Example: Find the product of $\left(y+\frac{2}{7}y^2\right)$ and $(7y - y^2)$ and verify the result for $y = 3$.

Solution: We have,

$$\left(y+\frac{2}{7}y^2\right) \times (7y - y^2)$$

$$= y\times(7y-y^2)+\frac{2}{7}y^2\times(7y-y^2)$$

$$= y\times 7y - y\times y^2 + \frac{2}{7}y^2\times 7y - \frac{2}{7}y^2\times y^2$$

$$= 7y^2 - y^3 + 2y^3 - \frac{2}{7}y^4$$

$$= 7y^2 + y^3 - \frac{2}{7}y^4$$

Verification: When $y = 3$, we have

L.H.S. $= \left(y+\frac{2}{7}y^2\right)\times(7y-y^2)$

$$= \left(3+\frac{2}{7}\times(3)^2\right)\times(7\times 3-(3)^2)$$

$$= \left(3+\frac{2}{7}\times 9\right)\times(21-9)$$

$$= \left(3+\frac{18}{7}\right)\times 12 = \left(\frac{21+18}{7}\right)\times 12 = \frac{39}{7}\times 12 = \frac{468}{7}$$

R.H.S. $= 7y^2 + y^3 - \frac{2}{7}y^4$

$$= 7\times(3)^2 + (3)^3 - \frac{2}{7}\times(3)^4$$

$$= 7\times 9 + 27 - \frac{2}{7}\times 81$$

$$= 63+27-\frac{162}{7}=90-\frac{162}{7}=\frac{630-162}{7}=\frac{468}{7}$$

$\therefore$ L.H.S. = R.H.S.

Example: Find the value of the following products:

(i) $(x + 2y)(x - 2y)$ at $x = 1, y = 0$

(ii) $(3m - 2n)(2m - 3n)$ at $m = 1, n = -1$

(iii) $(4a^2 + 3b)(4a^2 + 3b)$ at $a = 1, b = 2$

Solution: (i) We have,

$(x + 2y)(x - 2y)$

$= x(x - 2y) + 2y(x - 2y)$

$= x \times x - x \times 2y + 2y \times x - 2y \times 2y$

$= x^2 - 2xy + 2yx - 4y^2$

$= x^2 - 4y^2$

When $x = 1, y = 0$, we get

$(x + 2y)(x - 2y)$

$= x^2 - 4y^2 = (1)^2 - 4 \times (0)^2 = 1 - 0 = 1.$

(ii) We have,

$(3m - 2n)(2m - 3n)$

$= 3m(2m - 3n) - 2n(2m - 3n)$

$= 3m \times 2m - 3m \times 3n - 2n \times 2m + 2n \times 3n$

$= 6m^2 - 9mn - 4mn + 6n^2$

$= 6m^2 - 13mn + 6n^2$

When $m = 1, n = -1$, we get

$(3m - 2n)(2m - 3n)$

$= 6m^2 - 13mn + 6n^2$

$= 6 \times (1)^2 - 13 \times 1 \times (-1) + 6 \times (-1)^2 = 6 + 13 + 6 = 25$

(iii) We have,

$(4a^2 + 3b)(4a^2 + 3b)$

$= 4a^2 \times (4a^2 + 3b) + 3b \times (4a^2 + 3b)$

$= 4a^2 \times 4a^2 + 4a^2 \times 3b + 3b \times 4a^2 + 3b \times 3b$

$= 16a^4 + 12a^2b + 12a^2b + 9b^2$

$= 16a^4 + 24a^2b + 9b^2$

When, a = 1, b = 2, we get

$(4a^2 + 3b)(4a^2 + 3b)$

$= 16a^4 + 24a^2b + 9b^2$

$= 16 \times (1)^4 + 24 \times (1)^2 \times 2 + 9(2)^2$

$= 16 + 48 + 36 = 100$

Example: Simplify the following :

(i) $\frac{1}{3}(6x^2 + 15y^2)(6x^2 - 15y^2)$

(ii) $9x^4(2x^3 - 5x^4) \times 5x^6(x^4 - 3x^2)$

Solution: (i) We have,

$\frac{1}{3}(6x^2 + 15y^2)(6x^2 - 15y^2)$

$= \left\{\frac{1}{3} \times (6x^2 + 15y^2)\right\} \times (6x^2 - 15y^2)$ [By using associativity of multiplication]

$= \left(\frac{1}{3} \times 6x^2 + \frac{1}{3} \times 15y^2\right) \times (6x^2 - 15y^2)$ [By using distributivity of multiplication over addition]

$= (2x^2 + 5y^2) \times (6x^2 - 15y^2)$

$= 2x^2 \times (6x^2 - 15y^2) + 5y^2 \times (6x^2 - 15y^2)$

$= 2x^2 \times 6x^2 - 2x^2 \times 15y^2 + 5y^2 \times 6x^2 - 5y^2 \times 15y^2$

$= 12x^4 - 30x^2y^2 + 30x^2y^2 - 75y^4$

$= 12x^4 - 75y^4$

(ii) We have,

$9x^4 (2x^3 - 5x^4) \times 5x^6 (x^4 - 3x^2)$

$= 9x^4 \times (2x^3 - 5x^4) \times 5x^6 \times (x^4 - 3x^2)$

$= \{9x^4 \times (2x^3 - 5x^4)\} \times \{5x^6 \times (x^4 - 3x^2)\}$ $\left[\begin{array}{c}\text{By using associativity} \\ \text{of multiplication}\end{array}\right]$

$= (9x^4 \times 2x^3 - 9x^4 \times 5x^4) \times (5x^6 \times x^4 - 5x^6 \times 3x^2)$

$= (18x^7 - 45x^8) \times (5x^{10} - 15x^8)$

$= 18x^7 \times (5x^{10} - 15x^8) - 45x^8 (5x^{10} - 15x^8)$

$= 18x^7 \times 5x^{10} - 18x^7 \times 15x^8 - 45x^8 \times 5x^{10} + 45x^8 \times 15x^8$

$= 90x^{17} - 270x^{15} - 225x^{18} + 675x^{16}$

$= -225x^{18} + 90x^{17} + 675x^{16} - 270x^{15}$

Example: Simplify the following:

(i) $(2x + 5) (3x - 2) + (x + 2) (2x - 3)$

(ii) $(3x + 2) (2x + 3) - (4x - 3) (2x - 1)$

(iii) $(2x + 3y) (3x + 4y) - (7x + 3y) (x + 2y)$

Solution: (i) We have,

$(2x + 5) (3x - 2) + (x + 2) (2x - 3)$

$= 2x(3x - 2) + 5 (3x - 2) + x (2x - 3) + 2(2x - 3)$

$= 6x^2 - 4x + 15x - 10 + 2x^2 - 3x + 4x - 6$

$= (6x^2 + 2x^2) + (-4x + 15x - 3x + 4x) + (-10 - 6)$

$= 8x^2 + 12x - 16$

(ii) We have,

(3x + 2) (2x + 3) – (4x – 3) (2x – 1)

= {3x(2x + 3) + 2(2x + 3)} – {4x(2x – 1) – 3 (2x – 1)}

= $(6x^2 + 9x + 4x + 6) – (8x^2 – 4x – 6x + 3)$

= $(6x^2 + 13x + 6) – (8x^2 – 10x + 3)$

= $6x^2 + 13x + 6 – 8 x^2 + 10x – 3$

= $-2x^2 + 23x + 3$

(iii) We have,

(2x + 3y) (3x + 4y) – (7x+ 3y) (x + 2y)

= {2x (3x + 4y) + 3y (3x + 4y) – {7x(x + 2y) + 3y (x + 2y)}

= $\{6x^2 + 8xy + 9xy + 12y^2\} – (7x^2 + 14xy + 3xy + 6y^2)$

= $(6x^2 + 17xy + 12y^2) – (7x^2 + 17xy + 6y^2)$

= $6x^2 + 17xy + 12y^2 – 7x^2 – 17xy – 6y^2$

= $6x^2 – 7x^2 + 17xy – 17xy + 12y^2 – 6y^2$

= $– x^2 + 6y^2$.

Example: If $x + \frac{1}{x} = 4$, find the value of

(i) $x^2 + \frac{1}{x^2}$ (ii) $x^4 + \frac{1}{x^4}$

Solution: (i) We have,

$$x+\frac{1}{x}=4$$

On squaring both sides, we get

$$\left(x+\frac{1}{x}\right)^2=4^2$$

$\Rightarrow$ $x^2+2\times x\times\frac{1}{x}+\left(\frac{1}{x}\right)^2=16$

$\Rightarrow$ $x^2+2+\frac{1}{x^2}=16$

$\Rightarrow$ $x^2+\frac{1}{x^2}=16-2$ [On transposing 2 on RHS]

$\Rightarrow$ $x^2+\frac{1}{x^2}=14$

(ii) We have,

$$x^2+\frac{1}{x^2}=14$$

On squaring both sides, we get

$$\left(x^2+\frac{1}{x^2}\right)^2=14^2$$

$\Rightarrow$ $(x^2)^2+\left(\frac{1}{x^2}\right)^2+2\times x^2\times\frac{1}{x^2}=196$

$\Rightarrow$ $x^4+\frac{1}{x^4}+2=196$

$\Rightarrow$ $x^4+\frac{1}{x^4}=196-2$ [On transposing 2 on RHS]

$\Rightarrow$ $x^4+\frac{1}{x^4}=194$

Example: If $x-\frac{1}{x}=9$, find the value of $x^2+\frac{1}{x^2}$.

Solution: We have,

$$x-\frac{1}{x}=9$$

On squaring both sides, we get

$$\left(x-\frac{1}{x}\right)^2=81$$

$$\Rightarrow \quad x^2-2\times x\times\frac{1}{x}+\left(\frac{1}{x}\right)^2=81$$

$$\Rightarrow \quad x^2-2+\frac{1}{x^2}=81$$

$$\Rightarrow \quad x^2+\frac{1}{x^2}=81+2$$ [On transposing – 2 on RHS]

$$\Rightarrow \quad x^2+\frac{1}{x^2}=83$$

Example: If $x^2+\frac{1}{x^2}=27$, find the values of each of the following :

(i) $x+\frac{1}{x}$ (ii) $x-\frac{1}{x}$

Solution: (i) We have

$$\left(x+\frac{1}{x}\right)^2=x^2+2\times x\times\frac{1}{x}+\frac{1}{x^2}$$

$$\Rightarrow \quad \left(x+\frac{1}{x}\right)^2=x^2+2+\frac{1}{x^2}$$

$$\Rightarrow \quad \left(x+\frac{1}{x}\right)^2=x^2+\frac{1}{x^2}+2$$

$$\Rightarrow \quad \left(x+\frac{1}{x}\right)^2 = 27+2 \qquad \left[\therefore x^2+\frac{1}{x^2}=27\,(\text{given})\right]$$

$$\Rightarrow \quad \left(x+\frac{1}{x}\right)^2 = 29$$

$$\Rightarrow \quad x+\frac{1}{x} = \pm\sqrt{29}$$ [Taking square root of both sides]

(ii) We have,

$$\left(x-\frac{1}{x}\right)^2 = x^2 - 2\times x\times\frac{1}{x}+\left(\frac{1}{x}\right)^2$$

$$\Rightarrow \quad \left(x-\frac{1}{x}\right)^2 = x^2-2+\frac{1}{x^2}$$

$$\Rightarrow \quad \left(x-\frac{1}{x}\right)^2 = x^2+\frac{1}{x^2}-2$$

$$\Rightarrow \quad \left(x-\frac{1}{x}\right)^2 = 27-2 \qquad \left[\because x^2+\frac{1}{x^2}=27\,(\text{given})\right]$$

$$\Rightarrow \quad \left(x-\frac{1}{x}\right)^2 = 25$$

$$\Rightarrow \quad \left(x-\frac{1}{x}\right)^2 = 5^2$$

$$\Rightarrow \quad x-\frac{1}{x} = \pm 5$$ [Taking square root of both sides]

Example: If x + y = 12 and xy = 14, find the value of $x^2 + y^2$.

Solution: We have,

$(x + y)^2 = x^2 + y^2 + 2xy$

Putting the values of x + y and xy, we obtain

$12^2 = x^2 + y^2 + 2 \times 14$

$\Rightarrow \quad 144 = x^2 + y^2 + 28$

$\Rightarrow \quad 144 - 28 = x^2 + y^2$

$\Rightarrow \quad x^2 + y^2 = 116$

Example: If $4x^2 + y^2 = 40$ and $xy = 6$, find the value of $2x + y$.

Solution: We have,

$(2x + y)^2 = (2x)^2 + y^2 + 2 \times 2x \times y$

$\Rightarrow \quad (2x + y)^2 = (4x^2 + y^2) + 4xy$

$\Rightarrow \quad (2x + y)^2 = 40 + 4 \times 6$ [Using $4x^2 + y^2 = 40$ and $xy = 6$]

$\Rightarrow \quad 2x + y = \pm\sqrt{64}$

$\Rightarrow \quad 2x + y = \pm 8$ [Taking square root of both sides]

Example: Prove that:

$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = (a - b)^2 + (b - c)^2 + (c - a)^2$

Solution: We have,

LHS $= 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca$

$= (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ca + a^2)$ [Re-arranging the terms]

$= (a - b)^2 + (b - c)^2 + (c - a)^2$

= R.H.S.

Hence, $2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = (a - b)^2 + (b - c)^2 + (c - a)^2$

Example: If $a^2 + b^2 + c^2 - ab - bc - ca = 0$, prove that $a = b = c$.

Solution: We have,

$a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\Rightarrow \quad 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca = 2 \times 0$ [Multiplying both sides by 2]

$\Rightarrow$ $(a^2 - 2b + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ac + a^2) = 0$

$\Rightarrow$ $(a - b)^2 + (b - c)^2 + (c - a)^2 = 0$

$$\left[\begin{array}{l}\because \text{Sum of positive quantities is zero if} \\ \text{and only if each quantity is zero}\end{array}\right]$$

$\Rightarrow$ $a - b = 0,\ b - c = 0,\ c - a = 0$

$\Rightarrow$ $a = b,\ b = c$ and $c = a$

$\Rightarrow$ $a = b = c.$

Example: Expand:

(i) $(2x + 3y)^3$ (ii) $(3x - 2y)^3$

Solution: We have :

(i) $(2x + 3y)^3 = (2x)^3 + 3 \times 2x \times 3y \times (2x + 3y) + (3y)^3$

$= 8x^3 + 27y^3 + 18xy\,(2x + 3y)$

$= 8x^3 + 27y^3 + 36x^2y + 54xy^2.$

(ii) $(3x - 2y)^3 = (3x)^3 - (2y)^3 - 3 \times 3x \times 2y \times (3x - 2y)$

$= 27x^3 - 8y^3 - 18xy\,(3x - 2y)$

$= 27x^3 - 8y^3 - 54x^2y + 36xy^2.$

Example: If $\left(x - \frac{1}{x}\right) = 5$, find the value of $\left(x^3 - \frac{1}{x^3}\right)$.

Solution: $\left(x - \frac{1}{x}\right) = 5 \Rightarrow \left(x - \frac{1}{x}\right)^3 = 5^3$ [Cubing both sides]

$\Rightarrow$ $x^3 - \frac{1}{x^3} - 3x \times \frac{1}{x} \times \left(x - \frac{1}{x}\right) = 125$

$\Rightarrow$ $x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right) = 125$

$$\Rightarrow \quad x^3 - \frac{1}{x^3} - 3 \times 5 = 125$$

$$\Rightarrow \quad \left(x^3 - \frac{1}{x^3}\right) = (125 + 15) = 140.$$

Example: If a + b = 7 and ab = 12, find the value of $a^3 + b^3$.

Solution: a + b = 7 $\Rightarrow$ $(a + b)^3 = (7)^3$ [Cubing both sides]

$\Rightarrow$ $a^3 + b^3 + 3ab\,(a + b) = 343$

$\Rightarrow$ $a^3 + b^3 + 252 = 343$

$\Rightarrow$ $a^3 + b^3 = (343 - 252) = 91.$

Example: If a + b + c = 11 and ab + bc + ca = 36, find the value of $(a^2 + b^2 + c^2)$.

Solution: a + b + c = 11

$\Rightarrow$ $(a + b + c)^2 = 121$ [Squaring both, sides]

$\Rightarrow$ $(a^2 + b^2 + c^2) + 2(ab + bc + ca) = 121$

$\Rightarrow$ $(a^2 + b^2 + c^2) = 49$

Example: If $\left(x + \frac{1}{x}\right) = 4$, find the value of $\left(x^3 + \frac{1}{x^3}\right)$.

Solution: $\left(x + \frac{1}{x}\right) = 4$ $\Rightarrow$ $\left(x + \frac{1}{x}\right)^3 = 4^3$

$$\Rightarrow \quad x^3 + \frac{1}{x^3} + 3 \times x \times \frac{1}{x} \times \left(x + \frac{1}{x}\right) = 64$$

$$\Rightarrow \quad x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = 64$$

$$\Rightarrow \quad x^3 + \frac{1}{x^3} + 3 \times 4 = 64$$

$$\Rightarrow \quad x^3 + \frac{1}{x^3} = (64 - 12) = 52$$