Side Notes for Hidden Markov Chains
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General setting:

Parameters: μ_0, σ_0^2, and σ^2 are known

Prior: $\mu \ \sim \ \text{Normal}\,(\mu_0, \sigma_0^2)$

Data: $X \ \sim \ \text{Normal}\,(\mu, \sigma^2)$

Posterior: $X = x \quad \Longrightarrow \quad \mu \ \sim \ \text{Normal}\left(\frac{\sigma^2\mu_0 + \sigma_0^2 x}{\sigma^2+\sigma_0^2}, \frac{\sigma^2\sigma_0^2}{\sigma^2+\sigma_0^2}\right)$

Constant beta application (model is $R_t = \beta m_t + \epsilon_t$):

Prior: $\beta \ \sim \ \text{Normal}\,(\mu, \sigma^2)$, as updated through time $t-1$

Data: $X_t \ = \ \frac{R_t}{m_t} \ = \ \frac{\beta m_t + \epsilon_t}{m_t} \ = \ \beta + \frac{\epsilon_t}{m_t} \ \sim \ \text{Normal}\left(\beta, \sigma_\epsilon^2/m_t^2\right)$

Posterior: $R_t = r_t \quad \Longrightarrow \quad X_t = \frac{r_t}{m_t} \quad \Longrightarrow$

$$\begin{aligned}\beta \ &\sim \ \text{Normal}\left(\frac{(\sigma_\epsilon^2/m_t^2)\mu + \sigma^2(r_t/m_t)}{(\sigma_\epsilon^2/m_t^2)+\sigma^2}, \frac{(\sigma_\epsilon^2/m_t^2)\sigma^2}{(\sigma_\epsilon^2/m_t^2)+\sigma^2}\right)\\ &\sim \ \text{Normal}\left(\frac{\sigma_\epsilon^2\mu + \sigma^2 r_t m_t}{\sigma_\epsilon^2 + m_t^2\sigma^2}, \frac{\sigma_\epsilon^2\sigma^2}{\sigma_\epsilon^2+m_t^2\sigma^2}\right)\end{aligned}$$

Full model ($R_t = \beta_t m_t + \epsilon_t$) with known σ_δ^2 and σ_ϵ^2:

Prior: $\beta_{t-1} \ \sim \ \text{Normal}\,(\mu, \sigma^2)$, so $\beta_t \ = \ \beta_{t-1} + \delta_t \ \sim \ \text{Normal}\,(\mu, \sigma^2 + \sigma_\delta^2)$

Data: $X_t \ = \ \frac{R_t}{m_t} \ = \ \frac{\beta_t m_t + \epsilon_t}{m_t} \ = \ \beta_t + \frac{\epsilon_t}{m_t} \ \sim \ \text{Normal}\left(\beta_t, \sigma_\epsilon^2/m_t^2\right)$

Posterior: $R_t = r_t \quad \Longrightarrow \quad X_t = \frac{r_t}{m_t} \quad \Longrightarrow$

$$\beta_t \ \sim \ \text{Normal}\left(\frac{\sigma_\epsilon^2\mu + (\sigma^2+\sigma_\delta^2) r_t m_t}{\sigma_\epsilon^2 + m_t^2(\sigma^2+\sigma_\delta^2)}, \frac{\sigma_\epsilon^2(\sigma^2+\sigma_\delta^2)}{\sigma_\epsilon^2+m_t^2(\sigma^2+\sigma_\delta^2)}\right)$$

Estimating σ_δ and σ_ϵ via Maximum Likelihood:

• We seek to find the value of these parameters that maximizes the likelihood function evaluated at the observed data:

$$L(\sigma_\epsilon, \sigma_\delta) \;=\; f(r_1, \ldots, r_{1258}),$$

where $f(\cdot)$ is the joint density function of $(R_1, \ldots, R_{1258})$ with the values σ_ϵ and σ_δ.

• We do this by writing $L(\cdot)$ as the product of conditional density functions:

$$L(\sigma_\epsilon, \sigma_\delta) \;=\; \prod_{t=1}^{1258} f_t(r_t),$$

where $f_t(\cdot)$ is the conditional density function of R_t given $D_{t-1} = \{R_1 = r_1, \ldots, R_{t-1} = r_{t-1}\}$.

• With a trial choice of the two parameters in hand suppose at time $t-1$ we deduce that:

$$\begin{aligned} \beta_{t-1} &\sim \text{Normal}\,(\mu, \sigma^2)\,; \quad \text{so} \\ \beta_t \;=\; \beta_{t-1} + \delta_t &\sim \text{Normal}\,(\mu, \sigma^2 + \sigma_\delta^2)\,; \quad \text{and} \\ \beta_t m_t &\sim \text{Normal}\,(\mu m_t, (\sigma^2 + \sigma_\delta^2) m_t^2). \end{aligned}$$

• Recalling that the independent $\epsilon_t \sim \text{Normal}\,(0, \sigma_\epsilon^2)$ we see that

$$R_t \;=\; \beta_t m_t + \epsilon_t \;\sim\; \text{Normal}\,\left(\mu m_t, (\sigma^2 + \sigma_\delta^2) m_t^2 + \sigma_\epsilon^2\right) \sim\; \text{Normal}\,(\mu m_t, v),$$

where $v = (\sigma^2 + \sigma_\delta^2) m_t^2 + \sigma_\epsilon^2$.

• Conditioned on the data through day $t-1$, the density function for R_t is therefore

$$f_t(r) \;=\; \frac{1}{\sqrt{2\pi v}} \exp\left[-(r-\mu m_t)^2/2v\right].$$

• We repeat this exercise over a grid of values for $(\sigma_\delta, \sigma_\epsilon)$ and observe that L is maximized at $\sigma_\delta = 0.015$ and $\sigma_\epsilon = 0.968$.