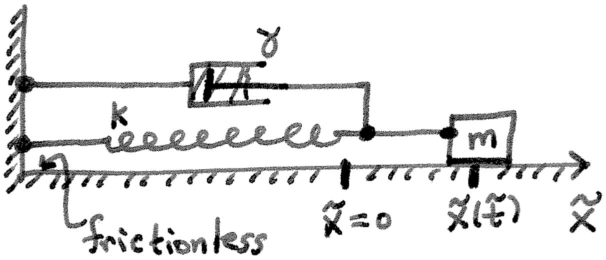
Higher Order Matched Asymptotic Expansions

Example: Mass-Spring-Damper System



$\tilde{t}$ = time (s)
$\tilde{x}$ = displacement (m)
m = mass (kg)
γ = damper constant (kg/s)
K = spring constant (kg/s²)

$\tilde{x}=0$ corresponds to the equilibrium position, where the spring is neither stretched nor compressed.

Spring Force: Hooke's Law $F_s = -K\tilde{x}$

Damper Force: $F_D = -\gamma \dfrac{d\tilde{x}}{d\tilde{t}}$

$\Sigma F = ma$

$F_s + F_D = m\dfrac{d^2\tilde{x}}{d\tilde{t}^2}$

$$\Rightarrow \boxed{m\frac{d^2\tilde{x}}{d\tilde{t}^2} + \gamma\frac{d\tilde{x}}{d\tilde{t}} + K\tilde{x} = 0}$$

Suppose that the mass is initially at the equilibrium position ($\tilde{x}=0$). At $\tilde{t}=0$, the mass is struck with an impulse which puts it in motion with an initial velocity V_0.

$$\Longrightarrow \boxed{\begin{array}{l}\tilde{x}(0) = 0\\ \dfrac{d\tilde{x}}{d\tilde{t}}(0) = V_0\end{array}}$$

Perturbation methods can be used to consider the limits of a relatively small (or large) m, γ, K, or V_0. Here, we will consider a relatively small **mass** m.

Nondimensionalization: To consider a relatively small mass, the appropriate scalings are $x = \dfrac{\tilde{x}}{x_*}$ where $x_* = \dfrac{mV_0}{\gamma}$.

$t = \dfrac{\tilde{t}}{t_*}$ where $t_* = \dfrac{\gamma}{K}$

$$\Longrightarrow \boxed{\varepsilon\ddot{x} + \dot{x} + x = 0} \text{ where } \varepsilon = \frac{mK}{\gamma^2} \text{ and } \dot{x} = \frac{dx}{dt}.$$

$$\boxed{\begin{array}{l}x(0) = 0\\ \dot{x}(0) = 1/\varepsilon\end{array}}$$

To consider a relatively small mass, assume $0 < \varepsilon \ll 1$.

$$\varepsilon \ddot{x} + \dot{x} + x = 0 \; ; \; \begin{array}{l} x(0) = 0 \\ \dot{x}(0) = 1/\varepsilon \end{array}$$

Outer Solution: $X_{OUT} = X_0(t) + \varepsilon X_1(t) + \cdots$

$$\varepsilon(\ddot{X}_0 + \cdots) + (\dot{X}_0 + \varepsilon \dot{X}_1 + \cdots) + (X_0 + \varepsilon X_1 + \cdots) = 0$$

$O(1)$: $\dot{X}_0 + X_0 = 0 \implies X_0(t) = C_0 e^{-t}$

$O(\varepsilon)$: $\ddot{X}_0 + \dot{X}_1 + X_1 = 0$

$$\dot{X}_1 + X_1 = -\ddot{X}_0 = -C_0 e^{-t}$$

$$\implies X_1(t) = (C_1 - C_0 t)e^{-t}$$

$$\implies \boxed{X_{OUT} = C_0 e^{-t} + \varepsilon(C_1 - C_0 t)e^{-t} + \cdots}$$

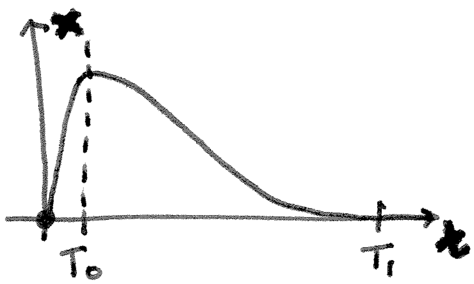
Notes: 1. X_{OUT} cannot satisfy the initial conditions $\implies$ Need an Initial Layer.

2. X_{OUT} is not valid for large t.

X_{OUT} is valid when $\frac{\varepsilon X_1}{X_0} = \varepsilon\left(\frac{C_1}{C_0} - t\right) << 1$.

$\implies X_{OUT}$ is valid when $t << \frac{1}{\varepsilon}$.

Therefore, X_{OUT} is valid for $0 < T_0 < t < T_1 < \infty$ (i.e. for t bounded away from 0 and ∞.)



At $t = \frac{1}{\varepsilon}$, $X_{OUT} = \underbrace{C_0 e^{-1/\varepsilon}}_{TST} + \underbrace{\varepsilon\left(C_1 - \frac{C_0}{\varepsilon}\right)e^{-\frac{1}{\varepsilon}}}_{TST} + \cdots \sim TST$

$\implies X_{OUT} \sim TST$ for large t

Though X_{OUT} is not well-ordered for large t, it is TST here, as is the exact solution, so X_{OUT} is sufficient for all $t > T_0$.

Initial Layer at $t=0$

$$\boxed{\tau = t/\varepsilon} \Longrightarrow \boxed{Y_{\tau\tau} + Y_\tau + \varepsilon Y = 0; \quad \begin{matrix} Y(0)=0 \\ Y_\tau(0)=1 \end{matrix}}$$

$Y'(0) = \frac{1}{\varepsilon} Y_\tau(0) = \frac{1}{\varepsilon}$

$Y_\tau(0) = 1$

$$Y_{IN} = y_0(\tau) + \varepsilon y_1(\tau) + \cdots$$

$$\Rightarrow \underline{(y_{0\tau\tau} + \varepsilon y_{1\tau\tau} + \cdots) + (y_{0\tau} + \varepsilon y_{1\tau} + \cdots) + \varepsilon (y_0 + \cdots) = 0}$$

$\underline{O(1):}$ $\boxed{y_{0\tau\tau} + y_{0\tau} = 0; \quad \begin{matrix} y_0(0)=0 \\ y_{0\tau}(0)=1 \end{matrix}}$ $\Rightarrow y_0 = D_0 + D_1 e^{-\tau}$

$y_0(0) = D_0 + D_1 = 0 \quad \underline{D_0 = 1}$

$y_{0\tau}(0) = -D_1 = 1 \quad \underline{D_1 = -1}$

$$\boxed{y_0(\tau) = 1 - e^{-\tau}}$$

$\underline{O(\varepsilon):}$ $\boxed{y_{1\tau\tau} + y_{1\tau} = -y_0 = -(1 - e^{-\tau}); \quad \begin{matrix} y_1(0) = 0 \\ y_{1\tau}(0) = 0 \end{matrix}}$

Integrate $\Rightarrow y_{1\tau} = (\tau + D_2) e^{-\tau} - 1 \qquad y_{1\tau}(0) = (0 + D_2) - 1 = 0$

$\underline{D_2 = 1}$

$$\underline{y_{1\tau} = (\tau + 1) e^{-\tau} - 1}$$

Integrate $\Rightarrow y_1 = (\tau+1)(-e^{-\tau}) - \int (-e^{-\tau})\, d\tau - \tau + D_3$

$= -(\tau+1)e^{-\tau} - e^{-\tau} - \tau + D_3 = (D_3 - 2e^{-\tau}) - \tau(1 + e^{-\tau})$

$y_1(0) = D_3 - 2 = 0$

$\underline{D_3 = 2}$ $\Longrightarrow \boxed{y_1(\tau) = 2(1 - e^{-\tau}) - \tau(1 + e^{-\tau})}$

Then, $\boxed{Y_{IN}(\tau) = (1 - e^{-\tau}) + \varepsilon\left[2(1 - e^{-\tau}) - \tau(1 + e^{-\tau})\right] + \cdots}$

The expansion is not valid when τ is large.

$1 \sim \varepsilon\tau$

$\underline{\tau \sim \frac{1}{\varepsilon}}$

Not valid when $\tau = O(\frac{1}{\varepsilon})$ or larger, or equivalently, when $t = O(1)$ or larger.

Not valid for $t = O(1) \quad (\tau = O(\frac{1}{\varepsilon}))$ or larger

Note that C_0 may be determined by primitive matching, whereas a more advanced matching technique is needed to determine the constants which are involved in the higher order terms (e.g. C_1).

Exact Solution:

$\varepsilon y'' + y' + y = 0$; $y(0) = 0$, $y'(0) = 1/\varepsilon$

$$\Rightarrow \boxed{Y_{ex}(t) = \frac{1}{\sqrt{1-4\varepsilon}}\left[e^{\frac{-1+\sqrt{1-4\varepsilon}}{2\varepsilon}t} - e^{\frac{-1-\sqrt{1-4\varepsilon}}{2\varepsilon}t}\right]} \quad (1)$$

OR in terms of $\tau = t/\varepsilon$,

$$\boxed{Y_{ex}(\tau) = \frac{1}{\sqrt{1-4\varepsilon}}\left[e^{\frac{-1+\sqrt{1-4\varepsilon}}{2}\tau} - e^{\frac{-1-\sqrt{1-4\varepsilon}}{2}\tau}\right]}$$

Compare the expansions of (1) and (2) to the Outer and Inner expansions

Recall: $\sqrt{1-x} \sim 1 - \frac{x}{2} - \frac{x^2}{8} + \cdots$ as $x \to 0$

$$\Rightarrow \quad \sqrt{1-4\varepsilon} \sim 1 - 2\varepsilon - 2\varepsilon^2 + \cdots$$

$$\frac{1}{\sqrt{1-4\varepsilon}} \sim 1 + 2\varepsilon + \cdots$$

$$\frac{-1 \pm \sqrt{1-4\varepsilon}}{2} \sim \frac{-1 \pm (1-2\varepsilon-\varepsilon^2+\cdots)}{2} \qquad + \Rightarrow \frac{-1+1-2\varepsilon-2\varepsilon^2+\cdots}{2} \sim -\varepsilon-\varepsilon^2$$

$$- \Rightarrow \frac{-1-1+2\varepsilon+\cdots}{2} \sim -1+\varepsilon$$

Outer expansion of $Y_{ex}(t)$

$$(1) \Rightarrow Y_{ex}(t) \sim (1+2\varepsilon)\left[e^{(-\varepsilon-\varepsilon^2)\frac{t}{\varepsilon}} - e^{(-1+\varepsilon)t/\varepsilon}\right] + \cdots$$

$$\sim (1+2\varepsilon)\left[e^{-t}e^{-\varepsilon t} - \cancel{e^{-t/\varepsilon}e^{-t}}\right] + \cdots \quad \text{TST}$$

$$\sim (1+2\varepsilon)e^{-t}(1-\varepsilon t + \cdots) \sim e^{-t}(1-\varepsilon t + 2\varepsilon)$$

$$\Rightarrow \boxed{Y_{ex}(t) \sim e^{-t} + \varepsilon e^{-t}(2-t) + \cdots}$$

Compare to Y_{OUT}

$$Y_{OUT}(t) \sim C_0 e^{-t} + \varepsilon e^{-t}(C_1 - C_0 t) + \cdots$$

The expansion agree when

$$\boxed{\begin{aligned} C_0 &= 1 \\ C_1 &= 2 \end{aligned}}$$

Inner expansion of $Y_{ex}(\tau)$.

$$② \Rightarrow Y_{ex}(\tau) \sim (1+2\varepsilon)\left[e^{(-\varepsilon-\varepsilon^2)\tau} - e^{(-1+\varepsilon)\tau}\right] + \cdots$$
$$\sim (1+2\varepsilon)\left[e^{-\varepsilon\tau}e^{-\varepsilon^2\tau} - e^{-\tau}e^{\varepsilon\tau}\right] + \cdots$$
$$\sim (1+2\varepsilon)\left[(1-\varepsilon\tau+\cdots)(1+\cdots) - e^{-\tau}(1+\varepsilon\tau+\cdots)\right] + \cdots$$
$$\sim (1-\varepsilon\tau) - e^{-\tau}(1+\varepsilon\tau) + 2\varepsilon(1-e^{-\tau}) + \cdots$$
$$\Rightarrow \boxed{Y_{ex}(\tau) \sim (1-e^{-\tau}) + \varepsilon\left[2(1-e^{-\tau}) - \tau(1+e^{-\tau})\right] + \cdots}$$

Compare to $Y_{IN}(\tau)$

$$Y_{IN}(\tau) \sim (1-e^{-\tau}) + \varepsilon\left[2(1-e^{-\tau}) - \tau(1+e^{-\tau})\right] + \cdots$$

The expansions agree.

We need to determine C_0 and C_1 without knowing the exact solution. Since C_0 is part of the leading order term, it can be determined by primitive matching. $\lim_{t \to 0^+} Y_0(t) = \lim_{\tau \to +\infty} y_0(\tau)$

$C_0 = 1$

To determine the constants which are involved in the higher order terms (e.g. C_1), we need to consider methods for

Higher Order Asymptotic Matching.

We'll consider the two most common methods.

1. Intermediate Variable Matching ($\tau \ll s \ll t$)
2. Van Dyke Matching Principle.

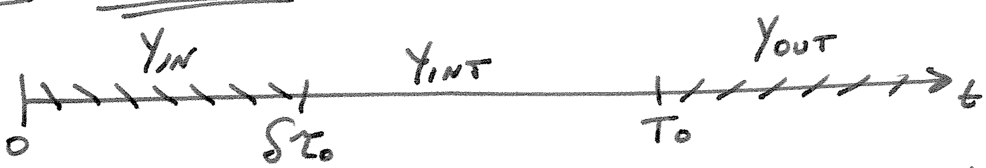
Method 1. always works whenever matching is possible, but method 2. is usually more convenient to use and it works most of the time.

Asymptotic Matching

Consider $y_{OUT}(t)$ and $y_{IN}(\tau)$, where $\tau = t/\delta(\varepsilon)$.

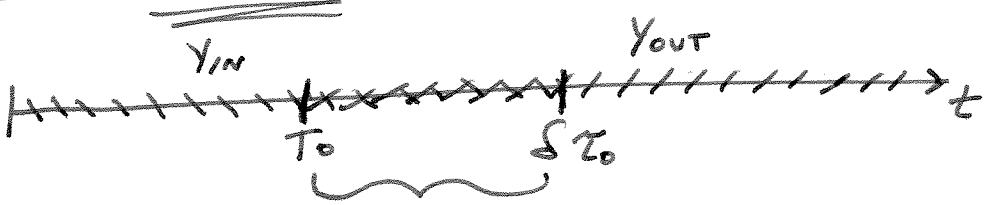
Suppose that $y_{OUT}(t)$ is valid for $t > T_0$ and $y_{IN}(\tau)$ is valid for $\tau < \tau_0$, or equivalently, for $t < \delta\tau_0$. The domains of validity of y_{OUT} and y_{IN} must overlap if matching is to be successful.

Case 1: $\delta\tau_0 < T_0$



There is no overlap region where y_{OUT} and y_{IN} are both valid, and therefore, they cannot be matched. An intermediate layer solution y_{INT} must be found, as in the two-ply layer example. The three expansions must then be matched as discussed below.

Case 2: $\delta\tau_0 > T_0$



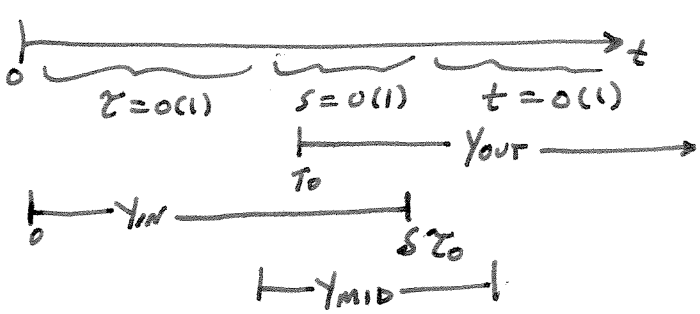
There is an overlap region, so matching may be possible.

y_{OUT} and y_{IN} can be matched if they agree in the overlap region. That is, they must agree when expressed in terms of an intermediate variable s, where $t \ll s \ll \tau$.

y_{OUT} = expansion of $y_{ex}(t)$

y_{IN} = expansion of $y_{ex}(\tau)$

y_{MID} = expansion of $y_{ex}(s)$



Intermediate Variable Matching Method

Suppose we have $y_{OUT}(t)$ and $y_{IN}(\tau)$, where $\tau = \frac{t}{\delta(\varepsilon)}$, $\delta(\varepsilon) \ll 1$.

Introduce an arbitrary intermediate variable s.

$$\boxed{s = \frac{t}{\eta(\varepsilon)} = \frac{\delta(\varepsilon)\tau}{\eta(\varepsilon)}, \text{ where } \delta(\varepsilon) \ll \eta(\varepsilon) \ll 1}$$

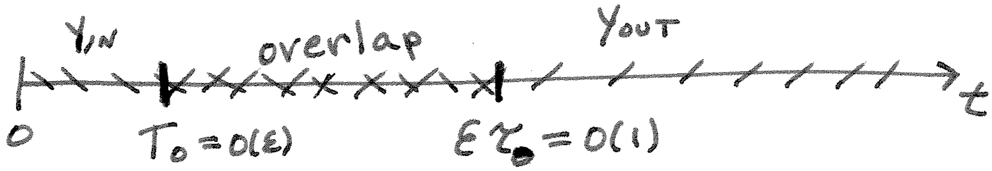
To match y_{OUT} and y_{IN}, expand each in terms of the intermediate variable s. Then equate the two expansions to determine the unknown constants.

Recall the Mass-Spring-Damper example.

$$y_{OUT}(t) \sim C_0 e^{-t} + \varepsilon e^{-t}(C_1 - C_0 t) + \cdots, \quad t > T_0 = O(\varepsilon)$$

$$y_{IN}(\tau) \sim (1 - e^{-\tau}) + \varepsilon\left[2(1-e^{-\tau}) - \tau(1+e^{-\tau})\right] + \cdots, \quad \tau < \tau_0 = O\left(\tfrac{1}{\varepsilon}\right).$$

$\tau = \frac{t}{\varepsilon}$, $\delta(\varepsilon) = \varepsilon$



Introduce the intermediate variable s.

$$s = \frac{t}{\eta(\varepsilon)} = \frac{\varepsilon\tau}{\eta(\varepsilon)}, \text{ where } \varepsilon \ll \eta(\varepsilon) \ll 1$$

Expand $y_{OUT}(s)$ and $y_{IN}(s)$ and keep enough terms so that all unknown constants appear in the expansions.

Ordering of the Gauge Functions

The gauge functions of the expansions of $y_{OUT}(s)$ and $y_{IN}(s)$ will be

$$1, \eta, \varepsilon, \eta^2, \varepsilon\eta, \varepsilon^2, \eta^3, \varepsilon\eta^2, \varepsilon^2\eta, \varepsilon^3, \eta^4, \varepsilon\eta^3, \ldots$$

$\varepsilon \ll \eta \ll 1$

$$\Rightarrow \underbrace{1 \gg \eta \gg \varepsilon} \quad \underbrace{\eta^2 \gg \varepsilon\eta \gg \varepsilon^2} \quad \underbrace{\eta^3 \gg \varepsilon\eta^2 \gg \varepsilon^2\eta \gg \varepsilon^3} \quad \underbrace{\eta^4 \gg \varepsilon\eta^3 \gg \cdots}$$

These orderings hold for all choices of η.

Question: How do ε^k and η^{k+1} compare?

For example, consider ε and η^2.

If $\eta = \varepsilon^{1/4}$, then $\varepsilon \ll \eta^2$.

If $\eta = \varepsilon^{3/4}$, then $\varepsilon \gg \eta^2$.

$\Rightarrow$ The comparison depends on the choice of η.

Since $y_{OUT}(s)$ and $y_{IN}(s)$ must agree for all intermediate variables s, η can be chosen so that $\varepsilon^k \gg \eta^{k+1}$ for all k, in which case the ordering is as follows.

$$\boxed{1 \gg \eta \gg \varepsilon \gg \eta^2 \gg \varepsilon\eta \gg \varepsilon^2 \gg \eta^3 \gg \varepsilon\eta^2 \gg \varepsilon^2\eta \gg \varepsilon^3 \gg \eta^4 \gg \varepsilon\eta^3 \gg \cdots}$$

e.g. Pick $\eta = \varepsilon \ln(\frac{1}{\varepsilon})$. Then, $\varepsilon \ll \eta \ll 1$ and $\varepsilon^k \gg \eta^{k+1}$. [redacted]

Note: We do not need to actually specify η. It suffices to know that η can be chosen so that the above ordering holds.

Outer Expansion: $\boxed{t = \eta s}$ $\quad \varepsilon \ll \eta \ll 1$

$$Y_{OUT} \sim C_0 e^{-t} + \varepsilon e^{-t}(C_1 - C_0 t) + \cdots$$

$$\sim C_0 e^{-\eta s} + \varepsilon e^{-\eta s}(C_1 - C_0 \eta s) + \cdots$$

$$\sim C_0\left(1 - \eta s + \frac{\eta^2 s^2}{2} + \cdots\right) + \varepsilon\left(1 - \eta s + \frac{\eta^2 s^2}{2} + \cdots\right)(C_1 - C_0 \eta s) + \cdots$$

Note: C_1 comes in at $O(\varepsilon)$ $\Rightarrow$ neglect higher order terms ($O(\eta^2)$ and smaller)

$$\sim C_0(1 - \eta s + \cdots) + \varepsilon(1 + \cdots)(C_1 + \cdots) + \cdots$$

$$\sim C_0(1 - \eta s) + \varepsilon C_1 + \cdots$$

$$\boxed{Y_{OUT}(s) \sim C_0 - C_0 \eta s + \varepsilon C_1 + O(\eta^2)}$$

Inner Expansion: $\boxed{\tau = \frac{\eta s}{\varepsilon}}$ $\quad \frac{\eta}{\varepsilon} \gg 1$

$$Y_{IN} \sim (1 - e^{-\tau}) + \varepsilon\left[2(1 - e^{-\tau}) - \tau(1 + e^{-\tau})\right] + \cdots$$

$$\sim (1 - \underbrace{e^{-\frac{\eta s}{\varepsilon}}}_{TST}) + \varepsilon\left[2(1 - \underbrace{e^{-\frac{\eta s}{\varepsilon}}}_{TST}) - \frac{\eta s}{\varepsilon}(1 + \underbrace{e^{-\eta s/\varepsilon}}_{TST})\right] + \cdots$$

$$\sim 1 + \varepsilon\left(2 - \frac{\eta s}{\varepsilon}\right) + \cdots$$

$$\boxed{Y_{IN}(s) \sim 1 - \eta s + 2\varepsilon + O(\eta^2)}$$

We have

$$Y_{OUT}(s) \sim C_0 - C_0\eta s + \varepsilon C_1 + O(\eta^2)$$
$$Y_{IN}(s) \sim 1 - \eta s + 2\varepsilon + O(\eta^2)$$

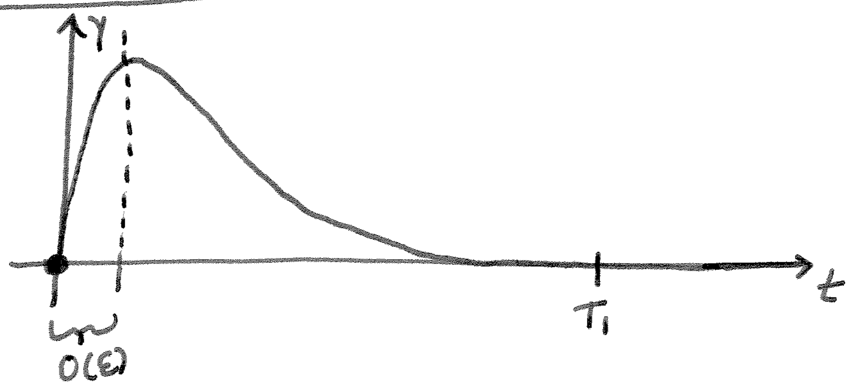
Equate the two expansions:

$$Y_{OUT}(s) = Y_{IN}(s)$$
$$C_0 - C_0\eta s + \varepsilon C_1 = 1 - \eta s + 2\varepsilon = \begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}$$
$$\Rightarrow \boxed{\begin{matrix} C_0 = 1 \\ C_1 = 2 \end{matrix}} \qquad \boxed{\begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix} = 1 - \eta s + 2\varepsilon}$$

Composite Expansion

$$Y_{comp} = Y_{OUT} + Y_{IN} - \begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}$$
$$\sim \left[e^{-t} + \varepsilon e^{-t}(2-t)\right] + \left[(1-e^{-\tau}) + \varepsilon\left(2(1-e^{-\tau}) - \tau(1+e^{-\tau})\right)\right] - (1 - \eta s + 2\varepsilon)$$
$$\sim e^{-t} + \varepsilon e^{-t}(2-t) + \cancel{1} - e^{-t/\varepsilon} + \varepsilon\left(2(1-e^{-t/\varepsilon}) - \tfrac{t}{\varepsilon}(1+e^{-t/\varepsilon})\right) - (\cancel{1} - t + 2\varepsilon)$$
$$\sim e^{-t} + \varepsilon e^{-t}(2-t) - e^{-t/\varepsilon} + \cancel{2\varepsilon} - 2\varepsilon e^{-t/\varepsilon} - t(\cancel{1} + e^{-t/\varepsilon}) + \cancel{t} - \cancel{2\varepsilon}$$
$$\sim e^{-t} - e^{-t/\varepsilon} - te^{-t/\varepsilon} + \varepsilon\left[e^{-t}(2-t) - 2e^{-t/\varepsilon}\right]$$

$$\boxed{Y_{comp}(t) \sim e^{-t} - (1+t)e^{-t/\varepsilon} + \varepsilon\left[e^{-t}(2-t) - 2e^{-t/\varepsilon}\right] + O(\varepsilon^2)}$$



Recall that the expansion is not valid for $t > T_1 = O(\frac{1}{\varepsilon})$, but it is sufficient since Y_{ex} and Y_{comp} are both transcendentally small when $t = O(\frac{1}{\varepsilon})$, so the expansion agrees with the exact solution to all orders in this region.

Van Dyke Matching Principle

Suppose we have an <u>n-term</u> outer expansion

$$Y_{OUT} \sim Y_0(x) + \varepsilon Y_1(x) + \cdots + \varepsilon^{n-1} Y_{n-1}(x)$$

and an <u>m-term</u> inner expansion

$$Y_{IN} \sim y_0(\xi) + \varepsilon y_1(\xi) + \cdots + \varepsilon^{m-1} y_{m-1}(\xi), \text{ where } \xi = \frac{x - x_0}{\delta(\varepsilon)}.$$

The Van Dyke Matching Principle is stated as

$$\boxed{\begin{array}{l}\text{the m-term inner expansion of}\\ \text{the n-term outer expansion.}\end{array} = \begin{array}{l}\text{the n-term outer expansion of}\\ \text{the m-term inner expansion.}\end{array}} = \begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}$$

or

$$\boxed{\text{the m-term expansion of } Y^{n}_{OUT}(x_0 + \delta\xi) = \text{the n-term expansion of } Y^{m}_{IN}\left(\frac{x - x_0}{\delta}\right)} = \begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}$$

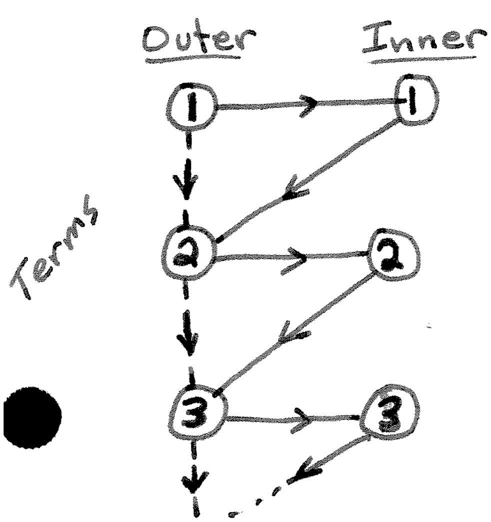
Matching in this way is called '<u>n-m matching</u>'.

Standard Matching Order

In general, it is required that $n \geq m$.

In practice, m is usually chosen to be n or $n-1$.

Matching must proceed step by step. The following diagram illustrates the order in which terms should be matched. Terms should be included in the matching sequence as indicated by the solid arrows.



Start with the first term of the outer expansion, and match with the first term of the inner expansion. Then, include the second outer term and match (if nec.), then include the second inner term and match (if nec.), etc.

Sometimes it is possible to short-circuit (dashed arrows) the standard matching order, in which case an unlimited number of terms of the outer expansion can be matched with an equivalent, or any lesser, number of terms of the inner expansion.

Recall the Mass-Spring-Damper example

$$Y_{OUT}(t) \sim C_0 e^{-t} + \varepsilon(C_1 - C_0 t)e^{-t}$$
$$Y_{IN}(\tau) \sim (1-e^{-\tau}) + \varepsilon\left[2(1-e^{-\tau}) - \tau(1+e^{-\tau})\right], \text{ where } \tau = t/\varepsilon.$$

Note: 1-1 matching gives $C_0 = 1$
2-1 matching gives $C_0 = 1$
2-2 matching gives $C_1 = 2$

For this example, we may short-circuit to 2-2 matching.

$\Rightarrow$ the $\underset{(m)}{2}$-term expansion of $Y_{OUT}(\varepsilon\tau)$ = the $\underset{(n)}{2}$-term expansion of $Y_{IN}(\frac{t}{\varepsilon})$

$$Y_{OUT}(\varepsilon\tau) \sim C_0 e^{-\varepsilon\tau} + \varepsilon(C_1 - C_0 t)e^{-\varepsilon\tau} \quad (\text{expand to } O(\varepsilon))$$
$$\sim C_0(1-\varepsilon\tau+\cdots) + \varepsilon(C_1+\cdots)(1+\cdots)$$
$$\sim C_0 + \varepsilon(C_1 - C_0\tau) + \cdots$$

$$Y_{IN}(\tfrac{t}{\varepsilon}) \sim (1-\underbrace{e^{-t/\varepsilon}}_{TST}) + \varepsilon\left[2(1-\underbrace{e^{-t/\varepsilon}}_{TST}) - \tfrac{t}{\varepsilon}(1+\underbrace{e^{-t/\varepsilon}}_{TST})\right] \quad (\text{expand to } O(\varepsilon))$$
$$\sim 1 + \varepsilon(2 - t/\varepsilon) + \cdots \sim 1 + 2\varepsilon - t + \cdots$$
$$\sim 1 - t + 2\varepsilon + \cdots$$

Equate: $C_0 + \varepsilon(C_1 - C_0\tau) = 1 - t + 2\varepsilon = \binom{\text{Common}}{\text{Part}}$
$C_0 + \varepsilon C_1 - C_0 t = 1 - t + 2\varepsilon$
$C_0(1-t) + C_1\varepsilon = (1-t) + 2\varepsilon$
$\Rightarrow$ $C_0 = 1$ $\quad$ $C_1 = 2$ $\quad$ $\binom{\text{Common}}{\text{Part}} = 1 - t + 2\varepsilon$

These results agree with those of the Intermediate Variable Method, and thus give the same composite expansion,

$$\boxed{Y_{comp}(t) = \left[e^{-t} - (t+1)e^{-t/\varepsilon}\right] + \varepsilon\left[(2-t)e^{-t} - 2e^{-t/\varepsilon}\right] + O(\varepsilon^2)}$$

Example: $(1+\varepsilon)x^2 y' = \varepsilon\left[(1-\varepsilon)xy^2 - (1+\varepsilon)x + y^3 + 2\varepsilon y^2\right]$, $y(1)=1$; $0<\varepsilon \ll 1$

Outer Solution: $Y_{OUT} = Y_0(x) + \varepsilon Y_1(x) + \varepsilon^2 Y_2(x) + \cdots$, $Y_{OUT}(1) = 1$

$$(1+\varepsilon)x^2(Y_0' + \varepsilon Y_1' + \varepsilon^2 Y_2') = \varepsilon\left[(1-\varepsilon)x(Y_0^2 + 2\varepsilon Y_0 Y_1) - (1+\varepsilon)x + (Y_0^3 + 3\varepsilon Y_0^2 Y_1) + 2\varepsilon Y_0^2\right] + \cdots$$

$O(1)$: $x^2 Y_0' = 0$, $Y_0(1) = 1$

$$\Rightarrow \boxed{Y_0(x) = 1}$$

$O(\varepsilon)$: $x^2 Y_1' + x^2 Y_0' = x Y_0^2 - x + Y_0^3$, $Y_1(1) = 0$

$$x^2 Y_1' + 0 = x - x + 1$$

$$Y_1' = \frac{1}{x^2} \qquad Y_1(1) = -1 + C_1 = 0$$

$$C_1 = 1 \Rightarrow \boxed{Y_1(x) = 1 - \frac{1}{x}}$$

$$Y_1 = -\frac{1}{x} + C_1$$

$O(\varepsilon^2)$: $x^2 Y_2' + x^2 Y_1' = 2x Y_0 Y_1 - x Y_0^2 - x + 3Y_0^2 Y_1 + 2Y_0^2$, $Y_2(1) = 0$

$$x^2 Y_2' + x^2 \cdot \frac{1}{x^2} = 2x \cdot 1 \cdot \left(1 - \frac{1}{x}\right) - x - x + 3\left(1 - \frac{1}{x}\right) + 2$$

$$x^2 Y_2' + 1 = \cancel{2x} - \cancel{2} - \cancel{2x} + 3 - 3/x + \cancel{2}$$

$$Y_2' = \frac{1}{x^2}\left(2 - \frac{3}{x}\right) = \frac{2}{x^2} - \frac{3}{x^3} \qquad Y_2(1) = -2 + \frac{3}{2} + C_2 = 0$$

$$C_2 = \frac{1}{2}$$

$$Y_2 = -\frac{2}{x} + \frac{3}{2x^2} + C_2 \qquad \boxed{Y_2(x) = \frac{1}{2} - \frac{2}{x} + \frac{3}{2x^2}}$$

$$\Rightarrow \boxed{Y_{OUT}(x) = 1 + \varepsilon\left(1 - \frac{1}{x}\right) + \varepsilon^2\left[\frac{1}{2} - \frac{2}{x} + \frac{3}{2x^2}\right] + \cdots}$$

Y_{OUT} is not valid for $x = O(\varepsilon)$.

$\Rightarrow$ Need a Boundary Layer at $x=0$ of thickness $O(\varepsilon)$.

Inner Solution: $\boxed{\zeta = \frac{x}{\varepsilon}}$ $\Rightarrow (1+\varepsilon)(\varepsilon\zeta)^2 \cdot \frac{1}{\varepsilon} Y_\zeta = \varepsilon\left[(1-\varepsilon)(\varepsilon\zeta)Y^2 - (1+\varepsilon)(\varepsilon\zeta) + Y^3 + 2\varepsilon Y^2\right]$

$$(1+\varepsilon)\zeta^2 Y_\zeta = (1-\varepsilon)\varepsilon\zeta Y^2 - (1+\varepsilon)\varepsilon\zeta + Y^3 + 2\varepsilon Y^2$$

$$Y_{IN} = y_0(\zeta) + \varepsilon y_1(\zeta) + \cdots$$

$$\Rightarrow (1+\varepsilon)\zeta^2(y_{0\zeta} + \varepsilon y_{1\zeta}) = \varepsilon\zeta y_0^2 - \varepsilon\zeta + (y_0^3 + 3\varepsilon y_0^2 y_1) + 2\varepsilon y_0^2$$

$O(1)$: $\zeta^2 y_{0\zeta} = y_0^3$

$$\Rightarrow y_0(\zeta) = \pm\left(\frac{\zeta}{D_0\zeta + 2}\right)^{1/2}$$

Primitive Matching

$$\lim_{x\to 0^+} Y_0(x) = \lim_{\zeta\to+\infty} y_0(\zeta)$$

$$1 = \pm\left(\frac{1}{D_0}\right)^{1/2}$$

need + $\quad$ $D_0 = 1$

1-1 matching

$n=1$: $Y_{OUT}(x) \sim 1$

$m=1$: $Y_{IN}(\zeta) \sim \pm\left(\frac{\zeta}{D_0\zeta+2}\right)^{1/2} = \pm\left(\frac{1/D_0}{1+2/D_0\zeta}\right)^{1/2}$

$$\underset{(m)}{1} \text{ term of } Y_{OUT}(\varepsilon\zeta) = \underset{(n)}{1} \text{ term of } Y_{IN}\left(\tfrac{x}{\varepsilon}\right)$$

$$Y_{OUT}(\varepsilon\zeta) \sim 1$$

$$Y_{IN}\left(\tfrac{x}{\varepsilon}\right) \sim \pm\left(\frac{1/D_0}{1+2\varepsilon/D_0 x}\right)^{1/2} \sim \pm\left(\frac{1}{D_0}\right)^{1/2}$$

Equate

$$1 = \pm\left(\frac{1}{D_0}\right)^{1/2}$$

need + $\quad$ $D_0 = 1$

2-1 matching

$n=2$: $Y_{OUT}(x) \sim 1 + \varepsilon\left(1 - \frac{1}{x}\right)$

$m=1$: $Y_{IN}(\zeta) \sim \pm\left(\frac{1/D_0}{1+2/D_0\zeta}\right)^{1/2}$

$$\underset{(m)}{1} \text{ term of } Y_{OUT}(\varepsilon\zeta) = \underset{(n)}{2} \text{ terms of } Y_{IN}\left(\tfrac{x}{\varepsilon}\right)$$

$$Y_{OUT}(\varepsilon\zeta) \sim 1 + \varepsilon\left(1 - \frac{1}{\varepsilon\zeta}\right) \sim 1 + \varepsilon - \frac{1}{\zeta} \sim 1 - \frac{1}{\zeta}$$

$$Y_{IN}\left(\tfrac{x}{\varepsilon}\right) \sim \pm\left(\frac{1/D_0}{1+2\varepsilon/D_0 x}\right)^{1/2} \sim \pm\left(\frac{1}{D_0}\right)^{1/2}\left(1 - \frac{\varepsilon}{D_0 x}\right)$$

Equate:

$$1 - \frac{1}{\zeta} = \pm\left(\frac{1}{D_0}\right)^{1/2}\left(1 - \frac{\varepsilon}{D_0 x}\right)$$

$$1 - \frac{1}{\zeta} = \pm\left(\frac{1}{D_0}\right)^{1/2}\left(1 - \frac{1}{D_0\zeta}\right)$$

need + $\quad$ $D_0 = 1$

$$\boxed{y_0(\zeta) = \left(\frac{\zeta}{\zeta+2}\right)^{1/2}}$$

$O(\varepsilon)$: $\zeta^2 y_{1\zeta\zeta} + \zeta^2 y_{0\zeta} = \zeta y_0^2 - \zeta + 3y_0^2 y_1 + 2y_0^2$

$$\Rightarrow y_{1\zeta} - \frac{3}{\zeta(\zeta+2)} y_1 = \frac{-1}{\zeta^{1/2}(\zeta+2)^{3/2}} \quad \text{Integrating Factor: } \mu = \left(\frac{\zeta+2}{2}\right)^{3/2}$$

$$y_1(\zeta) = \zeta^{1/2}\frac{D_1\zeta+1}{(\zeta+2)^{3/2}}$$

2-2 matching

$n=2$: $Y_{OUT}(x) \sim 1 + \varepsilon(1 - 1/x)$

$m=2$: $Y_{IN}(\zeta) \sim \left(\frac{\zeta}{\zeta+2}\right)^{1/2} + \varepsilon\zeta^{1/2}\frac{D_1\zeta+1}{(\zeta+2)^{3/2}} = \left(\frac{1}{1+2/\zeta}\right)^{1/2} + \varepsilon\frac{D_1+1/\zeta}{(1+2/\zeta)^{3/2}}$

$$\underset{(m)}{2} \text{ terms of } Y_{OUT}(\varepsilon\zeta) = \underset{(n)}{2} \text{ terms of } Y_{IN}\left(\frac{x}{\varepsilon}\right)$$

$$Y_{OUT}(\varepsilon\zeta) \sim 1 + \varepsilon\left(1 - \frac{1}{\varepsilon\zeta}\right) \sim 1 - \frac{1}{\zeta} + \varepsilon$$

$$Y_{IN}\left(\frac{x}{\varepsilon}\right) \sim \left(\frac{1}{1+2\varepsilon/x}\right)^{1/2} + \varepsilon\frac{D_1+\varepsilon/x}{(1+\frac{2\varepsilon}{x})^{3/2}} \sim \left(1-\frac{\varepsilon}{x}\right) + \varepsilon D_1 \sim 1 + \varepsilon\left(D_1 - \frac{1}{x}\right)$$

Equate: $\cancel{1 - \frac{1}{\zeta}} + \varepsilon = \cancel{1} + \varepsilon D_1 - \cancel{\frac{\varepsilon}{x}}$

$$D_1 = 1$$

3-2 matching

$n=3$: $Y_{OUT}(x) \sim 1 + \varepsilon\left(1-\frac{1}{x}\right) + \varepsilon^2\left(\frac{1}{2} - \frac{2}{x} + \frac{3}{2x^2}\right)$

$m=2$: $Y_{IN}(\zeta) \sim \left(\frac{1}{1+2/\zeta}\right)^{1/2} + \varepsilon\frac{D_1+1/\zeta}{(1+2/\zeta)^{3/2}}$

$$\underset{(m)}{2} \text{ terms of } Y_{OUT}(\varepsilon\zeta) = \underset{(n)}{3} \text{ terms of } Y_{IN}\left(\frac{x}{\varepsilon}\right)$$

$$Y_{OUT}(\varepsilon\zeta) \sim 1 + \varepsilon\left(1 - \frac{1}{\varepsilon\zeta}\right) + \varepsilon^2\left(\frac{1}{2} - \frac{2}{\varepsilon\zeta} + \frac{3}{2\varepsilon^2\zeta^2}\right) \sim 1 - \frac{1}{\zeta} + \frac{3}{2\zeta^2} + \varepsilon\left(1 - 2/\zeta\right)$$

$$Y_{IN}\left(\frac{x}{\varepsilon}\right) \sim \left(\frac{1}{1+2\varepsilon/x}\right)^{1/2} + \varepsilon\frac{D_1+\varepsilon/x}{(1+2\varepsilon/x)^{3/2}} \sim \left(1 - \frac{2\varepsilon}{x} + \frac{4\varepsilon^2}{x^2}\right) + \varepsilon\left(D_1 + \frac{\varepsilon}{x}\right)\left(1 - \frac{2\varepsilon}{x}\right)^{3/2}$$

$$\sim \left[1 + \frac{1}{2}\left(-\frac{2\varepsilon}{x} + \frac{4\varepsilon^2}{x^2}\right) - \frac{1}{8}\left(\frac{-2\varepsilon}{x}\right)^2\right] + \varepsilon(D_1 + \varepsilon/x)\left(1 - \frac{3\varepsilon}{x}\right)$$

$$\sim 1 - \varepsilon/x + \frac{2\varepsilon^2}{x^2} - \frac{\varepsilon^2}{2x^2} + \varepsilon D_1 - 3D_1\frac{\varepsilon^2}{x} + \frac{\varepsilon^2}{x} \sim 1 + \varepsilon\left(D_1 - \frac{1}{x}\right) + \varepsilon^2\left(\frac{3}{2x^2} - \frac{3D_1 - 1}{x}\right)$$

Equate: $\cancel{1 - \frac{1}{\zeta} + \frac{3}{2\zeta^2}} + \varepsilon(1 - 2/\zeta) = \cancel{1} + \varepsilon(D_1 \cancel{- \frac{1}{x}}) + \varepsilon^2\left(\cancel{\frac{3}{2x^2}} - \frac{3D_1-1}{x}\right)$

$$1 - 2/\zeta = D_1 - \frac{3D_1-1}{\zeta}$$

$$D_1 = 1 \quad \Rightarrow \quad y_1(\zeta) = \zeta^{1/2}\frac{\zeta+1}{(\zeta+2)^{3/2}}$$

$$\begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix} = 1 + \varepsilon\left(1 - \frac{1}{x}\right) + \varepsilon^2\left(\frac{3}{2x^2} - \frac{2}{x}\right)$$

All terms (3-2) must be considered to determine the appropriate common part.

We have $y_{OUT} \sim 1+\varepsilon(1-\frac{1}{x}) + \varepsilon^2(\frac{1}{2}-\frac{2}{x}+\frac{3}{2x^2})$

$y_{IN} \sim \left(\frac{\xi}{\xi+2}\right)^{1/2} + \varepsilon\, \xi^{1/2}\frac{\xi+1}{(\xi+2)^{3/2}}$

$\left(\begin{matrix}\text{Common}\\ \text{Part}\end{matrix}\right) \sim 1+\varepsilon(1-\frac{1}{x}) + \varepsilon^2(-\frac{2}{x}+\frac{3}{2x^2})$

$$y_{comp} = y_{OUT} + y_{IN} - \left(\begin{matrix}\text{Common}\\ \text{Part}\end{matrix}\right)$$

$$\sim \left[\cancel{1+\varepsilon(1-\tfrac{1}{x})}+\varepsilon^2\left(\tfrac{1}{2}\cancel{-\tfrac{2}{x}+\tfrac{3}{2x^2}}\right)\right] + \left[\left(\frac{\xi}{\xi+2}\right)^{1/2} + \varepsilon\,\xi^{1/2}\frac{\xi+1}{(\xi+2)^{3/2}}\right] - \left[\cancel{1+\varepsilon(1-\tfrac{1}{x})+\varepsilon^2\left(-\tfrac{2}{x}+\tfrac{3}{2x^2}\right)}\right]$$

$$\sim \frac{\varepsilon^2}{2} + \left(\frac{x/\varepsilon}{x/\varepsilon+2}\right)^{1/2} + \varepsilon\left(\frac{x}{\varepsilon}\right)^{1/2}\frac{x/\varepsilon+1}{(\frac{x}{\varepsilon}+2)^{3/2}}$$

$$\boxed{y_{comp}(x) = \left(\frac{x}{x+2\varepsilon}\right)^{1/2} + \varepsilon x^{1/2}\frac{x+\varepsilon}{(x+2\varepsilon)^{3/2}} + \varepsilon^2/2 + O(\varepsilon^3)}$$

<u>Outer expansion of y_{comp}</u>

$$y_{comp}(x) \sim \left(\frac{1}{1+2\varepsilon/x}\right)^{1/2} + \varepsilon\frac{1+\varepsilon/x}{(1+2\varepsilon/x)^{3/2}} + \frac{\varepsilon^2}{2} \sim \left(1-\frac{2\varepsilon}{x}+\frac{4\varepsilon^2}{x^2}\right)^{1/2} + \varepsilon\left(1+\frac{\varepsilon}{x}\right)\left(1-\frac{2\varepsilon}{x}\right)^{3/2} + \frac{\varepsilon^2}{2}$$

$$\sim 1+\frac{1}{2}\left(-\frac{2\varepsilon}{x}+\frac{4\varepsilon^2}{x^2}\right) - \frac{1}{8}\left(\frac{-2\varepsilon}{x}\right)^2 + \varepsilon\left(1+\frac{\varepsilon}{x}\right)\left(1-\frac{3\varepsilon}{x}\right) + \varepsilon^2/2$$

$$\sim 1 - \varepsilon/x + 2\varepsilon^2/x^2 - \varepsilon^2/2x^2 + \varepsilon - 3\varepsilon^2/x + \varepsilon^2/x + \varepsilon^2/2$$

$$\sim 1+\varepsilon(1-\tfrac{1}{x}) + \varepsilon^2\left(\tfrac{1}{2}-\tfrac{2}{x}+\tfrac{3}{2x^2}\right) + \cdots = \underline{y_{OUT}(x) + O(\varepsilon^3)}$$

<u>Inner expansion of y_{comp}</u>

$$y_{comp}(\xi) \sim \left(\frac{\xi}{\xi+2}\right)^{1/2} + \varepsilon\,\xi^{1/2}\frac{\xi+1}{(\xi+2)^{3/2}} + \frac{\varepsilon^2}{2} + \cdots = \underline{y_{IN}(\xi) + O(\varepsilon^2)}$$

---

y_{comp} is accurate to $O(\varepsilon^3)$ in the outer region, and accurate to $O(\varepsilon)$ in the inner region.

$\Longrightarrow$ $y_{comp}(x)$ is uniformly accurate to $O(\varepsilon)$

Neglecting the $\varepsilon^2/2$ term,

$$y_{comp}(x) = \left(\frac{x}{x+2\varepsilon}\right)^{1/2} + \varepsilon x^{1/2}\frac{x+\varepsilon}{(x+2\varepsilon)^{3/2}} + O(\varepsilon^2),$$

will reduce the accuracy of the approximation in the outer region, but it will not affect the overall accuracy of the approximation.

Logarithms / Van Dyke

Example: Consider $y(x) = 1 - \dfrac{\ln(x+\varepsilon)}{\ln 1/\varepsilon}$, $0 < \varepsilon \ll 1$

Outer Solution: $y_{OUT} = 1 - \dfrac{\ln(x(1+\varepsilon/x))}{\ln 1/\varepsilon} = 1 - \dfrac{\ln x + \ln(1+\varepsilon/x)}{\ln 1/\varepsilon} \sim 1 - \dfrac{\ln x + \varepsilon/x}{\ln 1/\varepsilon} + \cdots$

$$y_{OUT} \sim 1 - \frac{\ln x}{\ln 1/\varepsilon} + \frac{\varepsilon}{x \ln 1/\varepsilon} + \cdots$$

Gauge Functions: $1, \dfrac{1}{\ln 1/\varepsilon}, \dfrac{\varepsilon}{\ln 1/\varepsilon}, \ldots$

Inner Expansion: $\boxed{\zeta = \frac{x}{\varepsilon}}$ $y_{IN} = 1 - \dfrac{\ln(\varepsilon\zeta+\varepsilon)}{\ln 1/\varepsilon} = 1 - \dfrac{-\ln 1/\varepsilon + \ln(\zeta+1)}{\ln 1/\varepsilon}$

$$y_{IN} = 2 - \frac{\ln(\zeta+1)}{\ln 1/\varepsilon} \quad \text{(exact)}$$

Since the two expansions represent the same function, they should match.

1-1 matching: $y_{OUT} \sim 1$, $y_{IN} \sim 2$ $\quad 1 \neq 2 \quad$ 1-1 matching fails.

When logarithms are involved, terms should be counted according to powers of ε, while ignoring logarithms. *All* terms with the same power of ε (regardless of logarithms) should be counted as a single term.

$$y_{OUT} \sim 1 - \underbrace{\frac{\ln x}{\ln 1/\varepsilon}}_{1} + \underbrace{\frac{\varepsilon}{x \ln 1/\varepsilon}}_{2} + \cdots$$

$$y_{IN} = 2 - \underbrace{\frac{\ln(\zeta+1)}{\ln 1/\varepsilon}}_{1}$$

1-1 matching

$$y_{OUT}(\varepsilon\zeta) \sim 1 - \frac{\ln(\varepsilon\zeta)}{\ln 1/\varepsilon} \sim 1 - \frac{-\ln\frac{1}{\varepsilon} + \ln\zeta}{\ln 1/\varepsilon} \sim \underbrace{2 - \frac{\ln\zeta}{\ln 1/\varepsilon}}_{1}$$

$$y_{IN}\left(\frac{x}{\varepsilon}\right) = 2 - \frac{\ln(\frac{x}{\varepsilon}+1)}{\ln 1/\varepsilon} = 2 - \frac{\ln(\frac{x}{\varepsilon}(1+\varepsilon/x))}{\ln 1/\varepsilon} = 2 - \frac{\ln x + \ln\frac{1}{\varepsilon} + \ln(1+\varepsilon/x)}{\ln 1/\varepsilon}$$

$$\sim \underbrace{2 - \frac{\ln x}{\ln 1/\varepsilon} - 1}_{1} - \frac{\varepsilon/x}{\ln 1/\varepsilon} \sim \underbrace{1 - \frac{\ln x}{\ln 1/\varepsilon}}_{1}$$

Equate:

$$2 - \frac{\ln\zeta}{\ln 1/\varepsilon} = 1 - \frac{\ln x}{\ln 1/\varepsilon}$$

$$2 - \frac{\ln x + \ln\frac{1}{\varepsilon}}{\ln 1/\varepsilon} = 1 - \frac{\ln x}{\ln 1/\varepsilon}$$

$$1 - \frac{\ln x}{\ln 1/\varepsilon} = 1 - \frac{\ln x}{\ln 1/\varepsilon} \checkmark$$

1-1 matching is successful

Generalized Van Dyke Matching Principle

Sometimes it is not clear how terms should be counted.

e.g. 1. when terms have a coefficient of zero.

$$y_{OUT} \sim Y_0 + \varepsilon Y_1 + \varepsilon^2 Y_2 + \varepsilon^4 Y_4 + \varepsilon^5 Y_5 + \cdots$$

Does $0 \cdot \varepsilon^3$ count as a term?

2. when y_{OUT} and y_{IN} have different gauge functions.

$$y_{OUT} \sim Y_0 + \varepsilon^2 Y_1 + \varepsilon^4 Y_2 + \cdots$$
$$y_{IN} \sim \varepsilon^{1/2} y_0 + \varepsilon^{3/2} y_1 + \varepsilon^{5/2} y_2 + \cdots$$

When in doubt, the Generalized Van Dyke Matching Principle may be used.

The inner expansion to order Δ_I of the outer expansion to order Δ_0	=	The outer expansion to order Δ_0 of the inner expansion to order Δ_I

Instead of considering the number of terms in the expansions, consider the orders of the expansions.

Example:

$$y_{OUT} \sim Y_0 + \varepsilon^{1/2} Y_1 + \varepsilon^2 Y_2 + \cdots, \quad \Delta_0 = \varepsilon^2$$
$$y_{IN} \sim y_0 + \varepsilon y_1 + \varepsilon^{3/2} y_2 + \cdots, \quad \Delta_I = \varepsilon^{3/2}, \quad \xi = \frac{x - x_0}{\delta(\varepsilon)}$$

$$x = x_0 + \delta\xi$$

Then,

$y_{OUT}(x_0 + \delta\xi)$ expanded to order $\Delta_I = \varepsilon^{3/2}$ $=$ $y_{IN}\left(\frac{x - x_0}{\delta}\right)$ expanded to order $\Delta_0 = \varepsilon^2$

Example: $\varepsilon y''' - y' + xy = 0, \quad 0<x<1 \quad ; \quad 0<\varepsilon \ll 1$

$$y(0) = y'(0) = y(1) = 1$$

Find an asymptotic approximation of y which is accurate to $O(\varepsilon)$.

Outer Solution: $Y_{OUT} \sim Y_0(x) + \varepsilon Y_1(x) + \cdots$

$$\Rightarrow \varepsilon(Y_0''' + \cdots) - (Y_0' + \varepsilon Y_1' + \cdots) + x(Y_0 + \varepsilon Y_1 + \cdots) = 0$$

$O(1)$: $-Y_0' + xY_0 = 0$

$$Y_0' - xY_0 = 0$$

$$\mu = e^{-x^2/2} \Rightarrow e^{-x^2/2} Y_0 = C_0 \Rightarrow \boxed{Y_0(x) = C_0 e^{x^2/2}}$$

$Y_0' = C_0 e^{x^2/2} x$

$Y_0'(0) = 0 \neq 1$

Y_0 cannot satisfy the BC $y'(0) = 1$.

$\Rightarrow$ Try a Boundary Layer at $x = 0$.

$O(\varepsilon)$: $-Y_1' + xY_1 = -Y_0'''$

$$Y_1' - xY_1 = C_0 e^{x^2/2}(x^3 + 3x)$$

$$\mu = e^{-x^2/2} \Rightarrow e^{-x^2/2} Y_1 = C_0 \int (x^3 + 3x)\,dx$$

$$= C_0\left(\frac{x^4}{4} + \frac{3x^2}{2} + C_1\right)$$

$$\boxed{Y_1(x) = C_0 e^{x^2/2}\left(\frac{x^4}{4} + \frac{3x^2}{2} + C_1\right)}$$

$Y_0' = C_0 e^{x^2/2} x$

$Y_0'' = C_0 e^{x^2/2}(x^2 + 1)$

$Y_0''' = C_0 e^{x^2/2}(x^3 + 3x)$

$$\boxed{Y_{OUT} \sim C_0 e^{x^2/2}\left[1 + \varepsilon\left(\frac{x^4}{4} + \frac{3x^2}{2} + C_1\right)\right] + \cdots}$$

Y_{OUT} can't satisfy $y'(0) = 1$

$\Rightarrow$ Try a Boundary Layer at $x = 0$

Try a Boundary Layer at $x=0$

$$\xi = \frac{x}{\delta(\varepsilon)} \Longrightarrow \frac{\varepsilon}{\delta^3} Y_{\xi\xi\xi} - \frac{1}{\delta} Y_\xi + (\delta\xi) Y = 0$$

$$\delta = \varepsilon^{1/2} \Longrightarrow \boxed{Y_{\xi\xi\xi} - Y_\xi + \varepsilon\xi Y = 0}$$

Boundary Conditions:

$x=0$: $y(0)=1$ $\quad y'(0)=1$

$\xi=0$: $Y(0)=1$ $\quad \frac{1}{\varepsilon^{1/2}} Y_\xi(0) = 1$

$$Y_\xi(0) = \varepsilon^{1/2}$$

Expand Y_{IN} in powers of $\varepsilon^{1/2}$

$$Y_{IN} \sim y_0(\xi) + \varepsilon^{1/2} y_{\frac{1}{2}}(\xi) + \varepsilon y_1(\xi) + \cdots$$

$$\Rightarrow (y_{0\xi\xi\xi} + \varepsilon^{1/2} y_{\frac{1}{2}\xi\xi\xi} + \varepsilon y_{1\xi\xi\xi} + \cdots) - (y_{0\xi} + \varepsilon^{1/2} y_{\frac{1}{2}\xi} + \varepsilon y_{1\xi} + \cdots) + \varepsilon\xi(y_0 + \cdots) = 0$$

$O(1)$: $y_{0\xi\xi\xi} - y_{0\xi} = 0$; $y_0(0) = 1$, $y_{0\xi}(0) = 0$

$\Rightarrow y_0(\xi) = D_0 + D_1 e^{-\xi} + \cancel{D_2 e^{\xi}} \rightarrow D_2 = 0$ since e^{ξ} is not matchable as $\xi \to \infty$

$$\left.\begin{aligned} y_0(0) &= D_0 + D_1 = 1 \\ y_{0\xi}(0) &= -D_1 = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} D_0 &= 1 \\ D_1 &= 0 \end{aligned} \Longrightarrow \boxed{y_0(\xi) = 1}$$

$O(\varepsilon^{1/2})$: $y_{\frac{1}{2}\xi\xi\xi} - y_{\frac{1}{2}\xi} = 0$; $y_{\frac{1}{2}}(0) = 0$, $y_{\frac{1}{2}\xi}(0) = 1$

$\Rightarrow y_{\frac{1}{2}}(\xi) = E_0 + E_1 e^{-\xi} + \cancel{E_2 e^{\xi}} \rightarrow 0$

$$\left.\begin{aligned} y_{\frac{1}{2}}(0) &= E_0 + E_1 = 0 \\ y_{\frac{1}{2}\xi}(0) &= -E_1 = 1 \end{aligned}\right\} \Rightarrow \begin{aligned} E_0 &= 1 \\ E_1 &= -1 \end{aligned} \Longrightarrow \boxed{y_{\frac{1}{2}}(\xi) = 1 - e^{-\xi}}$$

$O(\varepsilon)$: $y_{1\xi\xi\xi} - y_{1\xi} = -\xi y_0 = -\xi$; $y_1(0) = y_{1\xi}(0) = 0$

$$y_1(\xi) = F_0 + F_1 e^{-\xi} + \cancel{F_2 e^{\xi}} + \xi^2/2$$

$$\left.\begin{aligned} y_1(0) &= F_0 + F_1 = 0 \\ y_{1\xi}(0) &= -F_1 = 0 \end{aligned}\right\} \Rightarrow F_0 = F_1 = 0 \Longrightarrow \boxed{y_1(\xi) = \xi^2/2}$$

$$\boxed{Y_{IN} \sim 1 + \varepsilon^{1/2}(1 - e^{-\xi}) + \varepsilon\,\xi^2/2 + \cdots}$$

We have $Y_{OUT} \sim C_0 e^{x^2/2}\left[1+\varepsilon\left(\frac{x^4}{4}+\frac{3x^2}{2}+C_1\right)\right]$ $\qquad \xi = x/\varepsilon^{1/2}$

$$Y_{IN} \sim 1+\varepsilon^{1/2}(1-e^{-\xi})+\varepsilon\,\xi^2/2$$

Matching

$$Y_{OUT}(\varepsilon^{1/2}\xi) \sim C_0 e^{\varepsilon\xi^2/2}\left[1+\varepsilon\left(\frac{\varepsilon^2\xi^4}{4}+\frac{3\varepsilon\xi^2}{2}+C_1\right)\right]$$

The expansion of $Y_{OUT}(\varepsilon^{1/2}\xi)$ involves only integral powers of ε (ε^n, $n=0,1,\ldots$).

$$Y_{IN}(x/\varepsilon^{1/2}) \sim 1+\varepsilon^{1/2}(1-\underbrace{e^{-x/\varepsilon^{1/2}}}_{TST})+\varepsilon\frac{x^2}{2\varepsilon} \sim 1+\frac{x^2}{2}+\underline{\underline{\varepsilon^{1/2}}}$$

The expansion of $Y_{IN}(x/\varepsilon^{1/2})$ involves half-powers of ε ($\varepsilon^{n/2}$, $n=0,1,\ldots$).

$\Rightarrow$ Y_{OUT} and Y_{IN} do not match.

Switchback

Revise the outer expansion in such a way so that matching is possible.

In hindsight, we see that Y_{OUT} may need to be expanded as

$$Y_{OUT} \sim Y_0(x)+\varepsilon^{1/2}Y_{\frac{1}{2}}(x)+\varepsilon Y_1(x)+\cdots$$

Insert this term

Recall: $\varepsilon y''' - y' + xy = 0$

$$Y_{OUT} \sim Y_0(x) + \varepsilon^{1/2} Y_{\frac12}(x) + \varepsilon Y_1(x) + \cdots$$

$$\Rightarrow \varepsilon(Y_0''' + \cdots) - (Y_0' + \varepsilon^{1/2} Y_{\frac12}' + \varepsilon Y_1' + \cdots) + x(Y_0 + \varepsilon^{1/2} Y_{\frac12} + \varepsilon Y_1 + \cdots) = 0$$

$O(1)$: $-Y_0' + x Y_0 = 0$ same as before $\Longrightarrow$ $\boxed{Y_0(x) = C_0 e^{x^2/2}}$

$O(\varepsilon^{1/2})$: $-Y_{\frac12}' + x Y_{\frac12} = 0$ Same as above $\Longrightarrow$ $\boxed{Y_{\frac12}(x) = C_{\frac12} e^{x^2/2}}$

$O(\varepsilon)$: $-Y_1' + x Y_1 = -Y_0'''$ same as before $\Longrightarrow$ $\boxed{Y_1(x) = C_0 e^{x^2/2}\left(\frac{x^4}{4} + \frac{3x^2}{2} + C_1\right)}$

$$\boxed{Y_{OUT} \sim e^{x^2/2}\left[C_0 + \varepsilon^{1/2} C_{1/2} + \varepsilon C_0\left(\frac{x^4}{4} + \frac{3x^2}{2} + C_1\right)\right] + \cdots}$$

$$Y_{IN} \sim 1 + \varepsilon^{1/2}(1 - e^{-\xi}) + \varepsilon \xi^2/2$$

3-3 Matching

$$Y_{OUT}(\varepsilon^{1/2}\xi) \sim e^{\varepsilon\xi^2/2}\left[C_0 + \varepsilon^{1/2} C_{\frac12} + \varepsilon C_0\left(\frac{\varepsilon^2\xi^4}{4} + \frac{3\varepsilon\xi^2}{2} + C_1\right)\right] \quad (\text{expand to } O(\varepsilon))$$

$$\sim \left(1 + \frac{\varepsilon\xi^2}{2}\right)(C_0 + \varepsilon^{1/2} C_{\frac12} + \varepsilon C_0 C_1) \sim C_0 + \varepsilon^{1/2} C_{\frac12} + \varepsilon C_0 (C_1 + \xi^2/2)$$

$$Y_{IN}\left(\frac{x}{\varepsilon^{1/2}}\right) \sim 1 + \varepsilon^{1/2}(1 - \underbrace{e^{-x/\varepsilon^{1/2}}}_{TST}) + \varepsilon \frac{x^2}{2\varepsilon} \sim 1 + \frac{x^2}{2} + \varepsilon^{1/2} \quad (\text{expand to } O(\varepsilon))$$

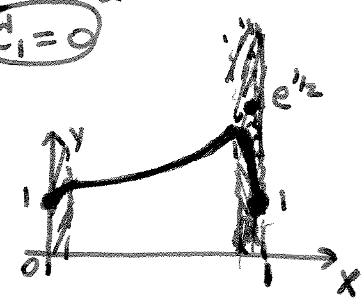
Equate: $C_0 + \varepsilon^{1/2} C_{\frac12} + \varepsilon C_0 (C_1 + \xi^2/2) = 1 + \frac{x^2}{2} + \varepsilon^{1/2}$

$$C_0 + \varepsilon^{1/2} C_{\frac12} + \varepsilon C_0 (C_1 + \xi^2/2) = 1 + \frac{\varepsilon\xi^2}{2} + \varepsilon^{1/2}$$

$O(1)$: $C_0 = 1$ $\quad$ $O(\varepsilon^{1/2})$: $C_{\frac12} = 1$ $\quad$ $O(\varepsilon)$: $C_1 + \frac{\xi^2}{2} = \frac{\xi^2}{2}$

$C_1 = 0$

$$\boxed{\begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}_{x=0} = 1 + \frac{x^2}{2} + \varepsilon^{1/2}}$$



$$\boxed{Y_{OUT} \sim e^{x^2/2}\left[1 + \varepsilon^{1/2} + \varepsilon\left(\frac{x^4}{4} + \frac{3x^2}{2}\right)\right] + \cdots}$$

Your cannot satisfy $Y(1) = 1$

$Y_{OUT}(1) \sim e^{1/2}[1 + \cdots] \sim e^{1/2} \neq 1$ $\Rightarrow$ Try a Boundary Layer at $x = 1$.

Try a Boundary Layer at $x=1$

$$\boxed{\eta = \frac{x-1}{\delta(\varepsilon)}} \Rightarrow \frac{\varepsilon}{\delta^3} Y_{\eta\eta\eta} - \frac{1}{\delta} Y_\eta + (1+\delta\eta) Y = 0$$

$$\boxed{\delta = \varepsilon^{1/2}} \Rightarrow Y_{\eta\eta\eta} - Y_\eta + \varepsilon^{1/2}(1+\varepsilon^{1/2}\eta) Y = 0, \quad Y(0) = 1$$

$$\bar{Y}_{in} \sim \bar{y}_0(\eta) + \varepsilon^{1/2}\bar{y}_{\frac{1}{2}}(\eta) + \varepsilon \bar{y}_1(\eta) + \cdots$$

$$\Rightarrow (\bar{y}_{0\eta\eta\eta} + \varepsilon^{1/2}\bar{y}_{\frac{1}{2}\eta\eta\eta} + \varepsilon \bar{y}_{1\eta\eta\eta} + \cdots) - (\bar{y}_{0\eta} + \varepsilon^{\frac{1}{2}}\bar{y}_{\frac{1}{2}\eta} + \varepsilon \bar{y}_{1\eta} + \cdots) + \varepsilon^{1/2}(1+\varepsilon^{1/2}\eta)(\bar{y}_0 + \varepsilon^{1/2}\bar{y}_{\frac{1}{2}} + \cdots) = 0$$

$O(1)$: $\bar{y}_{0\eta\eta\eta} - \bar{y}_{0\eta} = 0, \quad \bar{y}_0(0) = 1$

$\bar{y}_0(\eta) = a_0 + b_0 e^{\eta} + \cancel{c_0 e^{-\eta}}$ — $c_0 = 0$ since $e^{-\eta}$ is not matchable as $\eta \to -\infty$.

$\bar{y}_0(0) = a_0 + b_0 = 1$

$b_0 = -(a_0 - 1)$

$\Rightarrow \bar{y}_0(\eta) = a_0 - (a_0 - 1)e^{\eta}$

Primitive Matching

$$\lim_{x\to 1^-} Y_0(x) = \lim_{\eta \to -\infty} \bar{y}_0(\eta)$$

$$e^{1/2} = a_0$$

$$\Rightarrow \boxed{\bar{y}_0(\eta) = e^{1/2} - (e^{1/2} - 1)e^{\eta}}$$

$O(\varepsilon^{1/2})$: $\bar{y}_{\frac{1}{2}\eta\eta\eta} - \bar{y}_{\frac{1}{2}\eta} = -\bar{y}_0 = -e^{1/2} + (e^{1/2} - 1)e^{\eta}, \quad \bar{y}_{\frac{1}{2}}(0) = 0$

$$\bar{y}_{\frac{1}{2}}(\eta) = a_{\frac{1}{2}} + b_{\frac{1}{2}} e^{\eta} + \cancel{c_{\frac{1}{2}} e^{-\eta}}^{\,0} + e^{1/2}\eta + \frac{e^{1/2}-1}{2}\eta e^{\eta}$$

$Y_{\frac{1}{2}p} = A\eta + B\eta e^{\eta}$

$A = e^{\frac{1}{2}}$

$B = \frac{e^{\frac{1}{2}} - 1}{2}$

$\bar{y}_{\frac{1}{2}}(0) = a_{\frac{1}{2}} + b_{\frac{1}{2}} = 0$

$b_{\frac{1}{2}} = -a_{\frac{1}{2}}$

$$\Rightarrow \boxed{\bar{y}_{\frac{1}{2}}(\eta) = a_{\frac{1}{2}}(1 - e^{\eta}) + e^{1/2}\eta + \frac{e^{1/2}-1}{2}\eta e^{\eta}}$$

2-2 Matching: $\eta = \frac{x-1}{\varepsilon^{1/2}}$

$$Y_{OUT} \sim e^{x^2/2}(1+\varepsilon^{1/2}) + \cdots$$

$$\bar{Y}_{IN} \sim e^{1/2} - (e^{1/2}-1)e^{\eta} + \varepsilon^{1/2}\left[a_{\frac{1}{2}}(1-e^{\eta}) + e^{1/2}\eta + \frac{e^{\frac{1}{2}}-1}{2}\eta e^{\eta}\right] + \cdots$$

$$Y_{OUT}(1+\varepsilon^{1/2}\eta) \sim e^{\frac{1}{2}(1+\varepsilon^{1/2}\eta)^2}(1+\varepsilon^{1/2}) \sim e^{\frac{1}{2}(1+2\varepsilon^{1/2}\eta)}(1+\varepsilon) \sim e^{\frac{1}{2}}(1+\varepsilon^{1/2}\eta)(1+\varepsilon^{1/2}) \sim e^{1/2}(1+\varepsilon^{1/2}(\eta+1))$$

$$Y_{IN}\left(\frac{x-1}{\varepsilon^{1/2}}\right) \sim e^{1/2} - TST + \varepsilon^{1/2}\left[a_{\frac{1}{2}}(1-TST) + e^{1/2}\frac{x-1}{\varepsilon^{1/2}} + TST\right] \sim \cancel{e^{1/2}} + \varepsilon^{1/2}a_{\frac{1}{2}} + e^{\frac{1}{2}}(x\cancel{-1})$$

$$\sim e^{1/2}x + \varepsilon^{1/2}a_{\frac{1}{2}}$$

Equate:

$$e^{1/2}(1+\varepsilon^{1/2}(\eta+1)) = e^{1/2}x + \varepsilon^{\frac{1}{2}}a_{\frac{1}{2}}$$

$$\cancel{e^{1/2}} + \underline{e^{1/2}\varepsilon^{1/2}}\cancel{(\eta+1)} = e^{1/2}\cancel{(1+\varepsilon^{1/2}\eta)} + \varepsilon^{1/2}\underline{a_{\frac{1}{2}}}$$

$$a_{\frac{1}{2}} = e^{1/2}$$

$$\Longrightarrow \boxed{\bar{y}_{\frac{1}{2}}(\eta) = e^{\frac{1}{2}}(1+\eta-e^{\eta}) + \frac{e^{\frac{1}{2}}-1}{2}\eta e^{\eta}}$$

O(ε): $\bar{y}_{1\eta\eta\eta} - \bar{y}_{1\eta} = -\bar{y}_{\frac{1}{2}} - \eta\bar{y}_0$, $\bar{y}_1(0) = 0$

$$= -e^{1/2}(1+\eta-e^{\eta}) - \frac{e^{1/2}-1}{2}\eta e^{\eta} - \eta e^{1/2} + \eta(e^{1/2}-1)e^{\eta}$$

$$\bar{y}_{1\eta\eta\eta} - \bar{y}_{1\eta} = -e^{1/2}(1+2\eta-e^{\eta}) + \frac{e^{1/2}-1}{2}\eta e^{\eta}$$

$$y_{1P} = \eta(A\eta+B) + \eta(C\eta+D)e^{\eta}$$

$A = e^{1/2}$ $B = e^{1/2}$

$C = \frac{e^{1/2}-1}{8}$ $D = \frac{e^{1/2}+3}{8}$

$$\bar{y}_1(\eta) = a_1 + b_1e^{\eta} + \cancel{c_1e^{-\eta}}^{\,0} + e^{1/2}\eta(\eta+1) + \frac{\eta}{8}e^{\eta}\left((e^{1/2}-1)\eta + (e^{1/2}+3)\right)$$

$$\bar{y}_1(0) = a_1 + b_1 = 0$$

$$b_1 = -a_1$$

$$\Longrightarrow \boxed{\bar{y}_1 = a_1(1-e^{\eta}) + e^{1/2}\eta(\eta+1) + \frac{\eta}{8}e^{\eta}\left[(e^{\frac{1}{2}}-1)\eta + (e^{\frac{1}{2}}+3)\right]}$$

3-3 Matching:

$$Y_{OUT} \sim e^{x^2/2}\left[1+\varepsilon^{1/2}+\varepsilon\left(\frac{x^4}{4}+\frac{3x^2}{2}\right)\right]$$

$$\bar{Y}_{IN} \sim e^{1/2}-(e^{1/2}-1)e^{\eta}+\varepsilon^{1/2}\left[e^{1/2}(1+\eta-e^{\eta})+\frac{e^{1/2}-1}{2}\eta e^{\eta}\right]$$
$$+\varepsilon\left[a_1(1-e^{\eta})+e^{1/2}\eta(\eta+1)+\frac{\eta}{8}e^{\eta}\left((e^{1/2}-1)\eta+(e^{1/2}+3)\right)\right]$$

$$\eta=\frac{x-1}{\varepsilon^{1/2}}$$

$$Y_{OUT}(1+\varepsilon^{1/2}\eta) \sim e^{\frac{1}{2}(1+\varepsilon^{1/2}\eta)^2}\left(1+\varepsilon^{\frac{1}{2}}+\varepsilon\left(\tfrac{1}{4}+\tfrac{3}{2}\right)\right) \sim e^{\frac{1}{2}(1+2\varepsilon^{\frac{1}{2}}\eta+\varepsilon\eta^2)}\left(1+\varepsilon^{\frac{1}{2}}+\tfrac{7}{4}\varepsilon\right)$$
$$\sim e^{\frac{1}{2}}e^{\varepsilon^{1/2}\eta}e^{\varepsilon\eta^2/2}\left(1+\varepsilon^{\frac{1}{2}}+\tfrac{7}{4}\varepsilon\right) \sim e^{\frac{1}{2}}\left(1+\varepsilon^{\frac{1}{2}}\eta+\frac{\varepsilon\eta^2}{2}\right)\left(1+\frac{\varepsilon\eta^2}{2}\right)\left(1+\varepsilon^{1/2}+\tfrac{7}{4}\varepsilon\right)$$
$$\sim e^{\frac{1}{2}}\left(1+\frac{\varepsilon\eta^2}{2}+\varepsilon^{1/2}\eta+\frac{\varepsilon\eta^2}{2}\right)\left(1+\varepsilon^{1/2}+\tfrac{7}{4}\varepsilon\right) \sim e^{1/2}\left(1+\varepsilon^{1/2}+\tfrac{7}{4}\varepsilon+\varepsilon^{1/2}\eta+\varepsilon\eta+\varepsilon\eta^2\right)$$
$$\sim e^{1/2}\left[1+\varepsilon^{1/2}(\eta+1)+\varepsilon\left(\eta^2+\eta+\tfrac{7}{4}\right)\right]$$

$$\bar{Y}_{IN}\left(\frac{x-1}{\varepsilon^{1/2}}\right) \sim (e^{\frac{1}{2}}-TST)+\varepsilon^{1/2}\left[e^{1/2}\left(1+\frac{x-1}{\varepsilon^{1/2}}-TST\right)+TST\right]+\varepsilon\left[a_1(1-TST)+e^{1/2}\frac{x-1}{\varepsilon^{1/2}}\left(\frac{x-1}{\varepsilon^{1/2}}+1\right)+TST\right]$$
$$\sim e^{1/2}+\varepsilon^{1/2}e^{1/2}+e^{1/2}(x-1)+\varepsilon a_1+e^{1/2}(x-1)^2+e^{1/2}\varepsilon^{1/2}(x-1)$$
$$\sim e^{1/2}(x^2-x+1)+\varepsilon^{1/2}e^{1/2}x+\varepsilon a_1+e^{1/2}(x-1)^2+e^{1/2}\varepsilon^{1/2}(x-1)$$
$$\sim e^{1/2}(x^2-x+1)+\varepsilon^{1/2}e^{1/2}x+\varepsilon a_1$$

Equate:
$$e^{1/2}\left[\cancel{1}+\varepsilon^{1/2}(\eta+1)+\varepsilon\left(\eta^2+\eta+\tfrac{7}{4}\right)\right]=e^{\frac{1}{2}}(x^2-x+\cancel{1})+\varepsilon^{1/2}e^{1/2}x+\varepsilon a_1$$
$$\varepsilon^{1/2}e^{1/2}(\eta+1)+\varepsilon e^{1/2}\left(\eta^2+\eta+\tfrac{7}{4}\right)=e^{1/2}\left[(1+\varepsilon^{1/2}\eta)^2-(1+\varepsilon^{1/2}\eta)\right]+\varepsilon^{1/2}e^{1/2}(1+\varepsilon^{1/2}\eta)+\varepsilon a_1$$
$$\varepsilon^{1/2}(\cancel{\eta}+\cancel{1})+\varepsilon\left(\cancel{\eta^2}+\cancel{\eta}+\tfrac{7}{4}\right)=\cancel{1}+\cancel{2\varepsilon^{1/2}\eta}+\cancel{\varepsilon\eta^2}-\cancel{1}-\cancel{\varepsilon^{1/2}\eta}+\cancel{\varepsilon^{1/2}}+\cancel{\varepsilon\eta}+\varepsilon a_1 e^{-1/2}$$
$$\tfrac{7}{4}\varepsilon=\varepsilon a_1 e^{-1/2}$$
$$a_1=\tfrac{7}{4}e^{1/2}$$

$$\binom{\text{Common}}{\text{Part}}_{x=1}=e^{1/2}\left[(x^2-x+1)+\varepsilon^{1/2}x+\tfrac{7}{4}\varepsilon\right]$$

$$\bar{Y}_{IN} \sim e^{\frac{1}{2}}-(e^{1/2}-1)e^{\eta}+\varepsilon^{\frac{1}{2}}\left[e^{\frac{1}{2}}(1+\eta-e^{\eta})+\frac{e^{\frac{1}{2}}-1}{2}\eta e^{\eta}\right]$$
$$+\varepsilon\left[e^{1/2}\left(\eta^2+\eta+\tfrac{7}{4}-\tfrac{7}{4}e^{\eta}\right)+\frac{\eta}{8}e^{\eta}\left((e^{1/2}-1)\eta+(e^{1/2}+3)\right)\right]$$

$$y_{out} \sim e^{x^2/2}\left[1+\varepsilon^{1/2}+\varepsilon\left(\frac{x^4}{4}+\frac{3x^2}{2}\right)\right] \qquad \xi = x/\varepsilon^{1/2} \quad \eta = \frac{x-1}{\varepsilon^{1/2}}$$

$$y_{in} \sim 1+\varepsilon^{1/2}(1-e^{-\xi})+\varepsilon\,\xi^2/2$$

$$\bar{y}_{in} \sim e^{1/2}-(e^{1/2}-1)e^{\eta}+\varepsilon^{1/2}\left[e^{1/2}(1+\eta-e^{\eta})+\frac{e^{1/2}-1}{2}\eta e^{\eta}\right]+\varepsilon\left[e^{1/2}\left(\eta^2+\eta-\frac{7}{4}-\frac{7}{4}e^{\eta}\right)+\frac{\eta}{8}e^{\eta}\left((e^{1/2}-1)\eta+(e^{1/2}+3)\right)\right]$$

$$\begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}_{x=0} = 1+\frac{x^2}{2}+\varepsilon^{1/2} \qquad \begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}_{x=1} = e^{1/2}\left[(x^2-x+1)+\varepsilon^{1/2}+\frac{7}{4}\varepsilon\right]$$

$$y_{comp} = y_{out}+\left[y_{in}-\begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}_{x=0}\right]+\left[\bar{y}_{in}-\begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}_{x=1}\right]$$

$$\sim y_{out}+\left[\cancel{1}+\varepsilon^{1/2}(\cancel{1}-e^{-\xi})+\cancel{\varepsilon\xi^2/2}-\left(\cancel{1}+\cancel{\tfrac{x^2}{2}}+\cancel{\varepsilon^{1/2}}\right)\right]+\Big\{e^{1/2}-(e^{1/2}-1)e^{\eta}+\varepsilon^{1/2}\left[e^{1/2}(1+\eta-e^{\eta})+\frac{e^{1/2}-1}{2}\eta e^{\eta}\right]$$
$$+\varepsilon\left[e^{1/2}\left(\eta^2+\eta-\frac{7}{4}-\frac{7}{4}e^{\eta}\right)+\frac{\eta}{8}e^{\eta}\left((e^{1/2}-1)\eta+(e^{1/2}+3)\right)\right]-e^{1/2}\left[(x^2-x+1)+\varepsilon^{1/2}+\frac{7}{4}\varepsilon\right]\Big\}$$

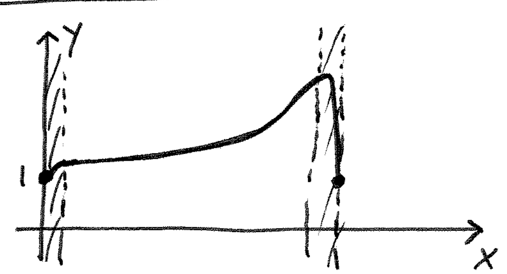
$$\sim y_{out}-\varepsilon^{1/2}e^{-\xi}+\Big\{\cancel{e^{1/2}}-(e^{1/2}-1)e^{\eta}+e^{1/2}(\cancel{\varepsilon^{1/2}}+x-\cancel{1}-\varepsilon^{1/2}e^{\eta})+\frac{e^{1/2}-1}{2}(x-1)e^{\eta}$$
$$+e^{1/2}\left[(x-1)^2+\varepsilon^{1/2}(x-\cancel{1})+\frac{7}{4}\varepsilon-\frac{7}{4}\varepsilon e^{\eta}\right]+\frac{x-1}{8}e^{\eta}\left((e^{1/2}-1)(x-1)+\varepsilon^{1/2}(e^{1/2}+3)\right)-e^{1/2}\left(x^2-x+1+\varepsilon^{1/2}+\frac{7}{4}\varepsilon\right)$$

$$\sim y_{out}-\varepsilon^{1/2}e^{-\xi}+e^{1/2}\left(\cancel{x}+\cancel{x^2}-\cancel{2x}+\cancel{1}+\cancel{\varepsilon^{1/2}x}+\cancel{\tfrac{7}{4}\varepsilon}-\cancel{x^2}+\cancel{x}-\cancel{1}-\cancel{\varepsilon^{1/2}x}-\cancel{\tfrac{7}{4}\varepsilon}\right)$$
$$+e^{\eta}\left[-(e^{1/2}-1)-\varepsilon^{1/2}e^{1/2}+\frac{e^{1/2}-1}{2}(x-1)-\frac{7}{4}e^{1/2}\varepsilon+\frac{e^{1/2}-1}{8}(x^2-2x+1)+\varepsilon^{1/2}\frac{e^{1/2}+3}{8}(x-1)\right]$$

$$\sim y_{out}-\varepsilon^{1/2}e^{-\xi}+\frac{e^{\eta}}{8}\left[(e^{1/2}-1)(-8+4x-4+x^2-2x+1)+\varepsilon^{1/2}\left(-8e^{1/2}+(e^{1/2}+3)(x-1)\right)-14e^{1/2}\varepsilon\right]$$

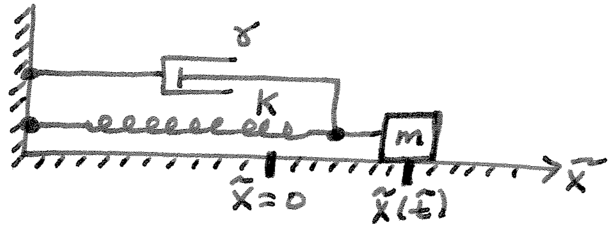
$$\sim y_{out}-\varepsilon^{1/2}e^{-\xi}+\frac{e^{\eta}}{8}\left[(e^{1/2}-1)(x^2+2x-11)+\varepsilon^{1/2}\left((e^{1/2}+3)x-3(3e^{1/2}+1)\right)-14e^{1/2}\varepsilon\right]$$

$$\boxed{\begin{aligned} y_{comp}(x) \sim\ & e^{x^2/2}\left[1+\varepsilon^{1/2}+\varepsilon\left(\frac{x^4}{4}+\frac{3x^2}{2}\right)\right]-\varepsilon^{1/2}e^{-x/\varepsilon^{1/2}} \\ & +\frac{1}{8}e^{-\frac{1-x}{\varepsilon^{1/2}}}\left[(e^{1/2}-1)(x^2+2x-11)+\varepsilon^{1/2}\left((e^{1/2}+3)x-3(3e^{1/2}+1)\right)-14e^{1/2}\varepsilon\right]+o(\varepsilon)\end{aligned}}$$



Mass-Spring-Damper System

$$\boxed{\begin{gathered} m\frac{d^2\tilde{x}}{d\tilde{t}^2} + \gamma\frac{d\tilde{x}}{d\tilde{t}} + k\tilde{x} = 0 \\ \tilde{x}(0) = 0, \quad \frac{d\tilde{x}}{d\tilde{t}}(0) = V_0 \end{gathered}}$$



Nondimensionalization : $t = \tilde{t}/t_*$, $x = \tilde{x}/x_*$

$$\Longrightarrow \quad m\frac{x_*}{t_*^2}\frac{d^2x}{dt^2} + \gamma\frac{x_*}{t_*}\frac{dx}{dt} + kx_*x = 0$$

$$\dot{x} = \frac{dx}{dt} \quad \Longrightarrow \quad \boxed{\begin{gathered} \frac{m}{\gamma t_*}\ddot{x} + \dot{x} + \frac{kt_*}{\gamma}x = 0 \\ x(0) = 0, \quad \dot{x}(0) = \frac{V_0 t_*}{x_*} \end{gathered}}$$

$$\frac{d\tilde{x}}{d\tilde{t}}(0) = V_0$$

$$\frac{x_*}{t_*}\frac{dx}{dt}(0) = V_0$$

$$\dot{x}(0) = \frac{V_0 t_*}{x_*}$$

---

① Relatively Small Mass (m)

$$t_* = \frac{\gamma}{k} \qquad \varepsilon = \frac{mk}{\gamma^2} \qquad \Longrightarrow \qquad \boxed{\begin{gathered} \varepsilon\ddot{x} + \dot{x} + x = 0 \\ x(0) = 0 \\ \dot{x}(0) = 1/\varepsilon \end{gathered}}$$

$$x_* = \varepsilon V_0 t_* \qquad x_* = \frac{mV_0}{\gamma}$$

---

② Relatively Small Spring Constant (k)

$$t_* = \frac{m}{\gamma} \qquad \varepsilon = \frac{mk}{\gamma^2} \qquad \Longrightarrow \qquad \boxed{\begin{gathered} \ddot{x} + \dot{x} + \varepsilon x = 0 \\ x(0) = 0 \\ \dot{x}(0) = 1 \end{gathered}}$$

$$x_* = V_0 t_* \qquad x_* = \frac{mV_0}{\gamma}$$

---

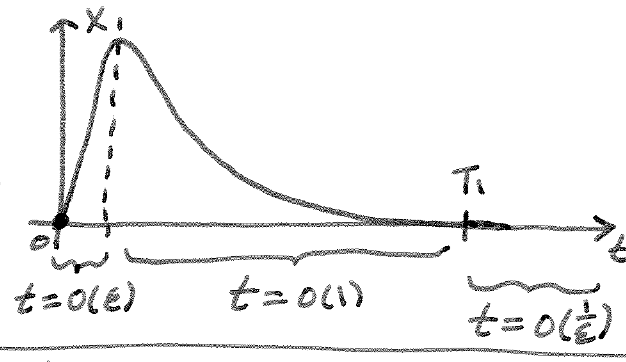
③ Relatively Small Damper Constant (γ)

$$t_* = \sqrt{\frac{m}{k}} \qquad \varepsilon = \frac{\gamma}{\sqrt{mk}} \qquad \Longrightarrow \qquad \boxed{\begin{gathered} \ddot{x} + \varepsilon\dot{x} + x = 0 \\ x(0) = 0 \\ \dot{x}(0) = 1 \end{gathered}}$$

$$x_* = V_0 t_* \qquad x_* = V_0\sqrt{\frac{m}{k}}$$

① $\boxed{\begin{aligned} &\varepsilon\ddot{x}+\dot{x}+x=0 \\ &x(0)=0 \\ &\dot{x}(0)=1/\varepsilon \end{aligned}}$ $\boxed{x(t)\sim e^{-t}-e^{-t/\varepsilon}}$

e^{-t}: O(1) Time (t) $e^{-t/\varepsilon}$: Fast Time $(\tau=t/\varepsilon)$



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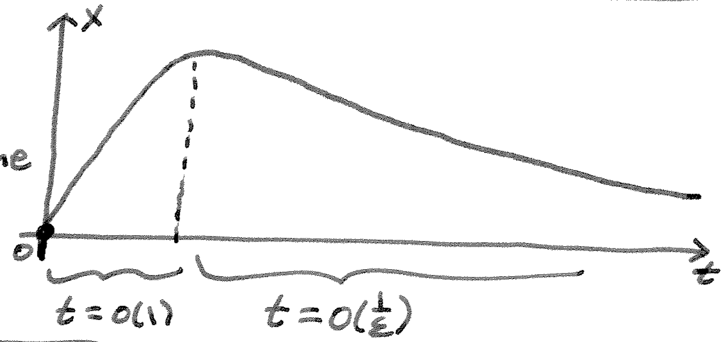
② $\boxed{\begin{aligned} &\ddot{x}+\dot{x}+\varepsilon x=0 \\ &x(0)=0 \\ &\dot{x}(0)=1 \end{aligned}}$ $\boxed{x(t)\sim e^{-\varepsilon t}-e^{-t}}$

$e^{-\varepsilon t}$: Slow Time $(\tau=\varepsilon t)$ e^{-t}: O(1) Time (t)



Naive Expansion: $x_{naive}\sim x_0(t)+\varepsilon x_1(t)+\cdots$

$$\Rightarrow (\ddot{x}_0+\varepsilon\ddot{x}_1+\cdots)+(\dot{x}_0+\varepsilon\dot{x}_1+\cdots)+\varepsilon(x_0+\cdots)=0$$

O(1): $\ddot{x}_0+\dot{x}_0=0$, $x_0(0)=0$, $\dot{x}_0(0)=1$

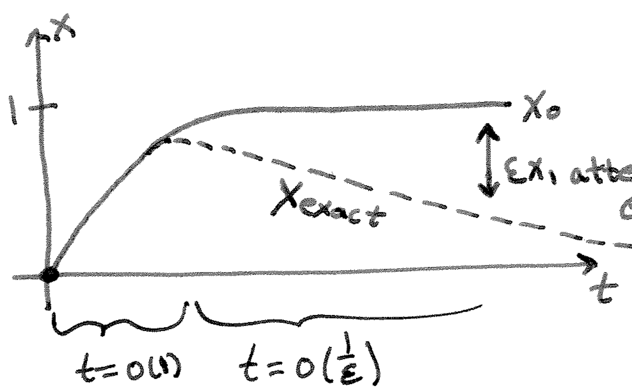
$$\Rightarrow \boxed{x_0(t)=1-e^{-t}}$$

O(ε): $\ddot{x}_1+\dot{x}_1=-x_0$, $x_1(0)=\dot{x}_1(0)=0$

$$\Rightarrow \boxed{x_1(t)=2(1-e^{-t})-t(1+e^{-t})}$$

$$\boxed{x_{naive}(t)\sim(1-e^{-t})-\varepsilon\left[(t-2)+(t+2)e^{-t}\right]+\cdots}$$

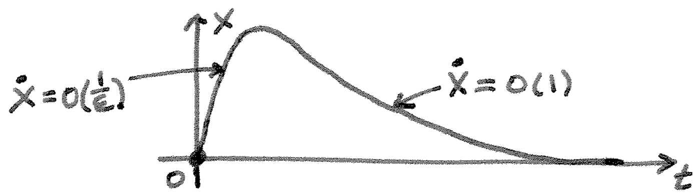
Not valid for $t=O(\frac{1}{\varepsilon})$ and larger



εx_1 attempts to correct for the O(1) error of the leading order term, which occurs for large t. Consequently, $x_1\to-\infty$ as $t\to\infty$.

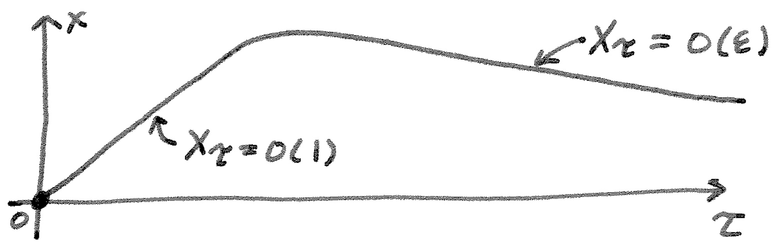
Far Field $(t \gg 1)$

- The naive expansion fails for initial (and boundary) layer problems since derivatives are <u>large</u> in the initial layer.

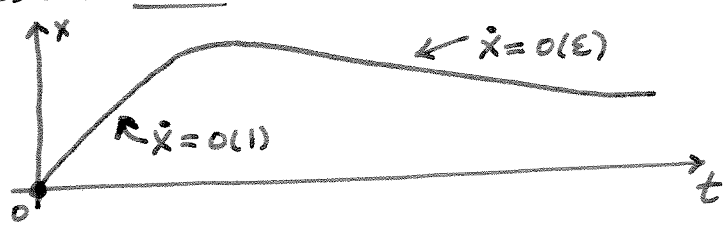


To find the inner expansion, we <u>stretch</u> time so that derivatives are $O(1)$ in the initial layer.

$$\boxed{\tau = t/\varepsilon}$$

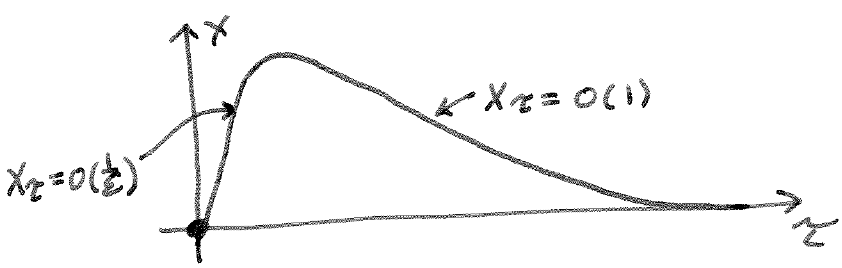


- In contrast, the naive expansion fails for <u>far field</u> problems since derivatives are <u>small</u> in the far field.



To find the far field expansion, we <u>compress</u> time so that derivatives are $O(1)$ in the far field

$$\boxed{\tau = \varepsilon t}$$



- Then, match and determine the composite expansion as usually.

We have $\ddot{x} + \dot{x} + \varepsilon x = 0$; $x(0) = 0$, $\dot{x}(0) = 1$

Naive Expansion: $x_{naive} \sim x_0(t) + \varepsilon x_1(t) + \cdots$

$O(1)$: $\ddot{x}_0 + \dot{x}_0 = 0$, $x_0(0) = 0$, $\dot{x}_0(0) = 1$

$$\Rightarrow \boxed{x_0(t) = 1 - e^{-t}}$$

Far Field: $\boxed{\tau = \varepsilon t} \Rightarrow \varepsilon^2 x_{\tau\tau} + \varepsilon x_\tau + \varepsilon x = 0$

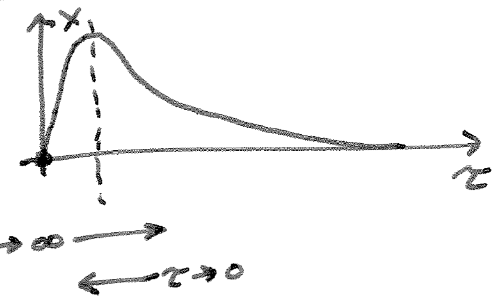
$$\boxed{\varepsilon x_{\tau\tau} + x_\tau + x = 0}$$

$$x_{far} \sim \chi_0(\tau) + \varepsilon \chi_1(\tau) + \cdots$$

$O(1)$: $\chi_{0\tau} + \chi_0 = 0$

$$\Rightarrow \boxed{\chi_0(\tau) = C_0 e^{-\tau}}$$

Primitive Matching



$$\lim_{t \to \infty} x_0(t) = \lim_{\tau \to 0} \chi_0(\tau)$$

$$\boxed{1 = C_0}$$

$$\boxed{\begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix} = 1}$$

$$x_{comp} = x_{naive} + x_{far} - \begin{pmatrix}\text{Common}\\ \text{Part}\end{pmatrix}$$

$$\sim 1 - e^{-t} + e^{-\tau} - 1$$

$$\boxed{x_{comp}(t) \sim e^{-\varepsilon t} - e^{-t}}$$

Alternately, we may set $s = \varepsilon t$, $\Rightarrow$ $\varepsilon x_{ss} + x_s + x = 0$
$x(0) = 0$, $x_s(0) = 1/\varepsilon$
and treat the IVP as a initial layer problem in terms of the s-variable.

③ $\ddot{x} + \varepsilon\dot{x} + x = 0 \; ; \; x(0) = 0, \; \dot{x}(0) = 1$

Naive Expansion: $x_{naive} \sim x_0(t) + \varepsilon x_1(t) + \cdots$

$$(\ddot{x}_0 + \varepsilon\ddot{x}_1 + \cdots) + \varepsilon(\dot{x}_0 + \cdots) + (x_0 + \varepsilon x_1 + \cdots) = 0$$

$O(1)$: $\ddot{x}_0 + x_0 = 0 \; ; \; x_0(0) = 0, \; \dot{x}_0(0) = 1$

$$\Rightarrow \boxed{x_0(t) = \sin t}$$

$O(\varepsilon)$: $\ddot{x}_1 + x_1 = -\dot{x}_0 \; ; \; x_1(0) = \dot{x}_1(0) = 0$

$$\Rightarrow x_1(t) = c_1\cos t + c_2\sin t - \tfrac{t}{2}\sin t \qquad c_1 = c_2 = 0$$

$$\boxed{x(t) = -\tfrac{t}{2}\sin t}$$

$$\Rightarrow \boxed{x_{naive}(t) \sim \sin t - \varepsilon\tfrac{t}{2}\sin t + \cdots}$$

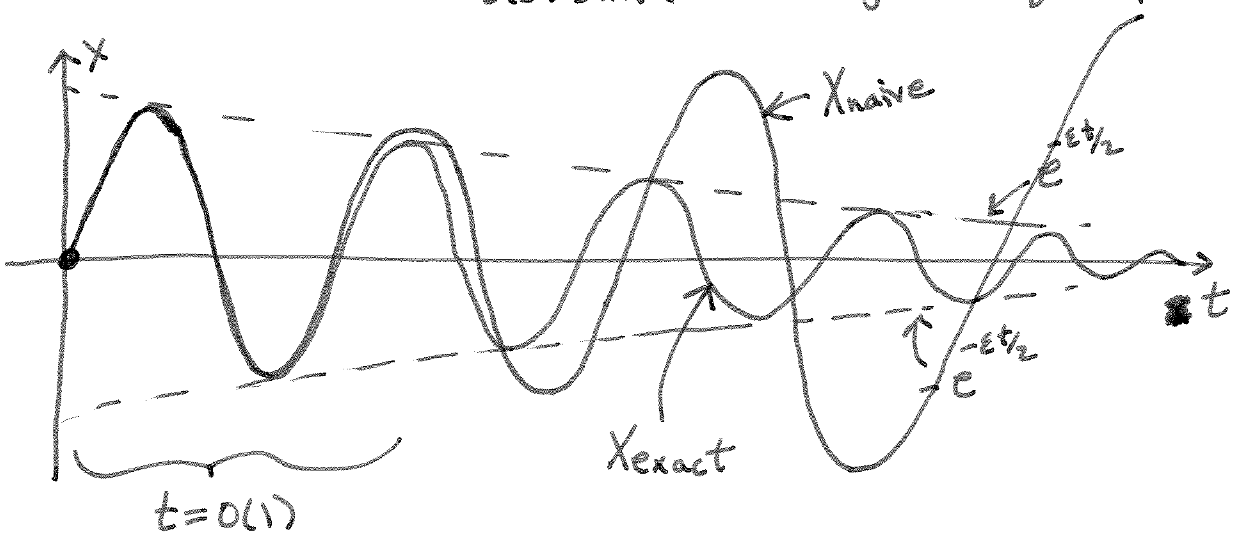
Not valid for $t = O(\frac{1}{\varepsilon})$ and larger

$$\boxed{x_{exact}(t) = e^{-\varepsilon t/2}\sin\left(\sqrt{1-\varepsilon^2/4}\; t\right)}$$

Slow Time ($\tau = \varepsilon t$)

O(1) Time (t)

$O(\varepsilon^2)$ shift in the angular frequency



Example: Far Field/Switchback/Logarithmic Van Dyke Matching

Consider the following dimensionless problem of radially symmetric steady heat conduction outside a unit sphere with a small heat source.

Approximate u to $O(\varepsilon)$.

$$\boxed{\begin{aligned}&\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) = -\varepsilon u\frac{du}{dr}\\ &u(1)=0,\quad u\to 1 \text{ as } r\to\infty\end{aligned}}$$

Naive Expansion: $u_{naive} \sim U_0(r) + \varepsilon U_1(r) + \cdots$

$$\Rightarrow \frac{1}{r^2}\left(r^2(U_{0r}+\varepsilon U_{1r})\right)_r = -\varepsilon U_0 U_{0r} + \cdots$$

$O(1)$: $\frac{1}{r^2}(r^2U_{0r})_r = 0$, $U_0(1) = 0$

$$r^2 U_{0r} = c_0 \qquad U_0(1) = -c_0 + c_1 = 0$$
$$U_{0r} = c_0/r^2 \qquad c_1 = c_0$$
$$U_0 = -\frac{c_0}{r} + c_1 \qquad \boxed{U_0(r) = c_0\left(1-\frac{1}{r}\right)}$$

$O(\varepsilon)$: $\frac{1}{r^2}(r^2U_{1r})_r = -U_0U_{0r}$, $U_1(1) = 0$

$$= -c_0\left(1-\frac{1}{r}\right)\cdot c_0\frac{1}{r^2}$$

$$U_1(1) = -c_0^2[0+1-c_2+c_3] = 0$$
$$c_3 = c_2 - 1$$

$$(r^2U_{1r})_r = -c_0^2\left(1-\frac{1}{r}\right)$$
$$r^2 U_{1r} = -c_0^2(r - \ln r + c_2)$$
$$U_{1r} = -c_0^2\left(\frac{1}{r} - \frac{\ln r}{r^2} + \frac{c_2}{r^2}\right)$$
$$U_1 = -c_0^2\left[\ln r - \left(-\frac{\ln r}{r} - \frac{1}{r}\right) - \frac{c_2}{r} + c_3\right]$$
$$\boxed{U_1(r) = -c_0^2\left[\left(1+\frac{1}{r}\right)\ln r + (c_2-1)\left(1-\frac{1}{r}\right)\right]}$$

U_1 is unbounded as $r\to\infty$, so the naive expansion cannot satisfy the boundary condition at infinity.

$\Longrightarrow$ Consider the Far Field.

$$\boxed{u_{naive} \sim c_0\left(1-\frac{1}{r}\right) - \varepsilon c_0^2\left[\left(1+\frac{1}{r}\right)\ln r + (c_2-1)\left(1-\frac{1}{r}\right)\right] + \cdots}$$

To see why the naive expansion fails for large r, rewrite the equation as
$$\underset{①}{u_{rr}} + \left(\underset{②}{\tfrac{2}{r}} + \underset{③}{\varepsilon u}\right) u_r = 0.$$
The naive expansion assumes that terms ① and ② dominate for all r.
However, as $r \to \infty$ term ② becomes smaller than term ③.
$$\frac{②}{③} = \frac{2/r}{\varepsilon u} \to \frac{0}{\varepsilon(1)} = 0 \quad \text{as } r \to \infty.$$

Far Field: $\rho = \varepsilon r$
$$\frac{1}{r^2}(r^2 u_r)_r = -\varepsilon u u_r$$
$$\frac{\cancel{\varepsilon^2}}{\rho^2}\cancel{\varepsilon}\left(\frac{\rho^2}{\cancel{\varepsilon^2}} \cdot \cancel{\varepsilon} u_\rho\right)_\rho = -\cancel{\varepsilon} u \cdot \cancel{\varepsilon} u_\rho$$
$$\frac{1}{\rho^2}(\rho^2 u_\rho)_\rho = -u u_\rho, \quad u \to 1 \text{ as } \rho \to \infty$$

$$u_{far} \sim u_0(\rho) + \varepsilon u_1(\rho) + \cdots$$
$$\Rightarrow \frac{1}{\rho^2}\left(\rho^2(u_{0\rho} + \varepsilon u_{1\rho})\right)_\rho = -(u_0 + \varepsilon u_1)(u_{0\rho} + \varepsilon u_{1\rho}) + \cdots$$

$O(1)$: $\frac{1}{\rho^2}(\rho^2 u_{0\rho})_\rho = -u_0 u_{0\rho}, \quad u_0 \to 1$ as $\rho \to \infty$

Though we have the full equation, it can be solved by inspection. $\Longrightarrow \boxed{u_0(\rho) = 1}$

$O(\varepsilon)$: $\frac{1}{\rho^2}(\rho^2 u_{1\rho})_\rho = -u_{1\rho}, \quad u_1 \to 0$ as $\rho \to \infty$

$$\frac{(\rho^2 u_{1\rho})_\rho}{\rho^2 u_{1\rho}} = -1$$
$$\ln(\rho^2 u_{1\rho}) = -\rho + D_0$$
$$u_{1\rho} = D_0 \frac{e^{-\rho}}{\rho^2}$$
$$\int_\rho^\infty u_{1\rho}\, d\rho = D_0 \int_\rho^\infty \frac{e^{-s}}{s^2}\, ds$$
$$\cancelto{0}{u_1\Big|_{\rho\to\infty}} - u_1(\rho) = D_0 \int_\rho^\infty \frac{e^{-s}}{s^2}\, ds$$
$$u_1(\rho) = -D_0 \int_\rho^\infty \frac{e^{-s}}{s^2}\, ds \Longrightarrow \boxed{u_{far} \sim 1 - \varepsilon D_0 \int_\rho^\infty \frac{e^{-s}}{s^2}\, ds}$$

2-2 Matching: $U_{naive} \sim C_0(1-\frac{1}{r}) - \varepsilon C_0^2\left[(1+\frac{1}{r})\ln r + (C_2-1)(1-\frac{1}{r})\right]$

$$U_{far} \sim 1 - \varepsilon D_0 \int_\rho^\infty \frac{e^{-s}}{s^2}\,ds \qquad \rho = \varepsilon r$$

To match as $\rho \to 0$, we need the known expansion

$$\int_\rho^\infty \frac{e^{-s}}{s^2}\,ds \sim \frac{1}{\rho} + \ln\rho + (\gamma - 1) - \frac{1}{2}\rho + o(\rho) \text{ as } \rho \to 0,$$

where γ = Euler's constant ≈ 0.5772

$$U_{naive} \sim C_0(1-\varepsilon/\rho) - \varepsilon C_0^2\left[(1+\frac{\varepsilon}{\rho})\ln\frac{\rho}{\varepsilon} + (C_2-1)(1-\varepsilon/\rho)\right] \sim C_0(1-\frac{\varepsilon}{\rho}) - \varepsilon C_0^2(\ln\rho + \ln\frac{1}{\varepsilon} + C_2 - 1)$$

$$U_{naive} \sim C_0 - C_0^2\varepsilon\ln\frac{1}{\varepsilon} - C_0^2\varepsilon\left(\frac{1}{C_0\rho} + \ln\rho + C_2 - 1\right)$$

(2-term)

$$U_{far} \sim 1 - \varepsilon D_0 \int_{\varepsilon r}^\infty \frac{e^{-s}}{s^2}\,ds \sim 1 - \varepsilon D_0\left[\frac{1}{\varepsilon r} + \ln(\varepsilon r) + (\gamma - 1)\right]$$

$$\sim 1 - \frac{D_0}{r} - \varepsilon D_0\left(-\ln\frac{1}{\varepsilon} + \ln r + \gamma - 1\right)$$

$$U_{far} \sim 1 - \frac{D_0}{r} + D_0\varepsilon\ln\frac{1}{\varepsilon} - \varepsilon D_0(\ln r + \gamma - 1)$$

(2-term)

Equate: $C_0 - C_0^2\varepsilon\ln\frac{1}{\varepsilon} - C_0^2\varepsilon\left(\frac{1}{C_0\rho} + \ln\rho + C_2 - 1\right) = 1 - \frac{D_0}{r} + D_0\varepsilon\ln\frac{1}{\varepsilon} - \varepsilon D_0(\ln r + \gamma - 1)$

$$C_0 - C_0^2\varepsilon\ln\frac{1}{\varepsilon} - C_0^2\varepsilon\left(\frac{1}{C_0\varepsilon r} + \ln(\varepsilon r) + C_2 - 1\right) = \quad ''$$

$$C_0 - \cancel{C_0^2\varepsilon\ln\frac{1}{\varepsilon}} - \frac{C_0}{r} - \varepsilon C_0^2\left(\cancel{-\ln\frac{1}{\varepsilon}} + \ln r + C_2 - 1\right) = \quad ''$$

$$C_0(1-\frac{1}{r}) - \varepsilon C_0^2(\ln r + C_2 - 1) = 1 - \frac{D_0}{r} + D_0\varepsilon\ln\frac{1}{\varepsilon} - \varepsilon D_0(\ln r + (\gamma - 1))$$

$O(1)$: $C_0(1-\frac{1}{r}) = 1 - \frac{D_0}{r}$

$\Rightarrow C_0 = D_0 = 1 \quad \Longrightarrow -\cancel{\varepsilon}(\cancel{\ln r} + C_2 - \cancel{1}) = \underline{\underline{\cancel{\varepsilon}\ln\frac{1}{\varepsilon}}} - \cancel{\varepsilon}(\cancel{\ln r} + \gamma - \cancel{1})$

This term cannot be matched.

The equation gives $\underline{C_2 = -\ln\frac{1}{\varepsilon} + \gamma}$, which is not asymptotically sound. Integration constants, such as C_2, should be independent of ε.

Though we used Van Dyke matching properly, matching still fails?

Switchback: Revise the naive expansion so that matching is possible. Though matching failed, it reveals the appropriate gauge functions.

Suppose we accept $C_2 = -\ln\frac{1}{\varepsilon} + \gamma$.

Then, $$u_{naive} \sim (1-\tfrac{1}{r}) - \varepsilon\left[(1+\tfrac{1}{r})\ln r + (-\ln\tfrac{1}{\varepsilon} + \gamma - 1)(1-\tfrac{1}{r})\right]$$

$$\sim \underbrace{(1-\tfrac{1}{r})}_{O(1)} + \underbrace{\varepsilon\ln\tfrac{1}{\varepsilon}(1-\tfrac{1}{r})}_{O(\varepsilon\ln\frac{1}{\varepsilon})} - \underbrace{\varepsilon\left[(1+\tfrac{1}{r})\ln r + (\gamma-1)(1-\tfrac{1}{r})\right]}_{O(\varepsilon)}$$

$\Rightarrow$ The appropriate gauge functions are $1, \varepsilon\ln\frac{1}{\varepsilon}, \varepsilon, \ldots$.

Expand u as $\boxed{\bar{U}_{naive} \sim U_0(r) + \varepsilon\ln\tfrac{1}{\varepsilon}\cdot\bar{U}_1(r) + \varepsilon U_1(r) + \cdots}$

$$\Rightarrow \frac{1}{r^2}\left(r^2(U_{0r} + \varepsilon\ln\tfrac{1}{\varepsilon}\bar{U}_{1r} + \varepsilon U_{1r})\right)_r = -\varepsilon U_0 U_{0r}, \quad U_0(1) = \bar{U}_1(1) = U_1(1) = 0$$

$O(1)$: $\frac{1}{r^2}(r^2 U_{0r})_r = 0$, $U_0(1) = 0 \Rightarrow \boxed{U_0(r) = C_0(1-\frac{1}{r})}$

$O(\varepsilon\ln\frac{1}{\varepsilon})$: $\frac{1}{r^2}(r^2\bar{U}_{1r})_r = 0$, $U_1(1) = 0 \Rightarrow \boxed{\bar{U}_1(r) = \bar{C}_1(1-\frac{1}{r})}$

$O(\varepsilon)$: $\frac{1}{r^2}(r^2 U_{1r})_r = -U_0U_{0r} = -C_0^2(1-\frac{1}{r})(\frac{1}{r^2})$, $U_1(1) = 0$

Same as before $\Rightarrow \boxed{U_1(r) = -C_0^2\left[(1+\frac{1}{r})\ln r + (C_2-1)(1-\frac{1}{r})\right]}$

$$\Rightarrow \boxed{\bar{U}_{naive}(r) \sim C_0(1-\tfrac{1}{r}) + \varepsilon\ln\tfrac{1}{\varepsilon}\cdot\bar{C}_1(1-\tfrac{1}{r}) - \varepsilon C_0^2\left[(1+\tfrac{1}{r})\ln r + (C_2-1)(1-\tfrac{1}{r})\right]}$$

- **2-2 Matching:** $\bar{u}_{naive} \sim C_0(1-\frac{1}{r}) + \epsilon\ln\frac{1}{\epsilon}\cdot\bar{C}_1(1-\frac{1}{r}) - \epsilon C_0^2\left[(1+\frac{1}{r})\ln r + (C_2-1)(1-\frac{1}{r})\right]$

  $u_{far} \sim 1 - \epsilon D_0\int_\rho^\infty \frac{e^{-s}}{s^2}ds \qquad \rho = \epsilon r$

$$\bar{u}_{naive} \sim C_0(1-\epsilon/\rho) + \epsilon\ln\tfrac{1}{\epsilon}\cdot\bar{C}_1(1-\epsilon/\rho) - \epsilon C_0^2\left[(1+\epsilon/\rho)\ln\tfrac{\rho}{\epsilon} + (C_2-1)(1-\epsilon/\rho)\right]$$
$$\sim C_0(1-\epsilon/\rho) + \epsilon\ln\tfrac{1}{\epsilon}\cdot\bar{C}_1 - \epsilon C_0^2(\ln\rho + \ln\tfrac{1}{\epsilon} + C_2 - 1)$$
$$\bar{u}_{naive} \sim C_0 + \epsilon\ln\tfrac{1}{\epsilon}(\bar{C}_1 - C_0^2) - \epsilon C_0^2\left(\tfrac{1}{C_0\rho} + \ln\rho + C_2 - 1\right)$$

$$u_{far} \sim 1 - \epsilon D_0\int_{\epsilon r}^\infty \frac{e^{-s}}{s^2}ds \sim 1 - \epsilon D_0\left[\frac{1}{\epsilon r} + \ln(\epsilon r) + (\gamma - 1)\right]$$
$$\sim 1 - \frac{D_0}{r} - \epsilon D_0(-\ln\tfrac{1}{\epsilon} + \ln r + \gamma - 1)$$
$$u_{far} \sim (1-\tfrac{D_0}{r}) + \epsilon\ln\tfrac{1}{\epsilon}\cdot D_0 - \epsilon D_0(\ln r + \gamma - 1)$$

- **Equate:**

$$C_0 + \epsilon\ln\tfrac{1}{\epsilon}\cdot(\bar{C}_1 - C_0^2) - \epsilon C_0^2\left(\tfrac{1}{C_0\rho} + \ln\rho + C_2 - 1\right) = (1-\tfrac{D_0}{r}) + \epsilon\ln\tfrac{1}{\epsilon}\cdot D_0 - \epsilon D_0(\ln r + \gamma - 1)$$
$$C_0 + \epsilon\ln\tfrac{1}{\epsilon}\cdot(\bar{C}_1 - C_0^2) - \epsilon C_0^2\left(\tfrac{1}{C_0\epsilon r} + \ln(\epsilon r) + C_2 - 1\right) = \quad ''$$
$$C_0 + \epsilon\ln\tfrac{1}{\epsilon}\cdot(\bar{C}_1 - \cancel{C_0^2}) - \tfrac{C_0}{r} + \cancel{C_0^2\epsilon\ln\tfrac{1}{\epsilon}} - \epsilon C_0^2(\ln r + C_2 - 1) = \quad ''$$
$$C_0(1-\tfrac{1}{r}) + \epsilon\ln\tfrac{1}{\epsilon}\cdot\bar{C}_1 - \epsilon C_0^2(\ln r + C_2 - 1) = (1-\tfrac{D_0}{r}) + \epsilon\ln\tfrac{1}{\epsilon}\cdot D_0 - \epsilon D_0(\ln r + \gamma - 1)$$

$\underline{O(1)}$: $C_0(1-\frac{1}{r}) = 1 - \frac{D_0}{r} \Rightarrow C_0 = D_0 = 1$

$\underline{O(\epsilon\ln\frac{1}{\epsilon})}$: $\bar{C}_1 = D_0 = 1$ $\quad \bar{C}_1 = 1$

$\underline{O(\epsilon)}$: $C_2 - 1 = \gamma - 1$ $\quad C_2 = \gamma$

$\Rightarrow$ Matching is successful.

- Had matching failed again, we could try additional revisions to the expansions, depending on how the failure occurred. Perhaps the far field expansion may need a $\epsilon\ln\frac{1}{\epsilon}$ term also. If matching continues to fail, the problem may not have a solution.

We have

$$\bar{u}_{naive}(r) \sim \left(1-\tfrac{1}{r}\right) + \varepsilon \ln\tfrac{1}{\varepsilon}\cdot\left(1-\tfrac{1}{r}\right) - \varepsilon\left[\left(1+\tfrac{1}{r}\right)\ln r + (\gamma-1)\left(1-\tfrac{1}{r}\right)\right]$$

$$u_{far}(\rho) \sim 1 - \varepsilon\int_\rho^\infty \frac{e^{-s}}{s^2}\,ds$$

$$\binom{\text{Common}}{\text{Part}} \sim \left(1-\tfrac{1}{r}\right) + \varepsilon\ln\tfrac{1}{\varepsilon} - \varepsilon\left(\ln r + \gamma - 1\right)$$

Composite Expansion

$$u_{comp} = \bar{u}_{naive} + u_{far} - \binom{\text{Common}}{\text{Part}}$$

$$\sim \left\{\cancel{\left(1-\tfrac{1}{r}\right)} + \varepsilon\ln\tfrac{1}{\varepsilon}\cdot\left(\cancel{1}-\tfrac{1}{r}\right) - \varepsilon\left[\left(\cancel{1}+\tfrac{1}{r}\right)\ln r + (\gamma-1)\left(\cancel{1}-\tfrac{1}{r}\right)\right]\right\}$$

$$+ \left\{1 - \varepsilon\int_\rho^\infty \frac{e^{-s}}{s^2}\,ds\right\} - \left\{\cancel{\left(1-\tfrac{1}{r}\right)} + \cancel{\varepsilon\ln\tfrac{1}{\varepsilon}} - \cancel{\varepsilon\left(\ln r + \gamma - 1\right)}\right\}$$

$$\sim -\frac{\varepsilon}{r}\ln\frac{1}{\varepsilon} - \frac{\varepsilon}{r}\ln r + \varepsilon(\gamma-1)\cdot\frac{1}{r} + 1 - \varepsilon\int_\rho^\infty \frac{e^{-s}}{s^2}\,ds$$

$$\boxed{u_{comp}(r) \sim 1 - \frac{\varepsilon}{r}\ln\frac{1}{\varepsilon} - \varepsilon\left[\frac{1}{r}\left(\ln r - \gamma + 1\right) + \int_{\varepsilon r}^\infty \frac{e^{-s}}{s^2}\,ds\right] + \cdots}$$

The expansion is not valid for $r = O(\varepsilon \ln\frac{1}{\varepsilon})$, but it is not an issue since the domain of the problem is $1 \le r < \infty$.

Check:

$r = O(1)$ $(\varepsilon r \ll 1)$ $\Rightarrow$

$$u_{comp} \sim 1 - \frac{\varepsilon}{r}\ln\frac{1}{\varepsilon} - \varepsilon\left[\frac{1}{r}(\ln r - \gamma + 1) + \left(\frac{1}{\varepsilon r} + \ln(\varepsilon r) + \gamma - 1\right)\right]$$

$$\sim \left(1-\tfrac{1}{r}\right) - \frac{\varepsilon}{r}\ln\frac{1}{\varepsilon} - \frac{\varepsilon}{r}(\ln r - \gamma + 1) - \varepsilon\left(\ln r - \ln\tfrac{1}{\varepsilon} + \gamma - 1\right)$$

$$\sim \left(1-\tfrac{1}{r}\right) + \varepsilon\ln\tfrac{1}{\varepsilon}\cdot\left(1-\tfrac{1}{r}\right) - \varepsilon\left[\left(1+\tfrac{1}{r}\right)\ln r + (\gamma-1)\left(1-\tfrac{1}{r}\right)\right] \sim \bar{u}_{naive} \checkmark$$

$r = O(\frac{1}{\varepsilon})$ $(\varepsilon r = O(1))$ $\Rightarrow$

$$u_{comp} \sim 1 - \varepsilon\int_{\varepsilon r}^\infty \frac{e^{-s}}{s^2}\,ds + O\!\left(\varepsilon^2\ln\tfrac{1}{\varepsilon}\right) \sim u_{far} \checkmark$$

Weakly Nonlinear Oscillators

General Form:

$$\boxed{\begin{aligned} &\ddot{x} + x + \varepsilon h(x,\dot{x}) = 0, \quad t > 0\,;\ 0 < \varepsilon \ll 1 \\ &x(0) = A, \quad \dot{x}(0) = B \end{aligned}}$$

$\varepsilon h(x,\dot{x})$ represents a weak nonlinearity

Classical Examples

1. Harmonic Oscillator with a Perturbed Frequency

$\ddot{x} + (1+\varepsilon)x = 0$ (linear)

period $= \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{1+\varepsilon}}$ (perturbed from 2π)

$x(0)=1$, $\dot{x}(0)=0$ $\Rightarrow$ $x(t) = \cos(\sqrt{1+\varepsilon}\, t)$

angular frequency $= \omega = \sqrt{1+\varepsilon}$

2. Linear Spring with Weak Damping

$\ddot{x} + x + 2\varepsilon\dot{x} = 0$

$x(0)=1$, $\dot{x}(0)=0$ $\Rightarrow$ $x(t) = e^{-\varepsilon t}\cos($[redacted] $t)$, with $\sqrt{1-\varepsilon^2}$ written above

$e^{-\varepsilon t}$: slowly decaying amplitude; $\sqrt{1-\varepsilon^2}$: perturbed frequency

3. Duffing Equation

$\ddot{x} + x + \varepsilon x^3 = 0$

e.g. weakly nonlinear spring

$F_s = -kx - cx^3 \Longrightarrow m\ddot{x} + kx + cx^3 = 0$

where c is relatively small

4. van der Pol oscillator

$\ddot{x} + x + \varepsilon(x^2 - 1)\dot{x} = 0$

5. Rayleigh Equation

$\ddot{x} + x - \varepsilon(\dot{x} - \frac{1}{3}\dot{x}^3) = 0$

Perturbation Methods

1. Strained Coordinate Methods (frequency corrections)
2. Multiple Scales Methods (frequency + amplitude corrections, initial/boundary layers, etc.)

Though multiple scale methods are the methods of choice, we'll consider strained coordinate methods as well since such methods appear in scientific literature, and thus, should be understood.

Strained coordinates corrects frequencies, but cannot capture slowly varying amplitudes, as does multiple scales.

Example: Harmonic oscillator with a perturbed frequency

$$\ddot{x}+(1+\varepsilon)x=0\,; \quad x(0)=1, \quad \dot{x}(0)=0$$

Exact Solution

$x(t)=\cos(\sqrt{1+\varepsilon}\,t)$

period $= T_{ex} = \frac{2\pi}{\sqrt{1+\varepsilon}} < 2\pi$

Naive Expansion: $x(t) \sim x_0(t)+\varepsilon x_1(t)+\varepsilon^2 x_2(t)+\cdots$

$$\Rightarrow (\ddot{x}_0+\varepsilon\ddot{x}_1+\varepsilon^2\ddot{x}_2+\cdots)+(1+\varepsilon)(x_0+\varepsilon x_1+\varepsilon^2 x_2+\cdots)=0$$

$O(1)$: $\ddot{x}_0+x_0=0\,;\; x_0(0)=1,\; \dot{x}_0(0)=0$

$$\Rightarrow \boxed{x_0(t)=\cos t}$$

goes out of phase with the exact solution as t becomes large.

period $= T_0 = 2\pi$

$O(\varepsilon)$: $\ddot{x}_1+x_1=-x_0=-\cos t\,;\; x_1(0)=\dot{x}_1(0)=0$

homog. $x_{1h}(t)=C_1\cos t+C_2\sin t$

$-\cos t$ is a Resonant forcing term

particular: $x_{1p}=t[A\cos t+B\sin t]$

$$+\quad \ddot{x}_{1p}=-t[A\cos t+B\sin t]+2[-A\sin t+B\cos t]$$

$$\ddot{x}_{1p}+x_{1p}=2[-A\sin t+B\cos t]=-\cos t$$

$$A=0 \quad B=-\tfrac{1}{2} \quad \Rightarrow \quad x_{1p}(t)=-\frac{t}{2}\sin t$$

$x_1(t)=C_1\cos t+C_2\sin t-\frac{t}{2}\sin t$

$x_1(0)=\dot{x}_1(0)=0 \Rightarrow C_1=C_2=0 \Rightarrow \boxed{x_1(t)=-\frac{t}{2}\sin t}$

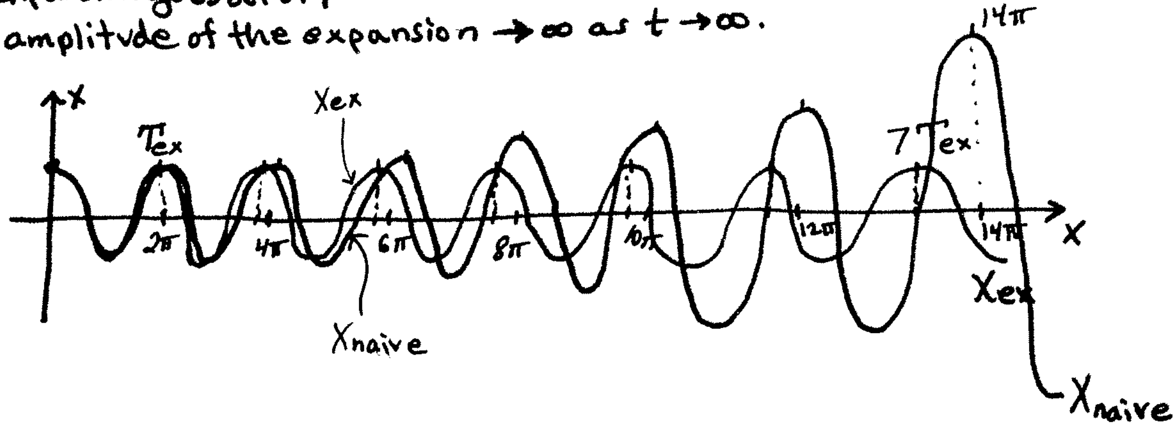
$O(\varepsilon^2)$: $\ddot{x}_2+x_2=-x_1=\frac{t}{2}\sin t\,;\; x_2(0)=\dot{x}_2(0)=0$

$$\Rightarrow \boxed{x_2(t)=-\frac{1}{8}(t^2\cos t-t\sin t)}$$

$$\Rightarrow \boxed{x(t)\sim \cos t-\varepsilon\frac{t}{2}\sin t-\frac{\varepsilon^2}{8}(t^2\cos t-t\sin t)+\cdots}$$

Not valid for $t=O(1/\varepsilon)$ and larger

The expansion goes out of phase with the exact solution as t becomes large. The amplitude of the expansion $\to\infty$ as $t\to\infty$.



$$X_{naive}(t) \sim \cos t - \frac{\varepsilon t}{2}\sin t - \frac{\varepsilon^2}{8}(t^2\cos t - t\sin t) + \cdots$$

Exact Solution: $X_{ex}(t) = \cos(\sqrt{1+\varepsilon}\, t)$

~~X_{ex} and X_0 both oscillate with an amplitude of one~~, but their frequencies differ slightly. Thus, X_{ex} and X_0 will slowly go out of phase. The cummulative phase error becomes significant when $t = O(\frac{1}{\varepsilon})$.

Taylor expand $X_{ex}(t)$ for small ε, with t fixed $(t = O(1))$.

$$X_{ex}(t) = \cos(\sqrt{1+\varepsilon}\, t) \sim \cos\left(\left(1 + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + \cdots\right)t\right) \sim \cos\left(t + \left(\frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + \cdots\right)t\right)$$

$$\sim \cos t \cos\left(\left(\frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + \cdots\right)t\right) - \sin t \sin\left(\left(\frac{\varepsilon}{2} - \frac{\varepsilon^2}{8} + \cdots\right)t\right)$$

$$\sim \cos t\left[1 - \frac{(\varepsilon t/2)^2}{2!} + \cdots\right] - \sin t\left[\left(\frac{\varepsilon}{2} - \frac{\varepsilon^2}{8}\right)t + \cdots\right]$$

$$X_{ex}(t) \sim \cos t - \frac{\varepsilon t}{2}\sin t - \frac{\varepsilon^2}{8}(t^2\cos t - t\sin t) + \cdots$$

Notes:

1) The naive expansion agrees with the Taylor expansion of the exact solution, as it should.

2) Though the infinite series associated with the naive expansion converges to the exact solution for all t, a truncation of the naive expansion gives an asymptotic approximation which is not valid when $t = O(\frac{1}{\varepsilon})$ and larger. That is, the naive expansion converges for all t, but it is asymptotic only when $t = O(1)$.

3) Terms (such as $\frac{t}{2}\sin t$) which yield nonuniformities in the naive expansion are called <u>secular</u> terms.

Perturbation Methods aim to "suppress secular terms".

The naive expansion fails for large t because it is not capable of making frequency corrections.

$\omega =$ angular frequency

$$\omega_{ex} = \sqrt{1+\varepsilon} \sim 1 + \varepsilon/2 - \varepsilon^2/8 + \cdots$$

$$\omega_{naive} = 1 = \omega_{ex} + O(\varepsilon)$$ at all orders of the expansion.

The solution oscillates on an $O(1)$ timescale, whereas the perturbed frequency acts on an $O(\frac{1}{\varepsilon})$ time scale. The naive expansion can only capture behavior which occurs on an $O(1)$ time scale.

One fix is to rewrite the naive expansion as follows.

$$X_{naive}(t) \sim \cos t - \frac{\varepsilon t}{2}\sin t + O(\varepsilon^2)$$

$$\sim \cos t\,(\underbrace{1 + O(\varepsilon^2)}_{\sim}) - \sin t\,(\underbrace{\tfrac{\varepsilon t}{2} + O(\varepsilon^2)}_{\sim}) + O(\varepsilon^2)$$

$$\sim \cos t \cos(\varepsilon t/2) - \sin t \sin(\varepsilon t/2) + O(\varepsilon^2)$$

$$\Rightarrow \boxed{X_{naive}(t) \sim \cos((1+\varepsilon/2)t)} \quad \boxed{\omega = 1 + \frac{\varepsilon}{2} = \omega_{ex} + O(\varepsilon^2)}$$

The approximation is now valid for $t = O(\frac{1}{\varepsilon})$, but not for $t = O(\frac{1}{\varepsilon^2})$. We extended the region of validity, but the frequency is still not exact. The region of validity can be extended as far as we wish by finding more terms of the naive expansion and combining them appropriately. However, the region of validity cannot be extended to infinity since that would require infinitely many terms.

Renormalization: Find the naive expansion and 'algebraically' combine the nonuniform terms with lower order terms to make frequency corrections, as above.

This can be tedious when applied to more complicated problems, or when more terms are required.

There are more efficient methods for making frequency corrections.

Without knowing the exact frequency of a solution it is impossible to find an asymptotic approximation which is uniformly valid for all $t > 0$. However, the more accurate is the approximation of the frequency, the larger is the time interval of validity. The naive expansion captures only the leading order approximation of the frequency ($\omega_{naive} = 1 = \omega_{ex} + O(\varepsilon)$) at all orders of the expansion. It makes sense to determine an asymptotic approximation of ω_{ex}.

Idea: Retain the naive expansion, but scale time in such a way that the exact solution X_{ex} has a frequency of exactly 1 (period $= 2\pi$) in the new coordinate system.

Let $\boxed{T = \omega t}$

($\omega = \omega_{ex}$ = angular frequency of the exact solution),

t-coordinates	T-coordinates
$X_{ex}(t) = \cos(\sqrt{1+\varepsilon}\, t)$	$X_{ex}(T) = \cos T$
$\omega_{ex} = \sqrt{1+\varepsilon}$	$\omega_{ex} = 1$
period $= \dfrac{2\pi}{\sqrt{1+\varepsilon}}$	period $= 2\pi$

and expand ω as

$$\boxed{\omega \sim \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots}.$$

Then determine $\omega_1, \omega_2, \ldots$ so as to suppress secularities. Consider the naive expansion $X(T) \sim X_0(T) + \varepsilon X_1(T) + \cdots$, and choose $\omega_1, \omega_2, \ldots$ in such a way to eliminate secular terms such as $T\sin T$, $T^2\cos T$, etc.

Recall the Example: Clearly, $\omega_0 = 1$.

$$X_{naive}(t) \sim \cos t - \varepsilon \frac{t}{2}\sin t + \cdots \sim \cos\big((\underbrace{1}_{\omega_0 = 1} + \underbrace{\tfrac{\varepsilon}{2}}_{\omega_1 = \frac{1}{2}})\, t\big) + \cdots$$

The secularity is suppressed by picking $\omega_1 = \frac{1}{2}$.

In terms of T-coordinates,

$$X_{naive}(T) \sim \cos T + \cdots,$$

where $T = \omega t \sim (1 + \frac{\varepsilon}{2} + \cdots)\, t$.

Example: $\ddot{x} + (1+\varepsilon)x = 0$; $x(0) = 1$, $\dot{x}(0) = 0$

To leading order, $\ddot{x} + x = 0 \Rightarrow \omega_0 = 1$

Let $\boxed{T = \omega t}$, where $\omega \sim 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots$

Then, $\dot{x} = \omega x_T$

$$\ddot{x} = \omega^2 x_{TT} \sim (1 + 2\varepsilon\omega_1 + \varepsilon^2(\omega_1^2 + 2\omega_2) + \cdots) x_{TT}$$

$$\Rightarrow (1 + 2\varepsilon\omega_1 + \varepsilon^2(\omega_1^2 + 2\omega_2)) x_{TT} + (1+\varepsilon)x = 0 \,;\; x(0) = 1,\; x_T(0) = 0$$

Naive Expansion: $x \sim x_0(T) + \varepsilon x_1(T) + \varepsilon^2 x_2(T) + \cdots$

$$\Rightarrow (1 + 2\varepsilon\omega_1 + \varepsilon^2(\omega_1^2 + 2\omega_2))(x_{0TT} + \varepsilon x_{1TT} + \varepsilon^2 x_{2TT}) + (1+\varepsilon)(x_0 + \varepsilon x_1 + \varepsilon^2 x_2) \sim 0$$

$O(1)$: $x_{0TT} + x_0 = 0$; $x_0(0) = 1$, $x_{0T}(0) = 0$

$$\Rightarrow \boxed{x_0(T) = \cos T}$$

$O(\varepsilon)$: $x_{1TT} + x_1 = -2\omega_1 x_{0TT} - x_0 = 2\omega_1 \cos T - \cos T$

$$x_{1TT} + x_1 = (2\omega_1 - 1)\cos T$$

Leads to secular terms since $\cos T$ is a solution of the associated homogeneous equation.

Pick ω_1 to suppress the secularity

$$\Rightarrow \omega_1 = \frac{1}{2} \Rightarrow x_{1TT} + x_1 = 0 \,;\; x_1(0) = x_{1T}(0) = 0$$

$$\Rightarrow \boxed{x_1(T) = 0}$$

$O(\varepsilon^2)$: $x_{2TT} + x_2 = \cancel{-2\omega_1 x_{1TT}} - (\omega_1^2 + 2\omega_2)x_{0TT} - \cancel{x_1} = (\omega_1^2 + 2\omega_2)\cos T$

$$x_{2TT} + x_2 = \left(\tfrac{1}{4} + 2\omega_2\right)\cos T$$

suppress the secularities $\Rightarrow \boxed{\omega_2 = -\frac{1}{8}}$

$$\Rightarrow \omega \sim 1 + \varepsilon/2 - \varepsilon^2/8 + \cdots$$

$$x \sim x_0(T) + \varepsilon \cancelto{0}{x_1(T)} + \cdots \sim \cos T + O(\varepsilon^2)$$

$$\boxed{x(t) \sim \cos\left(\left(1 + \varepsilon/2 - \varepsilon^2/8\right)t\right) + O(\varepsilon^2)}$$

Not valid for $t = O(\frac{1}{\varepsilon^3})$ and larger

Taylor expansion of $\omega_{ex} = \sqrt{1+\varepsilon}$

n-Term Expansions / Accuracy and Validity

Convention: An n-term expansion of a solution is considered to be one which has an error $O(\varepsilon^n)$ and is valid for $t = O(1/\varepsilon^n)$.

Consider $\ddot{x} + \omega_0^2 x + \varepsilon h(x, \dot{x}) = 0$

Let $T = \omega t$, $\omega \sim \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots$

$$\Rightarrow \omega^2 x_{TT} + \omega_0^2 x + \varepsilon h(x, \omega x_T) = 0$$

Naive Expansion: $x \sim x_0(T) + \varepsilon x_1(T) + \varepsilon^2 x_2(T) + \cdots$

$O(1)$: $\omega_0^2 x_{0TT} + \omega_0^2 x_0 = 0 \Rightarrow x_0(T) = \cdots$

$x(t) \sim x_0(T) + O(\varepsilon) \sim x_0(\omega_0 t) + O(\varepsilon)$

error $= O(\varepsilon)$
Validity: $t = O(1)$

Not sufficient for a 1-term expansion

$O(\varepsilon)$: $\omega_0^2 x_{1TT} + \omega_0^2 x_1 = f_1(x_0, x_{0T}, x_{0TT}; \omega_1)$

Pick ω_1 to suppress secularities

$\Rightarrow \omega_1 = \cdots$

$$\Rightarrow \boxed{x(t) \sim x_0\big((\omega_0 + \varepsilon\omega_1)t\big) + O(\varepsilon)}$$

error $= O(\varepsilon)$
validity: $t = O(1/\varepsilon)$

One-Term Expansion

Solve for $x_1 \Rightarrow x_1(T) = \cdots$

$x(t) \sim x_0\big((\omega_0 + \varepsilon\omega_1)t\big) + \varepsilon x_1\big((\omega_0 + \varepsilon\omega_1)t\big) + O(\varepsilon^2)$

error $= O(\varepsilon^2)$
Validity: $t = O(1/\varepsilon)$

Not sufficient for a 2-term expansion

$O(\varepsilon^2)$: $\omega_0^2 x_{2TT} + \omega_0^2 x_2 = f_2(x_0, x_{0T}, x_{0TT}, x_1, x_{1T}, x_{1TT}; \omega_2)$

Pick ω_2 to suppress secularities

$\Rightarrow \omega_2 = \cdots$

$$\Rightarrow \boxed{x(t) \sim x_0\big((\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2)t\big) + \varepsilon x_1\big((\omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2)t\big) + O(\varepsilon^2)}$$

error $= O(\varepsilon^2)$
validity: $t = O(1/\varepsilon^2)$

Two-Term Expansion

Next Term: Find x_2 and ω_3 for a 3-term expansion.

To find an n-term expansion, x_{n-1} and ω_n must be determined.

Perturbation Methods

I. Renormalization

Determine the naive expansion and eliminate the secularities through algebraic manipulation.

II. Strained Coordinates

A. Poincaré – Lindstedt (PL) Method

ω = angular frequency of the exact solution.

Let $\boxed{T = \omega t, \quad \omega \sim \omega_0 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots}$ $\Rightarrow$ $X_{ex}(T)$ has a period of 2π.

$$\boxed{X \sim X_0(T) + \varepsilon X_1(T) + \varepsilon^2 X_2(T) + \cdots}$$

Pick ω_n's to suppress secularities

PL works only for oscillatory solutions with constant amplitude.

B. Poincaré – Lighthill – Kao (PLK) Method

Expand t as $\boxed{t \sim T + \varepsilon S_1(T) + \varepsilon^2 S_2(T) + \cdots}$

$$\boxed{X \sim X_0(T) + \varepsilon X_1(T) + \varepsilon^2 X_2(T) + \cdots}$$

Pick $S_n(T)$'s to suppress secularities

PLK is more versitile than PL.

PLK can be used for problems with non-oscillatory solutions.

PLK can be very tedious to use.

III. Multiple Scales (many variations)

A. Define two time scales: $\boxed{T = (1+\varepsilon^2\omega_2 + \varepsilon^3\omega_3 + \cdots)\, t}$ ← strained time

$\boxed{\tau = \varepsilon t}$ ← slow time

(This choice is appropriate for problems with solutions such as $x(t) = e^{-\varepsilon t}\cos(\sqrt{1-\varepsilon^2}\, t)$)

Consider T and τ to be independent variables.
(Clearly not true, but it works)

Write the ODE as a PDE, with the independent variables being T and τ, and expand the solution as

$$\boxed{x \sim x_0(T,\tau) + \varepsilon x_1(T,\tau) + \varepsilon^2 x_2(T,\tau) + \cdots}$$

This approach allows us the freedom to suppress secularities.

B. A problem may call for the following expansions:

$$\boxed{T = (1+\varepsilon\omega_1 + \varepsilon^3\omega_3 + \varepsilon^4\omega_4 + \cdots)\, t}$$

$$\boxed{\tau = \varepsilon^2 t}$$

$$\boxed{x \sim x_0(T,\tau) + \varepsilon x_1(T,\tau) + \varepsilon^2 x_2(T,\tau) + \cdots}$$

(This choice is appropriate for problems with solutions such as $x(t) = e^{-\varepsilon^2 t}\cos(\sqrt{1+\varepsilon}\, t)$)

C. Define $N+1$ time scales (N may be infinite)

$$\boxed{T_n = \varepsilon^n t\ ;\ n = 0, 1, 2, \ldots, N}$$

$\Rightarrow$ Get a PDE with $N+1$ independent variables.

$$\boxed{x \sim x_0(T_0, T_1, \ldots, T_N) + \varepsilon x_1(T_0, T_1, \ldots, T_N) + \cdots}$$

D. etc.

Notes: 1) C. is most general, but A. is most commonly used.

2) For a leading order solution approximation, A. reduces to $\boxed{\begin{matrix} T = t \\ \tau = \varepsilon t \end{matrix}}$.

3) Multiple scales methods work for a variety of problems which involve more than one scale in the independent variable.

e.g. – weakly nonlinear oscillators
– boundary layer problems
– far field problems

Compare the following methods for the Duffing Equation.

0. Naive Expansion
I. Renormalization
IIA. PL Strained Coordinates
IIB. PLK Strained Coordinates
IIIA. Multiple Scales

Duffing Equation: $\ddot{x} + x + \varepsilon x^3 = 0$; $x(0)=\alpha$, $\dot{x}(0)=0$

0. Naive Expansion:

$x \sim x_0(t) + \varepsilon x_1(t) + \cdots$

$\Rightarrow (\ddot{x}_0 + \varepsilon\ddot{x}_1) + (x_0 + \varepsilon x_1) + \varepsilon x_0^3 + \cdots = 0$

$O(1)$: $\ddot{x}_0 + x_0 = 0$; $x_0(0)=\alpha$, $\dot{x}_0(0)=0$

$\Rightarrow x_0(t) = \alpha\cos t$

$O(\varepsilon)$: $\ddot{x}_1 + x_1 = -x_0^3$; $x_1(0) = \dot{x}_1(0) = 0$

$\ddot{x}_1 + x_1 = -\alpha^3\cos^3 t = -\frac{\alpha^3}{4}(3\cos t + \cos(3t))$

$\Rightarrow x_1(t) = -\frac{\alpha^3}{32}\left[\cos t + 12t\sin t - \cos(3t)\right]$

$$\boxed{x(t) \sim \alpha\cos t - \varepsilon\frac{\alpha^3}{32}\left[\cos t + 12t\sin t - \cos(3t)\right] + \cdots}$$

Not Valid for $t = O(1/\varepsilon)$ (secular term)

I. Renormalization:

Combine the secular term with the leading order term.

$$x(t) \sim \underbrace{\left[\alpha\cos t - \varepsilon\frac{3\alpha^3}{8}t\sin t\right]}_{} - \varepsilon\frac{\alpha^3}{32}\left[\cos t - \cos(3t)\right] + \cdots$$

$$\boxed{x(t) \sim \alpha\cos\left(\left(1 - \varepsilon\frac{3\alpha^2}{8}\right)t\right) + O(\varepsilon)}$$

Valid for $t = O(1/\varepsilon)$, but not for $t = O(1/\varepsilon^2)$.

The $O(\varepsilon)$ term cannot be kept unless the secularities are suppressed at $O(\varepsilon^2)$. To do so, three terms of the naive expansion are needed.

IIA. Poincaré-Lindstedt (PL) Method of Strained Coordinates

Consider the general weakly nonlinear oscillator: $\ddot{x} + \omega_0^2 x + \varepsilon h(x, \dot{x}) = 0$; $x(0) = A$, $\dot{x}(0) = B$

Let $\boxed{\tau = \omega t}$ and $\boxed{\omega \sim \omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots}$.

Then,

$$\dot{x} = \omega x_\tau \sim (\omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots) x_\tau$$

$$\ddot{x} = \omega^2 x_{\tau\tau} \sim \left(\omega_0^2 + 2\varepsilon \omega_0 \omega_1 + \varepsilon^2(\omega_1^2 + 2\omega_0 \omega_2) + \cdots\right) x_{\tau\tau}$$

$$\Rightarrow \underline{\omega^2 x_{\tau\tau} + \omega_0^2 x + \varepsilon h(x, \omega x_\tau) = 0} \; ; \quad x(0) = A, \quad \omega x_\tau(0) = B$$

Expand x: $\boxed{x \sim x_0(\tau) + \varepsilon x_1(\tau) + \varepsilon^2 x_2(\tau) + \cdots}$

$$\Rightarrow \left(\omega_0^2 + 2\varepsilon \omega_0 \omega_1 + \varepsilon^2(\omega_1^2 + 2\omega_0\omega_2)\right)\left(x_{0\tau\tau} + \varepsilon x_{1\tau\tau} + \varepsilon^2 x_{2\tau\tau}\right) + \omega_0^2\left(x_0 + \varepsilon x_1 + \varepsilon^2 x_2\right)$$
$$+ \varepsilon h\left(x_0 + \varepsilon x_1, (\omega_0 + \varepsilon \omega_1)(x_{0\tau} + \varepsilon x_{1\tau})\right) + \cdots = 0$$

Initial Conditions:

$$x(0) \sim x_0(0) + \varepsilon x_1(0) + \varepsilon^2 x_2(0) + \cdots = A$$

$$\Rightarrow \boxed{x_0(0) = A \; ; \; x_n(0) = 0, \; n = 1, 2, \ldots}$$

$$\omega x_\tau(0) = (\omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots)(x_{0\tau}(0) + \varepsilon x_{1\tau}(0) + \varepsilon^2 x_{2\tau}(0) + \cdots) = B$$

$\underline{O(1)}$: $\omega_0 x_{0\tau}(0) = B \Longrightarrow \boxed{x_{0\tau}(0) = B/\omega_0}$

$\underline{O(\varepsilon)}$: $\omega_0 x_{1\tau}(0) + \omega_1 x_{0\tau}(0) = 0$

$$x_{1\tau}(0) = -\frac{\omega_1}{\omega_0} \cdot \frac{B}{\omega_0} \Longrightarrow \boxed{x_{1\tau}(0) = -\frac{\omega_1 B}{\omega_0^2}}$$

$\underline{O(\varepsilon^2)}$: $\omega_0 x_{2\tau}(0) + \omega_1 x_{1\tau}(0) + \omega_2 x_{0\tau}(0) = 0$

$$\omega_0 x_{2\tau}(0) + \omega_1\left(\frac{-\omega_1 B}{\omega_0^2}\right) + \omega_2\left(\frac{B}{\omega_0}\right) = 0$$

$$\Longrightarrow \boxed{x_{2\tau}(0) = \left(-\omega_2 + \frac{\omega_1^2}{\omega_0}\right) B/\omega_0^2}$$

Recall: $\ddot{x} + x + \epsilon x^3 = 0$; $x(0) = \alpha$, $\dot{x}(0) = 0$ $\rightarrow$ $\boxed{\omega_0 = 1}$

Let $\boxed{T = \omega t}$, $\boxed{\omega \sim \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \cdots}$,

and $\boxed{x \sim X_0(T) + \epsilon X_1(T) + \epsilon^2 X_2(T) + \cdots}$.

$$\Rightarrow \left(1 + 2\epsilon\omega_1 + \epsilon^2(\omega_1^2 + 2\omega_2)\right)\left(X_{0TT} + \epsilon X_{1TT} + \epsilon^2 X_{2TT}\right) + \left(X_0 + \epsilon X_1 + \epsilon^2 X_2\right) + \epsilon\left(X_0^3 + 3\epsilon X_0^2 X_1\right) + \cdots = 0$$

$O(1)$: $X_{0TT} + X_0 = 0$; $X(0) = \alpha$, $X_{0T}(0) = 0$

$$\Rightarrow \boxed{X_0(T) = \alpha\cos T}$$

$O(\epsilon)$: $X_{1TT} + X_1 = -2\omega_1 X_{0TT} - X_0^3$; $X_1(0) = X_{1T}(0) = 0$

$$= -2\omega_1(-\alpha\cos T) - \alpha^3\cos^3 T$$

$$= 2\alpha\omega_1\cos T - \frac{\alpha^3}{4}\left(3\cos T + \cos(3T)\right)$$

$$X_{1TT} + X_1 = \underbrace{\left(2\alpha\omega_1 - \frac{3\alpha^3}{4}\right)\cos T}_{\text{Secular}} - \frac{\alpha^3}{4}\cos(3T)$$

Secular $\Rightarrow$ $\boxed{\omega_1 = \frac{3\alpha^2}{8}}$

$$\Rightarrow \boxed{x(t) \sim \alpha\cos T \sim \alpha\cos\left(\left(1 + \epsilon\frac{3\alpha^2}{8}\right)t\right) + O(\epsilon)} \quad \left(\text{Leading Order}\right)$$

Not Valid for $t = O(1/\epsilon^2)$

Higher Order: Find $X_1(T)$ and ω_2.

$$X_{1TT} + X_1 = -\frac{\alpha^3}{4}\cos(3T) \; ; \quad X_1(0) = X_{1T}(0) = 0$$

$$\Rightarrow \boxed{X_1(T) = -\frac{\alpha^3}{32}\left(\cos T - \cos(3T)\right)}$$

Pick ω_2 to suppress secularities at $O(\epsilon^2)$.

$O(\varepsilon^2)$: $X_{2TT} + X_2 = -2\omega_1 X_{1TT} - (\omega_1^2 + 2\omega_2) X_{0TT} - 3X_0^2 X_1$; $X_2(0) = X_{2T}(0) = 0$

$$= -2\omega_1 \cdot \frac{\alpha^3}{32}\left(\cos T - 9\cos(3T)\right) - (\omega_1^2 + 2\omega_2)(-\alpha\cos T)$$
$$- 3(\alpha^2\cos^2 T) \cdot -\frac{\alpha^3}{32}\left(\cos T - \cos(3T)\right)$$

Note: To identify the secular terms, trig. functions must be reduced to first powers.

$$= -\frac{\omega_1\alpha^3}{16}\left(\cos T - 9\cos(3T)\right) + (\omega_1^2 + 2\omega_2)(\alpha\cos T)$$
$$+ \frac{3\alpha^5}{32}\Big[\underbrace{\tfrac{1}{4}\left(3\cos T + \cos(3T)\right)}_{\cos^3 T} \underbrace{- \tfrac{1}{2}\cos(3T) - \tfrac{1}{4}\left(\cos T + \cos 5T\right)}_{-\cos^2 T\cos(3T)}\Big]$$

$$X_{2TT} + X_2 = \underbrace{\left[-\frac{\omega_1\alpha^3}{16} + \alpha(\omega_1^2 + 2\omega_2) + \frac{3\alpha^5}{32}\left(\frac{3}{4} - \frac{1}{4}\right)\right]}_{=0}\cos T + \text{h.h.}$$

(higher harmonics e.g. $\cos 3T$, $\cos 5T$ (non-secular))

Pick ω_2 to suppress secularities

$$\Rightarrow -\frac{\omega_1\alpha^3}{16} + \alpha(\omega_1^2 + 2\omega_2) + \frac{3\alpha^5}{32}\left(\frac{3}{4} - \frac{1}{4}\right) = 0$$

$\omega_1 = \frac{3\alpha^2}{8}$ $\Rightarrow$ $$-\frac{\alpha^3}{16}\left(\frac{3\alpha^2}{8}\right) + \alpha\left(\frac{3\alpha^2}{8}\right)^2 + 2\alpha\omega_2 + \frac{3\alpha^5}{64} = 0$$

$$\frac{\alpha^5}{128}(-3 + 18 + 6) + 2\alpha\omega_2 = 0$$

$$\Rightarrow \boxed{\omega_2 = \frac{-21\alpha^4}{256}}$$

Then,

$$X \sim X_0(T) + \varepsilon X_1(T) + \cdots, \quad T \sim \left(1 + \varepsilon\frac{3\alpha^2}{8} - \varepsilon^2\frac{21\alpha^4}{256} + \cdots\right)t$$

$$\Rightarrow \boxed{X(t) \sim \alpha\cos(\hat{\omega}t) - \varepsilon\frac{\alpha^3}{32}\left[\cos(\hat{\omega}t) - \cos(3\hat{\omega}t)\right] + O(\varepsilon^2)}$$ Valid for $t = O(\frac{1}{\varepsilon^2})$ but not for $t = O(\frac{1}{\varepsilon^3})$.

where $\boxed{\hat{\omega} = 1 + \varepsilon\frac{3\alpha^2}{8} + \varepsilon^2\frac{21\alpha^4}{256}}$

Need to suppress secularities at $O(\varepsilon^3)$ to include the $O(\varepsilon^2)$ term.

Typically, one term is sufficient. The approximation $X(t) \sim \alpha\cos\left(\left(1 + \frac{3\alpha^2}{8}\right)t\right)$ captures the behavior on both time scales. Though the approximation eventually goes out of phase with the exact solution, it retains the behavior of the exact solution. The phase of the solution at large times is usually not important. It is the amplitude and the frequency which are of interest. With the frequency correction, the leading order approximation captures the amplitude and frequency with reasonable accuracy.

II B. Poincaré-Lighthill-Kao (PLK) Method of Strained Coordinates

Expand time as $\boxed{t \sim T + \varepsilon S_1(T) + \varepsilon^2 S_2(T) + \cdots}$

where $S_n(0) = 0$ for each n. $\Rightarrow T = 0$ when $t = 0$.

The $S_n(T)$'s are chosen to suppress secularities.

$$\frac{dt}{dT} \sim 1 + \varepsilon S_{1T} + \varepsilon^2 S_{2T} + \cdots$$

$$\frac{dT}{dt} \sim \frac{1}{1 + (\varepsilon S_{1T} + \varepsilon^2 S_{2T} + \cdots)} \sim 1 - (\varepsilon S_{1T} + \varepsilon^2 S_{2T} + \cdots) + (\varepsilon S_{1T} + \cdots)^2 + \cdots$$

$$\frac{dT}{dt} \sim 1 - \varepsilon S_{1T} + \varepsilon^2 (S_{1T}^2 - S_{2T}) + \cdots$$

Then, $\dot{x} = x_T \frac{dT}{dt} \sim \left[1 - \varepsilon S_{1T} + \varepsilon^2 (S_{1T}^2 - S_{2T}) + \cdots\right] x_T$

$$\ddot{x} = \frac{d\dot{x}}{dt} \sim \frac{d}{dt}\left\{\left[1 - \varepsilon S_{1T} + \varepsilon^2 (S_{1T}^2 - S_{2T}) + \cdots\right] x_T\right\}$$

To $O(\varepsilon)$, $\ddot{x} \sim \frac{d}{dt}\left[(1 - \varepsilon S_1) x_T\right] \sim \left[(1 - \varepsilon S_{1T}) x_T\right]_T \frac{dT}{dt}$

$$\sim (1 - \varepsilon S_{1T})\left[(1 - \varepsilon S_{1T}) x_{TT} - \varepsilon S_{1TT} x_T\right]$$

$$\sim (1 - 2\varepsilon S_{1T}) x_{TT} - \varepsilon S_{1TT} x_T$$

$$\ddot{x} \sim x_{TT} - \varepsilon (2 S_{1T} x_{TT} + S_{1TT} x_T) + \cdots$$

Initial Conditions: $x(0) = A \Rightarrow x(0) = A$

$\dot{x}(0) = B \Rightarrow \dot{x}(0) = \frac{dT}{dt}(0)\, x_T(0) \sim (1 - \varepsilon S_{1T}(0))\, x_T(0) = B$

$$x_T(0) \sim \frac{B}{1 - \varepsilon S_{1T}(0)} \sim B(1 + \varepsilon S_{1T}(0))$$

$$x_T(0) \sim B(1 + \varepsilon S_{1T}(0)) + \cdots$$

Expand x as $\boxed{x \sim x_0(T) + \varepsilon x_1(T) + \cdots}$

$$\Rightarrow x_0(0) = A, \quad x_1(0) = 0$$

$$x_{0T}(0) = B, \quad x_{1T}(0) = B S_{1T}(0)$$

Recall: $\ddot{x} + x + \varepsilon x^3 = 0$; $\begin{array}{l} x(0)=\alpha \\ \dot{x}(0)=0 \end{array}$

Let $\boxed{t \sim T + \varepsilon S_1(T) + \cdots}$

$$\Rightarrow x_{TT} - \varepsilon(2S_{1T}x_{TT} + S_{1TT}x_T) + x + \varepsilon x^3 + \cdots = 0; \quad \begin{array}{l} x(0)=\alpha \\ x_T(0)=0 \end{array}$$

Expand x: $\boxed{x \sim x_0(T) + \varepsilon x_1(T) + \cdots}$.

$$\Rightarrow (x_{0TT} + \varepsilon x_{1TT}) - \varepsilon(2S_{1T}x_{0TT} + S_{1TT}x_{0T}) + (x_0 + \varepsilon x_1) + \varepsilon x_0^3 + \cdots = 0$$

$O(1)$: $x_{0TT} + x_0 = 0$; $x_0(0) = \alpha$, $x_{0T}(0) = 0$

$$\Rightarrow \boxed{x_0(T) = \alpha\cos T}$$

$O(\varepsilon)$:
$$\begin{aligned} x_{1TT} + x_1 &= 2S_{1T}x_{0TT} + S_{1TT}x_{0T} - x_0^3 \\ &= 2S_{1T}(-\alpha\cos T) + S_{1TT}(-\alpha\sin T) - \alpha^3\cos^3 T \\ &= -2\alpha S_{1T}\cos T - \alpha S_{1TT}\sin T - \frac{\alpha^3}{4}(3\cos T + \cos(3T)) \\ &= \underbrace{\left(-2\alpha S_{1T} - \frac{3\alpha^3}{4}\right)}_{=0}\cos T - \underbrace{\alpha S_{1TT}}_{=0}\sin T + \text{h.h.} \end{aligned}$$

(non-secular terms)

Pick $S_1(T)$ to suppress secularities.

$\cos T$: $-2\alpha S_{1T} - \frac{3\alpha^3}{4} = 0$ $\quad$ $\sin T$: $-\alpha S_{1TT} = 0$

$\boxed{S_{1T} = \frac{-3\alpha^2}{8}}$ $\quad$ $\boxed{S_{1TT} = 0}$

$S_1(0) = 0$

$$\Rightarrow \boxed{S_1(T) = -\frac{3\alpha^2}{8}T}$$

Then, $t \sim T + \varepsilon S_1(T) \sim T\left(1 - \varepsilon\frac{3\alpha^2}{8}\right)$

$$T \sim \frac{t}{1 - \varepsilon\frac{3\alpha^2}{8}} \sim \left(1 + \varepsilon\frac{3\alpha^2}{8}\right)t$$

$$\boxed{T \sim \left(1 + \varepsilon\frac{3\alpha^2}{8}\right)t}$$

$$x \sim x_0(T) \sim x_0\left(\left(1 + \varepsilon\frac{3\alpha^2}{8}\right)t\right)$$

$$\Rightarrow \boxed{x(t) \sim \alpha\cos\left(\left(1 + \varepsilon\frac{3\alpha^2}{8}\right)t\right) + O(\varepsilon)}$$

Valid for $t = O(\frac{1}{\varepsilon})$, but not for $t = O(\frac{1}{\varepsilon^2})$.

Agrees with previous results.

III A. Multiple Scales

Introduce two time scales,

1. Fast Time: $T \sim (1 + \varepsilon^2\omega_2 + \varepsilon^3\omega_3 + \cdots)t$
2. Slow Time: $\tau = \varepsilon t$,

and consider $x(t)$ to be a composite function, $x(t) = X(T(t), \tau(t))$.
Treat T and τ as independent variables (though they are clearly dependent since $\tau \sim$ constant $\times T$) and write the ODE as a PDE with T and τ being the independent variables.

Expand x as $x \sim X_0(T,\tau) + \varepsilon X_1(T,\tau) + \cdots$.

Leading order: $\boxed{\begin{matrix} T \sim t \\ \tau = \varepsilon t \end{matrix}}$ $\quad x \sim X_0(T,\tau) + \varepsilon X_1(T,\tau) + \cdots$

Then, $\dfrac{d}{dt} = \dfrac{dT}{dt}\cdot\dfrac{\partial}{\partial T} + \dfrac{d\tau}{dt}\cdot\dfrac{\partial}{\partial \tau} \sim \dfrac{\partial}{\partial T} + \varepsilon\dfrac{\partial}{\partial \tau}$

$$\boxed{\frac{d}{dt} \sim \frac{\partial}{\partial T} + \varepsilon\frac{\partial}{\partial \tau}} \quad \underline{\text{OR}} \quad \dot{x} \sim X_T + \varepsilon X_\tau$$

$$\frac{d^2}{dt^2} \sim \frac{d}{dt}\left(\frac{\partial}{\partial T} + \varepsilon\frac{\partial}{\partial \tau}\right) \sim \frac{dT}{dt}\cdot\frac{\partial}{\partial T}\left(\frac{\partial}{\partial T} + \varepsilon\frac{\partial}{\partial \tau}\right) + \frac{d\tau}{dt}\cdot\frac{\partial}{\partial \tau}\left(\frac{\partial}{\partial T} + \varepsilon\frac{\partial}{\partial \tau}\right)$$

$$\sim \frac{\partial^2}{\partial T^2} + \varepsilon\frac{\partial^2}{\partial T\partial\tau} + \varepsilon\frac{\partial^2}{\partial\tau\partial T} + \cdots$$

$$\boxed{\frac{d^2}{dt^2} \sim \frac{\partial^2}{\partial T^2} + 2\varepsilon\frac{\partial^2}{\partial T\partial\tau}} \quad \underline{\text{OR}} \quad \ddot{x} \sim X_{TT} + 2\varepsilon X_{T\tau}$$

Initial Conditions: $t = 0 \Rightarrow T = \tau = 0$

$x(0) = A$ $\qquad x(0) \sim X_0(0,0) = A$

$\dot{x}(0) = B$ $\qquad X_0(0,0) = A$

$\dot{x}(0) \sim X_{0T}(0,0) = B$

$X_{0T}(0,0) = B$

Recall: $\ddot{x} + x + \varepsilon x^3 = 0$; $x(0)=\alpha$, $\dot{x}(0)=0$

Let $T \sim t$ and $\tau = \varepsilon t$.

$$\Rightarrow X_{TT} + 2\varepsilon X_{T\tau} + X + \varepsilon X^3 + \cdots = 0 \; ; \quad X(0,0)=\alpha, \; X_T(0,0)=0$$

Expand X: $X \sim X_0(T,\tau) + \varepsilon X_1(T,\tau) + \cdots$

$$\Rightarrow X_{0TT} + \varepsilon X_{1TT} + 2\varepsilon X_{0T\tau} + X_0 + \varepsilon X_1 + \varepsilon X_0^3 + \cdots = 0$$

$O(1)$: $X_{0TT} + X_0 = 0$; $X_0(0,0)=\alpha$, $X_{0T}(0,0)=0$

$$\Rightarrow \boxed{X_0(T,\tau) = A(\tau)\cos T + B(\tau)\sin T}$$

$$X_{0T} = -A\sin T + B\cos T$$

$$X_{0T\tau} = -A_\tau \sin T + B_\tau \cos T$$

$X_0(0,0) = A(0) + 0 = \alpha$

$X_{0T}(0,0) = 0 + B(0) = 0$

$$\Rightarrow \boxed{\begin{array}{l} A(0)=\alpha \\ B(0)=0 \end{array}}$$

Pick $A(\tau)$ and $B(\tau)$ to suppress secularities at the next order.

$O(\varepsilon)$:
$$\begin{aligned} X_{1TT} + X_1 &= -2X_{0T\tau} - X_0^3 \\ &= -2\left[-A'\sin T + B'\cos T\right] - \left[A\cos T + B\sin T\right]^3 \\ &= 2A'\sin T - 2B'\cos T - \left[A^3\cos^3 T + 3A^2B\cos^2 T\sin T + 3AB^2\cos T\sin^2 T + B^3\sin^3 T\right] \\ &= 2A'\sin T - 2B'\cos T - \frac{A^3}{4}\left(3\cos T + \cos(3T)\right) - \frac{3A^2B}{4}\left(\sin T + \sin(3T)\right) \\ &\quad + \frac{3AB^2}{4}\left(-\cos T + \cos(3T)\right) - \frac{B^3}{4}\left(3\sin T - \sin(3T)\right) \end{aligned}$$

$$X_{1TT} + X_1 = \underbrace{\left(-2B' - \frac{3A^3}{4} - \frac{3AB^2}{4}\right)}_{=0}\cos T + \underbrace{\left(2A' - \frac{3A^2B}{4} - \frac{3B^3}{4}\right)}_{=0}\sin T + \text{h.h.}$$

Suppress secularities $\Rightarrow \boxed{B' = -\frac{3}{8}A(A^2+B^2)}$ $\quad \boxed{A' = \frac{3}{8}B(A^2+B^2)}$

$B(0)=0 \qquad A(0)=\alpha$

We have a system of ODEs for $A(\tau)$ and $B(\tau)$.

We have ① $A' = \frac{3}{8}B(A^2+B^2)$, $A(0)=\alpha$

② $B' = -\frac{3}{8}A(A^2+B^2)$, $B(0)=0$

Convert to Polar: $\begin{pmatrix} A(\tau) \\ B(\tau) \end{pmatrix} \longrightarrow \begin{pmatrix} R(\tau) \\ \theta(\tau) \end{pmatrix}$

Let $\boxed{\begin{array}{l} R = \sqrt{A^2+B^2} \\ \theta = \tan^{-1}\frac{B}{A} \end{array}} \Rightarrow \begin{array}{l} A = R\cos\theta \\ B = R\sin\theta \end{array}$

$A(0) = R(0)\cos\theta(0) = \alpha$

$B(0) = R(0)\sin\theta(0) = 0$

$\Rightarrow \boxed{\begin{array}{l} R(0) = \alpha \\ \theta(0) = 0 \end{array}}$

① $\Rightarrow R'\cos\theta - R\theta'\sin\theta = \frac{3}{8}R\sin\theta \cdot R^2$

② $\Rightarrow R'\sin\theta + R\theta'\cos\theta = -\frac{3}{8}R\cos\theta \cdot R^2$

$\cos\theta \cdot ① + \sin\theta \cdot ② \Rightarrow \underset{R(0)=\alpha}{\boxed{R' = 0}} \Longrightarrow \boxed{R(\tau) = \alpha}$

Then, ① $\Rightarrow 0 - \cancel{R}\theta'\cancel{\sin\theta} = \frac{3}{8}\cancel{R}\cancel{\sin\theta} \cdot R^2$

$\underset{\theta(0)=0}{\boxed{\theta' = -\frac{3\alpha^2}{8}}} \Longrightarrow \boxed{\theta(\tau) = -\frac{3\alpha^2}{8}\tau}$

Therefore,

$A(\tau) = R(\tau)\cos\theta(\tau) = \alpha\cos\left(-\frac{3\alpha^2}{8}\tau\right) \Rightarrow \boxed{A(\tau) = \alpha\cos\left(\frac{3\alpha^2}{8}\tau\right)}$

$B(\tau) = R(\tau)\sin\theta(\tau) = \alpha\sin\left(-\frac{3\alpha^2}{8}\tau\right) \Rightarrow \boxed{B(\tau) = -\alpha\sin\left(\frac{3\alpha^2}{8}\tau\right)}$

$X_0(T,\tau) = A(\tau)\cos T + B(\tau)\sin T$

$= \alpha\cos\left(\frac{3\alpha^2}{8}\tau\right)\cos T - \alpha\sin\left(\frac{3\alpha^2}{8}\tau\right)\sin T$

$\boxed{X_0(T,\tau) = \alpha\cos\left(T + \frac{3\alpha^2}{8}\tau\right)}$

$\begin{array}{l} T \sim t \\ \tau = \varepsilon t \end{array} \Longrightarrow \boxed{X(t) \sim \alpha\cos\left(\left(1+\varepsilon\frac{3\alpha^2}{8}\right)t\right) + \cdots}$ Agrees with previous results.

Valid for $t = O(1/\varepsilon)$, but not for $t = O(1/\varepsilon^2)$

Alternate Forms of the Homogeneous Solution

When using multiple scales, expressing the homogeneous solution as $X_0(T,\tau) = A(\tau)\cos T + B(\tau)\sin T$ is not usually a convenient choice. There are alternatives which typically lead to less algebra.

Consider $X_{TT} + \omega_0^2 X = 0$, where $X = X(T,\tau)$

General Solutions

1. $$\boxed{X(T,\tau) = C_1(\tau)\cos(\omega_0 T) + C_2(\tau)\sin(\omega_0 T)}$$

2. $$\boxed{X(T,\tau) = R(\tau)\cos(\omega_0 T - \phi(\tau))}$$ $R(\tau) \geq 0$, $0 \leq \phi(\tau) < 2\pi$ for all $\tau \geq 0$.

$$= R\cos(\omega_0 T)\cos\phi + R\sin(\omega_0 T)\sin\phi$$

$$= (R\cos\phi)\cos(\omega_0 T) + (R\sin\phi)\sin(\omega_0 T)$$

$$\Rightarrow \begin{array}{l} C_1 = R\cos\phi \\ C_2 = R\sin\phi \end{array} \Rightarrow \boxed{\begin{array}{l} R = \sqrt{C_1^2 + C_2^2} \\ \phi = \tan^{-1}\frac{C_2}{C_1} \end{array}}$$

R and ϕ are the integration constants.

3. $$\boxed{X(T,\tau) = A(\tau)e^{i\omega_0 T} + \bar{A}(\tau)e^{-i\omega_0 T}} = A(\tau)e^{i\omega_0 T} + \text{c.c.}$$

(complex conjugate)

$$= 2\left[\mathrm{Re}\{A\}\cos(\omega_0 T) - \mathrm{Im}\{A\}\sin(\omega_0 T)\right]$$

$$\Rightarrow \begin{array}{l} C_1 = 2\mathrm{Re}\{A\} \\ C_2 = -2\mathrm{Im}\{A\} \end{array} \Rightarrow A = \frac{C_1}{2} + i\left(-\frac{C_2}{2}\right)$$

$$A = \frac{1}{2}(C_1 - iC_2)$$

$$X(T,\tau) = 2\mathrm{Re}\{Ae^{i\omega_0 T}\}$$

$\mathrm{Re}\{A\}$ and $\mathrm{Im}\{A\}$ are the integration constants.

Recall: $\ddot{x} + x + \varepsilon x^3 = 0$; $x(0) = \alpha$, $\dot{x}(0) = 0$

Let $T \sim t$, $\tau = \varepsilon t$ and $x \sim X_0(T,\tau) + \varepsilon X_1(T,\tau) + \cdots$

$$\Rightarrow (X_{0TT} + \varepsilon X_{1TT} + 2\varepsilon X_{0T\tau}) + (X_0 + \varepsilon X_1) + \varepsilon X_0^3 + \cdots = 0$$

$O(1)$: $X_{0TT} + X_0 = 0$; $X_0(0,0) = \alpha$, $X_{0T}(0,0) = 0$

$$\Rightarrow \boxed{X_0(T,\tau) = R_0(\tau)\cos(T + \phi_0(\tau))}$$

$X_0(0,0) = R_0(0)\cos\phi_0(0) = \alpha$

$X_{0T}(0,0) = -R_0(0)\sin\phi(0) = 0$

$$\Rightarrow \boxed{\begin{array}{l} R_0(0) = \alpha \\ \phi_0(0) = 0 \end{array}}$$

$X_{0T} = -R_0 \sin(T + \phi_0)$

$X_{0T\tau} = -R_0' \sin(T+\phi_0) - R\phi_0' \cos(T + \phi_0)$

$O(\varepsilon)$:
$$\begin{aligned}
X_{1TT} + X_1 &= -2X_{0T\tau} - X_0^3 \\
&= -2\left[-R_0'\sin(T+\phi_0) - R_0\phi_0'\cos(T+\phi_0)\right] - R_0^3\cos^3(T+\phi_0) \\
&= 2R_0'\sin(T+\phi_0) + 2R_0\phi_0'\cos(T+\phi_0) - \frac{R_0^3}{4}\left[3\cos(T+\phi_0) + \cos(3(T+\phi_0))\right] \\
&= 2R_0'\sin(T+\phi_0) + \left(2R_0\phi_0' - \frac{3R_0^3}{4}\right)\cos(T+\phi_0) + \text{h.h.}
\end{aligned}$$

Suppress Secularities $\Rightarrow$ $R_0' = 0$, $R_0(0) = \alpha$ $\Rightarrow \boxed{R_0(\tau) = \alpha}$

$\phi_0' = \frac{3R_0^2}{8} = \frac{3\alpha^2}{8}$, $\phi_0(0) = 0$ $\Rightarrow \boxed{\phi_0(\tau) = \frac{3\alpha^2}{8}\tau}$

Then, $X_0(T,\tau) = \alpha\cos\left(T + \frac{3\alpha^2}{8}\tau\right)$

$$\boxed{x(t) \sim \alpha\cos\left(\left(1 + \varepsilon\frac{3\alpha^2}{8}\right)t\right)}$$

Recall: $\ddot{x} + x + \varepsilon x^3 = 0$; $\begin{array}{l} x(0) = \alpha \\ \dot{x}(0) = 0 \end{array}$

Let $\begin{array}{l} T \sim t \\ \tau = \varepsilon t \end{array}$ and $x \sim x_0(T,\tau) + \varepsilon x_1(T,\tau) + \cdots$

$$\Rightarrow (x_{0TT} + \varepsilon x_{1TT} + 2\varepsilon x_{0T\tau}) + (x_0 + \varepsilon x_1) + \varepsilon x_0^3 + \cdots = 0 \; ; \; \begin{array}{l} x_0(0,0) = \alpha \\ x_{0T}(0,0) = 0 \end{array}$$

$O(1)$: $x_{0TT} + x_0 = 0$; $\begin{array}{l} x_0(0,0) = \alpha \\ x_{0T}(0,0) = 0 \end{array}$

$$\boxed{x_0(T,\tau) = A(\tau)e^{iT} + \bar{A}(\tau)e^{-iT}}$$

$$x_{0T} = iAe^{iT} - i\bar{A}e^{-iT}$$

$$x_{0T\tau} = iA'e^{iT} - i\bar{A}'e^{-iT}$$

$$x_0(0,0) = A(0) + \bar{A}(0) = \alpha$$

$$\boxed{2\,\mathrm{Re}\{A(0)\} = \alpha}$$

$$x_{0T}(0,0) = iA(0) - i\bar{A}(0) = 0$$

$$\boxed{\cancel{2}\,\mathrm{Im}\{A(0)\} = 0}$$

$O(\varepsilon)$:

$$\begin{aligned} x_{1TT} + x_1 &= -2x_{0T\tau} - x_0^3 \\ &= -2\left[iA'e^{iT} - i\bar{A}e^{-iT}\right] - \left[Ae^{iT} + \bar{A}e^{-iT}\right]^3 \\ &= -2iA'e^{iT} + 2i\bar{A}e^{-iT} - \left[A^3e^{3iT} + 3A^2\bar{A}e^{iT} + 3A\bar{A}^2e^{-iT} + \bar{A}^3e^{-3iT}\right] \\ &= (-2iA' - 3A^2\bar{A})e^{iT} + (2i\bar{A}' - 3A\bar{A}^2)e^{-iT} + h.h. \end{aligned}$$

suppress secularities $\Rightarrow$ $-2iA' - 3A^2\bar{A} = 0$ $\quad$ $2i\bar{A}' - 3A\bar{A}^2 = 0$

These equations are conjugates, and thus equivalent ($z = 0 \Leftrightarrow \bar{z} = 0$). Therefore, the second equation provides no additional information. Consequently, it is sufficient to track only the coefficient of e^{iT}.

e.g.

$$\begin{aligned} x_{1TT} + x_1 &= -2x_{0T\tau} - x_0^3 = -2\left[iA'e^{iT} - i\bar{A}e^{-iT}\right] - \left[Ae^{iT} + \bar{A}e^{-iT}\right]^3 \\ &= (-2iA' - 3A^2\bar{A})e^{iT} + \underbrace{c.c.}_{\text{complex conjugate}} + h.h. \end{aligned}$$

suppress secularities $\Rightarrow$ $-2iA' - 3A^2\bar{A} = 0$

$$A' = \frac{3}{2}iA^2\bar{A}$$

$$A\bar{A} = \bar{A}A = |A|^2 \Rightarrow \boxed{A' = \frac{3}{2}A|A|^2}$$

We have $A' = \frac{3}{2} i A |A|^2$; $Re\{A(0)\} = \alpha/2$, $Im\{A(0)\} = 0$

Convert to Polar: $A(\tau) = R(\tau)e^{i\theta(\tau)}$, where R and θ are real-valued functions.

$$\Rightarrow R'\cancel{e^{i\theta}} + iR\theta'\cancel{e^{i\theta}} = \frac{3}{2} i R\cancel{e^{i\theta}} \cdot R^2$$

$$\boxed{R' + iR\theta' = \frac{3}{2} i R^3}$$

Initial Conditions: $Re\{A(0)\} = R(0)\cos\theta(0) = \alpha/2$
$Im\{A(0)\} = R(0)\sin\theta(0) = 0$

$$\Rightarrow \boxed{\begin{aligned} R(0) &= \alpha/2 \\ \theta(0) &= 0 \end{aligned}}$$

Real Part: $R' = 0$, $R(0) = \alpha/2$ $\Rightarrow \boxed{R(\tau) = \alpha/2}$

Imaginary Part: $\theta' = \frac{3}{2} R^2 = \frac{3\alpha^2}{8}$, $\theta(0) = 0$ $\Rightarrow \boxed{\theta(\tau) = \frac{3\alpha^2}{8}\tau}$

$$\Rightarrow \boxed{A(\tau) = \frac{\alpha}{2} e^{i\frac{3\alpha^2}{8}\tau}}$$

Then,

$$X_0(T,\tau) = Ae^{iT} + \bar{A}e^{-iT} = 2Re\{Ae^{iT}\}$$

$$= 2Re\left\{\frac{\alpha}{2} e^{i\frac{3\alpha^2}{8}\tau} e^{iT}\right\} = \alpha\cos\left(T + \frac{3\alpha^2}{8}\tau\right)$$

$T \sim t$, $\tau = \varepsilon t$ $\Rightarrow \boxed{X(t) \sim \alpha\cos\left(\left(1 + \varepsilon\frac{3\alpha^2}{8}\right)t\right)}$

Example: Weakly Damped Mass-Spring System

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0; \quad \begin{matrix} x(0) = 0 \\ \dot{x}(0) = 1 \end{matrix}$$

Exact Solution:

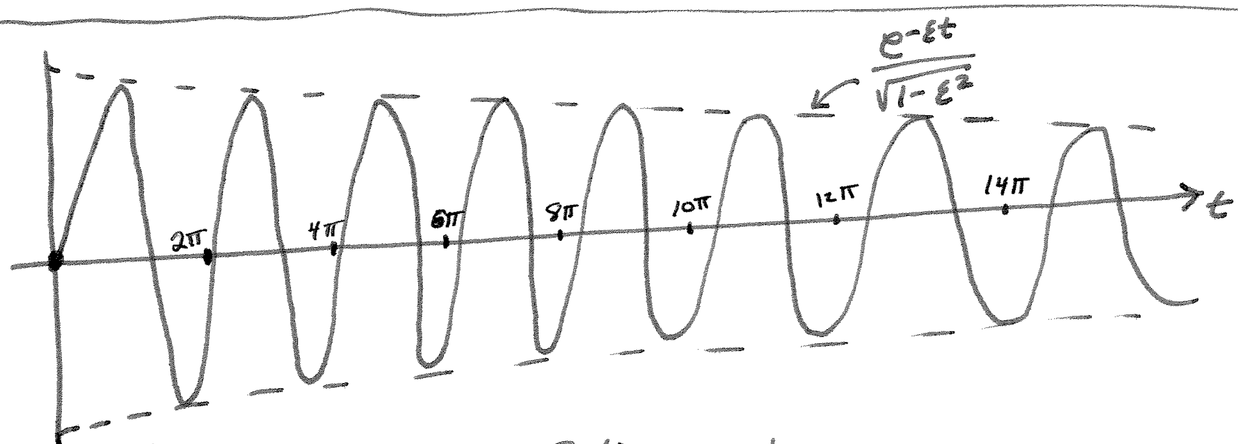
$$\boxed{x(t) = \frac{e^{-\varepsilon t}}{\sqrt{1-\varepsilon^2}} \sin\left(\sqrt{1-\varepsilon^2}\, t\right)}$$

The undamped ($\varepsilon = 0$) system oscillates with a frequency of $\omega_0 = 1$ and an amplitude of $A_0 = 1$. Weak damping ($0 < \varepsilon \ll 1$) causes the amplitude to decay slowly, $A(t) = \frac{e^{-\varepsilon t}}{\sqrt{1-\varepsilon^2}}$, and a slight decrease in frequency, $\omega = \sqrt{1-\varepsilon^2}$.

The naive expansion agrees with the Taylor expansion of the exact solution,

$$x(t) \sim \sin t - \varepsilon t \sin t + \frac{\varepsilon^2}{2}\left(t^2 \sin t - t\cos t + \sin t\right) + \cdots.$$

The series is convergent for all t, but asymptotic only for $t = O(1)$. Infinitely many terms are needed for the expansion to be bounded for all t.



The solution involves 3 time scales.

1) $x(t)$ oscillates on an $O(1)$ time scale.

$$T = \frac{2\pi}{\sqrt{1-\varepsilon^2}} \sim 2\pi\left(1 + \varepsilon^2/2 + \cdots\right) = O(1)$$

2) The amplitude decays on an $O(\frac{1}{\varepsilon})$ time scale $\left(A(t) = \frac{e^{-\varepsilon t}}{\sqrt{1-\varepsilon^2}} \sim 1 - \varepsilon t + \cdots\right)$.

In time $\Delta t = \frac{1}{\varepsilon}$, the amplitude decays by a factor of e^{-1}!

$$A\left(t_0 + \tfrac{1}{\varepsilon}\right) = e^{-1} A(t_0).$$

3) The frequency shift acts on an $O(1/\varepsilon^2)$ time scale $\left(\omega = \sqrt{1-\varepsilon^2} \sim 1 - \frac{\varepsilon^2}{2} + \cdots\right)$.

At $t = 1/\varepsilon^2$, the phase difference between the damped and undamped solutions is $\frac{1}{2}$ radian (to leading order).

$(\approx \pi/6)$

$$\ddot{x} + 2\epsilon\dot{x} + x = 0; \quad \begin{matrix} x(0)=0 \\ \dot{x}(0)=1 \end{matrix}$$

Multiple Scales: $T \sim (1+\epsilon^2\omega_2+\cdots)t$

$\tau = \epsilon t$

$$\dot{x} = x_T T_t + x_\tau \tau_t \sim (1+\epsilon^2\omega_2+\cdots)x_T + \epsilon x_\tau$$

$$\dot{x} \sim (1+\epsilon^2\omega_2)x_T + \epsilon x_\tau + \cdots$$

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d}{dt}\left[(1+\epsilon^2\omega_2)x_T + \epsilon x_\tau + \cdots\right]$$

$$\sim \left[(1+\epsilon^2\omega_2)x_T + \epsilon x_\tau + \cdots\right]_T T_t + \left[(1+\epsilon^2\omega_2)x_T + \epsilon x_\tau + \cdots\right]_\tau \tau_t$$

$$\sim (1+\epsilon^2\omega_2)\left[(1+\epsilon^2\omega_2)x_T + \epsilon x_\tau\right]_T + \epsilon\left[(1+\epsilon^2\omega_2)x_T + \epsilon x_\tau\right]_\tau + \cdots$$

$$\ddot{x} \sim (1+2\epsilon^2\omega_2)x_{TT} + 2\epsilon x_{T\tau} + \epsilon^2 x_{\tau\tau} + \cdots$$

Expand x: $x \sim x_0(T,\tau) + \epsilon x_1(T,\tau) + \epsilon^2 x_2(T,\tau) + \cdots$

$$\Rightarrow (1+2\epsilon^2\omega_2)(x_{0TT} + \epsilon x_{1TT} + \epsilon^2 x_{2TT} + \cdots) + 2\epsilon(x_{0T\tau} + \epsilon x_{1T\tau} + \cdots) + \epsilon^2(x_{0\tau\tau} + \cdots)$$
$$+ 2\epsilon\left[(x_{0T} + \epsilon x_{1T} + \cdots) + \epsilon(x_{0\tau} + \cdots)\right] + (x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots) = 0$$

Initial Conditions: $x(t=0) \sim x_0(0,0) + \epsilon x_1(0,0) + \cdots = 0$

$$\Rightarrow x_n(0,0) = 0 \text{ for all } n$$

$$\dot{x} \sim x_{0T} + \epsilon x_{1T} + \epsilon x_{0\tau} + \cdots$$

$$\dot{x}(t=0) \sim x_{0T}(0,0) + \epsilon x_{1T}(0,0) + \epsilon x_{0\tau}(0,0) + \cdots = 1$$

$O(1)$: $x_{0T}(0,0) = 1$

$O(\epsilon)$: $x_{1T}(0,0) = -x_{0\tau}(0,0)$

$O(1)$: $X_{0TT} + X_0 = 0$; $X_0(0,0) = 0$, $X_{0T}(0,0) = 1$

$$\Rightarrow \boxed{X_0(T,\tau) = R_0(\tau)\sin(T + \phi_0(\tau))}$$

$X_0(0,0) = R_0(0)\sin\phi_0(0) = 0$

$X_{0T}(0,0) = R_0(0)\cos\phi_0(0) = 1$

$$\Rightarrow \boxed{\begin{array}{l} R_0(0) = 1 \\ \phi_0(0) = 0 \end{array}}$$

$X_{0T} = R_0\cos(T+\phi_0)$

$X_{0T\tau} = R_0'\cos(T+\phi_0) - R_0\phi_0'\sin(T+\phi_0)$

$O(\varepsilon)$: $X_{1TT} + X_1 = -2X_{0T\tau} - 2X_{0T}$; $X_1(0,0) = 0$, $X_{1T}(0,0) = -X_{0\tau}(0,0)$

$$= -2\left[R_0'\cos(T+\phi_0) - R_0\phi_0'\sin(T+\phi_0)\right] - 2R_0\cos(T+\phi_0)$$

$$= -2\underline{(R_0' + R_0)}\cos(T+\phi_0) + 2\underline{R_0\phi_0'}\sin(T+\phi_0)$$

Suppress secularities $\Rightarrow$ $R_0' = -R_0$, $R_0(0) = 1$; $\phi_0' = 0$, $\phi_0(0) = 0$

$$\Rightarrow \boxed{R_0(\tau) = e^{-\tau}} \qquad \Rightarrow \boxed{\phi_0(0) = 0}$$

$$\Rightarrow \boxed{X_0(T,\tau) = e^{-\tau}\sin T}$$

$$\boxed{X(t) \sim e^{-\varepsilon t}\sin t}$$

Leading order Approximation

Valid for $t = O(1/\varepsilon)$

$X_{1TT} + X_1 = 0$; $X_1(0,0) = 0$, $X_{1T}(0,0) = -X_{0\tau}(0,0) = -e^{-\tau}\sin T\Big|_{(0,0)} = 0$

$$\Rightarrow \boxed{X_1(T,\tau) = R_1(\tau)\sin(T + \phi_1(\tau))}$$

$X_{1T} = R_1\cos(T+\phi_1)$

$X_{1T\tau} = R_1'\cos(T+\phi_1) - R_1\phi_1'\sin(T+\phi_1)$

$X_1(0,0) = R_1(0)\sin\phi_1(0) = 0$

$X_{1T}(0,0) = R_1(0)\cos\phi_1(0) = 0$

$$\Rightarrow \boxed{\begin{array}{l} R_1(0) = 0 \\ \phi_1(0) = ? \end{array}}$$

$\underline{O(\varepsilon^2)}$: $$X_{2TT} + X_2 = -2\omega_2 X_{0TT} - 2X_{1T\tau} - X_{0\tau\tau} - 2X_{1T} - 2X_{0\tau}$$

$$= -2\omega_2\left[-e^{-\tau}\sin T\right] - 2\left[R_1'\cos(T+\phi_1) - R_1\phi_1'\sin(T+\phi_1)\right]$$
$$- e^{-\tau}\sin T - 2R_1\cos(T+\phi_1) - 2\left[-e^{-\tau}\sin T\right]$$
$$= \underline{(2\omega_2+1)}e^{-\tau}\sin T - 2\underline{(R_1'+R_1)}\cos(T+\phi_1) + \underline{2R_1\phi_1'}\sin(T+\phi_1)$$

Suppress secularities: $\boxed{\omega_2 = -\frac{1}{2}}$ $\quad R_1' + R_1 = 0 \quad \phi_1' = 0$

$R_1(0) = 0 \quad \phi_1(0) = ?$

$\Rightarrow \boxed{R_1(\tau) = 0}$ $\quad$ ϕ_1 is not needed since $R_1(\tau) = 0$.

$\Rightarrow \boxed{X_1(T,\tau) = 0}$

Then,

$$X \sim X_0(T,\tau) + \varepsilon X_1(T,\tau) + O(\varepsilon^2)$$
$$\sim e^{-\tau}\sin T + \varepsilon \cdot 0 + O(\varepsilon^2)$$

$X \sim e^{-\tau}\sin T + O(\varepsilon^2)$, where $T \sim \left(1 - \frac{\varepsilon^2}{2} + \cdots\right)t$ and $\tau = \varepsilon t$.

$$\Longrightarrow \boxed{X(t) \sim e^{-\varepsilon t}\sin\left(\left(1-\tfrac{\varepsilon^2}{2}\right)t\right) + O(\varepsilon^2)}$$ Two-Term Approximation

Valid for $t = o(1/\varepsilon^2)$

All three time scales have been captured.

Example: $\overset{①}{\varepsilon\ddot{x}} + \overset{②}{2\varepsilon\dot{x}} + x + \overset{③}{\varepsilon x^3} = 0;\quad \begin{matrix} x(0)=0 \\ \dot{x}(0)=1 \end{matrix}$

The equation describes a weakly ③ nonlinear spring system with a small ① mass and weak ② damping.

① $\Rightarrow$ Initial Layer Problem

②, ③ $\Rightarrow$ Secular Problem (Oscillatory)

Solution Type: Initial Layer or Oscillatory?

Naive Expansion: $x \sim x_0(t) + \varepsilon x_1(t) + \cdots$

$$\Rightarrow \varepsilon\ddot{x}_0 + 2\varepsilon\dot{x}_0 + x_0 + \varepsilon x_1 + \varepsilon x_0^3 + \cdots = 0$$

$O(1)$: $x_0 = 0;\quad \begin{matrix} x_0(0)=0 \\ \dot{x}_0(0)=1 \end{matrix}$

x_0 cannot satisfy the initial conditions $\Rightarrow$ Try an Initial Layer

$O(\varepsilon^n)$: $x_n = 0$ for each n.

Method of Matched Asymptotic Expansions

Outer: $x_{OUT} \sim X_0(t) + \varepsilon X_1(t) + \cdots \sim 0$ (to all orders)

Inner: $\boxed{s = t/\delta(\varepsilon)} \Rightarrow \frac{\varepsilon}{\delta^2} x_{ss} + 2\frac{\varepsilon}{\delta} x_s + x + \varepsilon x^3 = 0$

$\boxed{\delta = \varepsilon^{1/2}} \Rightarrow \boxed{x_{ss} + 2\varepsilon^{1/2} x_s + x + \varepsilon x^3 = 0;\quad \begin{matrix} x(0)=0 \\ x_s(0)=\varepsilon^{1/2} \end{matrix}}$

$x_{IN} \sim \chi_0(s) + \varepsilon^{1/2}\chi_1(s) + \cdots$

$$\Rightarrow \chi_{0ss} + \varepsilon^{1/2}\chi_{1ss} + 2\varepsilon^{1/2}\chi_{0s} + \chi_0 + \varepsilon^{1/2}\chi_1 + \varepsilon\chi_0^3 + \cdots = 0$$

$O(1)$: $\chi_{0ss} + \chi_0 = 0;\quad \begin{matrix} \chi_0(0)=0 \\ \chi_{0s}(0)=0 \end{matrix} \Rightarrow \boxed{\chi_0(s) = 0}$

$O(\varepsilon^{1/2})$: $\chi_{1ss} + \chi_1 = 0;\quad \begin{matrix} \chi_1(0)=0 \\ \chi_{1s}(0)=1 \end{matrix} \Rightarrow \boxed{\chi_1(s) = \sin s}$

$\Rightarrow \boxed{x_{IN} \sim \varepsilon^{1/2}\sin s}$

$x_{IN} \sim \varepsilon^{1/2}\sin s$ cannot be matched to $x_{OUT} \sim 0$. $\Rightarrow$ Not an Initial Layer Problem.

However, we did find a scaling ($s = t/\varepsilon^{1/2}$) which yields an oscillatory solution, $x(t) \sim \varepsilon^{1/2}\sin(t/\varepsilon^{1/2})$, of small amplitude and high frequency.

$\Rightarrow$ Try Multiple Scales

$$\varepsilon\ddot{x}+2\varepsilon\dot{x}+x+\varepsilon x^3=0;\quad \begin{matrix}x(0)=0\\ \dot{x}(0)=1\end{matrix}$$

Let $\boxed{s=t/\varepsilon^{1/2}}$ $\Rightarrow$ $x_{ss}+2\varepsilon^{1/2}x_s+x+\varepsilon x^3=0;\quad \begin{matrix}x(0)=0\\ x_s(0)=\varepsilon^{1/2}\end{matrix}$

Let $\boxed{x=\varepsilon^{1/2}y}$ $\Rightarrow$ $\boxed{y_{ss}+2\varepsilon^{1/2}y_s+y+\varepsilon^2y^3=0;\quad \begin{matrix}y(0)=0\\ y_s(0)=1\end{matrix}}$

(weakly nonlinear oscillator)

Multiple Scales

Let $\boxed{\begin{matrix}T\sim s\\ \tau=\varepsilon^{1/2}s\end{matrix}}$ $\Rightarrow$ $\begin{matrix}y_s\sim y_T+\varepsilon^{1/2}y_\tau+\cdots\\ y_{ss}\sim y_{TT}+2\varepsilon^{1/2}y_{T\tau}+\cdots\end{matrix}$

$$y\sim y_0(T,\tau)+\varepsilon^{1/2}y_1(T,\tau)+\cdots$$

$$\Rightarrow\ y_{0TT}+\varepsilon^{1/2}y_{1TT}+2\varepsilon^{1/2}y_{0T\tau}+2\varepsilon^{1/2}y_{0T}+y_0+\varepsilon^{1/2}y_1+\cdots=0$$

$O(1)$: $y_{0TT}+y_0=0;\quad \begin{matrix}y_0(0,0)=0\\ y_{0T}(0,0)=1\end{matrix}$

$\Rightarrow$ $\boxed{y_0(T,\tau)=R(\tau)\sin(T+\phi(\tau))}$

$y_0(0,0)=R(0)\sin\phi(0)=0$

$y_{0T}(0,0)=R(0)\cos\phi(0)=1$

$\Rightarrow$ $\boxed{\begin{matrix}R(0)=1\\ \phi(0)=0\end{matrix}}$

$O(\varepsilon)$: $y_{1TT}+y_1=-2y_{0T\tau}-2y_{0T}$

$$=-2\left[R'\cos(T+\phi)-R\phi'\sin(T+\phi)\right]-2R\cos(T+\phi)$$

$$=-2\underbrace{(R'+R)}\cos(T+\phi)+\underline{2R\phi'}\sin(T+\phi)$$

Suppress secularities $\Rightarrow$

$R'+R=0$ $\qquad$ $\phi'=0$

$R(0)=1$ $\qquad$ $\phi(0)=0$

$\Rightarrow$ $\boxed{R(\tau)=e^{-\tau}}$ $\qquad$ $\boxed{\phi(\tau)=0}$

$$\Rightarrow\ y\sim y_0(T,\tau)+\cdots=e^{-\tau}\sin T+\cdots$$

$\Longrightarrow$ $\boxed{y(s)\sim e^{-\varepsilon^{1/2}s}\sin s}$ Valid for $s=O(1/\varepsilon^{1/2})$

Then,
$$\boxed{x(t)=\varepsilon^{1/2}y(s(t))\sim\varepsilon^{1/2}e^{-t}\sin(t/\varepsilon^{1/2})}$$

General Multiple Scales

Let $t_n = (\delta(\varepsilon))^n t$ for $n = 0, 1, \ldots, N$
and $x \sim X_0(t_0, t_1, \ldots, t_N) + \varepsilon X_1(t_0, t_1, \ldots, t_N) + \cdots$

Example: $\ddot{x} + 2\varepsilon\dot{x} + x = 0$; $x(0) = 0$, $\dot{x}(0) = 1$ (Weakly Damped Mass-Spring System)

Consider $\delta(\varepsilon) = \varepsilon$ and $N = 2$.

$$\Rightarrow \quad T = t_0 \sim t \qquad (T \sim (1 + \varepsilon^3\omega_3 + \varepsilon^4\omega_4 + \cdots)t)$$
$$\tau = t_1 = \varepsilon t$$
$$\sigma = t_2 = \varepsilon^2 t$$

$$\Rightarrow \quad \dot{x} = X_T T_t + X_\tau \tau_t + X_\sigma \sigma_t$$
$$\dot{x} \sim X_T + \varepsilon X_\tau + \varepsilon^2 X_\sigma + \cdots$$

$$\ddot{x} = \frac{d\dot{x}}{dt} = \dot{x}_T T_t + \dot{x}_\tau \tau_t + \dot{x}_\sigma \sigma_t$$
$$\sim [X_T + \varepsilon X_\tau + \varepsilon^2 X_\sigma]_T \cdot (1) + [X_T + \varepsilon X_\tau + \varepsilon^2 X_\sigma]_\tau \cdot (\varepsilon) + [X_T + \varepsilon X_\tau + \varepsilon^2 X_\sigma]_\sigma \cdot (\varepsilon^2) + \cdots$$
$$\sim (X_{TT} + \varepsilon X_{\tau T} + \varepsilon^2 X_{\sigma T}) + \varepsilon(X_{T\tau} + \varepsilon X_{\tau\tau}) + \varepsilon^2(X_{T\sigma}) + \cdots$$
$$\ddot{x} \sim X_{TT} + 2\varepsilon X_{T\tau} + \varepsilon^2(2X_{T\sigma} + X_{\tau\tau}) + \cdots$$

Initial Conditions: $X(0,0,0) = 0$
$X_T(0,0,0) \sim 1$ (leading order)

Expand X: $X \sim X_0(T,\tau,\sigma) + \varepsilon X_1(T,\tau,\sigma) + \varepsilon^2 X_2(T,\tau,\sigma) + \cdots$

$$\Rightarrow \quad \left[(X_{0TT} + \varepsilon X_{1TT} + \varepsilon^2 X_{2TT}) + 2\varepsilon(X_{0T\tau} + \varepsilon X_{1T\tau}) + \varepsilon^2(X_{0T\sigma} + X_{0\tau\tau})\right]$$
$$+ 2\varepsilon\left[(X_{0T} + \varepsilon X_{1T}) + \varepsilon X_{0\tau}\right] + \left[X_0 + \varepsilon X_1 + \varepsilon^2 X_2\right] + \cdots = 0$$

$$X_0(0,0,0) = 0$$
$$X_{0T}(0,0,0) = 1$$

$O(1)$: $X_{0TT} + X_0 = 0$; $X_0(0,0,0) = 0$, $X_{0T}(0,0,0) = 1$

$$\Rightarrow \boxed{X_0(T,\tau,\sigma) = A_0(\tau,\sigma)\sin T + B_0(\tau,\sigma)\cos T}$$

$$X_0(0,0,0) = 0 + B_0(0,0) = 0$$
$$X_{0T}(0,0,0) = A_0(0,0) + 0 = 1 \quad \Rightarrow \quad \boxed{\begin{matrix} A_0(0,0) = 1 \\ B_0(0,0) = 0 \end{matrix}}$$

$O(1)$: $X_{1TT} + X_1 = -2\left[X_{0T\tau} + X_{0T}\right]$

$$= -2\left[(A_{0\tau}\cos T - B_{0\tau}\sin T) + (A_0\cos T - B_0 \sin T)\right]$$

$$= -2\underline{(A_{0\tau} + A_0)}\cos T + 2\underline{(B_{0\tau} + B_0)}\sin T$$

Suppress Secularities $\Rightarrow$

$A_{0\tau} + A_0 = 0$ $B_{0\tau} + B_0 = 0$

$A_0(0,0) = 1$ $B_0(0,0) = 0$

$\Rightarrow \boxed{A_0(\tau,\sigma) = a_0(\sigma)e^{-\tau}}$ $\boxed{B_0(\tau,\sigma) = b_0(\sigma)e^{-\tau}}$

$A_0(0,0) = \boxed{a_0(0) = 1}$ $B_0(0,0) = \boxed{b_0(0) = 0}$

Then, $\boxed{X_0(T,\tau,\sigma) = e^{-\tau}\left[a_0(\sigma)\sin T + b_0(\sigma)\cos T\right]}$

$a_0(\sigma)$ and $b_0(\sigma)$ are determined at the next order.

$$X_{1TT} + X_1 = 0$$

$$\Rightarrow \boxed{X_1(T,\tau,\sigma) = A_1(\tau,\sigma)\sin T + B_1(\tau,\sigma)\cos T}$$

To capture the $O(\frac{1}{\varepsilon^2})$ time scale, we must suppress secularities at $O(\varepsilon^2)$. This can be achieved by choosing $a_0(\sigma)$ and $b_0(\sigma)$ appropriately.

$O(\varepsilon^2)$: $X_{2TT} + X_2 = -2\left[X_{1T\tau} + X_{0T\sigma} + \frac{1}{2}X_{0\tau\tau} + X_{1T} + X_{0\tau}\right]$

$$= -2\Big[(A_{1\tau}\cos T - B_{1\tau}\sin T) + e^{-\tau}(a_0'\cos T - b_0'\sin T) + \tfrac{1}{2}e^{-\tau}(a_0\sin T + b_0\cos T) + (A_1\cos T - B_1\sin T) - e^{-\tau}(a_0\sin T + b_0\cos T)\Big]$$

$$= -2(A_{1\tau} + a_0'e^{-\tau} + \tfrac{1}{2}b_0e^{-\tau} + A_1 - b_0e^{-\tau})\cos T - 2(-B_{1\tau} - b_0'e^{-\tau} + \tfrac{1}{2}a_0e^{-\tau} - B_1 - a_0e^{-\tau})\sin T$$

$$= -2\left[A_{1\tau} + A_1 + (a_0' - \tfrac{b_0}{2})e^{-\tau}\right]\cos T + 2\left[B_{1\tau} + B_1 + (b_0' + \tfrac{a_0}{2})e^{-\tau}\right]\sin T$$

suppress secularities $\Rightarrow$ $A_{1\tau} + A_1 = -(a_0' - \frac{b_0}{2})e^{-\tau}$, $B_{1\tau} + B_1 = -(b_0' + \frac{a_0}{2})e^{-\tau}$

homogeneous solutions A_1 and B_1 are still secular. Choose a_0 and b_0 to suppress the secularities.

$$\Rightarrow \boxed{\begin{aligned} a_0' &= \frac{b_0}{2}, \quad a(0) = 1 \\ b_0' &= -\frac{a_0}{2}, \quad b(0) = 0 \end{aligned}}$$

$b_0' = \frac{a_0}{2}$

$b_0'' = \frac{-a_0'}{2} = \frac{-b_0}{4}$

$b_0'' + \frac{b_0}{4} = 0$

$b_0(\sigma) = c_1\sin\sigma/2 + c_2\cos\sigma/2$

$b_0(0) = c_2 = 0 \Rightarrow b_0(\sigma) = c_1\sin\sigma/2$

$a_0 = -2b_0' = -2c_1 \cdot \frac{1}{2}\cos\sigma/2$

$a_0(\sigma) = -c_1\cos\sigma/2$

$a_0(0) = -c_1 = 1$

$c_1 = -1$

$$\Rightarrow \boxed{\begin{aligned} a_0(\sigma) &= \cos\sigma/2 \\ b_0(\sigma) &= -\sin\sigma/2 \end{aligned}}$$

Then, $X_0(T,\tau,\sigma) = e^{-\tau}\left[\cos\frac{\sigma}{2}\sin T - \sin\frac{\sigma}{2}\cos T\right] = e^{-\tau}\sin(T - \sigma/2)$

$$\boxed{X_0(T,\tau,\sigma) = e^{-\tau}\sin(T - \sigma/2)}$$

$T \sim t$

$\tau = \varepsilon t$

$\sigma = \varepsilon^2 t$

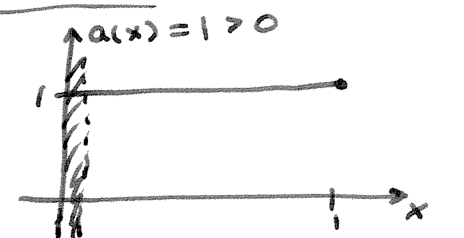
$$\Rightarrow \boxed{X(t) \sim e^{-\varepsilon t}\sin\left((1 - \tfrac{\varepsilon^2}{2})t\right)} + O(\varepsilon)$$

Valid for $t = O(\frac{1}{\varepsilon^2})$

Multiple Scales and Boundary Layer Problems

Example: $\varepsilon y'' + y' - y^2 = 0 \; ; \quad y(0)=0, \; y(1)=1$

Method of Matched Asymptotic Expansions



Boundary Layer at $x=0$.
thickness $= O(\varepsilon)$

Outer: $Y_{OUT} \sim Y_0(x) + \varepsilon Y_1(x) + \cdots$

$O(1)$: $Y_0' - Y_0^2 = 0 \; ; \; Y_0(1) = 1$

$$\Rightarrow \boxed{Y_0(x) = \frac{1}{2-x}}$$

Inner: $\boxed{\xi = \frac{x}{\varepsilon}} \Rightarrow y_{\xi\xi} + y_\xi - \varepsilon y^2 = 0 , \; y(0) = 0$

$$y_{IN} \sim y_0(\xi) + \varepsilon y_1(\xi) + \cdots$$

$O(1)$: $y_{0\xi\xi} + y_{0\xi} = 0 \; ; \; y_0(0) = 0$

$$\Rightarrow \boxed{y_0(\xi) = C_0(1 - e^{-\xi})}$$

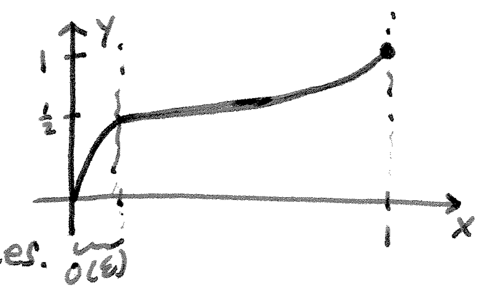
Primitive Matching: $\lim_{x \to 0^+} Y_0(x) = \lim_{\xi \to \infty} y_0(\xi)$

$$\boxed{\tfrac{1}{2} = C_0 = \text{Common Part}}$$

$$y_{comp} = Y_{OUT} + y_{IN} - (\text{Common Part})$$

$$\sim \frac{1}{2-x} + \frac{1}{2}(1 - e^{-\xi}) - \frac{1}{2} = \frac{1}{2-x} - \frac{1}{2} e^{-\xi}$$

$$\boxed{y_{comp}(x) \sim \frac{1}{2-x} - \frac{1}{2} e^{-x/\varepsilon}}$$



The solution varies on two spatial scales.

1) In the boundary layer, y varies on the 'Fast' scale ($\xi = x/\varepsilon$).

2) In the outer region, y varies on the 'Slow' scale (x).

Method of Multiple Scales

Define 'Fast' variable: $\boxed{\xi = \frac{x}{\varepsilon}}$ $y = y(\xi(x), \rho(x))$

'Slow' variable: $\boxed{\rho = x}$

Then,

$$y' = \frac{dy}{dx} = y_\xi \xi_x + y_\rho \rho_x = \frac{1}{\varepsilon} y_\xi + y_\rho$$

$$\boxed{y' = \frac{1}{\varepsilon} y_\xi + y_\rho}$$

$$y'' = \frac{dy'}{dx} = y'_\xi \xi_x + y'_\rho \rho_x = \frac{1}{\varepsilon}\left(\frac{1}{\varepsilon} y_\xi + y_\rho\right)_\xi + \left(\frac{1}{\varepsilon} y_\xi + y_\rho\right)_\rho$$

$$\boxed{y'' = \frac{1}{\varepsilon^2} y_{\xi\xi} + \frac{2}{\varepsilon} y_{\xi\rho} + \cdots}$$

Boundary Conditions:

$$y(0) = y(\xi(0), \rho(0)) = y(0,0) = 0$$
$$y(1) = y(\xi(1), \rho(1)) = y\left(\tfrac{1}{\varepsilon}, 1\right) = 1$$
$$\Rightarrow \boxed{\begin{array}{c} y(0,0) = 0 \\ y\left(\tfrac{1}{\varepsilon}, 1\right) = 1 \end{array}}$$

Expand y: $y \sim y_0(\xi, \rho) + \varepsilon y_1(\xi, \rho) + \cdots$ Recall: $\varepsilon y'' + y' - y^2 = 0$

$$\Rightarrow \varepsilon\left[\frac{1}{\varepsilon^2}(y_{0\xi\xi} + \varepsilon y_{1\xi\xi}) + \frac{2}{\varepsilon} y_{0\xi\rho}\right] + \left[\frac{1}{\varepsilon}(y_{0\xi} + \varepsilon y_{1\xi}) + y_{0\rho}\right] - y_0^2 = 0$$

$$\times\varepsilon \Rightarrow \boxed{y_{0\xi\xi} + \varepsilon y_{1\xi\xi} + 2\varepsilon y_{0\xi\rho} + y_{0\xi} + \varepsilon y_{1\xi} + \varepsilon y_{0\rho} - \varepsilon y_0^2 = 0}$$

$$y(0,0) = 0 \Rightarrow y_0(0,0) = 0$$

$$y\left(\tfrac{1}{\varepsilon}, 1\right) = 0 \Rightarrow \boxed{\lim_{\xi\to\infty} y_0(\xi, 1) = 1} \text{ (leading order)}$$

O(1): $y_{0\xi\xi} + y_{0\xi} = 0$; $y_0(0,0) = 0$, $\lim_{\xi\to\infty} y_0(\xi,1) = 1$

$$\Rightarrow \boxed{y_0(\xi, \rho) = C_1(\rho) + C_2(\rho) e^{-\xi}}$$

$$y_0(0,0) = C_1(0) + C_2(0) e^0 = 0$$
$$\lim_{\xi\to\infty} y_0(\xi, 1) = C_1(1) + C_2(1)\cdot 0 = 1$$
$$\Rightarrow \boxed{\begin{array}{c} C_1(0) + C_2(0) = 0 \\ C_1(1) = 1 \end{array}}$$

$\underline{O(\varepsilon):}$ $Y_{1\xi\xi} + Y_{1\xi} = -2Y_{0\xi\rho} - Y_{0\rho} + Y_0^2$

$= -2[-C_2' e^{-\xi}] - [C_1' + C_2' e^{-\xi}] + [C_1^2 + 2C_1C_2 e^{-\xi} + C_2^2 e^{-2\xi}]$

$= (-C_1' + C_1^2) + (2C_2' - C_2' + 2C_1C_2)e^{-\xi} + C_2^2 e^{-2\xi}$

non-secular

homogeneous solutions

$Y_{h1} = a(\rho)$

$Y_{h2} = b(\rho)e^{-\xi}$

suppress secularities $\Rightarrow$ $\boxed{C_1' = C_1^2 \quad C_1(1) = 1}$ $\boxed{C_2' = -2C_1C_2 \quad C_1(0) + C_2(0) = 0}$

$\Rightarrow \boxed{C_1(\rho) = \frac{1}{2-\rho}}$ $\Rightarrow \boxed{C_2(\rho) = -\frac{1}{8}(2-\rho)^2}$

Recall: $Y_0(\xi,\rho) = C_1(\rho) + C_2(\rho)e^{-\xi}$

$\Rightarrow \boxed{Y_0(\xi,\rho) = \frac{1}{2-\rho} - \frac{1}{8}(2-\rho)^2 e^{-\xi}}$

Then,

$y(x) \sim Y_0(\xi(x), \rho(x)) = Y_0(\frac{x}{\varepsilon}, x)$

$\Rightarrow \boxed{Y_{ms}(x) \sim \frac{1}{2-x} - \frac{1}{8}(2-x)^2 e^{-x/\varepsilon}} + O(\varepsilon)$

multiple scales

Compare to the composite expansion obtained from the method of matched asymptotic expansions. Recall: $Y_{comp}(x) \sim \frac{1}{2-x} - \frac{1}{2}e^{-x/\varepsilon}$

The solutions are asymptotically equivalent (to leading order).

Outer region $(x = O(1))$: $Y_{comp}, Y_{ms} \sim \frac{1}{2-x}$

Inner region $(x = O(\varepsilon))$: $Y_{comp}, Y_{ms} \sim \frac{1}{2} - \frac{1}{2}e^{-x/\varepsilon}$
(boundary layer)

Alternate Approach: $\varepsilon y'' + y' - y^2 = 0$; $\begin{matrix} y(0)=0 \\ y(1)=1 \end{matrix}$

Let $\boxed{t = \frac{x}{\varepsilon}}$ $\Rightarrow Y_{tt} + Y_t - \varepsilon Y^2 = 0$; $\begin{matrix} Y(0)=0 \\ Y(\frac{1}{\varepsilon})=1 \end{matrix}$

Then, let $T = t$ (Fast)
$\tau = \varepsilon t$ (Slow) $\quad Y \sim Y_0(T,\tau) + \varepsilon Y_1(T,\tau) + \cdots$

$\Rightarrow Y_0(T,\tau) = \frac{1}{2-\tau} - \frac{1}{8}(2-\tau)^2 e^{-T}$

$T = t = \frac{x}{\varepsilon}$

$\tau = \varepsilon t = x$ $\quad \Rightarrow \boxed{y(x) \sim \frac{1}{2-x} - \frac{1}{8}(2-x)^2 e^{-x/\varepsilon}}$

Multiple Scales and Weakly Nonlinear Oscillators

Consider a general weakly nonlinear oscillator.

$$\boxed{\begin{array}{l}\ddot{x} + x + \varepsilon h(x,\dot{x}) = 0 \\ x(0)=A\,,\ \dot{x}(0)=B\end{array}}$$

Multiple Scales

Let $T \sim (1+\cdots)t$

$\tau = \varepsilon t$

and $x \sim x_0(T,\tau) + \varepsilon x_1(T,\tau) + \cdots$

$$\Rightarrow \boxed{(x_{0TT} + \varepsilon x_{1TT} + 2x_{0T\tau}) + (x_0 + \varepsilon x_1) + \varepsilon h(x_0, x_{0T}) + O(\varepsilon^2) = 0}$$

$O(1)$: $\quad x_{0TT} + x_0 = 0 \;;\quad \begin{array}{l} x_0(0,0)=A \\ x_{0T}(0,0)=B\end{array}$

$$\Rightarrow \boxed{x_0(T,\tau) = R(\tau)\cos(T+\phi(\tau))}$$

① $x_0(0,0) = R(0)\cos\phi(0) = A$

② $x_{0T}(0,0) = -R(0)\sin\phi(0) = B$

$①^2 + ②^2 \Rightarrow R^2(0) = A^2 + B^2$

$\dfrac{②}{①} \Rightarrow \tan\phi(0) = -\dfrac{B}{A}$

$$\Rightarrow \boxed{\begin{array}{l} R(0) = \sqrt{A^2+B^2} \\ \phi(0) = \tan^{-1}\left(-\frac{B}{A}\right)\end{array}}$$

$O(\varepsilon)$: $\quad x_{1TT} + x_1 = -2x_{0T\tau} - h(x_0, x_{0T})$

$$\Rightarrow \boxed{x_{1TT} + x_1 = 2R'\sin(T+\phi) + 2R\phi'\cos(T+\phi) - h\big(R\cos(T+\phi), -R\sin(T+\phi)\big)}$$

Need to suppress secularities here. $\begin{pmatrix}\cos(T+\phi) \\ \sin(T+\phi)\end{pmatrix}$

Though $\mathbf{h}(x,\dot{x})$ is not specified, we can still proceed further.

Since $h(R\cos(T+\phi), -R\sin(T+\phi))$ is 2π-periodic with respect to T, it can be represented by a Fourier Series.

$$\boxed{h(R\cos(T+\phi), -R\sin(T+\phi)) = \frac{a_0(\tau)}{2} + \sum_{n=1}^{\infty}\left[a_n(\tau)\cos(n(T+\phi)) + b_n(\tau)\sin(n(T+\phi))\right]}$$

where

$$a_n(\tau) = \frac{1}{\pi}\int_{T_0}^{T_0+2\pi} h(R\cos(T+\phi), -R\sin(T+\phi))\cdot\cos(n(T+\phi))\,dT \qquad n = 0, 1, 2, \ldots$$

$$b_n(\tau) = \frac{1}{\pi}\int_{T_0}^{T_0+2\pi} h(R\cos(T+\phi), -R\sin(T+\phi))\cdot\sin(n(T+\phi))\,dT \qquad n = 1, 2, \ldots$$

The secular terms correspond to $n=1$, ($\cos(T+\phi)$ and $\sin(T+\phi)$). Therefore, $a_1(\tau)$ and $b_1(\tau)$ are the coefficients of the secular terms coming from $h(x_0, x_{0T})$.

$$\Rightarrow \quad X_{1TT} + X_1 = 2R'\sin(T+\phi) + 2R\phi'\cos(T+\phi) - [a_1\cos(T+\phi) + b_1\sin(T+\phi)] + h.h.$$

$$= (2R' - b_1)\sin(T+\phi) + (2R\phi' - a_1)\cos(T+\phi) + h.h.$$

suppress secularities $\Rightarrow$ $\boxed{R' = \frac{1}{2}b_1 \quad R(0) = \sqrt{A^2+B^2}}$ $\quad$ $\boxed{\phi' = \frac{1}{2}\frac{a_1}{R} \quad \phi'(0) = \tan^{-1}\left(-\frac{B}{A}\right)}$

We cannot proceed further without specifying $h(x, \dot{x})$.

It is necessary to use the Fourier Series approach when it is not clear which terms are secular.

e.g. $h(x, \dot{x}) = \dot{x}|\dot{x}|$

If $X_0(T, \tau) = R(\tau)\cos(T + \phi(\tau))$, then

$$h(X_0, X_{0T}) = X_{0T}|X_{0T}| = \underbrace{-R\sin(T+\phi)\cdot|R\sin(T+\phi)|}_{\text{secularities} = ?}$$

Phase Plane / Limit Cycles

Example: Rayleigh's Equation

$$\boxed{\ddot{u} - \varepsilon\left(1 - \frac{\dot{u}^2}{3}\right)\dot{u} + u = 0\ ,\ t > 0\ ;\ 0 < \varepsilon \ll 1}$$

Goal: Plot the solution trajectories in the phase plane to illustrate the solution behavior for <u>all</u> initial conditions.

Phase Plane Analysis

The phase plane analysis does not exploit the assumption that ε is a small quantity. Here, ε is treated the same as an $O(1)$ constant.

1. Convert to a system of two first order ODEs. $\dot{u} = f(u,v)$, $\dot{v} = g(u,v)$

Let $v = \dot{u} \implies \dot{u} = v$

$$\dot{v} = \ddot{u} = -u + \varepsilon\left(1 - \frac{\dot{u}^2}{3}\right)\dot{u} = -u + \varepsilon\left(1 - \frac{v^2}{3}\right)v$$

$$\dot{v} = -u + \varepsilon\left(1 - \frac{v^2}{3}\right)v$$

Vector Form: $\boxed{\dot{\vec{u}} = \vec{f}(\vec{u}), \text{ where } \vec{u} = \begin{pmatrix} u \\ v \end{pmatrix} \text{ and } \vec{f}(\vec{u}) = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} = \begin{pmatrix} v \\ -u + \varepsilon\left(1 - \frac{v^2}{3}\right)v \end{pmatrix}}$

2. Critical Points: Points in the phase plane (uv-plane) where $\dot{u} = \dot{v} = 0$.

$$\dot{u} = v = 0 \implies v = 0$$

$$\dot{v} = -u + \varepsilon\left(1 - \frac{v^2}{3}\right)v \implies u = 0$$

The system has only one critical point, $(u,v) = (0,0)$.

$$\implies \boxed{\vec{u} = 0}$$

3. Linearize $\dot{\vec{u}} = \vec{f}(\vec{u})$ about the critical point.

A nonlinear system can be approximated by a linear system near each of its critical points.

Suppose $\vec{u}_0$ is a critical point of $\dot{\vec{u}} = \vec{f}(\vec{u})$. $\Longrightarrow$ $\vec{f}(\vec{u}_0) = \vec{0}$

Solutions which are arbitrarily close to $\vec{u}_0$ may be expressed as

$$\vec{u} = \vec{u}_0 + \delta \vec{u}_1, \qquad \left(\vec{u}(t) = \vec{u}_0 + \delta\, \vec{u}_1(t)\right)$$

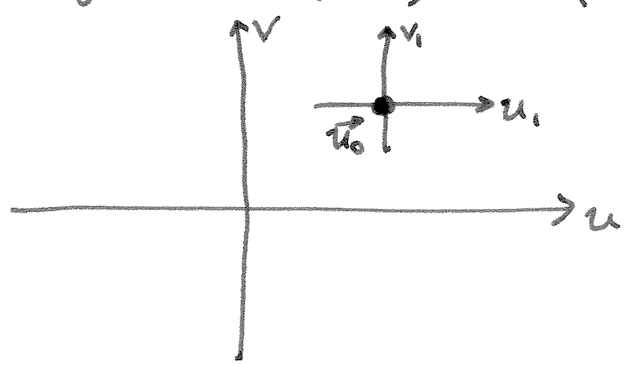
where δ is arbitrarily small. (It can be assumed that $\delta \ll \varepsilon$).

Then,
$$\dot{\vec{u}} = \vec{f}(\vec{u})$$
$$\cancel{\dot{\vec{u}}_0} + \delta \dot{\vec{u}}_1 = \vec{f}(\vec{u}_0 + \delta \vec{u}_1)$$
$$\delta \dot{\vec{u}}_1 = \cancel{\vec{f}(\vec{u}_0)} + \frac{d\vec{f}}{d\vec{u}}(\vec{u}_0)\cdot(\delta \vec{u}_1) + \cdots \quad \left(\begin{array}{c}\text{Vector Form}\\ \text{of the}\\ \text{Taylor}\\ \text{Expansion}\end{array}\right)$$
$$\dot{\vec{u}}_1 \approx \frac{d\vec{f}}{d\vec{u}}(\vec{u}_0)\cdot \vec{u}_1$$

where $\dfrac{d\vec{f}}{d\vec{u}} = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}$ is the Jacobian matrix of $\vec{f}(\vec{u})$.

$$\Rightarrow \boxed{\dot{\vec{u}}_1 \approx A\vec{u}_1, \text{ where } A = \frac{d\vec{f}}{d\vec{u}}(\vec{u}_0) = \begin{array}{c}\text{constant}\\ \text{matrix}\end{array}}$$

This solvable linear system approximates the nonlinear system sufficiently near the critical point $\vec{u}_0$. If $\det A \neq 0$, the linear system has a single critical point, $\vec{u}_1 = \vec{0}$ (which corresponds to $\vec{u} = \vec{u}_0$).



$\vec{u}_0 = \vec{0}$ is the only critical point of Rayleigh's equation

$$\vec{f}(\vec{u}) = \begin{pmatrix} v \\ -u + \varepsilon(1 - \frac{v^2}{3})v \end{pmatrix} \Rightarrow \frac{d\vec{f}}{d\vec{u}} = \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon(1-v^2) \end{pmatrix}$$

Then, $\boxed{\dot{\vec{u}}_1 \approx A\vec{u}_1, \text{ where } A = \frac{d\vec{f}}{d\vec{u}}(\vec{0}) = \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix}}$

$\det A \neq 0 \Rightarrow \vec{u}_1 = 0$ is the only critical point of the linear system.

4. Determine the eigenvalues of A

$$\det(A - \lambda I) = \lambda^2 - \varepsilon\lambda + 1 = 0$$

$$\lambda = \frac{\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}$$

$$0 < \varepsilon \ll 1 \Rightarrow \boxed{\lambda = \frac{\varepsilon \pm i\sqrt{4 - \varepsilon^2}}{2}}$$

5. Classify the critical point $\vec{u}_1 = \vec{0}$ by type and stability

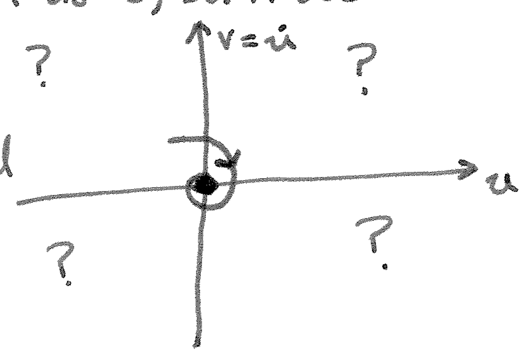
Since the eigenvalues of A are complex conjugates with positive real part, the critical point $\vec{u}_1 = \vec{0}$ is an unstable spiral point of the linear system. (The trajectories spiral clockwise).

6. Conclusion: If $\mathrm{Re}\,\lambda \neq 0$, then the critical points of the nonlinear and the linear systems are of the same type and stability.

$\Rightarrow$ $\boxed{\vec{u}_0 = \vec{0} \text{ is an unstable spiral point of Rayleigh's equation. (clockwise)}}$

The linear system approximates the solution behavior of the nonlinear system near the critical point $\vec{u}_0 = \vec{0}$, but it does not reveal what happens away from $\vec{u}_0 = \vec{0}$.

Multiple Scales may be used to complete the picture.



Multiple Scales

Recall: $\ddot{u} - \varepsilon\left(1 - \frac{\dot{u}^2}{3}\right)\dot{u} + u = 0, \quad t > 0\ ;\ 0 < \varepsilon \ll 1$

Let $T \sim t$, $\tau = \varepsilon t$ and $u \sim u_0(T,\tau) + \varepsilon u_1(T,\tau) + \cdots$

$$\Rightarrow \boxed{(u_{0TT} + \varepsilon u_{1TT} + 2\varepsilon u_{0T\tau}) - \varepsilon\left(1 - \frac{u_{0T}^2}{3}\right)u_{0T} + (u_0 + \varepsilon u_1) + \cdots = 0}$$

$O(1)$: $u_{0TT} + u_0 = 0$

$$\Rightarrow \boxed{u_0(T,\tau) = A(\tau)e^{iT} + \bar{A}(\tau)e^{-iT}} \qquad u_{0T} = i(Ae^{iT} - \bar{A}e^{-iT})$$

$O(\varepsilon)$: $u_{1TT} + u_1 = -2u_{0T\tau} + \left(1 - \frac{u_{0T}^2}{3}\right)u_{0T}$

$$= -2i(A'e^{iT} - \bar{A}'e^{-iT})$$
$$+ \left[1 - \tfrac{1}{3}(-1)(A^2e^{2iT} - 2|A|^2 + \bar{A}^2e^{-2iT})\right]i(Ae^{iT} - \bar{A}e^{-iT})$$

$$= -2i(A'e^{iT} - \bar{A}'e^{-iT}) + \left[(1 - \tfrac{2}{3}|A|^2) + \tfrac{1}{3}A^2e^{2iT} + \tfrac{1}{3}\bar{A}^2e^{-2iT}\right]i(Ae^{iT} - \bar{A}e^{-iT})$$

$$= \left[-2iA' + i\left((1 - \tfrac{2}{3}|A|^2)A - \tfrac{1}{3}|A|^2A\right)\right]e^{iT} + \text{c.c.} + \text{h.h.}$$

Suppress secularities: $-2\cancel{i}A' + \cancel{i}\left((1 - \tfrac{2}{3}|A|^2)A - \tfrac{1}{3}|A|^2A\right) = 0$

$$-2A' + (1 - |A|^2)A = 0$$

$$\boxed{A' = \tfrac{1}{2}A(1 - |A|^2)}$$

Convert to Polar

Let $A(\tau) = R(\tau)e^{i\theta(\tau)}$ $\Rightarrow$ $(R' + iR\theta')\cancel{e^{i\theta}} = \frac{1}{2}R\cancel{e^{i\theta}}(1 - R^2)$

$$R' + iR\theta' = \tfrac{1}{2}R(1 - R^2)$$

Real Part: $\boxed{R' = \tfrac{1}{2}R(1 - R^2)}$

Imaginary Part: $\boxed{\theta' = 0}$

Real Part: $R' = \frac{1}{2}R(1-R^2) \Rightarrow \boxed{R(\tau) = \dfrac{1}{\sqrt{1+Ce^{-\tau}}}}$

Imaginary Part: $\theta' = 0 \Rightarrow \boxed{\theta(\tau) = \theta_0 = \text{constant}}$

Then, $\boxed{A(\tau) = \dfrac{e^{i\theta_0}}{\sqrt{1+Ce^{-\tau}}}}$

$$u_0(T,\tau) = Ae^{iT} + \bar{A}e^{-iT} = 2\,\mathrm{Re}\{Ae^{iT}\} = 2\,\mathrm{Re}\left\{\frac{e^{i(T+\theta_0)}}{\sqrt{1+Ce^{-\tau}}}\right\}$$

$$\boxed{u_0(T,\tau) = \frac{2\cos(T+\theta_0)}{\sqrt{1+Ce^{-\tau}}}}$$

$$\Rightarrow \boxed{u(t) \sim \frac{2\cos(t+\theta_0)}{\sqrt{1+Ce^{-\varepsilon t}}}}$$

The integration constants C and θ_0 are determined by the initial conditions.

Phase Plane: Since u is oscillatory, the trajectories are spirals.

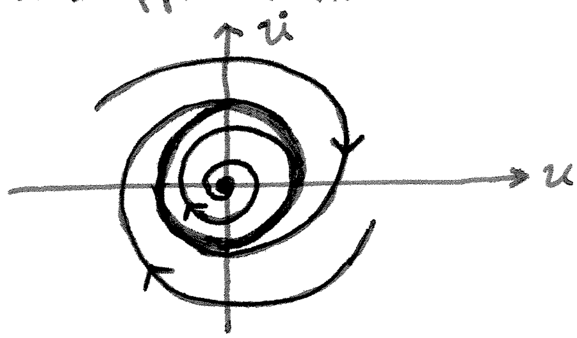
As $t \to \infty$, $u(t) \sim 2\cos(t+\theta_0)$ (u approaches simple harmonic motion)

$\dot{u}(t) \sim -2\cos(t+\theta_0)$

$\Rightarrow \boxed{u^2 + \dot{u}^2 = 4}$ (circle)

Limit Cycle

All trajectories approach the limit cycle as $t \to \infty$.



If the amplitude of u is initially small, it slowly increases to 2.
If the amplitude of u is initially large, it slowly decreases to 2.

If $\varepsilon < 0$, the trajectories spiral away from the circle $u^2 + \dot{u}^2 = 4$, and the critical point $(0,0)$ is an asymptotically stable spiral point.

Slowly Varying Frequency

Consider $\boxed{\ddot{X} + \omega^2(\varepsilon t) X = 0}$ $\omega(\varepsilon t) \sim$ frequency

Naive Expansion: Leads to secularities.

Multiple Scales: The usual scalings $\binom{T \sim t}{\tau = \varepsilon t}$ are not sufficient for this equation since the secularities cannot be suppressed with this choice. Suppressing the secularities leads to $X_0(T, \tau) = 0$, which is not acceptable.

Try a more general scaling, $\boxed{\begin{array}{l} T = T(t;\varepsilon) \\ \tau = \varepsilon t \end{array}}$ with $T \gg \tau$,

and choose T so that we have $X_{TT} + X = 0$ at leading order.

Then, $\boxed{\dot{X} = T_t X_T + \varepsilon X_\tau}$

$$\ddot{X} = \frac{d\dot{X}}{dt} = \frac{d}{dt}(T_t X_T + \varepsilon X_\tau) = T_{tt} X_T + T_t (X_{TT} T_t + X_{T\tau} \tau_t) + \varepsilon (X_{\tau T} T_t + X_{\tau\tau} \tau_t)$$

$$\boxed{\ddot{X} = T_t^2 X_{TT} + T_{tt} X_T + 2\varepsilon T_t X_{T\tau} + \varepsilon^2 X_{\tau\tau}}$$

The equation becomes,

$$(T_t^2 X_{TT} + T_{tt} X_T + 2\varepsilon T_t X_{T\tau} + \cdots) + \omega^2(\tau) X = 0$$

Pick $T(t;\varepsilon)$ so that the terms $T_t^2 X_{TT}$ and $\omega^2(\tau) X$ balance.

$$\Longrightarrow \quad T_t = \omega(\tau) = \omega(\varepsilon t)$$

$$\boxed{T(t;\varepsilon) = \int_0^t \omega(\varepsilon t)\, dt}$$

Note: The term $T_{tt} X_T$ is not part of the leading order balance since $T_{tt} = \varepsilon \omega'(\varepsilon t) \ll 1$.

With this choice of $T(t;\varepsilon)$, the equation is

$$X_{TT} + X = 0$$

to leading order.

Example: $\ddot{x} + \omega^2(\varepsilon t)x = 0$; $x(0) = A$, $\dot{x}(0) = B$

Let $\boxed{\begin{aligned} T &= \int_0^t \omega(\varepsilon t)\,dt \\ \tau &= \varepsilon t \end{aligned}}$ $\Rightarrow$ $\boxed{\dot{x} = \omega X_T + \varepsilon X_\tau}$

$\boxed{\ddot{x} = \omega^2 X_{TT} + \varepsilon\omega' X_T + 2\varepsilon X_{T\tau} + \varepsilon^2 X_{\tau\tau}}$

$T_t = \omega(\varepsilon t) = \omega(\tau)$

$T_{tt} = \varepsilon\omega'(\varepsilon t) = \varepsilon\omega'(\tau)$

$x(0) = X(0,0) = A$

$\dot{x}(0) = \omega(0)X_T(0,0) + \varepsilon X_\tau(0,0) = B$

Expand x : $X \sim X_0(T,\tau) + \varepsilon X_1(T,\tau) + \cdots$

$$\Rightarrow \left[\omega^2(X_{0TT} + \varepsilon X_{1TT}) + \varepsilon\omega' X_{0T} + 2\varepsilon\omega X_{0T\tau}\right] + \omega^2(X_0 + \varepsilon X_1) + \cdots = 0$$

$O(1)$: $\cancel{\omega^2}(X_{0TT} + X_0) = 0$; $X_0(0,0) = A$, $X_{0T}(0,0) = B/\omega(0)$

$X_0(0,0) = R(0)\cos\phi(0) = A$

$X_{0T}(0,0) = -R(0)\sin\phi(0) = B/\omega(0)$

$\Rightarrow \boxed{X_0(T,\tau) = R(\tau)\cos(T + \phi(\tau))}$ $\Rightarrow$ $\boxed{\begin{aligned} R(0) &= \sqrt{A^2 + \frac{B^2}{\omega^2(0)}} \\ \phi(0) &= \tan^{-1}\left(\frac{-B}{\omega(0)A}\right) \end{aligned}}$

$O(\varepsilon)$: $\omega^2(X_{1TT} + X_1) = -\omega' X_{0T} - 2\omega X_{0T\tau}$

$$= -\omega'(-R\sin(T+\phi)) - 2\omega(-R'\sin(T+\phi) - R\phi'\cos(T+\phi))$$

$$= (\omega' R + 2\omega R')\sin(T+\phi) + 2\omega R\phi'\cos(T+\phi)$$

Suppress Secularities $\Rightarrow$ $\frac{R'}{R} = -\frac{1}{2}\frac{\omega'}{\omega}$ $\qquad \phi' = 0$

$\Rightarrow \boxed{R(\tau) = R(0)\sqrt{\frac{\omega(0)}{\omega(\tau)}}}$ $\qquad \Rightarrow \boxed{\phi(\tau) = \phi(0)}$

Then,

$$X_0(T,\tau) = R(0)\sqrt{\frac{\omega(0)}{\omega(\tau)}}\cos(T + \phi(0))$$

$\Rightarrow \boxed{X(t) \sim R(0)\sqrt{\frac{\omega(0)}{\omega(\varepsilon t)}}\cos(T + \phi(0))}$ where $T = \int_0^t \omega(\varepsilon t)\,dt$

By plugging $R(0)$ and $\phi(0)$, the leading order approximation can be written as

$$\boxed{X(t) \sim \frac{A\omega(0)\cos T + B\sin T}{\sqrt{\omega(0)\cdot\omega(\varepsilon t)}}}$$