \frac{1}{2c}(2ct) = t$

$u(x,t) = t$



Domain of Dependence and Range of Influence



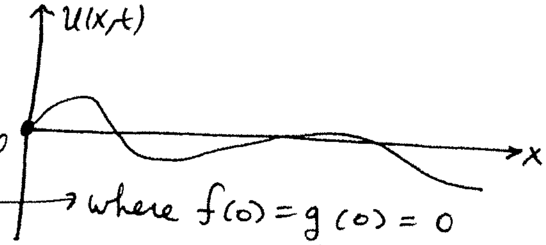


Semi-Infinite String

PDE: $u_{tt} = c^2 u_{xx}, \quad 0<t<\infty, \; 0<x<\infty$

BC: $u(0,t) = 0, \quad 0<t<\infty$

ICs: $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $0<x<\infty$



where $f(0) = g(0) = 0$

Note: $f(x)$ and $g(x)$ are defined only for $x \geq 0$.
$\Rightarrow f(x-ct)$ is not defined for all $x, t>0$

General Solution: $u(x,t) = \phi_1(x+ct) + \phi_2(x-ct)$

$$u(x,0) = \phi_1(x) + \phi_2(x) = f(x) \text{ for } x>0 \quad (1)$$

$$u_t(x,t) = c\,\phi_1'(x+ct) - c\,\phi_2'(x-ct)$$

$$u_t(x,0) = c\,\phi_1'(x) - c\,\phi_2'(x) = g(x)$$

$$\phi_1'(x) - \phi_2'(x) = \frac{1}{c} g(x) \text{ for } x>0$$

Integrate from 0 to x.

$$(\phi_1(x) - \phi_1(0)) - (\phi_2(x) - \phi_2(0)) = \frac{1}{c}\int_0^x g(s)\,ds$$

$$\phi_1(x) - \phi_2(x) = 2K + \frac{1}{c}\int_0^x g(s)\,ds \text{ for } x>0 \quad (2)$$

where $K = \frac{\phi_1(0) - \phi_2(0)}{2}$

Add (1) and (2) $\Rightarrow$ $\phi_1(x) = K + \frac{1}{2} f(x) + \frac{1}{2c}\int_0^x g(s)\,ds$ for $x>0$

Note: $x+ct>0$ for all x and t in the domain $(x,t>0)$

$$\Rightarrow \boxed{\phi_1(x+ct) = K + \frac{1}{2} f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds \text{ for all } x,t>0}$$

Subtract (2) from (1) $\Rightarrow$ $\phi_2(x) = -K + \frac{1}{2} f(x) - \frac{1}{2c}\int_0^x g(s)\,ds$ for $x>0$

Note: $-\infty < x-ct < \infty$ for $x,t>0$

$$\Rightarrow \boxed{\phi_2(x-ct) = -K + \frac{1}{2} f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)\,ds \text{ for } x-ct>0}$$

What if $x-ct<0$?

$f(x-ct)$ is not defined for $x-ct<0$, nor is g.

The boundary condition determines $\phi_2(x)$ for $x<0$.

BC: $u(0,t) = \phi_1(ct) + \phi_2(-ct) = 0$ for $t>0$

$$\phi_2(-ct) = -\phi_1(ct)$$

$-ct \to x<0 \Rightarrow \phi_2(x) = -\phi_1(-x)$ for $x<0$

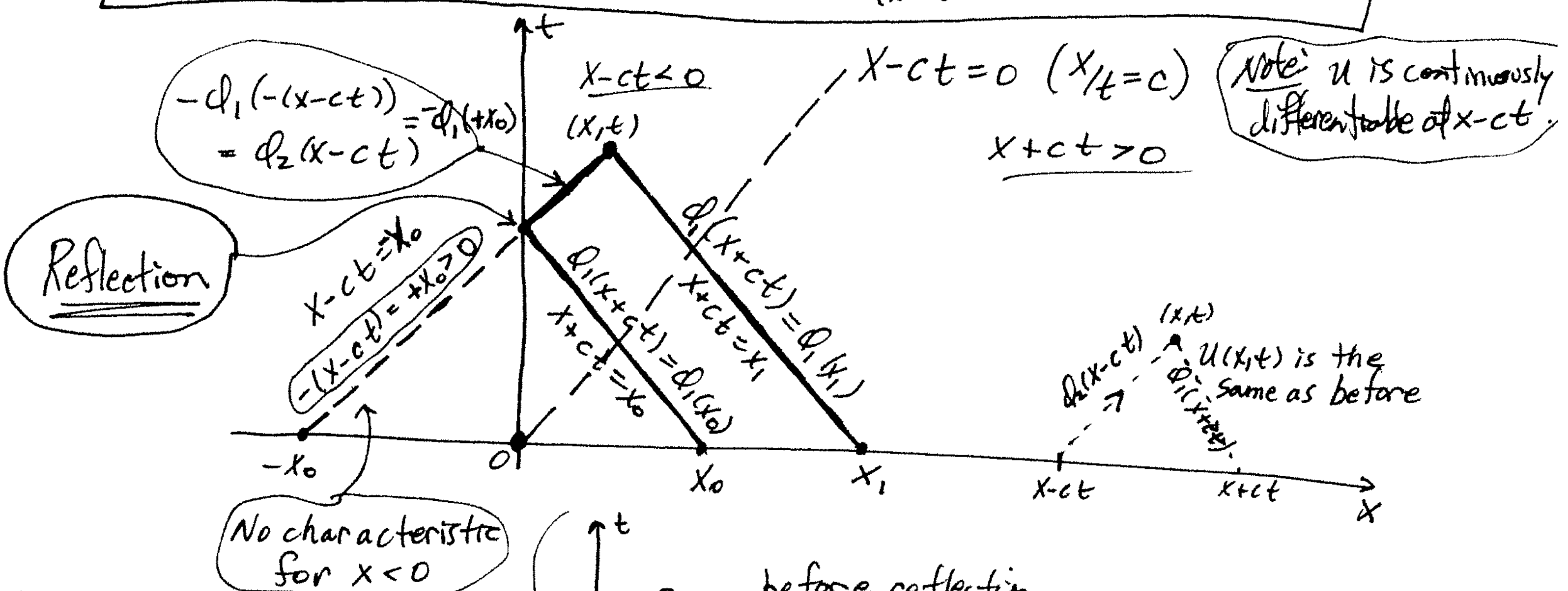
$$\Rightarrow \phi_2(x) = -\phi_1(-x) = -\left[K + \tfrac{1}{2}f(-x) + \tfrac{1}{2c}\int_0^{-x} g(s)\,ds\right] \text{ for } x<0$$

Then,

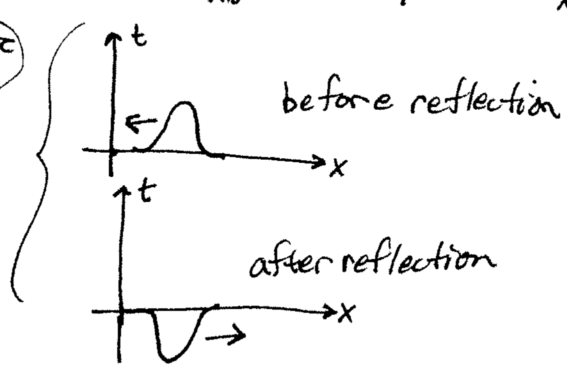
$$\boxed{\phi_2(x-ct) = -K - \tfrac{1}{2}f(-(x-ct)) - \tfrac{1}{2c}\int_0^{-(x-ct)} g(s)\,ds \text{ for } x-ct<0}$$ (reflected wave)

Finally, $u(x,t) = \phi_1(x+ct) + \phi_2(x-ct)$

$$\boxed{u(x,t) = \begin{cases} \frac{1}{2}\left[f(x+ct)+f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds, & x-ct>0 \ (x/t>c) \\ \frac{1}{2}\left[f(x+ct)-f(-(x-ct))\right] + \frac{1}{2c}\int_{-(x-ct)}^{x+ct} g(s)\,ds & x-ct<0 \ \left(\frac{x}{t}<c\right) \end{cases}}$$



$g(x) \equiv 0 \Rightarrow$

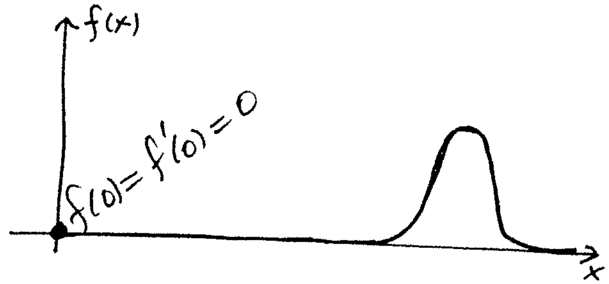


Example: $u_{tt} = c^2 u_{xx}$

$u(0,t) = 0 \qquad 0 < x < \infty$

$u(x,0) = f(x) \qquad 0 < t < \infty$

$u_t(x,0) = c f'(x)$



$$d_1(x+ct) = K + \frac{1}{2} f(x+ct) + \frac{1}{2\cancel{c}} \int_0^{x+ct} \cancel{c} f'(s)\, ds$$

$$= K + \frac{1}{2} f(x+ct) + \frac{1}{2}\left[f(x+ct) - \cancel{f(0)}\right]$$

$$d_1(x+ct) = K + f(x+ct) \text{ for all } x,t > 0$$

$x - ct > 0$

$$d_2(x-ct) = -K + \frac{1}{2} f(x-ct) - \frac{1}{2\cancel{c}} \int_0^{x-ct} \cancel{c} f'(s)\, ds$$

$$= -K + \frac{1}{2} \cancel{f(x-ct)} - \frac{1}{2}\left[\cancel{f(x-ct)} - \cancel{f(0)}\right]$$

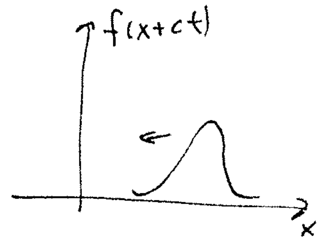
$$d_2(x-ct) = -K \text{ for } x - ct > 0$$

$x - ct < 0$ $\quad d_2(x-ct) = -d_1(-(x-ct)) = -K - f(-(x-ct))$

$$d_2(x-ct) = -K - f(-(x-ct)) \text{ for } x - ct < 0$$

Then, $u(x,t) = d_1(x+ct) + d_2(x-ct)$

$$\boxed{u(x,t) = \begin{cases} f(x+ct) & , \ x - ct > 0 \\ f(x+ct) + \left(-f(-(x-ct))\right) & , \ x - ct < 0 \end{cases}}$$



u is continuously differentiable at $x = ct$

$f(x+ct)$ is a left-traveling wave

$-f(-(x-ct))$ is a right-traveling inverted reflected wave

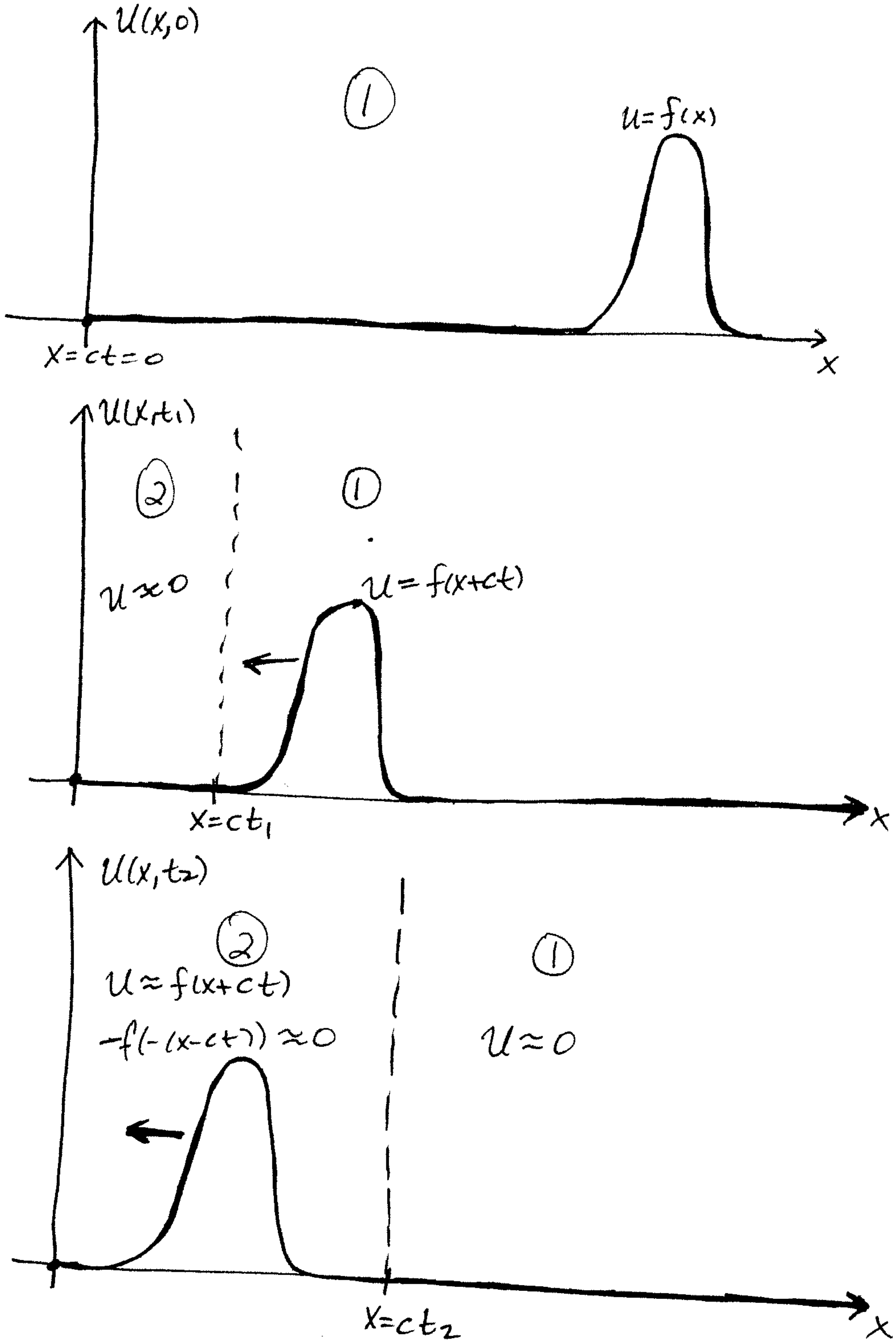
$$u(x,t) = \begin{cases} f(x+ct) & , x-ct>0 \quad (1) \\ f(x+ct)+(-f(-(x-ct))) & , x-ct<0 \quad (2) \end{cases}$$

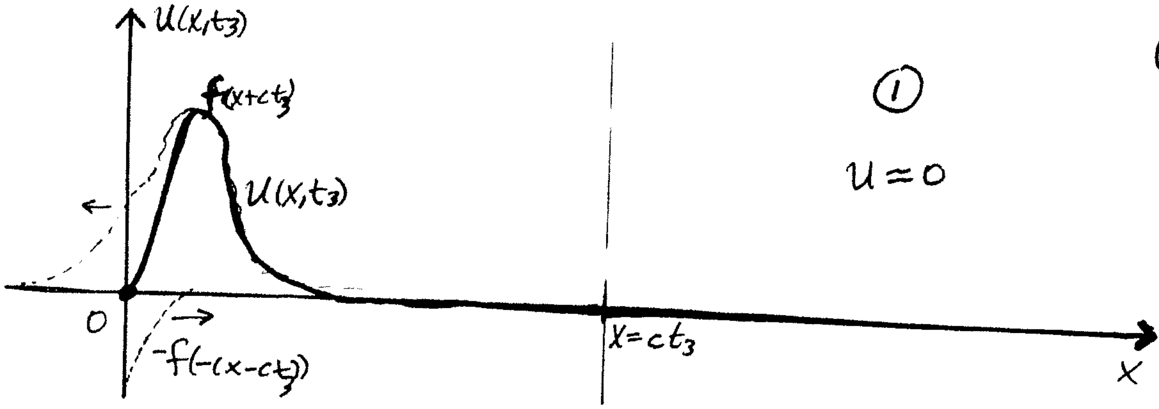
Solution plots at various times $(0<t_1<t_2<\cdots)$

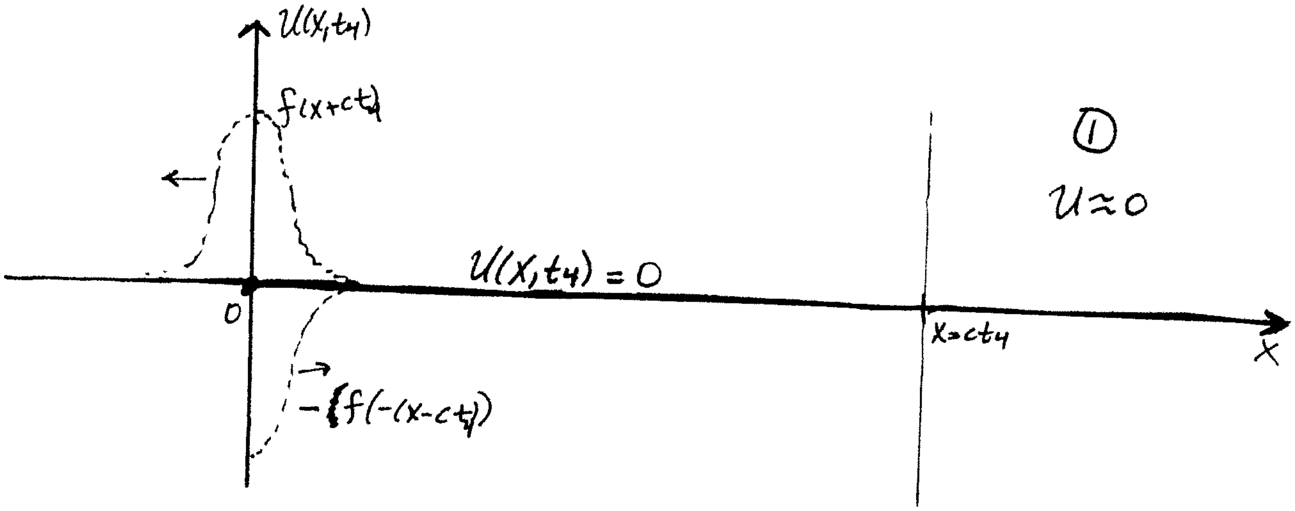
$t=0$

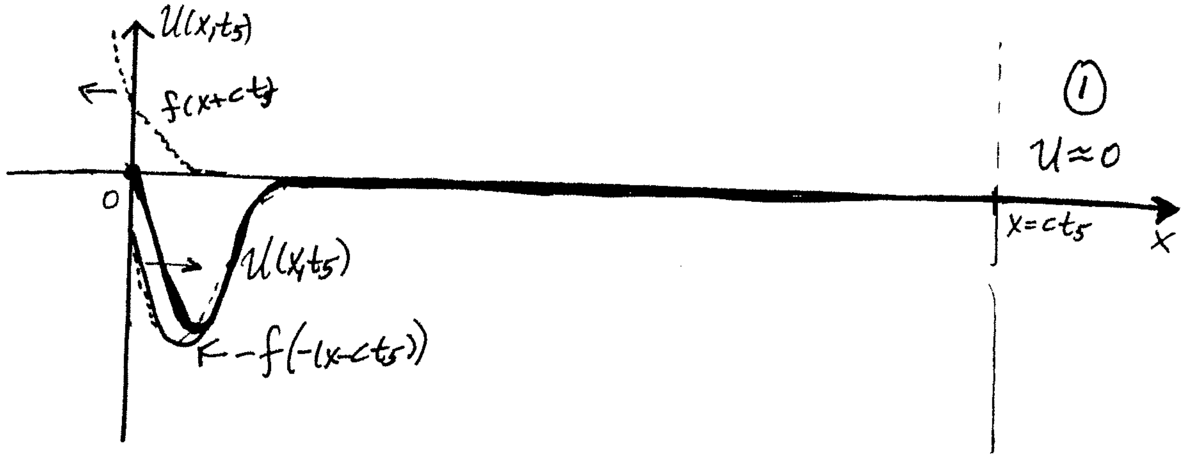
$t=t_1$

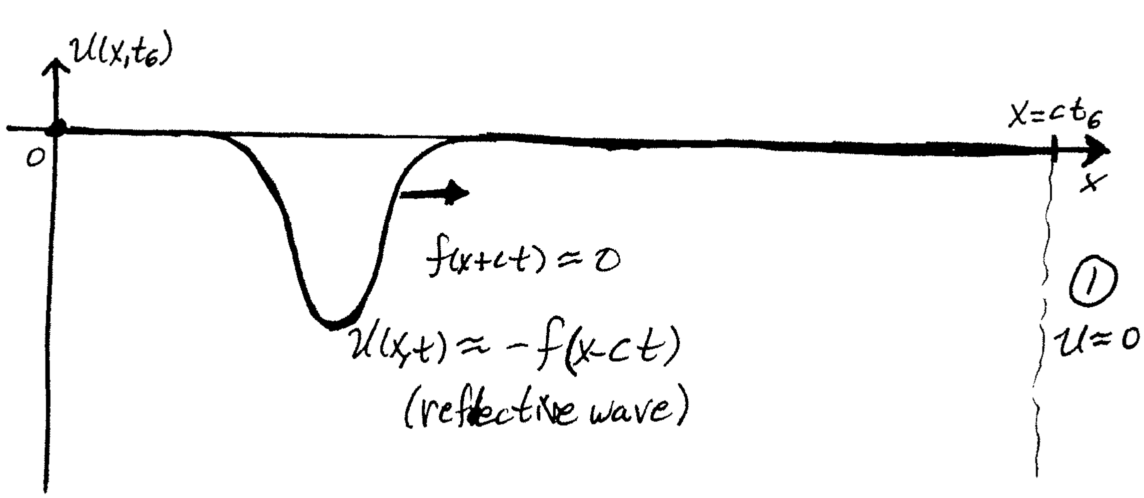
$t=t_2$




u(x,t3)
f(x+ct3)
u(x,t3)
1
u≈0
0
-f(-(x-ct3))
x=ct3
x


u(x,t4)
f(x+ct4)
1
u≈0
u(x,t4) = 0
0
x=ct4
x
-f(-(x-ct4))

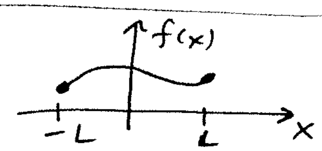

u(x,t5)
f(x+ct5)
1
u≈0
0
x=ct5
x
u(x,t5)
-f(-(x-ct5))


u(x,t6)
x=ct6
0
x
f(x+ct)≈0
1
u(x,t)≈-f(x-ct)
(reflective wave)
u≈0

Fourier Transform

- The Fourier Series is used to represent functions that are defined over a finite interval. The Fourier Transform is a related concept, but it applies to functions that are defined for all $x \in (-\infty, \infty)$. The Fourier Transform may be derived from the Fourier Series.

$$\text{Fourier Series} \xrightarrow[\text{complex form}]{\text{write in}} \text{Complex Fourier Series} \xrightarrow[\text{length} \to \infty]{\text{let the interval}} \text{Fourier Transform}$$



Fourier Series: Let $f(x)$ be defined on an interval $[-L, L]$.

then, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$,

where $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ and $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$.

Complex Fourier Series: $\cos\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\left(e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}\right)$, $\sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2i}\left(e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}\right)$

- Plug into the Fourier Series $\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[\underbrace{\frac{1}{2}(a_n - ib_n)}_{c_n} e^{i\frac{n\pi x}{L}} + \underbrace{\frac{1}{2}(a_n + ib_n)}_{d_n} e^{-i\frac{n\pi x}{L}}\right]$

$$c_n = \frac{1}{2}(a_n - ib_n) = \frac{1}{2}\left[\frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx - i\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right]$$

$$= \frac{1}{2L}\int_{-L}^{L} f(x)\left(\cos\left(\frac{n\pi x}{L}\right) - i\sin\left(\frac{n\pi x}{L}\right)\right)dx$$

$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi x}{L}}dx$ Similarly, $d_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{i\frac{n\pi x}{L}}dx$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(c_n e^{i\frac{n\pi x}{L}} + d_n e^{-i\frac{n\pi x}{L}}\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n e^{i\frac{n\pi x}{L}} + \sum_{n=-\infty}^{-1} d_{-n} e^{i\frac{n\pi x}{L}}$$

Let $C_n = \begin{cases} c_n, & n > 0 \\ d_{-n}, & n < 0 \\ a_0/2, & n = 0 \end{cases} \Longrightarrow \boxed{C_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-i\frac{n\pi x}{L}}dx;\ n = 0, \pm 1, \pm 2, \ldots}$

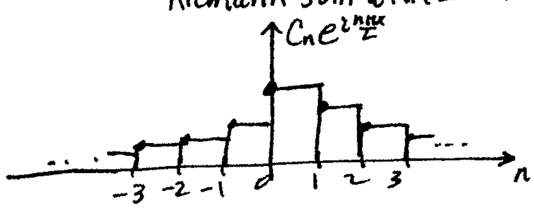
- then, $\boxed{f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{L}}}$

Complex Fourier Series of a function $f(x)$ that is defined on an interval $[-L, L]$.

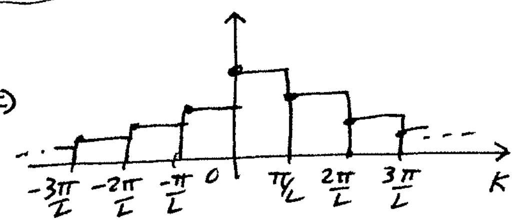
The Fourier Transform corresponds to the Complex Fourier Series in the limit $L \to \infty$.

We have $f(x) = \underbrace{\sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{L}}}$ where $C_n = \frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}} dx$

Consider this to be a Riemann sum with $\Delta n = 1$ $\Rightarrow$ $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{L}} \Delta n$



Let $k = \frac{n\pi}{L}$ $\Rightarrow$



$\Delta k = \frac{\Delta n \pi}{L} \to 0$ as $L \to \infty$

$\Delta n = \frac{L}{\pi}\Delta k$

Then, $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{i\frac{n\pi x}{L}} \cdot \frac{L}{\pi}\Delta k$

$$\Rightarrow \quad f(x) = \sum_{n=-\infty}^{\infty} \left(\frac{L}{\pi}C_n\right) e^{i\frac{n\pi x}{L}} \Delta k$$

$$\frac{L}{\pi}C_n = \frac{L}{\pi}\cdot\frac{1}{2L}\int_{-L}^{L} f(x) e^{-i\frac{n\pi x}{L}} dx = \frac{1}{2\pi}\int_{-L}^{L} f(x) e^{-ikx} dx$$

Let $F(k) = \frac{L}{\pi}C_n = \frac{1}{2\pi}\int_{-L}^{L} f(x) e^{-ikx} dx$

In the limit as $L \to \infty$, we have

$$\boxed{F(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x) e^{-ikx} dx} \quad \text{(Fourier Transform)}$$

Observe that $\Delta k = \frac{\pi}{L}\Delta n \to 0$ as $L \to \infty$.

$$f(x) = \lim_{\Delta k \to 0} \underbrace{\sum_{n=-\infty}^{\infty} F(k) e^{ikx} \Delta k}_{\text{Riemann Sum} \to \text{Integral as } \Delta k \to 0} = \int_{-\infty}^{\infty} F(k) e^{ikx} dk$$

$$\boxed{f(x) = \int_{-\infty}^{\infty} F(k) e^{ikx} dk} \quad \text{(Inverse Fourier Transform)}$$

Notes: 1) The Fourier Transform $F(k)$ corresponds to the coefficients of the complex Fourier series in the limit $L \to \infty$.

2) The Inverse Fourier Transform is the complex Fourier Series in the limit $L \to \infty$.

3) The Complex Fourier Series represents functions defined on a finite interval, whereas the Fourier Transform represents functions that are defined for all $x \in (-\infty, \infty)$.

There is some flexibility in defining the Fourier Transform in that the constants appearing in the formulas may be adjusted appropriately.

More generally, and equivalently, the Fourier Transform may be defined by

$$F(k) = \alpha \int_{-\infty}^{\infty} f(x) e^{\delta i k x} dx$$
$$f(x) = \beta \int_{-\infty}^{\infty} F(k) e^{-\delta i k x} dk$$

where α, β, and δ are any constants such that $\boxed{\alpha\beta = \frac{|\delta|}{2\pi}}$.

Consequently, there is an inconsistancy in how people choose to define the Fourier Transform. When using a Table of Fourier Transforms, one must first determine how the author chose to define the Fourier Transform (i.e. $\alpha, \beta, \delta = ?$).

Common Choices:

i) $\alpha = \frac{1}{2\pi}$, $\beta = 1$, $\delta = \pm 1$

ii) $\alpha = 1$, $\beta = \frac{1}{2\pi}$, $\delta = \pm 1$

iii) $\alpha = 1/\sqrt{2\pi}$, $\beta = 1/\sqrt{2\pi}$, $\delta = \pm 1$

iv) $|\delta| \cdot k$ = angular frequency of the oscillations of $e^{\delta i k x}$

$\delta = \pm 1 \Rightarrow k$ = angular frequency $\left(\frac{\text{rad.}}{\text{sec.}}\right)$

$\delta = \pm 2\pi \Rightarrow k$ = frequency $\left(\frac{\text{osc.}}{\text{sec.}}\right)$

$$\begin{pmatrix}\text{angular}\\ \text{frequency}\end{pmatrix} = 2\pi(\text{frequency})$$
$$\omega = 2\pi f$$

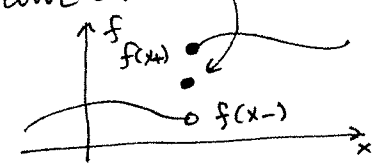
- We'll choose $\alpha = 1$, $\beta = 1/2\pi$, and $\delta = 1$.

$$\Rightarrow \boxed{F(k) = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx} \quad \underline{\text{Fourier Transform}}$$

$$\boxed{f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\, e^{-ikx}\, dk} \quad \underline{\text{Inverse Fourier Transform}}$$

This is a fairly common choice. It seems that it is most convenient to have the forward Fourier Transform formula as simple as possible since it is used more often than the Inverse Fourier Transform formula.

- At points where $f(x)$ has a jump discontinuity, the Inverse Fourier Transform converges to the average value of f.



$$\Rightarrow \frac{f(x-) + f(x+)}{2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\, e^{-ikx}\, dk$$

for all x.

<u>Notation:</u> $\boxed{\begin{aligned} F(k) &= \mathcal{F}\{f(x)\} \\ f(x) &= \mathcal{F}^{-1}\{F(k)\} \end{aligned}}$

<u>Terminology:</u> i) $f(x)$ and $F(k)$ are called a <u>Fourier Transform pair</u>.

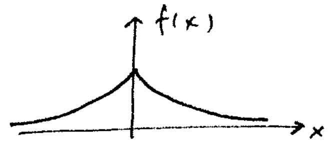
ii) k = transform variable

Often the transform variable is denoted by $\alpha, k, s, \omega, \ldots$

- <u>Theorem:</u> The Fourier Transform of a function f exists if f is absolutely integrable, i.e. if $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges.

Therefore, $\mathcal{L}\{f(x)\}$ exists $\Rightarrow f(x) \to 0$ as $x \to \pm\infty$

Example: $f(x) = e^{-|x|}$



$$F(k) = \int_{-\infty}^{\infty} e^{-|x|} e^{ikx}\, dx = \int_{-\infty}^{0} e^{x(1+ik)}\, dx + \int_{0}^{\infty} e^{-x(1-ik)}\, dx$$

$$= \left.\frac{e^{x(1+ik)}}{1+ik}\right|_{-\infty}^{0} - \left.\frac{e^{-x(1-ik)}}{1-ik}\right|_{0}^{\infty} = \left(\frac{1}{1+ik} - 0\right) - \left(0 - \frac{1}{1-ik}\right)$$

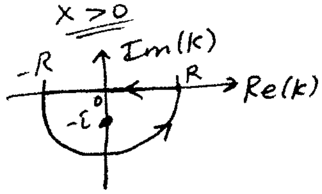
$$= \frac{(1-\cancel{ik}) + (1+\cancel{ik})}{(1+ik)(1-ik)} = \frac{2}{1+k^2}$$

$$\boxed{F(k) = \frac{2}{1+k^2}}$$

$$f(x) = \frac{1}{\cancel{2}\pi} \int_{-\infty}^{\infty} \frac{\cancel{2}}{1+k^2} e^{-ikx}\, dk = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{-ikx}}{1+k^2}\, dk \qquad 1+k^2 = (k+i)(k-i)$$

Contour Integration

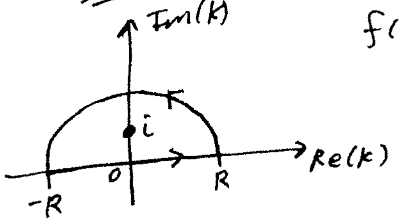
$x > 0$



$$f(x) = \frac{1}{\pi} \cdot -2\pi i \; \mathrm{Res}\left(\frac{e^{-ikx}}{(k+i)(k-i)},\ k = -i\right)$$

$$= -2i\, \frac{e^{-i(-i)x}}{-i-i} = e^{-x}$$

$x < 0$



$$f(x) = \frac{1}{\pi} \cdot 2\pi i \; \mathrm{Res}\left(\frac{e^{-ikx}}{(k+i)(k-i)},\ k = i\right)$$

$$= 2i \cdot \frac{e^{-i(i)x}}{i+i} = e^{x}$$

$$\Rightarrow \boxed{f(x) = \begin{cases} e^{-x}, & x > 0 \\ e^{x}, & x < 0 \end{cases} = e^{-|x|}}$$

Fourier Transform of Derivatives

$$\mathcal{F}\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)\, e^{ikx}\, dx = \cancel{f(x)e^{ikx}\Big|_{-\infty}^{\infty}}^{=0} - ik\int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx$$

Integrate by Parts: $u = e^{ikx}$ $\quad v = f(x)$ $\quad = -ik\,\mathcal{F}\{f(x)\}$

$du = ike^{ikx}dx \quad dv = f'(x)\,dx$

$$\boxed{\mathcal{F}\{f'(x)\} = -ik\,\mathcal{F}\{f(x)\} = -ik\,F(k)}$$

$\mathcal{F}\{f^{(n)}(x)\} = \int_{-\infty}^{\infty} f^{(n)}(x)\, e^{ikx}\, dx$ Integrate by parts n times. Each iteration produces a factor of $-ik$.

$$\Rightarrow \boxed{\mathcal{F}\{f^{(n)}(x)\} = (-ik)^n\,\mathcal{F}\{f(x)\} = (-ik)^n F(k)}$$

Note: More generally, $\mathcal{F}\{f^{(n)}(x)\} = (-i\delta k)^n F(k)$

for any admissible choice of α, β, and δ.

Example: Consider the heat equation on an infinite domain with a prescribed IC.

$$u_t = \lambda u_{xx}, \quad 0<t<\infty$$
$$u(x,0) = f(x), \quad -\infty < x < \infty$$

Let $U(k,t) = \mathcal{F}\{u(x,t)\} = \int_{-\infty}^{\infty} u(x,t)\, e^{ikx}\, dx$

and $F(k) = \mathcal{L}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx$

Fourier Transform the initial value problem

$$\mathcal{F}\{u_t(x,t)\} = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}(x,t)\, e^{ikx}\, dx = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)\, e^{ikx}\, dx = \frac{\partial}{\partial t} U(k,t)$$

$$\Rightarrow \boxed{\mathcal{F}\{u_t(x,t)\} = U_t(k,t)}$$

$$\mathcal{F}\{u_{xx}(x,t)\} = (-ik)^2 U(k,t) = -k^2 U(k,t)$$

$$\mathcal{F}\{u_t = \lambda u_{xx}\} \Longrightarrow \boxed{U_t = -\lambda k^2 U} \ (\text{ODE})$$

$$U(k,t) = \mathcal{F}\{u(x,t)\}$$
$$\Rightarrow U(k,0) = \mathcal{F}\{u(x,0)\} = \mathcal{F}\{f(x)\} = F(k)$$
$$\boxed{U(k,0) = F(k)} \ (\text{IC})$$

Solve: $U(k,t) = C(k)\, e^{-\lambda k^2 t}$ (general solution)

$U(k,0) = C(k) = F(k)$

$$\Rightarrow U(k,t) = F(k)\, e^{-\lambda k^2 t}$$

Then, $u(x,t) = \mathcal{F}^{-1}\{F(x)e^{-\lambda k^2 t}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{-\lambda k^2 t}\cdot e^{-ikx}\,dk$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\xi)e^{ik\xi}\,d\xi\right)e^{-\lambda k^2 t}e^{-ikx}\,dk$$

Interchange the order of integration: $= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\int_{-\infty}^{\infty} e^{-\lambda k^2 t - ik(x-\xi)}\,dk\,d\xi$

(*) $$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\underbrace{\int_{-\infty}^{\infty} e^{-\lambda t\left(k^2 + i\frac{x-\xi}{\lambda t}k\right)}\,dk}\,d\xi$$

$$\int_{-\infty}^{\infty} e^{-\lambda t\left(k^2+i\frac{x-\xi}{\lambda t}k\right)}dk = \int_{-\infty}^{\infty} e^{-\lambda t(k^2+2i\omega k)}dk \text{ where } \omega = \frac{x-\xi}{2\lambda t}$$

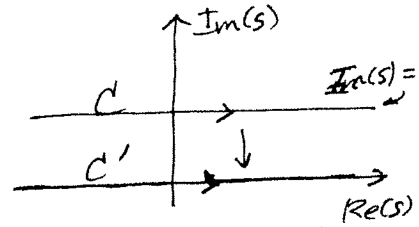
Complete the Square: $k^2+2i\omega k = (k^2+2i\omega k-\omega^2)+\omega^2 = (k+i\omega)^2+\omega^2$

$$= \int_{-\infty}^{\infty} e^{-\lambda t[(k+i\omega)^2+\omega^2]}dk = e^{-\lambda t\omega^2}\int_{-\infty}^{\infty} e^{-\lambda t(k+i\omega)^2}dk$$

Error Function: $\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}ds$, $\operatorname{erf}(\infty)=1$

Let $s = \sqrt{\lambda t}\,(k+i\omega)$.

$$= e^{-\lambda\omega^2 t}\int_{-\infty+i\omega\sqrt{\lambda t}}^{\infty+i\omega\sqrt{\lambda t}} e^{-s^2}\cdot\frac{ds}{\sqrt{\lambda t}}$$



$$= \frac{e^{-\lambda\omega^2 t}}{\sqrt{\lambda t}}\int_{-\infty}^{\infty} e^{-s^2}ds \;(\text{even}) = 2\frac{e^{-\lambda\omega^2 t}}{\sqrt{\lambda t}}\int_0^{\infty} e^{-s^2}ds$$

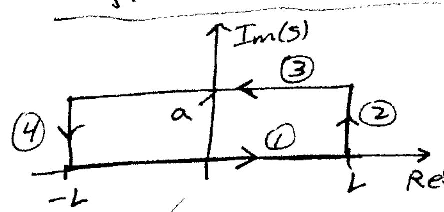
$$= \frac{2e^{-\lambda\omega^2 t}}{\sqrt{\lambda t}}\cdot\frac{\sqrt{\pi}}{2}\overbrace{\operatorname{erf}(\infty)}^{=1}$$

$$\int_{-\infty}^{\infty} e^{-\lambda t\left(k^2+i\frac{x-\xi}{\lambda t}k\right)}dk = \sqrt{\frac{\pi}{\lambda t}}\,e^{-\lambda\omega^2 t}$$

$$\int_{-\infty}^{\infty} e^{-\lambda t\left(k^2+i\frac{x-\xi}{\lambda t}k\right)}dk = \sqrt{\frac{\pi}{\lambda t}}\,e^{-\lambda t\left(\frac{x-\xi}{2\lambda t}\right)^2}$$

$$\boxed{\int_{-\infty}^{\infty} e^{-\lambda t\left(k^2+i\frac{x-\xi}{\lambda t}k\right)}dk = \sqrt{\frac{\pi}{\lambda t}}\,e^{-\frac{(x-\xi)^2}{4\lambda t}}}$$

Since the integrand has no singularities we can deform the integration contour from C to C', where Im(s) = 0



$$\oint = \int_{①} + \cancel{\int_{②}} + \int_{③} + \cancel{\int_{④}} = 0$$

As $L\to\infty$, $\to 0$ as $L\to\infty$

$$\int_{①} + \int_{③} = \int_{-L}^{L} - \int_{-L+ia}^{L+ia} = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} = \int_{-\infty+ia}^{\infty+ia}$$

then, $\circledast \Rightarrow u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\cdot\sqrt{\frac{\pi}{\lambda t}}\, e^{\frac{-(x-\xi)^2}{4\lambda t}}\, d\xi$

$$\boxed{u(x,t) = \frac{1}{2\sqrt{\pi\lambda t}}\int_{-\infty}^{\infty} f(\xi)\, e^{\frac{-(x-\xi)^2}{4\lambda t}}\, d\xi}$$

Note: The formula holds true even if f does not have a Fourier transform, i.e. $\int_{-\infty}^{\infty}|f(x)|\,dx$ does not converge.

OR $$u(x,t) = \int_{-\infty}^{\infty} f(\xi)\cdot \underbrace{\frac{e^{\frac{-(x-\xi)^2}{4\lambda t}}}{2\sqrt{\pi\lambda t}}}_{\text{Green's Function}}\, d\xi$$

Free (Infinite $-\infty<x<\infty$) Space Green's Function for the Heat Equation : $G(x,t\,|\,x_0,t_0) = \dfrac{e^{\frac{-(x-x_0)^2}{4\lambda(t-t_0)}}}{2\sqrt{\pi\lambda(t-t_0)}}$

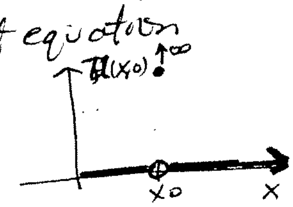
The Green's Function is a function of x and t, but the function varies from point to point (x_0,t_0) in the domain. The parameters (x_0,t_0) of the Green's Function vary through the domain of the initial data.

$$u(x,t) = \int_{-\infty}^{\infty} f(x_0)\, G(x,t\,|\,x_0,0)\, dx_0$$

Initial Data: $u(x,0)=f(x),\ -\infty<x<\infty \Rightarrow t_0=0,\ -\infty<x_0<\infty$

If t_0 varies as well, then the solution will involve a double integral.

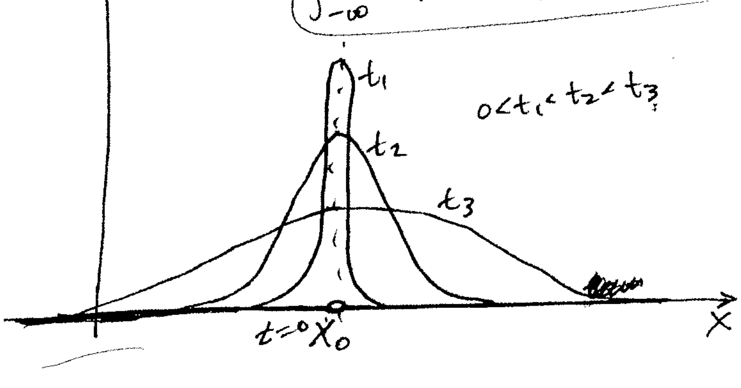
Note: The quantity $f(x_0)\,G(x,t\,|\,x_0,0)$ describes the influence of the initial data at a single point on the initial data curve.
That is, $f(x_0)\,G(x,t\,|\,x_0,0)$ is the solution of the heat equation subject to the initial data $u(x,0) = f(x)\,\delta(x-x_0)$.
$= f(x_0)\,\delta(x-x_0)$



Conservation of Energy $\Rightarrow \int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u(x,0)\,dx$ for all t
$= \int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$ for all t

$$u(x,0) = \begin{cases} \infty, & x = x_0 \\ 0, & x \neq x_0 \end{cases}$$

$\int_{-\infty}^{\infty} u(x,t)\,dx = f(x_0)$ for all t



Computation of the Green's Function

Given the initial value problem $u_t = \lambda u_{xx}$, $0<t<\infty$, $u(x,0)=f(x)$, $-\infty<x<\infty$,

the Green's Function G is the solution of the associated initial value problem

$$G_t = \lambda G_{xx}, \quad G(x,0) = \delta(x-x_0), \quad G = G(x,t|x_0,t_0),\ t_0=0,\ -\infty<x_0<\infty$$

Using the above solution formula we have

$$G(x,t|x_0,0) = \int_{-\infty}^{\infty} \delta(\xi - x_0)\frac{e^{-\frac{(x-\xi)^2}{4\lambda t}}}{2\sqrt{\pi\lambda t}}\,d\xi = \frac{e^{-\frac{(x-x_0)^2}{4\lambda t}}}{2\sqrt{\pi\lambda t}}$$

$$\boxed{G(x,t|x_0,0) = \frac{e^{-\frac{(x-x_0)^2}{4\lambda t}}}{2\sqrt{\pi\lambda t}}} = \begin{cases} \dfrac{e^{-\frac{(x-x_0)^2}{4\lambda t}}}{2\sqrt{\pi\lambda t}} \to 0 \text{ at } t\to0^+, & x\neq x_0 \\[2ex] \dfrac{1}{2\sqrt{\pi\lambda t}} \to \infty \text{ as } t\to 0^+, & x=x_0 \end{cases}$$

Therefore,

$$G(x,0^+|x_0,0) = \begin{cases} 0, & x\neq x_0 \\ \infty, & x=x_0 \end{cases}$$

Also, $\int_{-\infty}^{\infty} G(x,0^+|x_0,0)\,dx = 1$

$\Rightarrow G(x,0^+|x_0,0) = \delta(x-x_0)$

$\Rightarrow G$ satisfies the IC.

The integral $\int_{-\infty}^{\infty} f(x_0)G(x,t|x_0,0)\,dx_0$ is the sum of the influences of the initial data at each point on the initial data curve.

Note: The Green's can be used to treat source terms as well

e.g. forced mass-spring-damper

ODE: $mx'' + \beta x' + kx = F(t)$, $t>0$

$x(0)=x_0$, $x'(0)=v_0$

$x(t) = \int_0^{\infty} F(t_0)G(t|t_0)\,dt_0$

Green's Function

$mG'' + \beta G' + kG = \delta(t-t_0)$, $G = G(t|t_0)$

$G(0|t_0) = x_0$, $G'(0|t_0) = v_0$

Now t_0 varies over the domain of the source term $F(t)$.

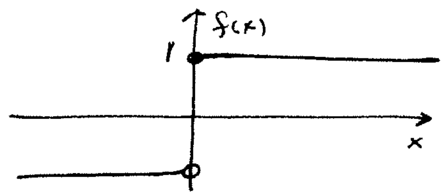
The Green's Function works for boundary value problems as well,

e.g. $y''+y' = F(x)$, $y(0)=0$, $y(1)=1$ $\Rightarrow$ $G''+G' = \delta(x-x_0)$, $G=G(x|x_0)$, $G(0|x_0)=0$, $G(1|x_0)=1$

We have $\begin{array}{ll} u_t = \lambda u_{xx} & 0<t<\infty \\ u(x,0)=f(x), & -\infty<x<\infty \end{array} \Rightarrow u(x,t) = \displaystyle\int_{-\infty}^{\infty} f(\xi)\, \frac{e^{-\frac{(x-\xi)^2}{4\lambda t}}}{2\sqrt{\pi\lambda t}}\, d\xi.$

<u>Example:</u> $u_t = \lambda u_{xx}$

$u(x,0) = f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x<0 \end{cases}$



Recall the similarity solution $u(x,t) = \mathrm{erf}\left(\frac{x}{2\sqrt{\lambda t}}\right)$.

Using the above solution formula, we have

$$u(x,t) = \int_{-\infty}^{0} -1 \cdot \frac{e^{-\frac{(x-\xi)^2}{4\lambda t}}}{2\sqrt{\pi\lambda t}}\, d\xi + \int_0^{\infty} 1 \cdot \frac{e^{-\frac{(x-\xi)^2}{4\lambda t}}}{2\sqrt{\pi\lambda t}}\, d\xi$$

$$s = \frac{x-\xi}{2\sqrt{\lambda t}}$$

$$= +\int_{\infty}^{\frac{x}{2\sqrt{\lambda t}}} \frac{e^{-s^2}}{\cancel{2}\sqrt{\pi\cancel{\lambda t}}} \cdot +\cancel{2\sqrt{\lambda t}}\, ds + \int_{\frac{x}{2\sqrt{\lambda t}}}^{-\infty} \frac{e^{-s^2}}{\cancel{2\sqrt{\pi\lambda t}}} \cdot -\cancel{2\sqrt{\lambda t}}\, ds$$

$$= \frac{1}{\sqrt{\pi}}\left[-\int_{\frac{x}{2\sqrt{\lambda t}}}^{\infty} e^{-s^2}\, ds + \int_{-\infty}^{\frac{x}{2\sqrt{\lambda t}}} e^{-s^2}\, ds \right]$$

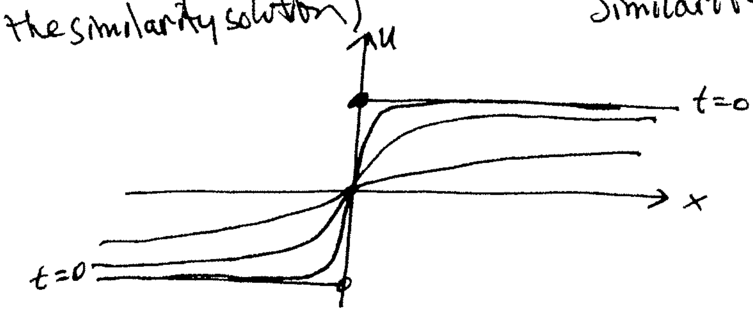
$$= \frac{1}{\sqrt{\pi}}\left[-\int_{\frac{x}{2\sqrt{\lambda t}}}^{\infty} e^{-s^2}\, ds + \left(\underbrace{\int_{-\infty}^{\infty} e^{-s^2}\, ds}_{\text{even}} - \int_{\frac{x}{2\sqrt{\lambda t}}}^{\infty} e^{-s^2}\, ds \right) \right]$$

$$= \frac{2}{\sqrt{\pi}}\left[\int_0^{\infty} e^{-s^2}\, ds - \int_{\frac{x}{2\sqrt{\lambda t}}}^{\infty} e^{-s^2}\, ds \right]$$

$$\boxed{u(x,t) = \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{\lambda t}}} e^{-s^2}\, ds = \mathrm{erf}\left(\frac{x}{2\sqrt{\lambda t}}\right)}$$

Similarity Variable

(agrees with the similarity solution)



Convolution

Convolution of f and g: $\boxed{(f*g)(x) = \int_{-\infty}^{\infty} f(\xi)\, g(x-\xi)\, d\xi \overset{\rho = x-\xi}{=} \int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi}$

$\Rightarrow \quad f*g = g*f$

Example: $f(x) = x, \quad g(x) = e^{-x^2}$

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-\xi)\, g(\xi)\, d\xi = \int_{-\infty}^{\infty} (x-\xi)\, e^{-\xi^2} d\xi$$

$$= x\int_{-\infty}^{\infty} \underbrace{e^{-\xi^2}}_{\text{even}} d\xi - \cancel{\int_{-\infty}^{\infty} \underbrace{\xi e^{-\xi^2}}_{\text{odd}} d\xi}^{=0} = x\cdot 2\int_0^{\infty} e^{-\xi^2} d\xi = 2x \cdot \frac{\sqrt{\pi}}{2}\operatorname{erf}(\infty)$$

$$= x\sqrt{\pi}\cdot 1$$

$\boxed{(f*g)(x) = \sqrt{\pi}\, x}$

Inverting Products of Transforms: $\mathcal{F}^{-1}\{F(k)G(k)\} \neq f(x)g(x)$

i.e. $F(k)G(k) \neq \mathcal{F}\{f(x)g(x)\}$

$$F(k)G(k) = \left(\int_{-\infty}^{\infty} f(\xi) e^{ik\xi} d\xi\right)\left(\int_{-\infty}^{\infty} g(\rho) e^{ik\rho} d\rho\right) = \int_{-\infty}^{\infty} f(\xi)\left(\int_{-\infty}^{\infty} g(\rho)\, e^{ik(\rho+\xi)} d\rho\right) d\xi$$

$x = \rho + \xi$

$$= \int_{-\infty}^{\infty} f(\xi)\left(\int_{-\infty}^{\infty} g(x-\xi) e^{ikx} dx\right) d\xi = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(\xi)\, g(x-\xi)\, d\xi\right) e^{ikx} dx$$

$$= \int_{-\infty}^{\infty} (f*g)(x)\cdot e^{ikx} dx = \mathcal{F}\{(f*g)(x)\}$$

$F(k)G(k) = \mathcal{F}\{(f*g)(x)\}$

OR $\boxed{\mathcal{F}^{-1}\{F(k)G(k)\} = (f*g)(x)}$

OR $\boxed{\mathcal{F}\{F(k)G(k)\} = \mathcal{F}^{-1}\{F(k)\} * \mathcal{F}^{-1}\{G(k)\}}$

$F(k) = \mathcal{F}\{f(x)\}$

$G(k) = \mathcal{F}\{g(x)\}$

Note: If we had defined the Fourier transform as $F(k) = \alpha\int_{-\infty}^{\infty} f(x)\, e^{\pm ikx} dx$,

e.g. $\alpha = 1, \frac{1}{2\pi}, \frac{1}{\sqrt{2\pi}}$

then, $F(k)G(k) = \mathcal{F}\left\{\alpha\int_{-\infty}^{\infty} f(\xi)\, g(x-\xi)\, d\xi\right\} = \mathcal{F}\{(f*g)(x)\}$ $\qquad \beta = \frac{1}{2\pi\alpha}$

Convolution is defined with an addition factor of α.

Recall: $u_t = \lambda u_{xx}$, $0 < t < \infty$, $-\infty < x < \infty$
$u(x,0) = f(x)$

$$U(k,t) = \mathcal{F}\{u(x,t)\} = \int_{-\infty}^{\infty} u(x,t)\, e^{ikx}\, dx$$

$$\Rightarrow \quad \begin{matrix} U_t = -\lambda k^2 U \\ U(k,0) = F(k) \end{matrix} \quad \Rightarrow \quad U(k,t) = F(k) e^{-\lambda k^2 t}$$

Invert: $u(x,t) = \mathcal{F}^{-1}\{U(k,t)\} = \mathcal{F}^{-1}\{F(k) e^{-\lambda k^2 t}\} = \mathcal{F}^{-1}\{F(k)\} * \mathcal{F}^{-1}\{e^{-\lambda k^2 t}\}$

$$= f(x) * \underbrace{\frac{e^{-x^2/4\lambda t}}{2\sqrt{\pi \lambda t}}}_{g}$$

$$= \int_{-\infty}^{\infty} f(\xi)\, g(x-\xi)\, d\xi$$

From Table: $\mathcal{F}\{e^{-ax^2}\} = \sqrt{\frac{\pi}{a}}\, e^{-k^2/4a}$

$\Rightarrow \quad \mathcal{F}^{-1}\{e^{-k^2/4a}\} = \sqrt{\frac{a}{\pi}}\, e^{-ax^2}$

$\frac{1}{4a} = \lambda t \qquad \mathcal{F}\{e^{-\lambda k^2 t}\} = \sqrt{\frac{1/4\lambda t}{\pi}}\, e^{-\frac{x^2}{4\lambda t}}$

$a = \frac{1}{4\lambda t} \qquad \mathcal{F}\{e^{-\lambda k^2 t}\} = \frac{e^{-x^2/4\lambda t}}{2\sqrt{\pi \lambda t}}$

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} f(\xi) \cdot \frac{e^{-\frac{(x-\xi)^2}{4\lambda t}}}{2\sqrt{\pi \lambda t}}\, d\xi}$$

Agrees with previous result.

Properties

1) $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\xi) e^{-i\xi x}\, d\xi$

$$f(-k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\xi) e^{ik\xi}\, d\xi = \frac{1}{2\pi}\mathcal{F}\{F(x)\}$$

$$\boxed{\mathcal{F}\{F(x)\} = 2\pi f(-k)}$$

Example: From Table: $\mathcal{F}\{e^{-a|x|}\} = \frac{2a}{k^2+a^2}, \quad a>0$

$f(x) = e^{-a|x|}$

$F(k) = \frac{2a}{k^2+a^2}$ $\Rightarrow$ $\mathcal{F}\left\{\frac{\cancel{2}a}{x^2+a^2}\right\} = \cancel{2}\pi e^{-a|k|}$

$$\mathcal{F}\left\{\frac{1}{x^2+a^2}\right\} = \frac{\pi}{a} e^{-a|k|}$$

2) $\mathcal{F}\{f(ax)\} = \int_{-\infty}^{\infty} f(ax) e^{ikx}\, dx = \int_{-\infty}^{\infty} f(\xi) e^{i(k/a)\xi} \cdot \frac{1}{a} d\xi = \frac{1}{a} F(k/a)$

$a>0$ $\quad \xi = ax$

$a<0 \Rightarrow \;= -\frac{1}{a} F(k/a)$

$$\boxed{\mathcal{F}\{f(ax)\} = \frac{1}{|a|} F(k/a)}$$

Example: $\mathcal{F}\{e^{-2a|x|}\} = \mathcal{F}\{e^{-a|2x|}\} = \frac{1}{|2|} F(k/2) = \frac{1}{\cancel{2}} \frac{\cancel{2}a}{(k/2)^2+a^2} = \frac{4a}{k^2+4a^2}$

$f(2x)$

$f(x) = e^{-a|x|} \Rightarrow F(k) = \frac{2a}{k^2+a^2}$

3) $\mathcal{F}\{f(x-x_0)\} = \int_{-\infty}^{\infty} f(x-x_0) e^{ikx}\, dx = \int_{-\infty}^{\infty} f(\xi) e^{ik(\xi+x_0)}\, dx = e^{ikx_0}\mathcal{F}\{f(\xi)\}$

$\xi = x - x_0$

$$\boxed{\mathcal{F}\{f(x-x_0)\} = e^{ikx_0} F(k)}$$ Translation (x-axis)

4) $\mathcal{F}\{f(x)e^{-ik_0x}\} = \int_{-\infty}^{\infty} f(x) e^{-ik_0x} e^{ikx}\, dx = \int_{-\infty}^{\infty} f(x) e^{i(k-k_0)x}\, dx = F(k-k_0)$

$$\boxed{\mathcal{F}\{f(x)e^{-ik_0x}\} = F(k-k_0)}$$ Translation (k-axis)

Multiple Transforms

A separate transform can be applied for each independent variable.

Example: $u_t = \lambda u_{xx}$, $0 < t < \infty$
$u(x,0) = f(x)$, $-\infty < x < \infty$

Fourier: $U(k,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{ikx}\, dx$

$$\Rightarrow \boxed{\begin{array}{l} U_t = -\lambda k^2 u_{xx} \\ U(k,0) = F(k) \end{array}}$$

Laplace: $V(k,s) = \int_0^{\infty} U(k,t)\, e^{-st}\, dt$

The IC gets applied here

$$\mathcal{L}\{U_t(k,t)\} = s\,V(k,s) - U(k,0) = s\,V(k,s) - F(k)$$

$$\Rightarrow \quad s\,V(k,s) - F(k) = -\lambda k^2 V(k,s)$$

$$\boxed{V(k,s) = \frac{F(k)}{s + \lambda k^2}} \qquad \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\Rightarrow U(k,t) = \mathcal{L}^{-1}\left\{\frac{F(k)}{s+\lambda k^2}\right\} = F(k)\,\mathcal{L}^{-1}\left\{\frac{1}{s+\lambda k^2}\right\} = F(k)\, e^{-\lambda k^2 t}$$

$$\boxed{U(k,t) = F(k)\, e^{-\lambda k^2 t}}$$

$$\Rightarrow u(x,t) = \mathcal{F}^{-1}\{F(k)\} * \mathcal{F}^{-1}\{e^{-\lambda k^2 t}\}$$

$$= f(x) * \frac{e^{-x^2/4\lambda t}}{2\sqrt{\pi \lambda t}}$$

$$\boxed{u(x,t) = \int_{-\infty}^{\infty} f(\xi)\, \frac{e^{-(x-\xi)^2/4\lambda t}}{2\sqrt{\pi \lambda t}}\, d\xi}$$

Example: $u_t = \lambda(u_{xx}+u_{yy})$, $0<t<\infty$
$u(x,y,0) = f(x,y)$, $-\infty < x,y<\infty$

Fourier(x): $U(k_1,y,t) = \mathcal{F}_1\{u(x,y,t)\} = \int_{-\infty}^{\infty} u(x,y,t)e^{ik_1x}\,dx$

$$\Rightarrow \quad \boxed{\begin{aligned} U_t &= \lambda(-k_1^2U + U_{yy}) \\ U(k_1,y,0) &= F_1(k_1,y) \end{aligned}} \text{ where } F_1(k_1,y) = \mathcal{F}_1\{f(x,y)\}$$

Fourier(y): $V(k_1,k_2,t) = \mathcal{F}_2\{U(k_1,y,t)\} = \int_{-\infty}^{\infty} U(k_1,y,t)e^{ik_2y}\,dy$

$$\Rightarrow V_t = \lambda(-k_1^2V - k_2^2V)$$

$$\boxed{\begin{aligned} V_t &= -\lambda(k_1^2+k_2^2)V \\ V(k_1,k_2,0) &= F_2(k_1,k_2) \end{aligned}} \text{ where } F_2(k_1,k_2) = \mathcal{F}_2\{F_1(k_1,y)\}$$

Solve: $V(k_1,k_2,t) = F_2(k_1,k_2)\,e^{-\lambda(k_1^2+k_2^2)t}$

Invert(y): $U(k_1,y,t) = \mathcal{F}_2^{-1}\{V(k_1,k_2,t)\} = e^{-\lambda k_1^2 t}\,\underbrace{\mathcal{F}_2^{-1}\{F_2(k_1,k_2)\}}_{F_1(k_1,y)} * \underbrace{\mathcal{F}_2^{-1}\{e^{-\lambda k_2 t}\}}_{\frac{e^{-\frac{y^2}{4\lambda t}}}{2\sqrt{\pi\lambda t}}}$

$$U(k_1,y,t) = e^{-\lambda k_1^2 t}\int_{-\infty}^{\infty} F_1(k_1,\eta)\frac{e^{-\frac{(y-\eta)^2}{4\lambda t}}}{2\sqrt{\pi\lambda t}}\,d\eta$$

Invert(x): $u(x,y,t) = \mathcal{F}_1^{-1}\{U(k_1,y,t)\} = \mathcal{L}^{-1}\{e^{-\lambda k_1^2 t}\} * \mathcal{L}^{-1}\left\{\int_{-\infty}^{\infty} F_1(k_1,\eta)\frac{e^{-\frac{(y-\eta)^2}{4\lambda t}}}{2\sqrt{\pi\lambda t}}\,d\eta\right\}$

$$= \underbrace{\frac{e^{-\frac{x^2}{4\lambda t}}}{2\sqrt{\pi\lambda t}}}_{f} * \underbrace{\left(\int_{-\infty}^{\infty} f(x,\eta)\frac{e^{-\frac{(y-\eta)^2}{4\lambda t}}}{2\sqrt{\pi\lambda t}}\,d\eta\right)}_{g} \quad = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi$$

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-\xi)^2}{4\lambda t}}}{2\sqrt{\pi\lambda t}}\left(\int_{-\infty}^{\infty} f(\xi,\eta)\frac{e^{-\frac{(y-\eta)^2}{4\lambda t}}}{2\sqrt{\pi\lambda t}}\,d\eta\right)d\xi$$

$$\boxed{u(x,y,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi,\eta)\,\frac{e^{-\frac{(x-\xi)^2-(y-\eta)^2}{4\lambda t}}}{4\pi\lambda t}\,d\xi\,d\eta}$$

2-D Free Space Green's Function

Observe that

$$V(k_1,k_2,t) = \int_{-\infty}^{\infty} U(k_1,y,t)e^{ik_2y}\,dy$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} u(x,y,t)\,e^{ik_1x}\,dx\right)e^{ik_2y}\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y,t)\,e^{i(k_1x+k_2y)}\,dx\,dy$$

Let $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{k} = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$.

$$\Rightarrow \boxed{\vec{V}(\vec{k},t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(\vec{x},t)\,e^{i\vec{k}\cdot\vec{x}}\,dx\,dy} \quad \text{(2-D Fourier Transform)}$$

<u>n-Dimensional Fourier Transform</u> $\quad \vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \vec{k} = \begin{pmatrix} k_1 \\ \vdots \\ k_n \end{pmatrix}$

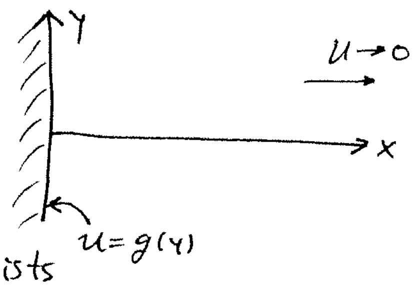
$$\boxed{\mathcal{F}\{f(\vec{x})\} = \int_{\mathbb{R}^n} f(\vec{x})\,e^{i\vec{k}\cdot\vec{x}}\,d\vec{x}}$$

$$\boxed{f(\vec{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} F(\vec{k})\,e^{-i\vec{k}\cdot\vec{x}}\,d\vec{k}}$$

$$\int_{\mathbb{R}^n} = \underbrace{\int_{-\infty}^{\infty}\cdot\ \cdots\ \cdot\int_{-\infty}^{\infty}}_{n\text{-fold}} \qquad \begin{aligned} d\vec{x} &= dx_1\cdot\ \cdots\ \cdot dx_n \\ d\vec{k} &= dk_1\cdot\ \cdots\ \cdot dk_n \end{aligned}$$

Example: The Laplace Equation on a half-plane.

$$u_{xx} + u_{yy} = 0, \quad 0 < x < \infty$$
$$u(0,y) = g(y) \quad -\infty < y < \infty$$
$$u \to 0 \text{ as } x \to \infty$$



where $\int_{-\infty}^{\infty} |g(y)|\,dy$ converges $\Rightarrow$ $G(k)$ exists

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Transform (y): $U(x,k) = \mathcal{F}\{u(x,y)\} = \int_{-\infty}^{\infty} u(x,y)e^{iky}\,dy$

$$\Rightarrow \boxed{\begin{array}{ll} U_{xx} - k^2 U = 0 & ① \\ U(0,k) = G(k) & ② \\ U \to 0 \text{ as } x \to \infty & ③ \end{array}}$$

① $\Rightarrow$ $U(x,k) = A(k)e^{kx} + B(k)e^{-kx}$

③ $\Rightarrow$ $U(x,k) = \begin{cases} A(k)e^{kx}, & k < 0 \\ B(k)e^{-kx}, & k > 0 \end{cases}$

Therefore, $\boxed{U(x,k) = C(k)e^{-|k|x}}$ where $C(k) = \begin{cases} A(k), & k < 0 \\ B(k), & k > 0 \end{cases}$

② $\Rightarrow$ $U(0,k) = C(k)e^{0} = G(k)$

$$\underline{\underline{C(k) = G(k)}}$$

$$\Rightarrow \boxed{U(x,k) = G(k)e^{-|k|x}}$$

convolution in the y-variable

Invert: $u(x,y) = \mathcal{F}^{-1}\{U(x,k)\} = \mathcal{F}^{-1}\{G(k)e^{-|k|x}\} = \mathcal{F}^{-1}\{G(k)\} * \mathcal{F}^{-1}\{e^{-|k|x}\}$

$$\boxed{u(x,y) = g(y) * \mathcal{F}^{-1}\{e^{-|k|x}\}}$$

$$\mathcal{F}^{-1}\{e^{-|k|x}\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|k|x} e^{-iky}\, dk$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{kx} e^{-iky}\, dk + \int_{0}^{\infty} e^{-kx} e^{-iky}\, dk\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{k(x-iy)}\, dk + \int_{0}^{\infty} e^{-k(x+iy)}\, dk\right]$$

$$= \frac{1}{2\pi}\left[\left.\frac{e^{k(x-iy)}}{x-iy}\right|_{-\infty}^{0} + \left.\frac{e^{-k(x+iy)}}{-(x+iy)}\right|_{0}^{\infty}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1-0}{x-iy} - \frac{0-1}{x+iy}\right] = \frac{1}{2\pi}\,\frac{(x+\cancel{iy}) + (x-\cancel{iy})}{x^2+y^2}$$

$$\boxed{\mathcal{F}^{-1}\{e^{-|k|x}\} = \frac{x/\pi}{x^2+y^2}}$$

OR use a Table

$$\mathcal{F}\left\{\frac{1}{x^2+a^2}\right\} = \frac{\pi}{a}e^{-a|k|}$$

$$\mathcal{F}^{-1}\{e^{-a|k|}\} = \frac{a/\pi}{x^2+a^2}$$

$a = x$

$x = y$

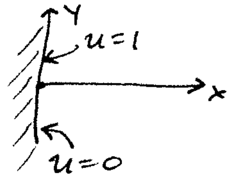
$$\boxed{\mathcal{F}^{-1}\{e^{-|k|x}\} = \frac{x/\pi}{x^2+y^2}}$$

$$u(x,y) = g(y) * \frac{x/\pi}{x^2+y^2}$$

$$\boxed{u(x,y) = \frac{x}{\pi}\int_{-\infty}^{\infty} \frac{g(\eta)}{x^2+(y-\eta)^2}\, d\eta}$$

Note: This solution is valid for a larger class of functions $g(y)$ than those for which the Fourier Transform exists.

Roughly speaking, this solution is valid provided that the integral exists. This fact can be verified by using a Green's Function approach to solving the problem. Such a case is demonstrated in the following example.



Example: Solve the above problem with $g(y)=\begin{cases}0, & y<0\\ 1, & y>0\end{cases}$.

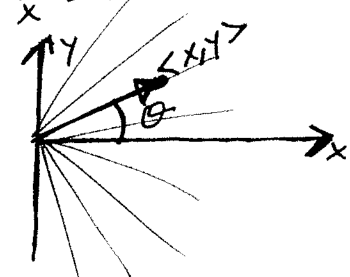
$$u(x,y)=\frac{x}{\pi}\int_{-\infty}^{\infty}\frac{g(\eta)\,d\eta}{x^2+(y-\eta)^2}=\frac{x}{\pi}\int_0^{\infty}\frac{d\eta}{x^2+(y-\eta)^2}=\frac{-1}{\pi}\int_0^{\infty}\frac{-\frac{1}{x}\,d\eta}{1+\left(\frac{y-\eta}{x}\right)^2}$$

$$=\frac{-1}{\pi}\tan^{-1}\left(\frac{y-\eta}{x}\right)\Big|_0^{\infty}=\frac{-1}{\pi}\left[\tan^{-1}(-\infty)-\tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$=\frac{-1}{\pi}\left[-\frac{\pi}{2}-\tan^{-1}\left(\frac{y}{x}\right)\right]=\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{y}{x}\right)$$

$$\boxed{u(x,y)=\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{y}{x}\right)}$$

Observe that the solution is constant along the lines $\frac{y}{x}=c$.
That is, the solution depends only on the angle $\theta=\tan^{-1}\left(\frac{y}{x}\right)$ that $\langle x,y\rangle$ makes with the positive x-axis.



Had we known this beforehand, we could have used polar coordinates with $\frac{\partial}{\partial r}=0$. $\Rightarrow$ $u=u(\theta)$

$$u_{xx}+u_{yy}=\cancel{u_{rr}+\frac{1}{r}u_r}+\frac{1}{r^2}u_{\theta\theta}=0$$

$$u_{\theta\theta}=0 \qquad u\left(-\frac{\pi}{2}\right)=0,\quad u\left(\frac{\pi}{2}\right)=1$$

Then, $u(\theta)=A\theta+B$

$$u(-\pi/2)=-\frac{\pi}{2}A+B=0$$
$$u(\pi/2)=\frac{\pi}{2}A+B=1$$

$+\Rightarrow \quad 2B=1 \Rightarrow B=1/2$

$-\Rightarrow \quad \pi A=1 \Rightarrow A=1/\pi$

$$\Rightarrow \boxed{u(\theta)=\frac{1}{\pi}\theta+\frac{1}{2}} \Rightarrow \boxed{u(x,y)=\frac{1}{\pi}\tan^{-1}\left(\frac{y}{x}\right)+\frac{1}{2}}$$

Example:

$$u_{tt} = c^2 u_{xx} \qquad 0<t<\infty$$
$$u(x,0) = f(x) \qquad -\infty < x < \infty$$
$$u_t(x,0) = 0 \qquad \text{d'Alembert} \Rightarrow u(x,t) = \tfrac{1}{2}[f(x-ct)+f(x+ct)]$$

Transform: $U(k,t) = \mathcal{F}\{u(x,t)\} = \int_{-\infty}^{\infty} u(x,t)e^{ikx}\,dx$

$$\boxed{\begin{array}{l} U_{tt} = -c^2k^2U \\ U(k,0) = F(k) \\ U_t(k,0) = G(k) \end{array}}$$

$\Rightarrow U(k,t) = A(k)\cos(ckt) + B(k)\sin(ckt)$

$U(k,0) = A(k) + 0 = F(k) \quad \Rightarrow \quad \underline{A(k) = F(k)}$

$U_t(k,0) = 0 + ck\,B(k) = 0 \quad \Rightarrow \quad \underline{B(k) = 0}$

$$\boxed{U(k,t) = F(k)\cos(ckt)}$$

Then,

$$u(x,t) = \mathcal{F}^{-1}\{U(k,t)\} = \mathcal{F}^{-1}\{F(k)\cos(ckt)\}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\cos(ckt)e^{-ikx}\,dk \qquad \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \theta = ckt$$

$$= \frac{1}{2}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)\left(e^{ickt}+e^{-ickt}\right)e^{-ikx}\,dk$$

$$= \frac{1}{2}\Bigg[\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{-ik(x-ct)}\,dk}_{f(x-ct)} + \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{-ik(x+ct)}\,dk}_{f(x+ct)}\Bigg]$$

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(k)e^{-ikx}\,dk \Rightarrow$

$$\boxed{u(x,t) = \frac{1}{2}[f(x-ct)+f(x+ct)]}$$

OR use the formula $\mathcal{F}^{-1}\{F(k)e^{ikx_0}\} = f(x-x_0)$

$$= \frac{1}{2}\Bigg[\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(F(k)e^{ickt}\right)e^{-ikx}\,dk + \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(F(k)e^{-ickt}\right)e^{-ikx}\,dk\Bigg]$$

$$x_0 = ct \qquad\qquad x_0 = -ct$$

$$= \frac{1}{2}[f(x-ct)+f(x+ct)]$$

Fourier Sine and Cosine Transforms

Fourier Transform: $F(k) = \alpha \int_{-\infty}^{\infty} f(x) e^{ikx} dx = \alpha \int_{-\infty}^{\infty} f(x)(\cos(kx) + i\sin(kx))\, dx$

$$\boxed{F(k) = \alpha \int_{-\infty}^{\infty} f(x)\cos(kx)\, dx + i\alpha \int_{-\infty}^{\infty} f(x)\sin(kx)\, dx}$$

$$f(x) = \beta \int_{-\infty}^{\infty} F(k) e^{-ikx} dk = \beta \int_{-\infty}^{\infty} F(k)(\cos(kx) - i\sin(kx))\, dk$$

$$\boxed{f(x) = \beta \int_{-\infty}^{\infty} F(k)\cos(kx)\, dk - i\beta \int_{-\infty}^{\infty} F(k)\sin(kx)\, dk} \text{ where } \boxed{\alpha\beta = \frac{1}{2\pi}}$$

Suppose f is even $\Rightarrow$ $f(x)\cos(kx)$ is even, $f(x)\sin(kx)$ is odd

then, $F(k) = 2\alpha \int_0^{\infty} f(x) \underbrace{\cos(kx)}_{\text{even}}\, dx + 0 \Rightarrow F(k)$ is even also

Pick $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{\pi}$ $\Rightarrow$ $\boxed{F_c(k) = \int_0^{\infty} f(x)\cos(kx)\, dx}$ Fourier Cosine Transform

$\boxed{f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(k)\cos(kx)\, dk}$ Inverse Fourier Cosine Transform

Suppose f is odd $\Rightarrow$ $f(x)\cos(kx)$ is odd, $f(x)\sin(kx)$ is even

Then, $F(k) = 0 + 2i\alpha \int_0^{\infty} f(x)\sin(kx)\, dx$

Pick $\alpha = \frac{1}{2i}$ and $\beta = \frac{i}{\pi}$ $\Rightarrow$ $\boxed{F_s(k) = \int_0^{\infty} f(x)\sin(kx)\, dx}$ Fourier Sine Transform

$\boxed{f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(k)\sin(kx)\, dk}$ Inverse Fourier Sine Transform

Note: On a semi-infinite domain, say $x \geq 0$, f may be defined only for $x > 0$. An even (odd) extension of f can be made and the Fourier Cosine (Sine) Transform can be used.

The Fourier Cosine (Sine) Transform of f is equivalent to the Fourier Transform of the even (odd) extension of f.

Derivatives

$$\mathcal{F}_c\{f'(x)\} = \int_0^\infty f'(x)\cos(kx)\,dx \qquad \begin{matrix} u=\cos(kx) & v=f(x) \\ du=-k\sin(kx) & dv=f'(x)\,dx \end{matrix}$$

$$= f(x)\cos(kx)\Big|_0^\infty + k\int_0^\infty f(x)\sin(kx)\,dx = (0-f(0)) + k\,\mathcal{F}_s\{f(x)\}$$

$$\boxed{\mathcal{F}_c\{f'(x)\} = k\,\mathcal{F}_s\{f(x)\} - f(0)}$$

$$\mathcal{F}_s\{f'(x)\} = \int_0^\infty f'(x)\sin(kx)\,dx \qquad \begin{matrix} u=\sin(kx) & v=f(x) \\ du=k\cos(kx)\,dx & dv=f'(x)\,dx \end{matrix}$$

$$= f(x)\sin(kx)\Big|_0^\infty - k\int_0^\infty f(x)\cos(kx)\,dx = (0-0) - k\,\mathcal{F}_c\{f(x)\}$$

$$\boxed{\mathcal{F}_s\{f'(x)\} = -k\,\mathcal{F}_c\{f(x)\}}$$

$$\mathcal{F}_c\{f''(x)\} = k\,\mathcal{F}_s\{f'(x)\} - f'(0) = k\left[-k\,\mathcal{F}_c\{f(x)\}\right] - f'(0)$$

$$\boxed{\mathcal{F}_c\{f''(x)\} = -k^2\,\mathcal{F}_c\{f(x)\} - f'(0)}$$

$$\mathcal{F}_s\{f''(x)\} = -k\,\mathcal{F}_c\{f'(x)\} = -k\left[k\,\mathcal{F}_s\{f(x)\} - f(0)\right]$$

$$\boxed{\mathcal{F}_s\{f''(x)\} = -k^2\,\mathcal{F}_s\{f(x)\} + k f(0)}$$

Note: $\mathcal{F}_c\{f'(x)\}$ depends on $\mathcal{F}_s\{f(x)\}$
and $\mathcal{F}_s\{f'(x)\}$ depends on $\mathcal{F}_c\{f(x)\}$.

$\Rightarrow$ The Fourier Sine and Cosine Transforms are usually not useful if the PDE involves first derivatives of the transformed variable.

Useful Laplace Transforms

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$$

The following two formulas can be found in tables of Laplace transforms.

1) $$\mathcal{L}\left\{\frac{e^{-x^2/4t}}{\sqrt{\pi t}}\right\} = \frac{e^{-|x|\sqrt{s}}}{\sqrt{s}}$$

2) $$\mathcal{L}\{f(at)\} = \frac{1}{a} F(s/a) \quad \text{for } a > 0$$

Let $f(t) = \dfrac{e^{-x^2/4t}}{\sqrt{\pi t}}$. $\Rightarrow$ $F(s) = \dfrac{e^{-|x|\sqrt{s}}}{\sqrt{s}}$

Then
$$\mathcal{L}\left\{\frac{e^{-x^2/4\lambda t}}{2\sqrt{\pi\lambda t}}\right\} = \frac{1}{2}\mathcal{L}\{f(\lambda t)\} = \frac{1}{2}\cdot\frac{1}{\lambda}F\left(\frac{s}{\lambda}\right)$$
$$= \frac{1}{2\lambda}\frac{e^{-|x|\sqrt{s/\lambda}}}{\sqrt{s/\lambda}} = \frac{e^{-|x|\sqrt{s/\lambda}}}{2\sqrt{\lambda s}}$$

$$\boxed{\mathcal{L}\left\{\frac{e^{-x^2/4\lambda t}}{2\sqrt{\pi\lambda t}}\right\} = \frac{e^{-|x|\sqrt{s/\lambda}}}{2\sqrt{\lambda s}}}$$

$$\Rightarrow \mathcal{L}\left\{\frac{e^{-\frac{(x-\xi)^2}{4\lambda t}}}{2\sqrt{\pi\lambda t}}\right\} = \frac{e^{-|x-\xi|\sqrt{s/\lambda}}}{2\sqrt{\lambda s}}$$

Free Space Green's Function for the 1-D Heat Equation.

Duhamel's Principle

ODEs: Let L be the linear differential operator defined by

$$Lu(t) = u^{(n)}(t) + a_{n-1}u^{(n-1)}(t) + \cdots + a_1 u'(t) + a_0 u(t).$$

Consider the IVP

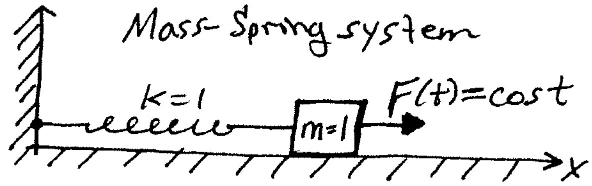
$$\boxed{\begin{aligned} &Lu(t) = f(t) \\ &u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0. \end{aligned}}$$ (n homogeneous BCs)

Let v be the solution of the IVP given by

$$\boxed{\begin{aligned} &Lv(t|s) = 0 \quad (s = \text{parameter}) \\ &v(s|s) = v'(s|s) = \cdots = v^{(n-2)}(s|s) = 0 \\ &v^{(n-1)}(s|s) = f(s). \end{aligned}}$$

Then, $\boxed{u(t) = \int_0^t v(t|s)\, ds}$

Example: Solve the IVP $x'' + x = \cos t$, $x(0) = x'(0) = 0$



Duhamel: Consider $y'' + y = 0$, $y = y(t|s)$, $y(s|s) = 0$, $y'(s|s) = \cos(s)$

$$\Rightarrow y(t|s) = C_1(s)\cos t + C_2(s)\sin t$$

$$y'(t|s) = -C_1(s)\sin t + C_2(s)\cos t$$

ICs:
$$y(s|s) = C_1\cos(s) + C_2\sin(s) = 0 \Rightarrow C_1 + C_2\tan(s) = 0$$
$$y'(s|s) = -C_1\sin(s) + C_2\cos(s) = \cos(s) \Rightarrow -C_1\tan(s) + C_2 = 1$$

$$C_1\tan(s) + C_2\tan^2(s) = 0$$
$$+(-C_1\tan(s) + C_2 = 1)$$
$$C_2(1+\tan^2(s)) = 1$$
$$C_2 \cdot \sec^2(s) = 1$$
$$C_2 = \cos^2(s)$$
$$C_1 = -C_2\tan(s) = -\cos^2(s)\cdot\frac{\sin(s)}{\cos(s)}$$
$$C_1 = -\cos(s)\sin(s)$$

$$\Rightarrow y(t|s) = (-\cos(s)\sin(s))\cos t + (\cos^2(s))\sin t$$
$$= \left(-\tfrac{1}{2}\sin(2s)\right)\cos t + \left(\frac{1+\cos(2s)}{2}\right)\sin t$$
$$= \tfrac{1}{2}\sin t + \tfrac{1}{2}\left(\sin t\cos(2s) - \sin(2s)\cos t\right)$$

$$\boxed{y(t|s) = \tfrac{1}{2}\left[\sin t + \sin(t-2s)\right]}$$

Then,

$$x(t) = \int_0^t y(t|s)\,ds = \frac{1}{2}\int_0^t \left[\sin t + \sin(t-2s)\right]ds$$
$$= \frac{1}{2}\left[s\cdot\sin t + \tfrac{1}{2}\cos(t-2s)\right]_0^t$$
$$= \tfrac{1}{2}\left[\left(t\sin t + \cancel{\tfrac{1}{2}\cos t}\right) - \left(0 + \cancel{\tfrac{1}{2}\cos t}\right)\right] = \tfrac{1}{2}t\sin t$$

$$\boxed{x(t) = \tfrac{1}{2}t\sin t}$$

PDEs:

Example: $u_t = \lambda u_{xx} + f(x,t)$
$u(x,0) = 0$

Duhamel: $v_t = \lambda v_{xx}, \quad v = v(x,t|s)$
$v(x,s|s) = f(x,s)$

Fourier Transform (Same as before, but with t replaced by $t-s$) $\Longrightarrow v(x,t|s) = \int_{-\infty}^{\infty} f(\xi,s) \dfrac{e^{\frac{-(x-\xi)^2}{4\lambda(t-s)}}}{2\sqrt{\pi\lambda(t-s)}}\, d\xi$

Then,

$$\boxed{u(x,t) = \int_0^t v(x,t|s)\,ds = \int_0^t \int_{-\infty}^{\infty} f(\xi,s) \underbrace{\frac{e^{\frac{-(x-\xi)^2}{4\lambda(t-s)}}}{2\sqrt{\pi\lambda(t-s)}}}_{\text{Free Space Green's Function for the 1-D Heat Equation}: \ G(x,t|\xi,s)}\, d\xi\, ds}$$

Example: $u_{tt} = c^2 u_{xx} + g(x,t)$
$u(x,0) = u_t(x,0) = 0$

Duhamel: $v_{tt} = c^2 v_{xx}, \quad v = v(x,t|s)$
$v(x,s|s) = 0$
$v_t(x,s|s) = g(x,s)$

d'Alembert (with t replaced by $t-s$) $\Longrightarrow v(x,t|s) = \dfrac{1}{2c}\displaystyle\int_{x-c(t-s)}^{x+c(t-s)} g(\xi,s)\, d\xi$

Then,

$$\boxed{u(x,t) = \int_0^t v(x,t|s)\,ds = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} g(\xi,s)\, d\xi\, ds}$$

Geometrical Interpretation of the Heat Equation

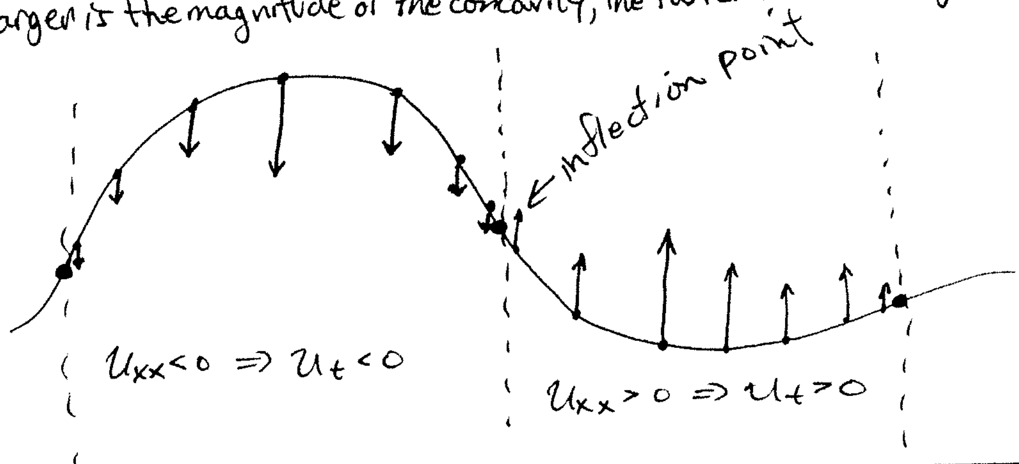
$$u_t = k\, u_{xx}, \quad k > 0$$ e.g. u = temperature

u_{xx} = concavity of u

$u_{xx} > 0 \Rightarrow u_t > 0 \Rightarrow$ temperature increases

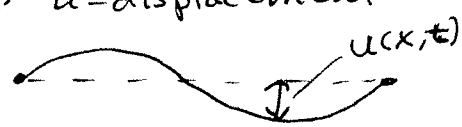
$u_{xx} < 0 \Rightarrow u_t < 0 \Rightarrow$ temperature decreases

The larger is the magnitude of the concavity, the faster is the change in temperature.



Geometrical Interpretation of the Wave Equation

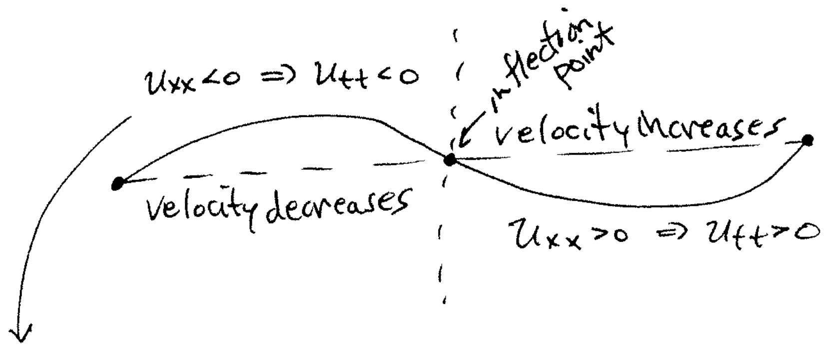
$$u_{tt} = c^2 u_{xx}$$ e.g. u = displacement



u_{xx} = concavity of u

$u_{xx} > 0 \Rightarrow u_{tt} > 0 \Rightarrow$ velocity increases (upward acceleration)

$u_{xx} < 0 \Rightarrow u_{tt} < 0 \Rightarrow$ velocity decreases (downward acceleration)



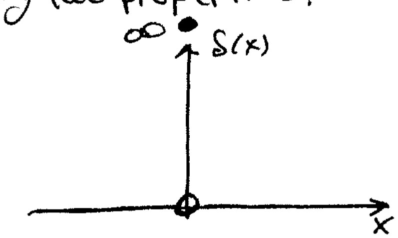
$u_{tt} < 0 \Rightarrow$ velocity decreases, even though the string may still be moving upward ($u_t > 0$).

Dirac Delta Function

The Dirac Delta Function is the function with the following two properties.

① $\boxed{\delta(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}}$

② $\boxed{\int_{-\infty}^{\infty} \delta(x)\,dx = 1} \Rightarrow \int_{0^-}^{0^+} \delta(x)\,dx = 1$

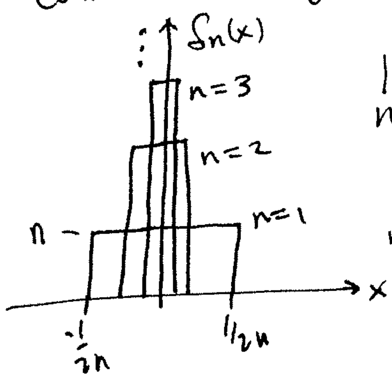


$\int_{-\infty}^{\infty} \delta(x)\,dx = \text{Area} = (\text{width}) \times (\text{height}) = 0 \cdot \infty$ (indeterminate form)

The value of the indeterminate form $0 \cdot \infty$ can be taken to be equal to 1 if we consider $\delta(x)$ to be the limit of a so-called δ-sequence of functions

δ-sequences

Consider the sequence of functions $\boxed{\delta_n(x) = \begin{cases} n, & |x| \le \frac{1}{2n} \\ 0, & |x| > \frac{1}{2n} \end{cases}}$



$$\lim_{n \to \infty} \delta_n(x) = \begin{cases} \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \delta_n(x)\,dx = \lim_{n \to \infty} n \cdot \left(\frac{1}{2n} - \frac{-1}{2n}\right) = \lim_{n \to \infty} 1 = 1$$

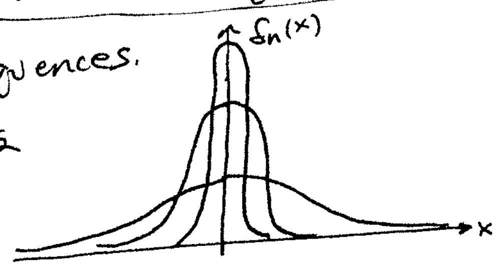
$$\Rightarrow \boxed{\lim_{n \to \infty} \delta_n(x) = \delta(x)}$$

The sequence $\{\delta_n(x)\}_{n=1,2,\ldots}$ is said to be a δ-sequence since $\delta_n(x) \to \delta(x)$.

There are infinitely many such sequences.

e.g. i) $\delta_n(x) = \frac{1}{\pi} \cdot \frac{n}{1 + n^2 x^2}$

ii) $\delta_n(x) = \frac{n}{\sqrt{\pi}} e^{-n^2 x^2}$

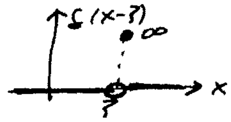


If $\delta(x)$ is considered to be the limit of such a sequence, then the above indeterminate form $0 \cdot \infty$ is clearly equal to 1, though this is an arbitrary (convenient) choice.

Sifting Property: $\boxed{\int_{-\infty}^{\infty} \delta(x - \xi) f(x)\,dx = f(\xi)}$

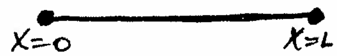
Proof: $\int_{-\infty}^{\infty} \delta(x-\xi) f(x)\,dx = \lim_{\varepsilon \to 0} \int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x-\xi) f(x)\,dx = \lim_{\varepsilon \to 0} f(\xi) \underbrace{\int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x-\xi)\,d\xi}_{=1} = f(\xi)$

$\delta(x - \xi) = \delta$-function centered at ξ



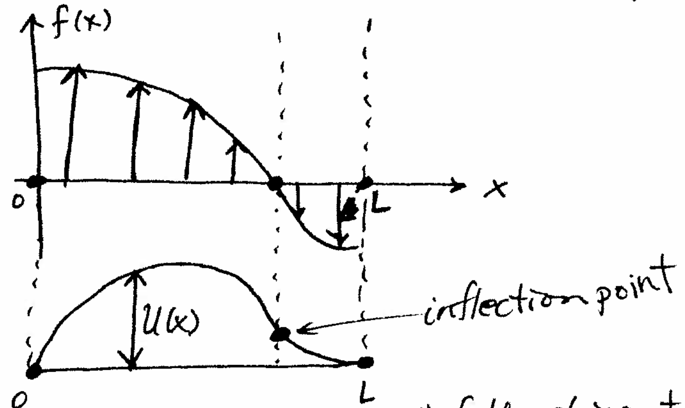
Static Deflection of a Taut String Subject to a Transverse Loading
(Introductory Example for Green's Function)

A string under tension T is stretched along the x-axis between fixed endpoints $x=0$ and $x=L$.



T = tension (constant force).

A transverse loading $f(x)$ is applied, thus deflecting the string.



$u(x)$ = deflection (or displacement) of the string at x.

The ODE describing this scenario is derived by balancing the forces (T and $\int f(x)dx$) exerted on the string. This approach is that same as that used to derive the wave equation, except here the string is stationary ($u_{tt}=0$) and the load $f(x)$ exerted on the string appears as a source term.

$$\Rightarrow \boxed{\begin{array}{c} -Tu_{xx} = f(x) \\ u(0) = u(L) = 0 \end{array}}$$

(Steady-State Wave Equation with a source term)

$f(x) > 0 \Rightarrow u_{xx} < 0$

$f(x) < 0 \Rightarrow u_{xx} > 0$

Dimensions: Notation: $[A]$ = dimensions of the quantity A

~~[T] = tension (force)~~ T = tension (force) $\Rightarrow [T] = N$

x, u = length $\Longrightarrow [x] = [u] = m$

Then, $[f(x)] = [Tu_{xx}] = N \cdot \frac{m}{m^2} = N/m \Rightarrow [f(x)] = \frac{N}{m} = \frac{\text{force}}{\text{length}}$

Therefore, $f(x)$ has the units of pressure, and the force exerted on the string between points x_1 and x_2 is given by $\text{Force} = \int_{x_1}^{x_2} f(x)\,dx$.

- Consider the case in which a unit force (magnitude = 1) is applied at a single point $x=\xi$. In this case, the pressure $f(x)$ is infinite at $x=\xi$ and zero elsewhere.

$$\text{Force} = \lim_{\Delta x \to 0} \int_{\xi-\Delta x}^{\xi+\Delta x} f(x)\,dx = 1 \implies f(\xi) \text{ is infinite}$$

Therefore, $$\boxed{f(x) = \begin{cases} \infty, & x=\xi \\ 0, & x \neq \xi \end{cases}} \quad (1)$$

Furthermore, $$\text{Force applied to the string} = \boxed{\int_0^L f(x)\,dx = 1} \quad (2)$$

$$(1) \text{ and } (2) \implies \boxed{f(x) = \delta(x-\xi)}$$

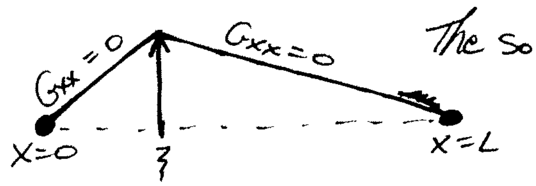
- Let $G(x|\xi)$ denote the deflection of the string subject to a load (with force = 1) exerted at a single point $x=\xi$.

$G(x|\xi)$ is a function of x with parameter ξ.

$$\implies \boxed{\begin{aligned} -TG_{xx} &= \delta(x-\xi) \\ G(0|\xi) &= G(L|\xi) = 0 \end{aligned}}$$

$x \neq \xi \implies G_{xx} = 0 \implies G$ is linear on each side of the applied load.

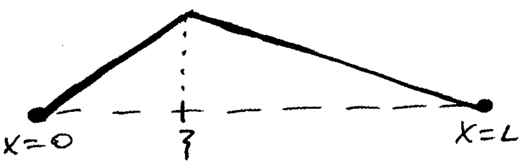
$x = \xi \implies G_{xx} = \infty \implies G_x$ experiences a jump at $x=\xi$
(i.e. There is a "Kink" separating the two linear pieces)



The solution $G(x|\xi)$ is a piecewise linear function.

- Note: $G(x/\xi)$ is continuous at $x=\xi$.
$G_x(x|\xi)$ experiences a jump at $x=\xi$.

Solve $\boxed{\begin{array}{l} -T G_{xx} = \delta(x-\xi) \\ G(0|\xi) = G(L|\xi) = 0 \end{array}}$



$x \neq \xi \Rightarrow G_{xx} = 0 \Rightarrow \boxed{G(x|\xi) = \begin{cases} A_1(\xi)x + B_1(\xi), & x<\xi \\ A_2(\xi)x + B_2(\xi), & x>\xi \end{cases}}$

4 Unknowns (A_1, B_1, A_2, B_2) $\Rightarrow$ Need 4 conditions

Boundary Conditions

$G(0|\xi) = A_1(\xi)\cdot 0 + B_1(\xi) = 0 \Rightarrow \boxed{B_1(\xi) = 0}$ ①

$G(L|\xi) = A_2(\xi)\cdot L + B_2(\xi) = 0 \Rightarrow \boxed{A_2(\xi) = \frac{-B_2(\xi)}{L}}$ ②

Conditions at $x=\xi$

G is continuous $x=\xi$ $\Rightarrow$ $A_1(\xi)\cdot\xi + \cancel{B_1(\xi)} = A_2(\xi)\cdot\xi + B_2(\xi)$

$\boxed{A_1(\xi) - A_2(\xi) = \frac{B_2(\xi)}{\xi}}$ ③

$G_x(x|\xi)$ has a jump at $x=\xi$

$[G_x]_\xi = ?$

Notation

$[h(x)]_\xi = h(\xi^+) - h(\xi^-)$

= jump in $h(x)$ at $x=\xi$

$-T G_{xx} = \delta(x-\xi)$

$-T\int_{\xi^-}^{\xi^+} G_{xx}\,dx = \int_{\xi^-}^{\xi^+} \delta(x-\xi)\,dx$

$-T[G_x]_\xi = 1$

$\boxed{[G_x]_\xi = \frac{-1}{T}}$

$G_x(x|\xi) = \begin{cases} A_1(\xi), & x<\xi \\ A_2(\xi), & x>\xi \end{cases}$

$[G_x]_\xi = A_2(\xi) - A_1(\xi) = \frac{-1}{T}$

$\boxed{A_1(\xi) - A_2(\xi) = \frac{1}{T}}$ ④

- We have
  1. $\boxed{B_1(\xi) = 0}$
  2. $A_2(\xi) = -B_2(\xi)/L$
  3. $A_1(\xi) - A_2(\xi) = B_2(\xi)/\xi$
  4. $A_1(\xi) - A_2(\xi) = 1/T$

---

(3) and (4) $\Rightarrow \dfrac{B_2(\xi)}{\xi} = \dfrac{1}{T} \Rightarrow \boxed{B_2(\xi) = \dfrac{\xi}{T}}$

(2) $\Rightarrow A_2(\xi) = -\dfrac{B_2(\xi)}{L} = \dfrac{-\xi}{LT} \Rightarrow \boxed{A_2(\xi) = \dfrac{-\xi}{LT}}$

(4) $\Rightarrow A_1(\xi) = A_2(\xi) + \dfrac{1}{T} = \dfrac{-\xi}{LT} + \dfrac{1}{T} = \dfrac{L-\xi}{LT} \Rightarrow \boxed{A_1(\xi) = \dfrac{L-\xi}{LT}}$

Then, $\underline{x < \xi} \Rightarrow G(x|\xi) = \dfrac{L-\xi}{LT} x + 0 = \dfrac{L-\xi}{LT} x$

$\underline{x > \xi} \Rightarrow G(x|\xi) = \dfrac{-\xi}{LT} x + \dfrac{\xi}{T} = \dfrac{L-x}{LT} \xi$

$$\boxed{G(x|\xi) = \begin{cases} \dfrac{L-\xi}{LT} x, & x < \xi \\ \dfrac{L-x}{LT} \xi, & x > \xi \end{cases}}$$

Green's Function: Describes the response of the ODE to a load applied to the string at a single point $x = \xi$.

$$G(\xi|x) = \begin{cases} \dfrac{L-x}{LT} \xi, & \xi < x \\ \dfrac{L-\xi}{LT} x, & \xi > x \end{cases} = \begin{cases} \dfrac{L-\xi}{LT} x, & x < \xi \\ \dfrac{L-x}{LT} \xi, & x > \xi \end{cases} = G(x|\xi)$$

$$\boxed{G(\xi|x) = G(x|\xi)}$$

This symmetry property of $G(x|\xi)$ holds for the Green's Function of a wide range of problems.

Arbitrary load ($f(x)$): $\boxed{\begin{array}{l} -Tu_{xx} = f(x) \\ u(0) = u(L) = 0 \end{array}}$ ①

Green's Function problem: $\boxed{\begin{array}{l} -TG_{xx} = \delta(x-\xi) \\ G(0|\xi) = G(L|\xi) = 0 \end{array}}$ ②

$$(u \times ②) - (G \times ①) \Rightarrow \quad -TG_{xx}u = \delta(x-\xi)u$$
$$-(-Tu_{xx}G = f(x)G)$$
$$T(u_{xx}G - G_{xx}u) = \delta(x-\xi)u - f(x)G$$

Integrate: $T\int_0^L (u_{xx}G - G_{xx}u)\,dx = \int_0^L (\delta(x-\xi)u - f(x)G)\,dx$

$u = G \quad v = u_x$
$du = G_x dx \quad dv = u_{xx}dx$

$u = u \quad v = G_x$
$du = u_x dx \quad dv = G_{xx}dx$

$$T\left[G(x|\xi)u_x(x)\Big|_0^L - \cancel{\int_0^L u_xG_x\,dx} - u(x)G_x(x|\xi)\Big|_0^L + \cancel{\int_0^L u_xG_x\,dx}\right] = \int_0^L \delta(x-\xi)u(x)\,dx - \int_0^L f(x)G(x|\xi)\,dx$$

$$T\left[\left(\underbrace{\cancel{G(L|\xi)}}_{=0}u_x(L) - \underbrace{\cancel{G(0|\xi)}}_{=0}u_x(0)\right) - \left(\underbrace{\cancel{u(L)}}_{=0}G_x(L|\xi) - \underbrace{\cancel{u(0)}}_{=0}G_x(0|\xi)\right)\right] = u(\xi) - \int_0^L f(x)G(x|\xi)\,dx$$

$$u(\xi) = \int_0^L f(x)G(x|\xi)\,dx$$

$$G(x|\xi) = G(\xi|x) \Rightarrow u(\xi) = \int_0^L f(x)G(\xi|x)\,dx$$

Interchange the roles of x and ξ $\Rightarrow$ $\boxed{u(x) = \int_0^L f(\xi)G(x|\xi)\,d\xi}$

The integral represents the sum of the contributions of the load at each point.

We have $G(x|\xi) = \begin{cases} \frac{L-\xi}{LT} x, & 0<x<\xi<L \\ \frac{L-x}{LT}\xi, & 0<\xi<x<L \end{cases}$

$$u(x) = \int_0^L f(\xi)\, G(x|\xi)\, d\xi = \int_0^x f(\xi) \frac{L-x}{LT} \cdot \xi \, d\xi + \int_x^L f(\xi) \frac{L-\xi}{LT} \cdot x \, d\xi$$

$$\boxed{u(x) = \frac{1}{LT}\left[(L-x)\int_0^x f(\xi)\cdot \xi \, d\xi + x\int_x^L f(\xi)\cdot(L-\xi)\, d\xi\right]}$$

Solution for an arbitrary load $f(x)$

For example, suppose $\underline{f(x)=1}$ for $0<x<L$.

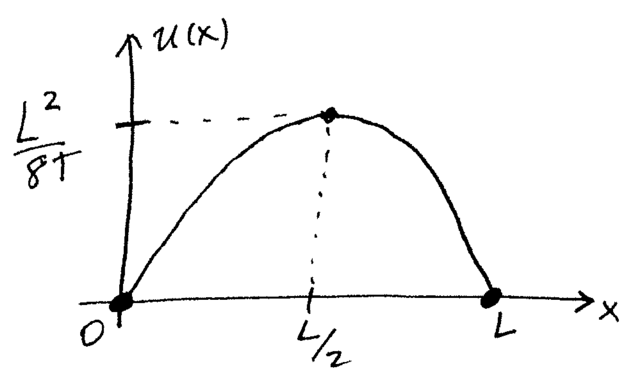
$$u(x) = \frac{1}{LT}\left[(L-x)\int_0^x \xi \, d\xi + x\int_x^L (L-\xi)\, d\xi\right]$$

$$= \frac{1}{LT}\left[(L-x)\cdot \frac{\xi^2}{2}\Big|_0^x - x \, \frac{(L-\xi)^2}{2}\Big|_x^L\right]$$

$$= \frac{1}{2LT}\left[(L-x)(x^2-0) - x\left(0-(L-x)^2\right)\right]$$

$$= \frac{x(L-x)}{2\cancel{L}T}\left[\cancel{x}+(\cancel{L}-\cancel{x})\right] = \frac{x(L-x)}{2T}$$

$$\boxed{u(x) = \frac{x(L-x)}{2T}}$$ Solution for $f(x)=1$.



Initial Value Problem

Consider $\begin{aligned} &u'' + p(t)u' + q(t)u = g(t), \quad t \ge 0 \\ &u(0) = u'(0) = 0, \end{aligned}$

- 2nd order
- linear
- nonhomogeneous
- variable coefficients

$\Rightarrow$ Variation of Parameters

where p, q, and g are sufficiently differentiable.

Let $u_1(t)$ and $u_2(t)$ be two linearly independent solutions of the associated homogeneous equation.

$\Rightarrow$ Wronskian $= W(u_1, u_2)(t) \neq 0$

$\Rightarrow u_{homog.} = C_1 u_1(t) + C_2 u_2(t)$ general solution of the associated homogeneous equation

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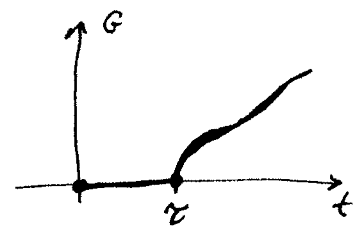
Green's Function Problem:

$G = G(t/\tau)$

$$\begin{aligned} &G'' + p(t)G' + q(t)G = \delta(t-\tau) \\ &G(0/\tau) = G'(0/\tau) = 0 \end{aligned}$$



$t \neq \tau$: $G'' + p(t)G' + q(t)G = 0$

$$\Rightarrow G(t/\tau) = \begin{cases} A_1(\tau)u_1(t) + A_2(\tau)u_2(t), & 0 \le t < \tau \\ B_1(\tau)u_1(t) + B_2(\tau)u_2(t) & \tau < t < \infty \end{cases}$$

4 Unknowns (A_1, A_2, B_1, B_2) $\Rightarrow$ Need 4 conditions

① $G(0/\tau) = A_1(\tau)u_1(0) + A_2(\tau)u_2(0) = 0$

② $G'(0/\tau) = A_1(\tau)u_1'(0) + A_2(\tau)u_2'(0) = 0$

$$\begin{pmatrix} u_1(0) & u_2(0) \\ u_1'(0) & u_2'(0) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} u_1(0) & u_2(0) \\ u_1'(0) & u_2'(0) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\det = W(u_1, u_2)(0) \neq 0$

$$\boxed{A_1 = A_2 = 0}$$

③ G is continuous at $t = \tau$: $[G]_\tau = 0$

$$\boxed{B_1(\tau)u_1(\tau) + B_2(\tau)u_2(\tau) = 0}$$

④ $[G']_\tau = ?$

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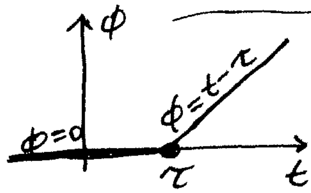
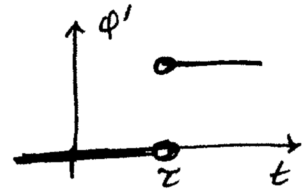
We have $G'' + p(t)G' + q(t)G = \delta(t-\tau)$

The left-hand side of the equation must behave like a δ-function near $t=\tau$. Since differentiation increases the severity of singularities, G'' is "more singular" than both G and G', and therefore, it must be G'' that behaves like a δ-function near $t=\tau$.

e.g. Suppose $\phi(t) = \begin{cases} 0, & t<\tau \\ t-\tau, & t>\tau \end{cases} \Rightarrow \phi'(t) = H(t-\tau) \Rightarrow \phi''(t) = \begin{cases} \infty, & t=\tau \\ 0, & t\neq\tau \end{cases}$





$$\int_{-\infty}^{\infty} \phi''(t)\,dt = \phi'(t)\Big|_{-\infty}^{\infty} = H(t-\tau)\Big|_{-\infty}^{\infty} = 1-0 = 1$$

$$\Rightarrow \boxed{\phi''(t) = \delta(t-\tau)}$$

Integrate from $t=\tau-\varepsilon$ to $t=\tau+\varepsilon$:

$$\int_{\tau-\varepsilon}^{\tau+\varepsilon} \Big(\underset{\approx\,\delta(t-\tau)}{G''} + \cancel{p(t)\underset{\approx\,H(t-\tau)}{G'} + q(t)\underset{\text{cont.}}{G}}\Big)\,dt = \int_{\tau-\varepsilon}^{\tau+\varepsilon} \delta(t-\tau)\,d\tau$$

Let $\varepsilon\to 0$: Since G is continuous and G' has only a jump discontinuity at $t=\tau$,

$$\int_{\tau-\varepsilon}^{\tau+\varepsilon} \big(p(t)G' + q(t)G\big)\,dt \to 0 \quad \text{as } \varepsilon\to 0.$$

Then,

$$\int_{\tau-\varepsilon}^{\tau+\varepsilon} G''\,dt = \int_{\tau-\varepsilon}^{\tau+\varepsilon} \delta(t-\tau)\,d\tau$$

$$G'\Big|_{\tau-\varepsilon}^{\tau+\varepsilon} = 1$$

$$\varepsilon\to 0 \Rightarrow \boxed{[G']_\tau = 1}$$

When integrating across $t=\tau$, it is always the term with the highest-ordered derivative that yields the only nonzero contribution to the left-hand side of the equation. All other terms of the left-hand side approach 0 as $\varepsilon\to 0$. It is the highest-ordered derivative term that behaves like a δ-function near $t=\tau$, whereas the singularities of the lower-ordered derivative terms are less severe.

$$G'(t|\tau) = \begin{cases} 0 & , \ 0 < t < \tau \\ B_1(\tau)u_1'(t) + B_2(\tau)u_2'(t), & \tau < t < \infty \end{cases}$$

④ $[G']_\tau = B_1(\tau)u_1'(\tau) + B_2(\tau)u_2'(\tau) = 1$

③ and ④ $\Rightarrow$ $\underbrace{\begin{pmatrix} u_1(\tau) & u_2(\tau) \\ u_1'(\tau) & u_2'(\tau) \end{pmatrix}}_{U} \begin{pmatrix} B_1(\tau) \\ B_2(\tau) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\det(U) = W(u_1,u_2)(\tau) \neq 0 \Rightarrow U^{-1}$ exists and $\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = U^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\Rightarrow \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} u_1(\tau) & u_2(\tau) \\ u_1'(\tau) & u_2'(\tau) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{W(u_1,u_2)(\tau)} \begin{pmatrix} u_2'(\tau) & -u_2(\tau) \\ -u_1'(\tau) & u_1(\tau) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow B_1 = \frac{-u_2(\tau)}{W(u_1,u_2)(\tau)} \qquad B_2 = \frac{u_1(\tau)}{W(u_1,u_2)(\tau)}$$

$$\Rightarrow \boxed{G(t|\tau) = \begin{cases} 0 & , \ 0 < t < \tau \\ \dfrac{-u_2(\tau)u_1(t) + u_1(\tau)u_2(t)}{W(u_1,u_2)(\tau)}, & \tau < t < \infty \end{cases}}$$

<u>Theorem:</u> Let L be the 2nd order linear differential operator defined by

$$Lu = a_2(x)u'' + a_1(x)u' + a_0(x)u,$$

where a_0, a_1, and a_2 are continuous on an interval $[a,b]$.

(a and/or b may be infinite, e.g. $[0,\infty)$)

Consider the ODE $Lu = f(x)$, where $f(x)$ is piecewise continuous on $[a,b]$, subject to the conditions

$$A(u) = \alpha_{11}u(a) + \alpha_{12}u'(a) + \beta_{11}u(b) + \beta_{12}u'(b) = 0$$
$$B(u) = \alpha_{21}u(a) + \alpha_{22}u'(a) + \beta_{21}u(b) + \beta_{22}u'(b) = 0.$$

(⊛)

If $Lu = 0$, $A(u) = B(u) = 0$ has only the trivial solution ($u = 0$), then ⊛ has a unique solution given by $u(x) = \int_a^b f(\xi)\, G(x|\xi)\, d\xi$, where $G(x|\xi)$ is the solution of the associated Green's Function problem: $LG = \delta(x-\xi)$, $A(G) = B(G) = 0$.

Since our problem is an ordinary linear initial value problem, the associated homogeneous problem is guaranteed to have a unique solution, which, because of the homogeneous initial conditions, is clearly the trivial solution.

Thus, the hypotheses of the theorem are satisfied.

For our problem, $a=0$ and $b=\infty$.

$$\Rightarrow u(t) = \int_0^\infty f(\tau)G(t/\tau)\,d\tau = \underbrace{\int_0^t f(\tau)\cdot \frac{-u_2(\tau)u_1(t)+u_1(\tau)u_2(t)}{W(u_1,u_2)(\tau)}\,d\tau}_{0<\tau<t} + \underbrace{\cancel{\int_t^\infty f(\tau)\cdot 0\,d\tau}}_{t<\tau<\infty}$$

$$\boxed{u(t) = -u_1(t)\int_0^t \frac{u_2(\tau)f(\tau)}{W(u_1,u_2)(\tau)}\,d\tau + u_2(t)\int_0^t \frac{u_1(\tau)f(\tau)}{W(u_1,u_2)(\tau)}\,d\tau}$$

(Variation of Parameters Formula)

Nonhomogeneous Initial Conditions

$$u'' + p(t)u' + q(t)u = f(t)$$
$$u(0)=\alpha\,,\quad u'(0)=\beta$$

$$\boxed{u(t) = \alpha u_1(t) + \beta u_2(t) - u_1(t)\int_0^t \frac{u_2(\tau)f(\tau)}{W(u_1,u_2)(\tau)}\,d\tau + u_2(t)\int_0^t \frac{u_1(\tau)f(\tau)}{W(u_1,u_2)(\tau)}\,d\tau}$$

where $u_1(t)$ and $u_2(t)$ are the two homogeneous solutions that satisfy the initial conditions $u_1(0)=1$, $u_1'(0)=0$ and $u_2(0)=0$, $u_2'(0)=1$.