Asymptotic Integration

Integrals which involve a small parameter can often be approximated by means of Taylor expansions

Example: $$I = \int_0^1 \frac{e^{\varepsilon x}}{1+x^2}\,dx \sim \int_0^1 \frac{1+\varepsilon x + \varepsilon^2 x^2/2 + \cdots}{1+x^2}\,dx$$

$$\sim \int_0^1 \frac{dx}{1+x^2} + \varepsilon \int_0^1 \frac{x\,dx}{1+x^2}dx + \frac{\varepsilon^2}{2}\int_0^1 \frac{x^2}{1+x^2}dx + O(\varepsilon^3)$$

$$\Rightarrow \boxed{I \sim \tan^{-1} 1 + \frac{\varepsilon}{2}\ln 2 + \frac{\varepsilon^2}{2}(1 - \tan^{-1} 1) + O(\varepsilon^3)}$$

Here, we expanded the integrand for small ε.

$$\boxed{\int_a^b f(x;\varepsilon)\,dx \sim \int_a^b \left[f(x;0) + \varepsilon \frac{df}{d\varepsilon}(x;0) + \frac{\varepsilon^2}{2}\frac{d^2 f}{d\varepsilon^2}(x;0) + \cdots\right]dx}$$

It may be necessary to expand $f(x;\varepsilon)$ in powers of ε^α, for some α.

Alternatively, we may consider the integral I to be a function of ε and Taylor expand $I(\varepsilon)$ for small ε.

$$I(\varepsilon) = \int_0^1 \frac{e^{\varepsilon x}}{1+x^2}dx \Rightarrow I(0) = \int_0^1 \frac{dx}{1+x^2} = \tan^{-1} 1$$

$$I'(\varepsilon) = \int_0^1 \frac{x e^{\varepsilon x}}{1+x^2}dx \Rightarrow I'(0) = \int_0^1 \frac{x\,dx}{1+x^2} = \frac{1}{2}\ln 2$$

$$I''(\varepsilon) = \int_0^1 \frac{x^2 e^{\varepsilon x}}{1+x^2}dx \Rightarrow I''(0) = \int_0^1 \frac{x^2 dx}{1+x^2} = 1 - \tan^{-1} 1$$

Then, $$\boxed{I(\varepsilon) \sim I(0) + \varepsilon I'(0) + \frac{\varepsilon^2}{2} I''(0) + \cdots}$$

$$\Rightarrow \boxed{I(\varepsilon) \sim \tan^{-1} 1 + \frac{\varepsilon}{2}\ln 2 + \frac{\varepsilon^2}{2}(1 - \tan^{-1} 1) + O(\varepsilon^3)}$$

It is necessary to take the later approach if ε appears in the limits of integration.

e.g. $$I(\varepsilon) = \int_0^{1-\varepsilon} \frac{dx}{1+x^2}$$

Example: $I(\varepsilon) = \int_0^{1-\varepsilon} \frac{dx}{1+x^2} \Rightarrow I(0) = \int_0^1 \frac{dx}{1+x^2} = \tan^{-1} 1$

$I'(\varepsilon) = \frac{-1}{1+(1-\varepsilon)^2} \Rightarrow I'(0) = -\frac{1}{2}$

Then, $I(\varepsilon) \sim I(0) + \varepsilon I'(0) + O(\varepsilon^2)$

$$\Rightarrow \boxed{I(\varepsilon) \sim \tan^{-1} 1 - \varepsilon/2 + O(\varepsilon^2)}$$

Exact: $I(\varepsilon) = \tan^{-1}(1-\varepsilon) \sim \tan^{-1} - \frac{\varepsilon}{2} + O(\varepsilon^2)$ ✓

Consider the integral $\boxed{I(\varepsilon) = \int_{a(\varepsilon)}^{b(\varepsilon)} f(x;\varepsilon)\,dx}$

$$\boxed{I'(\varepsilon) = \int_{a(\varepsilon)}^{b(\varepsilon)} \frac{df}{d\varepsilon}(x;\varepsilon)\,dx + b'(\varepsilon) f(b(\varepsilon);\varepsilon) - a'(\varepsilon) f(a(\varepsilon);\varepsilon)}$$ (Leibnitz Formula)

$I(\varepsilon) \sim I(0) + \varepsilon I'(0) + O(\varepsilon^2)$

$$\boxed{I(\varepsilon) \sim \int_{a(0)}^{b(0)} f(x;0)\,dx + \varepsilon\left[\int_{a(0)}^{b(0)} \frac{df}{d\varepsilon}(x;0)\,dx + b'(0) f(b(0);0) - a'(0) f(a(0);0)\right] + O(\varepsilon^2)}$$

Example: $I(\varepsilon) = \int_0^{e^\varepsilon} (x^2+\varepsilon)^{10}\,dx \Rightarrow I(0) = \int_0^1 x^{20}\,dx = \frac{1}{21}$

$I'(\varepsilon) = \int_0^{e^\varepsilon} 10(x^2+\varepsilon)^9\,dx + e^\varepsilon (e^{2\varepsilon}+\varepsilon)^{10} - 0$

$I'(0) = \int_0^1 10x^{18}\,dx + 1\cdot(1+0)^{10} = \frac{10}{19} + 1 = \frac{29}{19}$

Then, $I(\varepsilon) \sim I(0) + \varepsilon I'(0) + O(\varepsilon^2)$

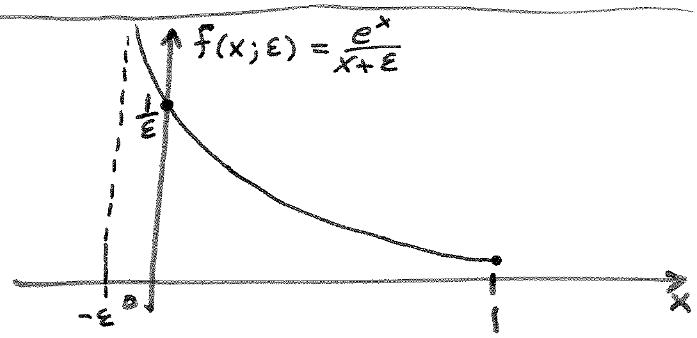
$$\Rightarrow \boxed{I(\varepsilon) \sim \frac{1}{21} + \frac{29}{19}\varepsilon + O(\varepsilon^2)}$$

ε	exact	approximate
.1	.9282	.20025
.01	.06543	.06288
.001	.049168	.049145
.0001	.04777191	.4777168

The Taylor expansion approach fails if

1) $I(\varepsilon)$ cannot be represented by a Taylor series.
2) the expansion of the integrand is not uniformly valid over the entire domain of integration.
3) the resulting integrals do not exist.

Example: $I(\varepsilon) = \int_0^1 \frac{e^x}{x+\varepsilon}\,dx$



Expand

$$I(\varepsilon) \sim \int_0^1 \frac{e^x}{x}\left(1 - \frac{\varepsilon}{x} + \frac{\varepsilon^2}{x^2} + \cdots\right) dx$$

Problems: 2) the expansion is not valid for $x = O(\varepsilon)$
3) the integrals $\int_0^1 \frac{e^x}{x^n}\,dx,\ n = 1, 2, \ldots$ do not exist.

There is no well-developed theory which applies in general to such difficulties.

Consider the more general integral $I(\varepsilon) = \int_0^1 \frac{f(x)}{x+\varepsilon}\,dx$, where $f(x)$ is a smooth, well-behaved, $O(1)$ function on $(0,1)$. $(f, f', f'', \ldots = O(1))$

Then,
$$I(\varepsilon) = \int_0^1 \frac{f(x)}{x+\varepsilon}\,dx = f(x)\ln(x+\varepsilon)\Big|_0^1 - \int_0^1 f'(x)\ln(x+\varepsilon)\,dx$$

Integrate by Parts: $u = f(x)$, $v = \ln(x+\varepsilon)$, $du = f'(x)\,dx$, $dv = \frac{dx}{x+\varepsilon}$

$$= \underbrace{f(1)\ln(1+\varepsilon)}_{= O(\varepsilon)} - \underbrace{f(0)\ln\varepsilon}_{= O(\ln\frac{1}{\varepsilon})} - \int_0^1 f'(x)\ln(x+\varepsilon)\,dx$$

$$\Rightarrow I(\varepsilon) \sim f(0)\ln\frac{1}{\varepsilon} + O(\varepsilon) - \underbrace{\int_0^1 f'(x)\ln(x+\varepsilon)\,dx}$$

$$\le \max_{0\le x\le 1}\{f'(x)\} \int_0^1 \ln(x+\varepsilon)\,dx$$

$$\le \max_{0\le x\le 1}\{f'(x)\} \cdot \left[(x+\varepsilon)\ln(x+\varepsilon) - x\right]\Big|_0^1$$

$$\le \max_{0\le x\le 1}\{f'(x)\} \cdot \Big[\underbrace{(1+\varepsilon)\ln(1+\varepsilon)}_{= O(\varepsilon)} - \underbrace{1}_{= O(1)} - \underbrace{\varepsilon\ln\varepsilon}_{= O(\varepsilon\ln\frac{1}{\varepsilon})}\Big]$$

$$\le \max_{0\le x\le 1}\{f'(x)\} \cdot O(1) = O(1)$$

$$I(\varepsilon) \sim f(0)\ln\frac{1}{\varepsilon} + O(\varepsilon) + O(1)$$

$$\boxed{I(\varepsilon) = \int_0^1 \frac{f(x)}{x+\varepsilon}\,dx \sim f(0)\cdot\ln\frac{1}{\varepsilon} + O(1)}$$

Methods of Integration

There are various methods of asymptotic integration for approximating integrals involving a large real parameter ($x \gg 1$) of the form

$$I(x) = \int_C f(z) e^{x\phi(z)} dz \; ; \; x \gg 1 ,$$

where f and ϕ are complex-valued functions of the complex variable z, and C is some contour in the complex plane.

General Case: **Method of Steepest Decent**

To approximate $\boxed{I(x) = \int_C f(z) e^{x\phi(z)} dz \; ; \; x \gg 1}$, as defined above.

Special Cases: **Method of Stationary Phase**

To approximate $\boxed{I(x) = \int_a^b f(t) e^{ix\phi(t)} dt \; ; \; x \gg 1}$,

where t, f, ϕ, a, b are real quantities. (a and b may be infinite)

Note: With $\phi(t) = t$, the Fourier Transform, $F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ixt} dt$, of a function $f(t)$ can be approximated for large values of the transform variable (x) by way of this method.

Laplace's Method

To approximate $\boxed{I(x) = \int_a^b f(t) e^{x\phi(t)} dt \; ; \; x \gg 1}$,

where t, f, ϕ, a, b are real quantities. (a and b may be infinite)

Notes:
1) With $\phi(t) = -t$, the Laplace Transform, $L(x) = \int_0^\infty f(t) e^{-xt} dt$, of a function $f(t)$ can be approximate for large values of the transform variable (x) by way of this method.
2) If $\phi(t)$ is strictly monotonic on $[a, b]$, then Laplace's method follows directly from integration by parts.
3) Laplace's method will be our main focus.

Integration by Parts

Example: Find an asymptotic expansion (as $x \to \infty$) of the integral

$$I(x) = \int_a^b e^{xt^2} dt, \quad 0 < a < b < \infty; \; x \gg 1$$

$$I(x) = \int_a^b e^{xt^2} \cdot \frac{2xt}{2xt} dt = \frac{1}{2x}\int_a^b \frac{1}{t} e^{xt^2} \cdot 2xt\,dt \qquad \begin{matrix} u = \frac{1}{t} & v = e^{xt^2} \\ du = \frac{-1}{t^2}dt & dv = e^{xt^2} \cdot 2xt\,dt \end{matrix}$$

$$= \frac{1}{2x}\left[\frac{1}{t} e^{xt^2}\Big|_a^b + \int_a^b \frac{1}{t^2} e^{xt^2} \cdot \frac{2xt}{2xt} dt\right]$$

$$= \frac{e^{xt^2}}{2xt}\Big|_a^b + \frac{1}{(2x)^2}\int_a^b \frac{1}{t^3} e^{xt^2} \cdot 2xt\,dt \qquad \begin{matrix} u = \frac{1}{t^3} & v = e^{xt^2} \\ du = \frac{-3}{t^4}dt & dv = e^{xt^2} \cdot 2xt\,dt \end{matrix}$$

$$= \frac{e^{xt^2}}{2xt}\Big|_a^b + \frac{1}{(2x)^2}\left[\frac{1}{t^3} e^{xt^2}\Big|_a^b + 3\int_a^b \frac{1}{t^4} e^{xt^2} \cdot \frac{2xt}{2xt} dt\right]$$

$$= \frac{e^{xt^2}}{2xt}\left(1 + \frac{1}{2xt^2}\right)\Big|_a^b + \frac{3}{(2x)^3}\int_a^b \frac{1}{t^5} e^{xt^2} \cdot 2xt\,dt \qquad \begin{matrix} u = \frac{1}{t^5} & v = e^{xt^2} \\ du = \frac{-5}{t^6}dt & dv = e^{xt^2} \cdot 2xt\,dt \end{matrix}$$

$$\vdots$$

$$I(x) \sim \frac{e^{xt^2}}{2xt}\left(1 + \frac{1}{2xt^2} + \frac{1\cdot 3}{(2xt^2)^2} + \frac{1\cdot 3\cdot 5}{(2xt^2)^3} + \frac{1\cdot 3\cdot 5\cdot 7}{(2xt^2)^4} + \cdots\right)\Bigg|_a^b$$

$$\sim \frac{e^{xt^2}}{2xt}\left(1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{(2xt^2)^n}\right)\Bigg|_a^b$$

$$\Rightarrow \boxed{I(x) \sim \frac{e^{xb^2}}{2xb}\left(1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{(2xb^2)^n}\right) - \frac{e^{xa^2}}{2xa}\left(1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2n-1)}{(2xa^2)^n}\right)}$$

On the next page, it is shown that all terms corresponding to the upper limit $(t=b)$ of integration are much larger than all terms which correspond to the lower limit $(t=a)$.

We have

$$I(x) \sim \frac{e^{xb^2}}{2xb}\left(1+\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}{(2xb^2)^n}\right) - \frac{e^{xa^2}}{2xa}\left(1+\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}{(2xa^2)^n}\right)$$

Now, compare an arbitrary term coming from the lower limit ($t=a$) to an arbitrary term coming from the upper limit ($t=b$) of integration.

Ignoring constants, we have

$$\frac{e^{xa^2}/x^m}{e^{xb^2}/x^n} = x^{n-m}e^{-(b^2-a^2)x} \longrightarrow 0 \text{ as } x\to\infty \quad \text{since } b^2>a^2.$$

Therefore, $\quad \frac{e^{xa^2}}{x^m} \ll \frac{e^{xb^2}}{x^n}$ for all $m,n=1,2,3,\dots$.

Every term corresponding to $t=a$ is (transcendentally) small in comparison to every term corresponding to $t=b$.

$$\Longrightarrow \boxed{I(x) \sim \frac{e^{xb^2}}{2xb}\left(1+\sum_{n=1}^{\infty}\frac{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}{(2xb^2)^n}\right)} + TST$$

Notice that the asymptotic expansion of $I(x)$ depends only on the right endpoint ($t=b$) of the interval of integration, where the integrand is largest. This is the essence of Laplace's method.

error function

Example:

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt, \quad -\infty < x < \infty$$

Determine asymptotic expansions of erf(x) for $|x| \ll 1$ and $|x| \gg 1$

Case: $|x| \ll 1$ $\qquad 0 \le |t| \le |x| \ll 1$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \sim \frac{2}{\sqrt{\pi}}\int_0^x \left(1 - \frac{t^2}{1!} + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots\right)dt$$

$$\sim \frac{2}{\sqrt{\pi}}\left(x - \frac{x^3}{3\cdot 1!} + \frac{x^5}{5\cdot 2!} - \frac{x^7}{7\cdot 3!} + \cdots\right)$$

$$\Longrightarrow \boxed{\mathrm{erf}(x) \sim \frac{2x}{\sqrt{\pi}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\cdot\frac{x^{2n}}{2n+1} \quad \text{as } x \to 0}$$

Case: $|x| \gg 1$

We may consider $x > 0$, and then obtain results for $x < 0$ by using the fact that erf(x) is an odd function.

Instead of attempting to expand erf(x) directly, it is convenient to determine the expansion of the complimentary error function erfc(x) first, and then use the relation $\mathrm{erf}(x) = 1 - \mathrm{erfc}(x)$ to get the expansion for erf(x).

Recall: $\boxed{\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}dt}$

$$\mathrm{erf}(x) + \mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt + \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\int_0^{\infty} e^{-t^2}dt = 1$$

$$\boxed{\mathrm{erf}(x) + \mathrm{erfc}(x) = 1}$$

Consider $\text{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt$ with $x \gg 1$

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\cdot\frac{-2t}{-2t}\,dt = \frac{-1}{\sqrt{\pi}}\int_x^\infty \frac{1}{t}e^{-t^2}(-2t)\,dt$$

$u = \frac{1}{t}$ $\quad v = e^{-t^2}$

$du = -\frac{1}{t^2}dt$ $\quad dv = e^{-t^2}(-2t)dt$

$$= \frac{-1}{\sqrt{\pi}}\left[\frac{e^{-t^2}}{t}\Big|_x^\infty + \int_x^\infty \frac{1}{t^2}e^{-t^2}\cdot\frac{-2t}{-2t}\,dt\right]$$

$$= \frac{1}{\sqrt{\pi}}\left[\frac{e^{-x^2}}{x} + \frac{1}{2}\int_x^\infty \frac{1}{t^3}e^{-t^2}(-2t)\,dt\right]$$

$u = \frac{1}{t^3}$ $\quad V = e^{-t^2}$

$du = -\frac{3}{t^4}dt$ $\quad dv = e^{-t^2}(-2t)dt$

$$\sim \frac{1}{\sqrt{\pi}}\left[\frac{e^{-x^2}}{x} + \frac{1}{2}\frac{e^{-t^2}}{t^3}\Big|_x^\infty + \cdots\right]$$

$$\sim \frac{1}{\sqrt{\pi}}\frac{e^{-x^2}}{x}\left(1 - \frac{1}{2x^2} + \cdots\right)$$

Repeated Integration by Parts $\Longrightarrow$ $\boxed{\text{erfc}(x) \sim \frac{e^{-x^2}}{\sqrt{\pi}\,x}\left(1 + \sum_{n=1}^\infty (-1)^n \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{(2x^2)^n}\right) \text{ as } x\to\infty}$

Then, $\boxed{\text{erf}(x) \sim 1 - \frac{e^{-x^2}}{\sqrt{\pi}\,x}\left(1 + \sum_{n=1}^\infty (-1)^n \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{(2x^2)^n}\right) \text{ as } x\to\infty}$

For $x < 0$, $\text{erf}(x) = -\text{erf}(-x)$

$$\Rightarrow \text{erf}(x) \sim -\left[1 - \frac{e^{-(-x)^2}}{\sqrt{\pi}(-x)}\left(1 + \sum_{n=1}^\infty (-1)^n \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{(2(-x)^2)^n}\right)\right]$$

$$\boxed{\text{erf}(x) \sim -1 - \frac{e^{-x^2}}{\sqrt{\pi}\,x}\left(1 + \sum_{n=1}^\infty (-1)^n \frac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{(2x^2)^n}\right) \text{ as } x\to-\infty}$$

Laplace's Integral Method

Consider integrals of the form

$$\boxed{I(x) = \int_a^b f(t)\, e^{x\phi(t)}\, dt}$$

where $x \gg 1$ and x, t, f, ϕ, a, b are real-valued.
(a and/or b may be infinite)

Notes:

1) $\phi(t) = -t \Rightarrow I(x)$ is the Laplace Transform of $f(t)$.

2) Integration by parts does not always work for such integrals. Laplace's method provides a 'fix' when it doesn't.

3) We'll consider only the leading order approximation of $I(x)$. Results can be generalized to higher order without too much difficulty, but it is algebraically tedious.

4) The leading order approximation of $I(x)$ is determined at the point(s) where $\phi(t)$ attains its absolute maximum over the interval $[a, b]$.

First consider the case in which $\phi'(t) \neq 0$ in $[a,b]$

In this case, $\phi(t)$ attains its absolute maximum on $[a,b]$ at one of the endpoints.

$$I(x) = \int_a^b f(t) e^{x\phi(t)} \cdot \frac{x\phi'(t)}{x\phi'(t)}\,dt = \frac{1}{x}\int_a^b \frac{f(t)}{\phi'(t)} e^{x\phi(t)} x\phi'(t)\,dt$$

$$u = \frac{f}{\phi'} \qquad v = e^{x\phi(t)}$$

$$du = \left(\frac{f}{\phi'}\right)' dt \quad dv = e^{x\phi(t)} x\phi'(t)\,dt$$

$$= \frac{1}{x}\left[\frac{f(t)}{\phi'(t)} e^{x\phi(t)}\Big|_a^b - \int_a^b \left(\frac{f}{\phi'}\right)' e^{x\phi(t)} \cdot \frac{x\phi'(t)}{x\phi'(t)}\,dt\right]$$

$$= \frac{1}{x}\frac{f(b)}{\phi'(b)} e^{x\phi(b)} - \frac{1}{x}\frac{f(a)}{\phi'(a)} e^{x\phi(a)} - \frac{1}{x^2}\int_a^b \left(\frac{f}{\phi'}\right)' \cdot \frac{1}{\phi'} \cdot e^{x\phi(t)} x\phi'(t)\,dt$$

$$\Rightarrow \boxed{I(x) \sim \underbrace{\frac{f(b)e^{x\phi(b)}}{x\phi'(b)}}_{①} - \underbrace{\frac{f(a)e^{x\phi(a)}}{x\phi'(a)}}_{②} + \cdots}$$

The ordering of ① and ② depends on the magnitudes of $\phi(a)$ and $\phi(b)$.

$$① = O\left(\frac{1}{x}e^{x\phi(b)}\right)$$

$$② = O\left(\frac{1}{x}e^{x\phi(a)}\right)$$

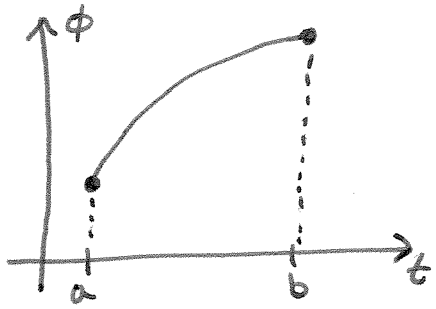
Then,

$\phi(b) > \phi(a) \Rightarrow ① \gg ②$ (provided $f(b) \neq 0$)

$\phi(a) > \phi(b) \Rightarrow ② \gg ①$ (provided $f(a) \neq 0$).

Conclusions for the case $\phi'(t) = 0$ in $[a,b]$

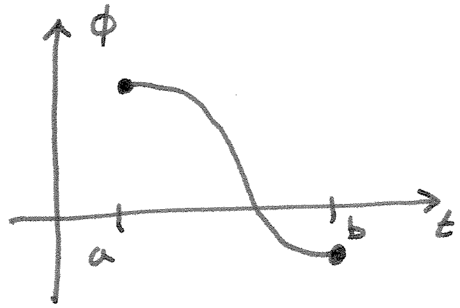
Case: $\phi(b) > \phi(a)$ (i.e. $\phi'(t) > 0$ on $[a,b]$)



$$\boxed{I(x) \sim \frac{f(b)e^{x\phi(b)}}{x\,\phi'(b)}} \quad \text{for } x \gg 1$$

provided $f(b) \neq 0$

Case: $\phi(a) > \phi(b)$ (i.e. $\phi'(t) < 0$ on $[a,b]$)



$$\boxed{I(x) = -\frac{f(a)\,e^{x\phi(a)}}{x\,\phi'(a)}} \quad \text{for } x \gg 1$$

provided $f(a) \neq 0$

Special Cases

1. a) $f(a) = 0$ when $\phi'(t) < 0$ on $[a,b]$
   b) $f(b) = 0$ when $\phi'(t) > 0$ on $[a,b]$

2. $\phi'(t) = 0$ in $[a,b]$

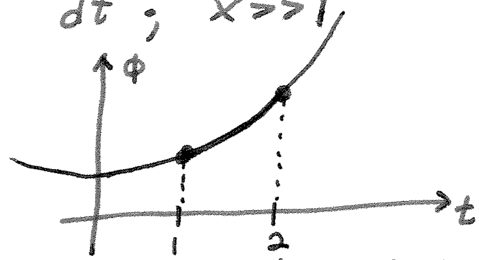
These cases require special treatment.

Example: $I(x) = \int_1^2 t e^{x\cosh t}\,dt$; $x \gg 1$

$f(t) = t$

$\phi(t) = \cosh t$

$\phi'(t) = \sinh t \neq 0$ in $[a,b]$



$\phi(2) > \phi(1)$

$$\Rightarrow \quad I(x) \sim \frac{f(2)e^{x\phi(2)}}{x\,\phi'(2)}$$

$$\boxed{I(x) \sim \frac{2e^{x\cosh 2}}{x\sinh 2}}$$

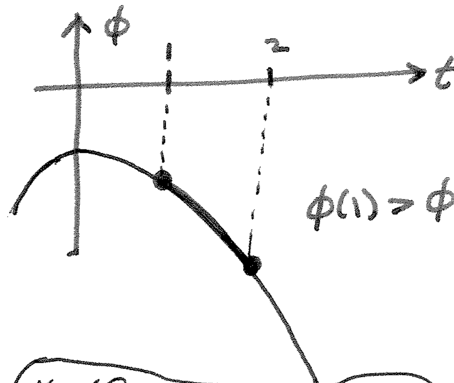
$x = 10$
exact $= 1.222 \times 10^{15}$
approx. $= 1.203 \times 10^{15}$

Example: $I(x) = \int_1^2 e^{-x\cosh^2 t}\,dt$; $x \gg 1$

$f(t) = 1$

$\phi(t) = -\cosh^2 t$

$\phi'(t) = -2\cosh t \cdot \sinh t \neq 0$ in $[1,2]$



$\phi(1) > \phi(2)$

$$\Rightarrow \quad I(x) \sim -\frac{f(1)e^{x\phi(1)}}{x\,\phi'(1)}$$

$$\boxed{I(x) \sim \frac{e^{-x\cosh^2 1}}{2x\cosh 1 \cdot \sinh 1}}$$

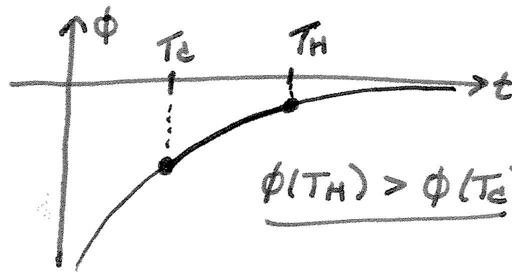
$x = 10$
exact $= 1.193 \times 10^{-12}$
approx. $= 1.257 \times 10^{-12}$

Example: $I(Z) = \int_{T_c}^{T_H} e^{-Z/T}\,dT$; $T_H > T_c$; $Z \gg 1$

$f(T) = 1$

$\phi(T) = \frac{-1}{T}$

$\phi'(T) = \frac{1}{T^2} \neq 0$



$\phi(T_H) > \phi(T_c)$

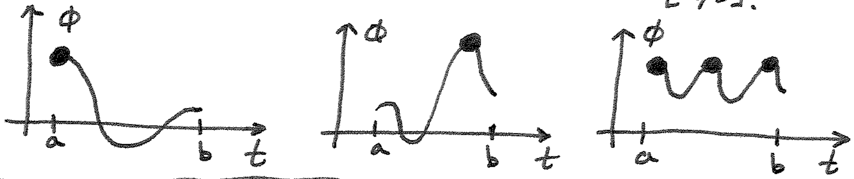
$$\Rightarrow \quad I(Z) \sim \frac{f(T_H)e^{Z\phi(T_H)}}{Z\,\phi'(T_H)}$$

$$\boxed{I(Z) \sim \frac{1}{Z} T_H^2 e^{-Z/T_H}}$$

In general, the asymptotic approximation of the integral

$$I(x) = \int_a^b f(t)\, e^{x\phi(t)}\, dt$$

is determined (to all orders) by the behavior of $\phi(t)$ near the point(s) where it attains its absolute maximum on $[a,b]$.

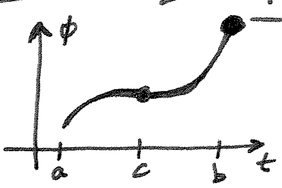


The formulas found above are valid if the absolute maximum of ϕ occurs only at boundary points, at which $f \neq 0$ and $\phi' \neq 0$. Additional formulas will be derived for other possibilities.

Case: $\phi'(c) = 0$ for a unique $c \in [a,b]$.

In this case, ϕ has either a local minimum, a local maximum, or an inflection point at $t=c$.

Subcases: I. ϕ has an inflection point at $t=c$.

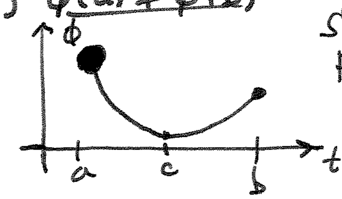


Since ϕ attains its absolute maximum at a boundary point, the formulas derived above can be used.

Note: If the inflection point occurs at a boundary, it is a special case which requires special treatment.

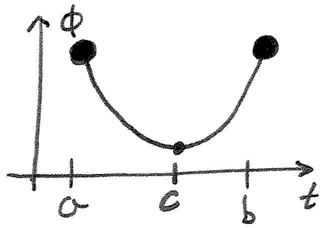
II. ϕ has a local minimum at $t=c$.

a) $\phi(a) \neq \phi(b)$



Since ϕ attains its absolute maximum at a boundary point, the formulas derived above may be used.

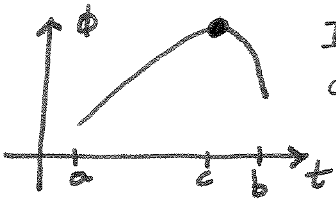
b) $\phi(a) = \phi(b)$



Here, there is a significant contribution to the value of the integral at each endpoint of the interval $[a,b]$. In this case, the contributions at both endpoints must be taken into account.

$$\Rightarrow I(x) \sim \frac{f(b)\, e^{x\phi(b)}}{x\,\phi'(b)} - \frac{f(a)\, e^{x\phi(a)}}{x\,\phi'(a)}$$

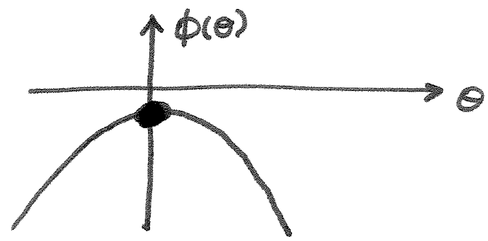
III. ϕ has a local maximum at $t=c$



If $\phi'(t)=0$ only at $t=c$, then $\phi(c)$ will be the absolute maximum of ϕ on $[a,b]$. The approximation of the integral is determined at $t=c$ rather than at a boundary point. The idea here is to Taylor expand f and ϕ about $t=c$, and proceed from there.

Example: $I(x)=\displaystyle\int_{-\infty}^{\infty} e^{-x\cosh\theta}\,d\theta, \quad x>>1$

$f(\theta)=1$
$\phi(\theta)=-\cosh\theta$
$\phi'(\theta)=-\sinh\theta=0$ at $\theta=0$
$\phi''(\theta)=-\cosh\theta<0 \Rightarrow \phi$ has a local maximum at $\theta=0$



Expand the integrand about $\theta=0$

$$I(x)\sim\int_{-\infty}^{\infty} e^{-x(1+\theta^2/2+\cdots)}\,d\theta$$

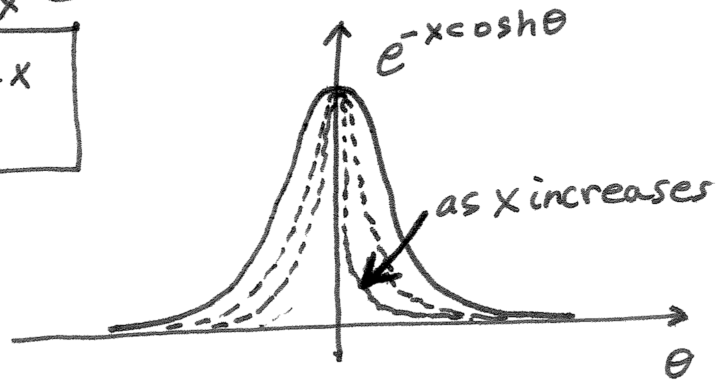
$$\sim e^{-x}\int_{-\infty}^{\infty} e^{-x\theta^2/2}\,d\theta = 2e^{-x}\int_0^{\infty} e^{-x\theta^2/2}\,d\theta$$

$$s^2 = x\frac{\theta^2}{2}$$
$$\theta=\sqrt{\frac{2}{x}}\,s$$
$$d\theta=\sqrt{\frac{2}{x}}\,ds$$

$$\sim 2e^{-x}\int_0^{\infty} e^{-s^2}\cdot\sqrt{\frac{2}{x}}\,ds$$

$$\sim 2e^{-x}\sqrt{\frac{2}{x}}\cdot\frac{\sqrt{\pi}}{2}=\sqrt{\frac{2\pi}{x}}\,e^{-x}$$

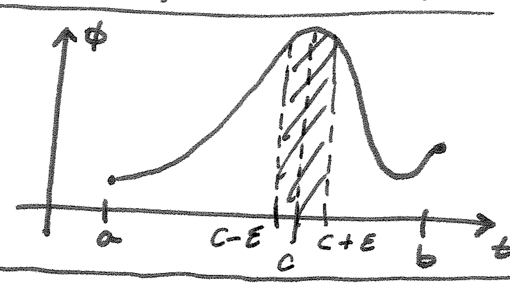
$$\boxed{I(x)\sim\sqrt{\frac{2\pi}{x}}\,e^{-x}}$$



General Formula for the case in which $\phi'(t)$ has a local maximum at a unique interior point, $t = c \in (a,b)$.

Consider $I(x) = \int_a^b f(t)e^{x\phi(t)}\,dt$

with $\phi'(c) = 0$ and $\phi''(c) < 0$ for some $c \in (a,b)$. $\Longrightarrow$



Fact: $\boxed{I(x) = \int_a^b f(t)e^{x\phi(t)}\,dt \sim \int_{c-\varepsilon}^{c+\varepsilon} f(t)e^{x\phi(t)}\,dt + TST \text{ as } x \to \infty}$

for each arbitrarily small $\varepsilon > 0$.

Note: If c is an endpoint, say $c = a$, then $I(x) \sim \int_a^{a+\varepsilon} f(t)e^{x\phi(t)}\,dt + TST$

Consequences:

1) The expansion of $I(x)$ is determined (to all orders) by the behavior of the integrand (i.e. f and ϕ) near $t = c$.
2) The expansion does not depend on a or b to all orders. Thus, a and b can be chosen (or adjusted) without affecting the expansion provided that c remains within the interior of the interval (a,b) and $\phi(c)$ is still the unique absolute maximum of ϕ on $[a,b]$.

Formula:

Expand f and ϕ about $t = c$.

$f(t) \sim f(c) + \cdots$

$\phi(t) \sim \phi(c) + \cancel{(t-c)\phi'(c)}^{=0} + \frac{1}{2}(t-c)^2\phi''(c) + \cdots$

Then,

$$I(x) \sim \int_a^b f(c)\,e^{x(\phi(c) + \frac{1}{2}(t-c)^2\phi''(c) + \cdots)} \sim f(c)e^{x\phi(c)}\int_{-\infty}^{\infty} e^{\frac{x}{2}(t-c)^2\phi''(c)}\,dt$$

$$\sim f(c)e^{x\phi(c)}\int_{-\infty}^{\infty} e^{-s^2}\cdot\frac{ds}{\sqrt{\frac{-x}{2}\phi''(c)}} = \frac{f(c)e^{x\phi(c)}}{\sqrt{\frac{-x}{2}\phi''(c)}}\cdot\sqrt{\pi}$$

$s^2 = -\frac{x}{2}(t-c)^2\phi''(c)$

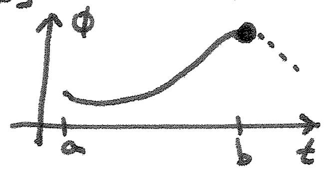
$ds = \sqrt{-\frac{x}{2}\phi''(c)}\,dt$

$$\Longrightarrow \boxed{I(x) \sim \sqrt{\frac{2\pi}{-x\phi''(c)}}\,f(c)\,e^{x\phi(c)}} \quad \text{as } x \to \infty$$

provided that $f(c) \neq 0$.

Other Related Cases

Case: The unique absolute maximum of ϕ on $[a,b]$ occurs at an endpoint c, with $\phi'(c)=0$.



The derivation of the formula is the same as above, except for the limits of integration.

If $c=a$, we have $I(x) \sim f(c)e^{x\phi(c)} \int_0^{\infty} \dfrac{e^{-s^2}\,ds}{\sqrt{-\frac{x}{2}\phi''(c)}} = \dfrac{f(c)e^{x\phi(c)}}{\sqrt{-\frac{x}{2}\phi''(c)}} \cdot \dfrac{\sqrt{\pi}}{2}$

If $c=b$, we have $I(x) \sim f(c)e^{x\phi(c)} \int_{-\infty}^{0} \dfrac{e^{-s^2}\,ds}{\sqrt{-\frac{x}{2}\phi''(c)}} = \dfrac{f(c)\,e^{x\phi(c)}}{\sqrt{-\frac{x}{2}\phi''(c)}} \cdot \dfrac{\sqrt{\pi}}{2}$

Both endpoints lead to the same formula.

$$\Rightarrow \boxed{I(x) \sim \sqrt{\frac{\pi}{-2x\,\phi''(c)}}\, f(c)\, e^{x\phi(c)}} \text{ as } x \to \infty \quad \begin{pmatrix} c=a \text{ or } c=b \\ \phi'(c)=0 \end{pmatrix}$$

provided $f(c) \neq 0$

Case: $\phi(t)$ has a unique absolute maximum at an interior point, $t=c \in (a,b)$, with $\phi'(c)=\phi''(c)=\cdots=\phi^{(p-1)}(c)=0$, and $\phi^{(p)}(c)<0$.

$$\boxed{I(x) \sim \frac{2\,\Gamma\!\left(\frac{1}{p}\right)(p!)^{1/p}}{p\left[-x\,\phi^{(p)}(c)\right]^{1/p}}\, f(c)\, e^{x\phi(c)}} \text{ as } x \to \infty$$

provided $f(c) \neq 0$.

Higher Order Terms

Suppose $\phi(t)$ has a unique absolute maximum at an interior point, $t=c \in (a,b)$, with $\phi'(c)=0$ and $\phi'(c)<0$.

Expand f and ϕ to higher order:

$$f(t) \sim f(c) + (t-c)f'(c) + \cdots$$

$$\phi(t) \sim \phi(c) + \cancel{(t-c)\phi'(c)} + \tfrac{1}{2}(t-c)^2\phi''(c) + \tfrac{1}{6}(t-c)^3\phi'''(c) + \tfrac{1}{24}(t-c)^4\phi^{iv}(c) + \cdots$$

Then, plug in to get

$$\boxed{I(x) \sim \sqrt{\frac{2\pi}{-x\phi''(c)}}\, e^{x\phi(c)} \left\{ f(c) - \frac{1}{2x\phi''(c)}\left[f''(c) - \frac{f(c)\phi^{iv}(c)}{4\phi''(c)} - \frac{f'(c)\phi'''(c)}{\phi''(c)} + \frac{5f(c)(\phi'''(c))^2}{12(\phi''(c))^2} \right] + O\!\left(\frac{1}{x^2}\right) \right\}}$$

as $x \to \infty$

[redacted]

Example: $I(x) = \int_{-2}^{0} e^{t + x(3t^2 + 2t^3)}\,dt, \quad x \gg 1$

$$I(x) = \int_{-2}^{0} e^{t} e^{x(3t^2+2t^3)}\,dt$$

$$\sim \sqrt{\frac{2\pi}{-x\phi''(-1)}}\, f(-1)\, e^{x\phi(-1)}$$

$$\sim \sqrt{\frac{2\pi}{-x(-6)}}\, e^{-1} e^{x}$$

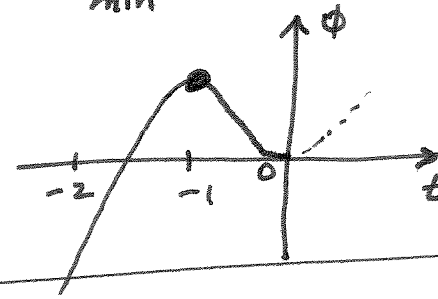
$$\boxed{I(x) \sim \sqrt{\frac{\pi}{3x}}\, e^{x-1}} \quad \text{as } x \to \infty$$

$f(t) = e^t$

$\phi(t) = 3t^2 + 2t^3$

$\phi'(t) = 6t + 6t^2 = 0$

$t=0$ local min $t=-1$ local max



If $\phi(t)$ attains its absolute maximum at multiple points in $[a,b]$, the contribution to the integral at each such point must be considered.

Example: $I(x) = \int_{-2}^{2} e^{x(t^3/3 - t)} \ln(1+t^2)\,dt, \quad x \gg 1$

$$I(x) \sim \sqrt{\frac{2\pi}{-x\phi''(-1)}}\, f(-1)\, e^{x\phi(-1)} + \frac{f(2)\, e^{x\phi(2)}}{x\phi'(2)}$$

$$\sim \underbrace{\sqrt{\frac{\pi}{x}}\, \ln 2\, e^{2x/3}}_{①} + \underbrace{\frac{\ln 5}{3x}\, e^{2x/3}}_{②}$$

① $\gg$ ②

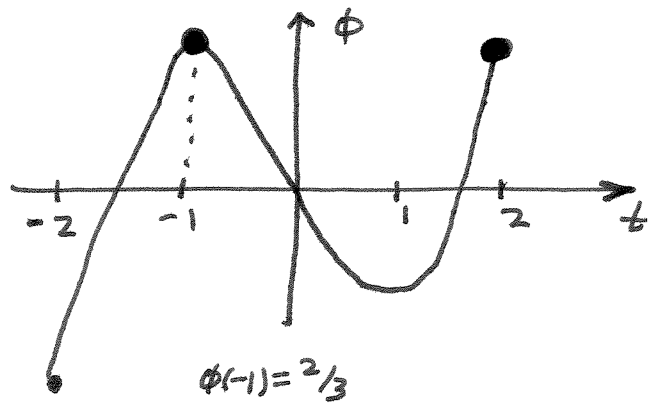
$$\Rightarrow \boxed{I(x) \sim \sqrt{\frac{\pi}{x}}\, (\ln 2)\, e^{2x/3}}$$

$f(t) = \ln(1+t^2)$

$\phi(t) = t^3/3 - t$

$\phi'(t) = t^2 - 1 = 0$

$t=-1$ local max $t=1$ local min



$\phi(-1) = 2/3$

$\phi(1) = -2/3$

$\phi(2) = 2/3$

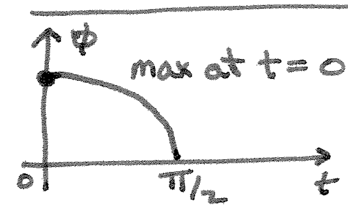
Case: The absolute maximum of $\phi(t)$ occurs at $t=c$, with $f(c)=0$.

The Laplace method formulas fail in this case.

Possible Remedies:
1) Abandon the formulas and try an alternate approach.
2) Write the integral in a form for which Laplace's method is applicable.

Example: $I(x) = \int_0^{\pi/2} e^{x\cos t} \ln(\lambda + \sin t)\,dt$, $\lambda > 0$; $x \gg 1$

$\phi(t) = \cos t$
$\phi'(t) = -\sin t$
$\phi''(t) = -\cos t$



$f(t) = \ln(\lambda + \sin t)$

$f(0) = \ln\lambda$

$\Rightarrow f(0) = 0$ when $\lambda = 1$

$\lambda = 1$ is a special case.

$\underline{\lambda \neq 1}$ $\Rightarrow$ $I(x) \sim \sqrt{\dfrac{\pi}{-2x\phi''(0)}}\, f(0) e^{x\phi(0)} = \sqrt{\dfrac{\pi}{-2x(-1)}}(\ln\lambda)e^x$

$$\boxed{I(x) \sim \sqrt{\frac{\pi}{2x}}(\ln\lambda)e^x, \ \lambda \neq 1}$$

$\underline{\lambda = 1}$ $\Rightarrow$ $f(0) = 0$ (try an alternate approach)

Expand ϕ and f about $t=0$

$\phi(t) = \cos t \sim 1 - t^2/2 + \cdots$

$f(t) \sim f(0) + t f'(0) + \cdots \sim 0 + t\cdot 1 + \cdots \sim t + \cdots$

Then,

$$I(x) \sim \int_0^{\pi/2} t e^{x(1-t^2/2)}\,dt \sim \frac{-1}{x} e^x \int_0^{\infty} e^{-x\frac{t^2}{2}}(-xt)\,dt$$

$$\sim \frac{-1}{x} e^x e^{-xt^2/2}\Big|_0^{\infty} = \frac{-1}{x} e^x (0-1) = e^x/x$$

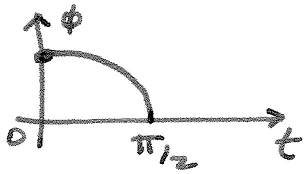
$$\boxed{I(x) \sim e^x/x, \ \lambda = 1}$$

$$\Rightarrow \boxed{I(x) \sim \begin{cases} \sqrt{\frac{\pi}{2x}}(\ln\lambda)e^x, & \lambda \neq 1 \\ e^x/x, & \lambda = 1 \end{cases} \quad \text{as } x \to \infty}$$

Take a closer look at the case of $\lambda = 1$

$$I(x) = \int_0^{\pi/2} e^{x\cos t} \ln(1+\sin t)\,dt$$

$\phi(t) = \cos t$

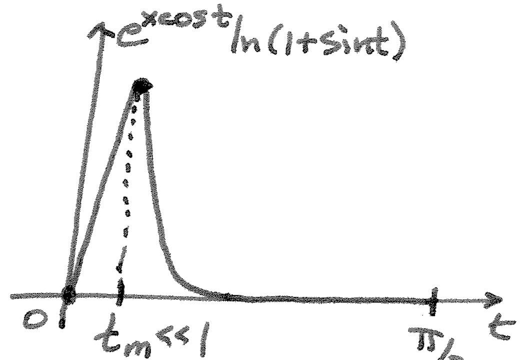


ϕ is maximal at $t=0$.

$f(t) = \ln(1+\sin t)$

$f(0) = \ln(1+0) = 0$

Integrand



Since $f(0)=0$, the maximum of the integrand does not occur at $t=0$. However, since $e^{x\cos t}$ dominates $\ln(1+\sin t)$, we can expect the maximum of the integrand to occur near $t=0$.

Expand the integrand, $g(t) = e^{x\cos t}\ln(1+\sin t)$, about $t=0$.

$$g(t) \sim e^{x(1-t^2/2+\cdots)} \ln(1+t) \sim e^x e^{-x\frac{t^2}{2}} \cdot t$$

$$\Rightarrow \boxed{g(t) \sim e^x e^{-xt^2/2} \cdot t \quad \text{for } t \ll 1}$$

Find the maximum of $g(t)$: $\quad g'(t) = e^x e^{-x\frac{t^2}{2}}(1-xt^2) = 0$

$$\Rightarrow \boxed{t_m = \frac{1}{\sqrt{x}} \ll 1}, \text{ with } g''(t_m) < 0.$$

Moving Maximum: The maximum of the integrand occurs at $t = t_m(x)$.

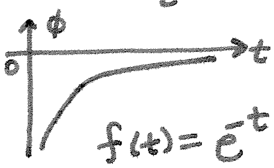
In the above example, the moving maximum occurs near $t=0$, so we may expand about $t=0$ accordingly.

Next, we'll consider the case of a large moving maximum, $t = t_m(x) \gg 1$.

Moving maximum at $t = t_m(x) \gg 1$

Example: $I(x) = \int_0^\infty e^{-t - x/t}\,dt = \int_0^\infty e^{-t} e^{-x/t}\,dt\,; \quad x \gg 1$

$\phi(t) = -\frac{1}{t}$



$f(t) = e^{-t}$

The maximum of ϕ occurs as $t \to \infty$, but $f \to 0$ as $t \to \infty$. $\Rightarrow$ formulas fail

The maximum of the integrand occurs when $g(t) = -t - x/t$ is maximal.

$$g(t) = -t - x/t$$

$$g'(t) = -1 + x/t^2 = 0$$

$$\Rightarrow \boxed{t_m = \sqrt{x} \gg 1}, \text{ with } g''(t_m) < 0.$$

Make an appropriate change of variable $(s = t/t_m)$.

Let $\boxed{s = t/\sqrt{x}}$. $\Rightarrow$ maximum occurs at $s = 1$.

Then,
$$I(x) = \int_0^\infty e^{-\sqrt{x}s - \frac{x}{\sqrt{x}s}} \cdot \sqrt{x}\,ds = \sqrt{x} \int_0^\infty e^{-\sqrt{x}\left(s + \frac{1}{s}\right)}\,ds$$

Laplace's method can now be applied.

$f(s) = 1$

$\phi(s) = -\left(s + \frac{1}{s}\right)$

$\phi'(s) = -1 + 1/s^2 = 0$

$\underline{s = 1}$

$\phi''(s) = \frac{-2}{s^3} < 0 \Rightarrow \underline{s = 1 \text{ is a max}}$ (interior)

Large parameter $= \sqrt{x}$

$$\Rightarrow \quad I(x) \sim \sqrt{x}\sqrt{\frac{2\pi}{-\sqrt{x}\,\phi''(1)}}\, f(1)\, e^{\sqrt{x}\,\phi(1)} = \sqrt{x}\sqrt{\frac{2\pi}{-\sqrt{x}(-2)}}\,(1)\, e^{-2\sqrt{x}}$$

$$\Rightarrow \quad \boxed{I(x) \sim \pi^{1/2} x^{1/4} e^{-2\sqrt{x}}} \quad \text{as } x \to \infty.$$

Gamma Function

Example: $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x \neq 0, -1, -2, \ldots$

Approximate $\Gamma(x)$ to leading order for $x \gg 1$.

It is convenient to consider $\Gamma(x+1)$.

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = \int_0^\infty e^{-t} e^{x \ln t} dt$$

$f(t) = e^{-t}$
$\phi(t) = \ln t$
ϕ is maximum as $t \to \infty$, but $f \to 0$ as $t \to \infty$.

Moving Maximum: The integrand is maximum when $g(t) = -t + x\ln t$ is maximum.

$$g(t) = -t + x \ln t$$
$$g'(t) = -1 + \frac{x}{t} = 0 \implies \boxed{t_m = x}$$
$$g''(t) = -x/t^2 < 0 \implies \text{maximum (interior)}$$

Change of Variable: Let $\boxed{s = \frac{t}{x}} \implies$ The maximum occurs at $s = 1$.

$$\Gamma(x+1) = \int_0^\infty e^{-sx} e^{x \ln(sx)} x\, ds = x e^{x \ln x} \int_0^\infty e^{-sx} e^{x \ln s} ds$$

$$\Gamma(x+1) = x^{x+1} \int_0^\infty e^{-x(s - \ln s)} ds$$

Laplace's method

$f(t) = 1$

$\phi(s) = -s + \ln s$

$\phi'(s) = -1 + \frac{1}{s} = 0 \implies \boxed{s = 1}$

$\phi''(s) = -\frac{1}{s^2} < 0 \implies$ maximum (interior)

$$\Gamma(x+1) \sim x^{x+1} \sqrt{\frac{2\pi}{-x\phi''(1)}} f(1) e^{x\phi(1)} = x^{x+1} \sqrt{\frac{2\pi}{x}} e^{-x}$$

$$\implies \boxed{\Gamma(x+1) \sim x^x \sqrt{2\pi x}\, e^{-x}}$$

$$\Gamma(x) = ?$$

Sterling's Formula: $n! = \Gamma(n+1)$, $n = 1, 2, \ldots$

$$\boxed{n! \sim \sqrt{2\pi n}\, n^n e^{-n},\ n = 1, 2, \ldots}$$

n	$n!$	$\sqrt{2\pi n}\, n^n e^{-n}$
1	1	.922
2	2	1.92
10	3.63×10^6	3.60×10^6
100	9.333×10^{157}	9.325×10^{157}

An approximation of $\Gamma(x)$ can be found by replacing x by $x-1$ in the above approximation of $\Gamma(x+1)$.

$$\Rightarrow \boxed{\begin{array}{l}\Gamma(x) \sim (x-1)^{x-1}\sqrt{2\pi(x-1)}\, e^{-(x-1)} \text{ as } x \to \infty \\ \sim (x-1)^{x-1}\sqrt{2\pi x}\, e^{-(x-1)}\end{array}} \quad (1)$$

Alternatively, we may use the formula $\Gamma(x+1) = x\,\Gamma(x)$.

$$\Rightarrow \Gamma(x) = \frac{1}{x}\Gamma(x+1) \sim \frac{1}{x} x^x \sqrt{2\pi x}\, e^{-x}$$

$$\boxed{\Gamma(x) \sim x^x \sqrt{\frac{2\pi}{x}}\, e^{-x} \text{ as } x \to \infty} \quad (2)$$

The approximation is not unique.

Exercise: Show that (1) and (2) are asymptotically equivalent as $x \to \infty$.
i.e. show that $\frac{(1)}{(2)} \to 1$ as $x \to \infty$.

$$\frac{(1)}{(2)} = \frac{(x-1)^{x-1}\sqrt{2\pi(x-1)}\, e^{-(x-1)}}{x^x\sqrt{\frac{2\pi}{x}}\, e^{-x}} = \frac{(x-1)^{x-\frac{1}{2}}\cancel{e^{-x}}\, e}{x^{x-\frac{1}{2}}\cancel{e^{-x}}} = \left(1-\frac{1}{x}\right)^{x-\frac{1}{2}} e$$

$$= \frac{\left(1-\frac{1}{x}\right)^x e}{\left(1-\frac{1}{x}\right)^{1/2}} \sim \left(1-\frac{1}{x}\right)^x e \to e^{-1} e = 1 \text{ as } x \to \infty$$

Examples

#1 $I(x)=\int_0^\infty \cos t \cdot e^{-x(t^2-2t)}dt \sim \sqrt{\dfrac{2\pi}{-x\phi''(1)}}\, f(1)\, e^{x\phi(1)}$

$f(t)=\cos t$
$\phi(t)=-t^2+2t$
$\phi'(t)=-2t$ interior
$\phi''(t)=-2$ max at $t=1$
$f(1)\neq 0$

$$\Rightarrow \boxed{I(x)\sim\sqrt{\frac{\pi}{x}}(\cos 1)e^x+O\left(\frac{e^x}{x^{3/2}}\right)}$$

x	Exact	$\sqrt{\frac{\pi}{x}}(\cos 1)e^x$	$\lvert\text{Error}\rvert$	$e^x/x^{3/2}$	Relative Error
2	4.212	5.004	.79	2.6	.19
5	60.37	63.56	3.2	13.3	.05
10	6506	6670	164	697	.02
100	2.568×10^{42}	2.574×10^{42}	6×10^{39}	2.7×10^{40}	.0025
1000	5.9647×10^{432}	5.9661×10^{432}	1.4×10^{429}	6.2×10^{429}	.00024

#2 $I(z)=\int_{.2}^{1} e^{z(1-\frac{1}{T})}dT = e^z\int_{.2}^{1} e^{-z/T}dT \sim e^z\cdot\dfrac{f(1)\,e^{z\phi(1)}}{z\,\phi'(1)}$

$f(t)=1$ endpoint
$\phi(t)=-\frac{1}{T}$ max at $t=1$
$f(1)\neq 0$

$\sim e^z\,\dfrac{1\cdot e^{-z}}{z\cdot 1}=\dfrac{1}{z}$

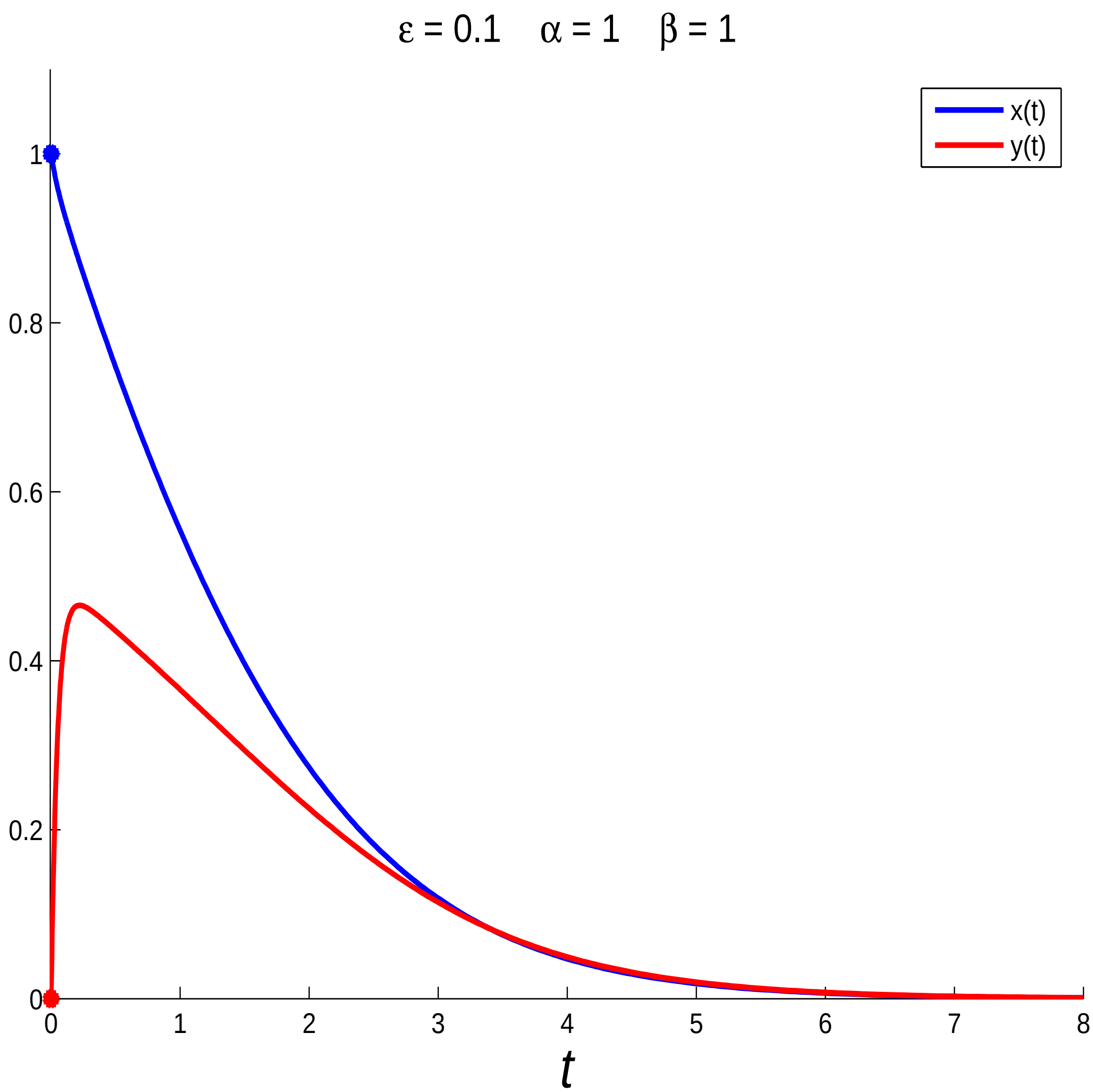
$$\boxed{I(z)\sim\frac{1}{z}+O\left(\frac{1}{z^2}\right)}$$

z	Exact	$\frac{1}{z}$	$\lvert\text{Error}\rvert$	$1/z^2$	Relative Error
2	.2773	.5	.22	.25	.80
5	.1479	.2	.052	.04	.35
10	.08436	.1	.016	.01	.19
100	.009806	.01	1.94×10^{-4}	10^{-4}	.020
1000	$.9980\times10^{-3}$	.001	2.0×10^{-6}	10^{-6}	.002
10000	$.9998\times10^{-4}$	.0001	2.0×10^{-8}	10^{-8}	.0002 $\approx 2/z^2$

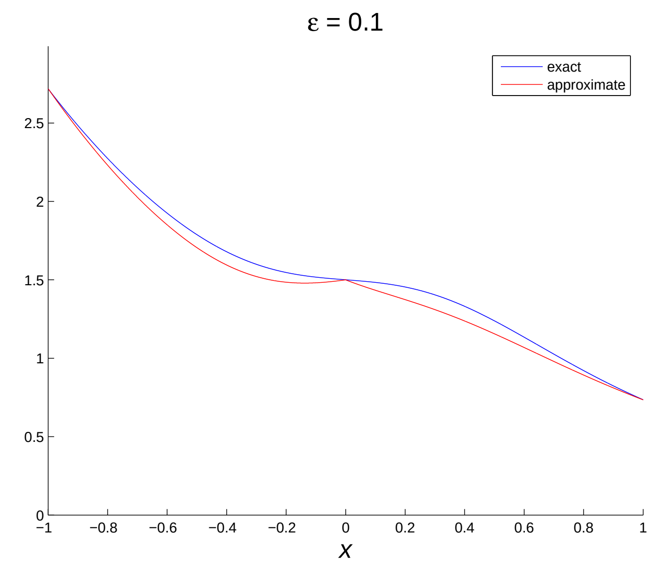
Higher Order

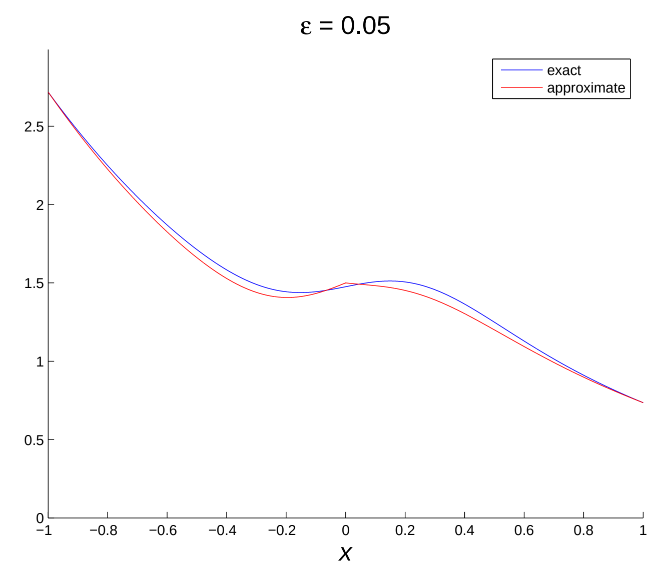
Integration by Parts yields

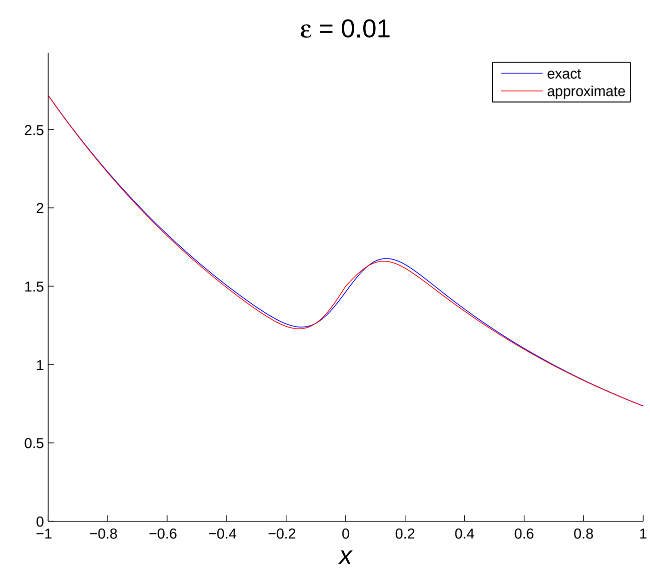
$$\boxed{I(z)\sim\frac{1}{z}+\frac{2}{z^2}+O\left(\frac{1}{z^3}\right)}$$

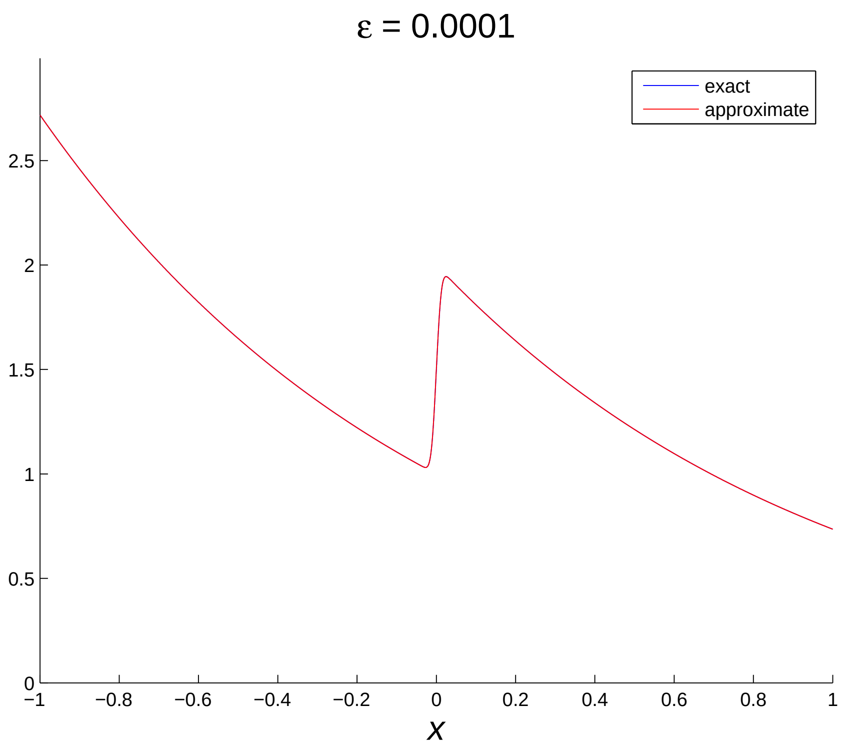


ε = 0.1 α = 1 β = 1
x(t)
y(t)
1
0.8
0.6
0.4
0.2
0
0
1
2
3
4
5
6
7
8
t

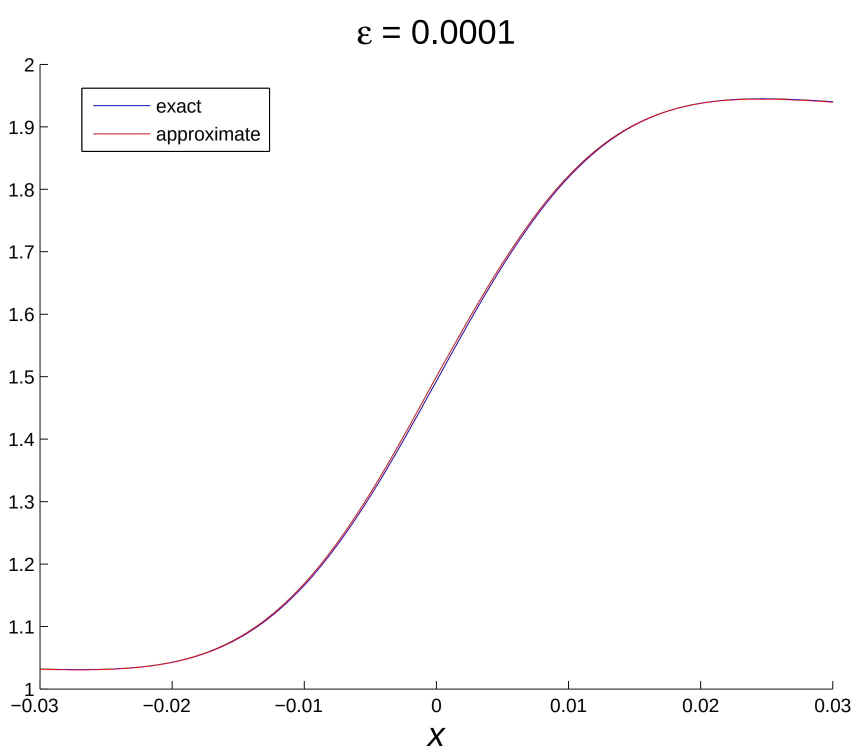

ε = 0.1
exact
approximate
0
0.5
1
1.5
2
2.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
x


ε = 0.05
exact
approximate
0
0.5
1
1.5
2
2.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
x


ε = 0.01
exact
approximate
0
0.5
1
1.5
2
2.5
−1
−0.8
−0.6
−0.4
−0.2
0
0.2
0.4
0.6
0.8
1
x



Zoom in on the interior layer.



Laplace Method Formulas

Laplace's method is used to approximate integrals of the form

$$I(x) \;=\; \int_a^b f(t)\, e^{x\phi(t)}\, dt \quad \text{as} \quad x \to \infty.$$

The asymptotic approximation is determined at the point(s) where ϕ attains its absolute maximum on $[a, b]$.

Case: $\phi'(t) \neq 0$ in $[a, b]$

In this case, ϕ has no local maximums in $[a, b]$ so the absolute maximum must occur at an endpoint.

max at t = a: $$\boxed{I(x) \;\sim\; -\frac{f(a)}{x\,\phi'(a)}\, e^{x\phi(a)}}$$ provided $f(a) \neq 0$

max at t = b: $$\boxed{I(x) \;\sim\; \frac{f(b)}{x\,\phi'(b)}\, e^{x\phi(b)}}$$ provided $f(b) \neq 0$

Case: $\phi'(t) = 0$ at a unique interior point, say at $t = c \in (a, b)$

In this case $\phi(c)$ is either a local maximum or a local minimum (or an inflection point).

local min: If $\phi(c)$ is a local minimum (or an inflection point), then the absolute maximum of $\phi(t)$ occurs at an endpoint and the above formulas can be used.

- There is one *exception*: If $\phi(t)$ has a local minimum at an interior point, then it may be that $\phi(a) = \phi(b)$, in which case there is a significant contribution to the value of the integral at each endpoint. The contributions given by the above formulas must be added.

local max: If $\phi''(c) < 0$, then $\phi(c)$ must be the absolute maximum of ϕ on $[a, b]$. In this case, the dominant contribution to the integral occurs at $t = c$.

$$\boxed{I(x) \;\sim\; \sqrt{\frac{2\pi}{-x\,\phi''(c)}}\; f(c) e^{x\phi(c)}}$$ provided $f(c) \neq 0$

Case: $\phi'(d) = 0$ where d is an endpoint and the absolute maximum of ϕ

$$\boxed{I(x) \;\sim\; \sqrt{\frac{\pi}{-2x\,\phi''(d)}}\; f(d) e^{x\phi(d)}}$$ provided $f(d) \neq 0$

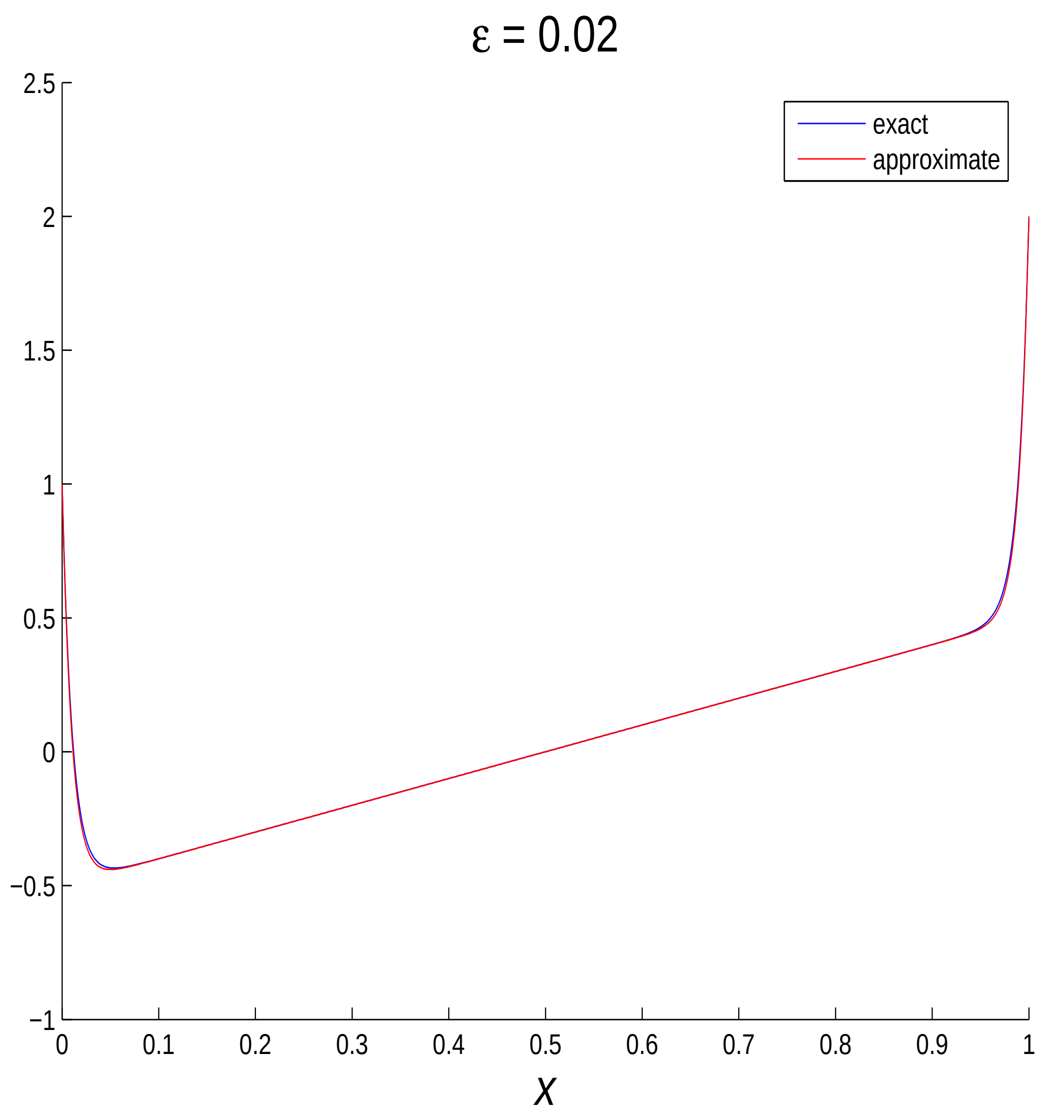
Case: Interior absolute max at $t = c$ with $\phi'(c) = \phi''(c) = \cdots = \phi^{(p-1)}(c) = 0$

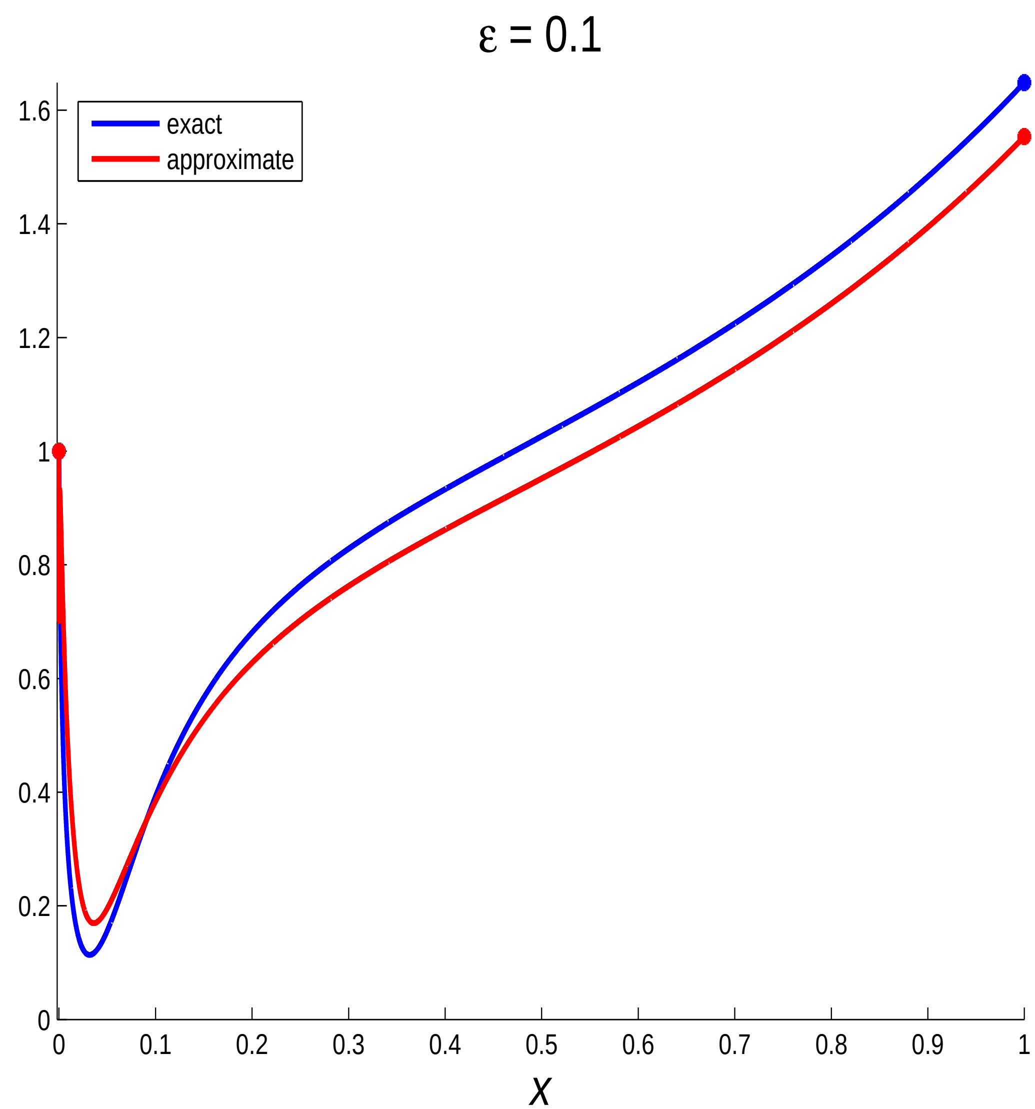
$$\boxed{I(x) \;\sim\; \frac{2\,\Gamma\!\left(\frac{1}{p}\right)(p!)^{1/p}}{p\left[-x\,\phi^{(p)}(c)\right]^{1/p}}\; f(c) e^{x\phi(c)}}$$ provided $f(c) \neq 0$

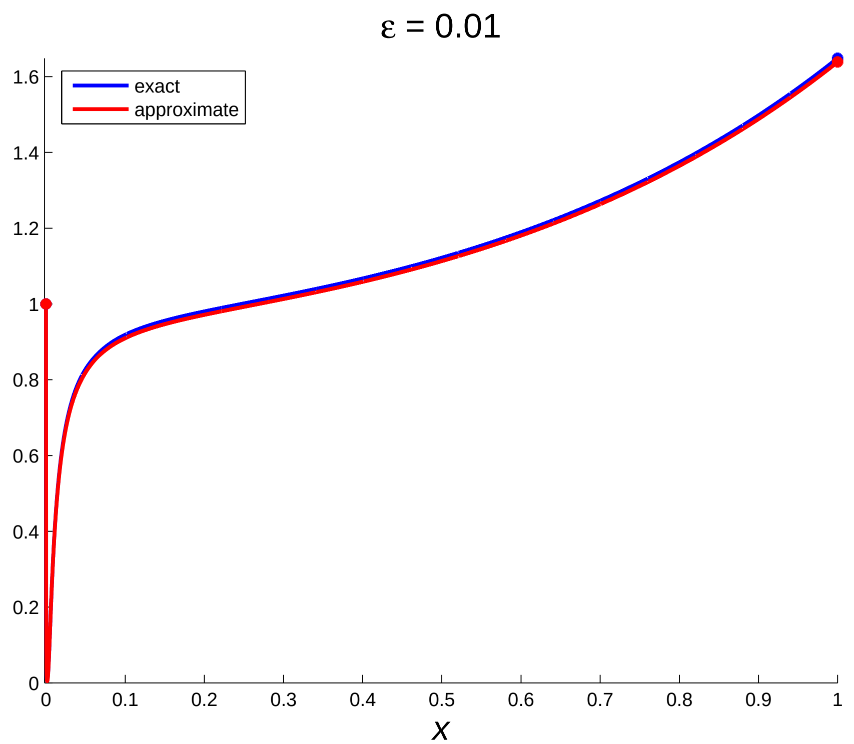
Case: The absolute maximum occurs at multiple points: $t = c_1, c_2, \ldots, c_n$

$$I(x) \;\sim\; I_1(x) + I_2(x) + \cdots + I_n(x),$$

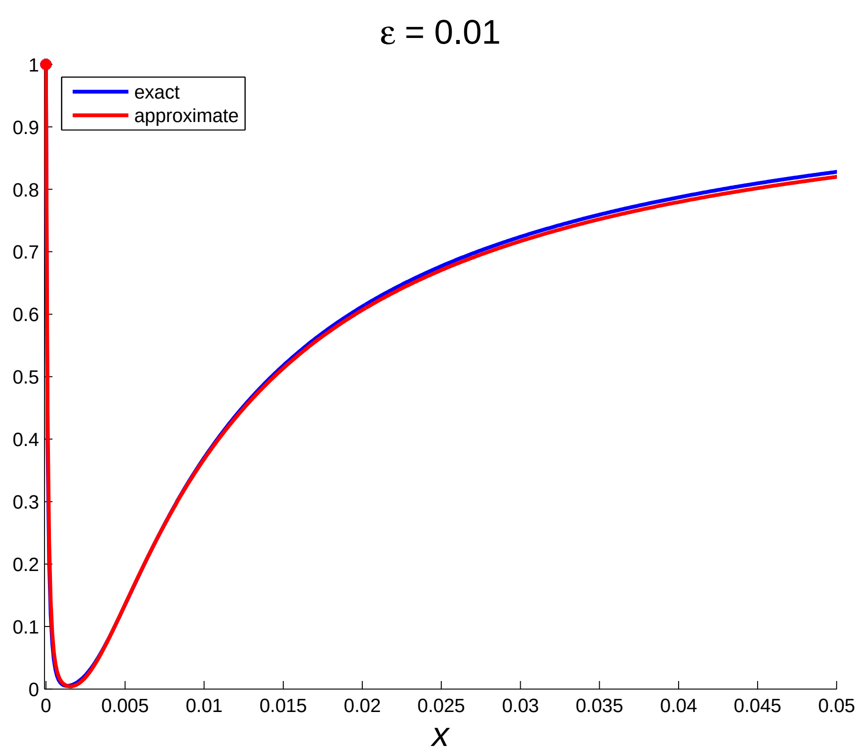
where $I_k(x)$ denotes the contribution to the value of the integral at $t = c_k$. $I_k(x)$ can be determined from the above formulas. One example of this case is when ϕ is periodic.


ε = 0.02
exact
approximate
2.5
2
1.5
1
0.5
0
−0.5
−1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x


ε = 0.1
exact
approximate
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x



Zoom in on the two-ply boundary layer.



Trigonometric Identities

$\cos^2 A = \frac{1}{2}(1 + \cos 2A)$

$\sin^2 A = \frac{1}{2}(1 - \cos 2A)$

$\cos A\ \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$\sin A\ \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$\sin A\ \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$

$\cos^3 A = \frac{1}{4}\left[3\cos A + \cos 3A\right]$

$\sin^3 A = \frac{1}{4}\left[3\sin A - \sin 3A\right]$

$\cos^2 A\ \cos B = \frac{1}{4}\left[\cos(2A+B) + \cos(2A-B) + 2\cos B\right]$

$\cos^2 A\ \sin B = \frac{1}{4}\left[\sin(2A+B) - \sin(2A-B) + 2\sin B\right]$

$\sin^2 A\ \cos B = -\frac{1}{4}\left[\cos(2A+B) + \cos(2A-B) - 2\cos B\right]$

$\sin^2 A\ \sin B = -\frac{1}{4}\left[\sin(2A+B) - \sin(2A-B) - 2\sin B\right]$