MTH 4010 PMWA Homework 1 due Wednesday, February 5

You are encouraged to collaborate with other students in this class, but you must write up your answers IN YOUR OWN WORDS. Do not look at the solutions of others while writing your own. You are required to list and identify clearly all sources (e.g. a particular theorem from a reference book) and collaborators. ("Wikipedia" is too vague.) On the other hand, you don't need to list as sources the instructor or the textbook. Do list classmates, tutors and any other source animate or inanimate. Your grade will not count unless you submit this information. You also need to SHOW YOUR WORK IN DETAILS (e.g. proofs, steps of calculation, reference theorems) to receive credit. The "Recommended problems" below will NOT be collected, but may be tested in exams. All the other problems will be collected.

Exercise 3.2, #3 (a, c):

Solution. Part (a): Recall that in class, we have already proved that for all $x \in \mathbb{R}$, there is a unique number $-x \in \mathbb{R}$ such that $x + (-x) = 0$. Therefore, to show $-(-x) = x$, it suffices to verify that x is the negative of $-x$, in other words, $(-x) + x = 0$ by definition. Note that this is a consequence of $x + (-x) = 0$ and the property (A2). The proof is hence complete.

Part (c): We first prove $\frac{1}{x} \neq 0$. To see this, suppose $\frac{1}{x} = 0$, then according to Theorem 3.2.2(b), for all $y \in \mathbb{R}$, there should hold $y \cdot \frac{1}{x} = y \cdot 0 = 0$. However, if we take $y = x$, this implies $x \cdot \frac{1}{x} = 0$. Since we already know by property (M5) that $x \cdot \frac{1}{x} = 1$, this is a contradiction. Therefore, one should have $\frac{1}{x} \neq 0$.

We now turn to prove $\frac{1}{1/x} = x$. By the uniqueness of the reciprocal, it suffices to show that $\frac{1}{x} \cdot x = 1$, which is obviously implied by $x \cdot \frac{1}{x} = 1$ and property (M2). $\square$

Exercise 3.2, #4:

Solution. According to Theorem 3.2.8 (taking $y = 0$), one has from the assumption $x \leq 0 + \epsilon$, $\forall \epsilon > 0$ that $x \leq 0$. Since we also have $x \geq 0$, there must hold $x = 0$ by property (O1).

Alternatively, if you don't want to use Theorem 3.2.8, this can be proved directly. Suppose $x \neq 0$, since $x \geq 0$, one must have $x > 0$. Define $\epsilon = \frac{x}{2}$. Then obviously $\epsilon > 0$ by property (O4). However, since

$$x \overset{\text{(DL)}}{=} \frac{x}{2} + \frac{x}{2} = \epsilon + \epsilon \overset{\text{(O3)}, \epsilon > 0}{>} \epsilon,$$

one arrives at a contradiction, which implies $x = 0$.

$\square$

Exercise 3.2, #5:

Solution. First, if either one of x, y is 0, then it is direct to see that $|xy| = |0| = 0$ and $|x| \cdot |y| = 0$, so we are done.

Now, assume neither of x, y is 0. There are four cases left to consider:

If $x > 0$ and $y > 0$, then by definition, $|x| = x$, $|y| = y$. Moreover, by property (O4), one has

$$x \cdot y > x \cdot 0 = 0,$$

which implies $|xy| = xy$. Hence, the desired result reduces to $xy = xy$, which is obviously true.

If $x > 0$ and $y < 0$ (and symmetrically if $x < 0$ and $y > 0$, which we omit), then by definition, $|x| = x$, $|y| = -y$. Moreover,

$$x \cdot y \overset{(O4)}{<} x \cdot 0 = 0,$$

which implies $|xy| = -xy = x \cdot (-y) = |x| \cdot |y|$.

The last case is $x < 0$ and $y < 0$. According to Theorem 3.2.2(g), one has

$$x \cdot y > 0 \cdot y = 0.$$

Therefore,

$$|xy| = xy \overset{\text{Exercise3(a)}}{=} [-(-x)] \cdot [-(-y)] = (-|x|) \cdot (-|y|).$$

According to Theorem 3.2.2(d), $-|x| = (-1) \cdot |x|$, same for the other term. Hence, the above is further equal to

$$= (-1) \cdot |x| \cdot (-1) \cdot |y| \overset{(M2)}{=} (-1) \cdot (-1)|x||y| = (-(-1))|x||y| = |x||y|.$$

□

Exercise 3.2, #6 (a, c):

Solution. Part (a): By Theorem 3.2.10(b), the desired result is equivalent to

$$-|x-y| \leq |x| - |y| \leq |x-y|.$$

By definition of the absolute value, for all $z \in \mathbb{R}$, $|z| = |-z|$. So the above is further equivalent to

$$-|y-x| \leq |x| - |y| \leq |x-y|.$$

To obtain the first inequality, define $y' = y - x$, then Theorem 3.2.10(d) (Triangle inequality) implies that

$$|y| = |x + y'| \leq |x| + |y'| = |x| + |y - x|,$$

which implies $-|y-x| \leq |x| - |y|$ due to (O3) (for "<") and Theorem 3.2.2(a) (for "="). Similarly, to prove the second inequality, define $x' = x - y$, then again the Triangle inequality implies that

$$|x| = |x' + y| \leq |x'| + |y| = |x - y| + |y|,$$

which is the same as $|x| - |y| \leq |x - y|$.

Part (c): By definition, $|z| \geq 0$, $\forall z \in \mathbb{R}$. In particular, $|x-y| \geq 0$. Since $|x-y| < \epsilon$, $\forall \epsilon > 0$, according to Exercise #4, one has $|x - y| = 0$.

To show that this implies $x = y$, it suffices to prove that $x - y = 0$ (by Theorem 3.2.2(a)). Suppose this is not true, then one either has $x - y > 0$ or $x - y < 0$ according to (O1). If $x - y > 0$, then by definition, $|x-y| = x-y > 0$, which is impossible. If $x-y<0$, definition implies that $|x-y| = -(x-y) > 0$ by Theorem 3.2.2(c, d, f), which again is a contradiction. Therefore, one has $x-y=0$.

$\square$

Exercise 3.3, #3, #4 (c, h, k, n):

Solution. Part (c): the set $S = [0,4]$, it is direct to see that $\sup S = \max S = 4$ and $\inf S = \min S = 0$.

Part (h): the set $S = \{(-1)^n(1+\frac{1}{n}) : n \in \mathbb{N}\}$. It is easy to check that S is bounded from above and below, so according to the Completeness Axiom, both $\sup S$ and $\inf S$ exist. Denote $a_n = (-1)^n(1+\frac{1}{n})$. Then one has $a_1 = -(1+1)$, $a_2 = 1+\frac{1}{2}$, $a_3 = -(1+\frac{1}{3})$, ... One observes that the number a_n oscillates between positive and negative, and as n increases, $|a_n|$ approaches 1. Hence, $\inf S = \min S = a_1 = -2$, and $\sup S = \max S = a_2 = \frac{3}{2}$.

Part (k): the set $S = \bigcap_{n=1}^{\infty}(1-\frac{1}{n}, 1+\frac{1}{n})$. It is obvious that $S \subset (1-1, 1+1) = (0,2)$, so S is bounded both from above and below. According to the Completeness Axiom, $\sup S$ and $\inf S$ both exist in $\mathbb{R}$. For any $x\mathbb{R}$ such that $x \neq 0$, by Theorem 3.3.10(c) (and of course, its symmetric statement for negative numbers), there exists $n \in \mathbb{N}$ such that $x \notin (1-\frac{1}{n}, 1+\frac{1}{n})$, which implies that $x \notin S$. Hence, $S = \{0\}$. It is thus obvious that $\sup S = \max S = 0 = \min S = \inf S$.

Part (n): the set $S = \{r \in \mathbb{Q} : r^2 \leq 5\}$. This is very similar to the example we discussed at the end of Lecture #2. One can first show that $r^2 \leq 5$ iff $-\sqrt{5} \leq r \leq \sqrt{5}$. (To see this, suppose $r^2 \leq 5$, then if r is not in the claimed range, one must have $r > \sqrt{5}$ or $r < -\sqrt{5}$. In either case, by (O2), (O4), or Theorem 3.2.2(g), there must hold $r^2 > \sqrt{5}\cdot\sqrt{5} = 5$, contradiction. On the other hand, suppose $-\sqrt{5} \leq r \leq \sqrt{5}$, then by a case by case study depending on whether $r > 0$ or not, one can similarly prove as above that $r^2 \leq 5$, we omit the details.)

Then, $S = \{r \in \mathbb{Q} : -\sqrt{5} \leq r \leq \sqrt{5}\}$. One can see that neither $\max S$ nor $\min S$ exists (since $\sqrt{5}$, hence $-\sqrt{5}$ as well, is irrational), and $\sup S = \sqrt{5}$, $\inf S = -\sqrt{5}$.

$\square$

Exercise 3.3, #6 (a):

Solution. We prove by contradiction. Suppose there are two different real numbers x, y, both are the supremum of S. Without loss of generality, assume $x > y$. By definition of $x = \sup S$, for all $x' < x$, there must exist $s_0 \in S$ such that $x' < s_0$. Take $x' = y$, then one obtains an element $s_0 \in S$ with $y < s_0$.

On the other hand, by definition of $y = \sup S$, one should have $y \geq s$, $\forall s \in S$, which is a contradiction. Therefore, the assumption is false, i.e. there is only a unique $\sup S$. $\square$

Exercise 3.3, #8:

Solution. Since S, T are nonempty and bounded, by the Completeness Axiom, all the supremums/infimums exist. Let $s \in S$, by definition, there holds $s \leq \sup S$ and $\inf S \leq s$. Hence, using (O2), one gets $\inf S \leq \sup S$.

Now let's compare the two supremums. Since $S \subset T$, for all $s \in S$, one has $s \in T$, therefore, $s \leq \sup T$, in other words, $\sup T$ is an upper bound of S. Since $\sup S$ is defined to be the least upper bound of S, one automatically obtains $\sup S \leq \sup T$. Following the exact same argument, one can show that $\inf T$ is an lower bound of S, hence the greatest lower bound of S satisfies $\inf S \geq \inf T$. Combining everything together, we have proved that

$$\inf T \leq \inf S \leq \sup S \leq \sup T.$$

□

Recommended problems:

Problem 1: Prove part (e) of Theorem 3.2.2.

Solution. This is precisely Practice 3.2.3. □

Exercise 3.2, #3 (b, d, k):

Solution. Part (b): We first prove $(-x) \cdot y = -(xy)$. By definition of negative and its uniqueness, it suffices to show that $xy + (-x) \cdot y = 0$. This can be deduced as the following:

$$xy + (-x) \cdot y \overset{\text{(DL)}}{=} [x + (-x)] \cdot y = 0 \cdot y \overset{\text{Theorem3.2.2(b)}}{=} 0.$$

Next, to prove $(-x) \cdot (-y) = xy$. We argue as follows.

$$\begin{aligned}(-x) \cdot (-y) &\overset{\text{Theorem3.2.2(d)}}{=} (-1) \cdot x \cdot (-1) \cdot y \overset{\text{(M2),(M3)}}{=} (-1) \cdot (-1) \cdot (xy) \\ &\overset{\text{Theorem3.2.2(d)}}{=} (-(-1)) \cdot (xy) \overset{\text{Exercise \#3(a)}}{=} 1 \cdot (xy) \overset{\text{(M4)}}{=} xy.\end{aligned}$$

Part (d): Since $z \neq 0$, by (M5), there is $\frac{1}{z} \in \mathbb{R}$ and $\frac{1}{z} \neq 0$ because of Exercise #3(c). Since $xz = yz$, one has

$$xz \cdot \frac{1}{z} = yz \cdot \frac{1}{z}.$$

Applying (M3), (M4) and (M5), one obtains $x = x \cdot 1 = y \cdot 1 = y$.

Part (k): If the desired result is not true, then either $x > 0, y < 0$ or $x < 0, y > 0$. By symmetry, it suffices to show the first case is impossible. If $x > 0$ and $y < 0$, by (O4), one should have $xy = y \cdot x < 0 \cdot x = 0$, which contradicts $xy > 0$. □

Exercise 3.2, #6 (b):

Solution. Define $x' = x - y$, then by Triangle inequality, one has

$$|x| = |x' + y| \leq |x'| + |y| = |x - y| + |y|.$$

Since $|x - y| < c$, by (O3), one obtains $|x - y| + |y| < c + |y|$. Therefore, $|x| < |y| + c$, where we have used (O2). $\square$

Exercise 3.3, #3, #4 (i, m):

Solution. Part (i): Let $a_n = n + \frac{(-1)^n}{n}$. It is easy to see that the set S is unbounded from above, so there is no supremum, no maximum. Observe that $a_1 = 1 - 1 = 0$, $a_2 = 2 + \frac{1}{2}$, $a_3 = 3 - \frac{1}{3}$, and the rest of the elements are obviously all greater than 0. Therefore, $\inf S = \min S = a_1 = 0$.

Part (m): $S = \{r \in \mathbb{Q} : r < 5\}$. It is obvious that the set is not bounded from below, hence neither $\inf S$ nor $\min S$ exists. For the upper bound, it is direct to see that $\sup S = 5$, however, since $5 \notin S$, there is no $\max S$. $\square$

Exercise 3.3, #6 (b):

Solution. If m, n are both maxima of the set S, then in particular $m, n \in S$. Moreover, by definition of m being a maximum of S, for all $s \in S$, there holds $m \geq s$. In particular, $m \geq n$. The exact same argument shows that one also has $n \geq m$. Therefore, one must have $m = n$. $\square$

Exercise 3.3, #7:

Solution. Part (a): Since S is bounded, one must have kS is also bounded, hence the Completeness Axiom tells that both supremum/infimum exist.

In the case that $k = 0$, one has $kS = \{0\}$, then the desired equalities obviously hold true. From now on, it suffices to consider the case $k > 0$.

First, to show that $\sup(kS) = k \cdot \sup S$. There are two things to check: 1. $k \cdot \sup S$ is an upper bound of the set kS; 2. it is the least upper bound.

By definition of the set kS, for all $x \in kS$, it must be of the form $x = ks$, for some $x \in S$. Then, since $s \leq \sup S$, one has by (O4) that $x = ks \leq k \cdot \sup S$, in other words, $k \cdot \sup S$ is indeed an upper bound of the set kS. Now, let m be any real number smaller than $k \cdot \sup S$. Since $k > 0$, one can define $m' = \frac{m}{k}$. Since $m < k \cdot \sup S$, using (O4), one has $m' < \sup S$. Hence, there exists $s \in S$ such that $m' < s$, which then implies by (O4) again that $m = m' \cdot k < sk \in kS$. In other words, we have found an element in kS that is larger than m. Therefore, by definition, $\sup(kS) = k \cdot \sup S$.

The conclusion regarding infimum can be proved in the completely symmetric way, which we omit.

Part (b): again, we only prove the first equality $\sup(kS) = k \cdot \inf S$ since the other one is symmetric.

Similarly as in part (a), one needs to verify that: 1. $k \cdot \inf S$ is an upper bound of the set kS; 2. it is the least upper bound.

For all $x \in kS$, there exists $s \in S$ such that $x = ks$. Since $s \geq \inf S$ and $k < 0$, one has by Theorem 3.2.2(g) that $x = ks \leq k \cdot \inf S$, i.e. $k \cdot \inf S$ is an upper bound of kS. For any real number $m < k \cdot \inf S$, define $m' = \frac{m}{k}$. Since $k < 0$, one must have, by Exercise #3(k) of Section 3.2 that $\frac{1}{k} < 0$, hence $m' > \inf S$. By definition of the infimum, one can thus find $s \in S$ satisfying $s < m'$, which then implies $ks > m'k = m$. Therefore, $k \cdot \inf S$ must be the least upper bound of kS and the proof is complete. $\square$