# 306: Differential Equations I

## Chapter 1: Introduction to Differential Equations

**Definition:** A Differential Equation (DE) is an equation which involves derivatives

**Examples:** All quantities, other than the dependent and independent variables, are constant.

### ① Exponential Growth/Decay

$$\frac{dx}{dt} = kx, \quad x \geq 0$$

Goal: Find $x(t)$

$x$ = amount of some quantity

$k > 0 \Rightarrow$ growth (e.g. bacterial populations)

$k < 0 \Rightarrow$ decay (e.g. radioactive decay, carbon dating)

### ② Logistic Population Model

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P, \quad P \geq 0$$

Goal: Find $P(t)$

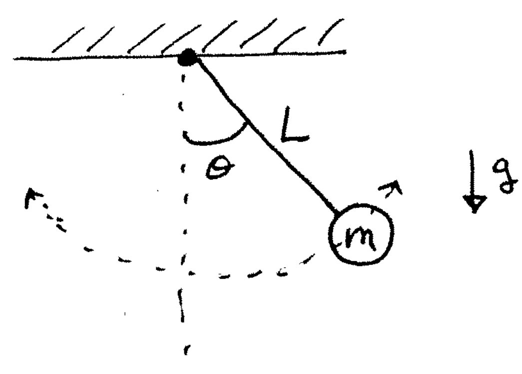
$P$ = size of population

$r, K > 0$ are real constants

### ③ Pendulum

Newton's 2nd Law ($F = ma$) $\Rightarrow$ $\dfrac{d^2\theta}{dt^2} + \dfrac{g}{L}\sin\theta = 0$

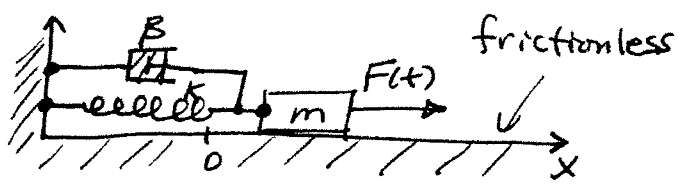
Goal: Find $\theta(t)$



## 4 Forced Mass-Spring-Damper System

Newton's 2nd Law $\Rightarrow$ $mx'' + \beta x' + kx = F(t)$

Goal: Find $x(t)$




$x$ = displacement from the relaxed state
$m > 0$ (mass)
$\beta \geq 0$ (damper constant)
$k > 0$ (spring constant)

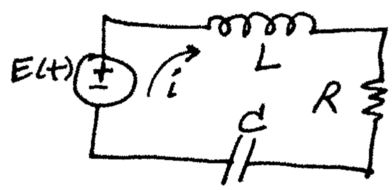
## 5 Circuit Analysis

Kirchoff's Law $\Rightarrow$ $Lq'' + Rq' + \frac{1}{C}q = E(t)$

Goal: Find $q(t)$

Once $q(t)$ is known, voltages and the current can be determined.

RLC circuit




$E$ = voltage source (e.g. battery, wall socket)
$R$ = resistance
$L$ = inductance
$C$ = capacitance
$q$ = charge on the capacitor

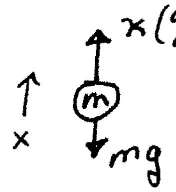
## 6 Falling Object subject to Air Resistance

Newton's 2nd Law $\Rightarrow$ $m\frac{d^2x}{dt^2} = -mg + k\left(\frac{dx}{dt}\right)^2$

could be 1st power

Goal: Find $x(t)$

$k\left(\frac{dx}{dt}\right)^2$ (air resistance)

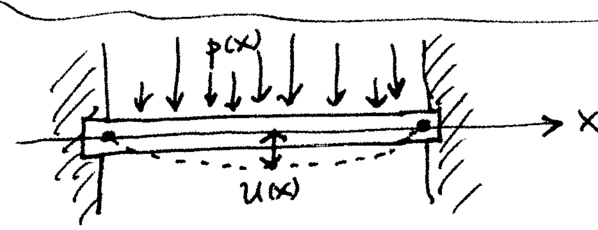



$x$ = altitude
$m$ = [redacted] mass
$k > 0$ is a constant

## 7 Elastic Beam

$$\alpha \frac{d^4u}{dx^4} - \beta \frac{du^2}{dx^2} = -p(x)$$

Goal: Find $u(x)$




$u$ = displacement from the relaxed state
$p(x)$ = load on beam
$\alpha, \beta > 0$ are constants

More examples are given in section 1.3.

DEs arise frequently in scientific and engineering applications.

To investigate a physical system (such as in the above examples),

1. derive the DEs which describe the system using
   i) the laws of physics,
   and/or ii) reasonable assumptions.
2. Solve and/or analyze the DEs to obtain information about the behavior of the system.

Our main focus will be on step 2.

In this course, we'll mainly consider given DEs, and attempt to describe the solution behavior by either
   i) finding the solution,
   or ii) by other means (e.g. – direction fields – numerical methods)

Possibility i) is usually not possible for real-life problems.

DEs may be classified in various ways.

e.g. – separable/non-separable
– linear/non-linear
– etc.

We'll discuss several methods for solving DEs. The solution method to be used when solving a particular DE follows from the classification of the DE. i.e. The classification of a DE indicates which solution method to use.

e.g. Example ④ is a $2^{nd}$ order, linear, non-homogeneous, scalar, ordinary DE with constant coefficients

$\Longrightarrow$ Use Method X

# Section 1.1: Classification of DEs/Terminology

## Classifications

**I. Order:** The *order* of a DE is the order of the highest-ordered derivative appearing in the DE.

Examples:
1. 1st order
2. 1st order
3. 2nd order
4. 2nd order
5. 2nd order
6. 2nd order
7. 4th order

In this course, we'll focus almost exclusively on 1st and 2nd order DEs.

1st order DEs: Chapters 2 and 3

2nd order DEs: Chapters 4 and 5

Laplace Transforms: Chapter 7

## II Linear/Nonlinear:

A DE is linear if it has a linear dependence on the dependent variable (unknown function) and its derivatives.

General Form of an $n^{th}$ order linear DE: $\boxed{a_n(x)\dfrac{d^ny}{dx^n}+a_{n-1}(x)\dfrac{d^{n-1}y}{dx^{n-1}}+\cdots+a_2(x)\dfrac{d^2y}{dx^2}+a_1(x)\dfrac{dy}{dx}+a_0(x)y=g(x)}$

Special Cases: $1^{st}$ order linear DE: $\boxed{a_1(x)\dfrac{dy}{dx}+a_0(x)y=g(x)}$

$2^{nd}$ order linear DE: $\boxed{a_2(x)\dfrac{d^2y}{dx^2}+a_1(x)\dfrac{dy}{dx}+a_0(x)y=g(x)}$

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Notes: 1) The dependent variable ($y$) and its derivatives appear everywhere to the first power. There are no products or transcendental functions. For example, the equation for a line, $ax+by=c$, is linear in $x$ and $y$.

2) The coefficients $a_j(x)$ $(j=0,1,\ldots,n)$ may be nonlinear functions of $x$.

e.g. $x^2y'+(\sin x)y=\ln x$ is a linear DE.

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If a DE is not linear, it is said to be nonlinear.

i.e. A DE is nonlinear if it cannot be put into the above general form.

some nonlinear terms: $y^2,\ \left(\frac{dy}{dx}\right)^2,\ y\frac{dy}{dx},\ \sin y,\ \sqrt{y},\ \ldots$

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Examples: ① Linear ⑤ Linear

② Nonlinear ⑥ Nonlinear

③ Nonlinear ⑦ Linear

④ Linear

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There are huge differences between linear and nonlinear DEs.
Linear DEs are often solvable, the theory is well-developed, and solutions are well-understood.
Nonlinear DEs are much less predictable than linear DEs.
Only the simplest nonlinear DEs are solvable. Nonlinear DEs can describe complex and interesting systems.
In this course, we'll consider both linear and nonlinear DEs.

## III. Ordinary/Partial

Ordinary Diff. Eq. (ODE): Derivatives are with respect to a single independent variable.

Examples ① – ⑦ are all ODEs.

Partial Diff. Eq. (PDE): Derivatives are with respect to 2 or more independent variables.

e.g. The one-dimensional 'heat conduction equation' is

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2}, \text{ where } u(x,t) = \text{temperature.}$$

partial derivatives

In this course, we'll consider only ODEs.

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## IV. Scalar/System

Scalar: A scalar DE is a single DE with a single unknown function

Examples ① – ⑦ are all scalar DEs.

System: A system of DEs is a set of $n$ DEs involving $n$ unknown functions.

e.g. Predator vs. Prey Population Model $(n=2)$

$x(t)$ = population of prey
$y(t)$ = population of predator

$$\begin{cases} \dfrac{dx}{dt} = 2x - xy \\ \dfrac{dy}{dt} = -3y + 5xy \end{cases}$$

(1st order nonlinear system of ODEs)

Since both equations depend on both unknowns ($x$ and $y$), they cannot be solved separately. Rather, they must be solved simultaneously.

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Note: Though we'll consider only scalar DEs, systems of DEs appear in many of the sections that we'll discuss. Any mention of systems may be ignored, but feel free to read about systems if interested.

# Solutions

**Definition**: A *solution* of an ODE is any continuous function that satisfies the ODE on some interval $I$.
The graph of a solution is called a *solution curve*.

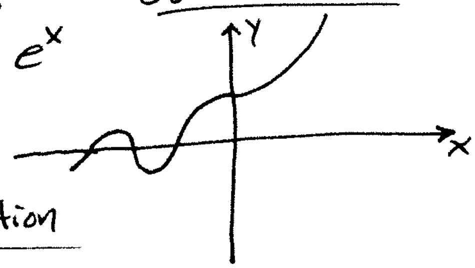
## Verification of Solutions

**Example**: Verify that $y(x) = \cos x + \frac{1}{2}e^x$ is a solution of the ODE
$$y'' + y = e^x. \quad (2^{\text{nd}} \text{ order, linear})$$

$$y = \cos x + \tfrac{1}{2}e^x$$
$$y' = -\sin x + \tfrac{1}{2}e^x$$
$$y'' = -\cos x + \tfrac{1}{2}e^x$$

$$y'' + y \overset{?}{=} e^x$$
$$(-\cancel{\cos x} + \tfrac{1}{2}e^x) + (\cancel{\cos x} + \tfrac{1}{2}e^x) \overset{?}{=} e^x$$
$$e^x \overset{?}{=} e^x$$
$$0 = 0 \checkmark$$
$$\Rightarrow y = \cos x + \tfrac{1}{2}e^x \text{ is a solution}$$

Solution Curve




**Note**: The equation has infinitely many solutions given by
$$y(x) = C_1 \cos x + C_2 \sin x + \tfrac{1}{2}e^x \text{ for any constants } C_1 \text{ and } C_2.$$

**Example**: Is $p(t) = 1 + e^{-2t}$ a solution of the ODE $\frac{dp}{dt} + 2p = 1$ ?

$$p = 1 + e^{-2t}$$
$$\frac{dp}{dt} = -2e^{-2t}$$

$$\frac{dp}{dt} + 2p \overset{?}{=} 1$$
$$(-2e^{-2t}) + 2(1 + e^{-2t}) \overset{?}{=} 1$$
$$-2\cancel{e^{-2t}} + 2 + 2\cancel{e^{-2t}} \overset{?}{=} 1$$
$$2 \neq 1 \Rightarrow \text{NOT a solution}$$

**Note**: The equation has infinitely many solutions given by
$$p(t) = \tfrac{1}{2} + Ce^{-2t} \text{ for any constant } C.$$

Observe that the ODEs in the above examples have infinitely many solutions. The same is true of all ODEs that we will consider in this course.

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**Definition:** The <u>general solution</u> of an ODE is a family of solutions which represent all possible solutions of the ODE.

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## $1^{st}$ order ODEs

The general solution of a $1^{st}$ order ODE is a 1-parameter family of solutions.

e.g. Consider the exponential growth/decay equation,

$$\frac{dx}{dt} = kx.$$

The ODE has infinitely many solutions given by $x(t) = Ce^{kt}$ for any constant $C$. Furthermore, all solutions have this form.

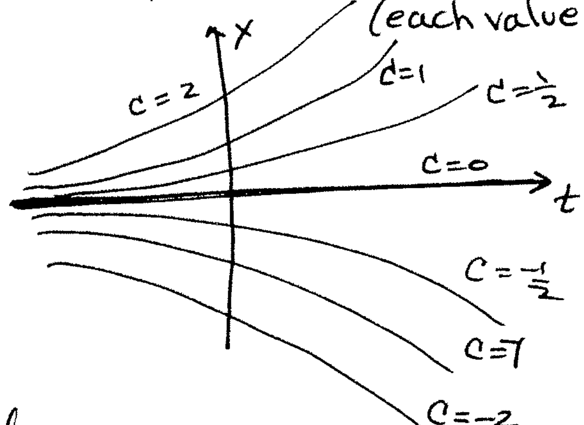
i.e. $x(t) = Ce^{kt}$ represents all possible solutions.

$\Rightarrow$ $\boxed{x(t) = Ce^{kt} \text{ is the general solution of } \frac{dx}{dt} = kx}$

$x(t) = Ce^{kt}$ is a 1-parameter family of solutions.

There are infinitely many solutions parameterized by $C$. (each value of $C$ gives a different solution)

$k > 0$




The solution curves fill out the entire plane.

To solve $\frac{dx}{dt} = kx$, we separate the variables and integrate once, thus yielding a single constant of integration,

$$x(t) = \underline{\underline{C}}e^{kt}$$

- To solve a 2nd order ODE, we must effectively integrate twice, thus yielding 2 constants of integration. It follows that the general solution of a 2nd order ODE is a 2-parameter family of solutions.

e.g. Consider the ODE $x''(t) = 1$

integrate once $x'(t) = t + c_1$

integrate again $x(t) = t^2/2 + \underline{c_1} t + \underline{c_2}$

$\uparrow \quad \uparrow$

2-parameter family of solutions.

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| |
|---|
| The general solution of an $n^{th}$ order ODE is an $n$-parameter family of solutions. |

- e.g. The general solution of $\frac{d^4y}{dx^4} - \frac{d^2y}{dx^2} = 0$ ($4^{th}$ order)

is $y = \underline{c_1} + \underline{c_2} x + \underline{c_3} e^x + \underline{c_4} e^{-x}$

4 constants (parameters)

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Definition: Any one of the infinitely many solutions of an ODE is called a particular solution.

e.g. In the above example, $y(x) = 1 + 2x - e^x + \frac{10}{3} e^{-x}$ is a particular solution.

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Definition: The trivial solution of an ODE is a solution which is identically equal to zero. (e.g. $y = 0$)

Some ODEs posses the trivial solution, others do not.

- e.g. Consider $y'' + 2y' + y \stackrel{?}{=} 0$ and $y'' + 2y' + y \stackrel{?}{=} 1$

$y = 0$, $y' = 0$, $y'' = 0$

$0 + 0 + 0 \stackrel{?}{=} 0$ $\qquad$ $0 + 0 + 0 \stackrel{?}{=} 1$

$0 = 0$ ✓ $\qquad$ $0 \neq 1$

$y = 0$ is a solution $\qquad$ $y = 0$ is not a solution.

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Explicit/Implicit Solutions: Postpone until section 2.2.

# Section 1.2: Initial Value Problems

Suppose a culture of bacteria is started at time $t=0$, and the population size obeys the exponential growth equation.

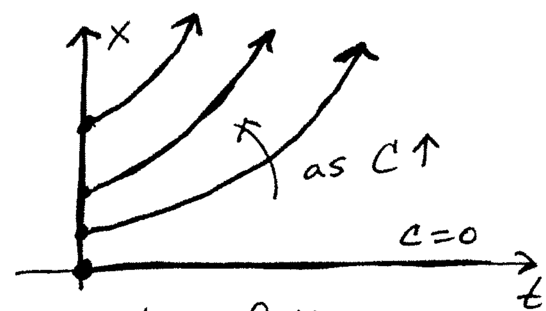
$$X(t) = \text{\# of bacteria at time } t$$

$$\Rightarrow \boxed{\frac{dx}{dt} = kx \ ; \ x \geq 0, \ t \geq 0, \ \underbrace{k>0}_{\text{growth}}} \quad \text{(assume } k \text{ is known)}$$

The general solution is

$$\boxed{X(t) = Ce^{kt}, \ C \geq 0}$$

$C = ?$




Question: Which curve does the population follow.

$C$ cannot be determined from the given information. That is, more information about the population is needed to select a single value of $C$. We must impose an additional condition on the general solution to select one of the infinitely many possible solutions.

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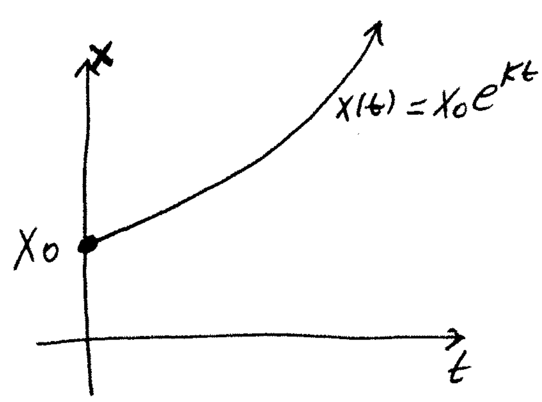
Suppose that the initial size of the population (at $t=0$) is known to be $X_0$. That is, $\boxed{X(0) = X_0}$. $X_0$ = some given constant. We can determine the appropriate value of $C$ by imposing this condition on the general solution.




general solution

$$X(t) = Ce^{kt} \longrightarrow \begin{array}{c} X(0) = X_0 \\ C\overbrace{\cancel{e^0}}^{=1} = X_0 \end{array}$$

$$\underline{C = X_0}$$

$$\Rightarrow \boxed{X(t) = X_0 e^{kt}} \quad \begin{pmatrix}\text{particular} \\ \text{solution}\end{pmatrix}$$

Satisfies both the ODE and the condition $X(0) = X_0$.

## Alternative Condition

Suppose that the initial population size is not known, but rather we measure the population size at time $t_1$ to be $x_1$.

That is, $\boxed{x(t_1) = x_1}$ $t_1$ and $x_1$ are given constants

general solution

$x(t) = Ce^{kt} \longrightarrow$ $x(t_1) = x_1$

$$Ce^{kt_1} = x_1$$

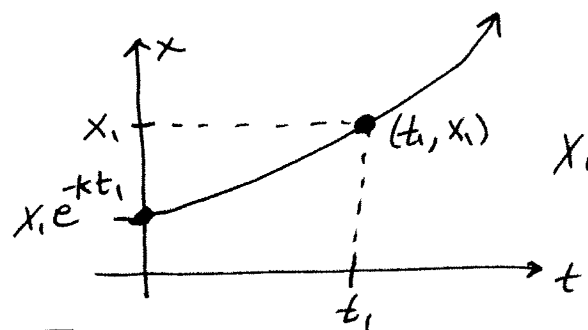
$$C = x_1 e^{-kt_1}$$

Then, $x(t) = Ce^{kt} = (x_1 e^{-kt_1}) e^{kt} = x_1 e^{k(t-t_1)}$

$$\boxed{x(t) = x_1 e^{k(t-t_1)}} \text{ (particular solution)}$$

satisfies both the ODE and the condition $x(t_1) = x_1$.

$x(t_1) = x_1 e^{k(t_1-t_1)} = x_1 e^0 = x_1$ ✓




$x(0) = x_1 e^{k(0-t_1)} = x_1 e^{-kt_1}$

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The ODE $\frac{dx}{dt} = kx$ together with an additional condition is called an <u>Initial Value Problem</u>.

$1^{st}$ order ODE $\Rightarrow$ The general solution involves 1 constant $\Rightarrow$ need 1 condition to determine the constant

The general solution of an $n^{th}$ order ODE involves $n$ arbitrary constants. To select just one of the infinitely many solutions, $n$ conditions must be imposed on the general solution to determine the $n$ constants.

$n^{th}$ order ODE $\Rightarrow$ The general solution involves $n$ constants $\rightarrow$ need $n$ conditions to determine the $n$ constants.

<u>Definition</u>: An $n^{th}$ order ODE together with a set of $n$ so-called <u>Initial Conditions</u> (ICs) of the form

$$y(x_0)=y_0,\ y'(x_0)=y_1,\ y''(x_0)=y_2,\ \ldots,\ y^{(n-1)}(x_0)=y_{n-1}$$

is called an <u>Initial Value Problem</u> (IVP).

<u>e.g.</u> <u>$1^{st}$ order IVP</u>: $y'+x^2y=e^x\ ;\ y(1)=5$

<u>$2^{nd}$ order IVP</u>: $t^2x''+2x^2=\cos t\ ;\ \begin{matrix} x(-1)=2 \\ x'(-1)=-5 \end{matrix}$

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<u>Notes</u>:

1) The ICs specify the value of $y$ and its derivatives at a single $x$-value, namely at $x_0$.
2) The term "Initial Condition" is a bit misleading
   - The ICs are not necessary specified at the initial time, that is, $x_0$ can be any value of the independent variable.
   - The term is used even if the independent variable is not time.

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To solve an IVP,

i) Find the general solution of the ODE, and then

ii) Impose the IC(s) on the general solution to determine the constants, and thus arriving at a particular solution.

<u>Example</u>: Given that the general solution of the ODE $y'' - 4y = 0$ is $y(x) = C_1 e^{2x} + C_2 e^{-2x}$, solve the IVP

$$\boxed{y'' - 4y = 0 \; ; \; y(0) = 1, \; y'(0) = 0}$$

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<u>general solution</u>:
$$y = C_1 e^{2x} + C_2 e^{-2x}$$
$$y' = 2C_1 e^{2x} - 2C_2 e^{2x}$$

<u>ICs</u>:

$$y(0) = 1 \implies y(0) = C_1 \overset{=1}{\cancel{e^0}} + C_2 \overset{=1}{\cancel{e^0}} = 1$$
$$\underline{\underline{C_1 + C_2 = 1}}$$

$$y'(0) = 0 \implies y'(0) = \cancel{2}C_1 e^0 - \cancel{2}C_2 e^0 = 0$$
$$\underline{\underline{C_1 - C_2 = 0}}$$

We have 2 equations to determine the 2 constants ($C_1$ and $C_2$).

$$\begin{array}{rl} & C_1 + C_2 = 1 \\ + & (C_1 - C_2 = 0) \\ \hline & 2C_1 = 1 \\ & C_1 = \frac{1}{2} \end{array} \qquad \begin{array}{l} C_2 = 1 - C_1 = 1 - \frac{1}{2} \\ C_2 = \frac{1}{2} \end{array}$$

OR

$$C_1 - C_2 = 0 \implies C_2 = C_1$$
$$C_1 + C_2 = C_1 + C_1 = 2C_1 = 1$$
$$C_1 = \frac{1}{2} = C_2$$

$$\implies y(x) = C_1 e^{2x} + C_2 e^{-2x} = \frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}$$

$$\boxed{y(x) = \frac{1}{2} e^{2x} + \frac{1}{2} e^{-2x}}$$ satisfies both the ODE and the 2 ICs.

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## <u>Existence and Uniqueness of Solutions</u>

Given an IVP, two important questions arise.

1. Does a solution exist?
2. If so, is it unique? (i.e. Is it the only solution, or are there others?)

We won't worry much about these issues. We'll discuss some theorems which state that the IVPs in this course will have unique solutions.

<u>Note</u>: Linear IVPs have unique solutions.

<u>Recall</u>: A solution of an ODE is a function which is continuous (and differentiable) and satisfies the ODE over some interval $I$.

($I$ = Interval of Definition)

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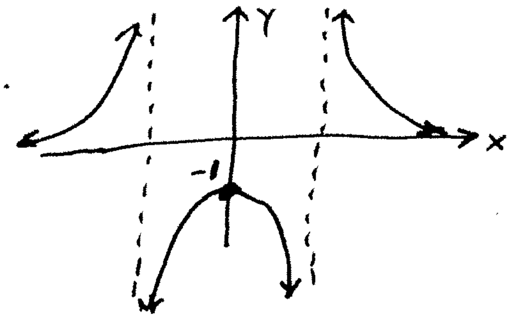
<u>EXAMPLE 2 (pg. 14)</u>: Consider $y(x) = \frac{1}{x^2-1}$.




1. Considered as a function, $y(x) = \frac{1}{x^2-1}$ is defined for all $x \neq \pm 1$. $\Rightarrow$ domain $= (-\infty,-1) \cup (-1,1) \cup (1,\infty)$

2. Now, consider $y(x) = \frac{1}{x^2-1}$ to be a solution of the ODE $y' + 2xy^2 = 0$.
   The interval $I$ of definition may be taken to be any interval over which $y$ is differentiable.
   The largest such intervals are $(-\infty,-1)$, $(-1,1)$, and $(1,\infty)$.
   Any of these three intervals can be taken to be $I$.
   That is, $y(x) = \frac{1}{x^2-1}$ is a solution of the ODE on $I = (-\infty,-1)$
   $y(x) = \frac{1}{x^2-1}$ is a solution of the ODE on $I = (-1,1)$
   $y(x) = \frac{1}{x^2-1}$ is a solution of the ODE on $I = (1,\infty)$
   However, the three branches together is not considered as a solution since there are discontinuities.

3. Now, consider $y(x) = \frac{1}{x^2-1}$ to be a solution of the IVP
$$y' + 2xy^2 = 0; \; y(0) = -1.$$
   The interval $I$ must contain the initial point ($x=0$). $\Rightarrow I = (-1,1)$
   To satisfy the IC, we must select the branch of the function which lies in the interval $(-1,1)$

$\Rightarrow \quad y(x) = \frac{1}{x^2-1}, \; -1 < x < 1$ is the unique solution of the IVP.

# Chapter 2: $1^{st}$ Order ODEs

## Section 2.1: Solution Curves Without a Solution

Even though a $1^{st}$ order ODE may not be solvable, we still can obtain approximate solution curves by the methods of this section.

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General Form of a $1^{st}$ order ODE: $F\left(x, y, \frac{dy}{dx}\right) = 0$   This denotes any expression involving $x$, $y$, and $dy/dx$.

e.g. $x e^{y^2} \sin\left(y \, dy/dx\right) = \sqrt{dy/dx}$

The general form is too general to be useful.

We'll consider the special case in which $\frac{dy}{dx}$ can be solved for explicitly.

Normal Form: $\boxed{\frac{dy}{dx} = f(x, y)}$   e.g. $\frac{dy}{dx} = e^y \tan^{-1}(xy^2)$ OR $\frac{dx}{dt} = kx$

In real-life problems, most $1^{st}$ order ODEs fit the normal form.

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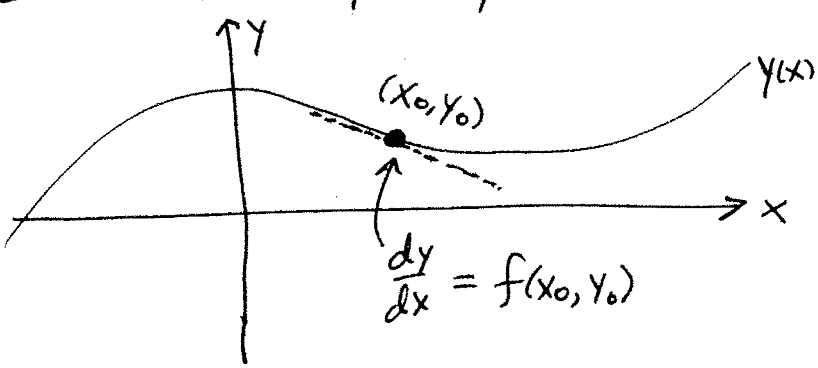
The function $f(x, y)$ is called the slope function since it gives the slope of the solution curves at each point in the xy-plane.

e.g. Consider one of the infinitely many solutions of the ODE.

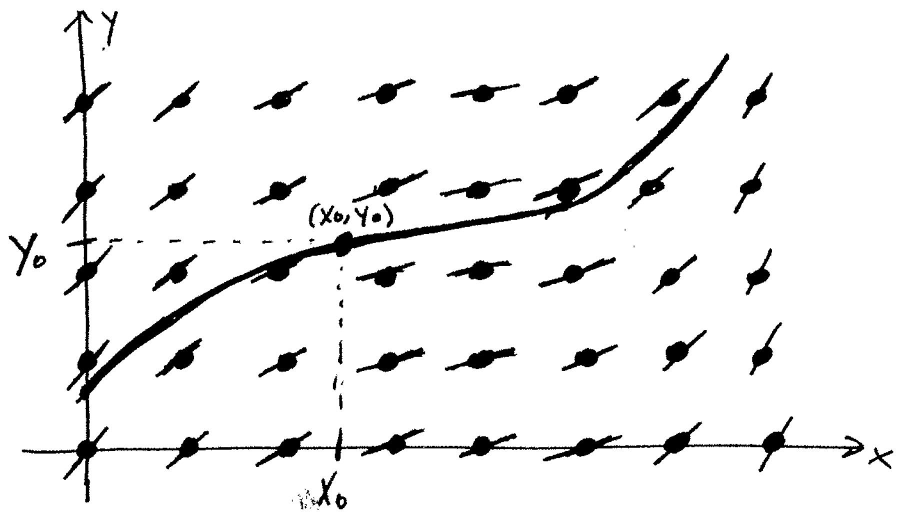


# Direction Fields (or Slope Fields)

Consider the ODE $\frac{dy}{dx} = f(x,y)$.

The idea of a direction field is to compute the slope $f(x,y)$ of the solution curves at several points in the $xy$-plane, and indicate each with a small line segment with the same slope.

e.g. $\frac{dy}{dx} = f(x,y)$ ; $x, y \geq 0$

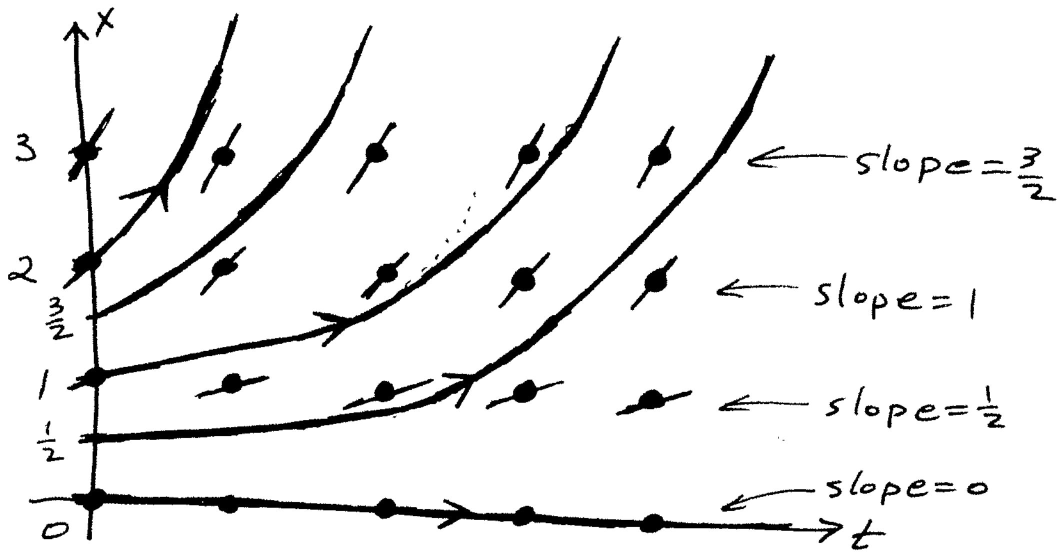



Then, given an IC, say $y(x_0) = y_0$, the solution curve passing through the point $(x_0, y_0)$ may be sketched by following the path indicated by the line segments.

<u>Example:</u> Exponential Growth with $k = \frac{1}{2}$

$$\frac{dx}{dt} = \frac{1}{2}x; \quad t, x \geq 0$$

<u>Plot the direction field and sketch some solution curves.</u>




Slope $= f(t,x) = \frac{1}{2}x$ (independent of $t$)

$$\left.\frac{dx}{dt}\right|_{x=0} = f(t,0) = \frac{1}{2}\cdot 0 = 0$$

$$\left.\frac{dx}{dt}\right|_{x=1} = f(t,1) = \frac{1}{2}\cdot 1 = \frac{1}{2}$$

$$\left.\frac{dx}{dt}\right|_{x=2} = f(t,2) = \frac{1}{2}\cdot 2 = 1$$

$$\left.\frac{dx}{dt}\right|_{x=3} = f(t,3) = \frac{1}{2}\cdot 3 = 3/2$$

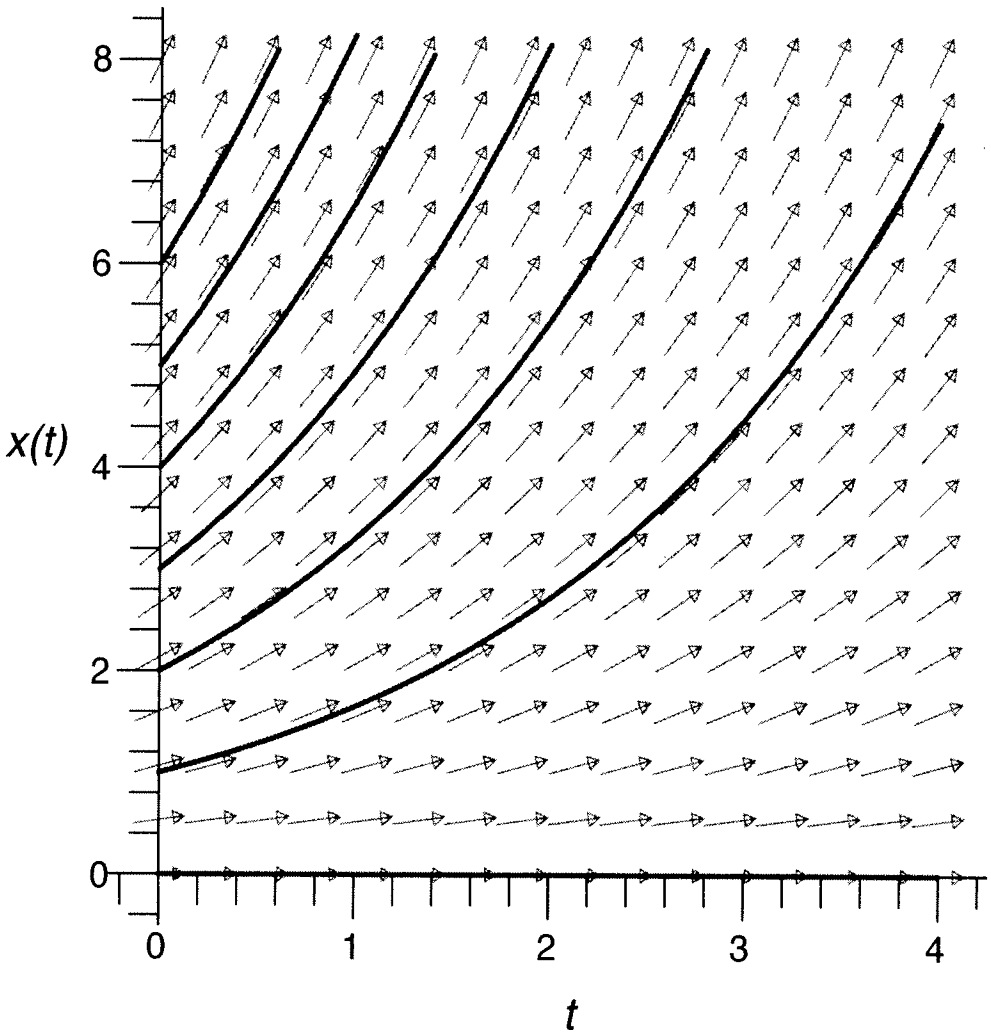
<u>General Solution</u>

$$x(t) = Ce^{\frac{1}{2}t}$$

# Direction Field Plot for the Exponential Growth Equation with $k = 1/2$

$$\frac{dx}{dt} = \frac{1}{2}x, \quad t, x \geq 0$$

The curves correspond to the initial conditions $x(0) = i, \quad i = 0, 1, \ldots, 6.$




**Maple commands**

```
> restart:
> with(DEtools):
> DEplot(diff(x(t),t)=.5*x(t),x(t),t=0..4,x=0..8,[seq([0,i],i=0..6)],linecolor=blue,
        thickness=3,arrows=medium,dirgrid=[16,16],axesfont=[helvetica,helvetica,20],
        labelfont=[italic,italic,24]);
```

# Overview of Chapter 2: 1st Order ODEs

General Form: $F(x, y, dy/dx) = 0$ (Too general)

2.1a: Normal Form: $\frac{dy}{dx} = f(x,y) \Rightarrow$ Direction Fields

---

Special Cases of the Normal Form

2.1b: $\frac{dy}{dx} = f(y)$ (autonomous)

2.2: $\frac{dy}{dx} = f(x)g(y)$ (separable)

2.3: $\frac{dy}{dx} = -a_0(x)y + g(x)$ (linear)

$$1\frac{dy}{dx} + a_0(x)y = g(x)$$ $\Rightarrow$ Integrating Factor Method (IFM)

Need a coefficient of 1 here to use the IFM.

# Autonomous 1st Order ODEs

Recall the Normal Form: $\frac{dy}{dt} = f(t, y)$, $y = y(t)$ ← switch to $t$

An autonomous ODE is a special case of the normal form in which $f$ does not depend explicitly on $t$.

$$\Rightarrow \boxed{\frac{dy}{dt} = f(y)} \text{ (autonomous)}$$

Note: Though not explicitly, $f$ does depend on $t$ through $y$. The function $f(y)$ is actually a composite function since $y$ is a function of $t$. $\Rightarrow f = f(y(t))$

e.g. $\frac{dy}{dt} = (1-y)e^y + \sin y + \cancel{t}$ ← explicit

---

Autonomous ODEs give rise to the notion of critical points:

Definition: The critical points (CPs) of an autonomous ODE, $\frac{dy}{dt} = f(y)$ are the constant solutions of the equation

$$f(y) = 0.$$

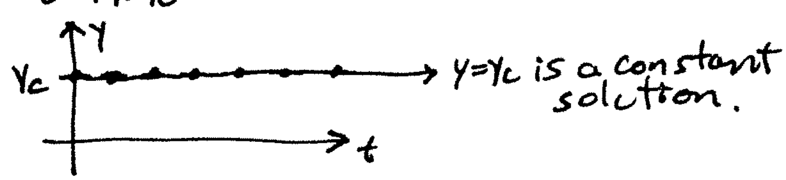
i.e. $y_c$ is a CP if $f(y_c) = 0$.
↑ constant $y$-value

---

The slope of the solution curves is zero at the CPs.

That is, $\left.\frac{dy}{dt}\right|_{y=y_c} = f(y_c) = 0$

Direction Field:




Thus, $y(t) = y_c$ for all $t$ is a constant solution of the ODE.

Note: CPs are also called stationary points or equilibrium solutions.

(Introductory)
**Example**: Consider the autonomous ODE

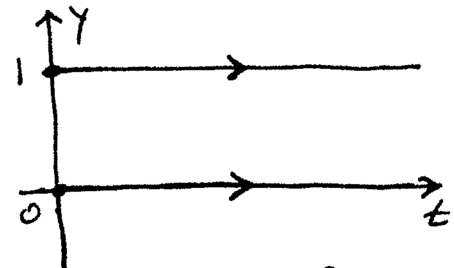
$$\frac{dy}{dt} = y(1-y), \quad t \geq 0.$$

Sketch some solution curves.

---

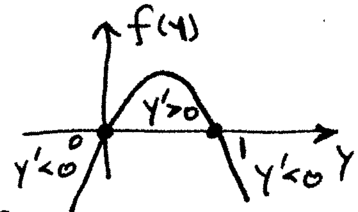
1. **CPs** : $f(y) = y(1-y) = 0$

   $\underline{y_c = 0}$, $\underline{y_c = 1}$




What happens elsewhere?

2. **Look at $dy/dt$** : $\dfrac{dy}{dt} = f(y) = y(1-y)$

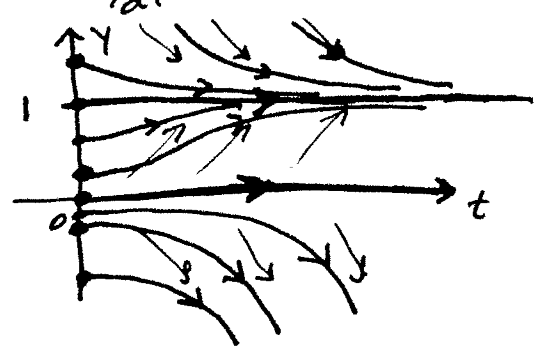



$\dfrac{dy}{dt}$ is zero only at the CPs, and thus the slope of the solution $(dy/dt)$ can change sign only at the CPs. (i.e. only at $y=0, 1$)

If $\underline{y<0}$, then $\dfrac{dy}{dt} = y(1-y) = (<0)(>0) < 0 \Rightarrow \underline{y'<0}$

If $\underline{0<y<1}$, then $\dfrac{dy}{dt} = y(1-y) = (>0)(>0) > 0 \Rightarrow \underline{y'>0}$

If $\underline{y>1}$, then $dy/dt = y(1-y) = (>0)(<0) < 0 \Rightarrow \underline{y'<0}$




3. **Direction Field** : See the plot on the next page.

Observe that as $t \to \infty$, solutions converge toward the CP $y_c = 1$, and diverge away from the CP $y_c = 0$.

$y_c = 1$ is called an asymptotically stable CP

$y_c = 0$ is called an unstable CP.

Definition: A critical point $y_c$ of the ODE $\frac{dy}{dt} = f(y)$ is said to be asymptotically stable (AS) if

$$\lim_{t \to \infty} y(t) = y_c$$

whenever $y$ is initially sufficiently close to $y_c$.

That is, all solutions which are sufficiently close to an asymptotically Stable CP will converge to that CP as $t \to \infty$.

If solution diverge away from a critical point as $t \to \infty$, the critical point is said to be unstable.

In the above example,

$y_c = 1$ is an asymptotically stable CP.

$$\lim_{t \to \infty} y(t) = 1 \quad \text{if} \quad \underbrace{0 < y(0) < \infty}_{\text{sufficiently close to } y_c = 1}$$

may be any finite time.

$y_c = 0$ is an unstable CP.

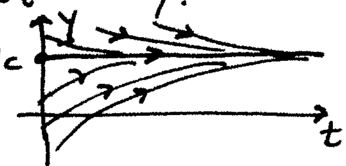
$$\lim_{t \to \infty} y(t) = 0 \text{ only if } y(0) = 0.$$

In real-life problems, $\lim_{t \to \infty} y(t)$ is an important quanity because it corresponds to the state of the system.

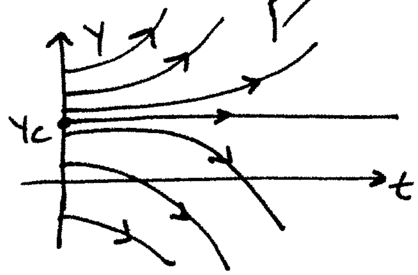
# Notes on Stability

1. There are actually four types of stability.

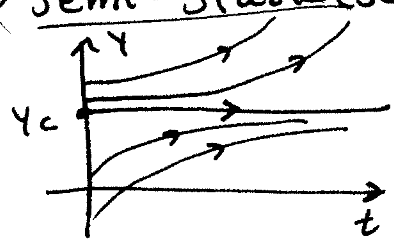
(i) Asymptotically Stable (AS)



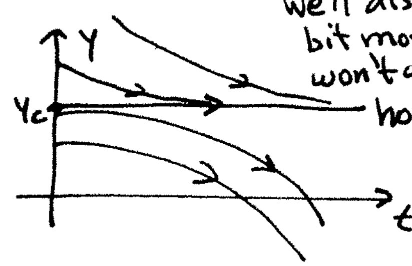

(ii) Unstable (U)




(iii) Semi-Stable (SS)




OR




We'll discuss these a bit more, but they won't appear on the homework or exams.

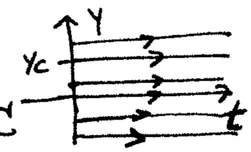
(iv) Stable (S) – solution stay near the CP as $t \to \infty$, but don't necessarily converge to the CP.

$AS \Rightarrow S$, but $S \not\Rightarrow AS$

Autonomous 1st order ODEs don't have (non-asymptotically) stable CPs, except for the one special case in which $f(y) \equiv 0$.

Stable CPs occur in systems of autonomous ODES

$\Rightarrow \frac{dy}{dt} = 0$ ; general solution : $y(t) = C$

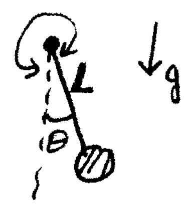



Here, all values of $y$ are (non-asymptotically) stable CPs.

---

2. Only stable, semi-stable, and asymptotically stable equilibrium solutions occur in reality, while unstable solutions do not occur.

e.g. Consider a frictionless pendulum : $\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin\theta = 0$




Equilibrium Solutions (constant) : Set $\frac{d^2\theta}{dt^2} = 0 \Rightarrow \sin\theta = 0$

$\theta_c = 0, \pm\pi, \pm 2\pi, \ldots$

Stable (but not asymptotically)

$\theta_c = 0, \pm 2\pi, \pm 4\pi, \ldots$

$\theta = \pm\pi, \pm 3\pi, \ldots$

Unstable

These eq. sols. are not observed in real life.

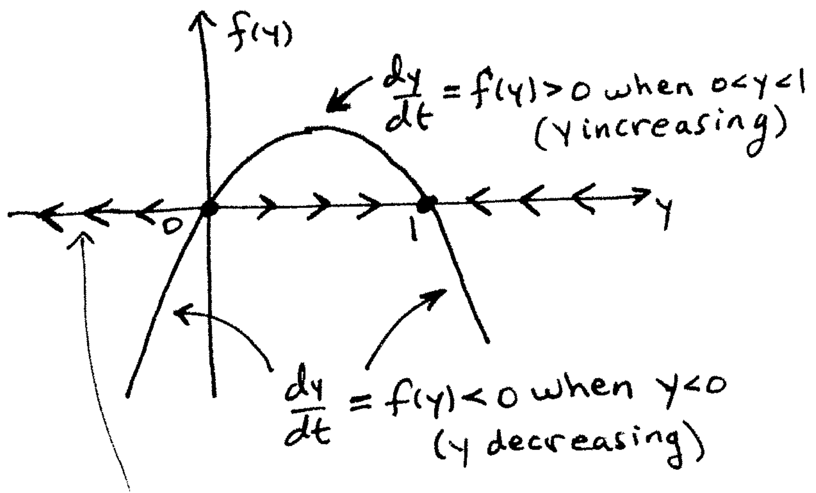
## Three Techniques for Classifying CPs by Stability (Asymptotically Stable OR Unstable)

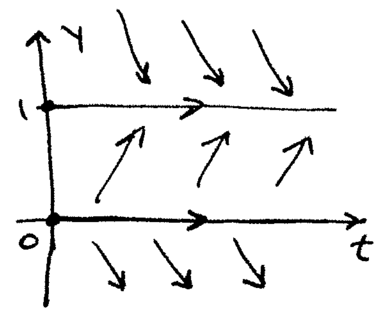
**Method 0:** Sketch some solution curves as above and visually identify the stability status of the CPs.

---

**Method 1:** Consider the graph of $f(y)$ vs. $y$

e.g. $\dfrac{dy}{dt} = f(y) = y(1-y)$




The arrows indicate whether $y$ increases or decreases as $t$ increases.

The arrows point to the left wherever $f(y) < 0$,
and to the right wherever $f(y) > 0$.

The arrows suggest that as $t \to \infty$, solutions converge to $y_c = 1$ and diverge away from $y_c = 0$.

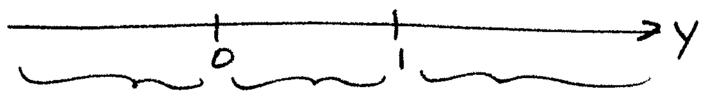
$$\Rightarrow \boxed{\begin{array}{l} y_c = 1 \text{ is Asymptotically Stable} \\ y_c = 0 \text{ is Unstable} \end{array}}$$

**Method 2:** Modification of Method 1

It is not necessary to actually plot $f(y)$.

e.g. $\frac{dy}{dt} = f(y) = y(1-y)$

1. Draw the phase line ($y$-axis) and indicate the CPs.

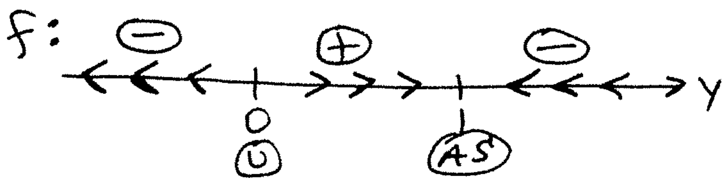



Note that $f(y)$ can change sign only at the CPs. Thus, $f(y)$ must be of the same sign throughout each of the three intervals $(-\infty,0)$, $(0,1)$, and $(1,\infty)$

2. Pick a point from each interval and determine the sign of $f(y)$ at that point. This gives the sign of $f$ over the entire interval.

| Interval | pick a point | sign of $f(y)$ |
|---|---|---|
| $(-\infty, 0)$ | $y=-1$ | $f(-1) = (-1)(1-(-1)) = -2 < 0$ |
| $(0,1)$ | $y=\frac{1}{2}$ | $f(\frac{1}{2}) = \frac{1}{2}(1-\frac{1}{2}) = \frac{1}{4} > 0$ |
| $(1,\infty)$ | $y=2$ | $f(2) = 2(1-2) = -2 < 0$ |

3. Draw the phase portrait

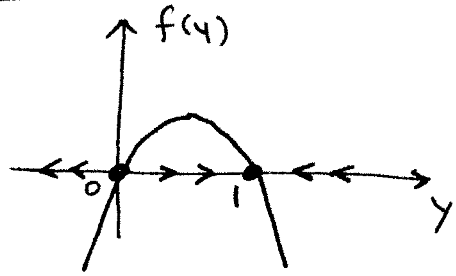



phase portrait

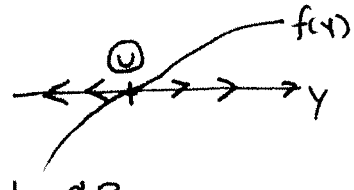
(diagram which depicts the behavior of $y$ as $t$ increases)

$\Rightarrow$ $\boxed{\begin{array}{l} y_c = 1 \text{ is Asymptotically Stable} \\ y_c = 0 \text{ is Unstable} \end{array}}$

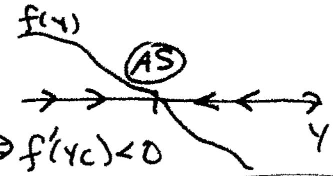
## Method 3: Preferred Method (This method can be generalized for more complicated types of ODEs)




Notice that $f'(y) > 0$ at the unstable CP and $f'(y) < 0$ at the asymptotically stable CP. *This is true in general*




Since the arrows point away from unstable CPs, it must be that $f(y) < 0$ to the left and $f(y) > 0$ to the right of an unstable CP. Thus, f increases through an unstable CP. $\Rightarrow f'(y_c) > 0$




Likewise, since arrows point toward asymptotically stable CPs, it must be that $f(y) > 0$ to the left and $f(y) < 0$ to the right of an asymptotically stable CP. Thus, f decreases through an asymptotically stable CP. $\Rightarrow f'(y_c) < 0$

---

**Theorem:** Let $y_c$ be a CP of the ODE $\frac{dy}{dt} = f(y)$.

If $f'(y_c) < 0$, then $y_c$ is asymptotically stable.

If $f'(y_c) > 0$, then $y_c$ is unstable

If $f'(y_c) = 0$, then the test is inconclusive

---

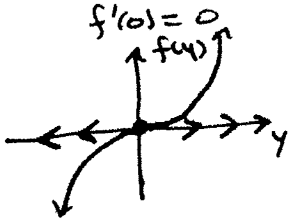
Note: If $f'(y_c) = 0$, $y_c$ may be either unstable, asymptotically stable, semi-stable, or stable. We'll avoid this case.

e.g.

$\frac{dy}{dt} = f(y) = y^3$

$y_c = 0$

$f'(0) = 0$

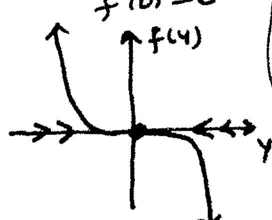



$y_c = 0$ is unstable

$\frac{dy}{dt} = f(y) = -y^3$

$y_c = 0$

$f'(0) = 0$

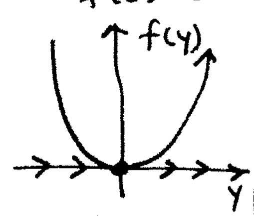



$y_c = 0$ is asymptotically stable

$\frac{dy}{dt} = f(y) = y^2$

$y_c = 0$

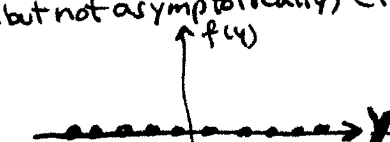
$f'(0) = 0$




$y_c = 0$ is semi-stable

$\frac{dy}{dt} = f(y) = 0$

All points are stable (but not asymptotically) CPs.

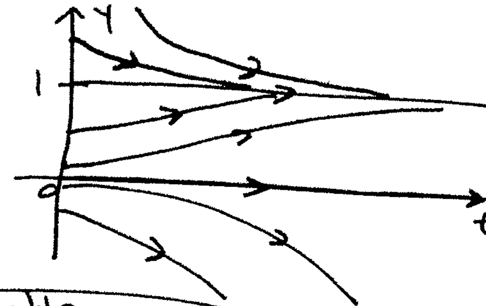



no arrows

All real $y_c$ are stable

For the example, $\frac{dy}{dt} = f(y) = y(1-y)$

$\underline{y_c = 0}$ $\underline{y_c = 1}$




$f(y) = y(1-y) = y - y^2$

$f'(y) = 1 - 2y$

$f'(0) = 1 - 0 > 0 \Rightarrow$ $y_c = 0$ is Unstable

$f'(1) = 1 - 2 < 0 \Rightarrow$ $y_c = 1$ is Asymptotically stable

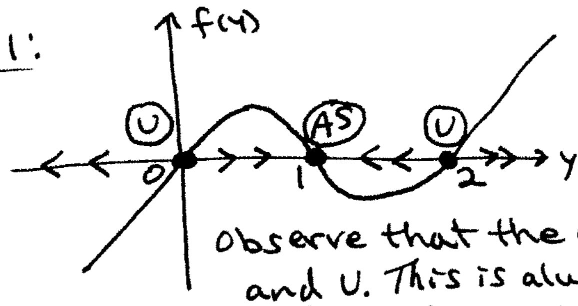
---

Example: Consider the autonomous ODE $\frac{dy}{dt} = y(y-1)(y-2)$.
Find all CPs and classify each as either asymptotically stable (AS) or unstable (U). Then plot some solution curves.

CPs: Set $f(y) = y(y-1)(y-2) = 0$

$\boxed{y_c = 0}$ $\boxed{y_c = 1}$ $\boxed{y_c = 2}$

Method 1:




$\Rightarrow$ $y_c = 0$ is Unstable
$y_c = 1$ is Asymptotically stable
$y_c = 2$ is unstable

Observe that the CPs alternate between AS and U. This is always the case in the absence of semi-stable CPs.

Method 2: $f$: U at 0, AS at 1, U at 2

$\longleftarrow\!\!\longleftarrow \underset{0}{|} \longrightarrow\!\!\longrightarrow \underset{1}{|} \longleftarrow\!\!\longleftarrow \underset{2}{|} \longrightarrow\!\!\longrightarrow y$

| Interval | pick a point | sign of $f(y)$ |
|---|---|---|
| $(-\infty, 0)$ | $y = -1$ | $f(-1) = (-1)(-1-1)(-1-2) = (<0)(<0)(<0) < 0$ |
| $(0, 1)$ | $y = \frac{1}{2}$ | $f(\frac{1}{2}) = \frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2) = (>0)(<0)(<0) > 0$ |
| $(1, 2)$ | $y = \frac{3}{2}$ | $f(\frac{3}{2}) = \frac{3}{2}(\frac{3}{2}-1)(\frac{3}{2}-2) = (>0)(>0)(<0) < 0$ |
| $(2, \infty)$ | $y = 3$ | $f(3) = 3(3-1)(3-2) = (>0)(>0)(>0) > 0$ |

$\Rightarrow$ $y_c = 0$ is Unstable
$y_c = 1$ is Asymptotically Stable
$y_c = 2$ is Unstable

Method 3: $\frac{dy}{dt} = f(y) = y(y-1)(y-2)$ ; $y_c = 0, 1, 2$

$$f(y) = y(y-1)(y-2) = y^3 - 3y^2 + 2y$$
$$f'(y) = 3y^2 - 6y + 2$$

OR When $f(y)$ consists of products of functions, it may be more convenient to use a product rule instead.

$$(fgh)' = f'gh + fg'h + fgh' \qquad ((fg)'' = f''g + 2f'g' + fg'')$$

$f(y) = y(y-1)(y-2)$

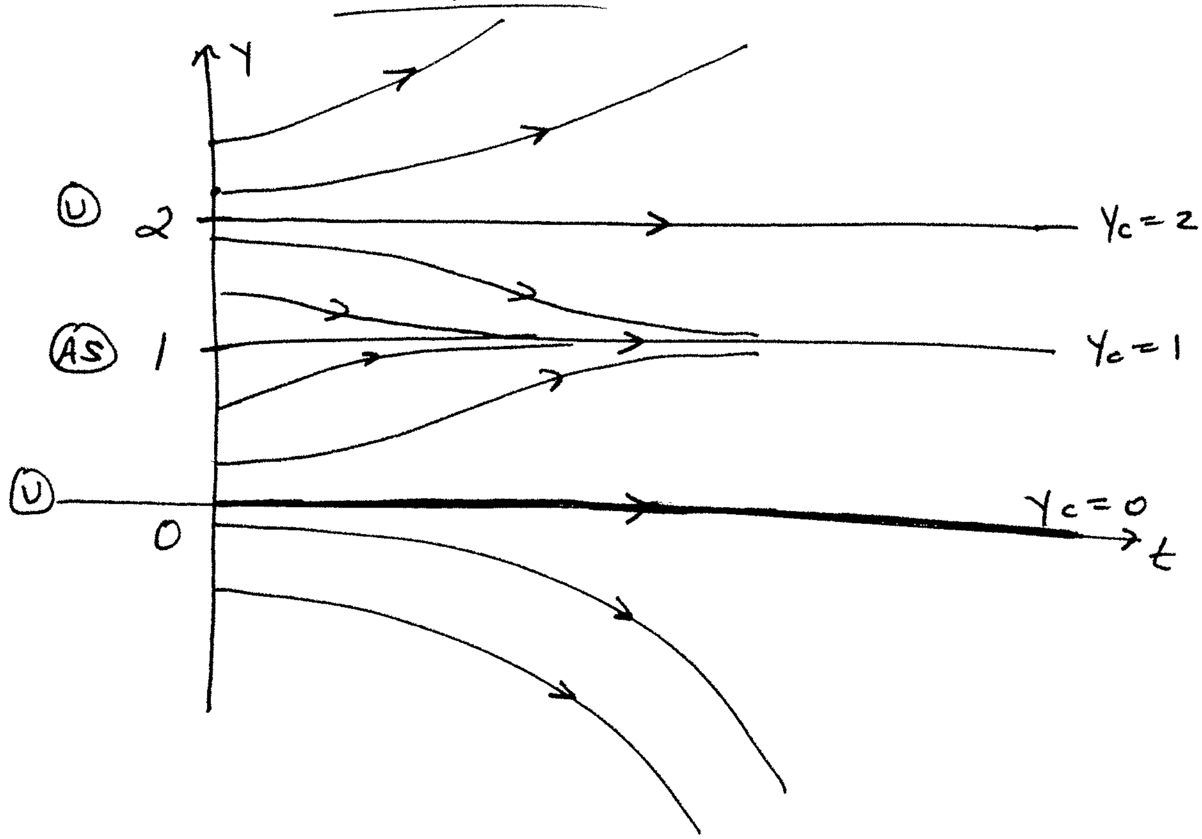
$f'(y) = (y-1)(y-2) + y(y-2) + y(y-1)$

$f'(0) = (0-1)(0-2) + 0 + 0 = 2 > 0 \Rightarrow$ $y_c = 0$ is Unstable

$f'(1) = 0 + (1)(1-2) + 0 = -1 < 0 \Rightarrow$ $y_c = 1$ is Asymptotically Stable

$f'(2) = 0 + 0 + (2)(2-1) = 2 > 0 \Rightarrow$ $y_c = 2$ is Unstable

Solution Curves



**Example:** Suppose a population of fish is described by the autonomous ODE

$$\frac{dP}{dt} = P(P-3)(2-e^P) \; ; \; t \geq 0, \; P \geq 0$$

(a) Find all CPs.

(b) Classify each CP as either Asymptotically Stable (AS) or Unstable (U)

(c) Determine the ultimate fate of the population ($\lim_{t\to\infty} P(t)$) if (i) $P(0) = \frac{1}{2}\ln 2$, (ii) $P(0) = \ln 2$, (iii) $P(0) = 1$, (iv) $P(0) = 4$.

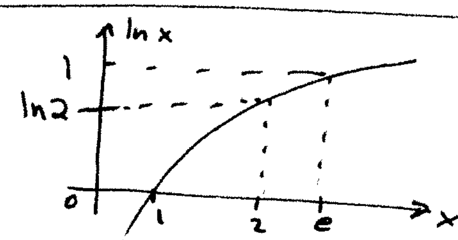
(d) Sketch some solution curves.

---

(a) Set $f(P) = P(P-3)(2-e^P) = 0$

$\boxed{P_c = 0}$ $\boxed{P_c = 3}$ $e^P = 2$ $\boxed{P_c = \ln 2}$

$0 < \ln 2 < 3$




$0 < \ln 2 < 1$

(b) $f'(P) = (P-3)(2-e^P) + P(2-e^P) - P(P-3)e^P$

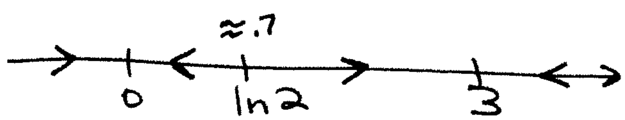
$f'(0) = (0-3)(2-1) + 0 + 0 = -3 < 0 \Rightarrow$

$f'(\ln 2) = 0 + 0 - \underset{(<0)(>0)\,(<0)(>0)}{(\ln 2)(\ln 2 - 3)e^{\ln 2}} > 0 \Rightarrow$

$f'(3) = 0 + (3)\underbrace{(2-e^3)}_{<0} + 0 < 0 \Rightarrow$

$$\boxed{\begin{array}{l} P_c = 0 \text{ is Asymptotically Stable} \\ P_c = \ln 2 \text{ is Unstable} \\ P_c = 3 \text{ is Asymptotically Stable} \end{array}}$$

(c)




Phase Portrait

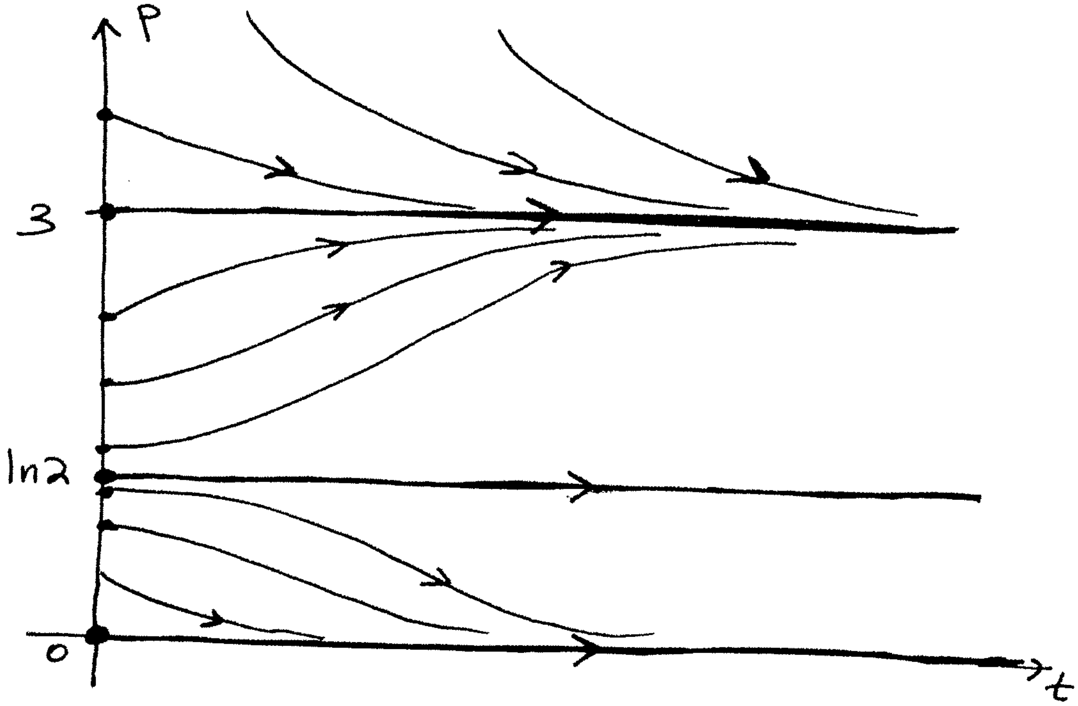
(i) $P(0) = \frac{1}{2}\ln 2 \Rightarrow \boxed{\lim_{t\to\infty} P(t) = 0}$

(ii) $P(0) = \ln 2 \Rightarrow \boxed{\lim_{t\to\infty} P(t) = \ln 2}$

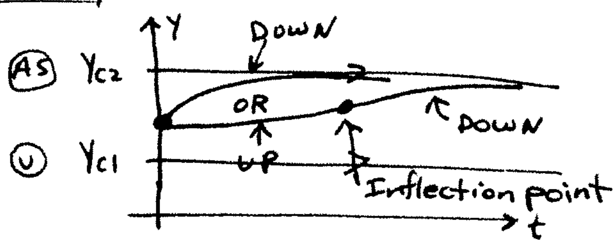
(iii) $P(0) = 1 \Rightarrow \boxed{\lim_{t\to\infty} P(t) = 3}$

(iv) $P(0) = 4 \Rightarrow \boxed{\lim_{t\to\infty} P(t) = 3}$

(d)




Note: The concavity of the solution curves may be determined.




The slope of the solution curves is given by $\frac{dy}{dt} = f(y)$,

The concavity of the solution curves is given by $\frac{d^2y}{dt^2} = ?$

Common mistake

$$\frac{dy}{dt} = f(y)$$

$$\cancel{\frac{d^2y}{dt^2} = f'(y)}$$

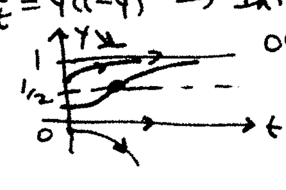
Consider $f$ to be a composite function.

$$\frac{dy}{dt} = f(y(t))$$

$$\frac{d^2y}{dt^2} = f'(y)\,\frac{dy}{dt} = f'(y)\,f(y)$$

$$\boxed{\frac{d^2y}{dt^2} = f'(y)\,f(y)}$$

Since $f(y)=0$ only at the CPs, inflection points ($d^2y/dt^2=0$) occur where $f'(y)=0$. e.g. $\frac{dy}{dt} = y(1-y) \Rightarrow$ Inflection points occur at $y=\frac{1}{2}$.



# Section 2.2: Separable 1st Order ODE

Definition: The ODE $\frac{dy}{dx} = f(x,y)$ is separable if $f$ can be written as $f(x,y) = g(x)h(y)$.

That is, $f$ is the product of two functions, one depending only on $x$, and the other depending only on $y$.

$$\Rightarrow \boxed{\frac{dy}{dx} = g(x)\,h(y)}$$

The general solution is found by "separating" the variables, integrating, and solving for $y$, if possible.

"separating" the variables $\Rightarrow$ get all the $y$'s on one side of the equation, and all of the $x$'s on the other. (Typically, the dependent variable $y$ is put on the left-hand side.)

$$\frac{dy}{dx} = g(x)\,h(y)$$

Separate: $\frac{dy}{h(y)} = g(x)\,dx$

Integrate: (if possible) $\int \frac{dy}{h(y)} = \int g(x)\,dx$

$$H(y) + C_1 = G(x) + C_2$$

$$H(y) = G(x) + C$$

Solve for $y$: (if possible) $y = \ldots$

(This is the Method of Separable Variables)

**Example:** Solve the IVP

$$\frac{dx}{dt} = kx \; ; \; x(0) = x_0$$

$x_0$ and $k$ are given constants

---

Solving an IVP:
1. Find the general solution
2. Impose the IC to determine the constant.

1. General Solution: $\frac{dx}{dt} = kx$

$x$ = dependent variable
$t$ = independent variable

Separate: $\frac{dx}{x} = k\,dt$ ($k$ can go on either side of the equation)

Integrate: $\int \frac{dx}{x} = \int k\,dt$

$$\ln|x| = kt + c_1$$

Solve for $x$: $\cancel{e}^{\ln|x|} = e^{kt+c_1}$

$$|x| = e^{c_1} e^{kt} > 0$$

$$\Rightarrow \quad x = \pm e^{c_1} e^{kt}$$

$c = \pm e^{c_1}$

Common Mistake

$e^{kt+c} \neq e^{kt} + c$

$\quad = ce^{kt}$

$$\boxed{x(t) = ce^{kt}}$$

(general solution)

2. Impose the IC to find $c$

$$x(0) = x_0$$

$$c\,\cancel{e^0} = x_0$$

$$c = x_0$$

$$\Rightarrow \quad \boxed{x(t) = x_0 e^{kt}}$$

**Example:** Solve the IVP $\frac{dy}{dx} = \frac{x-5}{y^2}$ ; $y(0)=2$

Separate: $y^2 dy = (x-5)dx$

Integrate: $\int y^2 dy = \int (x-5)dx$

$$y^3/3 = x^2/2 - 5x + C$$

Solve for y: $y^3 = \frac{3}{2}x^2 - 15x + C$ (Replace 3C by C)

$$\boxed{y = \left[\tfrac{3}{2}x^2 - 15x + C\right]^{1/3}}$$ (general solution)

Find C: $y(0) = [0 - 0 + C]^{1/3} = 2$

$$C = 8$$

$$\Rightarrow \boxed{y(x) = \left[\tfrac{3}{2}x^2 - 15x + 8\right]^{1/3}}$$ (particular solution)

Check: ODE: $\frac{dy}{dx} = \frac{1}{3}\left[\frac{3}{2}x^2 - 15x + 8\right]^{-2/3}(3x - 15) = \frac{x-5}{\left[\frac{3}{2}x^2 - 15x + 8\right]^{2/3}} = \frac{x-5}{y^2}$ ✓

IC: $y(0) = [0 - 0 + 8]^{1/3} = 2$ ✓

---

A common mistake is to ignore C while solving for y, and then adding it to the solution in the end.

e.g. $\frac{y^3}{3} = \frac{x^2}{2} - 5x \cancel{+C}$

$$y = \left[\tfrac{3}{2}x^2 - 15x\right]^{1/3} + C \quad \text{✗}$$

Doesn't satisfy the ODE

C must be included in the calculation when solving for y.

Note: Constants can be absorbed into C while finding the general solution, but not while imposing the IC.

That is, C can be adjusted while finding the general solution, but once the general solution is specified, C becomes fixed.

- Sometime an ODE doesn't appear to be separable at first sight, but with some manipulation it can be written in separable form.

---

Example: Find the general solution of the ODE

$$\frac{dy}{dx} = xy + 2 + 2y + x \quad \text{(Separable ?)}$$

Factor the RHS: $\frac{dy}{dx} = y(x+2) + (x+2) = (y+1)(x+2)$

$$\int \frac{dy}{y+1} = \int (x+2)\, dx$$

$$\ln|y+1| = x^2/2 + 2x + C_1$$

$$e^{\ln|y+1|} = e^{x^2/2 + 2x + C_1}$$

$$|y+1| = e^{C_1} e^{x^2/2 + 2x}$$

$$y+1 = \pm e^{C_1} e^{x^2/2 + 2x}$$

$C = \pm e^{C_1}$

$$\boxed{y = Ce^{x^2/2 + 2x} - 1}$$

---

Example: Find the general solution of the ODE

$$y = \ln(xy'), \quad \text{(separable ?)}$$

$dx$ is hidden inside $y'$
$\Rightarrow$ Replace $y'$ by $\frac{dy}{dx}$ :

$$y = \ln\left(x \frac{dy}{dx}\right)$$

$$e^y = \cancel{e^{\ln}}(x\, dy/dx)$$

$$e^y = x\, dy/dx$$

$$\int e^{-y} dy = \int \frac{dx}{x}$$

$$-e^{-y} = \ln|x| + C$$

$$e^{-y} = C - \ln|x|$$

$$-y = \ln(C - \ln|x|)$$

$$\boxed{y(x) = -\ln(C - \ln|x|)}$$

**Example:** Solve the IVP $(y^3+y+1)y' = \dfrac{y^2+1}{x^2+1}$, $y(1)=0$

$$y' = dy/dx \Rightarrow \int \frac{y^3+y+1}{y^2+1}\,dy = \int \frac{dx}{x^2+1}$$

Rational Function

$\dfrac{P(y)}{Q(y)}$ : P, Q are polynomials

If deg(P) ≥ deg(Q), divide first.

$$\begin{array}{r} y + \frac{1}{y^2+1} \\ y^2+1 \,\overline{)\, y^3+y+1} \\ \underline{y^3+y} \\ 0+1 \end{array}$$

$$\frac{y^3+y+1}{y^2+1} = y + \frac{1}{y^2+1}$$

$$\int \left(y + \frac{1}{y^2+1}\right) dy = \int \frac{dx}{x^2+1}$$

$$\boxed{y^2/2 + \tan^{-1}y = \tan^{-1}x + C}$$ (Implicit General Solution)

Can't solve for y.

Find C: Impose $y(1)=0$

$x=1$, $y=0$ $\Rightarrow$ $0^2/2 + \tan^{-1}0 = \tan^{-1}1 + C$

$0 + 0 = \pi/4 + C$

$C = -\pi/4$

$$\Rightarrow \boxed{y^2/2 + \tan^{-1}y = \tan^{-1}x - \pi/4}$$ (Implicit Particular Solution)

Check: Implicit Differentiation

$\frac{d}{dx}$: $yy' + \dfrac{y'}{1+y^2} = \dfrac{1}{1+x^2}$

$\left(y + \frac{1}{1+y^2}\right)y' = \dfrac{1}{1+x^2}$

$\dfrac{y^3+y+1}{1+y^2}\,y' = \dfrac{1}{1+x^2}$

$(y^3+y+1)y' = \dfrac{y^2+1}{x^2+1}$ ✓

---

Explicit Solution : $y = f(x)$

e.g. $y = x^2 + e^x\cos x$

Implicit Solution : $f(x,y) = 0$

e.g. $xy^2\sec y = e^{y-x} + 5/y$

**Example: Solve the IVP $\frac{dy}{dx} = \frac{x}{y+1}$, $y(0) = 2$**

$$\int (y+1)\,dy = \int x\,dx$$

$$\frac{y^2}{2} + y = \frac{x^2}{2} + c$$

$$y^2 + 2y = x^2 + c \qquad (2c \to c)$$

Solve for y?

**Method 1: quadratic formula**

$$y^2 + 2y - (x^2+c) = 0$$

$$y = \frac{-2 \pm \sqrt{4 + 4(x^2+c)}}{2}$$

$$y = -1 \pm \sqrt{1 + x^2 + c}$$

$$y = -1 \pm \sqrt{x^2+c} \qquad (c+1 \to c)$$

**Method 2: complete the square**

$$y^2 + 2y + 1 = x^2 + c + 1$$

$$(y+1)^2 = x^2 + c \qquad (c+1 \to c)$$

$$y + 1 = \pm\sqrt{x^2+c}$$

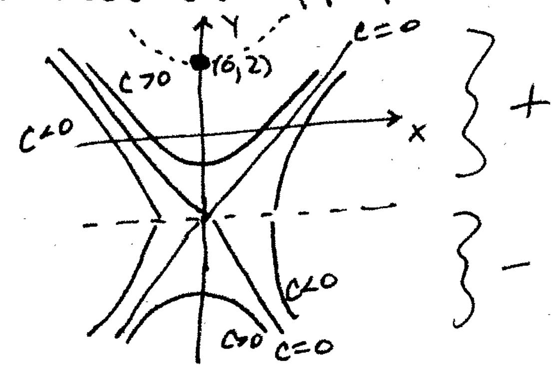
$$y = -1 \pm \sqrt{x^2+c}$$

$$\boxed{y = -1 \pm \sqrt{x^2+c}} \qquad \text{(general solution)}$$

This is a solution whether the + or − is used. When finding the particular solution, we must choose the appropriate sign.

$$y = -1 + \sqrt{x^2+c} \geq -1$$

$$y = -1 - \sqrt{x^2+c} \leq -1$$




**Find C and choose + or −**

$$y(0) = -1 \pm \sqrt{0 + c} = 2$$

$\pm\sqrt{c} = 3$ only the + sign can satisfy the IC.

$+\sqrt{c} = 3$ ($y(0)$ can be $>-1$ only if the + sign is used)

$c = 9$ $\Rightarrow$ $\boxed{y = -1 + \sqrt{x^2+9}}$

**Example:** Solve the IVP $\dfrac{dy}{dx} = \dfrac{3x^2 y}{2(1+x^3)}$ ; $y(-2) = 2$

$$\int \frac{dy}{y} = \frac{3}{2}\int \frac{x^2\,dx}{1+x^3}$$

$$\ln|y| = \cancel{\frac{3}{2}}\cdot\cancel{\frac{1}{3}}\ln|1+x^3| + C_1$$

$$\cancel{e}^{\ln|y|} = e^{\frac{1}{2}\ln|1+x^3| + C_1}$$

$$|y| = e^{C_1}\, e^{\frac{1}{2}\ln|1+x^3|}$$

$$y = \pm e^{C_1}\, e^{\ln|1+x^3|^{1/2}}$$

$C = \pm e^{C_1}$ $\Rightarrow$ $y = C\,|1+x^3|^{1/2}$ (general solution)

$$y(-2) = C\,|1+(-2)^3|^{1/2} = 2$$

$$C\,|-7|^{1/2} = 2$$

$$C = 2/7^{1/2}$$

$$\Rightarrow \quad y = C\,|1+x^3|^{1/2} = \frac{2}{7^{1/2}}\,|1+x^3|^{1/2}$$

$$\boxed{y(x) = 2\left|\frac{1+x^3}{7}\right|^{1/2}}$$

## Section 2.3 : Linear 1st Order ODEs

Definition: The ODE $\frac{dy}{dx} = f(x,y)$ is linear if $f$ has a linear dependence on $y$. That is, if $f(x,y) = -a(x)y + g(x)$.

General Form of a 1st Order ODEs :
$$\boxed{\frac{dy}{dx} + a(x)y = g(x)}$$

Note: Linear 1st order ODEs may or may not be separable, depending on the coefficient functions, $a(x)$ and $g(x)$. Separation of variables is possible if

i) $a \equiv 0$ or $g \equiv 0$, ii) $a$ and $g$ are constant, or iii) $g = ca$, $c$=constant.

---

Consider the exponential decay ODE with $k=-1$

$$\frac{dy}{dx} = -y \quad \text{(separable and linear)}$$

The general solution may be found by the Method of Separable Variables.

$$\int \frac{dy}{y} = -\int dx$$

$$\ln|y| = -x + c_1$$

$$|y| = e^{c_1}e^{-x}$$

$$y = \pm e^{c_1}e^{-x}$$

$$C = \pm e^{c_1} \Rightarrow \boxed{y(x) = Ce^{-x}}$$

## Alternative Solution Method

$$\frac{dy}{dx} = -y$$

Recall

$$\frac{d}{dx}(e^x y) = e^x\frac{dy}{dx} + e^x y$$

Rewrite the ODE as $\dfrac{dy}{dx} + y = 0$

Multiply be $e^x$: $\quad e^x\dfrac{dy}{dx} + e^x y = 0$

Write the LHS as $\frac{d}{dx}(e^x y)$ : $\quad \dfrac{d}{dx}(e^x y) = 0$

Integrate: $\quad \int d(e^x y) = \int 0 \, dx$

$$e^x y = C$$

$$\Rightarrow \boxed{y(x) = Ce^{-x}}$$

---

The LHS was made to be integrable by multiplying the ODE by $e^x$.

i.e. $\quad LHS = \dfrac{d}{dx}(e^x y)$ (a perfect derivative)

In this example, the function $e^x$ is called the

**Integrating Factor (IF).**

Note: $e^x$ does not work in general.
The IF differs from problem to problem.

The IF, denoted by $\mu(x)$, can be determined for a general linear 1st order ODE.

# Integrating Factor Method (IFM)

The above technique can be used to solve a general linear $1^{st}$ order ODE, $\boxed{\frac{dy}{dx} + a(x)y = g(x)}$.

Idea of the IFM: Multiple the ODE by an IF, say $\mu(x)$, to make the LHS integrable. In particular, we want $LHS = \frac{d}{dx}(\mu(x)y)$. $\mu(x) = ?$

Determine an expression for $\mu(x)$

$$\frac{dy}{dx} + a(x)y = g(x)$$

Multiply by $\mu(x)$: $\underbrace{\mu(x)\frac{dy}{dx} + \mu(x)a(x)y}_{LHS} = \mu(x)g(x)$ ①

Want $LHS = \frac{d}{dx}(\mu(x)y)$

$$\cancel{\mu(x)\frac{dy}{dx}} + \mu(x)a(x)\cancel{y} = \cancel{\mu(x)\frac{dy}{dx}} + \frac{d\mu(x)}{dx}\cancel{y}$$

Find $\mu(x)$ so that this equation holds true

$$\mu(x)a(x) = \frac{d\mu(x)}{dx} \quad \text{(separable)}$$

Drop the argument: $\frac{d\mu}{dx} = a(x)\mu$ (Similar to the exponential ODE but with a variable growth rate.)

Separate: $\frac{d\mu}{\mu} = a(x)\,dx$

Integrate: $\ln|\mu| = \int a(x)\,dx + C$ (or pick $C=0$ here)

Solve for $\mu$: $\boxed{\mu(x) = C e^{\int a(x)dx}}$

This expression for $\mu(x)$ works for any non-zero choice of $C$.

Pick $C = 1$ for convenience $\Longrightarrow$ $\boxed{\mu(x) = e^{\int a(x)dx}}$ (Integrating Factor)

Note: The integration constant has already been accounted for.

With this choice of $\mu(x)$, equation ① becomes

$$\frac{d}{dx}(\mu(x)\,y) = \mu(x)\,g(x)$$

Multiply by dx and Integrate: $\int d(\mu(x)y) = \int \mu(x)g(x)\,dx$

$$\mu(x)y = \int \mu(x)g(x)\,dx + C$$

Solve for y: $y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)g(x)\,dx + C\right]$

Plug in $\mu(x) = e^{\int a(x)dx}$: $\Rightarrow$ $\boxed{y(x) = e^{-\int a(x)dx}\left[\int g(x)\,e^{\int a(x)dx}\,dx + C\right]}$

(General Solution of $\frac{dy}{dx} + a(x)y = g(x)$)

Notes: 1. When evaluating the integrals appearing in this formula, the corresponding integration constants may be ignored since they have already been taken into account.

2. The equation is written in poor notation. Nested integrals should not have the same integration variable.

3. The integrals may not be evaluable, in which case there is no choice but to leave them as integrals in the general solution.

Theorem 4.1 guarantees that the IVP

$$\frac{dy}{dx} + a(x)y = g(x), \quad y(x_0) = y_0$$

has a unique solution provided $a(x)$ and $g(x)$ are continuous on some interval $I$ containing $x_0$.

## Step-By-Step Procedure

Rather than memorizing and using the above formula to find a general solution, it is highly recommended to start from scratch and proceed according to the following steps.

---

To find the general solution of the ODE

$$B(x)\frac{dy}{dx} + A(x)y = G(x) \quad \ldots$$

1. Write the ODE as $\boxed{\underline{\underline{1}}\frac{dy}{dx} + a(x)y = g(x)}$ ② Need a coefficient of $\underline{\underline{1}}$ here

   i.e. Divide by $B(x) \Rightarrow a(x) = \frac{A(x)}{B(x)}$ and $g(x) = \frac{G(x)}{B(x)}$,

   and identify the function $a(x)$.

2. Calculate the IF : $\boxed{\mu(x) = e^{\int a(x)\,dx}}$ (ignore the integration constant)

3. Multiply the ODE ② by $\mu(t)$.

$$\Rightarrow \mu(x)\frac{dy}{dx} + \mu(x)a(x)y = \mu(x)g(x)$$

4. Write the ODE as $\frac{d}{dx}(\mu(x)y) = \mu(x)g(x)$

5. Integrate: $\int d(\mu(x)y) = \int \mu(x)g(x)\,dx$

$$\mu(x)y = \int \mu(x)g(x)\,dx + C$$

6. Solve for $y$: $\boxed{y(x) = \frac{1}{\mu(x)}\left[\int \mu(x)g(x)\,dx + C\right]}$

These are precisely the same steps as those used in deriving the formula for the general solution.

7. Find $C$ as usual if an IC is specified.

<u>Example:</u> Solve the IVP

$$y' = e^{-x} + x - y \ ; \ y(2) = 1$$

1. Write the ODE as $y' + y = e^{-x} + x$

$a(x) = 1$

2. Calculate the IF: $\mu(x) = e^{\int a(x)dx} = e^{\int 1 \cdot dx} = e^{x}$

$\mu(x) = e^{x}$

3. Multiply the ODE by $\mu(x) = e^{x}$

$$e^{x}y' + e^{x}y = 1 + xe^{x}$$

4. Write the ODE as $\frac{d}{dx}(e^{x}y) = 1 + xe^{x}$

5. Integrate: $\int d(e^{x}y) = \int (1 + xe^{x})dx$

$$e^{x}y = \int dx + \int xe^{x}dx$$

$$e^{x}y = x + (x-1)e^{x} + C$$

<u>Integrate by Parts</u>

$u = x \quad v = e^{x}$

$du = dx \quad dv = e^{x}dx$

$= xe^{x} - \int e^{x}dx$

$= xe^{x} - e^{x}$

$= (x-1)e^{x}$

6. Solve for $y$: $\boxed{y(x) = (x-1) + (x+C)e^{-x}}$ (general solution)

7. <u>Find C</u>: $y(2) = \cancel{(2-1)} + (2+C)e^{-2} = \cancel{1}$

$$\underline{\underline{C = -2}}$$

$$\Rightarrow \boxed{y(x) = x - 1 + (x-2)e^{-x}}$$ (particular solution)

**Example:** Solve the IVP

$$3ty + ty' = 3t^3 + 2y \; ; \quad y(1) = 3$$

1. $ty' + (3t-2)y = 3t^3$

$$y' + (3 - 2/t)\,y = 3t^2 \;\Rightarrow\; a(t) = 3 - 2/t$$

2. $$\mu(t) = e^{\int a(t)dt} = e^{\int (3-2/t)\,dt} = e^{3t - 2\ln|t|} = e^{3t}e^{-2\ln|t|}$$

$$= e^{3t}\cancel{e}^{\cancel{\ln}|t|^{-2}} = e^{3t}|t|^{-2} = e^{3t}/t^2$$

$$\mu(t) = e^{3t}/t^2$$

3. $$\frac{e^{3t}}{t^2}\left[y' + (3 - 2/t)y\right] = \frac{e^{3t}}{\cancel{t^2}} \cdot 3\cancel{t^2}$$

4. $$\frac{d}{dt}\left(\frac{e^{3t}}{t^2}y\right) = 3e^{3t}dt$$

5. $$\int d\left(\frac{e^{3t}}{t^2}y\right) = \int 3e^{3t}dt$$

$$\frac{e^{3t}}{t^2}y = e^{3t} + c$$

6. $$y(t) = t^2\left[1 + ce^{-3t}\right]$$ (general solution)

7. $$y(1) = 1^2\left[1 + ce^{-3}\right] = 3$$

$$ce^{-3} = 2$$

$$c = 2e^3$$

$$\Rightarrow y(t) = t^2\left[1 + 2e^3e^{-3t}\right]$$

$$\boxed{y(t) = t^2\left[1 + 2e^{3(1-t)}\right]}$$

**Example:** Find the general solution of $\frac{dy}{dt} + ty = 1$.

$a(t) = t$

$\mu(t) = e^{t^2/2}$ $\Rightarrow$ $\boxed{y(t) = e^{-t^2/2}\left[\int e^{t^2/2}\,dt + C\right]}$

The integral cannot be evaluated.

All integrals in our problems will be evaluable, so be sure to evaluate all integrals on the homework and exams.

---

## Summary of Solution Methods for 1st Order ODEs

We disussed solution methods for two special cases of $\frac{dy}{dx} = f(x,y)$. (1)

If (1) is separable, we may use the method of separable variables (SV)

If (1) is linear, we may use the integrating factor method (IFM).

These are the only types of 1st order ODEs ~~that we can solve~~ for which we have solution methods

If (1) is both separable and linear, we may use either method

| 1st Order ODEs | Linear | Nonlinear |
|---|---|---|
| Separable | SV or IFM | SV |
| Non-separable | IFM | ? |

Unfortunately, most 1st order ODEs are non-separable and non-linear

# Chapter 3: Modeling with $1^{st}$ Order ODEs

## Section 3.1: Modeling with Linear $1^{st}$ Order ODEs

In this section, we'll consider

1. Newton's Law of Heating/Cooling
2. Radioactive Decay
3. Exponential Population Growth.

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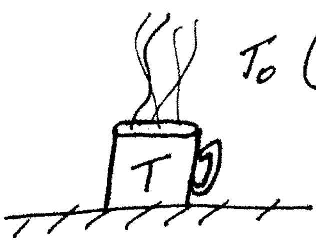
### Newton's Law of Heating/Cooling

The law states that the rate of change of an object's temperature $T$ is proportional to the difference between $T$ and the surrounding temperature $T_0$.




$T_0$ (e.g. room temperature) $\Rightarrow \dfrac{dT}{dt} \propto T - T_0$

$$\Rightarrow \boxed{\frac{dT}{dt} = -k(T - T_0),\ k > 0}$$

Heating: $\dfrac{dT}{dt} > 0$ and $T < T_0 \Rightarrow (>0) = (<0)(<0)$ ✓

Cooling: $\dfrac{dT}{dt} < 0$ and $T > T_0 \Rightarrow (<0) = (<0)(>0)$ ✓

---

Notes: 1. The constant of proportionality $k$ depends only on the properties of the object. Typically it is not known a-priori and it must be determined experimentally.

2. Though the model is crude, it provides a reasonable approximation in some case when $T - T_0$ is not too large.

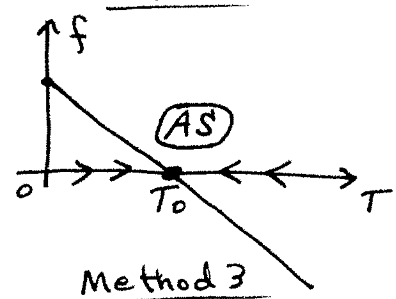
We have $\frac{dT}{dt} = -k(T - T_0)$, $k > 0$.

The ODE is autonomous with $f(T) = -k(T - T_0)$.

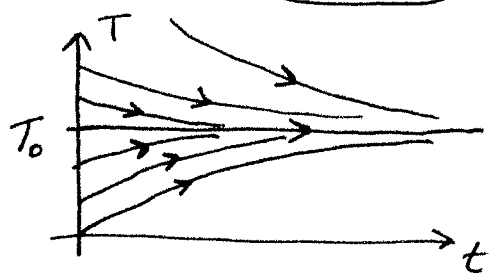
Critical Point (Eq. Sol.): $f(T) = -k(T - T_0) = 0$

$$\boxed{T_c = T_0}$$

Method 1




Solution Curves:




Method 3

$f'(T) = -k < 0 \Rightarrow$ AS

The critical point (or, eq. sol.) corresponds to the temperature at which the object is in thermal equilibrium with its surroundings.

---

Solution: $\frac{dT}{dt} = -k(T - T_0)$ (separable and linear)

$$\int \frac{dT}{T - T_0} = \int -k\,dt$$

$$\ln|T - T_0| = -kt + c$$

$$|T - T_0| = e^{-kt + c}$$

$$T - T_0 = \pm e^{c} e^{-kt}$$

$$\Rightarrow \boxed{T(t) = T_0 + Ce^{-kt}}$$

$$\lim_{t \to \infty} T(t) = \lim_{t \to \infty} (T_0 + Ce^{-kt}) = T_0 + 0 = T_0 \checkmark$$

More information is needed to determine $T_0$, $C$, and $k$.

<u>Example</u>: At $t=0$, a cup of coffee at $100°C$ is poured and left to cool in a room which is kept at a constant temperature of $20°C$. The coffee cools to $60°C$ in two minutes. Assuming that the coffee cools according to Newton's law of cooling, at what time ($t_*$) will the temperature of the coffee be $40°C$.

---

$\Rightarrow \quad \frac{dT}{dt} = -K(T-T_0)$, $K>0$, $t$ is in minutes.

General Solution: $\boxed{T(t) = T_0 + Ce^{-kt}}$

There are 4 quantities that need to be found, $T_0, C, k$, and $t_*$

<u>Find the time $t_*$ at which $T = 40°C$</u>

Need 4 conditions to determine $T_0, C, K$, and $t_*$

<u>Given</u>:
1. $T_0 = 20°C$
2. $T(0) = 100°C$
3. $T(2) = 60°C$
4. $T(t_*) = 40°C$

---

1. $\underline{T_0 = 20°C} \Rightarrow \underline{T(t) = 20 + Ce^{-kt}}$

2. $T(0) = 20 + Ce^0 = 100$
   $\underline{C = 80} \Rightarrow \underline{T(t) = 20 + 80e^{-kt}}$

3. $T(2) = 20 + 80e^{-2k} = 60$
   $e^{-2k} = \frac{1}{2}$
   $\underline{k = \frac{1}{2}\ln 2}$
   $T(t) = 20 + 80e^{-\frac{t}{2}\ln 2}$
   $\underline{T(t) = 20 + 80 \cdot 2^{-t/2}}$

4. $T(t_*) = 20 + 80 \cdot 2^{-t_*/2} = 40$
   $2^{-t_*/2} = \frac{1}{4}$
   $2^{t_*/2} = 4 = 2^2$
   $\frac{t_*}{2} = 2$
   $\boxed{t_* = 4 \text{ minutes}}$

---

Observe that $T$ decreases by $40°C$ in the first two minutes, but decreases by only $20°C$ in the second two minutes.

$\Rightarrow$ The decrease in $T$ is slower as $T - T_0$ gets smaller.

## Exponential Growth/Decay

The underlying assumption of exponential growth/decay is that the rate of change of a quantity $x$ is proportional to the amount, or size, of that quantity.

Assumption: $\frac{dx}{dt} \propto x \implies \boxed{\frac{dx}{dt} = kx, \ x \geq 0}$ (linear and separable)

$k$ = constant of proportionality

Typically, $t \geq 0$ with $t=0$ corresponding to the initial time.

General Solution: $\boxed{x(t) = Ce^{kt}}$ $k>0 \Rightarrow$ growth, $k<0 \Rightarrow$ decay

By evaluating the general solution at $t=0$, $C$ is found to be the initial value of $x$, i.e. $C = x(0)$

$x(0) = C\cancel{e^0}$ Any IC will yield the same

$\underline{C = x(0)}$

$$\implies \boxed{x(t) = x(0)\,e^{kt}} \text{ (general solution)}$$

An initial condition $x(t_1) = x_1$ is needed to find $x(0)$. That is, $x(0)$ must be determined to specify a particular solution.

---

Note: The units of $k$ are $\frac{1}{\text{time}}$.

$k$ = growth/decay rate

# Radioactive Decay

$Q(t)$ = mass of a radioactive substance at time $t$.

Assumption: The rate of decay in mass is proportional to the amount present

$$\Rightarrow \frac{dQ}{dt} \propto Q \Rightarrow \boxed{\frac{dQ}{dt} = -kQ, \quad t \geq 0;\ k > 0}$$

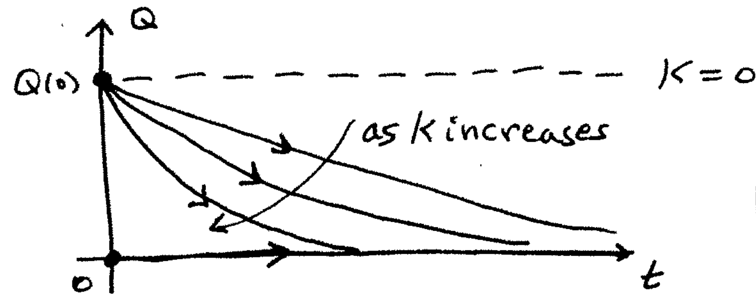
$k$ depends on the properties of the substance. The units of $k$ are $\frac{1}{time}$

Exponential Decay $\Rightarrow$ $\boxed{Q(t) = Q(0)\, e^{-kt}}$

$k$ = decay rate

$Q(0)$ = initial mass (at $t=0$)

Note: $Q_c = 0$ is an asymptotically stable critical point.



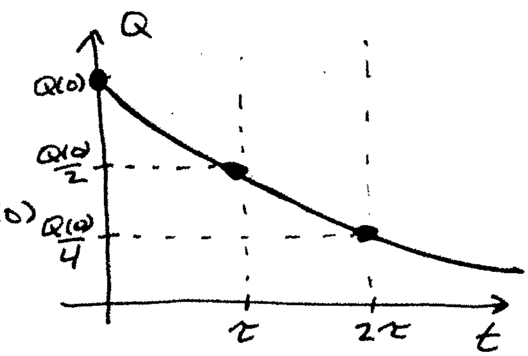

(These are not the solution curves, but rather a single solution curve shown for various values of $k$.)

The larger is $k$, the faster is the decay.

---

Definition: The half-life ($\tau$) of a radioactive substance is the time required for a sample to decay to half of its initial amount

i.e. $\tau$ is defined by $\boxed{Q(\tau) = \frac{Q(0)}{2}}$




Notes: 1) $\tau$ is independent of the initial mass $Q(0)$

$$Q(\tau) = Q(0)e^{-k\tau} = \frac{Q(0)}{2}$$

$$e^{-k\tau} = \frac{1}{2}$$

$$-k\tau = \ln\frac{1}{2} = -\ln 2$$

$$\boxed{\tau = \frac{\ln 2}{k}}$$

2) Each radioactive substance has its own half-life, which depends only on $k$.

**Example:** A radioactive sample of Thorium-234 ($Th^{234}$) decays from 100 mg to 82 mg in one week. Find the half-life $\tau$ of $Th^{234}$.

Radioactive Decay $\Longrightarrow$ $Q(t) = Q(0)\,e^{-kt}$, $k > 0$

$t$ is in weeks
$Q$ is in mg
$k$ is in $\frac{1}{\text{weeks}}$

Need 3 conditions to find $Q(0)$, $k$, and $\tau$

1. $Q(0) = 100\,mg$
2. $Q(1) = 82\,mg$
3. $Q(\tau) = \frac{Q(0)}{2} = 50\,mg$

---

1. $Q(0) = 100\,mg \Longrightarrow Q(t) = 100e^{-kt}$
2. $Q(1) = 82\,mg$

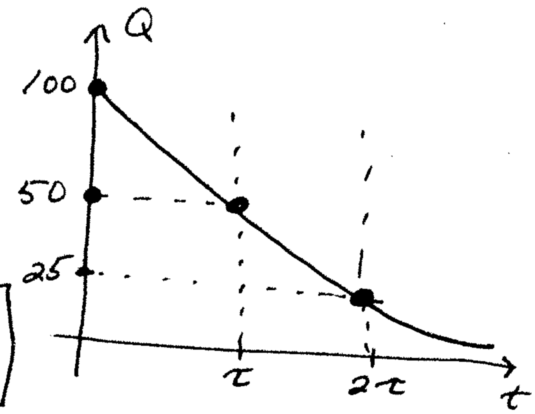
$$100e^{-k} = 82$$
$$k = -\ln(.82)$$

$\Longrightarrow Q(t) = 100e^{t\ln(.82)} = 100\,e^{\ln(.82^t)}$

$$Q(t) = 100\cdot(.82)^t$$

3. $Q(\tau) = \frac{Q(0)}{2} = 50\,mg$

$$Q(\tau) = 100\cdot(.82)^\tau = 50$$
$$(.82)^\tau = \tfrac{1}{2}$$
$$\tau\ln(.82) = \ln(\tfrac{1}{2})$$
$$\boxed{\tau = \frac{-\ln 2}{\ln(.82)} \approx 3.5\text{ weeks}}$$



**Example:** Suppose that in the above example $Th^{234}$ is added to the sample at a continuous rate of $\lambda \ \frac{mg}{week}$.

The ODE becomes $\boxed{\dfrac{dQ}{dt} = -kQ + \lambda}$ (linear and separable)

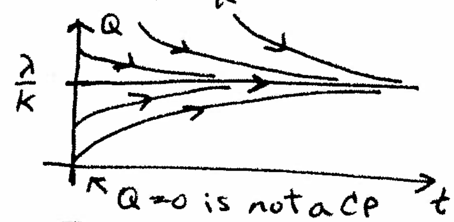
Critical Point (Eq. Sol.): $f(Q) = -kQ + \lambda = 0$

$$Q_c = \frac{\lambda}{k}$$

Method 3

$f'(Q) = -k < 0$

$\Rightarrow Q_c = \frac{\lambda}{k}$ is AS.




Solve by the Integrating Factor Method

$$\frac{dQ}{dt} + kQ = \lambda$$

$$\mu(t) = e^{\int k\,dt} = e^{kt}$$

$$\Rightarrow e^{kt}Q = \int \lambda e^{kt}\,dt$$

$$e^{kt}Q = \frac{\lambda}{k}e^{kt} + C$$

$$Q(t) = \frac{\lambda}{k} + Ce^{-kt}$$

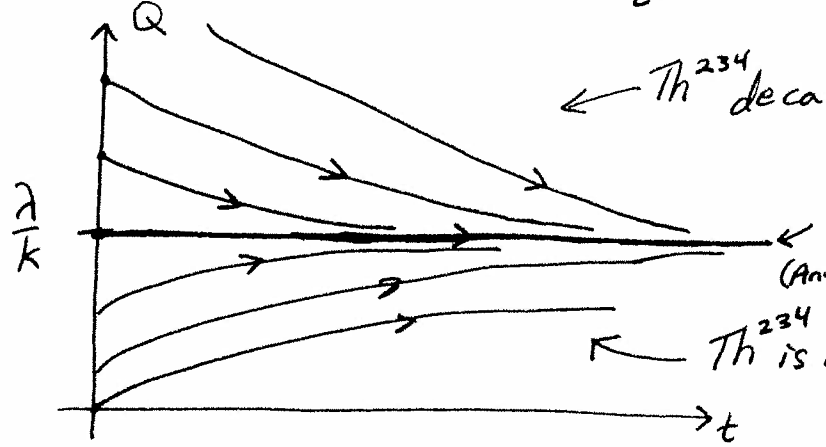
At $t=0$,

$$Q(0) = \frac{\lambda}{k} + Ce^{0}$$

$$C = Q(0) - \frac{\lambda}{k}$$

$$\Rightarrow \boxed{Q(t) = \frac{\lambda}{k} + \left(Q(0) - \frac{\lambda}{k}\right)e^{-kt}}$$

$$\lim_{t\to\infty} Q(t) = \frac{\lambda}{k} \ \checkmark$$




← $Th^{234}$ decays faster than it is added

← $Th^{234}$ decays at the same rate as it is added (Any initial mass will limit toward this balance)

← $Th^{234}$ is added faster than it decays

# Section 3.2: Modeling with Nonlinear 1st Order ODEs.

## Population Models

Population models serve as good examples. Not only are population trends intuitive, but many population models are autonomous.

---

## Exponential Population Model (EPM)

The EPM describes unlimited population growth.

- no limitations on resources (e.g. plenty of food, water, shelter, ...)
- the population has an unlimited potential to grow ($P \to \infty$ at $t \to \infty$)

In the short-term, the **EPM** may describe, for example, populations of bacteria or mice in a large field, with some accuracy.

**Model:** (IVP) $$\boxed{\frac{dP}{dt} = rP, \quad P(0) = P_0 \; ; \; r > 0}$$

$P$ = population size (or biomass, e.g. kgs of fish in a lake)
$t$ = time
$r$ = population growth rate $\quad r = \underbrace{\begin{pmatrix}\text{birth}\\ \text{rate}\end{pmatrix}}_{r_b} - \underbrace{\begin{pmatrix}\text{death}\\ \text{rate}\end{pmatrix}}_{r_d}$

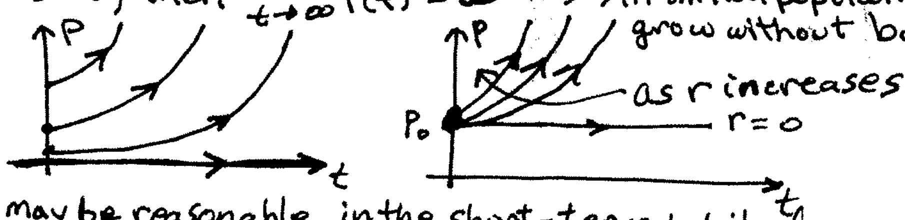
Assumption: $r_b$ and $r_d$ are constant with $r_b > r_d \Rightarrow r > 0$

Note: $r = \dfrac{dP/dt}{P}$ = "relative" growth rate
i.e. The rate of change of $P$ scaled by the size of $P$.

---

Solution of the IVP: $\boxed{P(t) = P_0 e^{rt}}$ $\quad P_c = 0$ is an unstable CP.

If $P_0 > 0$, then $\lim\limits_{t \to \infty} P(t) = \infty \Rightarrow$ All initial populations grow without bound.




NOT REALISTIC

The EPM may be reasonable in the short-term, but it does not accurately predict population trends in the long-term.

The EPM applies only to a few mice in a large field, for example.

A few mice will not feel limitations. They have plenty of food and space. However, at some point, the field will become crowded and the mice will have to compete for resources. As the size of the population increases (as P becomes large), a lack of resources will hinder the growth of the population, and the growth rate will decrease. The population model should account for the eventuality that the field will become over-populated.

---

Realistic populations cannot grow indefinitely at an exponential rate. Eventually, the population will feel limitations.

---

It is more realistic to consider a growth rate which depends on the size of the population P. The growth rate should decrease, and even become negative, as the population size P increases.

---

Now, let $R=R(P)$ denote the growth rate

$$\Rightarrow \boxed{\frac{dP}{dt} = R(P)\cdot P \ ; \ P(0)=0}$$

If P is relatively small, $R(P)>0 \Rightarrow$ growth.

If P is relatively large, $R(P)<0 \Rightarrow$ decay.

## Logistic Population Model (LPM)

The LPM describes populations that are limited by the available resources.

Consider $\frac{dP}{dt} = R(P) \cdot P$.

The growth rate $R(P)$ should be

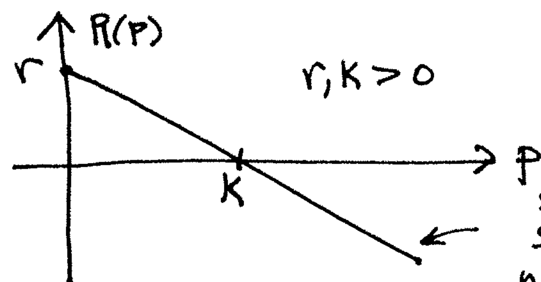
i) a decreasing function of $P$

ii) positive if $P$ is relatively small $\Rightarrow$ growth

iii) negative if $P$ is relatively large $\Rightarrow$ decay

(e.g. more mice than resources)

The LPM corresponds to the simplest growth rate function which satisfies the criteria. That is, the LPM assumes that $R(P)$ is a linear function of $P$.




simplest choice
straight line with negative slope.

$$\Rightarrow \boxed{R(P) = r\left(1 - P/K\right); \quad r, K > 0}$$

Thus, the LPM is

$$\boxed{\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P, \quad P(0) = P_0\ ; \quad r, K > 0}$$

(nonlinear, but separable)

The values of $r$ and $K$ depend on the specific population and environment under consideration.

---

$r$ = <u>instrinsic growth rate</u> = growth rate in the absence of limitations. (same as that of the EPM)

$K$ = <u>environmental carrying capacity</u> = maximum population size that the environment can comfortably accommodate.

$P < K \Rightarrow R(P) > 0 \Rightarrow$ growth

$P > K \Rightarrow R(P) < 0 \Rightarrow$ decay.

Most useful ODEs cannot be solved exactly, so it is necessary to describe the solution behavior without actually knowing the solution. The LPM is separble and can be solved exactly. However, before finding the exact solution, it is instructive to investigate the solution behavior by other means.

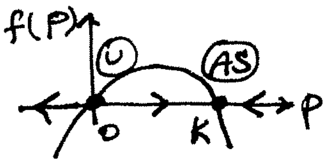
---

$$\frac{dP}{dt} = r\left(1-\frac{P}{K}\right)P$$

Critical Points: $f(P) = r\left(1-\frac{P}{K}\right)P = 0$

$\boxed{P_c = K}$ $\boxed{P_c = 0}$

Stability of CPs: Method 1




Method 3

$f'(P) = r\left[\frac{-P}{K} + (1-P/K)\right] = r\left(1-\frac{2P}{K}\right)$

$f'(0) = r(1-0) = r > 0$ U

$f'(K) = r(1-2) = -r < 0$ AS

$\boxed{\begin{array}{l} P_c = 0 \text{ is Unstable} \\ P_c = K \text{ is Asymptotically Stable} \end{array}}$

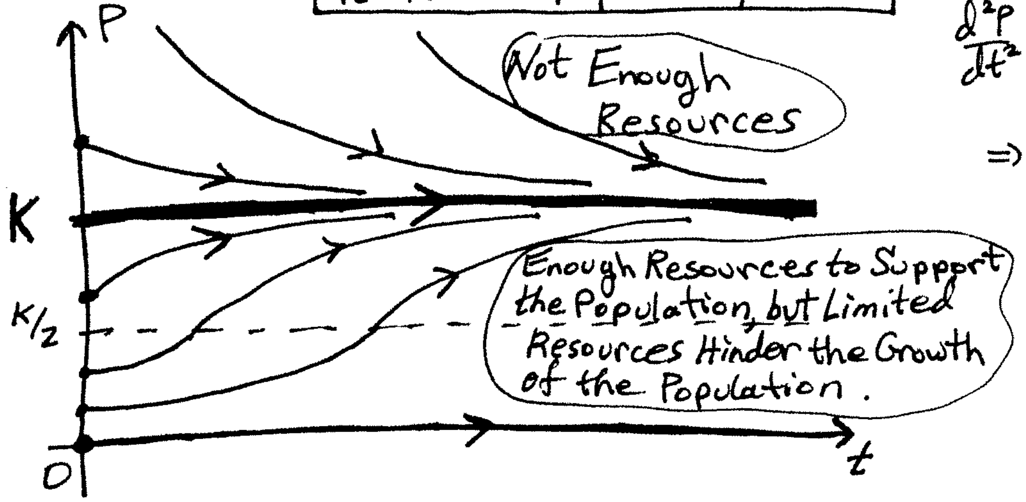
inflection points

$\frac{d^2P}{dt^2} = f'(P)f(P) = 0$

$f'(P) = 0$

$\Rightarrow r\left(1-\frac{2P}{K}\right) = 0$

$P = K/2$




K = environmental carrying capacity
(= maximum population size that can be supported by the environment)

Suppose P is much less than K ($P \ll K$, or $\frac{P}{K} \ll 1$).

$\Longrightarrow \frac{dP}{dt} \approx rP \Rightarrow$ If P is relatively small, the population growth is approximately exponential.

# Solution of the LPM

$\frac{dP}{dt} = r\left(1-\frac{P}{K}\right)P, \quad P(0)=P_0 \; ; \; r, K > 0$ (separable)

$$\frac{dP}{dt} = \frac{r}{-K}(P-K)P$$

$$\int \frac{-K\,dP}{P(P-K)} = \int r\,dt$$

partial fractions

$$\frac{-K}{P(P-K)} = \frac{A}{P} + \frac{B}{P-K}$$

$$-K = A(P-K) + BP$$

$$\Rightarrow \underline{A=1}, \; \underline{B=-1}$$

$$\Rightarrow \frac{-K}{P(P-K)} = \frac{1}{P} - \frac{1}{P-K}$$

$$\int\left(\frac{1}{P} - \frac{1}{P-K}\right)dP = \int r\,dt$$

$$\ln|P| - \ln|P-K| = rt + C$$

$$\ln\left|\frac{P}{P-K}\right| = rt + C$$

$$\left|\frac{P}{P-K}\right| = e^{rt+C}$$

$$\frac{P}{P-K} = \pm e^{C} e^{rt}$$

$$\frac{P}{P-K} = Ce^{rt}$$

$\left(\frac{1}{C} \to C\right)$ $\quad PCe^{-rt} = P - K$

$$P(1 - Ce^{-rt}) = K$$

$$P(t) = \frac{K}{1-Ce^{-rt}}$$

general solution

$$P(0) = \frac{K}{1-C\cancel{e^{0}}} = P_0$$

$$\frac{K}{P_0} = 1 - C$$

$$C = 1 - \frac{K}{P_0}$$

$$\Rightarrow \boxed{P(t) = \frac{K}{1+\left(\frac{K}{P_0}-1\right)e^{-rt}}, \quad P_0 \neq 0}$$

$$P(t) = 0 \text{ if } P_0 = 0$$

$$\lim_{t\to\infty} P(t) = \frac{K}{1+0} = K$$

(Asymptotically Stable Critical Point)

## Logistic Population Model with a Critical Threshold

Some populations die out if the population size is too small.

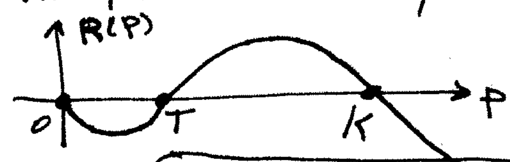
e.g. 1. If a population size is small, chance encounters may be infrequent, thus limiting the opportunities for reproduction. If chance encounters are too infrequent, the death rate will exceed the birth rate, and extinction will occur.

2. Many herding animals find safety from predators by living in large groups. If the population size is too small, individual members will have to fend for themselves and not have the necessary security to reproduce. Again, the death rate will exceed the birth rate and extinction will occur.

Now $R(P)$ should be (Recall: $R(P)$ = growth/decay rate)

i) negative if $P$ is relatively large $\Rightarrow$ decay
ii) positive if $P$ is moderately sized $\Rightarrow$ growth
iii) negative if $P$ is relatively small $\Rightarrow$ decay

Choose the simplest such $R(P)$.

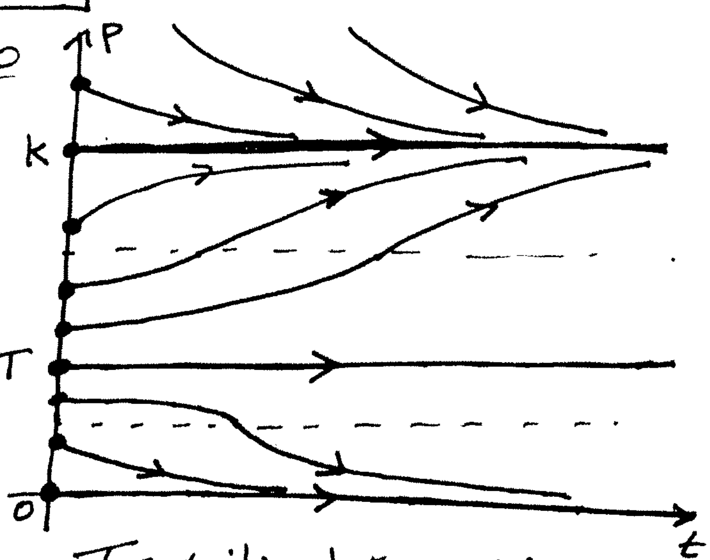



$$\Rightarrow \boxed{\frac{dP}{dt} = r\left(\frac{P}{T}-1\right)\left(1-\frac{P}{K}\right)P, \quad P(0)=P_0 \qquad r>0, \quad 0<T<K}$$

$R(P) = r\left(\frac{P}{T}-1\right)\left(1-\frac{P}{K}\right); \; r>0, \; 0<T<K$

(nonlinear, but separable)

Critical Points: $P_c = T$ (U) $\quad P_c = K$ (AS) $\quad P_c = 0$ (AS)




Given a specific population, $r$, $T$, and $K$ can be determined (or estimated), and then predictions can be made about the fate of the population.

$T$ = critical threshold

# Other Variations of the Logistic Population Model

1. Logistic Population Model with Immigration/Emmigration

$$\frac{dP}{dt} = r\left(1-\frac{P}{K}\right)P \pm I(P)$$

2. Harvesting a Renewable Resource (e.g. fish, crops, ...)

$$\frac{dP}{dt} = r\left(1-\frac{P}{K}\right)P - h(P)$$

harvesting function (typically an increasing function of P. e.g. $h(P) = cP$)

---

In examples 1 and 2, the critical points and their stability depends on the form of the functions $I(P)$ and $h(P)$, respectively.

---

In all of the above population models (EPM, LPM, ...), P does not necessarily represent the raw number of a species. P may denote some other measure of population size. For example, for a population of fish, it may be more appropriate to let

$$P = \text{biomass} = \text{total mass of the fish population.}$$

# Chapter 4: Linear Second-Order ODEs

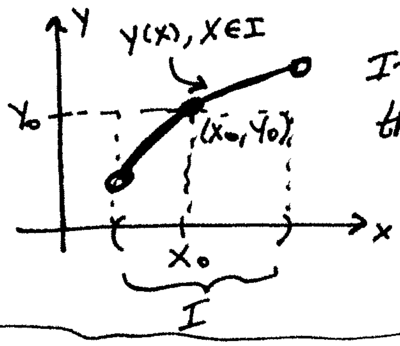
General Form: $\boxed{y'' + p(x)y' + q(x)y = g(x)}$ ①

IVP: $\boxed{y(x_0) = y_0, \quad y'(x_0) = y_1}$ ②

## Section 4.1: Theory and Terminology

**Theorem: Existence and Uniqueness**

Consider the IVP given by ① and ②. If the functions $p, q$, and $g$ are continuous on an interval $I$ which contains $x_0$, then a unique solution exists on $I$.




If $p, q$, and $g$ are continuous for all $x$, then a unique solution exists for all $x$.
$\Rightarrow I = (-\infty, \infty)$

---

**Example:** Consider the IVP $(x-3)y'' + (\ln x)y' + x^2 y = \frac{1}{x-2}$; $y(1) = 5$, $y'(1) = 2$

Find the largest interval ($I_{max}$) over which a unique solution is guaranteed to exist.

i.e. Find the largest interval containing $x_0$ over which $p, q$, and $g$ are all continuous.

Put the ODE in the form of ①, as required by the theorem $\Rightarrow$ $y'' + \underbrace{\frac{\ln x}{x-3}}_{p} y' + \underbrace{\frac{x^2}{x-3}}_{q} y = \underbrace{\frac{1}{(x-2)(x-3)}}_{g}$; $x_0 = 1$

$\ln x \Rightarrow x > 0$
$\frac{1}{x-3} \Rightarrow x \neq 3$
$p$ is continuous on $(0,3)$ and $\cancel{(3,\infty)}$
contains $x_0 = 1$
$I_p = (0,3)$

$\frac{1}{x-3} \Rightarrow x \neq 3$
$q$ is continuous on $(-\infty,3)$ and $\cancel{(3,\infty)}$
contains $x_0$
$I_q = (-\infty,3)$

$\frac{1}{x-2} \Rightarrow x \neq 2$
$\frac{1}{x-3} \Rightarrow x \neq 3$
$g$ is continuous on $(-\infty,2)$, $\cancel{(2,3)}$, and $\cancel{(3,\infty)}$.
contains $x_0 = 1$
$I_g = (-\infty,2)$

$I_{max}$ is the intersection of the intervals $I_p$, $I_q$, and $I_g$


$$\boxed{I_{max} = I_p \cap I_q \cap I_g = (0,2)}$$

**Note:** A unique solution may exist over an interval which is large than $I_{max}$, but there are no guarantees.

---

**Note:** The above theorem and ideas can be easily generalized to linear IVPs of any order $n$ $(n = 1, 2, 3, \ldots)$.

**Definition:** The ODE $y''+p(x)y'+q(x)y=g(x)$ is **homogeneous** if $g(x)=0$ for all $x$.
Otherwise, the ODE is **nonhomogeneous**.
$g(x)$ is called the **nonhomogeneous term**.

Homogeneous ODEs: Sections 4.1–4.3
Nonhomogeneous ODEs: Sections 4.4, 4.6

For now, we'll consider only homogeneous ($g \overset{\text{for all } x}{\equiv} 0$) ODEs

$$\Rightarrow \boxed{y''+p(x)\,y'+q(x)\,y=0} \quad \text{(H)}$$

We need to discuss some preliminary concepts concerning the general solution of (H).

**Linear Combination**: A linear combination of $n$ quantities $A_1, A_2, \ldots, A_n$ is a sum of the form $c_1A_1+c_2A_2+\cdots+c_nA_n$, where $c_1,\ldots,c_n$ are arbitrary constants.

**Principle of Superposition**: If $y_1$ and $y_2$ are solutions of (H), then the linear combination $y=c_1y_1+c_2y_2$ is also a solution of (H) for any constants $c_1$ and $c_2$. (but not necessarily the general solution)

i.e. The linear combination of any two solutions of (H) is also a solution of (H).

**Proof:** Suppose $y_1$ and $y_2$ are solutions of (H) and let $y=c_1y_1+c_2y_2$.

Then,
$$\begin{aligned} y''+py'+qy &= (c_1y_1+c_2y_2)''+p(c_1y_1+c_2y_2)'+q(c_1y_1+c_2y_2)\\ &= (c_1y_1''+c_2y_2'')+p(c_1y_1'+c_2y_2')+q(c_1y_1+c_2y_2)\\ &= c_1\underbrace{(y_1''+py_1'+qy_1)}_{\substack{=0\\ \text{since } y_1 \text{ is a}\\ \text{solution of (H)}}}+c_2\underbrace{(y_2''+py_2'+qy_2)}_{\substack{=0\\ \text{since } y_2 \text{ is a}\\ \text{solution of (H)}}}=0 \end{aligned}$$

$$\Rightarrow y''+py'+qy=0 \checkmark$$

Therefore, $y=c_1y_1+c_2y_2$ satisfies the ODE (H), and thus is a solution of (H).

Q.E.D.

**Note:** Equation (H) possesses the trivial solution ($y \equiv 0$ for all $x$).

# Linear Independence

**Fact:** $\boxed{\text{If two solutions, } y_1 \text{ and } y_2\text{, of (H) are linearly independent, then the linear combination } y = c_1y_1 + c_2y_2 \text{ is the general solution of (H).}}$

i.e. Given two solutions of (H), $y = c_1y_1 + c_2y_2$ is the general solution of (H) only if $y_1$ and $y_2$ are linearly independent.

$\Rightarrow$ To find the general solution of (H), it suffices to find any two linearly independent particular solutions of (H).

---

**Definition:** Two functions, $y_1$ and $y_2$, are linearly dependent if one is a constant multiple of the other, that is, if

$$y_1 = ky_2 \quad (\text{or } y_2 = \tfrac{1}{k}y_1) \text{ for some nonzero constant } k.$$

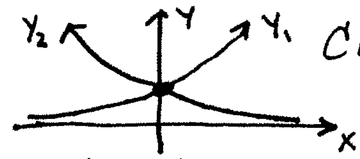
Otherwise, $y_1$ and $y_2$ are linearly independent.

i.e. $\boxed{y_1 \neq ky_2 \Rightarrow y_1 \text{ and } y_2 \text{ are linearly independent}}$

**Notation:** $y_1 \perp\!\!\!\perp y_2 \Rightarrow y_1$ and $y_2$ are linearly independent.
$y_1 \not\perp\!\!\!\perp y_2 \Rightarrow y_1$ and $y_2$ are linearly dependent.

---

**Examples:** 1) $y'' - y = 0$; $y_1 = e^x$, $y_2 = e^{-x}$.




Clearly $y_1 \neq ky_2 \Rightarrow y_1 \perp\!\!\!\perp y_2$

$y_1 \perp\!\!\!\perp y_2 \Rightarrow y(x) = c_1e^x + c_2e^{-x}$ is the general solution

2) $y'' + y = 0$; $y_1 = \cos x$, $y_2 = \sin x$. Clearly $y_1 \neq ky_2 \Rightarrow y_1 \perp\!\!\!\perp y_2$

$y_1 \perp\!\!\!\perp y_2 \Rightarrow y(x) = c_1\cos x + c_2\sin x$ is the general solution.

3) $y'' + y = 0$; $y_1 = \cos x$, $y_2 = 2\cos x$. Clearly $y_1 = \frac{1}{2}y_2$ $(y_2 = 2y_1)$ $\Rightarrow y_1 \not\perp\!\!\!\perp y_2$

$y_1 \not\perp\!\!\!\perp y_2 \Rightarrow y(x) = c_1\cos x + c_2 \cdot 2\cos x$ is not the general solution.

$$y = c_1y_1 + c_2y_2 = c_1y_1 + c_2 2y_1 = (c_1 + 2c_2)y_1 = Cy_1$$

The linear combination of $y_1$ and $y_2$ is equivalent to a single solution $\Rightarrow y_1$ and $y_2$ yield only one of the two necessary linearly independent solutions.

Often linear independence is not so obvious.

Example: $y''+y=0$; $y_1=\cos(x+\pi/4)$, $y_2=\sin(x-\pi/4)$

Are $y_1$ and $y_2$ linearly independent?

Trig. identities reveal that $y_1=-y_2 \Rightarrow y_1 \not\perp\!\!\!\perp y_2$.

$\Rightarrow$ $y(x)=C_1\cos(x+\pi/4)+C_2\sin(x-\pi/4)$ is not the general solution.

$$y=C_1y_1+C_2y_2=C_1y_1+C_2(-y_1)=(C_1-C_2)y_1=Cy_1$$

$\Rightarrow$ $y_1$ and $y_2$ give only one of the two necessary $\perp\!\!\!\perp$ solutions.

Example: $y_1=\left(\frac{1}{\cos x}+1\right)\left(\frac{1}{\cos x}-1\right)$

$y_2=\tan^2 x$

Are $y_1$ and $y_2$ linearly independent?

Trig. identities $\Rightarrow$ $y_1=y_2 \Rightarrow$ NO

Example: For what values of $a$ and $b$ are the functions

$f(x)=\cos(x+a)$ and $g(x)=\sin(x+b)$

linearly independent?

It's not so easy to answer this question while thinking in terms of one function being a multiple of the other.

The point is that it would be useful to have a method to establish the linear independence (or dependence) of two functions, namely, two solutions of (H).

Fortunately, such a method does exist.

The method depends on a function called 'the Wronskian of $y_1$ and $y_2$'.

# Wronskian

Recall: $y'' + p(x)y' + q(x)y = g(x)$ (H)

The Wronskian may be written as a determinant.

The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is defined as

$$\boxed{\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc}$$

---

**Definition:** The Wronskian of two functions, $y_1(x)$ and $y_2(x)$, is the function defined by

$$\boxed{W = W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)\,y_2'(x) - y_1'(x)\,y_2(x)}$$

---

**Informally:** Suppose $y_1$ and $y_2$ are solutions of (H) on an interval $I$.

Then, $W \neq 0$ on $I \Rightarrow y_1 \perp\!\!\!\perp y_2$ on $I \Rightarrow y = c_1 y_1 + c_2 y_2$ is the general solution of (H) on $I$.

On the other hand, $W = 0 \Rightarrow y_1 \not\perp\!\!\!\perp y_2 \Rightarrow y = c_1 y_1 + c_2 y_2$ is not the general solution.

---

**Theorem:** Let $y_1$ and $y_2$ be solutions of the ODE

$$y'' + p(x)y' + q(x)y = \overset{0}{\cancel{g(x)}}, \quad \text{(H)} \quad \text{on an interval } I.$$

Then, the following statements are equivalent.

1. $W(y_1, y_2)(x_0) \neq 0$ for some $x_0$ in $I$.
2. $W(y_1, y_2)(x) \neq 0$ for all $x$ in $I$.
3. $y_1(x) \perp\!\!\!\perp y_2(x)$ for all $x$ in $I$.
4. $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is the general solution of (H) on $I$.

---

**Notes:** 1. and 2. imply that either i) $W$ is nonzero everywhere in $I$, or ii) $W = 0$ everywhere in $I$.

1) i.e. $W$ cannot be zero at some points in $I$ and nonzero at others.

Abel's Formula: $\boxed{W(y_1, y_2)(x) = Ce^{-\int p(x)\,dx}}$ $\quad C = 0 \Rightarrow W = 0$ on $I$; $\quad C \neq 0 \Rightarrow W \neq 0$ on $I$

2) If $y_1$ and $y_2$ are solutions of (H) on $I$, with $W(y_1, y_2)(x) \neq 0$ on $I$, then $y(x) = c_1 y_1(x) + c_2 y_2(x)$ is the general solution of (H) on $I$.

**Examples**: Determine whether or not the two solutions, $y_1$ and $y_2$, of the given ODE are linearly independent. If so, write down the general solution.

1) $y''+y=0$, $y_1=\cos x$, $y_2=\sin x$

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}=\cos^2 x+\sin^2 x=1\neq 0$$

$$W\neq 0 \Rightarrow \boxed{y_1 \perp\!\!\!\perp y_2}$$

General Solution: $\boxed{y(x)=C_1\cos x+C_2\sin x}$

2) $y''+y=0$, $y_1=\cos(x+\pi/4)$, $y_2=\sin(x-\pi/4)$

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} \cos(x+\pi/4) & \sin(x-\pi/4) \\ -\sin(x+\pi/4) & \cos(x-\pi/4) \end{vmatrix}$$

$$=\cos(x+\tfrac{\pi}{4})\cos(x-\pi/4)+\sin(x+\tfrac{\pi}{4})\sin(x-\pi/4)$$

$$=\cos[(\cancel{x}+\pi/4)-(\cancel{x}-\pi/4)]=\cos\pi/2=0$$

$$W=0 \Rightarrow \boxed{y_1 \not\perp\!\!\!\perp y_2}$$

The linear combination of $y_1$ and $y_2$ does not give the general solution.

Note: Between parts 1) and 2), we have four solutions of $y''+y=0$. The linear combination of any two linearly independent solution gives the general, e.g. $y(x)=C_1\cos x+C_2\cos(x+\pi/4)$. The general solution may be expressed in many ways, but they are all equivalent.

3) $y''-2y'+y=0$, $y_1=e^t$, $y_2=te^t$

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix}=(\cancel{t}+1)e^{2t}-\cancel{te^{2t}}=e^{2t}\neq 0 \text{ for all } t$$

$$W\neq 0 \Rightarrow \boxed{y_1 \perp\!\!\!\perp y_2}$$

General Solution: $y(t)=C_1e^t+C_2te^t$

$$\boxed{y(t)=(C_1+C_2t)e^t}$$

We'll discuss how to find the given solutions of the above ODEs

## Section 4.2: Reduction of Order (Postpone. Insert into section 4.3)

## Section 4.3: Homogeneous, Linear, 2nd Order ODEs with constant coefficients

$\Rightarrow \quad y'' + py' + qy = 0$, where $p$ and $q$ are constants

For convenience, let $p = \frac{b}{a}$ and $q = \frac{c}{a}$, with $a \neq 0$

$$\Rightarrow \boxed{ay'' + by' + cy = 0, \ a \neq 0} \quad \text{HC}$$

Recall: The general solution is the linear combination of any two linearly independent solutions.

i.e. If $y_1$ and $y_2$ are solutions of HC, and $y_1 \perp\!\!\!\perp y_2$, then

$y = c_1 y_1 + c_2 y_2$ is the general solution of HC

Thus the goal is to find two solutions of HC and show that they are linearly independent. Then we'll have the general solution.

Fact: All homogeneous, linear ODEs with constant coefficients, such as HC, have solutions of the form $y = e^{rx}$ for some constant $r$.
(r may be a complex number)

Consider the 1st order Case

$$Ay' + By = 0, \ A \neq 0 \quad \left(\text{Exponential ODE}: \ y' = -\frac{B}{A} y\right)$$

To solve, plug in $y = e^{rx}$, and solve for $r$.

$y = e^{rx}$
$y' = re^{rx}$ $\Rightarrow$ $A(re^{rx}) + B(e^{rx}) = 0$

$(Ar + B)\cancel{e^{rx}} = 0$ (linear $\Rightarrow e^{rx}$ can be factored from each term)

$e^{rx} \neq 0 \Rightarrow Ar + B = 0$ (homogeneous $\Rightarrow e^{rx}$ can be cancelled)

$r = -\frac{B}{A}$ (constant coefficients $\Rightarrow$ get an algebraic equation for $r$ which is independent of $x$)

$\Rightarrow$ $y(x) = e^{-B/A\, x}$ is a solution

linear combination $\Rightarrow$ $\boxed{y(x) = Ce^{-B/A\, x} \text{ is the general solution of } Ay' + By = 0}$

Take the same approach for the $2^{nd}$ order case.

$$ay'' + by' + cy = 0 \quad \text{HC}$$

Try $y = e^{rt}$ as a solution.

$$y' = re^{rt}$$
$$y'' = r^2 e^{rt}$$

Then, $ay'' + by' + cy = 0$

$$a(r^2 e^{rt}) + b(re^{rt}) + c(e^{rt}) = 0$$

$$(ar^2 + br + c)\cancel{e^{rt}} = 0$$

$$\boxed{ar^2 + br + c = 0}$$ Characteristic Equation

(Textbook: Auxiliary Equation)

If $r$ is a root of the characteristic equation, then $y = e^{rt}$ is a solution of HC.

---

Notice that the characteristic equation has the same coefficients as HC.

$\Rightarrow$ To obtain the characteristic equation of HC, replace the $n^{th}$ derivative of $y$ with the $n^{th}$ power of $r$.

That is, replace $\frac{d^n y}{dx^n}$ by $r^n$ $\left(\frac{d^0 y}{dx^0} \to r^0\right)$

$$y \to r^0 = 1$$
$$y' \to r^1$$
$$y'' \to r^2$$

Characteristic Equation : $ar^2+br+c=0$

The characteristic equation has two roots. (2nd degree polynomial)

Quadratic Formula $\Rightarrow$ $\boxed{r_1 = \dfrac{-b+\sqrt{b^2-4ac}}{2a}}$ $\boxed{r_2 = \dfrac{-b-\sqrt{b^2-4ac}}{2a}}$

Two linearly independent solution can be determined from the roots, thus giving the general solution of HC

---

There are 3 cases to consider.

1. $\underline{b^2-4ac>0}$ $\Longrightarrow$ Real Distinct Roots $(r_1 \neq r_2)$

2. $\underline{b^2-4ac=0}$ $\Longrightarrow$ Real Repeated Roots $(r_1 = r_2)$

3. $\underline{b^2-4ac<0}$ $\Longrightarrow$ Complex Conjugate Roots
$$r_1 = \alpha + i\beta, \quad r_2 = \alpha - i\beta, \quad \beta \neq 0$$

---

Case 1: gives to real solutions $y_1 = e^{r_1 x}$ and $y_2 = e^{r_2 x}$

$y_1 \perp\!\!\!\perp y_2 \Rightarrow y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ is the general solution of HC

Case 2: gives only one real solution $y_1 = e^{r_1 x}$

$\Rightarrow$ need to find another solution

Case 3: gives two complex solutions $\Rightarrow$ need to find two real solutions

## Case 1: Real Distinct Roots ($r_1 \neq r_2$)

ODE: $ay'' + by' + cy = 0, \; a \neq 0 \quad (b^2 - 4ac > 0)$

Characteristic Equation: $ar^2 + br + c = 0$

Roots: $r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

If $b^2 - 4ac > 0$, then $r_1$ and $r_2$ are real and unequal.

$\Rightarrow$ Two solutions are $\boxed{\begin{array}{l} y_1 = e^{r_1 x} \\ y_2 = e^{r_2 x} \end{array}}$

---

Check for linear independence

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix} = (e^{r_1 x})(r_2 e^{r_2 x}) - (e^{r_2 x})(r_1 e^{r_1 x})$$

$$= \underbrace{e^{(r_1 + r_2)x}}_{\substack{\neq 0 \text{ since} \\ e^x \neq 0 \text{ for} \\ \text{all } x}} \underbrace{(r_2 - r_1)}_{\substack{\neq 0 \text{ since} \\ r_1 \neq r_2}} \neq 0 \Rightarrow y_1 \perp\!\!\!\perp y_2$$

---

Then, the general solution is

$$\boxed{y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}} \quad \left(\begin{array}{c}\text{Real} \\ \text{Distinct} \\ \text{Roots}\end{array}\right)$$

**Example:** Find the general solution of $y''+8y'-9y=0$. $(y'=\frac{dy}{dx})$

Characteristic Equation: $r^2+8r-9=0$

$(r-1)(r+9)=0$

$r=1$ $r=-9$ $\Longrightarrow$ $y_1=e^x$, $y_2=e^{-9x}$

General Solution: $\boxed{y(x)=C_1e^x+C_2e^{-9x}}$

---

**Example:** Solve the IVP $2\frac{d^2y}{dt^2}-3\frac{dy}{dt}+y=0$ ; $y(0)=0$, $y'(0)=1$

Characteristic Equation: $2r^2-3r+1=0$

$$r=\frac{-(-3)\pm\sqrt{(-3)^2-4(2)(1)}}{2(2)}=\frac{3\pm1}{4}=1,\tfrac{1}{2}$$

$\Rightarrow$ $r=1$ $r=\frac{1}{2}$ $y_1=e^t$, $y_2=e^{\frac{1}{2}t}$

General Solution: $\boxed{y(t)=C_1e^t+C_2e^{\frac{1}{2}t}}$

$y'(t)=C_1e^t+\frac{C_2}{2}e^{\frac{1}{2}t}$

ICs:

$y(0)=C_1+C_2=0$

$y'(0)=C_1+C_2/2=1$

Subtract: $0+C_2/2=-1$

$C_2=-2$

$C_1=-C_2=-(-2)=2$

$C_1=2$

$y(t)=2e^t-2e^{\frac{1}{2}t}$

$$\boxed{y(t)=2(e^t-e^{\frac{1}{2}t})}$$

# Real Repeated Roots

ODE: $ay'' + by' + cy = 0$, where $b^2 - 4ac = 0$, $a \neq 0$

Characteristic Equation: $ar^2 + br + c = 0$

Roots: $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\dfrac{b}{2a} \Rightarrow \boxed{r = \dfrac{-b}{2a}}$

This gives only one solution $\boxed{y_1 = e^{rx} = e^{\frac{-b}{2a}x}}$

We need a second linearly independent solution.

---

## Reduction of Order (section 4.2)

Idea: By knowing one solution, the $2^{nd}$ order ODE can be reduced to a $1^{st}$ order ODE as follows.

Try $y_2(x) = v(x)\, y_1(x)$ as a solution.

Substitute into the ODE and determine $v(x)$.

This substitution leads to a $1^{st}$ order equation for $v'(x)$.
$\underbrace{\phantom{xx}}_{\text{reduction of order}}$

---

$$y_2 = v y_1$$

$$y_2' = v y_1' + v' y_1 \qquad (fg)'' = fg'' + 2f'g' + f''g$$

$$y_2'' = v y_1'' + 2v' y_1' + v'' y_1$$

Choose $v(x)$ so that $ay_2'' + by_2' + cy_2 = 0$.

(i.e. so that $y_2$ is a solution)

$$ay_2'' + by_2' + cy_2 = 0$$

$$a(vy_1'' + 2v'y_1' + v''y_1) + b(vy_1' + v'y_1) + c(vy_1) = 0$$

$$v(\cancel{ay_1'' + by_1' + cy_1}) + v'(2ay_1' + by_1) + v''(ay_1) = 0$$

$= 0$ since $y_1$ is a solution

$$\boxed{v'(2ay_1' + by_1) + v''(ay_1) = 0}$$

Separable 1st order ODE for $v'$

$$\frac{v''}{v'} = -\frac{2ay_1' + by_1}{ay_1} = -\left(2\frac{y_1'}{y_1} + \frac{b}{a}\right)$$

Plug in $y_1 = e^{-\frac{b}{2a}x}$ $\Rightarrow$ $v'\left[2a\left(\frac{-b}{2a}\cancel{e^{-\frac{b}{2a}x}}\right) + b\cancel{e^{-\frac{b}{2a}x}}\right] + v''\left(a\cancel{e^{-\frac{b}{2a}x}}\right) = 0$

$y_1' = \frac{-b}{2a}e^{-\frac{b}{2a}x}$

$$v'(\cancel{-b+b}) + \cancel{a}v'' = 0$$

$$v'' = 0$$

$$v' = A$$

$$\boxed{v(x) = Ax + B}$$

We have $y_1 = e^{rx} = e^{-\frac{b}{2a}x}$ and $y_2 = v(x)y_1(x) = (Ax+B)e^{-\frac{b}{2a}x}$.

$y_2$ is a solution for all $A$ and $B$.

Pick $A$ and $B$ so that $y_1 \perp\!\!\!\perp y_2$ (if possible).

$\boxed{y_1 = e^{rx},\ r = \frac{-b}{2a}}$ $\boxed{y_2 = v(x)y_1 = (Ax+B)y_1}$

Pick $A$ and $B$ so that $y_1 \perp\!\!\!\perp y_2$ (if possible)

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} y_1 & (Ax+B)y_1 \\ y_1' & (Ax+B)y_1' + Ay_1 \end{vmatrix}$$

$$= (y_1)\left[\cancel{(Ax+B)y_1'} + Ay_1\right] - (y_1')\left[\cancel{(Ax+B)y_1}\right] = Ay_1^2$$

$\boxed{W = Ay_1^2}$, $y_1^2 = e^{\frac{-b}{a}x} > 0$

$\Rightarrow$ $W \neq 0$ when $A \neq 0$ (only restriction on $A$ and $B$)

i.e. $y_1 \perp\!\!\!\perp y_2$ for all $A \neq 0$ and all $B$.

Any choice of $A$ and $B$ with $A \neq 0$ will yield a second linearly independent solution.

For convenience, pick $A = 1$ and $B = 0$.

$\Rightarrow v(x) = x$

$\Rightarrow$ $\boxed{y_2(x) = x\,y_1(x) = xe^{rx},\ r = \frac{-b}{2a}}$

$y_1 \perp\!\!\!\perp y_2 \Rightarrow$ The general solution is

$$y(x) = C_1y_1(x) + C_2y_2(x) = C_1e^{rx} + C_2xe^{rx}$$

$$\boxed{y(x) = (C_1 + C_2x)e^{rx},\ r = \frac{-b}{2a}}$$ (Real Repeated Roots)

---

Note: Reduction of order works for variable coefficients as well. $y'' + p(x)y' + q(x)y = 0$.

Given a solution $y_1(x)$, let $y_2(x) = v(x)y_1(x)$. Then plug $y_2(x)$ into the ODE and solve for $v(x)$.

**Example:** Find the general solution of $4y'' + 4y' + y = 0$, $y = y(x)$.

Characteristic Equation: $4r^2 + 4r + 1 = 0$

$$(2r+1)^2 = 0$$

$$r = -\tfrac{1}{2}, -\tfrac{1}{2}$$

$$\Rightarrow \boxed{y(x) = (c_1 + c_2 x)e^{-x/2}}$$

---

**Example:** Find the general solution of $9\frac{d^2x}{dt^2} + 12\frac{dx}{dt} + 4x = 0$.

Characteristic Equation: $9r^2 + 12r + 4 = 0$

$$(3r+2)^2 = 0$$

$$r = -2/3, -2/3$$

$$\Rightarrow \boxed{x(t) = (c_1 + c_2 t)e^{-2/3\, t}}$$

# Complex Conjugate Roots

$i = \sqrt{-1}$ e.g. $\sqrt{-2} = \sqrt{(-1)(2)} = \sqrt{-1}\cdot\sqrt{2} = i\sqrt{2} \Rightarrow \sqrt{-2} = i\sqrt{2}$

**ODE:** $ay'' + by' + cy$, $b^2 - 4ac < 0$, $a \neq 0$

**Characteristic Equation:** $ar^2 + br + c = 0$

$4ac - b^2 > 0$

**Roots:** $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{-b \pm \sqrt{-(4ac - b^2)}}{2a} = \dfrac{-b \pm \sqrt{-1}\cdot\sqrt{4ac - b^2}}{2a}$

$$r = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}$$

For convenience, write

$\boxed{r = \alpha \pm i\beta}$ where $\boxed{\alpha = \dfrac{-b}{2a}} = Re(r)$ $\quad \beta \neq 0$

$\boxed{\beta = \dfrac{\sqrt{4ac - b^2}}{2a}} = \pm Im(r)$

Note: $\alpha$ and $\beta$ are real numbers.

---

The two distinct roots yield two linearly independent solutions,

Check the Wronskian

$V_1(x) = e^{(\alpha + i\beta)x}$ and $V_2(x) = e^{(\alpha - i\beta)x}$

thus giving the general solution $y(x) = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$.

Note: Given real ICs, $C_1$ and $C_2$ will be complex numbers with values such that $y(x)$ turns out to be real.

---

The general solution given above is not a convenient choice.
It is difficult to extract information from the complex expression.
e.g. $V_1$ and $V_2$ cannot be plotted.

---

It is desirable to express the general solution as a linear combination of two real-valued solutions.

e.g. $y(x) = C_1 y_1(x) + C_2 y_2(x)$, where
$y_1$ and $y_2$ are real-valued functions.

# Euler's Formula

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Derivation: Taylor series approach: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ Radius of Convergence $= (-\infty,\infty)$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!} = \frac{(i\theta)^0}{0!} + \frac{(i\theta)^1}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots$$

$i = \sqrt{-1}$

$i^2 = -1$

$i^3 = i^2 \cdot i = -i$

$i^4 = i^3 \cdot i = -i \cdot i = -i^2 = -(-1) = 1$

$i^5 = i^4 \cdot i = 1 \cdot i = i$

$$= 1 + i\theta - \theta^2/2! - i\,\theta^3/3! + \theta^4/4! + i\,\theta^5/5! + \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}$$

$$= \cos\theta + i\sin\theta \checkmark$$

---

Then, the two solutions, $V_1$ and $V_2$, can be written as

$$V_{1,2} = e^{(\alpha \pm i\beta)x} = e^{\alpha x} e^{\pm i\beta x} = e^{\alpha x}\left[\cos(\pm\beta x) + i\sin(\pm\beta x)\right]$$

Euler's formula with $\theta = \pm\beta x$

Recall: cosine is an even function $\Rightarrow$ $\cos(-x) = \cos x$

sine is an odd function $\Rightarrow$ $\sin(-x) = -\sin x$

$$\Rightarrow V_{1,2} = e^{\alpha x}\left[\cos(\beta x) \pm i\sin(\beta x)\right]$$

So,

$$V_1(x) = e^{\alpha x}\left[\cos(\beta x) + i\sin(\beta x)\right]$$

$$V_2(x) = e^{\alpha x}\left[\cos(\beta x) - i\sin(\beta x)\right]$$

These solutions are still complex-valued.

- The Principal of Superposition can be used to find two real-valued solutions.

  In the context of the current task, the principal of superposition states that if $V_1$ and $V_2$ are solutions of $ay''+by'+cy=0$, then the linear combination $C_1V_1+C_2V_2$ is also a solution for all $C_1$ and $C_2$.

  may be complex-valued

  Idea: By choosing $C_1$ and $C_2$ appropriately, the two complex solutions, $V_1$ and $V_2$, can be combined to yield two real-valued solutions.

  $$\text{Let } Y_1 = \frac{V_1+V_2}{2} = e^{\alpha x}\cos(\beta x)$$

  $$\text{and } Y_2 = \frac{V_1-V_2}{2i} = e^{\alpha x}\sin(\beta x)$$

- We now have two real-valued solutions.

  $$\boxed{\begin{aligned} Y_1(x) &= e^{\alpha x}\cos(\beta x) \\ Y_2(x) &= e^{\alpha x}\sin(\beta x) \end{aligned}}$$

  It remains to be shown that $Y_1 \perp\!\!\!\perp Y_2$.

  If $Y_1 \perp\!\!\!\perp Y_2$, we have a general solution expressed as a linear combination of two real-valued functions.

$y_1(x) = e^{\alpha x}\cos(\beta x)$ $\quad$ $y_2 = e^{\alpha x}\sin(\beta x)$, $\beta \neq 0$

- Check for linear independence.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{\alpha x}\cos(\beta x) & e^{\alpha x}\sin(\beta x) \\ e^{\alpha x}[\alpha\cos(\beta x) - \beta\sin(\beta x)] & e^{\alpha x}[\alpha\sin(\beta x) + \beta\cos(\beta x)] \end{vmatrix}$$

$$= e^{2\alpha x}\cos(\beta x)[\cancel{\alpha\sin(\beta x)} + \beta\cos(\beta x)] - e^{2\alpha x}\sin(\beta x)[\cancel{\alpha\cos(\beta x)} - \beta\sin(\beta x)]$$

$$= e^{2\alpha x}\cdot\beta\,[\cos^2(\beta x) + \sin^2(\beta x)] = \underbrace{\beta}_{\neq 0}\,\underbrace{e^{2\alpha x}}_{\neq 0} \neq 0$$

$W \neq 0 \Rightarrow y_1 \perp\!\!\!\perp y_2$

---

- Then, the general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$\boxed{y(x) = e^{\alpha x}[c_1\cos(\beta x) + c_2\sin(\beta x)]}$$ (Complex Conjugate Roots)

Recall: $ar^2 + br + c = 0$

$\boxed{r_{1,2} = \alpha \pm i\beta}$ where $\alpha = \frac{-b}{2a}$

and $\beta = \frac{\sqrt{4ac - b^2}}{2a}$

-

**Example:** Solve the IVP $y''+2y'+5y=0$; $y(0)=0$, $y'(0)=1$

Characteristic Equation: $r^2+2r+5=0$

Roots: $r=\frac{-2\pm\sqrt{4-20}}{2}=\frac{-2\pm\sqrt{-16}}{2}=\frac{-2\pm 4i}{2}$

$r=-1\pm 2i \Rightarrow \alpha=-1,\ \beta=2$

$\alpha\pm i\beta$

General Solution: $\boxed{y(x)=e^{-x}\left[C_1\cos(2x)+C_2\sin(2x)\right]}$

$y(0)=\cancel{e^0}\left[C_1\cos 0+\cancel{C_2\sin 0}\right]=C_1=0 \Rightarrow C_1=0$

$\Rightarrow y(x)=C_2e^{-x}\sin(2x)$

$y'(x)=C_2e^{-x}\left[2\cos(2x)-\sin(2x)\right]$

$y'(0)=C_2\cancel{e^0}\left[2\cos 0-\cancel{\sin 0}\right]=2C_2=1$

$C_2=\frac{1}{2}$

$\Rightarrow \boxed{y(x)=\frac{1}{2}e^{-x}\sin(2x)}$

---

**Example:** Find the general solution of $x''+\omega^2x=0$, $x=x(t)$.

Characteristic Equation: $r^2+\omega^2=0$

$r^2=-\omega^2$

$r=\pm\sqrt{-\omega^2}=\pm i\omega$

$r=\pm i\omega$

$\alpha=0,\ \beta=\omega$

$\Rightarrow x(t)=e^0\left[C_1\cos(\omega t)+C_2\sin(\omega t)\right]$

$\boxed{x(t)=C_1\cos(\omega t)+C_2\sin(\omega t)}$ (Oscillatory Solution) (periodic)

# Summary

ODE: $\boxed{ay'' + by' + cy = 0}$

- 2nd order
- linear
- homogeneous
- constant coefficients

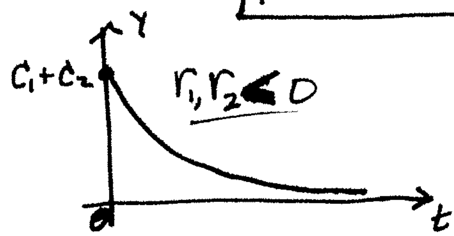
Try $y = e^{rt}$ as a solution.

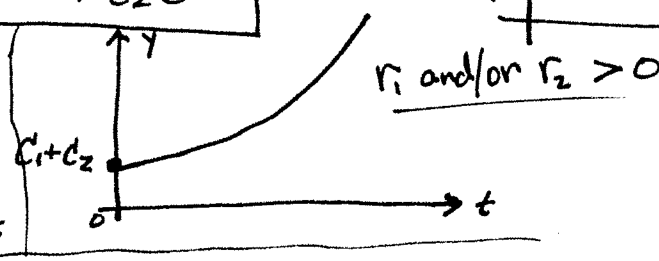
$\Rightarrow$ CE: $ar^2 + br + c = 0$

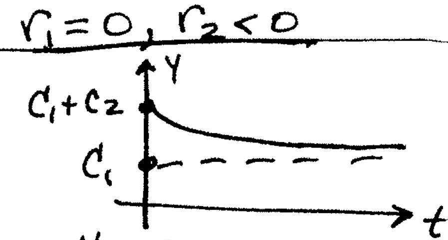
Roots: $r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

## 1. Real Distinct Roots ($r_1 \neq r_2$)

$$\boxed{y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}}$$

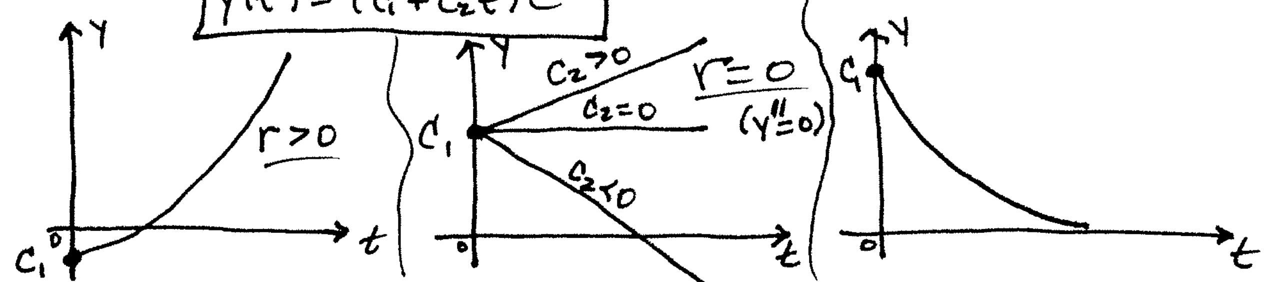



## 2. Real Repeated Roots ($r = r_1 = r_2$)

$$\boxed{y(t) = (C_1 + C_2 t)e^{rt}}$$

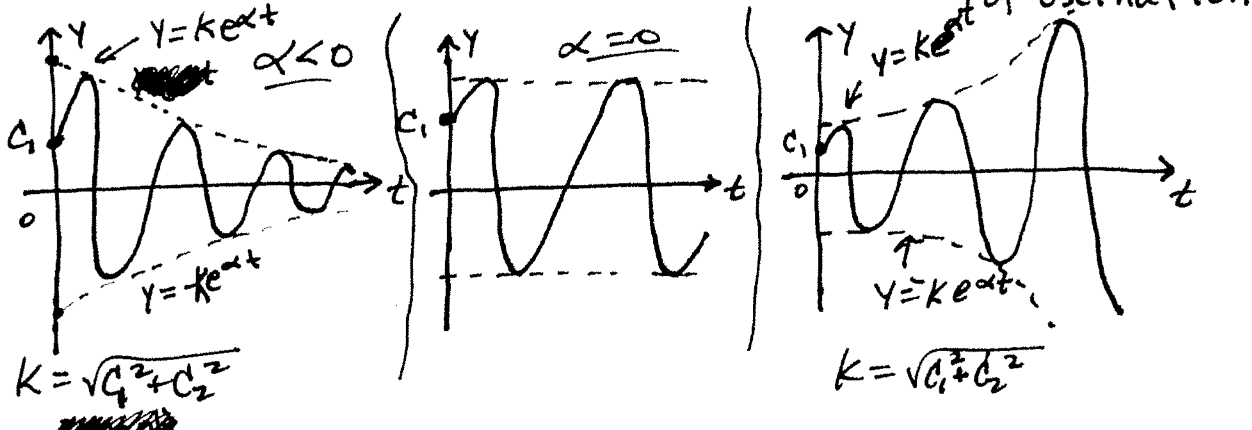



## 3. Complex Conjugate Roots ($r_{1,2} = \alpha \pm i\beta$)

$$\boxed{y(t) = e^{\alpha t}\left[C_1 \cos(\beta t) + C_2 \sin(\beta t)\right]}$$

$\alpha$ = decay/growth rate of amplitude

$\beta$ = angular frequency of oscillation



## $n^{th}$ Order Linear Homogeneous ODEs

$$\boxed{y^{(n)}+a_{n-1}(t)y^{(n-1)}+a_{n-2}(t)y^{(n-2)}+\cdots+a_2(t)y''+a_1(t)y'+a_0(t)y=0}$$

General Solution: $y(t)=C_1y_1(t)+C_2y_2(t)+\cdots+C_ny_n(t)$

where $y_1,y_2,\ldots,y_n$ are linearly independent solutions.

$$W=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$W\neq 0 \Rightarrow y_1 \perp\!\!\!\perp y_2 \perp\!\!\!\perp \cdots \perp\!\!\!\perp y_n$$

---

### Constant Coefficients

**Example:** $y^{VIII}-3y^{VII}+y^{VI}+7y^{V}-10y^{IV}-4y'''+12y''-4y'-8y=0$

CE: $r^8-3r^7+r^6+7r^5-10r^4-4r^3+12r^2-4r-8=0$

factor $\Rightarrow (r-2)(r+1)^3(r^2-2r+2)^2=0$

$r=2$ $\quad r=-1,-1,-1 \quad r=1\pm i, 1\pm i$

$y_1=e^{2t}$

$y_2=e^{-t}$, $y_3=te^{-t}$, $y_4=t^2e^{-t}$

$y_5=e^{t}\cos t$, $y_6=e^{t}\sin t$, $y_7=te^{t}\cos t$, $y_8=te^{t}\sin t$

$$\Rightarrow \boxed{y(t)=C_1e^{2t}+(C_2+C_3t+C_4t^2)e^{-t}+e^{t}\left[(C_5+C_7t)\cos t+(C_6+C_8t)\sin t\right]}$$

## Section 4.4: Nonhomogeneous Liner $2^{nd}$ Order ODEs and the Method of Undetermined Coefficients (MUC)

General Form: $\boxed{y'' + p(t)y' + q(t)y = g(t)}$ (N) $g(t) \not\equiv 0$

Goal: Determine the general solution of (N).

### Terminology / Notation

Corresponding to equation (N), there is an associated homogeneous ($g \equiv 0$) ODE given by

$$\boxed{y'' + p(t)y' + q(t)y = 0} \quad \text{(H)}$$

1. Let $y_h(t)$ be the general solution of the associated homogeneous ODE (H).

$$\Rightarrow \boxed{y_h'' + p y_h' + q y_h = 0}$$

and

$$\boxed{y_h(t) = C_1 y_{h1}(t) + C_2 y_{h2}(t)},$$

where $y_{h1}$ and $y_{h2}$ are linearly independent solutions of (H)

2. Let $y_p(t)$ be any solution of (N)

$$\Rightarrow \boxed{y_p'' + p y_p' + q y_p = g}$$

$y_p$ is called a particular solution.

e.g. $y'' + y' + y = 3e^t$

$y_p = e^t$ is a particular solution.

Fact: The general solution (N) is given by

$$y(t) = y_h(t) + y_p(t)$$
$$= C_1 y_{h1}(t) + C_2 y_{h2}(t) + y_p(t)$$

Clearly, $y = y_h + y_p$ is a solution of (N).

$$y'' + py' + qy = (y_h'' + y_p'') + p(y_h' + y_p') + q(y_h + y_p)$$
$$= \underbrace{(y_h'' + p y_h' + q y_h)}_{\substack{=0 \\ \text{since } y_h \text{ is a solution of (H)}}} + \underbrace{(y_p'' + p y_p' + q y_p)}_{\substack{= g \text{ since } y_p \text{ is a} \\ \text{solution of (N)}}} = g \checkmark$$

It can also be shown that all solutions have this form, and $C_1$ and $C_2$ can be uniquely determined so that $y = y_h + y_p$ can satisfy any set of initial conditions ($y(t_0) = y_0$, $y(t_1) = y_1$).

---

To find the general solution of (N),

1. Find the general solution $y_h$ of (H).
   $$y_h = C_1 y_{h1} + C_2 y_{h2} \quad \text{(as in section 4.3)}$$
2. Find a particular solution $y_p$ of (N) (sections 4.4 and 4.6)
3. Write down the general solution $y$ of (N)
   $$y(t) = y_h(t) + y_p(t).$$

(4. Find $C_1$ and $C_2$ if initial conditions are given.)

## Solution Strategy

**Example:** Solve the IVP $y'' - y = e^{2x}$ ; $y(0) = \frac{1}{3}$, $y'(0) = 1$

1. Find $y_h$ : $y_h'' - y_h = 0$ (associated homogeneous equation)

$$r^2 - 1 = 0$$

$$r = \pm 1 \Rightarrow \boxed{y_h(x) = c_1 e^x + c_2 e^{-x}}$$ ← DO NOT apply the ICs here

2. Find $y_p$ : (we'll discuss how to find $y_p$, but for now we'll guess)

Try $y_p = Ae^{2x}$ (plug into the ODE and determine A)

$$y_p' = 2Ae^{2x}$$
$$y_p'' = 4Ae^{2x}$$
$$\Rightarrow y_p'' - y_p = 4Ae^{2x} - Ae^{2x} = e^{2x}$$
$$3A\cancel{e^{2x}} = \cancel{e^{2x}}\ 1$$
$$A = \frac{1}{3}$$

$$\Rightarrow \boxed{y_p(x) = \frac{1}{3}e^{2x}}$$

3. General Solution: $y = y_h + y_p$

$$\boxed{y(x) = c_1 e^x + c_2 e^{-x} + \frac{1}{3}e^{2x}}$$ (general solution)

4. Find $c_1$ and $c_2$ :

$$y(0) = c_1 + c_2 + \frac{1}{3} = \frac{1}{3}$$
$$c_1 + c_2 = 0$$

$$y'(x) = c_1 e^x - c_2 e^{-x} + \frac{2}{3}e^{2x}$$

$$y'(0) = c_1 - c_2 + 2/3 = 1$$
$$c_1 - c_2 = 1/3$$

$$c_1 + c_2 = 0$$
$$+\ (c_1 - c_2 = 1/3)$$
$$2c_1 = \frac{1}{3}$$
$$c_1 = \frac{1}{6}$$

$$c_2 = -c_1$$
$$c_2 = -1/6$$

$$\Rightarrow y(x) = \frac{1}{6}e^x - \frac{1}{6}e^{-x} + \frac{1}{3}e^{2x}$$

$$\boxed{y(x) = \frac{1}{6}(e^x - e^{-x} + 2e^{2x})}$$ (solution of the IVP)

# Two Methods for Finding $y_p$

1. **Method of Undetermined Coefficients** (MUC) (section 4.4)

   MUC may be used when the ODE has constant coefficients,
   $$ay'' + by' + cy = g(t),$$
   and $g(t)$ is of a certain form.

2. **Variation of Parameters** (VP) (section 4.6)

   VP may be used for the general case with variable coefficients,
   $$y'' + p(t)y' + q(t) = g(t),$$
   where $g(t)$ is any function.

---

## Method of Undetermined Coefficients (MUC)

The MUC is a method for finding a particular solution $y_p$ in the case of constant coefficients, $\boxed{ay'' + by' + cy = g(t)}$. NC

The MUC may be used only if $g(t)$ is of a special form. That is, only if $g(t)$ consists of sums and products of

- exponential functions
- sines and cosines
- polynomials

The MUC works for these forms of $g(t)$ since the types of terms appearing in the derivatives of these functions can be listed.

e.g. i) $g(t) = 2e^{5t} + t^2 + 1$

ii) $g(t) = t\cos(2t) + e^{-t}\sin(5t)$

iii) $g(t) = t^2 + (t^5 + 3t^4 + 2)e^{t/2}\sin(\pi t) - e^{2t}$

---

Despite the requirement of constant coefficients and the limitations on $g(t)$, the MUC is useful in many practical applications. We'll consider some such applications in section 5.1.

## Procedure for finding $y_p$

1. Determine the appropriate form of $y_p$ (to be discussed), and write it with coefficients which are to be determined.

   e.g. i) $y_p = Ae^{2x}$ (as in the above example)

   ii) $y_p = (At+B)\cos t + (Ct+D)\sin t$

   iii) $y_p = (At^2+Bt+C)e^{5t}$.

   We'll discuss how to determine the form of $y_p$.

2. Plug $y_p$ into equation NC to arrive at a set of algebraic equations which determine the coefficients $A, B, C, \ldots$

3. Solve for $A, B, C, \ldots$, and plug back into the expression for $y_p$.

---

Idea: **Let $y_p$ be the linear combination of all terms which appear in the function $g$ and its derivatives.**

(Coefficients are irrelevant and may be ignored)

### Three Fundamental Cases

1) $g$ is an exponential function.

e.g. $g(t) = 2e^{5t}$

$\Rightarrow \boxed{y_p = Ae^{5t}}$

$g = \cancel{2}e^{5t}$
$g' = \cancel{10}e^{5t}$
$g'' = \cancel{50}e^{5t}$
$\vdots$

Terms: $e^{5t}$

2) $g$ is a sine or a cosine

e.g. $g(t) = 3\cos(4t)$

$\Rightarrow \boxed{y_p = A\cos(4t) + B\sin(4t)}$

$g = \cancel{3}\cos(4t)$
$g' = \cancel{-12}\sin(4t)$
$g'' = \cancel{-48}\cos(4t)$
$g''' = \cancel{192}\sin(4t)$
$\vdots$

Terms: $\cos(4t), \sin(4t)$

3) $g$ is a polynomial

e.g. $g(t) = 2t^3 + 5t$

$\Rightarrow \boxed{y_p = At^3 + Bt^2 + Ct + D}$

$g = 2t^3 + 5t$
$g' = 6t^2 + 5$
$g'' = 12t$
$g''' = 12$
$g^{iv} = 0$

Terms: $t^3, t^2, t, 1$

**Examples:** 1) $g(t) = \cos(3t) + 2\sin(2t)$

**Terms**

$\cos(3t)$ $\sin(2t)$
$\sin(3t)$ $\cos(2t)$

$g(t) = \boxed{\cos(3t)} + 2\boxed{\sin(2t)}$

$g'(t) = \sim\boxed{\sin(3t)} + \sim\boxed{\cos(2t)}$

some coefficient

$g''(t) = \sim\cancel{\cos}(3t) + \sim\cancel{\sin}(2t)$ (no new terms)

$\vdots$

$$\Rightarrow \boxed{Y_p(t) = A\cos(3t) + B\sin(3t) + C\cos(2t) + D\sin(2t)}$$

---

2) $g(t) = (t^2+1)e^t$

**Terms**

$t^2e^t$ $te^t$
$e^t$

$g(t) = (t^2+1)e^t$

$g'(t) = (t^2+1)e^t + 2te^t$

$g''(t) = (t^2+1)e^t + 2te^t + 2te^t + 2e^t$

$\vdots$ (no new terms)

$\Rightarrow Y_p(t) = At^2e^t + Bte^t + Ce^t$

$$\boxed{Y_p(t) = (At^2+Bt+C)e^t}$$

OR

$g$: $(t^2+1)e^t$ $\quad t^2e^t, e^t$

$g'$: $\cancel{(t^2+1)e^t}$ $\quad \cancel{2}te^t$ $\quad te^t$

$g''$: $t\cancel{e^t}$ $\quad \cancel{e^t}$

Terminate each branch when no new terms appear.

---

3) $g(t) = te^t\cos t$ $\qquad$ Recall: $(fgh)' = f'gh + fg'h + fgh'$

$g$: $te^t\cos t$

$g'$: $e^t\cos t$ $\quad te^t\cancel{\cos}t$ $\quad te^t\sin t$

$g''$: $e^t\cancel{\cos}t$ $\quad e^t\sin t$ $\quad \cancel{e^t\sin}t$ $\quad te^t\cancel{\sin}t$ $\quad te^t\cancel{\cos}t$

$e^t\cancel{\sin}t$ $\quad e^t\cancel{\cos}t$

$\Rightarrow Y_p(t) = Ate^t\cos t + Be^t\cos t + Cte^t\sin t + De^t\sin t$

$$\boxed{Y_p(t) = e^t\left[(At+B)\cos t + (Ct+D)\sin t\right]}$$

Compare to the three fundamental cases.

ii) There are two terms; one involving cosine and the other involving sine.

iii) The polynomial involves all powers of $t$ less than or equal to those appearing in $g$.

i) The exponential term appears just as it does in $g$.

**Example:** Find the general solution of $y'' + 2y' + y = 1 + \cos t$

homogeneous: $y_h'' + 2y_h' + y_h = 0$

$$r^2 + 2r + 1 = 0$$
$$(r+1)^2 = 0$$
$$r = -1, -1 \Rightarrow \boxed{y_h(t) = (C_1 + C_2 t)e^{-t}}$$

particular: $y_p = A + B\cos t + C\sin t \Leftarrow g(t) = 1 + \cos t$

$$y_p' = -B\sin t + C\cos t \qquad g'(t) = -\sin t$$
$$y_p'' = -B\cos t - C\sin t \qquad g''(t) = -\cancel{\cos t}$$

$$y_p'' + 2y_p' + y_p = (-\cancel{B\cos t} - \cancel{C\sin t}) + 2(-B\sin t + C\cos t) + (A + \cancel{B\cos t} + \cancel{C\sin t}) = 1 + \cos t$$

$$-2B\sin t + 2C\cos t + A = 1 + \cos t$$
$$(-2B\sin t + (2C-1)\cos t + (A-1) = 0)$$

Equate Coefficients:

$\sin t$: $-2B = 0 \Rightarrow B = 0$

$\cos t$: $2C = 1 \Rightarrow C = \frac{1}{2}$

$1$: $A = 1 \Rightarrow A = 1$

$$\Rightarrow \boxed{y_p(t) = 1 + \tfrac{1}{2}\sin t}$$

general: $y(t) = y_h(t) + y_p(t)$

$$\boxed{y(t) = (C_1 + C_2 t)e^{-t} + 1 + \tfrac{1}{2}\sin t}$$

Example: Find the general solution of $y''+y=5te^{2t}$.

homogeneous: $y_h''+y_h=0$ $\quad r=\pm i \Rightarrow \begin{matrix}\alpha=0\\ \beta=1\end{matrix}$

$r^2+1=0$

$r=\pm i$

$$\boxed{y_h(t)=C_1\cos t+C_2\sin t}$$

particular: $y_p=(At+B)e^{2t}$

$te^{2t}$

$\cancel{t}e^{2t}$ $\quad e^{2t}$

$\cancel{e}^{2t}$

$y_p'=(2At+2B+A)e^{2t}$

$y_p''=4(At+B+A)e^{2t}$

$\Rightarrow y_p=Ate^{2t}+Be^{2t}$

$$y_p''+y_p=4(At+B+A)\cancel{e^{2t}}+(At+B)\cancel{e^{2t}}=5t\cancel{e^{2t}}$$

$$5At+5B+4A=5t$$

Equate Coefficients: $t^1$: $5A=5 \Rightarrow A=1$

$t^0$: $5B+4A=0 \Rightarrow B=-\frac{4}{5}A=-\frac{4}{5}$

$$\boxed{y_p(t)=\left(t-\tfrac{4}{5}\right)e^{2t}}$$

general: $y(t)=y_h(t)+y_p(t)$

$$\boxed{y(t)=C_1\cos t+C_2\sin t+\left(t-4/5\right)e^{2t}}$$

There is a special case for which the above procedure fails, but it can be made to work with an appropriate modification.

The following example illustrates the difficulty.

**Example:** $y'' - y' - 6y = e^{\alpha t}$, $\alpha$ = constant

**homogeneous:** $y_h'' - y_h' - 6y_h = 0$

$$r^2 - r - 6 = 0$$

$$(r-3)(r+2) = 0$$

$$r=3 \quad r=-2 \quad \Rightarrow \quad \boxed{y_h(t) = C_1 e^{3t} + C_2 e^{-2t}}$$

**particular:**

$$y_p'' - y_p' - 6y_p = e^{\alpha t}$$

$$y_p = Ae^{\alpha t}$$

$$y_p' = A\alpha e^{\alpha t}$$

$$y_p'' = A\alpha^2 e^{\alpha t}$$

$$A\alpha^2 e^{\alpha t} - A\alpha e^{\alpha t} - 6Ae^{\alpha t} = e^{\alpha t}$$

$$A\cancel{e^{\alpha t}}(\alpha^2 - \alpha - 6) = \cancel{e^{\alpha t}},$$

$$A = \frac{1}{\alpha^2 - \alpha - 6} = \frac{1}{(\alpha-3)(\alpha+2)}$$

Same

$$\Rightarrow \boxed{y_p(t) = \frac{e^{\alpha t}}{(\alpha-3)(\alpha+2)}}$$

Observe that $y_p(t)$ has the assumed form $(Ae^{\alpha t})$ only if $\alpha \neq 3, -2$.

**Problem:** When $\alpha = 3, -2$, our guess $(y_p = Ae^{\alpha t})$ is a solution of the associated homogeneous equation. Plugging this guess into the nonhomogeneous equation yields $0 = g(t)$, and thus $y_p$ is not of the assumed form.

The guess for $y_p$ can be modified so that no term of $y_p$ is a solution of the homogeneous equation.

Loosely speaking, we can simply multiply $y_p$ by $t$.

Now suppose that $\alpha = 3 \Rightarrow \boxed{y'' - y' - 6y = e^{3t}}$

The guess $y_p = Ae^{3t}$ won't work since it is a solution of the homogeneous equation.

homogeneous: $y_h(t) = c_1 e^{3t} + c_2 e^{-2t}$

particular: Rather than $y_p = Ae^{3t}$, we can try

$$y_p = Ate^{3t}$$

← This choice is motivated by the case of real repeated roots ($e^{rt}$, $te^{rt}$) and the method of reduction of order. The choice is reasonable since $y_p'$ and $y_p''$ will also involve $e^{3t}$, which will then cancel when $y_p$ is plugged into the nonhomogenous ODE.

$$y_p' = A(3t+1)e^{3t}$$

$$y_p'' = A(9t+6)e^{3t}$$

$$y_p'' - y_p' - 6y_p = e^{3t}$$

$$A(9t+6)e^{3t} - A(3t+1)e^{3t} - 6Ate^{3t} = e^{3t}$$

$$At\underbrace{(\cancel{9e^{3t} - 3e^{3t} - 6e^{3t}})}_{= y_{h1}'' - y_{h1}' - 6y_{h1} = 0} + A\cancel{e^{3t}}(6-1) = \cancel{e^{3t}}$$

$y_{h1} = e^{3t}$

$$5A = 1$$

$$A = \frac{1}{5}$$

$$\Rightarrow \boxed{y_p(t) = \frac{1}{5}te^{3t}}$$

$y = y_h + y_p$

General Solution: $\boxed{y(t) = c_1 e^{3t} + c_2 e^{-2t} + \frac{1}{5}te^{3t}}$

The 'initial' guess $y_p = Ae^{3t}$ was modified by multiply it by $t$. This idea will work in general, but with a few issues which need to be addressed.

Note: In the above example, the $te^{3t}$ terms cancel. That will always be the case. i.e. If it is necessary to multiply by $t$, the terms with the factor $t$ will always cancel.

**Example:** $y'' + 4y = \cos(2t)$

homogeneous: $r^2 + 4 = 0$
$r = \pm 2i$ $\Rightarrow$ $Y_h(t) = C_1\cos(2t) + C_2\sin(2t)$

particular: Initial Guess: $Y_p = A\cancel{\cos(2t)} + B\sin(2t)$

According to g(t) alone

These are homogeneous solutions $\Rightarrow$ multiply by t

Modified Guess: $\boxed{Y_p = t\left[A\cos(2t) + B\sin(2t)\right]}$

---

**Example:** $y'' - y = te^t + t^2$

homogeneous: $r^2 - 1 = 0$
$r = \pm 1$ $\Rightarrow$ $Y_h(t) = C_1 e^t + C_2 e^{-t}$

particular: Initial Guess: $Y_p = (At+B)\cancel{e^t} + (Ct^2 + Dt + E)$

homogeneous solution

Only multiply the terms coming from $te^t$ by t.

modified Guess: $\boxed{Y_p = t(At+B)e^t + (Ct^2 + Dt + E)}$

---

**Example:** $y'' - 2y' + y = te^t$

homogeneous: $r^2 - 2r + 1 = 0$
$(r-1)^2 = 0$
$r = 1, 1$ $\Rightarrow$ $Y_h(t) = (C_1 + C_2 t)e^t$

particular: Initial Guess: $Y_p = (At + B)e^t$

homogeneous solutions

Modify (multiply by t): $Y_p = t(At + B)e^t$

homogeneous solution

Modified Guess (multiply by t again): $\boxed{Y_p = t^2(At + B)e^t}$

# Modified Procedure

- **Initially**: Guess $y_p$ according to the form of $g(t)$ as before.

  **Modify (if necessary)**: Compare the initial guess for $y_p$ to the solutions of the associated homogeneous ODE.

  If the initial guess for $y_p$ involves terms which are solutions of the associated homogeneous ODE, then multiply by $t$ (repeatedly, if necessary) until no term of $y_p$ is a solution of the associated homogeneous ODE.

---

**Example**: $y'' - 2y' = t^2 + e^t$

homogeneous: $r^2 - 2r = 0$

$r(r-2) = 0$

$r=0 \quad r=2$ $\Rightarrow$ $y_h(t) = C_1 + C_2 e^{2t}$

particular: Initial Guess: $y_p = At^2 + Bt + C + De^t$

$\uparrow$ homogeneous solution

- Modified Guess: $y_p = t(At^2 + Bt + C) + De^t$

$y_p = At^3 + Bt^2 + Ct + De^t$

$y_p' = 3At^2 + 2Bt + C + De^t$

$y_p'' = 6At + 2B + De^t$

$$y_p'' - 2y_p' = t^2 + e^t$$

$$(6At + 2B + De^t) - 2(3At^2 + 2Bt + C + De^t) = t^2 + e^t$$

$$-6At^2 + (6A - 4B)t + (2B - 2C) + (D - 2D)e^t = t^2 + e^t$$

Equate Coefficients:

$t^2$: $-6A = 1 \Rightarrow A = -1/6$

$t^1$: $6A - 4B = 0 \Rightarrow B = \frac{3}{2}A = -1/4$

$t^0$: $2B - 2C = 0 \Rightarrow C = B = -1/4$

$e^t$: $-D = 1 \Rightarrow D = -1$

$$\Rightarrow y_p(t) = -\frac{1}{6}t^3 - \frac{1}{4}t^2 - \frac{1}{4}t - e^t$$

$$y_p(t) = -\frac{1}{12}(2t^3 + 3t^2 + 3t) - e^t$$

- General Solution: $\boxed{y(t) = C_1 + C_2 e^{2t} - \frac{t}{12}(2t^2 + 3t + 3) - e^t}$

**Example:** $y''' - 2y'' = t + e^t$

homogeneous: $r^3 - 2r^2 = 0$

$$r^2(r-2) = 0 \Rightarrow \boxed{y_h(t) = C_1 + C_2 t + C_3 e^{2t}}$$

$$r = 0, 0 \quad r = 2$$

particular: $y_p = (\cancel{At + B}) + Ce^t$

homogeneous solutions $\Rightarrow$ multiply by $t$

$y_p = \cancel{t(At+B) + Ce^t}$

homogeneous solution $\Rightarrow$ multiply by $t$

$$y_p = \underbrace{t^2(At+B) + Ce^t}$$

$$y_p = At^3 + Bt^2 + Ce^t$$

$$y_p' = 3At^2 + 2Bt + Ce^t$$

$$y_p'' = 6At + 2B + Ce^t$$

$$y_p''' = 6A + Ce^t \qquad y_p''' - 2y_p'' = t + e^t$$

$$(6A + Ce^t) - 2(6At + 2B + Ce^t) = t + e^t$$

$$\boxed{(6A - 4B) - Ce^t - 12At = t + e^t}$$

Equate Coefficients: $t^1$: $-12A = 1 \Rightarrow A = -\frac{1}{12}$

$t^0$: $6A - 4B = 0 \Rightarrow B = \frac{3}{2}A = \frac{3}{2}\left(-\frac{1}{12}\right) = -\frac{1}{8}$

$e^t$: $-C = 1 \Rightarrow C = -1$ $\qquad B = -\frac{1}{8}$

$$\Rightarrow y_p = t^2\left(-\frac{1}{12}t - \frac{1}{8}\right) - e^t = -\frac{t^2}{24}(2t+3) - e^t$$

$$y_p(t) = -\frac{t^2}{24}(2t+3) - e^t$$

general: $y = y_h + y_p$

$$\boxed{y(t) = C_1 + C_2 t + C_3 e^{2t} - \frac{t^2}{24}(2t+3) - e^t}$$

Example: $y'' + y = \cos(\omega t),\ \omega > 0$

$g(t) = \cos(\omega t)$ is a homogeneous solution when $\omega = 1$

homogeneous: $y_h'' + y_h = 0$

$r^2 + 1 = 0$

$r = \pm i \Rightarrow y_h = A\cos(t) + B\sin(t)$

These are homogeneous solutions when $\omega = 1$.

$\omega \neq 1$: $y_p = A\cos(\omega t) + B\sin(\omega t)$

$y_p'' = -\omega^2[A\cos(\omega t) + B\sin(\omega t)]$

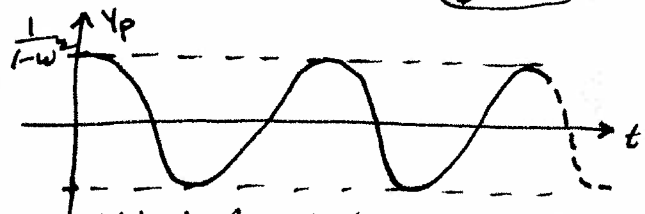
$$y_p'' + y_p = (1-\omega^2)[A\cos(\omega t) + B\sin(\omega t)] = \cos(\omega t)$$

Equate Coefficients: $\cos(\omega t)$: $(1-\omega^2)A = 1 \Rightarrow A = \frac{1}{1-\omega^2}$

$\sin(\omega t)$: $(1-\omega^2)B = 0 \Rightarrow B = 0$

$$\Rightarrow \boxed{y_p(t) = \frac{\cos(\omega t)}{1-\omega^2}}$$

$\omega \neq 1$




Note: Amplitude $\to \infty$ as $\omega \to 1$

$\omega = 1$: $y'' + y = \cos t$

$y_p = t[A\cos t + B\sin t]$

$y_p'' = -t[A\cos t + B\sin t] + 2[-A\sin t + B\cos t]$

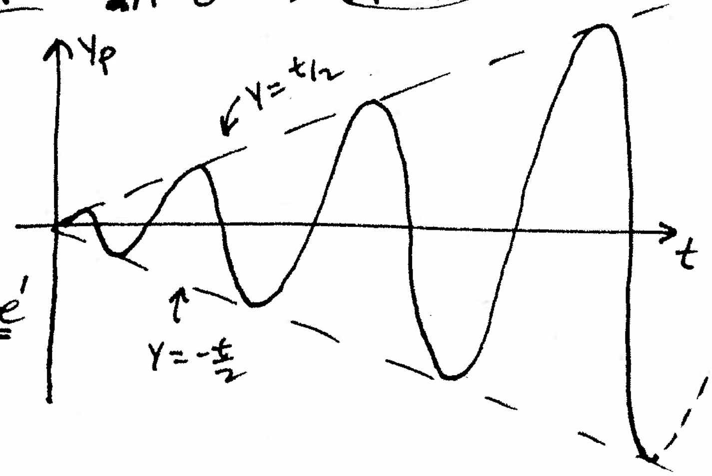
$$y_p'' + y_p = (t-t)\cancel{[A\cos t + B\sin t]} + 2[-A\sin t + B\cos t] = \cos t$$

Equate Coefficients: $\cos t$: $2B = 1 \Rightarrow B = \frac{1}{2}$

$\sin t$: $-2A = 0 \Rightarrow A = 0$

$$\Rightarrow \boxed{y_p(t) = \frac{t}{2}\sin t}$$

This is the idea behind the concept of 'resonance' (section 5.1)



## Section 4.6 : Variation of Parameters

Consider $\boxed{y''+p(t)y'+q(t)y=g(t)}$ (N) General form of a nonhomogeneous linear 2nd order ODE
$\Rightarrow$ variable coefficients

The general solution is $y=y_h+y_p = C_1y_1+C_2y_2+y_p$.

Variation of Parameters is a method for finding $y_p$. The method may be applied to the general equation (N) to arrive at a general formula for $y_p$.

---

### Derivation of a formula for $y_p$

The formula is expressed in terms of $y_1$, $y_2$, and $g$.

Assume that the general solution of the associated homogeneous ODE is known.

$\Rightarrow$ $y_h(t) = C_1y_1(t) + C_2y_2(t)$, where $y_1(t)$ and $y_2(t)$ are known.

Idea: Let $y_p(t)$ be of the same form as $y_h(t)$, but with variable coefficients.

$\Rightarrow$ Let $\boxed{y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)}$

Then $y_p$ into equation (N) and find formulas for $u_1(t)$ and $u_2(t)$.

---

Note: We began with one unknown function $y_p$, which is determined by equation (N). However, after making the substitution $y_p=u_1y_1+u_2y_2$, we are left with two unknown functions, $u_1$ and $u_2$, but only one equation to determine them.

Since we have only one equation (N) for two unknowns, $u_1$ and $u_2$, there will be infinitely many solutions which yield an acceptable particular solution. Consequently we may impose and additional constraint on $u_1$ and $u_2$ at our convenience to select just one of the infinitely many possibilities.

Plug $Y_p = u_1 Y_1 + u_2 Y_2$ into equation (N) and find $u_1$ and $u_2$.

$$Y_p = u_1 Y_1 + u_2 Y_2$$

$$Y_p' = u_1 Y_1' + \underline{u_1' Y_1} + u_2 Y_2' + \underline{u_2' Y_2}$$

This is a convenient point to impose a second constraint on $u_1$ and $u_2$.

If we impose the constraint

$$\boxed{u_1' Y_1 + u_2' Y_2 = 0}, \quad (2) \qquad \text{(Equation (N) yields (1))}$$

we can avoid 2nd derivatives of $u_1$ and $u_2$ when writing $Y_p''$.

Then, $\boxed{Y_p = u_1 Y_1 + u_2 Y_2}$

$$\boxed{Y_p' = u_1 Y_1' + u_2 Y_2'}$$

$$\boxed{Y_p'' = u_1 Y_1'' + u_1' Y_1' + u_2 Y_2'' + u_2' Y_2'}$$

Plug into (N): $\quad Y_p'' + p(t) Y_p' + q(t) Y_p = g(t)$

$$[u_1 Y_1'' + u_1' Y_1' + u_2 Y_2'' + u_2' Y_2'] + p(t)[u_1 Y_1' + u_2 Y_2'] + q(t)[u_1 Y_1 + u_2 Y_2] = g(t)$$

$$u_1 \underbrace{\cancel{[Y_1'' + p(t) Y_1' + q(t) Y_1]}}_{\substack{=0 \text{ since } Y_1 \text{ is a} \\ \text{homogeneous solution}}} + u_2 \underbrace{\cancel{[Y_2'' + p(t) Y_2' + q(t) Y_2]}}_{\substack{=0 \text{ since } Y_2 \text{ is a} \\ \text{homogeneous solution}}} + u_1' Y_1' + u_2' Y_2' = g(t)$$

$$\Rightarrow \boxed{u_1' Y_1' + u_2' Y_2' = g(t)} \quad (1)$$

We now have two equations to determine $u_1$ and $u_2$

① $\boxed{u_1' y_1' + u_2' y_2' = g}$ (derived from equation Ⓝ)

② $\boxed{u_1' y_1 + u_2' y_2 = 0}$ (imposed constraint)

---

Solve: ② $\Rightarrow$ $\boxed{u_2' = -\frac{y_1}{y_2} u_1'}$

plug into ① $\Longrightarrow$ $u_1' y_1' + \left(-\frac{y_1}{y_2} u_1'\right) y_2' = g$

$$\frac{u_1'}{y_2}\left(y_1' y_2 - y_1 y_2'\right) = g$$

$$u_1' = \frac{-y_2 g}{y_1 y_2' - y_1' y_2} \longleftarrow = W(y_1, y_2)(t)$$

(wronskian)

$$u_1'(t) = \frac{-y_2(t)\, g(t)}{W(y_1, y_2)(t)} \quad \text{(separable } 1^{st} \text{ order ODE)}$$

Integrate: $\boxed{u_1(t) = -\int \frac{y_2(t)\, g(t)}{W(y_1,y_2)(t)}\, dt}$

Then,

$$u_2' = -\frac{y_1}{y_2} u_1' = -\frac{y_1}{y_2}\left(\frac{-y_2 g}{W}\right) = \frac{y_1 g}{W}$$

Integrate: $\boxed{u_2(t) = \int \frac{y_1(t)\, g(t)}{W(y_1,y_2)(t)}\, dt}$

---

Then, $y_p = u_1 y_1 + u_2 y_2$

$$\boxed{y_p(t) = -y_1(t)\int \frac{y_2(t)\, g(t)}{W(y_1,y_2)(t)}\, dt + y_2(t)\int \frac{y_1(t)\, g(t)}{W(y_1,y_2)(t)}\, dt}$$

Integration constants may be ignored since any $y_p$ will do.

This is the particular solution of $y'' + p(t) y' + q(t) y = g(t)$ Ⓝ, where $y_1$ and $y_2$ are linearly independent solutions of the associated homogeneous ODE, $y'' + p(t) y' + q(t) y = 0$, and $W(y_1, y_2)(t)$ is the Wronskian of $y_1$ and $y_2$.

Using the above formula to find $y_p$ is called the Variation of Parameters method.

Origin of the name:

The general solution of the associated homogeneous ODE is $y_h = c_1 y_1 + c_2 y_2$, which may be thought of as a two parameter ($c_1$ and $c_2$) family of functions.

The above method assumes that $y_p$ has the same form as $y_h$, except the "parameters" are allowed to vary as a function of the independent variable. Hence, the method is called Variation of Parameters.

---

The above formula is in terms of $g, y_1, y_2$, and $W$.

To find $y_p$ using the above formula, the quantites $g, y_1, y_2$, and $W$ must first be determined.

To find $y_p$ for a given ODE:

1. Write the ODE in the form of equation Ⓝ.

$$\underline{\underline{1}}\, y'' + p(t)y' + q(t)y = g(t), \text{ and identify } g(t).$$

need a coefficient of 1 here

2. Find two linearly independent solutions ($y_1$ and $y_2$) of the associated homogeneous ODE, $y'' + p(t)y' + q(t)y = 0$.
3. Compute $W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$.
4. Plug $g, y_1, y_2$, and $W$ into the above formula to find $y_p$.

5. Write down the general solution $y(t) = c_1 y_1(t) + c_2 y_2(t) + y_p(t)$.
6. Impose IC's to find $c_1$ and $c_2$, if necessary.

Example: Find the general solution of $y''+y=\tan t$.

1. $\underline{1}y''+y=\tan t \Rightarrow \boxed{g(t)=\tan t} \Rightarrow$ MUC is not applicable.

2. $y_h''+y_h=0$

$r^2+1=0$
$\underline{r=\pm i}$ $\Longrightarrow$ $y_h = C_1\cos t + C_2\sin t$

$\boxed{\begin{matrix} y_1=\cos t \\ y_2=\sin t\end{matrix}}$

3. $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t+\sin^2 t = 1$

$\boxed{W=1}$

In general, W depends on t.

4. $y_p = -y_1\int \frac{y_2 g}{W}dt + y_2\int \frac{y_1 g}{W}dt$

$= -\cos t\int \frac{\sin t\cdot\tan t}{1}dt + \sin t\int\frac{\cos t\cdot \tan t}{1}dt$

$= -\cos t\int\frac{\sin^2 t}{\cos t}dt + \sin t\int \sin t\,dt$

$= -\cos t\int\frac{1-\cos^2 t}{\cos t}dt + \sin t\cdot(-\cos t)$ ← ignore the integration constant.

$= -\cos t\int(\sec t-\cos t)dt - \sin t\cos t$

$= -\cos t\left[\ln|\sec t+\tan t| - \cancel{\sin t}\right] - \cancel{\sin t\cos t}$

$\boxed{y_p = -\cos t\cdot\ln|\sec t+\tan t|}$

5. $y(t)=y_h(t)+y_p(t)$

$\Rightarrow \boxed{y(t) = C_1\cos t + C_2\sin t - \cos t\cdot\ln|\sec t+\tan t|}$

**Example:** Find the general solution $y''-y=e^t$.

Either the MUC (recommended) or V of P may be used to find $y_p$.

homogeneous: $y_h''-y_h=0$
$r^2-1=0$
$r=\pm 1$
$\Longrightarrow \boxed{y_h(t)=c_1e^t+c_2e^{-t}}$

Particular:

MUC: $y_p = Ate^t$
plug in to get $A=\frac{1}{2}$ $\Rightarrow \boxed{y_p=\frac{1}{2}te^t}$

V of P: $g(t)=e^t$ $\quad y_1(t)=e^t$ $\quad y_2(t)=e^{-t}$

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} = -1-1=-2 \Rightarrow W=-2$$

$$y_p = -y_1\int \frac{y_2 g}{W}dt + y_2\int\frac{y_1 g}{W}dt$$

$$= -e^t\int\frac{e^{-t}\cdot e^t}{-2}dt + e^{-t}\int\frac{e^t\cdot e^t}{-2}dt$$

$$= \frac{1}{2}e^t\int dt - \frac{1}{2}e^{-t}\int e^{2t}dt$$

$$= \frac{1}{2}e^t\cdot t - \frac{1}{2}e^{-t}\cdot\frac{1}{2}e^{2t}$$

$$y_p = \frac{1}{2}te^t - \frac{1}{4}\cancel{e^t}$$

This term alone is a particular solution

This term is a solution of the associated homogeneous ODE, and thus can be ignored, though it is not incorrect to retain this term.

$\Rightarrow \boxed{y_p=\frac{1}{2}te^t}$ (same as that found using the MUC)

$\Rightarrow \boxed{y(t)=c_1e^t+c_2e^{-t}+\frac{1}{2}te^t}$ (general solution)

OR If we keep the additional term,

$$y=c_1e^t+c_2e^{-t}+\frac{1}{2}te^t-\frac{1}{4}e^t$$

$$=(c_1-\tfrac{1}{4})e^t+c_2e^{-t}+\frac{1}{2}te^t$$

$c_1-\frac{1}{4}\to c_1$ $\qquad = c_1e^t+c_2e^{-t}+\frac{1}{2}te^t$ ✓ (same as above)

# MUC vs. V of P

**MUC**: The MUC may be used only for the case of constant coefficients,

$$ay'' + by' + cy = g(t),$$

where $g(t)$ is of an appropriate form.

We must guess the proper form of $y_p$.

restrictions: 1. constant coefficients
2. $g(t)$ must involve only sums and products of polynomials, sines and cosines, and exponential functions.

---

**V of P**: V of P can be used for the general equation Ⓝ, with variable coefficients and any function $g(t)$.

$$y'' + p(t)y' + q(t)y = g(t). \quad Ⓝ$$

V of P gives a formula for $y_p$

restrictions: none

---

It seems that V of P is the method of choice since it works for Equation Ⓝ in general, and since it provides a formula for $y_p$.

It is tempting to always use the formula.

However, the formula for $y_p$ may involve complicated integrals, which may or may not be evaluable. Those that are evaluable may be very time consuming. For these reasons, it may be substantially more convenient to use the MUC instead.

---

**Rule of Thumb**: Use MUC whenever possible!

That is, use MUC when the ODE has constant coefficients, and $g(t)$ is of an appropriate form.

Sums and products of polynomials, sines and cosines, and exponential functions.

**Example:** $y'' + 2y' + 5y = \cos t$ ; Find $y_p$.

homogeneous: $r^2 + 2r + 5 = 0$ $\qquad \alpha = -1,\ \beta = 2$

$$r = \frac{-2 \pm \sqrt{4-20}}{2} \Rightarrow y_h(t) = e^{-t}\left[C_1\cos(2t) + C_2\sin(2t)\right]$$

$$r = -1 \pm 2i$$

particular:

MUC : $y_p = A\cos t + B\sin t$

plug in to get $A = \frac{1}{5}$ and $B = \frac{1}{10}$

$$\Rightarrow \boxed{y_p(t) = \frac{1}{5}\cos t + \frac{1}{10}\sin t}$$

V of P: $g = \cos t$, $y_1 = e^{-t}\cos(2t)$, $y_2 = e^{-t}\sin(2t)$ $\Rightarrow$ $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 2e^{-2t}$

$$y_p = -y_1 \int \frac{y_2 g}{W}\,dt + y_2 \int \frac{y_1 g}{W}\,dt$$

$$y_p = -e^{-t}\cos(2t)\int \frac{e^{-t}\sin(2t)\cdot\cos t}{2e^{-2t}}\,dt + e^{-t}\sin(2t)\int \frac{e^{-t}\cos(2t)\cos t}{2e^{-2t}}\,dt$$

$$y_p = -\frac{1}{2}e^{-t}\cos(2t)\underbrace{\int e^{t}\sin(2t)\cos t\,dt} + \frac{1}{2}e^{-t}\sin(2t)\underbrace{\int e^{t}\cos(2t)\cos t\,dt}$$

These integrals are very tedious to evaluate.

Integrate $\Rightarrow$ $\boxed{y_p(t) = \frac{1}{5}\cos t + \frac{1}{10}\sin t}$

---

The MUC involves only simple differentiation and algebra, whereas V of P requires the evaluation of complicated integrals.

MUC is much more convenient than V of P for this problem.

This is often the case.

---

It is important to recognize which method is more appropriate for a particular problem.

# Section 5.1 : Modeling with Linear 2nd Order ODEs

Two prime examples:

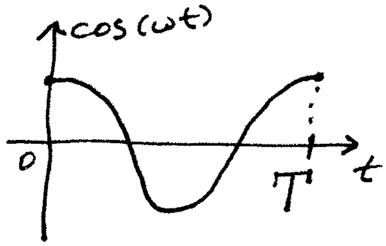
1. Forced Mass-Spring-Damper Systems
2. RLC circuits

---

## Angular Frequency

Consider the function $x(t) = \cos(\omega t)$, $t$ = time.
(or $\sin(\omega t)$)

Period ($T$):




$\omega T = 2\pi$

$$\boxed{T = \frac{2\pi}{\omega} \left(\frac{\text{seconds}}{\text{oscillation}}\right)}$$

Frequency ($f$): $\boxed{f = \frac{1}{T} = \frac{\omega}{2\pi} \left(\frac{\text{oscillations}}{\text{second}}\right)}$

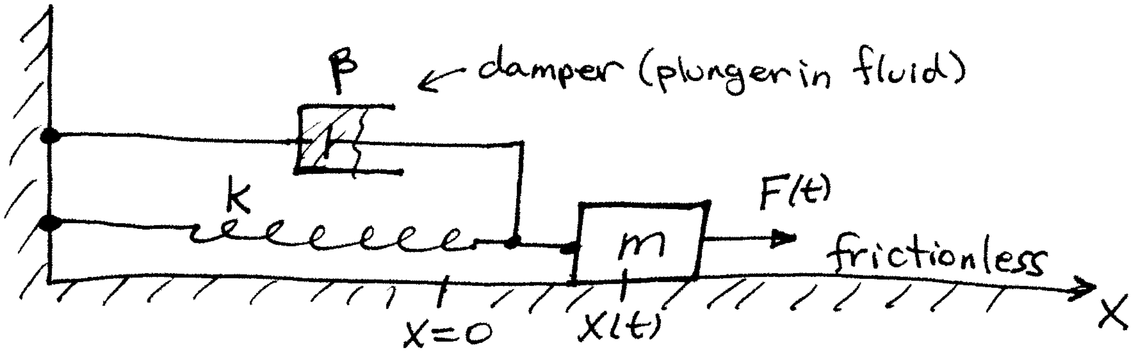
Angular Frequency ($\omega$): $\omega = \frac{2\pi}{T} = 2\pi f \left(\frac{\text{radians}}{\cancel{\text{oscillation}}}\right)\left(\frac{\cancel{\text{osillations}}}{\text{second}}\right)$

$$\boxed{\omega = 2\pi f \left(\frac{\text{radians}}{\text{second}}\right)}$$

---

Often physical systems are described by 2nd Order ODEs since the laws of physics (e.g. $F = ma = mx''$) often involve second order derivatives.

# Forced Mass-Spring-Damper Systems




$x=0$ corresponds to the relaxed state, in which the spring is neither stretched or compressed.

$x(t)$ = dispacement of the mass from the relaxed state.

$F(t)$ = external force applied to the mass

$m$ = mass
$k$ = spring constant
$\beta$ = damper constant

$m, k > 0$
$\beta \geq 0$

---

The spring force ($F_s$) acts in the direction opposite the displacement of the mass (i.e. $F_s$ acts to return the mass to the relaxed state), whereas the damping force ($F_D$) acts in the direction opposite the motion of the mass (i.e. $F_D$ acts to slow down the mass)

That is, $\text{sign}(F_s) = -\text{sign}(x)$

$\text{sign}(F_D) = -\text{sign}(v)$, $v = x'$

## Spring Force ($F_s$)

Assume that the spring is a linear spring. (simplest choice)

Hooke's Law: $\boxed{F_s = -kx}$

$k$ = spring constant $> 0$

The stiffer is the spring, the larger is $k$.

It is more accurate to consider the spring to be nonlinear.
e.g. cubic spring
$F_s = -kx - cx^3$

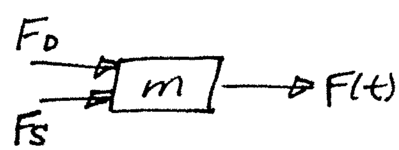
## Damper Force ($F_D$)

Assume that the damper is a linear damper. (simplest choice)

$\Rightarrow \boxed{F_D = -\beta v}$

$\beta$ = damper constant $\geq 0$

## Governing Equation




Newton's $2^{nd}$ Law (one-dimensional): $\sum F = ma$

$$F_s + F_D + F(t) = ma$$
$$-kx - \beta v + F(t) = ma$$
$$-kx - \beta x' + F(t) = mx''$$

$$\boxed{mx'' + \beta x' + kx = F(t)}$$

- $2^{nd}$ order
- linear
- nonhomogeneous
- constant coefficients

$m, k > 0; \quad \beta \geq 0$

Classifications:

$F(t) \equiv 0 \Rightarrow$ Free (homogeneous)

$F(t) \not\equiv 0 \Rightarrow$ Forced (nonhomogeneous)

$\beta = 0 \Rightarrow$ Undamped

$\beta > 0 \Rightarrow$ Damped

We'll consider all four combinations:

Free < Undamped / Damped

Forced < Undamped / Damped

# Free Mass-Spring-Damper Systems

**Undamped:** Free $\Rightarrow F(t) \equiv 0$, Undamped $\Rightarrow \beta = 0$ $\Bigg\} \Rightarrow \boxed{mx'' + kx = 0}$ $m, k > 0$

General Solution: $mr^2 + k = 0$

$r^2 = -k/m$

$r = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$

$$\Rightarrow \boxed{x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)}$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ = natural frequency

OR $x(t)$ may be written as

$$\boxed{x(t) = R\cos(\omega_0 t - \phi)}$$

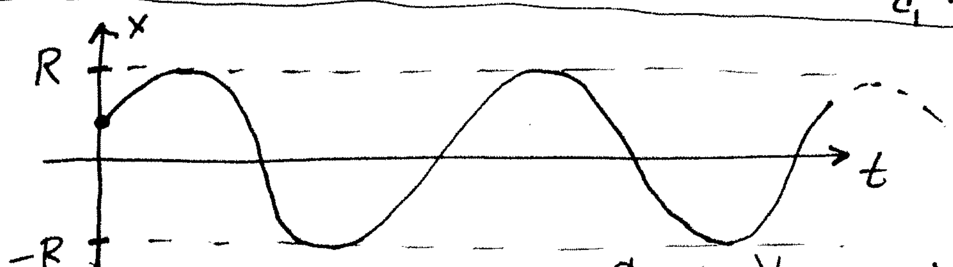
where $R = \sqrt{C_1^2 + C_2^2}$ = amplitude and $\phi = \tan^{-1} C_2/C_1$ = phase shift are the two integration constants.

To see this, $x(t) = R\cos(\omega_0 t - \phi) = R[\cos(\omega_0 t)\cos\phi + \sin(\omega_0 t)\sin\phi]$

$x(t) = (R\cos\phi)\cos(\omega_0 t) + (R\sin\phi)\sin(\omega_0 t)$

$C_1 = R\cos\phi$ $\qquad C_2 = R\sin\phi$ (similar to polar coordinates)

Solve for $R$ and $\phi \Rightarrow R = \sqrt{C_1^2 + C_2^2}$ and $\phi = \tan^{-1}\frac{C_2}{C_1}$.




Simple Harmonic Motion

The free-undamped case is not realistic for large times since it predicts perpetual motion in which the oscillations persist indefinitely with a constant amplitude. It is more realistic to include energy dissipation in the model.

e.g. friction or damping

**Damped:** Free $\Rightarrow F(t) \equiv 0$, Damped $\Rightarrow \beta > 0$ $\Rightarrow$ $\boxed{mx'' + \beta x' + kx = 0}$ $\quad m, k, \beta > 0$

Characteristic Equation: $mr^2 + \beta r + k = 0$

$$\Rightarrow \boxed{r = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m}}$$

There are three cases depending on the values of $m, k,$ and $\beta$.

1. $\beta^2 < 4mk \Rightarrow$ Complex Conjugate Roots $r_{1,2} = \alpha \pm i\omega$
2. $\beta^2 = 4mk \Rightarrow$ Real Repeated Roots $r_1 = r_2 = -\beta/2m$
3. $\beta^2 > 4mk \Rightarrow$ Real Distinct Roots $r_1 \neq r_2$

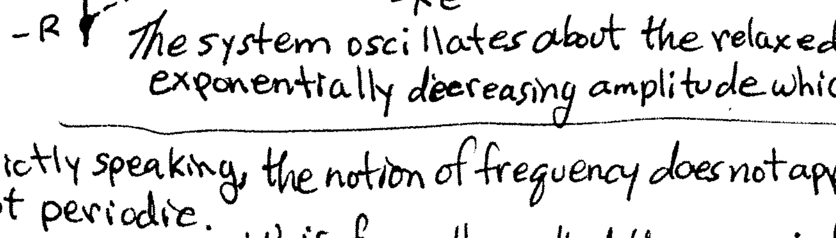
---

**Case 1:** $\beta^2 < 4mk$ : Under-Damped ($\beta$ is relatively small)

$$\Rightarrow r = -\frac{\beta}{2m} \pm i\omega, \text{ where } \omega = \frac{\sqrt{4mk - \beta^2}}{2m}$$

$$\Rightarrow \boxed{x(t) = e^{-\frac{\beta}{2m}t}\left[C_1 \cos(\omega t) + C_2 \sin(\omega t)\right]}$$

OR $\boxed{x(t) = Re^{-\beta/2m\, t} \cos(\omega t - \phi)}$ $\quad R = \sqrt{C_1^2 + C_2^2}$, $\phi = \tan^{-1}\frac{C_2}{C_1}$




The system oscillates about the relaxed state with an exponentially decreasing amplitude which approaches zero.

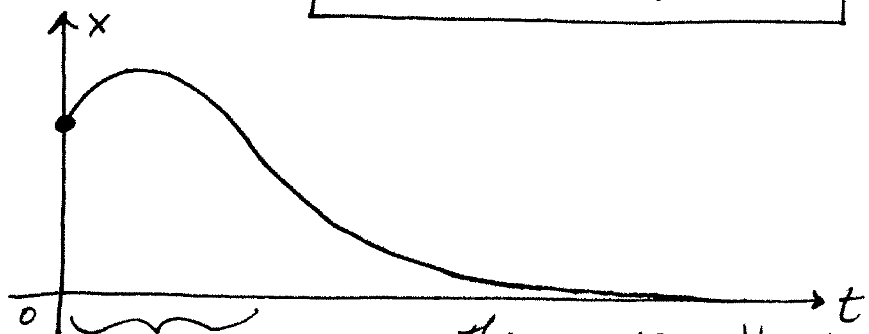
---

Note: Strictly speaking, the notion of frequency does not apply since $x(t)$ is not periodic. $\omega$ is formally called the quasi-frequency

**Case 2:** $\beta^2 = 4mk$ : Critically-Damped ($\beta$ is moderately sized)

The solution behavior changes at $\beta^2 = 4mk$.
Case 2 is the border-line between cases 1 and 3.

$$r_1 = r_2 = \frac{-\beta}{2m} \implies \boxed{x(t) = (C_1 + C_2 t)e^{-\frac{\beta}{2m}t}}$$



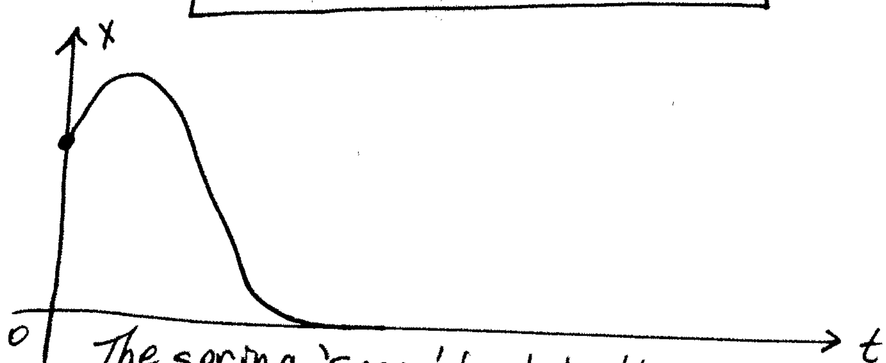

The mass is on the verge of oscillating, but instead it gently returns to the relaxed state.

---

**Case 3:** $\beta^2 > 4mk$ : Over-Damped ($\beta$ is relatively large)

$$r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m} < 0$$

$$\implies \boxed{x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}} \quad r_1, r_2 < 0$$




The spring 'snaps' back to the relaxed state

---

Note: In all three cases, $x(t) \to 0$ as $t \to \infty$.
$\implies$ The system returns to the relaxed state whenever $\beta > 0$.

$\implies$ $x = 0$ is an asymptotically stable equilibrium solution.

# Forced Mass-Spring-Damper System

$$\boxed{mx'' + \beta x' + kx = F(t)}$$

Since periodic forcing functions yield the most interesting behavior, we'll consider the special case in which $\boxed{F(t) = F_0 \cos(\omega t)}$ $\omega =$ forcing frequency

---

**Undamped ($\beta = 0$):** $\quad mx'' + kx = F_0 \cos(\omega t)$

Divide by $m \Rightarrow \boxed{x'' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)}$

where $\omega_0 = \sqrt{\frac{k}{m}}$ = natural frequency

homogeneous: $r^2 + \omega_0^2 = 0$

$r = \pm i\omega_0$

$\Rightarrow \boxed{x_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)}$

particular: 2 Cases:
1. $\omega \neq \omega_0$
2. $\omega = \omega_0$

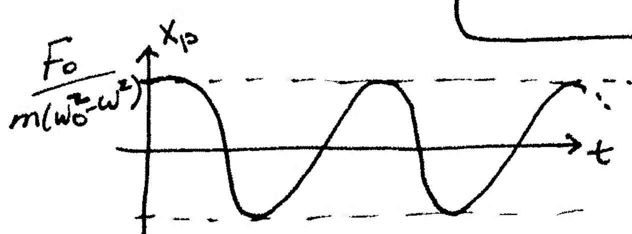
Case 1: $\omega \neq \omega_0 \Rightarrow F(t) = F_0\cos(\omega t)$ is not a solution of the associated homogeneous equation

MUC: $x_p = A\cos(\omega t) + B\sin(\omega t)$

$x_p'' = -\omega^2\left[A\cos(\omega t) + B\sin(\omega t)\right]$

plug in to get $A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$ and $B = 0$

$$\Rightarrow \boxed{x_p(t) = \frac{F_0 \cos(\omega t)}{m(\omega_0^2 - \omega^2)}} \quad \omega \neq \omega_0$$




$x_p$ oscillates with constant amplitude.
Amplitude $\to \infty$ as $\omega \to \omega_0$.

---

general: $x(t) = C_1\cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0 \cos(\omega t)}{m(\omega_0^2 - \omega^2)}$

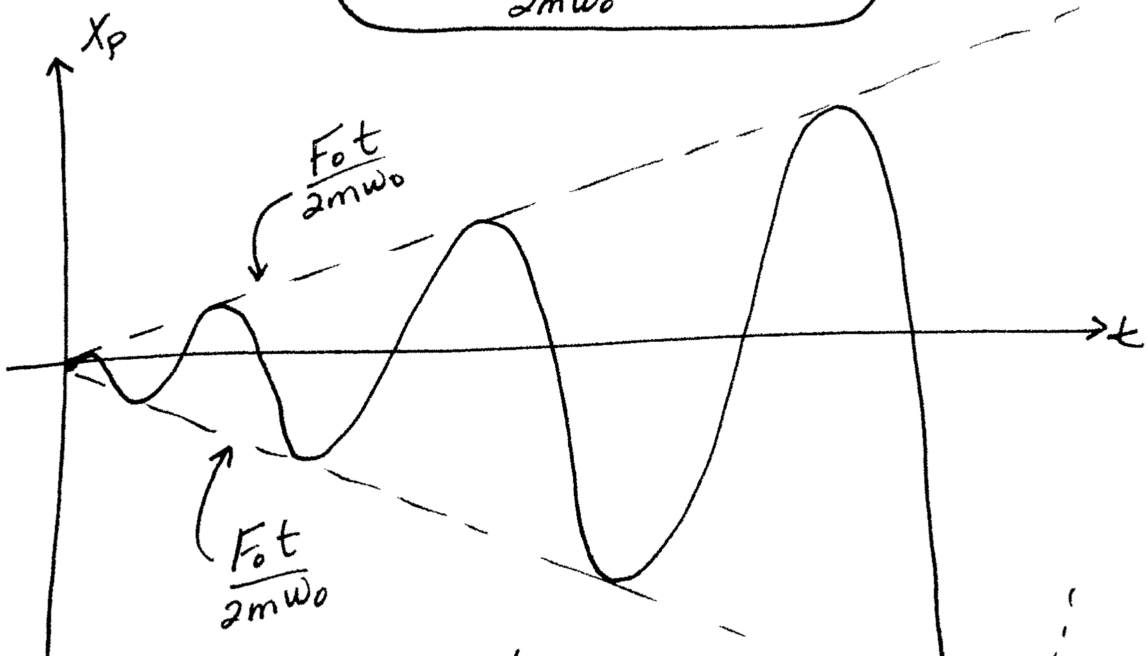
$x(t)$ is periodic and oscillated with ~~constant~~ bounded amplitude.

**Case 2:** $\omega = \omega_0 \Rightarrow F(t) = F_0\cos(\omega_0 t)$ is a solution of the associated homogeneous equation

**MUC:** $X_p = t\left[A\cos(\omega_0 t) + B\sin(\omega_0 t)\right]$

Plug in to get $A = 0$ and $B = \frac{F_0}{2m\omega_0}$

$$\Rightarrow \boxed{X_p(t) = \frac{F_0}{2m\omega_0}\, t\sin(\omega_0 t)}$$




Amplitude $\to \infty$ as $t \to \infty$

---

**Resonance:** forcing frequency $(\omega)$ = natural frequency $(\omega_0)$

In this case, the forcing is in phase with the motion of the mass-spring system. That is, the forcing always acts in the direction of motion, and thus it continually adds energy to the system. Without energy dissipation (e.g. friction or damping), the total energy of the system $\to \infty$ as $t \to \infty$, and consequently, so does the amplitude.

---

Again, without energy dissipation the results are not realistic for large times.

It is more realistic to include energy dissipation in the model.

e.g. friction or damping.

**Damped ($\beta > 0$):** $\boxed{mx'' + \beta x' + kx = F_0 \cos(\omega t)}$

homogeneous: $m x_h'' + \beta x_h' + k x_h = 0$ (Free-Damped Motion)

$$mr^2 + \beta r + k = 0 \implies \boxed{r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m}}$$

Recall the following results for the free-damped case.

1. Under-damped ($\beta^2 < 4mk$)

   $r_{1,2} = \alpha \pm i\beta$, where $\alpha = \frac{-\beta}{2m} < 0$

   $\implies x_h(t) = e^{\frac{-\beta t}{2m}}\left[C_1 \cos(\beta t) + C_2 \sin(\beta t)\right] \to 0$ as $t \to \infty$

2. Critically-damped ($\beta^2 = 4mk$)

   $r_{1,2} = \frac{-\beta}{2m} < 0$

   $\implies x_h(t) = (C_1 + C_2 t)\, e^{\frac{-\beta}{2m} t} \to 0$ as $t \to \infty$

3. Over-damped ($\beta^2 > 4mk$)

   $r_{1,2} = \frac{-\beta \pm \sqrt{\beta^2 - 4mk}}{2m} < 0, \quad r_1 \neq r_2$

   $x_h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \to 0$ as $t \to \infty$

---

Observations:

1) $x_h(t) \to 0$ as $t \to \infty$ in all three cases

2) $F(t) = F_0 \cos(\omega t)$ is NOT a solution of the associated homogeneous equation.

$\implies$ Resonance does not occur when there is damping.

homogeneous solution = transient solution

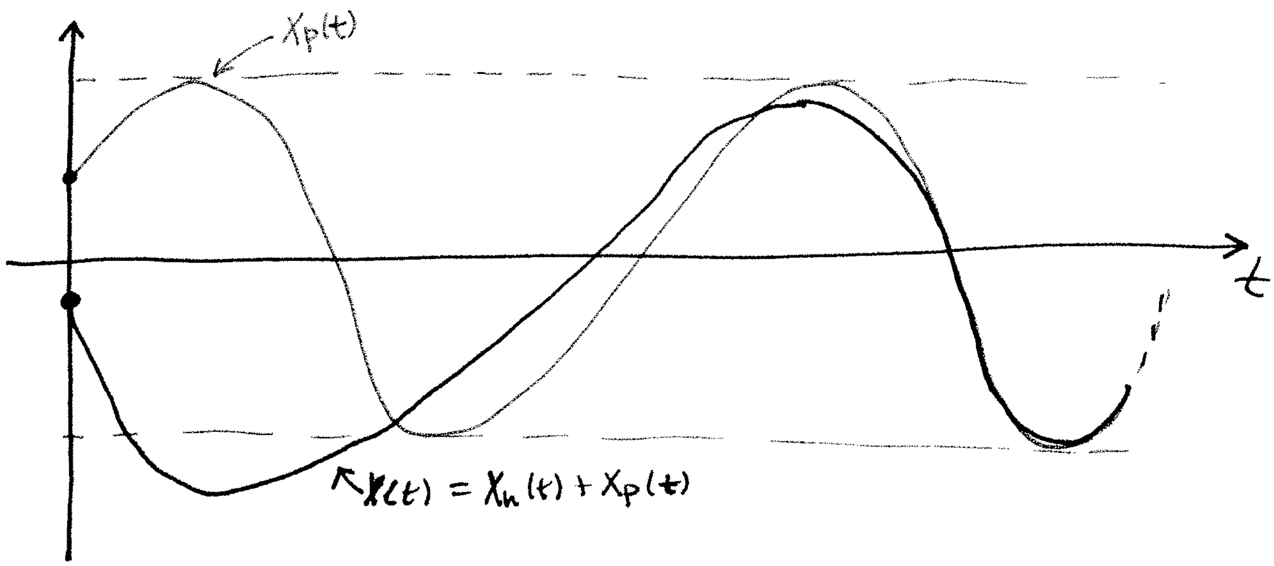
$X_h$ represents the effects of the initial conditions, which die out in time ($X_h \to 0$ as $t \to \infty$).

particular solution = steady-state solution (or forced response)

$X_p$ represents the effects of the forcing, which persist in time. ($X \to X_h$ as $t \to \infty$).

$$\boxed{X(t) = X_h(t) + X_p(t) \longrightarrow X_p(t) \text{ as } t \to \infty}$$

The larger is the damping ($\beta$), the faster does $X_h \to 0$.




Since $X_h \to 0$ as $t \to \infty$, the main interest lies in the forced response $X_p$, which represents the long-time solution behavior, after the transients have died out.

## Particular (Steady-State) Solution (or Forced Response)

MUC: $x_p = A\cos(\omega t) + B\sin(\omega t)$

These are not solutions of the associated homogeneous equation.

Plug in to get $A = \dfrac{m(\omega_0^2-\omega^2)F_0}{m^2(\omega_0^2-\omega^2)^2+(\beta\omega)^2}$ and $B = \dfrac{\beta\omega F_0}{m^2(\omega_0^2-\omega^2)^2+(\beta\omega)^2}$.

where $\omega_0 = \sqrt{\frac{k}{m}}$.

Note: $\omega = \omega_0$ does not cause difficulty here.

$$\Rightarrow \quad x_p(t) = \frac{F_0\left[m(\omega_0^2-\omega^2)\cos(\omega t) + \beta\omega\sin(\omega t)\right]}{m^2(\omega_0^2-\omega^2)^2 + (\beta\omega)^2}$$

$x_p$ may be written as

$$x_p(t) = R\cos(\omega t - \phi)$$

where $R = \sqrt{A^2+B^2} = \dfrac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+(\beta\omega)^2}}$

and $\phi = \tan^{-1}\dfrac{B}{A} = \tan^{-1}\left(\dfrac{\beta\omega}{m(\omega_0^2-\omega^2)}\right)$

$$\Rightarrow \quad \boxed{x_p(t) = \frac{F_0\cos(\omega t-\phi)}{\sqrt{m^2(\omega_0^2-\omega^2)^2+(\beta\omega)^2}}}$$

The key features of the forced response $X_p$ are the amplitude $R$ and the forcing frequency $\omega$. The phase shift $\phi$ is usually of little interest.

The forcing frequency is an input parameter which may be specified, whereas the amplitude is a consequence of the particular values of the input parameters.

The behavior of the forced response may be illustrated by plotting the amplitude $R$ as a function of the forcing frequency $\omega$ for various values of the damping constant $\beta$, while holding $m$, $k$, and $F_0$ fixed. $\Rightarrow \omega_0 = \sqrt{\frac{k}{m}}$ is fixed also.

$$\boxed{R(\omega) = \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+(\beta\omega)^2}}}$$

Observations:

1) $\omega = 0 \Rightarrow F(t) = F_0 \cos 0 = F_0$ (constant forcing)

$$R(0) = \frac{F_0}{m\omega_0^2} = \frac{F_0}{m \cdot k/m} = \frac{F_0}{k} \Rightarrow R(0) = F_0/k$$

2) $R(\omega) \to 0$ as $\omega \to \infty$ (high-frequency forcing)

3) $R(\omega) \to \dfrac{F_0}{m|\omega_0^2-\omega^2|}$ as $\beta \to 0$

Note: When $\beta = 0$, the homogeneous solution $X_h = C_1\cos(\omega_0 t) + C_2 \sin(\omega_0 t)$ and the effects of the initial conditions do not decay in time, in which case $X_p$ does not represent the long-time solution. However, the limit $\beta \to 0$ may be considered. The limit is not defined when $\omega = \omega_0$ since that corresponds to the resonant case, in which the amplitude of $X_p$ is not constant, but rather approaches infinity as $t \to \infty$.

4) $R(\omega)$ has a local maximum if $\beta$ is sufficiently small.

Set $R'(\omega)=0 \Rightarrow \boxed{\omega_m = \omega_0\sqrt{1-\frac{\beta^2}{2mk}} < \omega_0}$

$\omega_m$ is real only if $\beta^2 \le 2mk$ ($\beta^2 = 2mk \Rightarrow \omega_m = 0$)

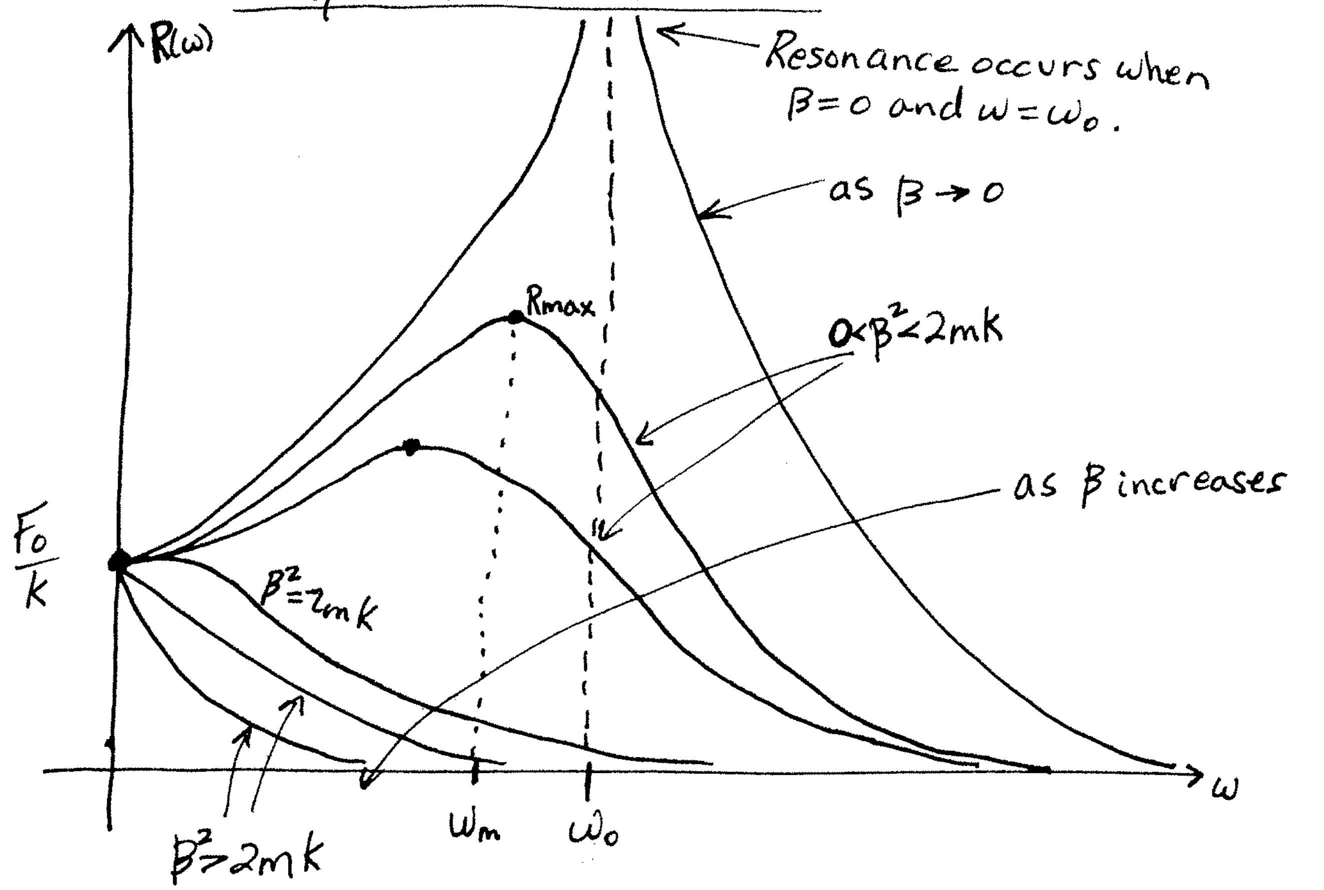
Case: $0 < \beta^2 \le 2mk$

$$R_{max} = R(\omega_m) = \frac{F_0}{\beta\omega_0\sqrt{1-\frac{\beta^2}{2mk}}}$$

Case: $\beta^2 > 2mk$

$R(\omega)$ is a decreasing function of $\omega \Rightarrow R_{max}$ occurs at $\omega = 0$.

Steady-State Solution Behavior




5) The amplitude $R$ increases when $\beta$ decreases or $F_0$ increases, with all else fixed

# Initial Conditions

**IVP:**
$$\boxed{\begin{gathered} mx'' + \beta x' + kx = F(t) \\ x(t_0) = x_0 \\ x'(t_0) = v_0 \end{gathered}}$$

$x(t_0) = x_0$ ← initial position

$x'(t_0) = v_0$ ← initial velocity

Usually $t_0 = 0$ for convenience

Two common possibilities:

1) The mass is displaced to a point $x = x_0$, and released from rest at time $t_0 = 0$.

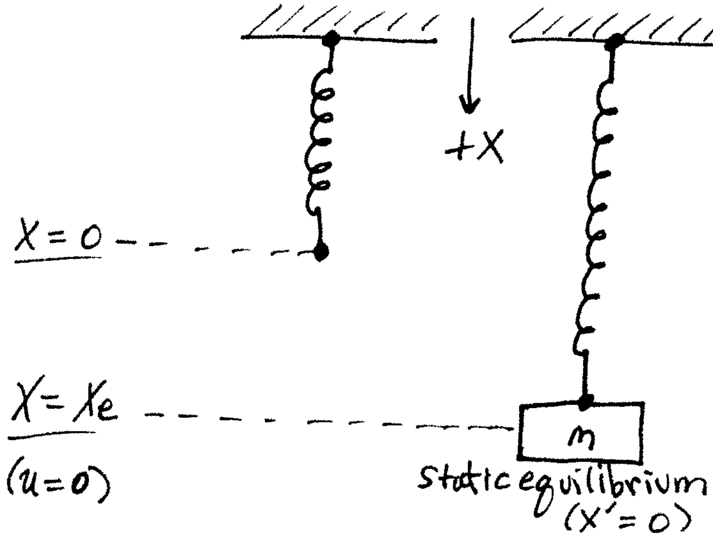
$$\Longrightarrow \boxed{\begin{gathered} x(0) = x_0 \\ x'(0) = 0 \end{gathered}}$$

2) The mass is at rest at $x = 0$ (the relaxed state). At time $t_0 = 0$, it is struck with an impulse which gives it an initial velocity of $v_0$.

$$\Longrightarrow \boxed{\begin{gathered} x(0) = 0 \\ x'(0) = v_0 \end{gathered}}$$

With these initial conditions, the Existence and Uniqueness Theorem applies, and a unique solution is guaranteed to exist.

# Hanging Spring Systems




$$\Sigma F = ma$$
$$F_s + F_g = 0$$
$$-kx_e + mg = 0$$
$$\boxed{k = \frac{mg}{x_e}}$$

(This is one method for finding k)

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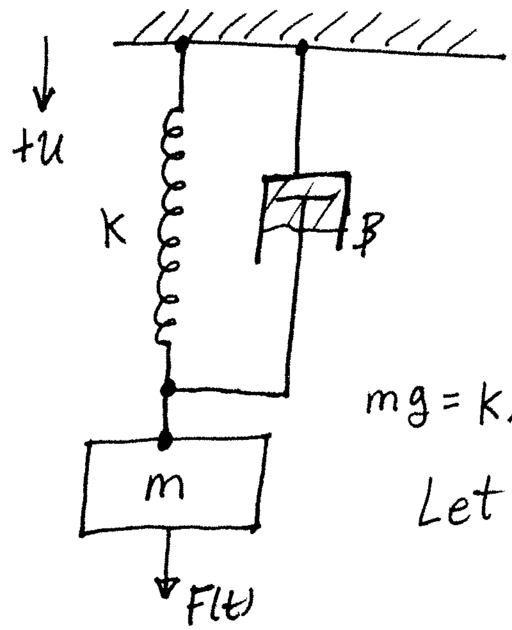



$$\Sigma F = ma$$
$$F_s + F_D + F_G + F(t) = ma$$
$$-kx - \beta x' + mg + F(t) = mx''$$
$$mx'' + \beta x' + kx - mg = F(t)$$
$$mg = kx_e \Rightarrow mx'' + \beta x' + k(x - x_e) = F(t)$$

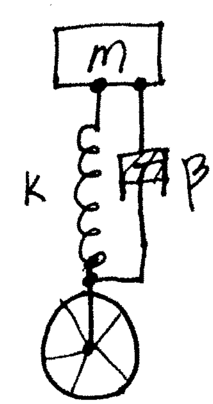
Let $u = x - x_e$. $x = x_e \Rightarrow u = 0$

$$u' = x'$$
$$u'' = x''$$

$$\Rightarrow \boxed{mu'' + \beta u' + ku = F(t)}$$

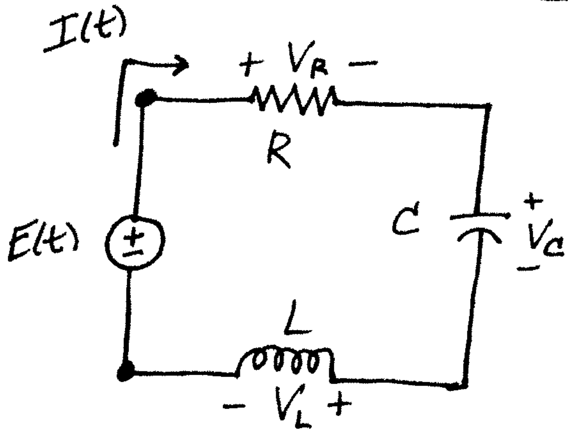
This is the same equation as before, but now the mass oscillates about $u=0$ ($x = x_e$). Otherwise, all of the above results apply.

Example: Shock Absorber



# RLC Circuits

The solution behavior of RLC Circuits is analogous to that of Forced Mass-Spring-Damper Systems.




$E(t)$ = electromotive force (Volts)

$I(t)$ = current (amps)

$V$ = voltage (Volts)

$R$ = resistance (ohms)

$C$ = capacitance (farads)

$L$ = inductance (henrys)

($R$, $C$, $L$: constants)

$\boxed{V_R = IR}$ (Ohm's Law)

$\boxed{V_C = Q/C}$ where $Q(t)$ is the electric charge on the capacitor (coulombs).

$\boxed{V_L = LI'}$

$\boxed{I = Q'} \Rightarrow I' = Q''$, $I'' = Q'''$

## Kirchoff's Voltage Law (KVL)

$$V_R + V_C + V_L = E(t)$$

$$IR + Q/C + LI' = E(t)$$

$$\Rightarrow \boxed{LQ'' + RQ' + \frac{Q}{C} = E(t)}$$

OR Differentiate with respect to $t$ to get

$$LQ''' + RQ'' + \frac{1}{C}Q' = E'(t)$$

$$\boxed{LI'' + RI' + \frac{1}{C}I = E'(t)}$$

(Analogous to the Forced Mass-Spring-Damper equation)

- 2nd Order
- linear
- nonhomogeneous
- constant coefficients.

# Analogy

| Spring Systems | RLC Circuits | |
|---|---|---|
| $mx'' + \beta x' + kx = F(t)$ | $LQ'' + RQ' + \frac{1}{C}Q = E(t)$ OR | $LI'' + RI' + \frac{1}{C}I = E'(t)$ |
| $t$ = time | $t$ = time | $t$ = time |
| $x$ = displacement | $Q$ = charge | $I$ = current |
| $x' = v$ = velocity | $Q' = I$ = current | $I'$ = rate of change of current |
| $m$ | $L$ | $L$ |
| $k$ | $1/C$ | $1/C$ |
| $\beta$ | $R$ | $R$ |
| $F(t)$ | $E(t)$ | $E'(t)$ |
| natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$ | $\omega_0 = \frac{1}{\sqrt{LC}}$ | $\omega_0 = \frac{1}{\sqrt{LC}}$ |

With the above substitutions, the results of the Spring Systems can be translate to results for RLC circuits.

natural frequency (no forcing, $E(t) \equiv 0$; no resistance, $R = 0$)

$$LQ'' + \tfrac{1}{C}Q = 0$$

$$Lr^2 + \tfrac{1}{C} = 0$$

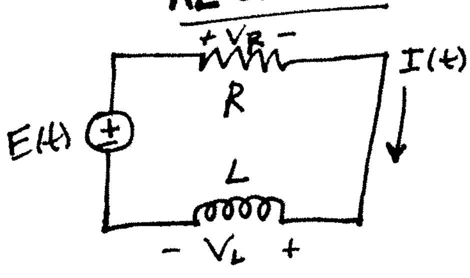
$$r = \pm i \frac{1}{\sqrt{LC}} = \pm i\omega_0$$

OR

$$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{1/C}{L}}$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

## RL Circuits



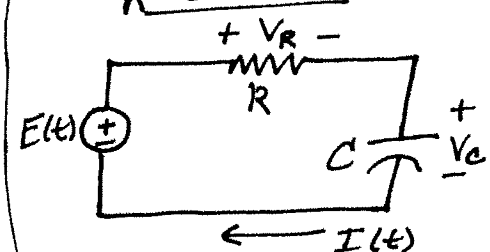

KVL: $V_R + V_L = E(t)$

$$IR + LI' = E(t)$$

$$\boxed{L\frac{dI}{dt} + RI = E(t)} \text{ (linear)}$$

(integrating factor method)

## RC Circuits




KVL: $V_R + V_C = E(t)$

$$IR + \frac{Q}{C} = E(t)$$

$I = Q' \Rightarrow$ $\boxed{R\frac{dQ}{dt} + \frac{1}{C}Q = E(t)}$ (linear)

OR $\boxed{R\frac{dI}{dt} + \frac{1}{C}I = E'(t)}$

(integrating factor method)

## LC circuits

Set $R = 0$ in the above RLC circuit.

# Chapter 7: Laplace Transform

## Motivation

The Laplace Transform may be used to solve linear nonhomogeneous ODEs with constant coefficients. We'll consider $1^{st}$ and $2^{nd}$ order ODEs of this type.

$1^{st}$ Order: $ay' + by = g(t)$ ← forcing function

$2^{nd}$ Order: $ay'' + by' + cy = g(t)$

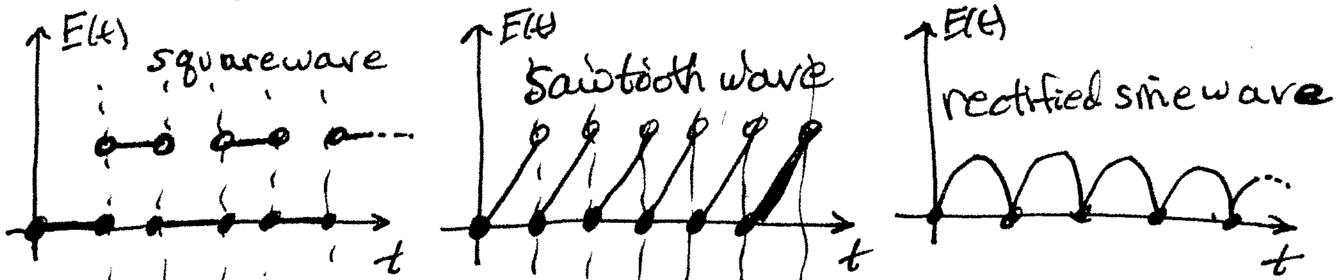
e.g. i) Forced Mass-Spring-Damper Systems

ii) RLC Circuits (other circuits also)

| a | b | c | g(t) |
|---|---|---|---|
| m | β | k | F(t) |
| L | R | 1/C | E(t) or E'(t) |

In principle, Variation of Parameters may be used for any forcing function $g(t)$. However, the integrals appearing in the formula may be difficult to evaluate. It is not uncommon for the forcing function $g(t)$ to be piecewise defined, in which case $g$ and its derivatives may have discontinuities.

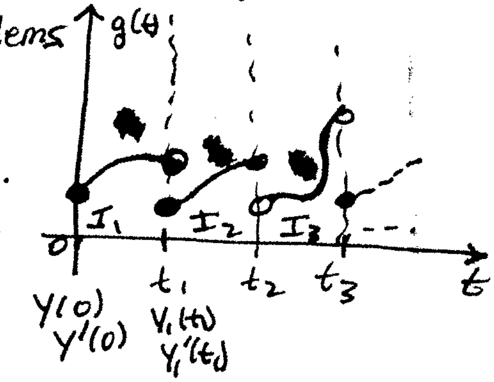
e.g. Circuits: $g(t) = E(t)$ = electromotive force




The Laplace Transform is ideal for treating such forcing functions.

Though $g(t)$ is not continuously differentiable, the Existence and Uniqueness Theorem (EUT) still applies for many practical problems

e.g. $ay'' + by' + cy = g(t)$; $y(0) = y_0$, $y'(0) = y_1$




By the EUT, the IVP has a unique solution on $I_1$, call it $y_1$. Then the solution at $t_1$ can be considered as the ICs for a second IVP on $I_2$. That is, $y(t_1) = \lim_{t \to t_1^-} y_1(t)$

$y'(t_1) = \lim_{t \to t_1^-} y_1'(t)$.

Then by the EUT, the second IVP will have a unique solution on $I_2$, and so on. The unique solutions on the various intervals can be pieced together to obtain a globally (valid for all $t$) unique solution.

## Section 7.1 : Laplace Transform (Definition/Theory)

Definition: Let $f$ be a function defined for $t \geq 0$. The Laplace Transform of $f$ is defined by the integral

$$\boxed{\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\,dt}$$, provided the integral converges.

To find the Laplace Transform of a function, multiply it by $e^{-st}$ and integrate with respect to $t$ from $0$ to $\infty$.

Notes: 1) The Laplace Transform may be considered as a function of $s$.

Notation: Functions are denoted by lower-case letters, and their Laplace Transforms are denoted by the respective upper-case letter. e.g. $\mathcal{L}\{f(t)\} = F(s)$

$\mathcal{L}\{g(t)\} = G(s)$

$\mathcal{L}\{y(t)\} = Y(s)$

2) Existence of the Laplace Transform

If $f(t)$ is piecewise continuous on $[0,\infty)$ and increases no faster than exponentially (i.e. $f$ can be bounded above by some exponential function), then, $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > c$, for some constant $c$.

Also, $\lim_{s\to\infty} F(s) = 0$ if $f$ is piecewise continuous on $(0,\infty)$.

This condition is sufficient, but not necessary.

e.g. $\mathcal{L}\{t^{-1/2}\}$ exists even though $t^{-1/2}$ is not continuous at $0$.

The Laplace Transform exist for practical piecewise functions (for $s > c$).

---

Recall: $\int_0^\infty f(t)\,dt = \lim_{b\to\infty}\int_0^b f(t)\,dt$.

It is understood that the upper integration limit is not equal to infinity, but rather it approaches infinity. Sometimes it is not sufficient to simply evaluate the antiderivative at infinity, but instead the limit must be considered.

e.g.

$$\int_0^\infty t e^{-t}\,dt = \overbrace{-te^{-t}\Big|_0^\infty}^{\infty\cdot 0 \text{ (indeterminant form)}} + \int e^{-t}\,dt = -\left(\lim_{t\to\infty} te^{-t} - 0\right) - e^{-t}\Big|_0^\infty$$

$u = t \quad v = -e^{-t}$

$du = dt \quad dv = e^{-t}dt$

$$= -(0-0) - (0-1) = 1$$

Note: $\boxed{\lim_{t\to\infty} t^p e^{-t} = 0 \text{ for all } p}$

## Solving Linear IVPs with Constant Coefficients

e.g. Consider the 2nd order case,

$$ay'' + by' + cy = g(t) \;;\; y(0) = y_0,\ y'(0) = y_1 .$$

1. Apply the Laplace Transform to both sides of the ODE.

$$\mathcal{L}\{ay'' + by' + cy\} = \mathcal{L}\{g(t)\}$$

Notes: 1) The ICs are imposed while finding the Laplace Transform of the LHS.
2) The Laplace Transform of the LHS involves only $s$ and $Y(s)$, with no derivatives.

$$\Rightarrow \quad T(s, Y(s)) = G(s)$$

2. Solve for $Y(s)$

3. Compute the Inverse Transform of $Y(s)$ to obtain the solution $y(t)$. $\Rightarrow$ $\boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\}}$ (solution of the IVP)

---

ODE $\rightarrow$ $\boxed{\mathcal{L}}$ $\rightarrow$ Solve for $Y(s)$ in Laplace Transform Space $\rightarrow$ $\boxed{\mathcal{L}^{-1}}$ $\rightarrow$ $y(t)$

---

Before doing so, we must first work out some formulas and properties of the Laplace Transform. The techniques for finding Laplace Transforms of functions parallel those of finding derivatives of functions, in the sence that we'll compile a list of Laplace Transforms for the elementary functions, such as $e^{kt}$, $\cos(kt)$, $t^n, \ldots$, and then derive a series of properties and formulas which allow us to compute the Laplace Transform for various forms and combinations of the elementary functions.

---

### Linearity

The Laplace Transform is a Linear Transform

$$\Rightarrow \boxed{\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}}$$

OR $\mathcal{L}\{af(t)\} = a\mathcal{L}\{f(t)\}$
and $\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$

The derivative is a linear operator
$(af + bg)' = af' + bg'$

# Laplace Transform of Some Common Functions

## 1. Polynomials

i) $f(t) = 1,\ t \geq 0$

converges for $s>0$

$$F(s) = \mathcal{L}\{1\} = \int_0^\infty 1 \cdot e^{-st}\,dt = -\frac{1}{s} e^{-st}\Big|_0^\infty = -\frac{1}{s}(0-1) = \frac{1}{s}$$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s},\ s>0}$$

ii) $f(t) = t^n,\ t \geq 0;\ n = 1, 2, \ldots$

(L'Hopital's Rule $\lim_{t\to\infty} t^n e^{-st} = 0$)

$$F(s) = \mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}\,dt = \underbrace{-\frac{t^n}{s} e^{-st}\Big|_0^\infty}_{\to 0,\ s>0} + \frac{n}{s}\int_0^\infty t^{n-1} e^{-st}\,dt$$

$u = t^n \quad v = -\frac{1}{s}e^{-st}$

$du = nt^{n-1}dt \quad dv = e^{-st}dt$

$$= \frac{n}{s}\mathcal{L}\{t^{n-1}\}$$

$$\Rightarrow \boxed{\mathcal{L}\{t^n\} = \frac{n}{s}\mathcal{L}\{t^{n-1}\},\ s>0},\ n = 1, 2, \ldots$$

Apply recursively $\Rightarrow$

$$\mathcal{L}\{t^n\} = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \cdots \cdot \underbrace{\frac{2}{s}}_{n=2} \cdot \underbrace{\frac{1}{s}}_{n=1}\mathcal{L}\{1\} = \frac{n!}{s^n} \cdot \frac{1}{s}$$

$$\boxed{\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}},\ s>0}\quad n = 0, 1, 2, \ldots$$

**Example:**

$$\mathcal{L}\{3t^2 - 5t + 2\} = 3\mathcal{L}\{t^2\} - 5\mathcal{L}\{t\} + 2\mathcal{L}\{1\}$$

$$= 3\frac{2!}{s^3} - 5\frac{1!}{s^2} + 2 \cdot \frac{1}{s} = \frac{6}{s^3} - \frac{5}{s^2} + \frac{2}{s}$$

$$\boxed{\mathcal{L}\{3t^2 - 5t + 2\} = \frac{6}{s^3} - \frac{5}{s^2} + \frac{2}{s}}$$

## 2. Exponential Functions: $f(t) = e^{kt},\ t \geq 0,\ k \neq 0.$

$$\mathcal{L}\{e^{kt}\} = \int_0^\infty e^{kt} e^{-st}\,dt = \int_0^\infty e^{-(s-k)t}\,dt = -\frac{1}{s-k} e^{-(s-k)t}\Big|_0^\infty$$

converges for $s>k$.

$$= -\frac{1}{s-k}(0-1) = \frac{1}{s-k}$$

$$\boxed{\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}}$$

**Example:**

$$\mathcal{L}\{e^{-2t} + 5\} = \mathcal{L}\{e^{-2t}\} + 5\mathcal{L}\{1\}$$

$$= \frac{1}{s-(-2)} + 5 \cdot \frac{1}{s} = \frac{1}{s+2} + \frac{5}{s}$$

$$\boxed{\mathcal{L}\{e^{-2t} + 5\} = \frac{1}{s+2} + \frac{5}{s}}$$

3. Sines and Cosines

$f(t) = \sin(kt),\ t \ge 0,\ k \ne 0$

$$\mathcal{L}\{\sin(kt)\} = \int_0^\infty \sin(kt)e^{-st}dt = -\frac{1}{s}\sin(kt)e^{-st}\Big|_0^\infty + \frac{k}{s}\int_0^\infty \cos(kt)e^{-st}dt$$

$u = \sin(kt)$ $\quad V = -\frac{1}{s}e^{-st}$
$du = k\cos(kt)$ $\quad dv = e^{-st}dt$

$u = \cos(kt)$ $\quad V = -\frac{1}{s}e^{-st}$
$du = -k\sin(kt)dt$ $\quad dv = e^{-st}dt$

$$= \frac{k}{s}\left[\frac{-1}{s}\cos(kt)e^{-st}\Big|_0^\infty - \frac{k}{s}\int_0^\infty \sin(kt)e^{-st}dt\right]$$

$$= \frac{k}{s}\left[\frac{-1}{s}(0-1) - \frac{k}{s}\mathcal{L}\{\sin(kt)\}\right]$$

$$\left(1+\frac{k^2}{s^2}\right)\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2}$$

$$\Rightarrow \boxed{\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2},\ s>0}\quad k \ne 0$$

Example:
$$\mathcal{L}\{\sin 5t\} = \frac{5}{s^2+25}$$

Similarly, $\boxed{\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2},\ s>0}$

---

4. Hyperbolic sines and cosines

$f(t) = \cosh(kt),\ t \ge 0,\ k \ne 0$

$\mathcal{L}\{\cosh(kt)\} = \int_0^\infty \cosh(kt)e^{-st}dt$ (can integrate by parts twice, as above)

OR $\mathcal{L}\{\cosh(kt)\} = \mathcal{L}\left\{\frac{e^{kt}+e^{-kt}}{2}\right\} = \frac{1}{2}\left[\mathcal{L}\{e^{kt}\} + \mathcal{L}\{e^{-kt}\}\right]$

$$= \frac{1}{2}\left[\frac{1}{s-k} + \frac{1}{s-(-k)}\right] \qquad s>k \qquad s>-k$$

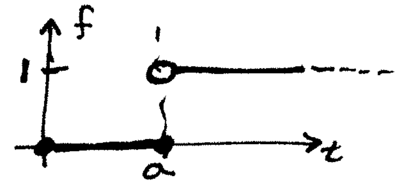
$$= \frac{1}{2}\,\frac{(s+k)+(s-k)}{(s-k)(s+k)} = \frac{1}{2}\,\frac{2s}{s^2-k^2}$$

$$\boxed{\mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2},\ s>|k|}\quad k \ne 0$$

Similarly, $\boxed{\mathcal{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2},\ s>|k|}$

Example: Let $f(t) = \begin{cases} 0, & 0 \le t \le a \\ 1, & t > a \end{cases}$

Find $\mathcal{L}\{f(t)\}$




$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^a 0 \cdot e^{-st}dt + \int_a^\infty 1 \cdot e^{-st}dt$$

$$= 0 + \left(\frac{-1}{s}e^{-st}\Big|_a^\infty\right) = \frac{-1}{s}(0 - e^{-as}) = \frac{1}{s}e^{-as}$$

$$\boxed{\mathcal{L}\{f(t)\} = \frac{1}{s}e^{-as}, \ s > 0}$$

Note: $\mathcal{L}\{f(t)\} = e^{-as}\mathcal{L}\{1\}$ (Translation property (or shifting))

---

The Laplace Transform of many common functions can be determined. There'll will be several more examples in later sections, as well as on the homework assignments and exams. Often Tables are used to find Laplace Transforms of Common functions, rather than redoing the same calculations for each problem.

So far we have the following known formulas.

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \ s > 0 \ ; \quad n = 0, 1, 2, \ldots$$

$$\mathcal{L}\{e^{kt}\} = \frac{1}{s-k}, \ s > k \ ; \ k \ne 0$$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}, \ s > 0; \ k \ne 0$$

$$\mathcal{L}\{\sin(kt)\} = \frac{k}{s^2+k^2}, \ s > 0; \ k \ne 0$$

$$\mathcal{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2}, \ s > |k|; \ k \ne 0$$

$$\mathcal{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2}, \ s > |k|; \ k \ne 0$$

See Theorem 7.1 on page 280 ~~of~~ the textbook.
An extensive list will be posted on the course web page.

## Section 7.2: Inverse Transforms and Transforms of Derivatives

Definition: If $F(s)=\mathcal{L}\{f(t)\}$, then $f(t)$ in the Inverse Laplace Transform of $F(s)$.

$$\boxed{f(t)=\mathcal{L}^{-1}\{F(s)\}}$$

Note:
$\mathcal{L}^{-1}$ is a linear transformation. $\mathcal{L}^{-1}\{aF(s)+bG(s)\}=a\mathcal{L}^{-1}\{F(s)\}+b\mathcal{L}^{-1}\{G(s)\}$
$= a f(t) + b g(t)$

---

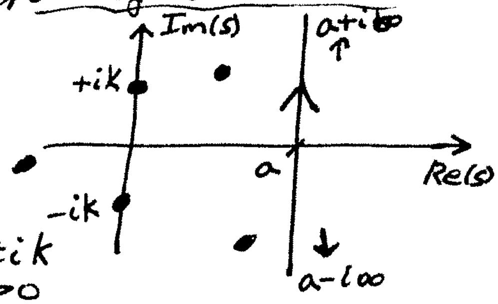
### Formula for $\mathcal{L}^{-1}\{F(s)\}$

$s$ is regarded as complex

$$f(t)=\mathcal{L}^{-1}\{F(s)\}=\frac{1}{2\pi i}\lim_{b\to\infty}\int_{a-ib}^{a+ib}F(s)e^{st}\,ds,$$

where $a$ is chosen sufficiently large that the line $Re(s)=a$ lies to the right of all singularities of $F(s)$.

e.g. $F(s)=\frac{s}{s^2+k^2}\Rightarrow s=\pm ik$
$\Rightarrow a>0$

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Instead of using the above formula for $\mathcal{L}^{-1}\{F(s)\}$, we'll simply invert the known already known Laplace Transforms.

e.g. $\mathcal{L}\{t\}=\frac{1}{s^2}\Rightarrow\mathcal{L}^{-1}\{\frac{1}{s^2}\}=t$

| $f(t)=\mathcal{L}^{-1}\{F(s)\}$ | $F(s)=\mathcal{L}\{f(t)\}$ |
|---|---|
| $t^n,\ t\geq 0$ | $n!/s^{n+1},\ s>0$ |
| $\cos(kt),\ t\geq 0$ | $\frac{s}{s^2+k^2},\ s>0$ |
| $\sin(kt),\ t\geq 0$ | $\frac{k}{s^2+k^2},\ s>0$ |

| $f(t)=\mathcal{L}^{-1}\{F(s)\}$ | $F(s)=\mathcal{L}\{f(t)\}$ |
|---|---|
| $e^{kt},\ t\geq 0$ | $\frac{1}{s-k},\ s>k$ |
| $\cosh(kt),\ t\geq 0$ | $\frac{s}{s^2-k^2},\ s>\lvert k\rvert$ |
| $\sinh(kt),\ t\geq 0$ | $\frac{k}{s^2-k^2},\ s>\lvert k\rvert$ |

To invert a Laplace Transform using a table of formulas, the transform must be written in such a way so as to match the form of one of the known formulas.

The idea is similar to that in using integral formulas.

e.g. $\int e^u du = e^u + C$

$$\Rightarrow \int e^{2x}dx=\frac{1}{2}\int \underbrace{e^{2x}}_{e^u}\cdot\underbrace{2dx}_{du}=\frac{1}{2}e^{2x}+C$$

Example: Find $\mathcal{L}^{-1}\{1/s^4\}$.

Compare $1/s^4$ to the Laplace Transforms given in the table and observe that $1/s^4$ can be written in the form $c\,\frac{n!}{s^{n+1}}$ for some $c$ and $n$.

$$1/s^4 = \frac{1}{3!}\frac{3!}{s^{3+1}} = \frac{1}{3!}\mathcal{L}\{t^3\} = \mathcal{L}\{t^3/3!\}$$

$n=3$

Invert $\Rightarrow$ $\boxed{\mathcal{L}^{-1}\{1/s^4\} = t^3/3!}$

In general, $\frac{1}{s^n} = \frac{1}{(n-1)!}\frac{(n-1)!}{s^{(n-1)+1}} = \mathcal{L}\{\frac{t^{n-1}}{(n-1)!}\}$ $n=1,2,\ldots$

Invert $\Rightarrow$ $\boxed{\mathcal{L}^{-1}\{\frac{1}{s^n}\} = \frac{t^{n-1}}{(n-1)!}}$ ← This formula can be included in the table.

---

Example: Find $\mathcal{L}^{-1}\{\frac{2s+k^2}{s^2+k^2}\}$

$$\frac{2s+k^2}{s^2+k^2} = 2\frac{s}{s^2+k^2} + k\frac{k}{s^2+k^2} = 2\mathcal{L}\{\cos(kt)\} + k\mathcal{L}\{\sin(kt)\}$$

$$\frac{2s+k^2}{s^2+k^2} = \mathcal{L}\{2\cos(kt) + k\sin(kt)\}$$

Invert $\Rightarrow$ $\boxed{\mathcal{L}^{-1}\{\frac{2s+k^2}{s^2+k^2}\} = 2\cos(kt) + k\sin(kt)}$

---

Example: Find $\mathcal{L}^{-1}\{\frac{1}{s^2-5}\}$. In general, $\mathcal{L}^{-1}\{\frac{1}{s^2-k^2}\} = \frac{1}{k}\sinh(kt)$

$$\frac{1}{s^2-5} = \frac{1}{\sqrt{5}}\frac{\sqrt{5}}{s^2-(\sqrt{5})^2} = \frac{1}{\sqrt{5}}\mathcal{L}\{\sinh(\sqrt{5}t)\} = \mathcal{L}\{\frac{1}{\sqrt{5}}\sinh(\sqrt{5}t)\}$$

Invert $\Rightarrow$ $\boxed{\mathcal{L}^{-1}\{\frac{1}{s^2-5}\} = \frac{1}{\sqrt{5}}\sinh(\sqrt{5}t)}$

---

Example: Find $\mathcal{L}^{-1}\{\frac{s-7}{(s+3)(s-2)}\}$.

$$\frac{s-7}{(s+3)(s-2)} = \frac{2}{s+3} - \frac{1}{s-2} = 2\mathcal{L}\{e^{-3t}\} - \mathcal{L}\{e^{2t}\}$$

$$\frac{s-7}{(s+3)(s-2)} = \mathcal{L}\{2e^{-3t} - e^{2t}\}$$

Invert $\Rightarrow$ $\boxed{\mathcal{L}^{-1}\{\frac{s-7}{(s+3)(s-2)}\} = 2e^{-3t} - e^{2t}}$

Partial Fractions

$$\frac{s-7}{(s+3)(s-2)} = \frac{A}{s+3} + \frac{B}{s-2}$$

$s-7 = A(s-2) + B(s+3)$

$s^1$: $A+B=1$

$s^0$: $-2A+3B=-7$

$\Rightarrow$ $B=-1$, $A=2$

## Transforms of Derivatives

$$\mathcal{L}\{f'(t)\} = \int_0^\infty f'(t)e^{-st}dt = f(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f(t)e^{-st}dt$$

$u = e^{-st}$ $\quad$ $v = f(t)$

$du = -se^{-st}dt$ $\quad$ $dv = f'(t)dt$

$$= (0 - f(0)) + s\mathcal{L}\{f(t)\}$$

$$\boxed{\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0) = sF(s) - f(0)}$$

$$\mathcal{L}\{f''(t)\} = \int_0^\infty f''(t)e^{-st}dt = f'(t)e^{-st}\Big|_0^\infty + s\int_0^\infty f'(t)e^{-st}dt$$

$u = e^{-st}$ $\quad$ $v = f'(t)$

$du = -se^{-st}dt$ $\quad$ $dv = f''(t)dt$

$$= (0 - f'(0)) + s\mathcal{L}\{f'(t)\}$$

$$\mathcal{L}\{f''(t)\} = s\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s\left[s\mathcal{L}\{f(t)\} - f(0)\right] - f'(0)$$

$$\boxed{\begin{aligned}\mathcal{L}\{f''(t)\} &= s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)\\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}}$$

In general, $\boxed{\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - s^0f^{(n-1)}(0)}$

---

**Example:** $f(t) = \cos(kt)$ $\quad$ $f(0) = 1$ $\quad$ $f''(t) = -k^2\cos(kt) = -k^2f(t)$

$f'(t) = -k\sin(kt)$ $\quad$ $f'(0) = 0$ $\quad$ $\Rightarrow \mathcal{L}\{f''(t)\} = -k^2\mathcal{L}\{f(t)\}$

$f''(t) = -k^2\cos(kt)$ $\quad$ Use the above formula to establish this equality.

$$\mathcal{L}\{f''(t)\} = s^2\mathcal{L}\{f(t)\} - sf(0) - f'(0)$$

$$= s^2 \cdot \frac{s}{s^2+k^2} - s(1) - 0$$

$$= \frac{s}{s^2+k^2}\left(s^2 - (s^2+k^2)\right) = -k^2\frac{s}{s^2+k^2} = -k^2\mathcal{L}\{f(t)\}$$

$$\Rightarrow \mathcal{L}\{f''(t)\} = -k^2\mathcal{L}\{f(t)\}.$$

# Solving Linear IVPs with Constant Coefficients

e.g. $ay' + by = g(t)$, $y(0) = y_0$

$$\text{ODE} \longrightarrow \boxed{\mathcal{L}} \longrightarrow \begin{matrix}\text{Solve}\\ \text{for } Y(s)\end{matrix} \longrightarrow \boxed{\mathcal{L}^{-1}} \longrightarrow y(t)$$

1. Apply the Laplace Transform to both sides of the ODE

$$\mathcal{L}\{ay' + by\} = \mathcal{L}\{g(t)\}$$

$$a\mathcal{L}\{y'\} + b\mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

$$a[sY(s) - y(0)] + bY(s) = G(s)$$

2. Solve for $Y(s)$: $\quad Y(s)(as+b) = G(s) + ay_0$

$$Y(s) = \frac{G(s) + ay_0}{as+b}$$

3. Invert: $\boxed{y(t) = \mathcal{L}^{-1}\{Y(s)\}}$ (Solution of the IVP)

---

**Example**: Solve the IVP $y' + y = 5\cos(2t)$, $y(0) = 2$ (could use the integrating factor method)

$\mathcal{L}$: $\quad \mathcal{L}\{y' + y\} = \mathcal{L}\{5\cos(2t)\}$ $\quad k=2$

$$[sY(s) - \underbrace{y(0)}_{=2}] + Y(s) = 5\frac{s}{s^2+4}$$

Solve for $Y(s)$: $\quad Y(s)(s+1) = \dfrac{5s}{s^2+4} + y(0) = \dfrac{5s + 2(s^2+4)}{s^2+4}$

$$\boxed{Y(s) = \frac{2s^2 + 5s + 8}{(s+1)(s^2+4)}}$$

Invert: Partial Fractions

$$Y(s) = \frac{2s^2+5s+8}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

$$2s^2 + 5s + 8 = A(s^2+4) + (Bs+C)(s+1)$$

$s^2$: $A + B = 2$ $\Rightarrow$ $C - A = 3$ $\Rightarrow$ $A - C = -3$

$s^1$: $B + C = 5$ $\qquad\qquad\qquad\qquad$ $4A + C = 8$

$s^0$: $4A + C = 8$ $\qquad\qquad\qquad\qquad$ $5A = 5$, $A = 1$

$C = 3 + A = 4$, $C = 4$

$B = 5 - C = 1$, $B = 1$

$$\Rightarrow Y(s) = \frac{1}{s+1} + \frac{s}{s^2+4} + \frac{4}{s^2+4} = \frac{1}{s+1} + \frac{s}{s^2+4} + 2\frac{2}{s^2+4}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1} + \frac{s}{s^2+4} + 2\cdot\frac{2}{s^2+4}\right\}$$

$$\boxed{y(t) = e^{-t} + \cos(2t) + 2\sin(2t)}$$

More Transform Formulas:

$$\boxed{\begin{aligned}\mathcal{L}\{e^{at}\cos(bt)\} &= \frac{s-a}{(s-a)^2+b^2}\\ \mathcal{L}\{e^{at}\sin(bt)\} &= \frac{b}{(s-a)^2+b^2}\end{aligned}}$$

---

Example: $y''-2y'+2y=e^{-t}$; $y(0)=0,\ y'(0)=-1$

$\mathcal{L}$: $\mathcal{L}\{y''\}-2\mathcal{L}\{y'\}+2\mathcal{L}\{y\}=\mathcal{L}\{e^{-t}\}$

$$[s^2Y(s)-sy(0)-y'(0)]-2[sY(s)-y(0)]+2Y(s)=\frac{1}{s+1}$$

Solve for $Y(s)$: $Y(s)(s^2-2s+2)=\frac{1}{s+1}+s\cancel{y(0)}+\underbrace{y'(0)}_{=-1}+2\cancel{y(0)}$

$$=\frac{1}{s+1}-1=\frac{1-(s+1)}{s+1}=\frac{-s}{s+1}$$

$$\boxed{Y(s)=\frac{-s}{(s+1)(s^2-2s+2)}}$$

Invert: Partial Fractions

$$Y(s)=\frac{-s}{(s+1)(s^2-2s+2)}=\frac{A}{s+1}+\frac{Bs+C}{s^2-2s+2}$$

$$-s=A(s^2-2s+2)+(Bs+C)(s+1)$$

Equate Coefficients:

$s^2$: $A+B=0 \Rightarrow B=-A$

$s^1$: $-2A+B+C=-1 \longrightarrow -2A-A-2A=-1$

$s^0$: $2A+C=0 \Rightarrow C=-2A$

$$\boxed{\begin{aligned}A&=\tfrac{1}{5}\\ B&=-1/5\\ C&=-2/5\end{aligned}}$$

$$\Rightarrow Y(s)=\frac{1}{5}\left[\frac{1}{s+1}-\frac{s+2}{s^2-2s+2}\right]$$

$s^2-2s+2=(s^2-2s+1)+2-1$
$=(s-1)^2+1$

$$Y(s)=\frac{1}{5}\left[\frac{1}{s+1}-\frac{s+2}{(s-1)^2+1}\right]$$

$$Y(s)=\frac{1}{5}\left[\frac{1}{s+1}-\frac{(s-1)+3}{(s-1)^2+1}\right]=\frac{1}{5}\left[\frac{1}{s+1}-\frac{s-1}{(s-1)^2+1}-\frac{3}{(s-1)^2+1}\right]$$

$$\boxed{Y(s)=\frac{1}{5}\left[\frac{1}{s+1}-\frac{s-1}{(s-1)^2+1}-\frac{3}{(s-1)^2+1}\right]}$$

$a=1,\ b=1$

$$\boxed{y(t)=\mathcal{L}^{-1}\{Y(s)\}=\frac{1}{5}\left[e^{-t}-e^{t}\cos t-3e^{t}\sin t\right]}$$

# Section 7.3: Operational Properties I

## First Translation Theorem

It is not convenient to evaluate the integral $\int_0^\infty f(t)e^{-st}dt$ each time a Laplace Transform is needed. The integral can be complicated/tedious even for elementary functions.

For example, consider homogeneous ODE $ay''+by'+cy=0$.

Real Distinct Roots: $y=C_1e^{r_1t}+C_2e^{r_2t}$ : $\mathcal{L}\{e^{rt}\}=\frac{1}{s-r}$

Real Repeated Roots: $y=(C_1+C_2t)e^{rt}$ : $\mathcal{L}\{t^ne^{rt}\}=?$

Complex Conjugate Roots: $(\alpha\pm i\beta)$ $y=e^{\alpha t}[C_1\cos(\beta t)+C_2\sin(\beta t)]$ : $\mathcal{L}\{e^{\alpha t}\cos(\beta t)\}=?$ $(e^{\alpha t}\sin(\beta t))$

It is helpful to derive properties of the Laplace Transform so as to handle various combinations of common functions without having to evaluate the integral $\int_0^\infty f(t)e^{-st}dt$ in each instance. The following theorem states one such useful property.

## First Translation (or Shifting) Theorem: (s-axis)

Suppose $\underset{(s>c)}{\mathcal{L}\{f(t)\}=F(s)}$ and let $a$ be any real number.

Then, $\boxed{\mathcal{L}\{e^{at}f(t)\}=\mathcal{L}\{f(t)\}_{s\to s-a}=\underset{(s-a>c)}{F(s-a)}}$

with s replaced by s-a

Proof: $\mathcal{L}\{e^{at}f(t)\}=\int_0^\infty e^{at}f(t)e^{-st}dt=\int_0^\infty f(t)e^{-(s-a)t}dt=F(s-a)$.

This formula can be used to find the Laplace Transform of the solutions in the three cases corresponding to the three types of roots of the homogeneous ODE described above.

**Example:** Find $\mathcal{L}\{t^n e^{at}\}$, $n=0,1,2,\ldots$

**Known:** $\mathcal{L}\{t^n\} = F(s) = \frac{n!}{s^{n+1}}$, $s>0$

$$\Rightarrow \mathcal{L}\{t^n e^{at}\} = \mathcal{L}\{t^n\}_{s\to s-a} = F(s-a), \quad s-a>0$$

$$\boxed{\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}, \quad s>a} \quad n=0,1,2,\ldots$$

special case: $n=0$

$\Rightarrow \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ ✓

---

**Example:** Find $\mathcal{L}\{e^{\alpha t}\cos(\beta t)\}$.

**Known:** $\mathcal{L}\{\cos(\beta t)\} = F(s) = \frac{s}{s^2+\beta^2}$, $s>0$

$$\Rightarrow \mathcal{L}\{e^{\alpha t}\cos(\beta t)\} = \mathcal{L}\{\cos(\beta t)\}_{s\to s-\alpha} = F(s-\alpha), \quad s-\alpha>0$$

$$\boxed{\mathcal{L}\{e^{\alpha t}\cos(\beta t)\} = \frac{s-\alpha}{(s-\alpha)^2+\beta^2}, \quad s>\alpha}$$

Similarly,

$$\boxed{\mathcal{L}\{e^{\alpha t}\sin(\beta t)\} = \frac{\beta}{(s-\alpha)^2+\beta^2}, \quad s>\alpha}$$

$s>|k|$

$s-\alpha>|\beta|$

$s>\alpha+|\beta|$

$$\boxed{\mathcal{L}\{e^{\alpha t}\cosh(\beta t)\} = \frac{s-\alpha}{(s-\alpha)^2-\beta^2}, \quad s>\alpha+|\beta|}$$

$$\boxed{\mathcal{L}\{e^{\alpha t}\sinh(\beta t)\} = \frac{\beta}{(s-\alpha)^2-\beta^2}, \quad s>\alpha+|\beta|}$$

---

The above formulas will typically be needed to invert Laplace Transforms.

**Example**: Solve the IVP $y''-2y'+y=1$; $y(0)=0, y'(0)=0$.

**$\mathcal{L}$**: $\mathcal{L}\{y''-2y'+y\}=\mathcal{L}\{1\}$

$$[s^2Y(s)-\cancel{sy(0)}-\cancel{y'(0)}]-2[sY(s)-\cancel{y(0)}]+Y(s)=1/s$$

**Solve for $Y(s)$**: $Y(s)(s^2-2s+1)=1/s$

$$Y(s)=\frac{1}{s(s-1)^2}$$

**Invert**: $Y(s)=\dfrac{1}{s(s-1)^2}=\dfrac{A}{s}+\dfrac{B}{s-1}+\dfrac{C}{(s-1)^2}$

$$1=A(s^2-2s+1)+Bs(s-1)+Cs$$

**Equate Coefficients**:

$s^0$: $A=1 \Rightarrow A=1$

$s^1$: $-2A-B+C=0 \Rightarrow C=1$

$s^2$: $A+B=0 \Rightarrow B=-1$

$$\Rightarrow Y(s)=\frac{1}{s}-\frac{1}{s-1}+\frac{1}{(s-1)^2}$$

$$=\frac{1}{s}-\frac{1}{s-1}+\mathcal{L}\{t\}_{s\to s-1}$$

$$y(t)=1-e^t+e^t\cdot t$$

$$\boxed{y(t)=\underbrace{(-1+t)e^t}_{\text{homogeneous}}+\underbrace{1}_{\text{particular}}}$$

**More Transform Formulas:**

$$\boxed{\begin{aligned}\mathcal{L}\{e^{at}\cos(bt)\} &= \frac{s-a}{(s-a)^2+b^2}\\ \mathcal{L}\{e^{at}\sin(bt)\} &= \frac{b}{(s-a)^2+b^2}\end{aligned}}$$

---

**Example:** $y''-2y'+2y=e^{-t}$; $y(0)=0$, $y'(0)=-1$

$\mathcal{L}$: $\mathcal{L}\{y''\}-2\mathcal{L}\{y'\}+2\mathcal{L}\{y\}=\mathcal{L}\{e^{-t}\}$

$$[s^2Y(s)-sy(0)-y'(0)]-2[sY(s)-y(0)]+2Y(s)=\frac{1}{s+1}$$

**Solve for $Y(s)$:** $Y(s)(s^2-2s+2)=\frac{1}{s+1}+\cancel{sy(0)}+\underbrace{y'(0)}_{=-1}+\cancel{2y(0)}$

$$=\frac{1}{s+1}-1=\frac{1-(s+1)}{s+1}=\frac{-s}{s+1}$$

$$\boxed{Y(s)=\frac{-s}{(s+1)(s^2-2s+2)}}$$

**Invert:** Partial Fractions

$$Y(s)=\frac{-s}{(s+1)(s^2-2s+2)}=\frac{A}{s+1}+\frac{Bs+C}{s^2-2s+2}$$

$$-s=A(s^2-2s+2)+(Bs+C)(s+1)$$

**Equate coefficients:**

$s^2$: $A+B=0 \Rightarrow B=-A$

$s^1$: $-2A+B+C=-1 \longrightarrow -2A-A-2A=-1$

$s^0$: $2A+C=0 \Rightarrow C=-2A$

$$\boxed{\begin{aligned}A&=\tfrac{1}{5}\\ B&=-\tfrac{1}{5}\\ C&=-\tfrac{2}{5}\end{aligned}}$$

$$\Rightarrow Y(s)=\frac{1}{5}\left[\frac{1}{s+1}-\frac{s+2}{s^2-2s+2}\right]$$

$s^2-2s+2=(s^2-2s+1)+2-1$
$=(s-1)^2+1$

$$Y(s)=\frac{1}{5}\left[\frac{1}{s+1}-\frac{s+2}{(s-1)^2+1}\right]$$

$$Y(s)=\frac{1}{5}\left[\frac{1}{s+1}-\frac{(s-1)+3}{(s-1)^2+1}\right]=\frac{1}{5}\left[\frac{1}{s+1}-\frac{s-1}{(s-1)^2+1}-\frac{3}{(s-1)^2+1}\right]$$

$$\boxed{Y(s)=\frac{1}{5}\left[\frac{1}{s+1}-\frac{s-1}{(s-1)^2+1}-\frac{3}{(s-1)^2+1}\right]}$$

$a=1,\ b=1$

$$\boxed{y(t)=\mathcal{L}^{-1}\{Y(s)\}=\frac{1}{5}\left[e^{-t}-e^{t}(\cos t+3\sin t)\right]}$$

# Alternative Method for Inverting $Y(s)$

$$Y(s) = \frac{-s}{(s+1)(s^2-2s+2)} = \frac{-s}{(s+1)(s-(1+i))(s-(1-i))} = \frac{A}{s+1} + \frac{B}{s-(1+i)} + \frac{C}{s-(1-i)}$$

$$s = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$$-s = A(s^2-2s+2) + B(s+1)(s-(1-i)) + C(s+1)(s-(1+i))$$

(0) $s^0$: $2A - (1-i)B - (1+i)C = 0$

(1) $s^1$: $-2A + iB - iC = -1$

(2) $s^2$: $A + B + C = 0 \Rightarrow$ $A = -(B+C)$

$2\times(2) + (1) \Rightarrow (2+i)B + (2-i)C = -1$

$-2\times(2) + (0) \Rightarrow (3-i)B + (3+i)C = 0 \Rightarrow B = -\frac{3+i}{3-i}C = -\frac{9+6i-1}{9+1} = -\frac{4+3i}{5}C$

$$B = -\frac{4+3i}{5}C$$

$$-(2+i)\frac{(4+3i)}{5}C + (2-i)C = -1$$

$$-(1+2i)C + (2-i)C = -1$$

$$(1-3i)C = -1$$

$$C = \frac{-1}{1-3i}\cdot\frac{1+3i}{1+3i} = -\frac{1}{10}(1+3i)$$

$$C = \frac{-1}{10}(1+3i)$$

$$B = -\frac{4+3i}{5}\cdot\frac{-1}{10}(1+3i) = \frac{1}{50}(-5+15i)$$

$$B = \frac{-1}{10}(1-3i)$$

$$A = \frac{1}{10}(1-3i+1+3i) = \frac{1}{5}$$

$$A = \frac{1}{5}$$

$$\Rightarrow \boxed{Y(s) = \frac{1/5}{s+1} - \frac{1}{10}\left(\frac{1-3i}{s-(1+i)} + \frac{1+3i}{s-(1-i)}\right)}$$

Invert:

$$y(t) = \frac{1}{5}e^{-t} - \frac{1}{10}\left((1-3i)e^{(1+i)t} + (1+3i)e^{(1-i)t}\right)$$

$$= \frac{1}{5}e^{-t} - \frac{e^t}{10}\left((1-3i)(\cos t + i\sin t) + (1+3i)(\cos t - i\sin t)\right)$$

$$= \frac{1}{5}e^{-t} - \frac{e^t}{10}\left[(\cos t + 3\sin t + \cos t + 3\sin t) + i(\cancel{\sin t} \cancel{- 3\cos t} \cancel{+ 3\cos t} \cancel{- \sin t})\right]$$

$$= \frac{1}{5}e^{-t} - \frac{e^t}{10}(2\cos t + 6\sin t)$$

$$\boxed{y(t) = \frac{1}{5}\left[e^{-t} - e^t(\cos t + 3\sin t)\right]}$$

**Example:** $g(t) = e^{-2t}\left[-4\cos(3t) + 3\sin(3t)\right]$

Find the Laplace Transform and invert back.

$$g(t) \xrightarrow{\mathcal{L}} G(s) \xrightarrow{\mathcal{L}^{-1}} g(t)$$

$$\mathcal{L}\{e^{at}f(t)\} = \mathcal{L}\{f(t)\}_{s \to s-a} = F(s-a)$$

**Transform:**

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{\underbrace{e^{-2t}}_{a=-2}\underbrace{\left[-4\cos(3t) + 3\sin(3t)\right]}_{f(t)}\} = \mathcal{L}\{-4\cos(3t) + 3\sin(3t)\}_{s \to s+2}$$

$$= -4\frac{s}{s^2+9} + 3\frac{3}{s^2+9}\Bigg|_{s \to s+2} = -4\frac{s+2}{(s+2)^2+9} + 3\cdot\frac{3}{(s+2)^2+9}$$

$$= \frac{-4s-8+9}{(s+2)^2+9} = \frac{1-4s}{s^2+4s+4+9} = \frac{1-4s}{s^2+4s+13}$$

$$\boxed{G(s) = \mathcal{L}\{g(t)\} = \frac{1-4s}{s^2+4s+13}}$$

**Invert:**

$$G(s) = \frac{1-4s}{s^2+4s+13} = \frac{1-4s}{(s^2+4s+4)+9} = \frac{1-4s}{(s+2)^2+9} = \frac{-4(s+2)+1+8}{(s+2)^2+9}$$

$$= -4\frac{s+2}{(s+2)^2+9} + 3\frac{3}{(s+2)^2+9}$$

$$= -4\mathcal{L}\{\cos(3t)\}_{s \to s+2} + 3\mathcal{L}\{\sin(3t)\}_{s \to s+2}$$

$$= -4\mathcal{L}\{e^{-2t}\cos(3t)\} + 3\mathcal{L}\{e^{-2t}\sin(3t)\}$$

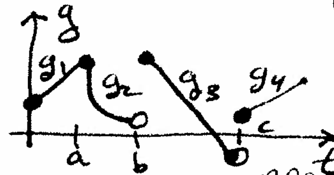
$$G(s) = \frac{1-4s}{s^2+4s+13} = \mathcal{L}\{e^{-2t}\left[-4\cos(3t) + 3\sin(3t)\right]\}$$

$$\boxed{g(t) = \mathcal{L}^{-1}\{G(s)\} = e^{-2t}\left[-4\cos(3t) + 3\sin(3t)\right]}$$

# Second Translation Theorem (STT)

The SST is useful in treating piecewise defined forcing functions, $g(t)$.

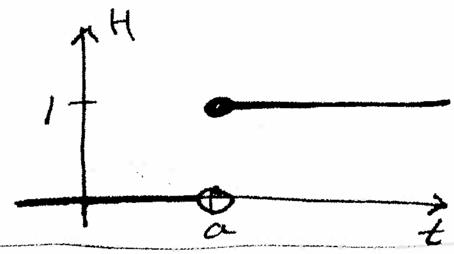
$$ay''+by'+cy=g(t)$$




The so-called Heaviside Unit Step Function is used to ~~define~~ express piecewise defined ~~in~~ as a compact single expression, rather than $g(t)=\begin{cases} g_1 & 0\le t<a \\ g_2 & a\le t<b \\ g_3 & b\le t<c \\ g_4 & c\le t<d \end{cases}$.

## Definition: Heaviside Unit Step Function (H)

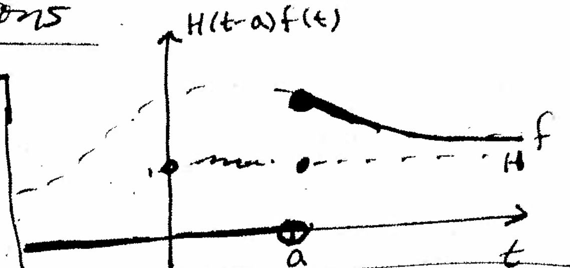
$$\boxed{H(t-a)=\begin{cases} 0, & t-a<0 \quad (t<a) \\ 1, & t-a\ge 0 \quad (t\ge a) \end{cases}}$$




## "Turning on" Functions

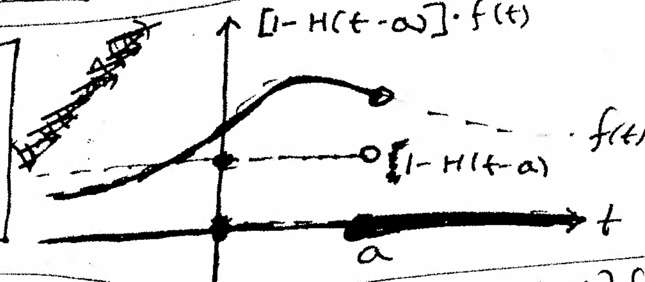
$$H(t-a)\cdot f(t)=\begin{cases} 0, & t<a \\ f(t), & t\ge a \end{cases}$$

"Turns on" f for $t\ge a$



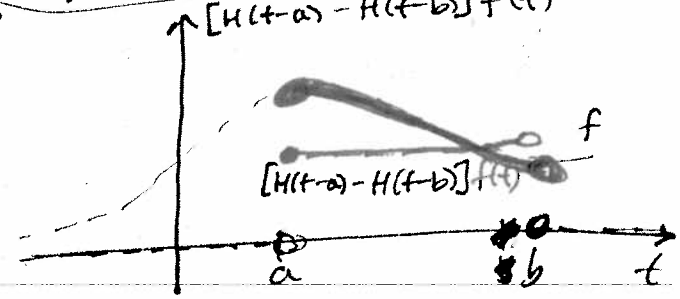

$$[1-H(t-a)]\cdot f(t)=\begin{cases} f(t), & t<a \\ 0, & t\ge a \end{cases}$$

"Turns on" f for $t<a$

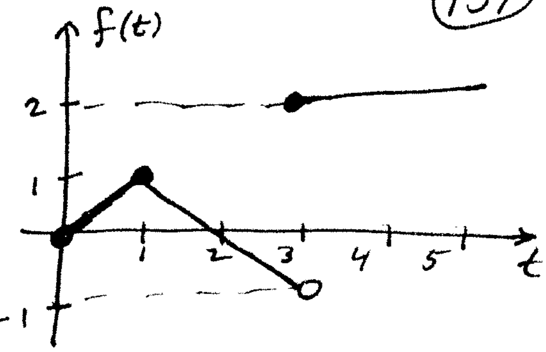



$$[H(t-a)-H(t-b)]f(t)=\begin{cases} 0, & -\infty<t<a \\ f(t), & a\le t<b \\ 0, & b\le t<\infty \end{cases}$$

"Turns on" f for $a\le t<b$



Example: Let $f(t) = \begin{cases} t, & 0 \le t < 1 \\ 2-t, & 1 \le t < 3 \\ 2, & 3 \le t < \infty \end{cases}$




turn on for $0 \le t < 1$

turn on for $1 \le t < 3$

$$f(t) = [H(t-0) - H(t-1)]\,t + [H(t-1) - H(t-3)](2-t) + H(t-3)\cdot 2$$

turn on for $3 \le t < \infty$

Note: $H(t-0) = H(t) = 1$ for all $t \ge 0$

If $f$ is defined for only $t \ge 0$, as is the case in which we're interested, then we may take $H(t)$ to be equal to 1 since $H(t) = 1$ for all $t \ge 0$, and the value of $H(t)$ for $t < 0$ is irrelevant

$$\Rightarrow \quad f(t) = \underbrace{H(t)}_{=1 \text{ for } t \ge 0}t + H(t-1)\big((2-t) - t\big) + H(t-3)\big(2 - (2-t)\big), \quad t \ge 0$$

$$\boxed{f(t) = t + H(t-1)\cdot 2(1-t) + H(t-3)\cdot t}, \quad t \ge 0$$

Jump locations

$\underline{0 \le t < 1:}$ $\quad f(t) = t + 0 + 0 = t$

$\underline{1 \le t < 3:}$ $\quad f(t) = t + 2(1-t) + 0 = 2-t$

$\underline{3 \le t < \infty:}$ $\quad f(t) = t + 2(1-t) + t = 2$

## Second Translation Theorem (STT) $\mathcal{L}\{H(t-a)f(t)\} = ?$

Suppose $a>0$.

$$\mathcal{L}\{H(t-a)f(t)\} = \int_0^\infty H(t-a)f(t)\cdot e^{-st}dt = \int_a^\infty f(t)e^{-st}dt$$

$$= \int_0^\infty f(\tau+a)e^{-s(\tau+a)}d\tau$$

$\tau = t-a$ $t=\tau+a$
$d\tau = dt$
$t=a \Rightarrow \tau=0$
$t=\infty \Rightarrow \tau=\infty$

$$= e^{-as}\int_0^\infty f(\tau+a)e^{-s\tau}d\tau$$

$\tau \to t$

$$= e^{-as}\int_0^\infty f(t+a)e^{-st}dt = e^{-as}\mathcal{L}\{f(t+a)\}$$

$$\boxed{\mathcal{L}\{H(t-a)f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\},\ a>0}$$ transform forward

This formula is convenient for finding forward LTs, but it is typically not so convenient for finding Inverse LTs. Rather, the formula can be written as...

$$\boxed{\mathcal{L}\{H(t-a)\cdot f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\},\ a>0}$$ transform inverse

---

**Special Case:** $\mathcal{L}\{H(t-a)\cdot f(t)\} = e^{-as}\mathcal{L}\{f(t+a)\}$

Take $f(t)=1 \Rightarrow \mathcal{L}\{H(t-a)\} = e^{-as}\mathcal{L}\{1\} = e^{-as}\cdot\frac{1}{s}$

$$\boxed{\mathcal{L}\{H(t-a)\} = e^{-as}/s}$$

**Example:** $G(s) = \dfrac{1-e^{-2s}}{s^2}$. Find $g(t) = \mathcal{L}^{-1}\{G(s)\}$

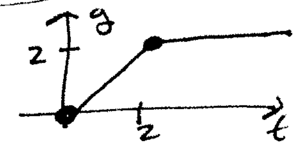
$$G(s) = \frac{1}{s^2} - \frac{e^{-2s}}{s^2} = \mathcal{L}\{t\} - e^{-2s}\mathcal{L}\{t\}$$

$e^{-as}\mathcal{L}\{f(t)\} = \mathcal{L}\{H(t-a)f(t-a)\}$; $a=2$ $f(t)=t$

$f(t-a) = f(t-2) = t-2$

$$= \mathcal{L}\{t\} - \mathcal{L}\{H(t-2)\cdot(t-2)\}$$

$$\boxed{g(t) = t - H(t-2)\cdot(t-2) = \begin{cases} t, & 0 \le t < 2 \\ 2, & 2 \le t < \infty \end{cases}}$$

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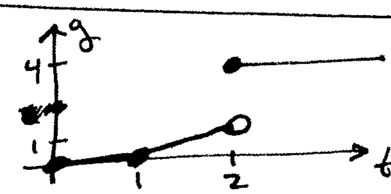
**Example:** Let $g(t) = \begin{cases} 0; & 0 \le t < 1 \\ t-1; & 1 \le t < 2 \\ 4; & 2 \le t < \infty \end{cases}$




Find $G(s) = \mathcal{L}\{g(t)\}$, and invert back, $g(t) = \mathcal{L}^{-1}\{G(s)\}$.

**Transform $g(t)$:** We must first write $g(t)$ in terms of Heaviside functions

$$g(t) = [H(t-1) - H(t-2)]\cdot(t-1) + H(t-2)\cdot 4 \qquad -t+\cancel{1}+4$$

$$g(t) = H(t-1)\cdot(t-1) - H(t-2)(t-1-4)$$

$$g(t) = H(t-1)(t-1) - H(t-2)(t-5)$$

Then,

$$G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{H(t-1)(t-1)\} - \mathcal{L}\{H(t-2)(t-5)\} = e^{-s}\underbrace{\mathcal{L}\{t\}}_{f(t+a)} - e^{-2s}\underbrace{\mathcal{L}\{t-3\}}_{f(t+a)}$$

$a=1$ $f(t)=t-1$ $\quad$ $a=2$ $f(t)=t-5$

$f(t+a) = (t+1)-1 = t$ $\quad$ $f(t+a) = (t+2)-5 = t-3$

$$\boxed{G(s) = e^{-s}\cdot\frac{1}{s^2} - e^{-2s}\left(\frac{1}{s^2} - \frac{3}{s}\right)}$$

$e^{-as}\mathcal{L}\{f(t)\} = \mathcal{L}\{H(t-a)f(t-a)\}$

$f(t)=t$, $a=1$ $\quad$ $f(t-1)=t-1$ $\quad$ $f(t)=t-3$, $a=2$ $\quad$ $f(t-2)=t-5$

**Invert:** $g(t) = \mathcal{L}^{-1}\left\{e^{-s}/s^2 - e^{-2s}(1/s^2 - 3/s)\right\}$

$$G(s) = e^{-s}/s^2 - e^{-2s}(1/s^2 - 3/s) = e^{-s}\mathcal{L}\{t\} - e^{-2s}\mathcal{L}\{t-3\}$$

$$= \mathcal{L}\{H(t-1)\cdot(t-1)\} - \mathcal{L}\{H(t-2)\cdot(t-5)\}$$

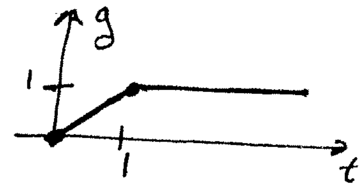
$$\Rightarrow \boxed{g(t) = H(t-1)(t-1) - H(t-2)(t-5) = \begin{cases} 0, & 0 \le t < 1 \\ t-1; & 1 \le t < 2 \\ 4, & 2 \le t < \infty \end{cases}}$$

$0 \le t < 1$: $g = 0 - 0 = 0$

$1 \le t < 2$: $g = (t-1) - 0 = t-1$

$2 \le t < \infty$: $g = (t-1) - (t-5) = 4$

**Example:** $y'' + y = g(t) = \begin{cases} t, & 0 \le t < 1 \\ 1 & 1 \le t < \infty \end{cases}$




$y(0) = 0, \; y'(0) = 0$

$g(t) = [H(t-0) - H(t-1)] \cdot t + H(t-1) \cdot (1)$

$\rightarrow H(t-0) = 1$ for all $t \ge 0$

$a = 1$
$f(t) = 1 - t$

$f(t+a) = 1 - (t+1) = t$ $\cancel{t}$

$g(t) = t + H(t-1)(1-t)$

$\mathcal{L}\{g(t)\} = \mathcal{L}\{t\} + \mathcal{L}\{H(t-1)(1-t)\} = \frac{1}{s^2} + e^{-s}\mathcal{L}\{1-(t+1)\}$

$= \frac{1}{s^2} + e^{-s}\mathcal{L}\{-t\} = \frac{1}{s^2} - e^{-s}/s^2 = \frac{1-e^{-s}}{s^2}$

$\mathcal{L}\{g(t)\} = \frac{1-e^{-s}}{s^2}$

$\mathcal{L}\{y'' + y\} = \mathcal{L}\{g(t)\}$

$[s^2 Y(s) - \cancel{s y(0)} - \cancel{y'(0)}] + [Y(s)] = \mathcal{L}\{g(t)\}$

$Y(s)(s^2+1) = \frac{1 - e^{-s}}{s^2}$

$\frac{1}{s^2(s^2+1)} = \frac{A}{s^2} + \frac{B}{s^2+1}$

linear in $s^2$

$1 = A(s^2+1) + Bs^2$

$A + B = 0$

$w = s^2$ $\frac{1}{w(w+1)}$

$A = 1$
$B = -1$

$Y(s) = (1 - e^{-s}) \cdot \frac{1}{s^2(s^2+1)} = (1 - e^{-s})\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)$

$= (1 - e^{-s})\mathcal{L}\{t - \sin t\}$ $\quad a = 1$

$= \mathcal{L}\{t - \sin t\} - e^{-s}\mathcal{L}\{t - \sin t\}$

$Y(s) = \mathcal{L}\{t - \sin t\} - \mathcal{L}\{H(t-1)((t-1) - \sin(t-1))\}$

Invert $\Rightarrow$ $\boxed{y(t) = t - \sin t - H(t-1)((t-1) - \sin(t-1))}$

OR $\quad 0 \le t < 1$: $y = t - \sin t - 0 = t - \sin t$

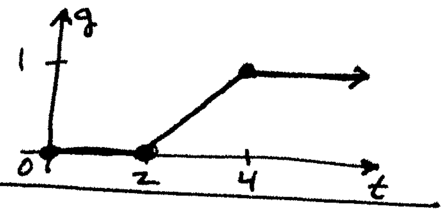
$1 \le t < \infty$: $y = \cancel{t} - \sin t - (\cancel{t}-1) + \sin(t-1)$

$y = 1 - \sin t + \sin(t-1)$

$$\boxed{y(t) = \begin{cases} t - \sin t, & 0 \le t < 1 \\ 1 - \sin t + \sin(t-1), & 1 \le t < \infty \end{cases}}$$

**Example**: Solve the IVP $y''+y = g(t) = \begin{cases} 0, & 0 \le t < 2 \\ \frac{1}{2}(t-2), & 2 \le t < 4 \\ 1, & t \ge 4 \end{cases}$

$y(0)=0,\ y'(0)=0$




$$g(t) = [H(t-2) - H(t-4)]\cdot \tfrac{1}{2}(t-2) + H(t-4)$$
$$= \tfrac{1}{2}H(t-2)\cdot(t-2) + H(t-4)\left(-\tfrac{1}{2}(t-2)+1\right)$$
$$= \tfrac{1}{2}H(t-2)(t-2) + H(t-4)\left(-\tfrac{1}{2}(t-2-2) - \cancel{1} + \cancel{1}\right)$$
$$g(t) = \tfrac{1}{2}\left[H(t-2)(t-2) - H(t-4)(t-4)\right]$$

$$\mathcal{L}\{g(t)\} = \tfrac{1}{2}\left[\mathcal{L}\{H(t-2)(t-2)\} - \mathcal{L}\{H(t-4)(t-4)\}\right]$$
$$= \tfrac{1}{2}\left[e^{-2s}\mathcal{L}\{t\} - e^{-4s}\mathcal{L}\{t\}\right] = \tfrac{1}{2}\left[\frac{e^{-2s}}{s^2} - \frac{e^{-4s}}{s^2}\right]$$
$$\boxed{\mathcal{L}\{g(t)\} = \frac{1}{2s^2}\left(e^{-2s} - e^{-4s}\right)}$$

$\underline{\mathcal{L}\{ODE\}}$: $\mathcal{L}\{y''+y\} = \mathcal{L}\{g(t)\}$

$$[s^2Y(s) - \cancel{sy(0)} - \cancel{y'(0)}] + Y(s) = \frac{1}{2s^2}\left(e^{-2s} - e^{-4s}\right)$$
$$Y(s)(s^2+1) = \frac{1}{2s^2}\left(e^{-2s} - e^{-4s}\right)$$
$$Y(s) = \tfrac{1}{2}\left(e^{-2s} - e^{-4s}\right)\frac{1}{s^2(s^2+1)}$$
$$= \tfrac{1}{2}\left(e^{-2s} - e^{-4s}\right)\left(\frac{1}{s^2} - \frac{1}{s^2+1}\right)$$
$$= \tfrac{1}{2}\left(e^{-2s} - e^{-4s}\right)\mathcal{L}\{t - \sin t\}$$

Partial Fractions

$$\frac{1}{s^2(s^2+1)} = \frac{A}{s^2} + \frac{B}{s^2+1}$$

↑ ↑ linear in $s^2$ $\quad 1 = A(s^2+1) + Bs^2$

$s^0$: $A = 1$

$s^1$: $B = -A = -1$

$$Y(s) = \tfrac{1}{2}\left[e^{-2s}\mathcal{L}\{t-\sin t\} - e^{-4s}\mathcal{L}\{t-\sin t\}\right]$$

$a_1 = 2$, $f_1(t) = t - \sin t$ $\qquad$ $a_2 = 4$, $f_2(t) = t - \sin t$

$$Y(s) = \tfrac{1}{2}\left[\mathcal{L}\{H(t-2)f_1(t-2)\} - \mathcal{L}\{H(t-4)f_2(t-4)\}\right]$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \tfrac{1}{2}\left[H(t-2)f_1(t-2) - H(t-4)f_2(t-4)\right]$$

$$\boxed{y(t) = \tfrac{1}{2}\left[H(t-2)\left((t-2) - \sin(t-2)\right) - H(t-4)\left((t-4) - \sin(t-4)\right)\right]}$$

$$\Rightarrow \boxed{y(t) = \begin{cases} 0, & 0 \le t < 2 \\ \frac{1}{2}\left[(t-2) - \sin(t-2)\right], & 2 \le t < 4 \\ \frac{1}{2}\left[2 - \sin(t-2) + \sin(t-4)\right], & t \ge 4 \end{cases}}$$

# More Properties of the Laplace Transform

Derivatives of Transforms: $\mathcal{L}\{t^n f(t)\} = ?$

Let $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt$.

Then, $F'(s) = \frac{d}{ds}\int_0^\infty f(t)\, e^{-st}\, dt = \int_0^\infty f(t)\, \frac{d}{ds}(e^{-st})\, dt$

$= \int_0^\infty f(t)(-t e^{-st})\, dt = -\int_0^\infty t f(t)\, e^{-st} = -\mathcal{L}\{t f(t)\}$

$F'(s) = -\mathcal{L}\{t f(t)\}$.

In general, $F^{(n)}(s) = \frac{d^n}{ds^n}\int_0^\infty f(t) e^{-st}\, dt = \int_0^\infty f(t)\, \frac{d^n}{ds^n}(e^{-st})\, dt = \int_0^\infty f(t)\left((-t)^n e^{-st}\right) dt$

$= (-1)^n \int_0^\infty t^n f(t)\, e^{-st}\, dt = (-1)^n \mathcal{L}\{t^n f(t)\}$

$F^{(n)}(s) = (-1)^n \mathcal{L}\{t^n f(t)\}$

$\Rightarrow \boxed{\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s)}$ $\quad F^{(n)}(s) = \frac{d^n}{ds^n} F(s)$

---

Example: Find $\mathcal{L}\{t^2 e^{3t}\}$

First Translation Theorem: $\mathcal{L}\{t^2 e^{3t}\} = \mathcal{L}\{t^2\}_{s \to s-3} = \left.\frac{2}{s^3}\right|_{s \to s-3} = \boxed{\frac{2}{(s-3)^3}}$

Above Formula: $\mathcal{L}\{t^2 \underset{\uparrow\, f(t)}{e^{3t}}\} = (-1)^2 \frac{d^2}{ds^2}\mathcal{L}\{e^{3t}\} = \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right) = \frac{d}{ds}\left(\frac{-1}{(s-3)^2}\right)$

$F(s) = \frac{1}{s-3}$

$= \boxed{\frac{2}{(s-3)^3}}$

---

Example: Find $\mathcal{L}\{t\cos(kt)\}$

$y'' + y = \cos t$
$y_p = t[A\cos t + B\sin t]$

$\mathcal{L}\{t\cos(kt)\} = (-1)^1 \frac{d^1}{ds^1}\mathcal{L}\{\cos(kt)\} = -\frac{d}{ds}\left(\frac{s}{s^2+k^2}\right) = -\frac{(s^2+k^2)(1) - (s)(2s)}{(s^2+k^2)^2} = -\frac{k^2 - s^2}{(s^2+k^2)^2}$

$f(t) = \cos(kt) \quad F(s) = \frac{s}{s^2+k^2}$

$\boxed{\mathcal{L}\{t\cos(kt)\} = \frac{s^2 - k^2}{(s^2+k^2)^2}}$

Similarly, $\boxed{\mathcal{L}\{t\sin(kt)\} = \frac{2ks}{(s^2+k^2)^2}}$

# Convolution

**Definition:** Let $f$ and $g$ be piecewise continuous functions on $[0,\infty)$.
The convolution of $f$ and $g$, denoted by $f*g$, is defined by

$$\boxed{(f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau} \quad \text{(function of } t\text{)}$$

**Note:** $(f*g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_t^0 f(t-\eta)g(\eta)\cdot -d\eta = \int_0^t g(\eta)\,f(t-\eta)\,d\eta$

$\eta = t-\tau$
$\tau = t-\eta$
$d\tau = -d\eta$

$$= (g*f)(t)$$

$$\Rightarrow \boxed{f*g = g*f}$$

---

**Convolution Theorem:** $\boxed{\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\{f(t)\}\cdot\mathcal{L}\{g(t)\}}$

OR $\boxed{\mathcal{L}\{f*g(t)\} = F(s)\cdot G(s)}$

This formula allows us to invert a product of transforms.

$$\circledast \quad \boxed{\mathcal{L}^{-1}\{F(s)\,G(s)\} = (f*g)(t)} \quad \circledast$$

---

**Example:** Find $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$

**Method 1:** Partial Fractions

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1-\cos t}$$

**method 2:** Convolution

$$\frac{1}{s(s^2+1)} = \frac{1}{s}\cdot\frac{1}{s^2+1} = \mathcal{L}\{1\}\cdot\mathcal{L}\{\sin t\}$$

$F(s)=\frac{1}{s}$ $\quad G(s)=\frac{1}{s^2+1}$ $\quad f(t)=1$ $\quad g(t)=\sin t$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\{\mathcal{L}\{1\}\cdot\mathcal{L}\{\sin t\}\}$$
$$= 1*\sin t = \sin t * 1$$
$$= \int_0^t 1\cdot\sin(t-\tau)\,d\tau = \int_0^t \sin\tau\cdot 1\,d\tau$$
$$= -\cos\tau\Big|_0^t = -(\cos t - 1)$$

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1-\cos t}$$

**Example:** Solve the IVP $\begin{array}{l} y''+4y = g(t) \\ y(0)=3,\ y'(0)=0 \end{array}$

$$\mathcal{L}\{y''+4y\} = \mathcal{L}\{g(t)\}$$

$$[s^2Y(s) - sy(0) - \cancel{y'(0)}] + 4[Y(s)] = G(s)$$

$$Y(s)\cdot(s^2+4) = 3s + G(s)$$

$$Y(s) = 3\,\frac{s}{s^2+4} + G(s)\cdot\frac{1}{s^2+4}$$

$$Y(s) = 3\mathcal{L}\{\cos(2t)\} + \mathcal{L}\{g(t)\}\cdot\mathcal{L}\{\sin(2t)\}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \cancel{\mathcal{L}^{-1}}\{3\cancel{\mathcal{L}}\{\cos(2t)\}\} + \mathcal{L}^{-1}\{\mathcal{L}\{g(t)\}\cdot\mathcal{L}\{\sin(2t)\}\}$$

$$= 3\cos(2t) + g(t) * \sin(2t)$$

$$\boxed{y(t) = \underbrace{3\cos(2t)}_{\text{homogeneous}} + \underbrace{\int_0^t g(\tau)\sin(2(t-\tau))\,d\tau}_{\text{particular}}}$$

OR $\displaystyle\int_0^t \sin(2\tau)\,g(t-\tau)\,d\tau$

# Transforms of Integrals

$$\mathcal{L}\{f*g\} = \mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)\,d\tau\right\} = F(s)\,G(s)$$

Let $g(t)=1$
$\rightarrow G(s) = 1/s \implies \boxed{\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{F(s)}{s}}$

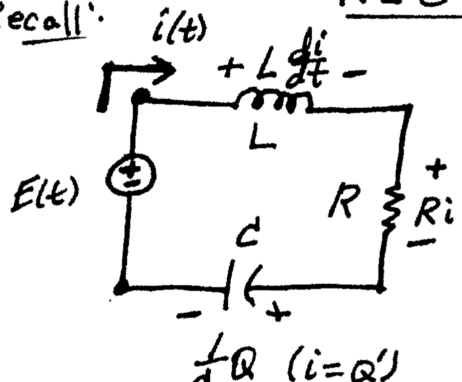
$$\mathcal{L}\left\{\int_a^t f(\tau)\,d\tau\right\} = \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau - \int_0^a f(\tau)\,d\tau\right\} = \frac{F(s)}{s} - \frac{1}{s}\int_0^a f(\tau)\,d\tau$$

$$\boxed{\mathcal{L}\left\{\int_a^t f(\tau)\,d\tau\right\} = \frac{1}{s}\left[F(s) - \int_0^a f(\tau)\,d\tau\right]}$$

---

# RLC Circuits

Recall:




KVL

$$L\frac{di}{dt} + Ri + \frac{1}{C}Q = E(t)$$

$$LQ'' + RQ' + \frac{1}{C}Q = E(t)$$

$$LQ''' + RQ'' + \frac{1}{C}Q' = E'(t)$$

$$Li'' + Ri' + \frac{1}{C}i = E'(t)$$

Same as in Section 5.1.

$$i(t) = Q'(t)$$

$$\int_0^t i(\tau)\,d\tau = Q(t) - Q(0)$$

$$Q(t) = Q(0) + \int_0^t i(\tau)\,d\tau$$

$$L\frac{di}{dt} + Ri + \frac{1}{C}\left[Q(0) + \int_0^t i(\tau)\,d\tau\right] = E(t)$$

$$\boxed{L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(\tau)\,d\tau = E(t) - \frac{Q(0)}{C}}$$

$\mathcal{L}\{ODE\}$:

$$L\left[sI(s) - i(0)\right] + RI(s) + \frac{I(s)}{Cs} = \mathcal{L}\left\{E(t) - \frac{Q(0)}{C}\right\}$$

$$I(s)\left(Ls + R + \frac{1}{Cs}\right) = \mathcal{L}\{E(t)\} - \frac{Q(0)}{Cs} + L\,i(0)$$

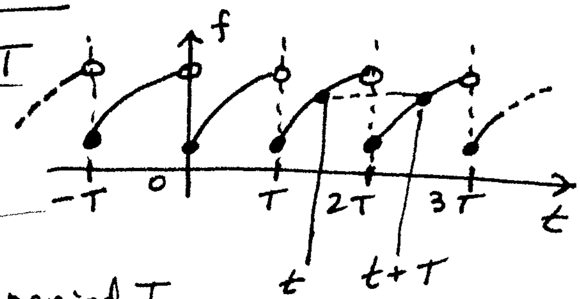
$$I(s)\left(Ls^2 + Rs + 1/C\right) = s\,\mathcal{L}\{E(t)\} - \frac{Q(0)}{C} + sL\,i(0)$$

$$I(s) = \frac{s\mathcal{L}\{E(t)\} + i(0)Ls + Q(0)/C}{Ls^2 + Rs + 1/C}$$

Then, $i(t) = \mathcal{L}^{-1}\{I(s)\}$

# Transforms of Periodic Functions

**Definition:** A function $f$ is periodic with period $T$ if $f(t+T) = f(t)$ for all $t$.




Let $f$ be a periodic function for $t \geq 0$ with period $T$.

$$\Rightarrow \boxed{f(t+T) = f(t) \text{ for all } t \geq 0}$$

Then,

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt = \int_0^T f(t)e^{-st}dt + \int_T^\infty f(t)e^{-st}dt$$

$$= I + \int_T^\infty f(t)e^{-st}dt, \text{ where } \boxed{I = \int_0^T f(t)e^{-st}dt}$$

Let $\tau = t - T$.

$$\Rightarrow t = \tau + T \qquad t = T \Rightarrow \tau = 0$$
$$dt = d\tau \qquad t = \infty \Rightarrow \tau = \infty$$

$$= I + \int_0^\infty f(\tau+T)e^{-s(\tau+T)}d\tau$$

$$= I + e^{-sT}\int_0^\infty f(\tau+T)e^{-s\tau}d\tau \qquad (f(\tau+T) = f(\tau))$$

$$= I + e^{-sT}\int_0^\infty f(\tau)e^{-s\tau}d\tau$$

$$\mathcal{L}\{f(t)\} = I + e^{-sT}\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f(t)\}(1 - e^{-sT}) = I$$

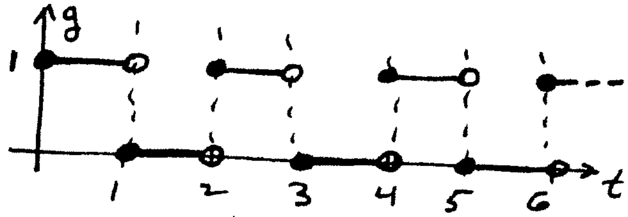
$$\mathcal{L}\{f(t)\} = \frac{I}{1-e^{-sT}}$$

$$\Rightarrow \boxed{\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}dt}$$

If $f$ is period, the Laplace Transform may be found by integrating over one period of $f$.

Example: Square Wave: $g(t) = \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t < 2 \end{cases}$ ; $g(t+2) = g(t)$ for $t \ge 0$.

Find $G(s) = \mathcal{L}\{g(t)\}$.




$g(t)$ is periodic with period $\underline{\underline{T=2}}$.

$$G(s) = \mathcal{L}\{g(t)\} = \frac{1}{1-e^{-2s}} \int_0^2 g(t)\, e^{-st}\, dt$$

$$= \frac{1}{1-e^{-2s}} \left[ \int_0^1 1 \cdot e^{-st}\, dt + \int_1^2 0 \cdot e^{-st}\, dt \right]$$

$$= \frac{1}{1-e^{-2s}} \left[ \frac{-1}{s} e^{-st} \Big|_0^1 + 0 \right]$$

$$= \frac{1}{1-e^{-2s}} \left[ \frac{-1}{s} (e^{-s} - 1) \right] = \frac{1-e^{-s}}{s(1-e^{-2s})}$$

$$= \frac{\cancel{1-e^{-s}}}{s(1+e^{-s})\cancel{(1-e^{-s})}} = \frac{1}{s(1+e^{-s})}$$

$$\boxed{G(s) = \mathcal{L}\{g(t)\} = \frac{1}{s(1+e^{-s})}}$$

Recall: Geometric Series

$$\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$$

Invert: $G(s) = \frac{1}{s} \cdot \frac{1}{1+e^{-s}} = \frac{1}{s}(1 - e^{-s} + e^{-2s} - e^{-3s} + \cdots) \leftarrow r = -e^{-s}$

$$= \frac{1}{s} - e^{-s} \cdot \frac{1}{s} + e^{-2s} \cdot \frac{1}{s} - e^{-3s} \cdot \frac{1}{s} + \cdots$$

$$= \mathcal{L}\{1\} - e^{-s}\mathcal{L}\{1\} + e^{-2s}\mathcal{L}\{1\} - e^{-3s}\mathcal{L}\{1\} + \cdots$$

Second Translation Theorem $\Rightarrow$ $G(s) = \mathcal{L}\{1\} - \mathcal{L}\{H(t-1)\} + \mathcal{L}\{H(t-2)\} - \mathcal{L}\{H(t-3)\} + \cdots$

$$\Rightarrow \boxed{g(t) = 1 - H(t-1) + H(t-2) - H(t-3) + \cdots}$$

$0 \le t < 1 \Rightarrow g(t) = 1 - 0 + 0 - 0 + \cdots = 1$

$1 \le t < 2 \Rightarrow g(t) = 1 - 1 + 0 - 0 + \cdots = 0$

$2 \le t < 3 \Rightarrow g(t) = 1 - 1 + 1 - 0 + 0 - 0 + \cdots = 1$

$3 \le t < 4 \Rightarrow g(t) = 1 - 1 + 1 - 1 + 0 - 0 + \cdots = 0$

$\vdots$

$G(s) = \sum_{n=0}^{\infty} (-1)^n \frac{e^{-ns}}{s}$

$= \sum_{n=0}^{\infty} (-1)^n e^{-ns} \mathcal{L}\{1\}$

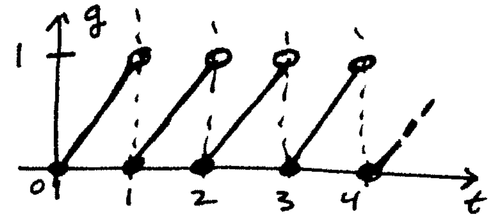
$= \sum_{n=0}^{\infty} (-1)^n \mathcal{L}\{H(t-n)\}$;

$= H(t) - H(t-1) + H(t-2) - H(t-3) \cdots$

$= 1$

**Example:** Sawtooth Wave : $g(t) = t, \ 0 \le t < 1$

$g(t+1) = g(t), \ t \ge 0$




**Find $G(s) = \mathcal{L}\{g(t)\}$.**

$g(t)$ is periodic with period $T=1$.

$$G(s) = \mathcal{L}\{g(t)\} = \frac{1}{1-e^{-s}} \int_0^1 g(t)\, e^{-st}\, dt$$

$$= \frac{1}{1-e^{-s}} \int_0^1 t\, e^{-st}\, dt$$

Integrate by Parts

$u = t \qquad V = -\frac{1}{s} e^{-st}$

$du = dt \qquad dv = e^{-st} dt$

$$= \frac{1}{1-e^{-s}} \left[ -\frac{t}{s} e^{-st} \Big|_0^1 + \frac{1}{s} \int_0^1 e^{-st} dt \right]$$

$$= \frac{1}{1-e^{-s}} \left[ \frac{-1}{s}(e^{-s} - 0) - \frac{1}{s^2} e^{-st} \Big|_0^1 \right]$$

$$= \frac{1}{1-e^{-s}} \left[ \frac{-1}{s} e^{-s} - \frac{1}{s^2}(e^{-s} - 1) \right]$$

$$= \frac{1}{1-e^{-s}} \left[ \frac{1}{s^2} - \frac{1}{s^2} e^{-s}(s+1) \right]$$

$$\boxed{G(s) = \mathcal{L}\{g(t)\} = \frac{1-(s+1)e^{-s}}{s^2(1-e^{-s})}}$$

To invert, write $\frac{1}{1-e^{-s}} = 1 + e^{-s} + e^{-2s} + e^{-3s} + \cdots$.

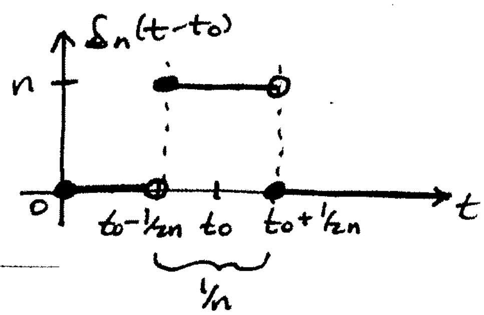
This will give $g(t)$ as a series of Heaviside functions.

## Section 7.5: Dirac Delta Function

The Dirac Delta Function may be used to model impulsive forces which act over a very short period of time (e.g. a hammer).

Consider the function

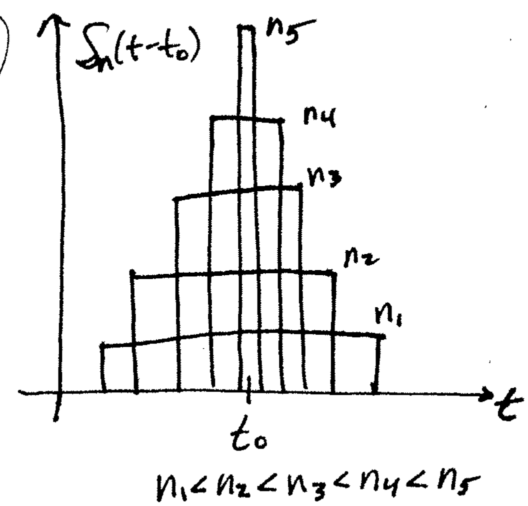
$$\delta_n(t-t_0) = \begin{cases} 0, & 0 \le t < t_0 - \frac{1}{2n} \\ n, & t_0 - \frac{1}{2n} \le t \le t_0 + \frac{1}{2n} \\ 0, & t \ge t_0 + \frac{1}{2n} \end{cases}$$




Note: $\displaystyle\int_0^\infty \delta_n(t-t_0)\,dt = \text{Area under the curve} = n \cdot \frac{1}{n} = 1$

Now, consider the sequence $\{\delta_n(t-t_0)\}$ of functions in the limit as $n \to \infty$.

Let $\delta(t-t_0) = \lim_{n\to\infty} \delta_n(t-t_0)$




$$\Rightarrow \boxed{\delta(t-t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \ne t_0 \end{cases}} \quad ① \quad t_0 \ge 0$$

and

$$\boxed{\int_0^\infty \delta(t-t_0)\,dt = 1} \quad ②$$

The Dirac Delta Function is defined by ① and ②.

The Dirac Delta Function is zero everywhere except at $t = t_0$, where it is so large that the area under this point is equal to 1.

The sequence $\{\delta_n(t-t_0)\}$ is called a $\delta$-sequence. There are many sequences of functions with similar behavior as $n \to \infty$.

$$\delta_n(t_0 - t_0) \to \infty$$
$$\delta_n(t-t_0) \to 0 \text{ for } t \ne t_0 \quad \text{and} \quad \int_0^\infty \delta_n(t-t_0)\,dt = 1$$

Property of $\delta(t-t_0)$: $\boxed{\int_0^\infty f(t)\,\delta(t-t_0)\,dt = f(t_0)}$

The integrand is zero everywhere, except at $t=t_0$.

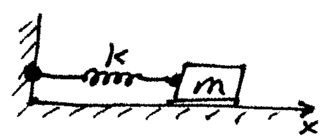
$$\Rightarrow \int_0^\infty f(t)\,\delta(t-t_0)\,dt = \int_0^\infty f(t_0)\,\delta(t-t_0)\,dt = f(t_0)\underbrace{\int_0^\infty \delta(t-t_0)\,dt}_{=1} = f(t_0).$$

then, $\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty \delta(t-t_0)\,e^{-st}\,dt = e^{-st_0}$

$$\boxed{\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}}$$

---

Example: Consider a Mass-Spring system with no damping. Suppose the mass is initially at rest in the relaxed state.




At time $t=3$, the mass is struck with an instantaneous impulsive force of magnitude 2 which sends the mass in motion. Find the position of the mass at time $t$.

IVP: $mx'' + kx = 2\delta(t-3)$ ; $x(0)=0$, $x'(0)=0$

Divide by $m$ $\Rightarrow$ $x'' + \omega_0^2 x = \frac{2}{m}\delta(t-3)$, where $\omega_0 = \sqrt{k/m}$

$\mathcal{L}\{\text{ODE}\}$: $\mathcal{L}\{x'' + \omega_0^2 x\} = \mathcal{L}\{\frac{2}{m}\delta(t-3)\}$

$$[s^2X(s) - \cancel{sx(0)} - \cancel{x'(0)}] + \omega_0^2 X(s) = \frac{2}{m}e^{-3s}$$

$$X(s)(s^2+\omega_0^2) = \frac{2}{m}e^{-3s}$$

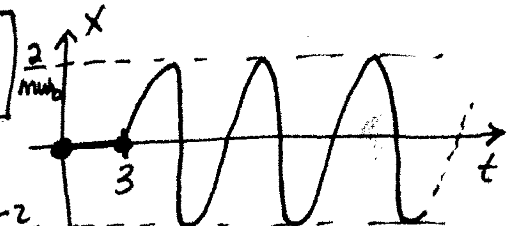
$$X(s) = \frac{2}{m}e^{-3s}\cdot\frac{1}{s^2+\omega_0^2} = \frac{2}{m\omega_0}e^{-3s}\cdot\frac{\omega_0}{s^2+\omega_0^2}$$

Second Translation Theorem

$$= \frac{2}{m\omega_0}e^{-3s}\mathcal{L}\{\sin(\omega_0 t)\} = \frac{2}{m\omega_0}\mathcal{L}\{H(t-3)\sin(\omega_0(t-3))\}$$

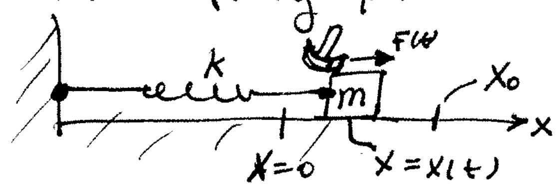
$$X(s) = \frac{2}{m\omega_0}\mathcal{L}\{H(t-3)\sin(\omega_0(t-3))\}$$

Invert: $\boxed{x(t) = \frac{2}{m\omega_0}H(t-3)\sin(\omega_0(t-3))}$

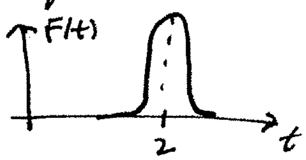



At $t=3$, the impulsive force sets the mass in motion with a speed of $2/m$, after which the mass oscillates harmonically with frequency $\omega_0$ and amplitude $2/m\omega_0$

**Example:** Consider a Mass-Spring System with no Damping.




Suppose the spring is stretch to the point $x=x_0$. At time $t=0$, the mass is released from rest, and it begins to oscillate. At time $t=2$, the mass is struck with a hammer. The impulsive force has a magnitude of 3 units. Find the position of the mass at time $t$.




$$F(t) \approx S\,\delta(t-2)$$

$$\int_{-\infty}^{\infty} F(t)\,dt = \int_{-\infty}^{\infty} S\,\delta(t-2)\,dt = S$$

$$S = \int_{-\infty}^{\infty} F(t)\,dt = 3$$

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IVP: $mx'' + kx = 3\,\delta(t-2)$, $x(0)=x_0$, $x'(0)=0$

Divide by m $\Rightarrow$ $x'' + \omega_0^2 x = \frac{3}{m}\delta(t-2)$, $\omega_0 = \sqrt{\frac{k}{m}}$

$$\mathcal{L}\{x'' + \omega_0^2 x\} = \frac{3}{m}\mathcal{L}\{\delta(t-2)\}$$

$$[s^2X(s) - s\,x(0) - \cancel{x'(0)}] + \omega_0^2 X(s) = \frac{3}{m}e^{-2s}$$

$$X(s)(s^2+\omega_0^2) = \frac{3}{m}e^{-2s} + s\,x(0) = s x_0 + \frac{3}{m}e^{-2s}$$

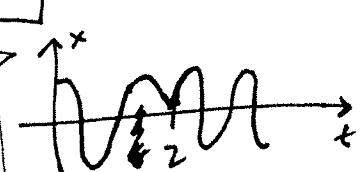
$$X(s) = x_0\frac{s}{s^2+\omega_0^2} + \frac{3}{m\omega_0}e^{-2s}\frac{\omega_0}{s^2+\omega_0^2}$$

$$= x_0\,\mathcal{L}\{\cos(\omega_0 t)\} + \frac{3}{m\omega_0}e^{-2s}\mathcal{L}\{\sin(\omega_0 t)\}$$

$$X(s) = x_0\,\mathcal{L}\{\cos(\omega_0 t)\} + \frac{3}{m\omega_0}\mathcal{L}\{H(t-2)\sin(\omega_0(t-2))\}$$

$$\Rightarrow \boxed{x(t) = x_0\cos(\omega_0 t) + \frac{3}{m\omega_0}H(t-2)\sin(\omega_0(t-2))}$$

$$\boxed{x(t) = \begin{cases} x_0\cos(\omega_0 t), & 0 \le t < 2 \\ x_0\cos(\omega_0 t) + \frac{3}{m\omega_0}\sin(\omega_0(t-2)), & t \ge 2 \end{cases}}$$




Effects of the h