MTH 4010 PMWA Homework 2 due Wednesday, February 19

You are encouraged to collaborate with other students in this class, but you must write up your answers IN YOUR OWN WORDS. Do not look at the solutions of others while writing your own. You are required to list and identify clearly all sources (e.g. a particular theorem from a reference book) and collaborators. ("Wikipedia" is too vague.) On the other hand, you don't need to list as sources the instructor or the textbook. Do list classmates, tutors and any other source animate or inanimate. Your grade will not count unless you submit this information. You also need to SHOW YOUR WORK IN DETAILS (e.g. proofs, steps of calculation, reference theorems) to receive credit. The "Recommended problems" below will NOT be collected, but may be tested in exams. All the other problems will be collected.

The class on Wednesday, Feb. 19 will be cancelled. Please email me (yumeng.ou@baruch.cuny.edu) a copy of your Homework 2 by 7pm on Feb. 19. I'll acknowledge receipt of your homework by replying your email.

Exercise 3.3, #10 (a):

Solution. Since $x < y$, according to the density theorem of rational numbers in $\mathbb{R}$, there exists $r_1 \in \mathbb{Q}$ such that $x < r_1 < y$. Apply the theorem again to the pair x, r_1, one can find $r_2 \in \mathbb{Q}$ such that $x < r_2 < r_1$. Repeating the process, one can find infinitely many rational numbers $r_1, r_2, r_3, \ldots$ that are all in the interval $[x, y]$ (in fact, (x, y)). $\square$

Exercise 3.3, #13:

Solution. It is obvious that the set $S := \{q \in \mathbb{Q} : q < x\}$ is bounded from the above, hence according to the Completeness Axiom, there exists $M := \sup S \in \mathbb{R}$. Our goal is to show that $x = M$.

First, assume that $x > M$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists $r \in (M, x) \cap \mathbb{Q}$. Hence, $r \in S$. But this contradicts the fact that $M = \sup S$.

Second, assume that $x < M$. We will show that this would also be impossible, hence we must have $x = M$.

To see this, apply the density theorem of $\mathbb{Q}$ in $\mathbb{R}$ again, one can find $r \in \mathbb{Q}$ so that $x < r < M$. (Note that, it is not important here that $r \in \mathbb{Q}$, we could have chosen an irrational number.) It is direct to see that r is an upper bound of S, but is smaller than its least upper bound M, hence we have a contradiction. $\square$

Exercise 3.4, #3, #4 (b, c):

Solution. Part (b): Note that the set $S := [0, 3] \cup (3, 5) = [0, 5)$, so its interior is $(0, 5)$. It is also easy to see that its boundary is bd$S = \{0, 5\}$.

Part (c): The interior is $\emptyset$. This is because irrational numbers are dense in $\mathbb{R}$. For any $x \in S := \{r \in \mathbb{Q} : 0 < r < \sqrt{2}\}$, for all $\epsilon > 0$, there must exist some irrational numbers contained in $N(x, \epsilon)$, hence $N(x, \epsilon) \not\subset S$. Therefore, no point in S is an interior point.

The above discussion also implies that all points in S must be boundary points. Moreover, since $\mathbb{Q}$ is dense in $\mathbb{R}$, 0 and $\sqrt{2}$ are also boundary points. Altogether, one has bd$S = S \cup \{0, \sqrt{2}\}$. □

Exercise 3.4, #5, #6 (a, d):

Solution. Part (a): the set $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. Since no point in S is an interior point, one has int$S = \emptyset \neq S$, hence S is not open. Moreover, consider the point 0. According to the corollary of Archimedean property, for all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \epsilon$, which means $N(0, \epsilon) \cap S \neq \emptyset$. Since $0 \notin S$, one also obviously has $N(0, \epsilon) \cap S^c \neq \emptyset$. This implies that $0 \in$ bd$S$, and in particular, bd$S \not\subset S$. This shows that S is also not closed.

We learnt from the discussion above that all points in S are boundary points and 0 is another boundary point of S. We claim that the closure $\bar{S} = S \cup \{0\}$. It is easy to see that no negative number could be a boundary point, and no point that is larger than 1 could be either. Now, given any $x \in S^c \cap (0, 1)$, we aim to prove that $x \notin$ bd$S$. To see this, apply Exercise #9 from Section 3.3, there exists a unique $n \in \mathbb{N}$ such that $n - 1 \leq \frac{1}{x} < n$. This is the same as saying $\frac{1}{n} < x \leq \frac{1}{n-1}$. Let $\epsilon = \frac{1}{2}\min(x - \frac{1}{n}, \frac{1}{n-1} - x) > 0$, one has that $N(x, \epsilon) \cap S = \emptyset$. Therefore, x is not a boundary point of S.

Part (d): the set $S = \bigcap_{n=1}^{\infty}(0, \frac{1}{n})$. We claim that $S = \emptyset$. Indeed, for all $x \leq 0$, it is obvious that $x \notin S$. If $x > 0$, one can apply Archimedean to find $n \in \mathbb{N}$ so that $0 < \frac{1}{n} < x$, hence $x \notin (0, \frac{1}{n})$ which means $x \notin S$.

From the discussion in class, we know that $S = \emptyset$ is both open and closed. Moreover, $\bar{S} = S = \emptyset$. □

Exercise 3.4, #7 (d):

Solution. Let $S = (0, 1) \cup (1, 2)$, then $\bar{S} = [0, 2]$. Hence, int$\bar{S} = (0, 2)$, which is different from S. □

Exercise 3.4, #9:

Solution. Part (a): It is easy to see from definition that interior points must all be accumulation points. Now, suppose x is an accumulation point of S but is not in int$S$, our goal is to show $x \in$ bd$S$. First, if $x \in S$, then from $x \notin$ int$S$ one knows immediately that $x \in$ bd$S$. Assume that $x \notin S$. For all $\epsilon > 0$, it is obvious that $N(x, \epsilon) \cap S^c \neq \emptyset$. Since x is an accumulation point, one also has

$$N(x, \epsilon) \cap S \supset N^*(x, \epsilon) \cap S \neq \emptyset.$$

Therefore, by definition, $x \in \text{bd}S$.

Part (b): First, one observes by definition that isolated points must all be boundary points. Let $x \in \text{bd}S$ and assume x is not isolated, then our goal is to show that $x \in S'$. For all $\epsilon > 0$, since $x \in \text{bd}S$, one has $N(x, \epsilon) \cap S \neq \emptyset$. Suppose $N^*(x, \epsilon) \cap S \neq \emptyset$, then we are done. Otherwise, the implication is that $N(x, \epsilon) \cap S = \{x\}$. However, this would imply that x is an isolated point of S, hence is a contradiction.

$\square$

Exercise 3.4, #11:

Solution. First, we prove $A \setminus B$ is open. It suffices to show that each point in $A \setminus B$ is an interior point. Let $x \in A \setminus B$, since A is open and $x \in A$, there exists $\epsilon_1 > 0$ such that $N(x, \epsilon_1) \subset A$. Also, since B is closed, one has B^c is open. Since $x \in B^c$, there exists $\epsilon_2 > 0$ such that $N(x, \epsilon_2) \subset B^c$. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$, then the above implies that $N(x, \epsilon) \subset A \cap B^c = A \setminus B$, which shows that $x \in \text{int}(A \setminus B)$, i.e. $A \setminus B$ is open.

Second, we prove $B \setminus A$ is closed. It suffices to show $(B \setminus A)^c = (B \cap A^c)^c = B^c \cup A$ is open. Since B is closed, we know B^c is open. Since A is also open, from Theorem 3.4.10, one has $B^c \cup A$ is open. Therefore, $B \setminus A$ is closed. $\square$

Exercise 3.4, #15:

Solution. Let $x \in S'$, for all $\epsilon > 0$, our goal is to show that $N(x, \epsilon) \cap S$ contains infinitely many points.

Suppose this is not true, i.e. $N(x, \epsilon) \cap S$ contains only finitely many points, hence, in particular only finitely many points that are different from x, say, $x_1, \cdots, x_k$. Then, let $\epsilon_1 = \min\{|x_i - x| : 1 \leq i \leq k\} > 0$, one has

$$N^*(x, \epsilon_1) \cap S \subset N(x, \epsilon_1) \cap S = \emptyset.$$

This contradicts the fact that x is an accumulation point of S. Therefore, the assumption is not true, i.e. $N(x, \epsilon) \cap S$ contains infinitely many points. $\square$

Exercise 3.5, #3 (a, c):

Solution. Part (a): the set $S = [1, 3)$. Define open set $O_n = (0, 3 - \frac{1}{n})$, $\forall n \in \mathbb{N}$. Then $\mathcal{F} = \{O_n\}_{n \in \mathbb{N}}$ form an open over of S. However, there is no subcover for S. Indeed, suppose $\{O_{n_1}, \cdots, O_{n_k}\}$ is such a subcover. Let $M = \max\{n_1, \cdots, n_k\}$, then $\bigcup_{i=1}^k O_{n_i} \subset (0, 3 - \frac{1}{M})$, which obviously doesn't cover S.

Part (c): the set $S = \mathbb{N}$. Define open set $O_n = (n - \frac{1}{2}, n + \frac{1}{2})$, $\forall n \in \mathbb{N}$, then $\mathcal{F} = \{O_n\}_{n \in \mathbb{N}}$ form an open cover of S. However, any finite subcollection of such open sets can only cover finitely many natural numbers, hence fails to form a subcover of $\mathbb{N}$. $\square$

Exercise 3.5, #5 (a):

Solution. Let $\mathcal{F}$ be any open cover of $S_1 \cup S_2$, our goal is to find a finite subcover of it. Since S_1 is compact and $\mathcal{F}$ is an open cover of S_1, there exists a finite subcover $\mathcal{F}_1 \subset \mathcal{F}$ for S_1. Similarly, there exists a finite subcover $\mathcal{F}_2 \subset \mathcal{F}$ for the compact set S_2. Now, it is easy to see that $\mathcal{F}_1 \cup \mathcal{F}_2$ is a finite subcollection of $\mathcal{F}$ and it covers $S_1 \cup S_2$. $\square$

Recommended problems:

Exercise 3.3, #9:

Solution. Part (a): Recall that when proving the density theorem of $\mathbb{Q}$ in $\mathbb{R}$ in class, as an intermediate step, we proved the following result: $\forall x \in \mathbb{R}$, $\exists m \in \mathbb{Z}$ so that $|x - m| < 1$.

Let $y > 0$, applying the result above, one can find $m \in Z$ such that $|y - m| < 1$. There are two cases.

First, assume $m \leq y$. Let $n = m + 1$. Then one must have $n \in \mathbb{N}$. Since otherwise one has $n \leq 0$, which together with $|y - m| < 1$ would contradict $y > 0$. Now, Triangle inequality implies that $y < m + 1 = n$. Combined with $y \geq m = n - 1$, this implies that $n - 1 \leq y < n$.

Second, assume $m > y$, then from $y > 0$ one knows that $m \in \mathbb{N}$. Let $n = m$. It is direct to see that $y < m = n$. On the other hand, $|y - n| < 1$ is the same as saying $n - y < 1$, which implies $n - 1 < y$. Altogether, we obtained that $n - 1 < y < n$.

Part (b): Suppose there are two different $n, n' \in \mathbb{N}$ both satisfying the claim in part (a). One has $n - 1 \leq y < n'$, which implies $n - n' < 1$. By symmetry, there also holds $n' - n < 1$. Therefore, one has $|n - n'| < 1$. Since natural numbers are apart from each other by at least distance 1, this implies that $n = n'$. $\square$

Exercise 3.3, #10 (b):

Solution. We already proved part (a) in the above. Part (b) here can be proved following the exact same strategy, except that one applies the density theorem of irrational numbers in $\mathbb{R}$ instead. We omit the details. $\square$

Exercise 3.4, #3, #4 (a, d, e):

Solution. Part (a): the set $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. No point in S is an interior point since all its neighborhood contains at least an irrational number, so int$S = \emptyset$. Also, from our discussion of Exercise #5 (a) above, the boundary of the set S is $S \cup \{0\}$.

Part (d): the set $S = \{r \in \mathbb{Q} : r \geq \sqrt{2}\}$. For all point in S, any of its neighborhood contains some irrational numbers, hence is not completely contained in S. Therefore, int$S = \emptyset$. This also implies that all points in S are boundary points. In addition, by density of $\mathbb{Q}$ in $\mathbb{R}$, for any $x \geq \sqrt{2}$, it is direct to see by definition that x is also a boundary point. Therefore, $\text{bd}S = [\sqrt{2}, \infty)$.

Part (e): the set $S = [0, 2] \cap [2, 4] = \{2\}$. Hence, $\text{int}S = \emptyset$ and $\text{bd}S = \{2\}$. $\square$

Exercise 3.4, #5, #6 (b, c, e, f):

Solution. Part (b): the set $S = \mathbb{N}$. As discussed in class, $\text{int}S = \emptyset \neq S$, so S is not open. Moreover, $\text{bd}S = S$, so $S \supset \text{bd}S$ which implies S is closed and $\bar{S} = S = \mathbb{N}$.

Part (c): the set $S = \mathbb{Q}$. Because of density of irrational numbers in $\mathbb{R}$, the interior of S is empty, hence S is not open. However, also by the density of $\mathbb{Q}$ in $\mathbb{R}$, one can check that $\text{bd}S = \mathbb{R}$. Therefore, $S \not\supset \text{bd}S$, i.e. S is not closed. And one has $\bar{S} = \mathbb{R}$.

Part (e): the set $S = \{x : |x - 5| \leq \frac{1}{2}\} = [\frac{9}{2}, \frac{11}{2}]$. It is direct to see that S is closed and is not open. Moreover, $\bar{S} = S$.

Part (f): the set $S = \{x : x^2 > 0\} = \mathbb{R} \setminus \{0\}$. One can see by definition that S is open. Also, since $0 \in \text{bd}S$, but $0 \notin S$, one has S is not closed and $\bar{S} = \mathbb{R}$. $\square$

Exercise 3.4, #7 (b, c, f):

Solution. Part (b): $S = \emptyset$ is an open set, as explained in class, but it contains less than two points.

Part (c): Take $S = \mathbb{N}$, we know that S is a closed set since it contains all its boundary points. However, $\text{int}S = \emptyset$ hence $\text{cl}(\text{int}S) = \emptyset \neq S$.

Part (f): Consider $S = \{r \in \mathbb{Q} : 0 < r < 1\}$. Then one can obtain from a similar discussion as in Exercise #6 (c) that $\text{bd}S = [0, 1]$. Hence, $\text{bd}(\text{bd}S) = \{0, 1\} \neq \text{bd}S$. $\square$

Exercise 3.4, #8:

Solution. Suppose first $x \in \text{int}S$. We need to prove $x \notin \text{int}(S^c)$ and $x \notin \text{bd}S$. It is direct to see that $x \in \text{int}S$ implies that $x \in S$, hence $x \notin S^c$, which means $x \notin \text{int}(S^c)$. Also, since $x \in \text{int}S$, there exists $\epsilon > 0$ such that $N(x, , \epsilon) \subset S$. In particular, $N(x, \epsilon) \cap S^c = \emptyset$, which implies $x \notin \text{bd}S$.

Now, assume $x \notin \text{int}S$. We need to prove that one either has $x \in \text{int}(S^c)$ or $x \in \text{bd}S$, but not both.

To see this, suppose $x \in S$. Then since $x \notin \text{int}S$, for all $\epsilon > 0$, $N(x, \epsilon) \not\subset S$, i.e. $N(x, \epsilon) \cap S^c \neq \emptyset$. It is also obvious that $N(x, \epsilon) \cap S \supset \{x\} \neq \emptyset$. So $x \in \text{bd}S$. Since $x \in S$, there is no way that x could be an interior point of S^c.

Next, assume $x \in S^c$. Suppose there exists $\epsilon > 0$ such that $N(x, \epsilon) \subset S^c$, then $x \in \text{int}(S^c)$. And in this case, $N(x, \epsilon) \cap S = \emptyset$, so $x \notin \text{bd}S$. Suppose this is not true, it means for all $\epsilon > 0$, $N(x, \epsilon) \cap S \neq \emptyset$. But one also has $N(x, \epsilon) \cap S^c \neq \emptyset$, so $x \in \text{bd}S$ but $x \notin \text{int}(S^c)$. $\square$

Exercise 3.4, #10:

Solution. If x is an isolated point of S, then there exists $\epsilon > 0$ such that $N^*(x,\epsilon) \cap S = \emptyset$. This implies that

$$N(x,\epsilon) \cap S^c \supset N^*(x,\epsilon) \cap S^c \neq \emptyset.$$

Since $x \in S$ by definition of isolated points, so $N(x,\epsilon) \cap S \neq \emptyset$. In conclusion, $x \in \mathrm{bd}S$. $\square$

Exercise 3.4, #13:

Solution. First, let $x \in \bar{S} \setminus \mathrm{int}S$, we prove that $x \in \mathrm{bd}S$. From Exercise #8, if $x \notin \mathrm{int}S$, there are only two possibilities, $x \in \mathrm{bd}S$ (which is what we want), or $x \in \mathrm{int}(S^c)$. Suppose $x \in \mathrm{int}(S^c)$, then $x \notin S$ and $x \notin \mathrm{bd}S$. Since $\bar{S} = S \cup \mathrm{bd}S$, this clearly contradicts $x \in \bar{S}$. Therefore, we should have $x \in \mathrm{bd}S$. This implies that $\bar{S} \setminus \mathrm{int}S \subset \mathrm{bd}S$.

Second, let $x \in \mathrm{bd}S$. By definition, this implies that $x \in \bar{S}$. Applying Exercise #8 again, one knows that $x \notin \mathrm{int}S$. Therefore, $x \in \bar{S} \setminus \mathrm{int}S$, which implies that $\mathrm{bd}S \subset \bar{S} \setminus \mathrm{int}S$.

In conclusion, $\bar{S} \setminus \mathrm{int}S = \mathrm{bd}S$. $\square$

Exercise 3.4, #16:

Solution. Part (a): since $\mathrm{bd}S \subset \bar{S}$, and $\mathrm{bd}S = \mathrm{bd}(S^c) \subset \overline{S^c}$, it is easy to see that $\mathrm{bd}S \subset \bar{S} \cap \overline{S^c}$. It suffices to prove the other inclusion.

Suppose $x \in \bar{S} \cap \overline{S^c}$. From $x \in \bar{S} = S \cup \mathrm{bd}S$, we know that $x \in \mathrm{bd}S$, since otherwise, we must have $x \in \mathrm{int}S$, which by Exercise #8 implies that $x \notin \mathrm{int}(S^c)$ and $x \notin \mathrm{bd}(S^c)$ hence $x \notin \overline{S^c}$. This is a contradiction. $\square$

Exercise 3.4, #17:

Solution. It suffices to show that $(S')^c$ is open. Suppose $x \in (S')^c$, by definition, there exists $\epsilon > 0$ such that $N^*(x,\epsilon) \cap S = \emptyset$. Our goal is to show that $N(x,\epsilon) \subset (S')^c$ (then it would imply x is an interior point of $(S')^c$, hence $(S')^c$ is open).

To see this, we need to prove that for all $y \in N(x,\epsilon)$, y is not an accumulation point of S. Choose $\epsilon_y = \epsilon - |x-y| > 0$, then one has $N(y,\epsilon_y) \subset N(x,\epsilon) \subset S^c$, in other words, $N(y,\epsilon_y) \cap S = \emptyset$. Hence, y is not an accumulation point of S, which completes the proof. $\square$

Exercise 3.4, #22 (a):

Solution. From the result we proved in class, $\bar{S}$ is a closed set. Moreover, we showed that a set A is closed if and only if $\bar{A} = A$. Hence, one has $\bar{\bar{S}} = \bar{S}$. $\square$

Exercise 3.4, #23 (a, b):

Solution. Part (a): this will follow from part (b) below and the definition of open set directly.

Part (b): let $x \in \text{int}S$, it suffices to show that x is an interior point of $\text{int}S$ (note, not S!). Since $x \in \text{int}S$, there exists $\epsilon > 0$ such that $N(x, \epsilon) \subset S$. Our goal is to show that $N(x, \epsilon) \subset \text{int}S$.

To see this, for any $y \in N(x, \epsilon)$, choose $\epsilon_y = \epsilon - |x - y| > 0$, then $N(y, \epsilon_y) \subset N(x, \epsilon) \subset S$. Therefore, $y \in \text{int}S$. This implies that $N(x, \epsilon) \subset \text{int}S$ and the proof is complete. $\square$