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DS METRIC

PIOTR RUDNICKI

Abstract. For any group G we introduce a "disjoint sum" metric and we relate boundedness of this metric to boundedness of Burnside group in the bi-invariant word metric. We also prove that d_{DS} is bounded for boundedly generated groups and estimate its diameter for finitely generated free abelian groups.

Notational remark. We use "topologist" notation for finite cyclic groups i.e., we write $\mathbb{Z}_k$ for the cyclic group of cardinality k. For two elements g, h is some group G we abbreaviate ${}^g h$ to be left conjugation of h by g, i.e., an element ghg^{-1}.

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1. Introduction and motivation

We collect basic definitions and properties of objects used in the thesis.

Suppose G is a group. We say that a function $|\cdot| \colon G \to [0, \infty)$ is a *norm* on a group G if the following conditions are satisfied

(1) $|g| = 0$ if and only if $g = e$,

(2) $|g| = |g^{-1}|$ for any $g \in G$,

(3) $|gh| \leqslant |g| + |h|$ for any $g, h \in G$

If moreover $|hgh^{-1}| = |g|$ for any choice of $g, h \in G$ then we say that $|\cdot|$ is *conjugation invariant*. If a group G admit an unbounded conjugation invariant norm then we say G is *unbounded*. Otherwise we say it is *bounded*.

For $S \subseteq G$ any symmetric set generating a group G we define an associated *word norm* $|\cdot|_S$ by

$$|g|_S := \min\{k \in \mathbb{N} \mid g = s_1 \cdots s_k, \text{ where } s_i \in S\}.$$

The *word metric* $d_S \colon G \times G \to [0, \infty)$ on Cayley graph $\mathrm{Cay}(G, S)$ is defined by

$$d_S(g, h) := |gh^{-1}|_S.$$

This metric is always right invariant. For any symmetric set S which generates G we put $\overline{S} := \bigcup_{g \in G} gSg^{-1}$. It is clear that $|\cdot|_{\overline{S}} \leqslant |\cdot|_S$ and $d_{\overline{S}}$ is bi-invariant (i.e., left and right invariant). To see left invariance consider $d_{\overline{S}}(gx, gy)$. By definition it is equal to $|gxy^{-1}g^{-1}|_{\overline{S}} = |xy^{-1}|_{\overline{S}} = d_{\overline{S}}(x, y)$ because the word norm associated to a conjugation invariant set is conjugation invariant.

Example 1.1. Let $G = F_{\{a,b\}}$ be the free group on generating set $S = \{a, b\}$ and let $w = a^n b a^{-n}$. Then $1 = |w|_{\overline{S}} \leqslant |w|_S = 2n + 1$.

We are interested in bi-invariant metrics on groups and whether they are bounded or not.

Definition 1.2. G is *normally finitely generated* if there exist a finite set $S \subseteq G$ such that G is generated by $\overline{S}$. In that situation we also say that G is *normally finitely generated by a set* S.

In Appendix (Proposition A.1) we prove fact which directly imply that for any two finite subsets $S, T \subseteq G$ such that $\langle \overline{S} \rangle = G = \langle \overline{T} \rangle$ induced norms $|\cdot|_{\overline{S}}, |\cdot|_{\overline{T}}$ are equivalent. So when G is normally finitely generated by set S we omit subscript $_{\overline{S}}$ in $|\cdot|_{\overline{S}}$.

In [1] BIP two open questions are stated one of which is motivation for our work. For basic definitions used throught all the thesis we refer to Appendix A. To state problems mentioned above we need to develop some theory about (stable) commutator length. At the end of this section we explain how the metric constructed by us might be helpful in proving that well known Burnside groups are examples of a group in the question in [1].

Commutator length. Suppose G is a group and let $G' = [G, G]$ be its commutator subgroup. Let $\mathrm{cl}(g)$ be *commutator length of* g, i.e. minimal number of commutators in G whose product is equal to g. For brief introduction to this topic we refer to [5] where various interpretations (for example in knot theory and theory of 4-manifolds) of this notion were presented.

The role of the commutator subgroup.

Proposition 1.3. If $H_1(G) := G/G'$ is infinite, then G is unbounded.

Corollary 1.4. An abelian group G is bounded if and only if it is finite.

In general it might not be possible to extend unbounded norm from a given normal subgroup to the whole group. Consider for example $\mathrm{Aff}(\mathbb{Z})$ which is an extension of $\mathbb{Z}$ by an element t of order 2 and with one additional relation $tz = z^{-1}t$. Thus $\mathbb{Z}$ is a normal subgroup of index 2 in $\mathrm{Aff}(\mathbb{Z})$. Clearly $\mathbb{Z}$ has an unbounded norm, while $\mathrm{Aff}(\mathbb{Z})$ admits no unbounded norm since t is conjugated to tz^{2n} (by z^n) for all $n \in \mathbb{Z}$. However the situation is different when we consider the commutator length on the commutator subgroup.

Proposition 1.5. Let G be any group. If the commutator length of G' is unbounded then G itself is unbounded.

We give the proofs of Proposition 1.3 and Proposition 1.5 in Appendix B.

Stably unbounded norms. Given a norm μ on G, we define its stabilization by

$$\mu_\infty(g) := \lim_{n\to\infty} \frac{\mu(g^n)}{n}$$

An unbounded norm μ is called *stably unbounded* if $\mu_\infty(g) \neq 0$ for some $g \in G$.

Example 1.6. Proposition 1.3 might be used to prove that an abelian torsion group is unbounded but never stably bounded.

Relation of stable commutator length with quasi-morphisms. Let G be any group. The commutator length cl on G' is stably unbounded if and only if G admits non-trivial homogeneous quasi-morphism [4].

Now we are ready to state two questions stated in [1]

Question (A). Does there exist a finitely presented group G with trivial first homology group $H_1(G)$ whose commutator length is unbounded but stably bounded?

Question (B). Does there exist an unbounded finitely presented group which admits no unbounded quasi-morphism?

In [2] Kędra and Branderbursky showed that the commutator subgroup of the infinite strand braid group $\mathbf{B}_\infty$ admits unbounded norm and $\widehat{\mathrm{QM}}(\mathbf{B}_\infty) = 0$, but this group is not finitely generated. In [3] Muranov showed there are such finitely generated groups using small cancellation theory, but these groups are not finitely presented. In brief construction of Muranov goes as follows. He constructed a sequence of simple groups G_i of finite commutator length n_i such that $n_i \to \infty$. The infinite direct product $\prod_i G_i$ is as required.

In this paper we focus on the following question: Are Burnside groups unbounded? Although Burnside groups are not finitely presented and strictly speaking they don't give an answer to the Question B, they are much easier to define than Muranov groups.

Now we describe our construction. Our motivation was to prove some Burnside groups $B(n, d)$ (parameters to be specified later) are also finitely generated groups which admits unbounded norm. Note that Burnside groups are torsion groups so they cannot posses (nontrivial) quasi-morphisms. For the sake of proving unboundedness we introduce some auxiliary metric defined for general group in Section 2. In short, for any group G we describe a norm $|\cdot|_{DS}$ on $\bigoplus_G \mathbb{Z}_2$. The induced metric is denoted by d_{DS}. At the end of this section (Proposition 2.6) we prove that boundedness of DS-metric behaves well after passing to finite index subgroups. We'll sketch consequences of this result after explaining the results obtained in later sections.

In Section 3 we specialize with our construction to Burnside groups. The main result is

Proposition 3.1.

$$\frac{1}{2} \operatorname{diam} \left(\bigoplus_{B(n,d)} \mathbb{Z}_2, |\cdot|_{DS} \right) \leqslant \operatorname{diam} \left(\mathbb{Z}_2 \wr B(n, d), |\cdot|_{\overline{S_1}} \right),$$

$$\operatorname{diam} \left(\mathbb{Z}_2 \wr B(n, d), |\cdot|_{\overline{S_1}} \right) \leqslant \operatorname{diam} \left(B(n+1, 2d), |\cdot| \right)$$

So immediate corollary is that if $\operatorname{diam}\left(\bigoplus_{B(n,d)} \mathbb{Z}_2, |\cdot|_{DS}\right) = \infty$ then $B(n+1, 2d)$ is unbounded in every bi-invariant word metric.

Unfortunately we were not able to prove that Burnside groups are unbounded in DS metric.

In Section 4 we compute diameters with respect to DS-metric for $\mathbb{Z}$ and we give an upper bound for this diameter for the groups $\mathbb{Z}^n$ for all $n \geqslant 1$. At the end we recall the notion of boundedly generated groups and we prove boundedly generated groups are bounded in DS-metric. More formally these two propositions are articulated as:

Proposition 4.2. Let $G = \bigoplus_{\mathbb{Z}^n} \mathbb{Z}_2$. Then $\operatorname{diam}_{DS}(G) \leqslant n + 1$.

Proposition ?? Suppose G is boundedly generated group by set $S = \{g_1, \ldots, g_k\}$. Then $\operatorname{diam}\left(\bigoplus_G \mathbb{Z}_2, d_{DS}\right) < \infty$.

This is appropriate moment to mark consequence of Proposition 4.2 and Proposition 2.6. Namely we obtained relatively wide class of groups which are bounded in their DS-metric. For example we see that any group H which sits in short exact sequence

$$0 \to \mathbb{Z} \to H \to \mathbb{Z}_2 \to 0$$

is bounded in its DS-metric. Using basic homological algebra one can prove that there are basically two isomorphism classes of such extension: $\mathbb{Z} \times \mathbb{Z}_2$ and semi-direct product $\mathbb{Z} \rtimes_\phi \mathbb{Z}_2 \cong D_\infty$ - the infinite Dihedral group. If we specialize to abelian groups then in general situation

$$0 \to G \to H \to K \to 0$$

isomorphism classes of (abelian) extensions are classified by the group $\mathrm{Ext}^1_{\mathbb{Z}}(G, K)$ where the zero element of this group corresponds to the isomorphism class $[G \times K]$ of the trivial (product) extension. The functor $\mathrm{Ext}^1_{\mathbb{Z}}(-,-)$ has many functorial properties with respect to two variables and because of that it is relatively easy to compute. For example $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}/(n), H)$ is isomorphic to H/nH for any $n > 0$ and any $\mathbb{Z}$-module H. Specializing to $G = \mathbb{Z}$ and $K = \mathbb{Z}_n$ we now see that abelian extensions

$$0 \to \mathbb{Z} \to H \to \mathbb{Z}_k \to 0$$

are parametrized by $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_k, \mathbb{Z}) \cong \mathbb{Z}_k$ and by Proposition 2.6 for each such H its group ring $\bigoplus_H \mathbb{Z}_2$ is bounded in its DS-metric.

Note however that infinite Burnside groups are not boundedly generated.

Summary. We were not able to prove that infinite Burnside groups are unbounded in DS metric. So our strategy was to study DS metric on simpler groups, but all of them turns out to be bounded in DS metric. So it is natural to ask the following:

Question (C). Is there any group G which is unbounded with respect to DS metric?

2. Construction of DS norm for general groups

Recall that for general two groups G, H their wreath product is defined to be the semidirect product $H \wr G := (\bigoplus_G H) \rtimes_\phi G$, where $\phi \colon G \to \mathrm{Aut}\,(\bigoplus_G H)$ is given by the translation action i.e. $\phi(g)(\sigma)(\overline{g}) := \sigma(g^{-1}\overline{g})$. We simply write $g \cdot \sigma$ instead of $\phi(g)(\sigma)$. In our cases H always equals $\mathbb{Z}_2$ and we write $\sigma = \delta_{g_1} + \cdots + \delta_{g_k}$.

Suppose group G is generated by set S and consider G as embedded in $\mathbb{Z}_2 \wr G$ by $g \mapsto (0, g)$. Put $S_1 := S \cup \{(\delta_e, e)\}$, where δ_e is the dirac delta the on identity element in G. It is clear that the set $\{(\delta_s, t) \colon s, t \in S\}$ generates $\mathbb{Z}_2 \wr G$.

Note that $(0, s) \cdot (\delta_e, e) = (0 + \phi(s)(\delta_e), s) = (\delta_s, s)$ and $(\delta_s, e) \cdot (0, t)$ is equal to $(\delta_s + \phi(e)(0), et) = (\delta_s, t)$ So we showed equality $\langle S_1 \rangle = \mathbb{Z}_2 \wr G$. Now we take $\overline{S_1} \subseteq \mathbb{Z}_2 \wr G$. Basic calculations demonstrate any $\overline{s} \in \overline{S_1}$ is of the form $(\sigma + g \cdot \sigma, h)$ or (δ_g, e).

Let $T = \{\sigma + g\sigma \mid \sigma \in \bigoplus_G \mathbb{Z}_2, g \in G\} \cup \{\delta_e\}$, where e denotes neutral element of G.

Remark 2.1. Note that $\delta_e + (\delta_g + g^{-1}\delta_g) = \delta_g$ so T generates $\bigoplus_G \mathbb{Z}_2$.

This motivates the following

Definition 2.2. Let G be a group and T be a generating subset of $\bigoplus_G \mathbb{Z}_2$ as described above. The *disjoint sum norm* is defined to be $|\cdot|_{DS} := |\cdot|_T$. The *disjoint sum metric* is the metric d_{DS} induced by $|\cdot|_{DS}$.

Remark 2.3. There is a natural embedding $\iota\colon \bigoplus_G \mathbb{Z}_2 \hookrightarrow \mathbb{Z}_2 \wr G$ defined by $\iota(f) := (f, e)$. By Remark 2.1 we see for $g \neq e$ $|\delta_g|_{DS} = 2$ and $|\iota(\delta_g)|_{\overline{S_1}} = 1$. Moreover let g be any element in G such that g is not in S and let $\sigma \in \bigoplus_G \mathbb{Z}_2$ be some nonzero element to be specified later. Clearly $|\sigma + g \cdot \sigma|_{DS} = 1$. It turns out that $|\iota(\sigma + g \cdot \sigma)|_{\overline{S_1}} > 1$ for carefully choosen σ. Indeed, suppose its norm equals 1. Then it can be written in the form $(\tau, h)\xi(\tau, h)^{-1}$ for some $\xi \in S_1$.

Suppose $\xi = (\delta_e, e)$. Then we obtain relation:

$$(\tau, h)(\delta_e, e)(\tau, h)^{-1} = (\sigma + g \cdot \sigma, e)$$

This implies $(\delta_{h^{-1}}, e) = (\sigma + g \cdot \sigma, e)$. It is clear that σ might be chosen to ensure this relation cannot be solved with respect to τ and h.

Now suppose $\xi = (0, s)$ for some $s \in S$. Here we use the fact that $g \notin S$. Indeed, now our relation is: $(\tau +^h s \cdot \tau, ^h s) = (\sigma + g \cdot \sigma, e)$.

This implies $s = e$ which is contradiction. So this equation also cannot be solved with respect to τ and h in general.

However there is relationship between norms $|\cdot|_{DS}$ and $|\cdot|_{\overline{S_1}}$ if we allow scalling factor.

Proposition 2.4. For every $\sigma \in \bigoplus_G \mathbb{Z}_2$ there is inequality:

$$\frac{1}{2}|\sigma|_{DS} \leqslant |\iota(\sigma)|_{\overline{S_1}}$$

Proof. Let us assume that $|\sigma|_{DS} = k$ and for the sake of contradiction suppose we have $|\iota(\sigma)|_{\overline{S_1}} = p < \frac{1}{2}k$. Then $\iota(\sigma)$ has decomposition as

$$(\tau_1, h_1)\xi_1(\tau_1, h_1)^{-1} \dots (\tau_p, h_p)\xi_p(\tau_p, h_p)^{-1}$$

Let $\mathcal{K} := \{1, \dots, p\}$ and $\mathcal{J}$ be subset of $\mathcal{K}$ defined as the set of these $i \in \mathcal{K}$ such that $\xi_i = (\delta_e, e)$. Straightforward computation shows that:

$$\iota(\sigma) = \left(\sum_{i \in K - J} \tau_i' + h_i' \cdot \tau_i' + \sum_{j \in J} \delta_{h_j^{-1}}, e \right)$$

So in particular we obtain new decomposition of σ in the first coordinate in $\iota(\sigma)$. Denote by ℓ the cardinality of $\mathcal{J}$. So we have $|\sigma|_{DS} \leqslant (p - \ell) + 2\ell = p + \ell$ which is strictly smaller than k by our assumption that $p < \frac{1}{2}k$. $\square$

Proposition 2.5. Suppose $f\colon G_1 \to G_2$ is an epimorphism. Assume also that $\bigoplus_{G_1} \mathbb{Z}_2$ is bounded with respect to DS metric. Then $\bigoplus_{G_2} \mathbb{Z}_2$ is bounded with respect to its DS metric as well.

Proof. Let $\bigoplus f\colon \bigoplus_{G_1} \mathbb{Z}_2 \to \bigoplus_{G_2} \mathbb{Z}_2$ be induced map which clearly is an epimorphism. Let $T_i = \{\sigma + g\sigma \mid \sigma \in \bigoplus_{G_i} \mathbb{Z}_2, g \in G_i\} \cup \{\delta_e^{(G_i)}\}$ be generating set of $\bigoplus_{G_i} \mathbb{Z}_2$ (see Remark 2.1). We claim $(\bigoplus f)(T_1) = T_2$. Inclusion $\subseteq$ is clear and we only need to establish the second one. Let us take $\sigma + g\sigma \in T_2$. Because $\bigoplus f$ is an epimorphism there are $\tilde{\sigma} \in \bigoplus_{G_1} \mathbb{Z}_2$ and $\tilde{g} \in G_1$ such that they are mapped by $\bigoplus f$ to σ and g respectively. Then $(\bigoplus f)(\tilde{\sigma} + \tilde{g}\tilde{\sigma}) = \sigma + g\sigma$. We also have $\bigoplus f(\delta_e^{(G_1)}) = \delta_e^{(G_2)}$ so we proved our claim that $(\bigoplus f)(T_1) = T_2$. From Lemma 3.2 we obtain:

$$\operatorname{diam}\left(\bigoplus_{G_2} \mathbb{Z}_2, |\cdot|_{T_2}\right) \leqslant \operatorname{diam}\left(\bigoplus_{G_1} \mathbb{Z}_2, |\cdot|_{T_1}\right)$$

Because G_1 is bounded with respect to DS metric the result follows from the equality above. $\square$

We summarize this section with the facts indicating that DS metric behaves nicely with respect to short exact sequences.

Proposition 2.6. Suppose H has finite index in G and $\bigoplus_H \mathbb{Z}_2$ is bounded with respect to DS-metric by N. Then

$$\operatorname{diam}\left(\bigoplus_{G} \mathbb{Z}_2, |\cdot|_{DS}\right) \leqslant [G : H] \cdot N$$

Sketch of proof. It is elementary. Take any $\sigma \in \bigoplus_G \mathbb{Z}_2$ and write it (uniquely) as $\sigma_1 + \cdots + \sigma_k$ where the supports of σ_i's lies in pairwise different cosets of H. Hence by assumption $k \leqslant [G : H]$. By assumption $\bigoplus_H \mathbb{Z}_2$ is bounded in its DS-metric by N so we can easily decompose each σ_i as a sum of at most N generators for DS-metric. Hence we can write σ using at most $[G : H] \cdot N$ generators. $\square$

3. Connection between conjugation invariant norms and d_{DS} for Burnside groups

Let $B(n, d)$ be the Burnside group on n generators and exponent d that is

$$B(n, d) = \left\langle x_1, \dots, x_n \mid \omega^d = 1 \right\rangle_{\omega \in (S \cup S^{-1})^*},$$

where $S = \{x_1, \dots, x_n\}$.

Proposition 3.1. There are inequalities

$$\frac{1}{2} \operatorname{diam}\left(\bigoplus_{B(n,d)} \mathbb{Z}_2, |\cdot|_{DS}\right) \leqslant \operatorname{diam}\left(\mathbb{Z}_2 \wr B(n, d), |\cdot|_{\overline{S_1}}\right),$$

$$\operatorname{diam}\left(\mathbb{Z}_2 \wr B(n, d), |\cdot|_{\overline{S_1}}\right) \leqslant \operatorname{diam}\left(B(n+1, 2d), |\cdot|\right)$$

Proof. Suppose $|\cdot|$ is bi-invariant metric on the group $B(n+1, 2d)$. Denote by $\{x_1, \dots, x_{n+1}\}$ and $\{y_1, \dots, y_n\}$ generators of $B(n+1, 2d)$ and $B(n, d)$ respectively. There is an epimorphism

$$B(n+1, 2d) \xrightarrow{q} \mathbb{Z}_2 \wr B(n, d)$$

given by $x_i \mapsto (0, y_i)$ for $1 \leqslant i \leqslant n$ and $x_{n+1} \mapsto (\delta_e, e)$. It is well defined because $q(x_i)^{2d} = (0, y_i)^{2d} = (0, (y_i^d)^2) = (0, e)$ and $q(x_{n+1})^{2d} = (\delta_e, e)^{2d} = (\underbrace{\delta_e + \cdots + \delta_e}_{2d-\text{times}}, e) = (0, e)$. Finally suppose ω is a word in alphabet $\{x_1^{\pm 1}, \ldots, x_{n+1}^{\pm 1}\}$ and is written over this alphabet as $\omega = x_{1,\omega}^{\epsilon_1} \ldots x_{k,\omega}^{\epsilon_k}$ with $\epsilon_j = \pm 1$.

Consider $\alpha := \prod_{r=1}^{2d} q(x_{1,\omega})^{\epsilon_1} \ldots q(x_{k,\omega})^{\epsilon_k}$. We claim α is equal to $(0, e)$. Indeed let $l := \sharp\{1 \leqslant i \leqslant k \colon x_{i,\omega} = x_{n+1}\}$. Using how group operation is defined in $\mathbb{Z}_2 \wr B(n, d)$ it is straightforward to see that α is equal to $(\underbrace{\delta_e + \cdots + \delta_e}_{2dl-\text{times}}, \omega_0^{2d})$ where ω_0 is word written over alphabet $\{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$, concretely ω_0 is formed from ω by cancelling all the letters $x_{n+1}^{\pm 1}$. This shows $\alpha = (0, e)$. By universal property of group presentations we obtain q is well defined. Surjectivity of q is immediate.

Consider the following diagram:

$$\begin{tikzcd}
(B(n+1, 2d), |\cdot|) \arrow[r, two heads, "q"] & \left(\mathbb{Z}_2 \wr B(n, d), |\cdot|_{\overline{S_1}}\right) \\
& \left(\bigoplus_{B(n,d)} \mathbb{Z}_2, |\cdot|_{DS}\right) \arrow[u, hook, "\iota"]
\end{tikzcd}$$

where ι is an embedding onto first coordinate defined abstractly in Remark 2.3.

The following lemma directly imply the second inequality in statement of the theorem, namely that:

$$\operatorname{diam}\left(\mathbb{Z}_2 \wr B(n, d), |\cdot|_{\overline{S_1}}\right) \leqslant \operatorname{diam}\left(B(n+1, 2d), |\cdot|\right)$$

Lemma 3.2. Suppose that groups G_i are generated by symmetric sets $S_i \subseteq G_i$ for $i \in \{1, 2\}$ and $q \colon G_1 \to G_2$ is an epimorphism such that $q(S_1) = S_2$. Then there is an inequality $\operatorname{diam}(G_2, |\cdot|_{S_2}) \leqslant \operatorname{diam}(G_1, |\cdot|_{S_1})$.

Proof of Lemma 3.2. Suppose $x \in G_1$, $|x|_{S_1} = k$ and $x = s_1 \ldots s_k$ with $s_i \in S_1$. Then we obtain $q(x) = q(s_1) \ldots q(s_k)$. Because $q(S_1) = S_2$ so one have $|q(x)|_{S_2} \leqslant k$. Now by taking any $y \in G_2$ we can choose $x \in q^{-1}(y)$ to obtain inequality $|y|_{S_2} = |q(x)|_{S_2} \leqslant |x|_{S_1}$. □

From Proposition 2.4 we know the first inequality in the statement holds. □

Corollary 3.3. Suppose that $\operatorname{diam}\left(\bigoplus_{B(n,d)} \mathbb{Z}_2, |\cdot|_{DS}\right) = \infty$. Then $B(n+1, 2d)$ is unbounded in every bi-invariant word metric.

4. Computations

The main result of this section is that DS-metrics on $\mathbb{Z}_2$-group rings on boundedly generated groups are bounded (Theorem 4.10). At first we prove it for $\mathbb{Z}$ (Proposition 4.1) and then for $\mathbb{Z}^n$ for any $n \geqslant 1$ (Propostion 4.2). Explicit bounds are provided.

Lemma 4.1. Let $G = \bigoplus_{\mathbb{Z}} \mathbb{Z}_2$. Then $\operatorname{diam}_{DS}(G) = 2$.

Proof. Let σ be any element of G of length k. Let's assume for a moment that k is even. We show inductively that in that case theorem holds. If $k = 2$ then $\sigma = \delta_{n_1} + \delta_{n_2}$. If $n_2 = n_1 + 1$ take $\tau := n_1$ and we have $\sigma = \tau + 1 \cdot \tau$. If more generally $n_2 = n_1 + r$ for some $r > 1$ then one simply takes $\tau := \delta_{n_1} + \delta_{n_1+1} + \cdots + \delta_{n_1+(r-1)}$. So theorem holds for any σ of length 2. Suppose $|\sigma| = 2r$ and $\sigma = \delta_{n_1} + \delta_{n_2} + \cdots + \delta_{n_{2r-1}} + \delta_{n_{2r}}$. Let $\sigma_i := \delta_{n_{2i-1}} + \delta_{n_{2i}}$ for $1 \leqslant i \leqslant r$. Let τ_i be such that $\sigma_i = \tau_i + 1 \cdot \tau_i$ as described in the base case of induction. One can easily check that if $\tau = \sum_{i=1}^{r} \tau_i$ then $\sigma = \tau + 1 \cdot \tau$. So we are done for even $|\sigma|$. If the length of σ is odd, say $\sigma = \delta_{n_1} + \cdots + \delta_{n_{2r}} + \delta_{n_{2r+1}}$ then take $\sigma_0 := \delta_{n_1} + \cdots + \delta_{n_{2r}}$ and by previous argument take τ_0 such that $\sigma_0 = \tau_0 + 1 \cdot \tau_0$ which exists because $|\sigma_0|$ is even. Then $\sigma = \tau_0 + 1 \cdot \tau_0 + \delta_{n_{2r+1}}$, so the diameter of the group G with respect to DS the metric equals at most 2. In fact it equals exactly 2 which is witnessed by $\delta_1 + \delta_2 + \delta_3$. □

Proposition 4.2. Let $G = \bigoplus_{\mathbb{Z}^n} \mathbb{Z}_2$. Then $\operatorname{diam}_{DS}(G) \leqslant n + 1$.

Proof. Lets denote by G the group $\bigoplus_{\mathbb{Z}^n} \mathbb{Z}_2$. We do induction on n. Base case $n = 1$ is the lemma above. So suppose $n > 1$ and we are given $\sigma \in G$. The idea of the proof is to divide σ into horizontal (verical?) parts and work on them using only one coordinate vector. More precisely we claim that there is $\tau \in G$ such that $\sigma = \tau + (-e_1) \cdot \tau + \eta$ where $\eta \in G$ is such that for every $(a_1, \ldots, a_n) \in \eta$ one has $a_1 = 0$. Now we use the inductive hypothesis to η and we are done. To obtain claim above consider division of σ into parts. For any sequence $\overline{a} = (a_2, \ldots, a_n) \in \mathbb{Z}^{n-1}$ let $\sigma_{\overline{a}}$ be the set of these $x \in \sigma$ such that $(x_2, \ldots, x_n) = \overline{a}$. Because σ is finite so there are only finitely many $\overline{a}$ such that $\sigma_{\overline{a}}$ is nonempty. Denote them by $\sigma_1, \ldots, \sigma_k$ and note that $\sigma = \sigma_1 + \cdots + \sigma_k$. Let σ' be any of them. Because elements of σ' differ only in its first coordinates we think of σ' as an element of $\bigoplus_{\mathbb{Z}} \mathbb{Z}_2$ by identification $\sigma' \ni (a_1, \overline{a}) \mapsto a_1 \in \mathbb{Z}$.

Case 1. If $\sharp\sigma'$ is even then we can decompose it as $\alpha + (-e_1) \cdot \alpha$.

Case 2. If $\sharp\sigma'$ is odd and $0 \notin \sigma'$. Let $\sigma'' = \sigma' + \delta_0$. So $\sharp\sigma''$ is even and we can decompose it as $\sigma'' = \alpha + (-e_1) \cdot \alpha$. Adding δ_0 to both sides we obtain $\sigma' = \alpha + (-e_1) \cdot \alpha + \delta_0$.

Case 3. If $\sharp\sigma'$ is odd and $0 \in \sigma'$. In this case we have $\sigma' = \sigma'' + \delta_0$. So $\sharp\sigma''$ is even and decomposes as $\alpha + (-e_1) \cdot \alpha$. In consequence $\sigma' = \alpha + (-e_1) \cdot \alpha + \delta_0$.

So for any $1 \leqslant i \leqslant k$ we have the decomposition

$$\sigma_i = \alpha_i + (-e_1) \cdot \alpha_i + \delta_{(0,\overline{a}_i)}^{\epsilon_i}$$

where $\epsilon_i \in \{0,1\}$. Let $\eta := \sum_{k=1}^{n} \delta^{\epsilon_i}_{(0,\overline{a}_i)}$, then:

$$\sigma = \sum_{i=1}^{k} \alpha_i + (-e_1)\cdot\alpha_i + \delta^{\epsilon_i}_{(0,\overline{a}_i)} = \sum_{i=1}^{k} \alpha_i + (-e_1)\cdot\alpha_i + \eta$$

□

While proving boundedness of DS-metric on $\bigoplus_{\mathbb{Z}^n}\mathbb{Z}_2$ reader might notice that every element σ was proved to have bounded at length decomposition as

$$\sigma = \sigma_1 + g_1\cdot\sigma_1 + \cdots + \sigma_k + g_k\cdot\sigma_k + \delta^{\epsilon}_{e}$$

and what important all of these "shifting elements" g_i's **might be choosen uniformly** from the set $\{e_1,\ldots,e_n\}$. This motivates the following

Definition 4.3. Assume $H = \bigoplus_G \mathbb{Z}_2$ is bounded in its DS-metric by n say. We say it is *strongly bounded* if there exist finite set $X = \{g_1,\ldots,g_l\}$ of elements from G such that every $\sigma\in H$ has decomposition

$$\sigma = \sigma_1 + g_{i_1}\cdot\sigma_1 + \cdots + \sigma_k + g_{i_k}\cdot\sigma_k + \delta^{\epsilon}_{e}$$

where $k \leqslant n$ and $1 \leqslant i_j \leqslant l$ for any $1\leqslant j\leqslant k$ and $\epsilon\in\{0,1\}$. We also say that H is strongly generated with respect to X.

Now we state and prove proposition which relies on a very simple idea. If a group G contains a (normal) subgroup K such that K is strongly bounded by X then G is bounded assuming that G/K is bounded. Notice that K might sit "very deeply" in G i.e. does not have to be a finite index subgroup (see Corollary 4.5).

Proposition 4.4. Let K be a subgroup of G and assume $\bigoplus_K \mathbb{Z}_2$ is strongly bounded with respect to set $X = \{k_1,\ldots,k_l\}$. Assume moreover that the DS-metric on $\bigoplus_{G/K}\mathbb{Z}_2$ is bounded by N say. Then $\bigoplus_G \mathbb{Z}_2$ is bounded in its DS-metric by $2l+N$.

Proof. Fix any $\sigma\in\bigoplus_G\mathbb{Z}_2$ and let Λ be the set of right cosets of K in G. For a coset $Kg\in\Lambda$ let $\sigma[Kg]\in\bigoplus_G\mathbb{Z}_2$ to be the part of σ with support in this particular coset Kg. We will find this notation convenient at the very end of this proof. Clearly we might write σ as $\sigma_1+\cdots+\sigma_m$ in a way σ_i and σ_j belongs to different cosets for $i\neq j$ and let enumerate these cosets as $Kg_1,\ldots,Kg_m\in\Lambda$. To keep notation simple we assume without significant effect for the general outline of the proof that each σ_i has decomposition as just $\tau_i g_i + k_{j_i}\cdot\tau_i g_i + \delta^{\epsilon_i}_{h_i}$ where $1\leqslant j_i\leqslant l$ and $\epsilon_i\in\{0,1\}$ (here we use the assumption that K is strongly bounded with respect to X). Assume for a moment all ϵ_i's are equal to 0. So we have partition $\{1,\ldots,m\} = A_1\sqcup\cdots\sqcup A_l$ where $A_t = \{1\leqslant i\leqslant m \mid j_i = t\}$. Let us also denote the cardinality of A_t by m_t. This partition naturally correspond to partition of the sum $\sigma_1+\cdots+\sigma_m$ as

$$\sigma^{(1)}_1+\cdots+\sigma^{(1)}_{m_1}+\cdots+\underbrace{\sigma^{(i)}_1+\cdots+\sigma^{(i)}_{m_i}}_{\eta_i}+\cdots+\sigma^{(l)}_1+\cdots+\sigma^{(l)}_{m_l}$$

Now for $1 \leqslant t \leqslant m_i$ we know each $\sigma_t^{(i)}$ is equal to

$$\tau_{\lambda_i(t)} \cdot g_{\lambda_i(t)} + k_i \cdot \tau_{\lambda_i(t)} \cdot g_{\lambda_i(t)}$$

where $\lambda_i(t) = m_1 + \cdots + m_{i-1} + t$. Note that

$$\eta_i = \underbrace{\sum_{t=1}^{m_i} \left(\tau_{\lambda_i(t)} \cdot g_{\lambda_i(t)}\right)}_{\omega_i} + k_i \cdot \sum_{t=1}^{m_i} \left(\tau_{\lambda_i(t)} \cdot g_{\lambda_i(t)}\right)$$

Finally we obtain wanted decomposition

$$\omega = \omega_1 + k_1 \cdot \omega_1 + \cdots + \omega_l + k_l \cdot \omega_l.$$

This ends the proof in the case where all ϵ_i's are equal 0. If some of the ϵ_i's are equal 1 then we have to do some combinatorial perturbation to reduce situation to the case where all ϵ_i's are 0. So assume ϵ_i's are equal 1, say for $i = 1, \ldots, k$. We collect them and define an element $\Delta := \delta_{h_1} + \cdots + \delta_{h_k}$. It leads to $\overline{\Delta} \in \bigoplus_{G/K} \mathbb{Z}_2$ and by assumption we have a decomposition

$$\overline{\Delta} = \overline{\sigma_1} + \overline{g_1'} \cdot \overline{\sigma_1} + \cdots + \overline{\sigma_s} + \overline{g_s'} \cdot \overline{\sigma_k} + \delta_K^{\epsilon}$$

for some $g_1', \ldots, g_s' \in G$ and $\epsilon \in \{0, 1\}$. Write $\overline{\sigma_i}$ as

$$\delta_{Kg_1^{(i)}} + \cdots + \delta_{Kg_{n_i}^{(i)}}$$

Let $\widetilde{\overline{\sigma_i}} := \delta_{g_1^{(i)}} + \cdots + \delta_{g_{n_i}^{(i)}}$. We obtain an element

$$\Delta' = \sum_{i=1}^{s} \left(\widetilde{\overline{\sigma_i}} + g_i' \cdot \widetilde{\overline{\sigma_i}}\right) + \delta_e^{\epsilon}$$

Clearly it might be the case that $\Delta \neq \Delta'$. All we know is that $\overline{\Delta - \Delta'} = 0$ in $\bigoplus_{G/K} \mathbb{Z}_2$. It means exactly that for any coset $Kg \in \Lambda$ a number $\sharp(\Delta - \Delta')[Kg]$ is even. So the corresponding sequence of ϵ's (defined exactly as for σ) for $\Delta - \Delta'$ has only zero terms. Now we are finally ready to decompose σ as

$$\sigma = (\sigma - \Delta) + (\Delta - \Delta') + \Delta'$$

First two terms in this decomposition have trivial sequence of ϵ's so by the first part of proof they admit wanted decomposition. We also obtained decomposition of Δ' so we are done.

In consequence we can decompose every element in $\bigoplus_G \mathbb{Z}_2$ using at most $2l+N$ elements where N is the diameter of DS-metric on $\bigoplus_{G/K} \mathbb{Z}_2$. $\square$

Corollary 4.5. From Proposition 4.4 we see that if any group G has $\mathbb{Z}^n$ as normal subgroup then assuming the quotient $G/\mathbb{Z}^n$ is DS-bounded we see $\bigoplus_G \mathbb{Z}_2$ is itself DS-bounded. One of such examples is discrete Heisenberg group $H_3(\mathbb{Z})$ which is the group of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

where a, b, c are integers. Its center is the set of matrices $\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ where z is an integer which is clearly isomorphic to $\mathbb{Z}$. The quotient of $H_3(\mathbb{Z})$ by $\mathbb{Z}$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$ so we are done.

Remark 4.6. To use our last Proposition we need some group extension. Now we describe some machinery which allow us to find *central* extensions. Using group cohomology theory we know that the set of central extensions $1 \to K \to G \to H \to 1$ is just $H^2(H, K)$ i.e. the second group cohomology of H with coefficients in K. We refer to the beautiful exposition of this fruitful research area to [15]. In the interesting to us case when $K = \mathbb{Z}$ one can use an isomorphism

$$H^2(H, \mathbb{Z}) \cong H^2_{\text{sing}}(K(H, 1), \mathbb{Z})$$

where $K(H, 1)$ is Eilenberg-Maclane space for H and H^*_{sing} is singular cohomology functor. Of course we cannot use Proposition 4.4 to randomly choosen group from $H^2(H, \mathbb{Z})$ because it might be the case that $\bigoplus_H \mathbb{Z}_2$ is unbounded in its DS-metric. One of the situation at which we avoid these troubles with possibly unbounded quotient is when H is finite. For the very example let us take $H = \mathbb{Z}_2$. We have

$$H^2(\mathbb{Z}_2, \mathbb{Z}) \cong H^2_{\text{sing}}(K(\mathbb{Z}_2, 1), \mathbb{Z})$$

It is well known that Eilenberg-Maclane space for $\mathbb{Z}_2$ is infinite dimensional real projective space $\mathbb{RP}^\infty$. Its integral cohomology ring $H^*_{\text{sing}}(\mathbb{RP}^\infty, \mathbb{Z})$ is isomorphic to $\mathbb{Z}[y]/(2y)$ where y has degree 2. In particular $H^2_{\text{sing}}(\mathbb{RP}^\infty, \mathbb{Z})$ is nontrivial and isomorphic to $\mathbb{Z}_2$. Nontrivial group which we localized is the infinite dihedral group D_∞ given explicitly by presentation $\langle x, y \mid x^2 = y^2 = 1 \rangle$.

Recall that G is said to be *boundedly generated* if there is a finite family $\langle g_1 \rangle, \dots, \langle g_m \rangle$ of not necessarily distinct cyclic subgroups such that $G = \langle g_1 \rangle \dots \langle g_m \rangle$.

Examples of boundedly generated groups. Now we present two examples of boundedly generated groups. The first example is classical result that $\mathrm{SL}(n, \mathbb{Z})$ is boundedly generated for $n \geqslant 3$. The second one have been hijacked from [13].

Example 4.7. Fix $n \geqslant 3$. Carter and Keller in [10] showed that there is constant μ_n such that every matrix in $\mathrm{SL}(n, \mathbb{Z})$ is a product of at most μ_n elementaty matrices. Their theorem provides, in fact an explicit bound for $\mu_n = \frac{1}{2}(3n^2 - n) + 36$. In [11] Carter and Keller extend their argument to rings of integers in algebraic number fields.

Example 4.8. Let $\pi \colon \mathfrak{g} \to \mathfrak{g}(V)$ be a representation of complex semisimple Lie algebra and $\mathcal{O}$ be the commutative ring with unit. Let Φ denotes the root system associated with a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. With these data there is associated group $E_\pi(\Phi, \mathcal{O})$ called *elementary Chevalley group* and it is defined as the subgroup of the automorphism group $\mathrm{Aut}(V_\mathbb{Z} \otimes_\mathbb{Z} \mathcal{O})$ generated by elements of the form

$$x_\alpha(t) = \exp(t\pi(x_\alpha))$$

where $\alpha \in \Phi$ is a root and $t \in \mathcal{O}$. Here $V_\mathbb{Z}$ is an admissible $\mathbb{Z}$-form of V that is an integral lattice preserved by the representation. It is a result of Tavgen' [14] that the group $E_\pi(\Phi, \mathcal{O})$ has bounded generation with respect to the set if elements of the form x_α.

Remark 4.9. Note that $\mathrm{SL}(2,\mathbb{Z})$ is not boundedly generated. The most basic argument is that it contains a free group on two generators as subgroup of index 12. It is relatively easy to see no nonabelian free group is boundedly generated. The argument might be completed using elementary fact that a group is boundedly generated if and only if its finite index subgroups are boundedly generated. See Corollary A.7 for a more geometrical argument which relies on the fact that space of homogoneous quasi-morphisms of a boundedly generated group is finitely dimensional.

Theorem below is a generalization of the Proposition 4.2.

Theorem 4.10. Suppose G is boundedly generated group by set $S = \{g_1, \dots, g_k\}$. Then $\operatorname{diam}(\bigoplus_G \mathbb{Z}_2, d_{DS}) < \infty$.

Proof. Suppose $G = \langle g_1 \rangle \dots \langle g_n \rangle$ and let $\phi \colon \mathbb{Z}^n \twoheadrightarrow G$ be set theoretical map given by

$$\phi(a_1, \dots, a_n) := g_1^{a_1} \dots g_n^{a_n}$$

Denote by e_i the i-th standard basis vector in $\mathbb{Z}^n$. Then following relation hold:

$$\begin{aligned} \phi(e_i + (0, \dots, 0, a_i, a_{i+1}, \dots, a_n)) &= g_i^{a_i+1} g_{i+1}^{a_{i+1}} \dots g_n^{a_n} \qquad (\dagger) \\ &= \phi(e_i)\phi(0, \dots, 0, , a_i, a_{i+1}, \dots, a_n) \end{aligned}$$

Consider induced map $\overline{\phi} \colon \bigoplus_{\mathbb{Z}^n} \mathbb{Z}_2 \twoheadrightarrow \bigoplus_G \mathbb{Z}_2$ which is a homomorphism of abelian groups. Consider $\overline{\sigma} \in \bigoplus_G \mathbb{Z}_2$ and let $\sigma \in \bigoplus_{\mathbb{Z}^n} \mathbb{Z}_2$ be such that $\overline{\phi}(\sigma) = \overline{\phi}$. By Proposition 4.2 there is decomposition

$$\sigma = \left(\sum_{i=1}^{n} \tau_i + (-e_i) \cdot \tau_i \right) + \delta_e^\epsilon$$

where $\epsilon \in \{0, 1\}$ and we understand δ_e^0 simply as lack of this element in decomposition. We also know from (proof of) Proposition 4.2 that if $(a_1, \dots, a_n) \in \tau_i$ then we see $a_1 = \dots = a_{i-1} = 0 \in \mathbb{Z}$. Using relation $(\dagger)$ we obtain decomposition of $\overline{\sigma}$:

$$\overline{\sigma} = \overline{\phi}(\sigma) = \sum_{i=1}^{n} \phi(\tau_i) + \phi(e_i \cdot \tau_i) + \phi(\delta_e^\epsilon) = \sum_{i=1}^{n} \phi(\tau_i) + \phi(e_i) \cdot \phi(\tau_i) + \phi(\delta_e^\epsilon)$$

□

Appendix A. Properties of norms and quasi-morphisms

In this Appendix we present the relation between quasi-morphisms and boundedness property on a group (see Proposition A.4). We present geometric flavored examples and give an upper bound to the dimension of the space of so called homogeneous quasi-morphisms. At the end we give more geometric argument that $\mathrm{SL}(2,\mathbb{Z})$ is not boundedly generated.

Proposition A.1. Suppose G is normally finitely generated by a set S, then for every conjugation invariant metric $|\cdot|$ on G there exists a constant $\lambda \geqslant 0$ such that for every $g \in G$ we have

$$\lambda \cdot |g|_{\overline{S}} \geqslant |g|$$

i.e. the identity map $\mathbb{1}\colon (G, |\cdot|_{\overline{S}}) \to (G, |\cdot|)$ is a Lipschitz transformation.

Proof. Let $S \subseteq G$ be a finite set such that $\overline{S}$ generates G and let

$$\lambda := \max\{|s| \mid s \in S \cup S^{-1}\}$$

Take $g \in G$ and let $k := |g|_{\overline{S}}$. Then there are $g_1, \dots, g_k \in G$ and $s_1, \dots, s_k \in S \cup S^{-1}$ such that $g = g_1 s_1 g_1^{-1} g_2 s_2 g_2^{-1} \cdot \dots \cdot g_k s_k g_k^{-1}$. We obtain:

$$\begin{aligned} |g| = \left|g_1 s_1 g_1^{-1} g_2 s_2 g_2^{-1} \cdot \dots \cdot g_k s_k g_k^{-1}\right| &\leqslant \left|g_1 s_1 g_1^{-1}\right| + \dots + \left|g_k s_k g_k^{-1}\right| \\ = |s_1| + \dots + |s_k| \leqslant \lambda \cdot k = \lambda \cdot |g|_{\overline{S}} \end{aligned}$$

□

Let us recall that a map $q\colon G \to \mathbb{R}$ is called a *quasi-morphism*, if there exists a real number $C \geqslant 0$ such that

$$|q(gh) - q(g) - q(h)| \leqslant C$$

for all $g, h \in G$. The smallest such C is called a *defect of* q and is denoted by D_ϕ. A quasi-morphism q is *homogeneous* if it additionally satisfies

$$q(g^n) = nq(g)$$

for all $n \in \mathbb{Z}$ and $g \in G$.

There is a procedure which transforms quasi-morphisms into homogeneous ones. Let us fix quasi-morphism q and consider:

$$\widehat{q}(g) := \lim_{n \to \infty} \frac{q^n(g)}{n}$$

This map $\widehat{q}$ is called *homogeneous quasi-morphism associated to* q and it is a nice exercise to show $\widehat{q}$ is well-defined homogeneous quasi-morphism.

Now we present two examples borrowed from [6].

Example A.2. This construction is due to Polterovich [8]. Let $g \geqslant 1$ and Σ_g be genus g surface with fixed Riemannian metric. For every point $x \in \Sigma_g$ choose path $\gamma_x\colon [0,1] \to \Sigma_g$ from z to x by choosing a measurable section of the map $\pi\colon R \to \Sigma_g$, where $\pi(\gamma) = \gamma(1)$ and:

$$R = \{\gamma\colon [0,1] \to \Sigma_g \mid \gamma(0) = z \text{ and } \gamma \text{ is geodesic}\}$$

Let $f \in \mathrm{Ham}(\Sigma_g)$ and let $\{f_t\}$ be Hamiltonian isotopy from the identity to f. For any $x \in \Sigma_g$ we obtain a loop based at z defined as $\gamma(f,x) := \overline{\gamma_{f(x)}} \cdot \{f_t(x)\}_{t=0}^{t=1} \cdot \gamma_x$. The fact that this loop is well defined follows from that $\pi_1(\mathrm{Ham}(\Sigma_g)) = 0$ (see [9]).

Let $\psi\colon \pi_1(\Sigma_g, z) \to \mathbb{R}$ be any quasi-morphism and let $f \in \mathrm{Ham}(\Sigma_g)$. Then the map $\Psi\colon \mathrm{Ham}(\Sigma_g) \to \mathbb{R}$ defined by:

$$\Psi(f) := \int_{\Sigma_g} \psi(\gamma(f, x))dx$$

is a well defined quasi-morphism. The induced homogeneous quasi-morphism is given by:

$$\widehat{\Psi}(f) = \lim_{p\to\infty} \frac{1}{p} \int_{\Sigma_g} \psi(\gamma(f^p, x))dx$$

Example A.3. This construction is due to Gambaudo and Ghys [7]. Let X_n denote the space of configurations of ordered n-tuples of points in two dimensional closed disk D^2. Its quotient by the n-th symmetric group $\mathcal{S}_n$ is denoted by C_n. Let $\mathbf{P}_n := \pi_1(X_n)$ and $\mathbf{B}_n := \pi_1(C_n)$. These groups are called the Artin pure braid group and the (full) Artin braid group respectively.

Let us fix $\bar{z} = (z_1, \ldots, z_n) \in X_n$. For every $x \in D^2$ let $\gamma_{i,x}$ be minimal Euclidean length geodesic from z_i to x.

For $f \in \mathrm{Ham}(D^2)$ let us choose Hamiltonian isotopy $f_t \in \mathrm{Ham}(D^2)$ from the identity to f and let $\bar{x} = (x_1, \ldots, x_n)$ be a point in the configuration space. We associate the brain $\gamma(f, \bar{x}) \in \mathbf{P}_n$ in the following way: using geodesic segments $\gamma_{i,x}$ we connect $\bar{z}$ with $\bar{x}$, then we connect $\bar{x}$ with $\overline{f(x)}$ using isotopy f_t and finally we connect $\overline{f(x)}$ with $\bar{z}$ using our geodesics once again.

Let $\psi\colon \mathbf{P}_n \to \mathbb{R}$ be a homogoneus quasimorphism. We define a map

$$\mathcal{G}\colon \widehat{\mathrm{QM}}(\mathbf{P}_n) \to \widehat{\mathrm{QM}}(\mathrm{Ham}(D^2))$$

be defined by $\mathcal{G}(\psi) := \widehat{\Psi}_n$ where

$$\widehat{\Psi}_n(f) := \lim_{n\to\infty} \frac{1}{p} \int_{X_n} \psi(\gamma(f^p, \bar{x}))dx_1 \wedge \cdots \wedge dx_n$$

The main tool in proving unboundedness of bi-invariant metrics on groups are quasi-morphisms. We have the following

Proposition A.4. Suppose G is generated by conjugation invariant set $\overline{S}$ and we are given $q\colon G \to \mathbb{R}$ a homogoneus quasi-morphism which is bounded on $\overline{S}$. Then $\mathrm{diam}(G, |\cdot|_{\overline{S}}) = \infty$.

At the end of this section we present how the fact that a group is boundedly generated have an impact on dimension of space of homogoneus quasi-morphisms on a given group.

Lemma A.5. Suppose ϕ is homogoneus quasi-morphism on group G which is bounded, i.e., there exists constant $M > 0$ such that for any $g \in G$ there is an inequality

$$|\phi(g)| < M.$$

Then $\phi = 0$.

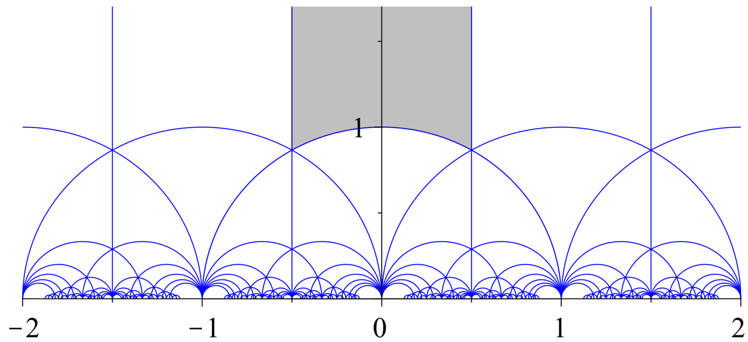

FIGURE A.1. Fundamental domain $\mathcal{D}$ (shaded) in the upper half plane. Image stolen from Wikipedia Commons.

Proof. Suppose *a contrario* there is a $g \in G$ such that $\phi(g) \neq 0$. Then for any integer k one has:

$$|\phi(g^k)| = |k \cdot \phi(g)| = |k| \cdot |\phi(g)| \to \infty$$

□

Proposition A.6. Let G be a boundedly generated group by elements $g_1, \ldots, g_n$. Then: $\dim_{\mathbb{R}} \widehat{\mathrm{QM}}(G) \leqslant n$.

Proof. Consider the linear map:

$$\widehat{\mathrm{QM}}(G) \longrightarrow \mathbb{R}^n$$

$$\phi \mapsto (\phi(g_1), \ldots, \phi(g_n))$$

We'll show this is an embedding. Assume we have homogoneus quasi-morphism ϕ such that $\phi(g_i) = 0$ for every $1 \leqslant i \leqslant n$. Take $k \in \mathbb{Z}$, then $\phi(g_i^k) = k \cdot \phi(g_i) = 0$. There is a constant $M \in \mathbb{N}$ such that any $g \in G$ might be written as $g = \prod_{j=1}^{M} g_{i_j}^{l_j}$ for some integers l_j. So we have:

$$|\phi(g)| = |\phi(g) - \phi(g_{i_1}^{l_1}) - \cdots - \phi(g_{i_M}^{l_M})| \leqslant (M-1) \cdot D_\phi$$

This computation shows ϕ is bounded homogoneus quasi-morphism and the Lemma A.5 implies $\phi = 0$. □

Corollary A.7. $\mathrm{SL}(2, \mathbb{Z})$ is not boundedly generated.

Sketch of proof. $G = \mathrm{SL}(2, \mathbb{Z})$ acts on upper half plane $\mathbb{H}$ with Poincare metric by Möbius transformations, i.e.,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$$

Any compactly supported 1-form α on the fundamental domain $\mathcal{D}$ (see Figure A) of this G-action extends uniquely to a G-invariant 1-form $\tilde{\alpha}$ on $\mathbb{H}$. If $z \in \mathbb{H}$ and γ is geodesic from z to $g \cdot z$ let

$$\phi_{\alpha,z}(g) \equiv \phi(g) := \int_\gamma \widetilde{\alpha}$$

We have

$$\phi(gh) - \phi(g) - \phi(h) = \int_\Delta d\widetilde{\alpha}$$

where Δ is a geodesic triangle with verices $z, g \cdot z$ and $h^{-1} \cdot z$. Geodesic triangles have the area bounded by π so

$$|\phi(gh) - \phi(g) - \phi(h)| \leqslant \pi \cdot ||d\widetilde{\alpha}||$$

Homogenisation $\widehat{\phi}_\alpha$ of $\phi_{\alpha,z}$ now depends only on α. Using dynamical systems theory it might be proven we are able to choose α in such a way $\widehat{\phi}_\alpha$ equals 1 on a given hyperbolic element and vanishes on a finite set of other hyperbolic elements with distinct fixed points. So G therefore has an infinite dimensional space of homogeneous quasi-morphisms and by Proposition A.6 it cannot be boundedly generated. □

Remark A.8. In [12] Brooks constructed infinitely many linearly independent homogoneus quasi-morphisms on free group F_2. So F_2 is not boundedly generated.

Appendix B. Proofs of Proposition 1.3 and Proposition 1.5

Definition B.1. Let G be a group. We say that a function $q\colon G \to [0, \infty)$ is a *quasi-norm* if:

(1) q is *quasi-subadditive*: there is constant C such that:

$$q(gh) \leqslant q(g) + q(h) + C$$

(2) q is *quasi-conjugation-invariant*: there is a constant C such that:

$$|q(g^{-1}hg) - q(h)| \leqslant C$$

(3) q is unbounded;

Remark B.2. Is is clear that *a priori* different constants from conditions (1) and (2) above might be choosen uniformly.

Example B.3. Let q be non trivial homogoneous quasi-morphism on group G. Then its absolute value $|q|$ is quasi-norm on G.

Every unbounded norm is clearly a quasi-norm. Now we sketch the argument of the fact that existence of quasi-norm implies existence of a conjugation invariant unbounded norm.

Lemma B.4. Assume q is a quasi-norm on G. Then G admits a conjugation invariant unbounded norm q^+.

Proof. We start to modify our q. First thing to do is to make it symmetric. It is simply achieved by exchanging the value of $q(g)$ by $\max\{q(g), q(g^{-1})\}$. So without loss of generality we assume q satisfies $q(g) = q(g^{-1})$ for all $g \in G$. To ensure conjugation-invariance we take any $h \in G$ and consider its conjugacy class $C_G(h) = \{ghg^{-1} \mid g \in G\}$. The image of $C_G(h)$ by q is bounded by condition (2) in the definition of quasi-norm above. Exchanging $q(h)$ by $\max_{g\in G} q(ghg^{-1})$ we ensure conjugation-invariance. These perturbations of q does not affect its unboundedness so condition (3) is still satisfied. The only thing to check which is not obvius is whether such modified quasi-norm still satisfies subadditivity condition. So we claim two modifications described above does not affect subadditivity. We prove it in the case of first modification. The second one is completely analogous.

Claim B.5. Assume $q\colon G \to [0, \infty)$ is a quasi-norm. Let

$$\widetilde{q}(g) = \max\{q(g), q(g^{-1})\}$$

Such defined $\widetilde{q}$ is subadditive.

Proof of Claim B.5. For $A, B, C, D > 0$ there are elementary properties of max operator: $\max\{A + C, B + C\} = \max\{A, B\} + C$ and $\max\{A + B, C + D\} = \max\{A, C\} + \max\{B, C\}$. Using them and the fact q is subadditive we obtain:

$$\begin{aligned}
&\widetilde{q}(gh) = \max\{q(gh), q(h^{-1}g^{-1})\} \leqslant \\
&\max\{q(g) + q(h) + C, q(h^{-1}) + q(g^{-1}) + C\} = \\
&\max\{q(g) + q(h), q(g^{-1}) + q(h^{-1})\} + C \leqslant \\
&\max\{q(g), q(g^{-1})\} + \max\{q(h), q(h^{-1})\} + C = \widetilde{q}(g) + \widetilde{q}(h) + C
\end{aligned}$$

□

We finally define:

$$q^{+}(g) = \begin{cases} 0 & g = e \\ q(g) + C & g \neq e \end{cases}$$

Let us see such defined q^{+} is subadditive. Take $g, h \in G$. Then $q^{+}(gh) = q(gh) + C \leqslant q(g) + C + q(h) + C = q^{+}(g) + q^{+}(h)$. Because the only interesting case is when $C > 0$ then is follows directly from definition of q^{+} that $q^{+}(g) = 0$ if and only if $g = e$. So q^{+} is indeed an unbounded norm on G. □

Suppose $f\colon G \to H$ is an epimorphism of groups and q is quasi-norm on H. Then we can consider $f^{*}(q)$ - the pullback of q defined to be $f^{*}(q)(g) := q(f(g))$.

Lemma B.6. $f^{*}(q)$ is quasi-norm on G.

Proof. It is straightforward computation. For example

$$\begin{aligned}
f^{*}(q)(gh) = q(f(gh)) = q(f(g)f(h) &\leqslant q(f(g)) + q(f(h)) + C \\
&= f^{*}(q)(g) + f^{*}(q)(h) + C
\end{aligned}$$

which shows $f^*(q)$ satisfies first condition of definition of quasi-norm. Checking second condition is similar in spirit so we skip it.

Taking sequence (h_n) of elements in H such that $\lim_{n\to\infty} q(h_n) = \infty$ we may pick $g_n \in f^{-1}(h_n) \neq \varnothing$ because of the fact f is an epimorphism. Then by the very definition of $f^*(q)$ one has $\lim_{n\to\infty} f^*(q)(g_n) = \infty$. $\square$

Now we are ready to provide a proofs of Proposition 1.3 and Proposition 1.5. Both of them may be found in [1].

Proof of Proposition 1.3. At the first step we are going to show that any infinite abelian group G admits an unbounded norm. If G is finitely generated, then it has $\mathbb{Z}$ factor as a direct summand. Hence it admits an epimorphism onto $\mathbb{Z}$. So by the Lemma B.6 above G admits an unbounded norm. For a countably but not finitely generated group $G = \langle g_1, g_2, \ldots \rangle$ let $q(g)$ to be smallest k such that $g \in \langle g_1, \ldots, g_k \rangle$. This norm is unbounded. For a general infinite abelian group G we take some infinite finitely or countably generated subgroup H. By previous part there is unbounded norm q_H on H. Taking any $g \in G \backslash H$ we consider $H' = \langle H \cup \{g\} \rangle_G$. This is easy to verify we might extend q_H to $q_{H'}$ and combining it with Zorn's lemma we infer this general case from the previous ones.

Now let G be any group with infinite first homology $H_1(G)$. By the first step above G/G' admits an unbounded norm. Now we invoke Lemma B.6 once again for an epimorphism $G \to G/G'$ to obtain that G is unbounded. $\square$

Proof of Proposition 1.5. If $[G, G]$ has an infinite index in G, look at the epimorphism $G \to H := G/G'$. So group H is infinite abelian hence by Proposition 1.3 H is unbounded. Then G is itself unbounded. So we assume H is finite. Let S be a finite set of coset representatives of G' in G. Hence every element g in G has (unique) decomposition as hs where $h \in G'$ and $s \in S$. Define a quasi-norm q on G by $q(g) := \mathrm{cl}_G(h)$. To see quasi-conjugation invariance note that if $b = hs$ and $a = h's'$ then $b^{-1}ab = [s^{-1}h^{-1}, h's']a$ hence $q(b^{-1}ab)$ differs from $q(a)$ at most by 1. Now we prove quasi-triangle inequality. Take $g_1 = h_1 s_1$ and $g_2 = h_2 s_2$. So $g_1 g_2 = h_1 h_2 [h_2^{-1}, s_1] s_1 s_2$. Write $s_1 s_2$ as $h(s_1, s_2) s(s_1, s_2)$ where $h(s_1, s_2) \in G'$ and $s(s_1, s_2) \in S$. Let us take $C := \max_{s_1, s_2 \in S} \mathrm{cl}_G(h(s_1, s_2))$. So we have

$$q(g_1 g_2) = \mathrm{cl}_G(h_1 h_2 [h_2^{-1}, s_1] h(s_1, s_2)) \leqslant \mathrm{cl}_G(h_1) + \mathrm{cl}_G(h_2) + 1 + C = q(g_1) + q(g_2) + 1 + C$$

$\square$

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