16. Let $\mathbb{F}$ be the ordered field of rational functions as given in Example 3.2.6, and note that $\mathbb{F}$ contains both $\mathbb{N}$ and $\mathbb{R}$ as subsets.

(a) Show that $\mathbb{F}$ does not have the Archimedean property. That is, find a member z in $\mathbb{F}$ such that $z > n$ for every $n \in \mathbb{N}$.

(b) Show that the property in Theorem 3.3.10(c) does not apply. That is, find a positive member z in $\mathbb{F}$ such that, for all $n \in \mathbb{N}$, $0 < z < 1/n$.

(c) Show that $\mathbb{F}$ does not satisfy the completeness axiom. That is, find a subset B of $\mathbb{F}$ such that B is bounded above, but B has no least upper bound. Justify your answer.

17. We have said that the real numbers can be characterized as a complete ordered field. This means that any other complete ordered field F is essentially the same as $\mathbb{R}$ in the sense that there exists a bijection $f: \mathbb{R} \to F$ with the following properties for all $a, b \in \mathbb{R}$.

(1) $f(a+b) = f(a) + f(b)$
(2) $f(a \cdot b) = f(a) \cdot f(b)$
(3) $a < b$ iff $f(a) < f(b)$

(Such a function is called an **order isomorphism**.) We can construct the function f by first defining $f(0) = 0_F$ and $f(1) = 1_F$, where 0_F and 1_F are the unique elements of F given in axioms A4 and M4. Then define $f(n+1) = f(n) + 1_F$ and $f(-n) = -f(n)$ for all $n \in \mathbb{N}$. This extends the domain of f to all of $\mathbb{Z}$.

Next we extend the domain of f to $\mathbb{Q}$ by defining $f(m/n) = f(m)/f(n)$ for $m, n \in \mathbb{Z}$ with $n \neq 0$. Since, for all $x \in \mathbb{R}$,

$$x = \sup \{q \in \mathbb{Q} : q < x\}$$

(Exercise 13), we can extend the domain of f to $\mathbb{R}$ by defining

$$f(x) = \sup \{f(q) : q \in \mathbb{Q} \text{ and } q < x\}.$$

Verify that the function f so defined is the required order isomorphism. [*Note*: When writing an equation such as $f(a+b) = f(a) + f(b)$, the "+" between a and b represents addition in $\mathbb{R}$ and the "+" between $f(a)$ and $f(b)$ represents addition in F. Similar comments apply to "·" and "<".]

Section 3.4 TOPOLOGY OF THE REAL NUMBERS

Many of the central ideas in analysis are dependent on the notion of two points being "close" to each other. We have seen that the distance between two points x and y in $\mathbb{R}$ is given by the absolute value of their difference: $|x-y|$. Thus, if we are given some positive measure of closeness, say ε, we may be interested in all points y that are less than ε away from x:

$$\{y : |x-y| < \varepsilon\}.$$

We formalize this idea in the following definition.

3.4.1 DEFINITION Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. A **neighborhood** of x (or an **ε-neighborhood** of x)[†] is a set of the form

$$N(x; \varepsilon) = \{y \in \mathbb{R} : |x - y| < \varepsilon\}.$$

The number ε is referred to as the **radius** of $N(x; \varepsilon)$.

Basically, a neighborhood of x of radius ε is the open interval $(x - \varepsilon, x + \varepsilon)$ of length 2ε centered at x. We prefer to use the term "neighborhood" in subsequent definitions and theorems because this terminology can be applied in more general settings. In this section we use neighborhoods to define the concepts of open and closed sets. The study of these sets is known as **point set topology**, and this explains the use of the word "topology" in the title of the section.

In some situations, particularly when dealing with limits of functions (Chapter 5), we shall want to consider points y that are close to x but different from x. We can accomplish this by requiring $|x - y| > 0$.

3.4.2 DEFINITION Let $x \in \mathbb{R}$ and let $\varepsilon > 0$. A **deleted neighborhood** of x is a set of the form

$$N^*(x; \varepsilon) = \{y \in \mathbb{R} : 0 < |x - y| < \varepsilon\}.^{‡}$$

Clearly, $N^*(x; \varepsilon) = N(x; \varepsilon) \setminus \{x\}$.

If $S \subseteq \mathbb{R}$, then a point x in $\mathbb{R}$ can be thought of as being "inside" S, on the "edge" of S, or "outside" S. Saying that x is "outside" S is the same as saying that x is "inside" the complement of S, $\mathbb{R} \setminus S$. Using neighborhoods, we can make the intuitive ideas of "inside" and "edge" more precise.

3.4.3 DEFINITION Let S be a subset of $\mathbb{R}$. A point x in $\mathbb{R}$ is an **interior point** of S if there exists a neighborhood N of x such that $N \subseteq S$. If for every neighborhood N of x, $N \cap S \neq \varnothing$ and $N \cap (\mathbb{R} \setminus S) \neq \varnothing$, then x is called a **boundary point** of S. The set of all interior points of S is denoted by int S, and the set of all boundary points of S is denoted by bd S.

It should be clear that every point x in a set S is either an interior point of S or a boundary point of S. Indeed, if $x \in S$, then every neighborhood of x will have a nonempty intersection with S. If some neighborhood of x is contained in S, then $x \in$ int S. If no neighborhood of x is contained in S, then every neighborhood of x will have nonempty intersection with $\mathbb{R} \setminus S$ and x must be a boundary point of S.

[†]In some advanced texts the set $N(x; \varepsilon)$ is called an ε-neighborhood of x, and a neighborhood of x is defined to be any set that contains an ε-neighborhood of x for some $\varepsilon > 0$. Since we shall not need this more general notion, we shall use the terms "ε-neighborhood" and "neighborhood" interchangeably.

[‡] Some authors use the name "punctured" neighborhood instead of "deleted" neighborhood.

3.4.4 EXAMPLE (a) Let S be the open interval $(0,5)$ and let $x \in S$. If $\varepsilon = \min\{x, 5-x\}$, then we claim that $N(x;\varepsilon) \subseteq S$. (See Figure 1.) Indeed, for all $y \in N(x;\varepsilon)$ we have $|y-x| < \varepsilon$, so that

$$-x \leq -\varepsilon < y-x < \varepsilon \leq 5-x.$$

Thus $0 < y < 5$ and $y \in S$. It follows that every point in S is an interior point of S. Since the inclusion $\operatorname{int} S \subseteq S$ always holds, we have $S = \operatorname{int} S$.

The point 0 is not a member of S, but every neighborhood of 0 will contain positive numbers in S. Thus 0 is a boundary point of S. Similarly, $5 \in \operatorname{bd} S$ and, in fact, $\operatorname{bd} S = \{0, 5\}$. Note that none of the boundary of S is contained in S. Of course, there is nothing special about the open interval $(0, 5)$ in this example. Similar comments would apply to any open interval.

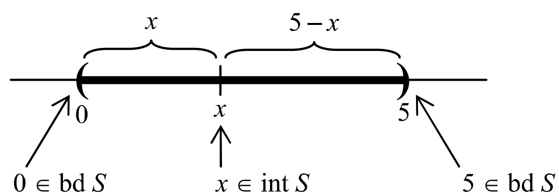

Figure 1 $S = (0,5)$

(b) Let S be the closed interval $[0, 5]$. The point 0 is still a boundary point of S, since every neighborhood of 0 will contain negative numbers not in S. We have $\operatorname{int} S = (0, 5)$ and $\operatorname{bd} S = \{0, 5\}$. This time S contains all of its boundary points, and the same could be said of any other closed interval.

(c) Let S be the interval $[0, 5)$. Then again $\operatorname{int} S = (0, 5)$ and $\operatorname{bd} S = \{0, 5\}$. We see that S contains some of its boundary, but not all of it.

(d) Let S be the interval $[2, \infty)$. Then $\operatorname{int} S = (2, \infty)$ and $\operatorname{bd} S = \{2\}$. Note that there is no "point" at ∞ to be included as a boundary point at the right end.

(e) Let $S = \mathbb{R}$. Then $\operatorname{int} S = S$ and $\operatorname{bd} S = \varnothing$.

3.4.5 PRACTICE Let $S = (1, 2) \cup (2, 3]$. Find $\operatorname{int} S$ and $\operatorname{bd} S$.

Closed Sets and Open Sets

We have seen that a set may contain all of its boundary, part of its boundary, or none of its boundary. Those sets in either the first or last category are of particular interest.

3.4.6 DEFINITION Let $S \subseteq \mathbb{R}$. If bd $S \subseteq S$, then S is said to be **closed**. If bd $S \subseteq \mathbb{R}\backslash S$, then S is said to be **open**.

If none of the points in S is a boundary point of S, then all the points in S must be interior points of S. On the other hand, if S contains its boundary, then since bd S = bd $(\mathbb{R}\backslash S)$, the set $\mathbb{R}\backslash S$ must not contain any of its boundary points. The converse implications also apply, so we obtain the following useful characterizations:

3.4.7 THEOREM
(a) A set S is open iff S = int S. Equivalently, S is open iff every point in S is an interior point of S.
(b) A set S is closed iff its complement $\mathbb{R}\backslash S$ is open.

3.4.8 EXAMPLE The interval $(0, 5)$ is open and the interval $[0, 5]$ is closed. Thus our present terminology is consistent with our interval notation in Section 2.1. That is, an "open interval" (a, b) is an open set and a "closed interval" $[a, b]$ is a closed set. In particular, this means that any neighborhood is an open set, since it is an open interval. The interval $[0, 5)$ is neither open nor closed, and the unbounded interval $[2, \infty)$ is closed.

The entire set $\mathbb{R}$ of real numbers is both open and closed! It is open since int $\mathbb{R} = \mathbb{R}$. It is closed since it contains its boundary: bd $\mathbb{R} = \varnothing$ and $\varnothing \subseteq \mathbb{R}$.

3.4.9 PRACTICE Is the empty set $\varnothing$ open? Is it closed?

Our next theorem shows how the set operations of intersection and union relate to open sets.

3.4.10 THEOREM
(a) The union of any collection of open sets is an open set.
(b) The intersection of any finite collection of open sets is an open set.

Proof: (a) Let $\mathscr{A}$ be an arbitrary collection of open sets and let $S = \bigcup \{A : A \in \mathscr{A}\}$. If $x \in S$, then $x \in A$ for some $A \in \mathscr{A}$. Since A is open, x is an interior point of A. That is, there exists a neighborhood N of x such that $N \subseteq A$. But $A \subseteq S$, so $N \subseteq S$ and x is an interior point of S. Hence, S is open.

(b) Let $A_1, \ldots, A_n$ be a finite collection of open sets and let $T = \bigcap_{i=1}^{n} A_i$. If $T = \varnothing$, we are done, since $\varnothing$ is open. If $T \neq \varnothing$, let $x \in T$. Then $x \in A_i$ for all $i = 1, \ldots, n$. Since each set A_i is open, there exist neighborhoods $N_i(x; \varepsilon_i)$ of x such that $N_i(x; \varepsilon_i) \subseteq A_i$. Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$. Then $N(x; \varepsilon) \subseteq A_i$ for each $i = 1, \ldots, n$, so $N(x; \varepsilon) \subseteq T$. Thus x is an interior point of T, and T is open. ♦

3.4.11 COROLLARY (a) The intersection of any collection of closed sets is closed.
(b) The union of any finite collection of closed sets is closed.

Proof: Both parts follow from Theorem 3.4.10 when combined with Theorem 3.4.7. Recall (Exercise 2.1.26) that $\mathbb{R}\backslash(\bigcup_{j\in J} A_j) = \bigcap_{j\in J}(\mathbb{R}\backslash A_j)$ and $\mathbb{R}\backslash(\bigcap_{j\in J} A_j) = \bigcup_{j\in J}(\mathbb{R}\backslash A_j)$. ♦

3.4.12 EXAMPLE For each $n \in \mathbb{N}$, let $A_n = (-1/n, 1/n)$. Then each A_n is an open set, but $\bigcap_{n=1}^{\infty} A_n = \{0\}$, which is not open. Thus we see that the restriction in Theorem 3.4.10(b) to intersections of *finitely* many open sets is necessary.

3.4.13 PRACTICE Find an example of a collection of closed sets whose union is not closed.

Accumulation Points

Our study of open and closed sets so far has been based on the notion of a neighborhood. By using deleted neighborhoods we can consider another property of points and sets.

3.4.14 DEFINITION Let S be a subset of $\mathbb{R}$. A point x in $\mathbb{R}$ is an **accumulation point** of S if every deleted neighborhood of x contains a point of S.† That is, for every $\varepsilon > 0$, $N^*(x; \varepsilon) \cap S \neq \varnothing$. The set of all accumulation points of S is denoted by S'. If $x \in S$ and $x \notin S'$, then x is called an **isolated point** of S.

An equivalent way of defining an accumulation point x of a set S would be to require that each neighborhood of x contain at least one point of S different from x. Note that an accumulation point of S may be, but does not have to be, a member of S.

3.4.15 EXAMPLE (a) If S is the interval $(0, 1]$, then $S' = [0, 1]$.
(b) If $S = \{1/n : n \in \mathbb{N}\}$, then $S' = \{0\}$.
(c) If $S = \mathbb{N}$, then $S' = \varnothing$. Thus $\mathbb{N}$ consists entirely of isolated points.
(d) If S is a finite set, then $S' = \varnothing$. Indeed, if $S = \{x_1, \ldots, x_n\}$ and $y \in \mathbb{R}$, then let $\varepsilon = \min\{|x_i - y| : x_i \neq y\}$. It follows that $\varepsilon > 0$ and $N^*(y; \varepsilon) \cap S = \varnothing$. Thus y is not an accumulation point of S.

3.4.16 DEFINITION Let $S \subseteq \mathbb{R}$. Then the **closure** of S, denoted cl S, is defined by

$$\text{cl } S = S \cup S',$$

where S' is the set of all accumulation points of S.

† Some authors use the name "limit point" or "cluster point" instead of "accumulation point."

In terms of neighborhoods, a point x is in cl S iff every neighborhood of x intersects S. To see this, let $x \in \text{cl } S$ and let N be a neighborhood of x. If $x \in S$, then $N \cap S$ contains x. If $x \notin S$, then $x \in S'$ and every deleted neighborhood intersects S. Thus in either case the neighborhood N must intersect S. Conversely, suppose that every neighborhood of x intersects S. If $x \notin S$, then every neighborhood of x intersects S in a point other than x. Thus $x \in S'$, and so $x \in \text{cl } S$.

The basic relationships between accumulation points, closure, and closed sets are presented in the following theorem.

3.4.17 THEOREM Let S be a subset of $\mathbb{R}$. Then

(a) S is closed iff S contains all of its accumulation points,

(b) cl S is a closed set,

(c) S is closed iff $S = \text{cl } S$,

(d) $\text{cl } S = S \cup \text{bd } S$.

Proof: (a) Suppose that S is closed and let $x \in S'$. We must show that $x \in S$. If $x \notin S$, then x is in the open set $\mathbb{R} \backslash S$. Thus there exists a neighborhood N of x such that $N \subseteq \mathbb{R} \backslash S$. But then $N \cap S = \varnothing$, which contradicts $x \in S'$. So we must have $x \in S$.

Conversely, suppose that $S' \subseteq S$. We shall show that $\mathbb{R} \backslash S$ is open. To this end, let $x \in \mathbb{R} \backslash S$. Then $x \notin S'$, so there exists a deleted neighborhood $N^*(x; \varepsilon)$ that misses S. Since $x \notin S$, the whole neighborhood $N(x; \varepsilon)$ misses S; that is, $N(x; \varepsilon) \subseteq \mathbb{R} \backslash S$. Thus $\mathbb{R} \backslash S$ is open and S is closed by Theorem 3.4.7(b).

(b) By part (a) it suffices to show that if $x \in (\text{cl } S)'$, then $x \in \text{cl } S$. So suppose that x is an accumulation point of cl S. Then every deleted neighborhood $N^*(x; \varepsilon)$ intersects cl S. We must show that $N^*(x; \varepsilon)$ intersects S. To this end, let $y \in N^*(x; \varepsilon) \cap \text{cl } S$. (See Figure 2.) Since $N^*(x; \varepsilon)$ is an open set (Exercise 12), there exists a neighborhood $N(y; \delta)$ contained in $N^*(x; \varepsilon)$. But $y \in \text{cl } S$, so every neighborhood of y intersects S. That is, there exists a point z in $N(y; \delta) \cap S$. But then $z \in N(y; \delta) \subseteq N^*(x; \varepsilon)$, so that $x \in S'$ and $x \in \text{cl } S$.

(c) and (d) are Exercise 18. ♦

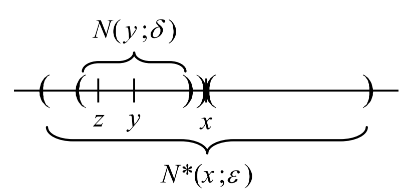

Figure 2 $x \in (\text{cl } S)'$, $y \in \text{cl } S$, and $z \in S$

Review of Key Terms in Section 3.4

Neighborhood	Boundary point	Accumulation point
Deleted neighborhood	Closed set	Isolated point
Interior point	Open set	Closure of a set

ANSWERS TO PRACTICE PROBLEMS

3.4.5 int $S = (1,2) \cup (2,3)$ and bd $S = \{1,2,3\}$.

3.4.9 The empty set $\varnothing$ is both open and closed, since it is the complement of the set $\mathbb{R}$, which is both open and closed. Or, to put it another way, $\varnothing$ is open since int $\varnothing = \varnothing$, and $\varnothing$ is closed since bd $\varnothing = \varnothing \subseteq \varnothing$.

3.4.13 There are many possibilities. For a simple one, let $A_n = [1/n, 2]$ for all $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} A_n = (0, 2]$, which is not closed.

3.4 *EXERCISES*

*Exercises marked with * are used in later sections, and exercises marked with ☆ have hints or solutions in the back of the book.*

1. Let $S \subseteq \mathbb{R}$. Mark each statement True or False. Justify each answer.

(a) int $S \cap$ bd $S = \varnothing$
(b) int $S \subseteq S$
(c) bd $S \subseteq S$
(d) S is open iff $S =$ int S.
(e) S is closed iff $S =$ bd S.
(f) If $x \in S$, then $x \in$ int S or $x \in$ bd S.
(g) Every neighborhood is an open set.
(h) The union of any collection of open sets is open.
(i) The union of any collection of closed sets is closed.

2. Let $S \subseteq \mathbb{R}$. Mark each statement True or False. Justify each answer.

(a) bd $S =$ bd $(\mathbb{R}\backslash S)$
(b) bd $S \subseteq \mathbb{R}\backslash S$
(c) $S \subseteq S' \subseteq$ cl S
(d) S is closed iff cl $S \subseteq S$.
(e) S is closed iff $S' \subseteq S$.
(f) If $x \in S$ and x is not an isolated point of S, then $x \in S'$.
(g) The set $\mathbb{R}$ of real numbers is neither open nor closed.
(h) The intersection of any collection of open sets is open.
(i) The intersection of any collection of closed sets is closed.

3. Find the interior of each set. ☆

(a) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

(b) $[0,3] \cup (3,5)$

(c) $\left\{r \in \mathbb{Q} : 0 < r < \sqrt{2}\right\}$

(d) $\left\{r \in \mathbb{Q} : r \geq \sqrt{2}\right\}$

(e) $[0,2] \cap [2,4]$

4. Find the boundary of each set in Exercise 3.

5. Classify each of the following sets as open, closed, neither, or both.

(a) $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$

(b) $\mathbb{N}$

(c) $\mathbb{Q}$

(d) $\bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right)$

(e) $\left\{x : |x-5| \leq \frac{1}{2}\right\}$

(f) $\{x : x^2 > 0\}$

6. Find the closure of each set in Exercise 5.

7. Let S and T be subsets of $\mathbb{R}$. Find a counterexample for each of the following.

(a) If P is the set of all isolated points of S, then P is a closed set.

(b) Every open set contains at least two points.

(c) If S is closed, then $\text{cl}(\text{int } S) = S$.

(d) If S is open, then $\text{int}(\text{cl } S) = S$.

(e) $\text{bd}(\text{cl } S) = \text{bd } S$

(f) $\text{bd}(\text{bd } S) = \text{bd } S$

(g) $\text{bd}(S \cup T) = (\text{bd } S) \cup (\text{bd } T)$

(h) $\text{bd}(S \cap T) = (\text{bd } S) \cap (\text{bd } T)$

8. Let S be a subset of $\mathbb{R}$ and let $x \in \mathbb{R}$. Prove that one and only one of the following three conditions holds:

(a) $x \in \text{int } S$

(b) $x \in \text{int}(\mathbb{R} \setminus S)$

(c) $x \in \text{bd } S = \text{bd}(\mathbb{R} \setminus S)$

9. Prove the following.

(a) An accumulation point of a set S is either an interior point of S or a boundary point of S.

(b) A boundary point of a set S is either an accumulation point of S or an isolated point of S.

10. Prove: If x is an isolated point of a set S, then $x \in \text{bd } S$.

11. If A is open and B is closed, prove that $A \backslash B$ is open and $B \backslash A$ is closed. ☆

12. Prove: For each $x \in \mathbb{R}$ and $\varepsilon > 0$, $N^*(x; \varepsilon)$ is an open set.

13. Prove: $(\text{cl } S) \backslash (\text{int } S) = \text{bd } S$. ☆

14. Let S be a bounded infinite set and let $x = \sup S$. Prove: If $x \notin S$, then $x \in S'$.

***15.** Prove: If x is an accumulation point of the set S, then every neighborhood of x contains infinitely many points of S. ☆

16. (a) Prove: $\text{bd } S = (\text{cl } S) \cap [\text{cl } (\mathbb{R} \backslash S)]$.

(b) Prove: $\text{bd } S$ is a closed set.

17. Prove: S' is a closed set. ☆

18. Prove Theorem 3.4.17(c) and (d).

19. Suppose S is a nonempty bounded set and let $m = \sup S$. Prove or give a counterexample: m is a boundary point of S.

20. Prove or give a counterexample: If a set S has a maximum and a minimum, then S is a closed set.

***21.** Let A be a nonempty open subset of $\mathbb{R}$ and let $\mathbb{Q}$ be the set of rationals. Prove that $A \cap \mathbb{Q} \neq \varnothing$.

22. Let S and T be subsets of $\mathbb{R}$. Prove the following.

(a) $\text{cl } (\text{cl } S) = \text{cl } S$

(b) $\text{cl } (S \cup T) = (\text{cl } S) \cup (\text{cl } T)$

(c) $\text{cl } (S \cap T) \subseteq (\text{cl } S) \cap (\text{cl } T)$

(d) Find an example to show that equality need not hold in part (c).

23. Let S and T be subsets of $\mathbb{R}$. Prove the following.

(a) $\text{int } S$ is an open set.

(b) $\text{int } (\text{int } S) = \text{int } S$

(c) $\text{int } (S \cap T) = (\text{int } S) \cap (\text{int } T)$

(d) $(\text{int } S) \cup (\text{int } T) \subseteq \text{int } (S \cup T)$

(e) Find an example to show that equality need not hold in part (d).

24. For any set $S \subseteq \mathbb{R}$, let $\overline{S}$ denote the intersection of all the closed sets containing S.

(a) Prove that $\overline{S}$ is a closed set.

(b) Prove that $\overline{S}$ is the smallest closed set containing S. That is, show that $S \subseteq \overline{S}$, and if C is any closed set containing S, then $\overline{S} \subseteq C$.

(c) Prove that $\overline{S} = \text{cl}\, S$.

(d) If S is bounded, prove that $\overline{S}$ is bounded.

25. For any set $S \subseteq \mathbb{R}$, let S^{o} denote the union of all the open sets contained in S.

(a) Prove that S^{o} is an open set.

(b) Prove that S^{o} is the largest open set contained in S. That is, show that $S^{\mathrm{o}} \subseteq S$, and if U is any open set contained in S, then $U \subseteq S^{\mathrm{o}}$.

(c) Prove that $S^{\mathrm{o}} = \text{int}\, S$.

26. In this exercise we outline a proof of the following theorem: A subset of $\mathbb{R}$ is open iff it is the union of countably many disjoint open intervals in $\mathbb{R}$.

(a) Let S be a nonempty open subset of $\mathbb{R}$. For each $x \in S$, let $A_x = \{a \in \mathbb{R} : (a, x] \subseteq S\}$ and let $B_x = \{b \in \mathbb{R} : [x, b) \subseteq S\}$. Use the fact that S is open to show that A_x and B_x are both nonempty.

(b) If A_x is bounded below, let $a_x = \inf A_x$. Otherwise, let $a_x = -\infty$. If B_x is bounded above, let $b_x = \sup B_x$; otherwise, let $b_x = \infty$. Show that $a_x \notin S$ and $b_x \notin S$.

(c) Let I_x be the open interval (a_x, b_x). Clearly, $x \in I_x$. Show that $I_x \subseteq S$. (*Hint*: Consider two cases for $y \in I_x : y < x$ and $y > x$.)

(d) Show that $S = \bigcup_{x \in S} I_x$.

(e) Show that the intervals $\{I_x : x \in S\}$ are pairwise disjoint. That is, suppose $x, y \in S$ with $x \neq y$. If $I_x \cap I_y \neq \varnothing$, show that $I_x = I_y$.

(f) Show that the set of distinct intervals $\{I_x : x \in S\}$ is countable.

Section 3.5 COMPACT SETS

In Section 3.4 we introduced several important topological concepts in $\mathbb{R}$. Some of these concepts related to points: interior points, boundary points, and accumulation points. Others related to sets: open sets and closed sets. In this section we define another type of set that occurs frequently in applications.

If we require a subset of $\mathbb{R}$ to be both closed and bounded, then it will have a number of special properties not possessed by sets in general. The first such property is called compactness, and although its definition may at first appear strange, it is really a widely used concept of analysis. (For example, see Theorems 5.3.2, 5.3.10, 5.4.6, and 5.5.9.)

3.5.1 DEFINITION A set S is said to be **compact** if whenever it is contained in the union of a family $\mathscr{F}$ of open sets, it is contained in the union of some finite number of the sets in $\mathscr{F}$. If $\mathscr{F}$ is a family of open sets whose union contains S, then $\mathscr{F}$ is called an **open cover** of S. If $\mathscr{G} \subseteq \mathscr{F}$ and $\mathscr{G}$ is also an open cover of S, then $\mathscr{G}$ is called a **subcover** of S. Thus S is compact iff every open cover of S contains a finite subcover.

3.5.2 EXAMPLE (a) Let $S = (0, 2)$ and for each $n \in \mathbb{N}$ let $A_n = (1/n, 3)$. (See Figure 1.) If $0 < x < 2$, then by the Archimedean property 3.3.10(c), there exists $p \in \mathbb{N}$ such that $1/p < x$. Thus $x \in A_p$, and $\mathscr{F} = \{A_n : n \in \mathbb{N}\}$ is an open cover for S. However, if $\mathscr{G} = \{A_{n_1}, \ldots, A_{n_k}\}$ is any finite subfamily of $\mathscr{F}$, and if $m =$ max $\{n_1, \ldots, n_k\}$, then

$$A_{n_1} \cup \cdots \cup A_{n_k} = A_m = \left(\frac{1}{m}, 3\right).$$

It follows that the finite subfamily $\mathscr{G}$ is not an open cover of $(0, 2)$. Since we have exhibited a particular open cover $\mathscr{F}$ that has no finite subcover, we conclude that the interval $(0, 2)$ is not compact.

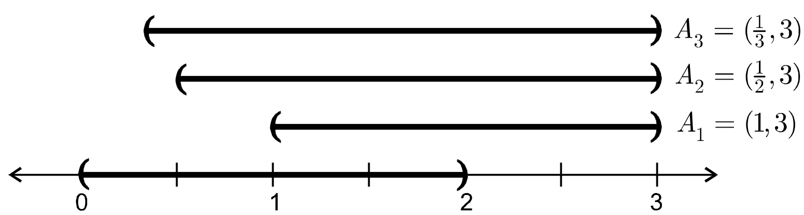

Figure 1 Part of an open cover of the interval (0, 2)

(b) Let $S = \{x_1, \ldots, x_n\}$ be a finite subset of $\mathbb{R}$, and let $\mathscr{F} = \{A_\alpha : \alpha \in \mathscr{A}\}$ be any open cover of S. For each $i = 1, \ldots, n$, there is a set A_{α_i} from $\mathscr{F}$ that contains x_i, since $\mathscr{F}$ is an open cover. It follows that the subfamily $\{A_{\alpha_1}, \ldots, A_{\alpha_n}\}$ also covers S. We conclude that any finite set is compact.

3.5.3 PRACTICE Show that $[0, \infty)$ is not compact by finding an open cover of $[0, \infty)$ that has no finite subcover.

Notice that in proving a set is compact we must show that any open cover (possibly containing uncountably many open sets) has a finite subcover. It is not sufficient to pick a particular open cover and extract a finite subcover. Because of this, it is often difficult to show directly that a given set satisfies the definition of being compact. Fortunately, the classical Heine-Borel theorem, which we prove following a preliminary lemma, gives us a much easier characterization to use for subsets of $\mathbb{R}$.

3.5.4 LEMMA If S is a nonempty closed bounded subset of $\mathbb{R}$, then S has a maximum and a minimum.

Proof: Since S is bounded above and nonempty, $m = \sup S$ exists by the completeness axiom. We want to show that $m \in S$. If m is an accumulation point of S, then since S is closed, we have $m \in S$ and $m = \max S$. If m is not an accumulation point of S, then for some $\varepsilon > 0$ we have $N^*(m;\varepsilon) \cap S = \varnothing$. But m is the *least* upper bound of S, so $N(m;\varepsilon) \cap S \neq \varnothing$. (If this intersection were empty, then $m - \varepsilon$ would be an upper bound of S.) Together these imply $m \in S$, so again we have $m = \max S$. Similarly, $\inf S \in S$, so $\inf S = \min S$. ♦

3.5.5 THEOREM (Heine–Borel) A subset S of $\mathbb{R}$ is compact iff S is closed and bounded.

Proof: First, let us suppose that S is compact. For each $n \in \mathbb{N}$, let $I_n = N(0;n) = (-n, n)$. Then each I_n is open and $S \subseteq \bigcup_{n=1}^{\infty} I_n$. Thus $\{I_n : n \in \mathbb{N}\}$ is an open cover of S. Since S is compact, there exist finitely many integers $n_1, \ldots, n_k$ such that

$$S \subseteq (I_{n_1} \cup \cdots \cup I_{n_k}) = I_m,$$

where $m = \max\{n_1, \ldots, n_k\}$. It follows that $|x| < m$ for all $x \in S$, and S is bounded.

To see that S must be closed, we suppose that it were not closed. Then there would exist a point $p \in (\mathrm{cl}\, S)\backslash S$. For each $n \in \mathbb{N}$, we let $U_n = \mathbb{R}\backslash[p - 1/n,\ p + 1/n]$. (The case where $p = 2$ is illustrated in Figure 2.) Now each U_n is an open set and we have

$$\bigcup_{n=1}^{\infty} U_n = \mathbb{R} \backslash \bigcap_{n=1}^{\infty} [p - 1/n, p + 1/n] = \mathbb{R} \backslash \{p\} \supseteq S,$$

by Exercise 2.1.26(d). Thus $\{U_n : n \in \mathbb{N}\}$ is an open cover of S. Since S is compact, there exist $n_1 < n_2 < \cdots < n_k$ in $\mathbb{N}$ such that $S \subseteq \bigcup_{i=1}^{k} U_{n_i}$. Furthermore, the U_n's are nested. That is, $U_m \subseteq U_n$ if $m \le n$. It follows that $S \subseteq U_{n_k}$. But then $S \cap N(p; 1/n_k) = \varnothing$, contradicting our choice of $p \in (\mathrm{cl}\, S)\backslash S$ and showing that S must be closed.

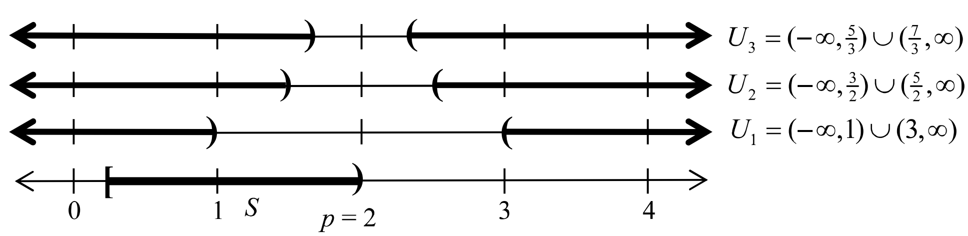

Figure 2 Part of an open cover of S

Conversely, suppose that S is closed and bounded. Let $\mathscr{F}$ be an open cover of S. For each $x \in \mathbb{R}$ let

$$S_x = S \cap (-\infty, x]$$

and let

$$B = \{x\colon S_x \text{ is covered by a finite subcover of } \mathscr{F}\}.$$

Since S is closed and bounded, Lemma 3.5.4 implies that S has a minimum, say d. Then $S_d = \{d\}$, and this is certainly covered by a finite subcover of $\mathscr{F}$. (See Figure 3.) Thus $d \in B$ and B is nonempty. If we can show that B is not bounded above, then it will contain a number z greater than sup S. But then $S_z = S$, and since $z \in B$, we can conclude that S is compact.

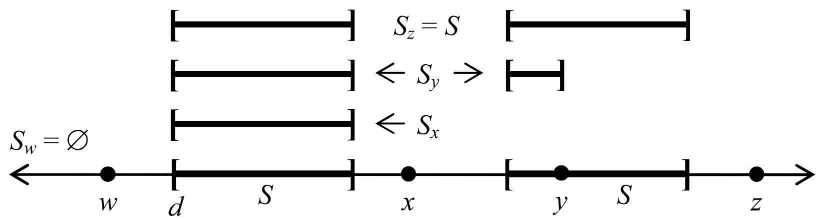

Figure 3 The set S_x for various values of x

To this end, we suppose that B is bounded above and let $m = \sup B$. We shall show that $m \in S$ and $m \notin S$ both lead to contradictions.

If $m \in S$, then since $\mathscr{F}$ is an open cover of S, there exists F_0 in $\mathscr{F}$ such that $m \in F_0$. Since F_0 is open, there exists an interval $[x_1, x_2]$ in F_0 such that

$$x_1 < m < x_2.$$

Since $x_1 < m$ and $m = \sup B$, there exist $F_1, \ldots, F_k$ in $\mathscr{F}$ that cover S_{x_1}. But then $F_0, F_1, \ldots, F_k$ cover S_{x_2}, so that $x_2 \in B$. This contradicts $m = \sup B$. (See Figure 4.)

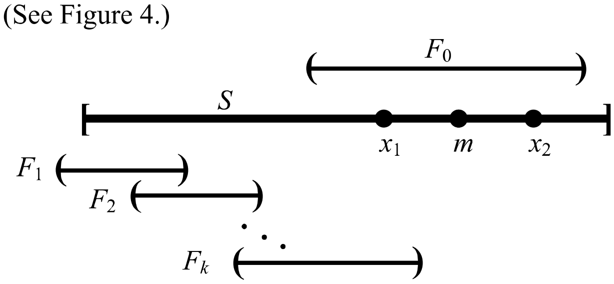

Figure 4 Showing $x_2 \in B$

On the other hand, if $m \notin S$, then since S is closed there exists an $\varepsilon > 0$ such that $N(m;\varepsilon) \cap S = \varnothing$. (See Figure 5.) But then

$$S_{m-\varepsilon} = S_{m+\varepsilon/2}.$$

Since $m-\varepsilon \in B$, we have $m+\varepsilon/2 \in B$, which again contradicts $m = \sup B$.

Since the possibility that B is bounded above leads to a contradiction, we must conclude that B is not bounded above, and hence S is compact. ♦

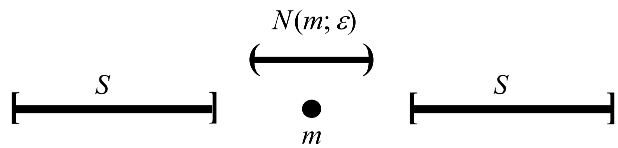

Figure 5 $S_{m-\varepsilon} = S_{m+\varepsilon/2}$

In Example 3.4.15 we showed that a finite set will have no accumulation points. We also saw that some unbounded sets (such as $\mathbb{N}$) have no accumulation points. As an application of the Heine–Borel theorem, we now derive the classical Bolzano–Weierstrass theorem, which states that these are the only conditions that can allow a set to have no accumulation points.

3.5.6 THEOREM (Bolzano–Weierstrass) If a bounded subset S of $\mathbb{R}$ contains infinitely many points, then there exists at least one point in $\mathbb{R}$ that is an accumulation point of S.

Proof: Let S be a bounded subset of $\mathbb{R}$ containing infinitely many points. Suppose that S has no accumulation points. Then S is closed by Theorem 3.4.17(a), so by the Heine–Borel theorem (3.5.5), S is compact. Since S has no accumulation points, given any $x \in S$, there exists a neighborhood $N(x)$ of x such that $S \cap N(x) = \{x\}$. (See Figure 6.) Now the family $\{N(x) : x \in S\}$ is an open cover of S, and since S is compact there exist $x_1, \ldots, x_n$ in S such that $\{N(x_1), \ldots, N(x_n)\}$ covers S. But

$$S \cap [N(x_1) \cup \cdots \cup N(x_n)] = \{x_1, \ldots, x_n\},$$

so $S = \{x_1, \ldots, x_n\}$. This contradicts S having infinitely many points. ♦

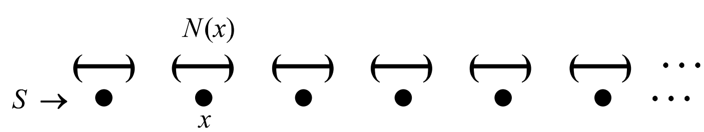

Figure 6 $S \cap N(x) = \{x\}$

We conclude this section with a result that illustrates an important property of compact sets.

3.5.7 THEOREM Let $\mathscr{F} = \{K_\alpha : \alpha \in \mathscr{A}\}$ be a family of compact subsets of $\mathbb{R}$. Suppose that the intersection of any finite subfamily of $\mathscr{F}$ is nonempty. Then $\bigcap \{K_\alpha : \alpha \in \mathscr{A}\} \neq \varnothing$.

Proof: For each $\alpha \in \mathscr{A}$, let $F_\alpha = \mathbb{R} \setminus K_\alpha$. Since each K_α is compact, it is closed and its complement F_α is open. Choose a member K of $\mathscr{F}$ and suppose that no point of K belongs to every K_α. Then every point of K belongs to some F_α. That is, the sets F_α form an open cover of K. Since K is compact, there exist finitely many indices $\alpha_1, \ldots, \alpha_n$ such that $K \subseteq (F_{\alpha_1} \cup \cdots \cup F_{\alpha_n})$. But

$$\begin{aligned} F_{\alpha_1} \cup \cdots \cup F_{\alpha_n} &= (\mathbb{R} \setminus K_{\alpha_1}) \cup \cdots \cup (\mathbb{R} \setminus K_{\alpha_n}) \\ &= \mathbb{R} \setminus (K_{\alpha_1} \cap \cdots \cap K_{\alpha_n}), \end{aligned}$$

by Exercise 2.1.26(d), so $K \cap (K_{\alpha_1} \cap \cdots \cap K_{\alpha_n}) = \varnothing$, a contradiction. Thus some point in K belongs to each K_α, and $\bigcap \{K_\alpha : \alpha \in \mathscr{A}\} \neq \varnothing$. ♦

3.5.8 COROLLARY (Nested Intervals Theorem) Let $\mathscr{F} = \{A_n : n \in \mathbb{N}\}$ be a family of closed bounded intervals in $\mathbb{R}$ such that $A_{n+1} \subseteq A_n$ for all $n \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty} A_n \neq \varnothing$.

Proof: Given any $n_1 < n_2 < \cdots < n_k$ in $\mathbb{N}$, we have $\bigcap_{i=1}^{k} A_{n_i} = A_{n_k} \neq \varnothing$. Thus Theorem 3.5.7 implies that $\bigcap_{n=1}^{\infty} A_n \neq \varnothing$. ♦

Review of Key Terms in Section 3.5

Compact set	Subcover	Bolzano–Weierstrass theorem
Open cover	Heine–Borel theorem	

ANSWERS TO PRACTICE PROBLEMS

3.5.3 One possibility is to let $A_n = (-1, n)$ for all $n \in \mathbb{N}$.

3.5 *EXERCISES*

*Exercises marked with * are used in later sections, and exercises marked with ☆ have hints or solutions in the back of the book.*

1. Mark each statement True or False. Justify each answer.

(a) A set S is compact iff every open cover of S contains a finite subcover.
(b) Every finite set is compact.
(c) No infinite set is compact.
(d) If a set is compact, then it has a maximum and a minimum.
(e) If a set has a maximum and a minimum, then it is compact.