$$\nabla^2 \varphi = 0$$

Solution of Laplace's Equation

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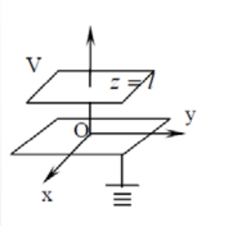
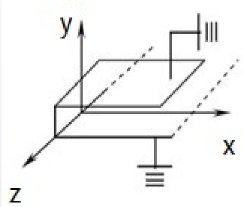
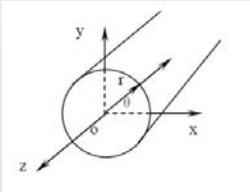
Structure







Derivation & Range

Analytical Solutions

Numerical Solutions

Equal Sapcing &
Unequal Spacings

Iteration &
Convergence

Program Realization

Derivation

Gauss's Law:

$$\left\{\begin{array}{l}\nabla \bullet \overrightarrow{D} = \rho_V \\ \overrightarrow{D} = \epsilon \overrightarrow{E}\end{array}\right. \Rightarrow \left\{\begin{array}{l}\nabla \bullet \left(\epsilon \overrightarrow{E}\right) = \rho_V \\ \overrightarrow{E} = -\nabla \varphi\end{array}\right. \Rightarrow \nabla \bullet (-\epsilon \nabla \varphi) = \rho_V$$

$$\Downarrow$$

$$\nabla^2 \varphi = -\frac{\rho_V}{\epsilon}$$

Poisson's Equation

Range

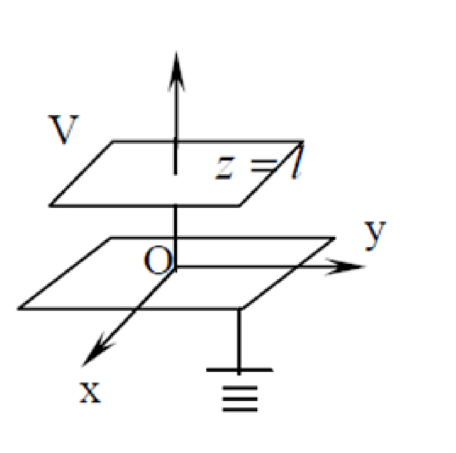
- Being analytical in the area D
- Space being free

In free space: $\rho_V = 0$

$$\therefore \quad \nabla^2 \varphi = 0$$

Analytical Solutions

Example 1

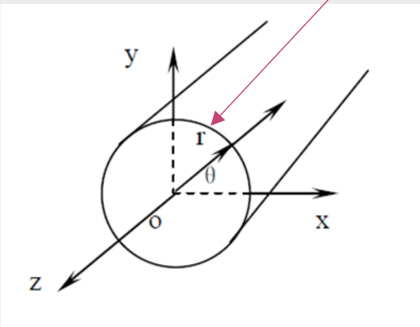


$$\begin{cases} \nabla^2 \varphi = 0 \\ \varphi(z=0) = 0 \\ \varphi(z=l) = V \end{cases} \Rightarrow \quad \varphi = \frac{V}{l} z$$

Analytical Solutions

Example 2

Uniform charge distribution: σ



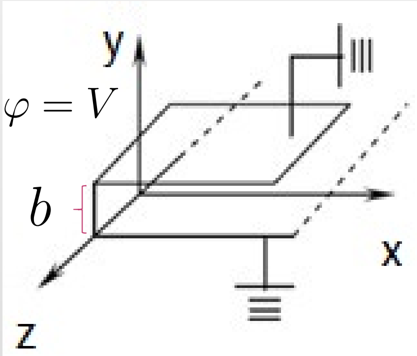
$$\begin{cases} \nabla^2 \varphi = 0 \\ \varphi(r = a) = 0 \end{cases} \Longrightarrow \quad \varphi(r) = -\frac{a\sigma}{\epsilon_0} \ln \frac{r}{a}$$

$\epsilon_0 = 8.85 \qquad pF/m$

Permittivity of free space

Analytical Solutions

Example 3

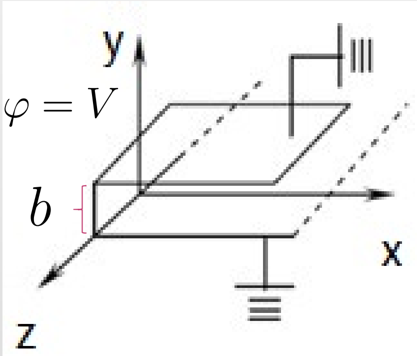


$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0$$

$$Let \qquad \varphi = X(x)Y(y)$$

Analytical Solutions

Example 3

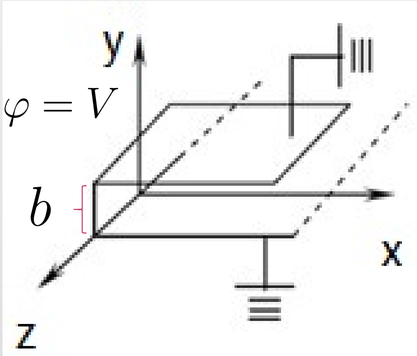


$$Let \qquad \varphi = X(x)Y(y)$$

$$where \qquad \begin{cases} X(x) = Ae^{kx} + Be^{-kx} \\ Y(y) = C\sin ky + D\cos ky \end{cases}$$

Analytical Solutions

Example 3



From boundary conditions:

$$y = 0, \varphi = 0 \Rightarrow D = 0 \qquad (AB \neq 0)$$

$$y = b, \varphi = 0 \Rightarrow \sin kb = 0 \qquad kb = n\pi$$

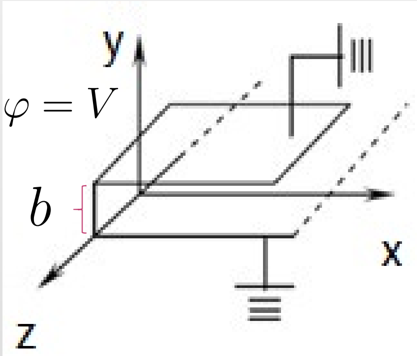
$$\therefore \quad \varphi \text{ is related to } n$$

$$\Rightarrow \varphi_n(x, y) = \left(A_n e^{kx} + B_n e^{-kx}\right)\left(C_n' \sin \frac{n\pi}{b} y\right)$$

$$B_n C_n' = C_n$$

Analytical Solutions

Example 3



Further:

$$x \to \infty \qquad \varphi = 0 \Rightarrow A_n = 0$$

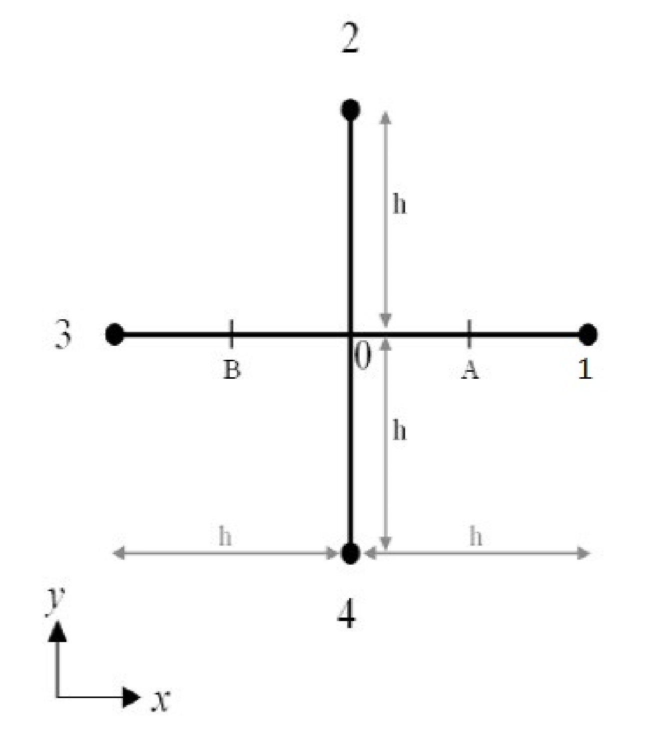
$$x = 0 \qquad \varphi = V \Rightarrow C_n = \begin{cases} \dfrac{4V}{m\pi} & m = oddnum. \\ 0 & m = evennum. \end{cases}$$

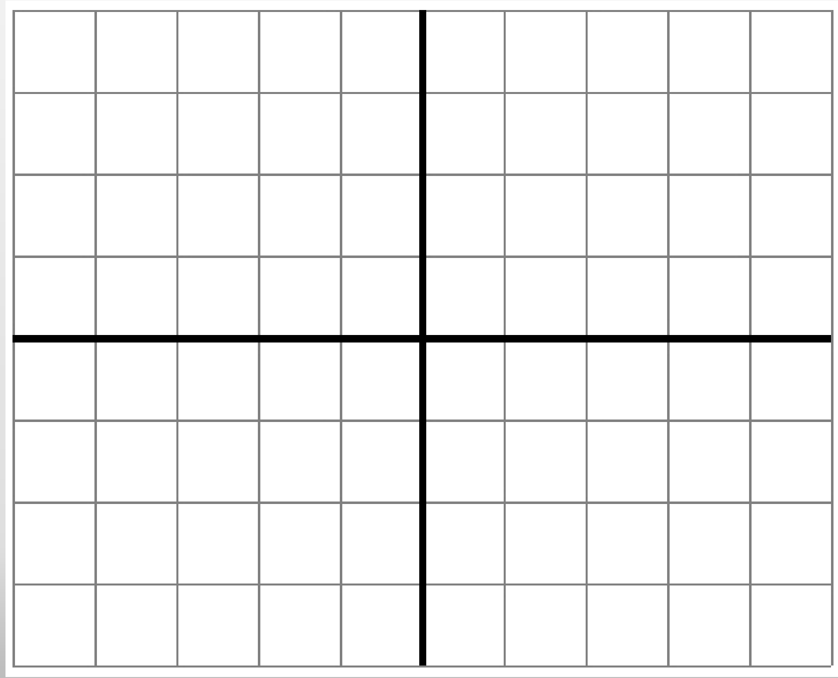
$$Let \qquad m = 2n+1, n = 0, 1, 2...$$

$$\boxed{\Rightarrow \varphi(x,y) = \frac{4V}{\pi} \sum_{m=0}^{\infty} \frac{1}{2n+1} \sin \frac{(m+1)\pi y}{b} e^{-\frac{(2n+1)\pi x}{b}}}$$

Numerical Solutions

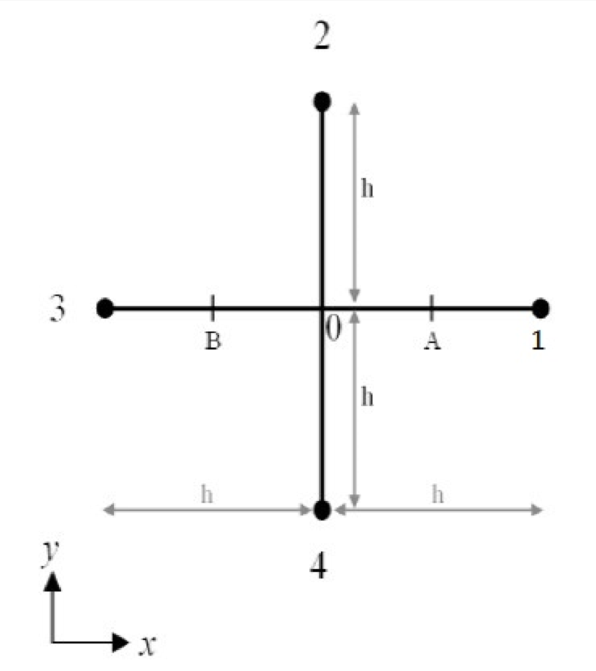
Equal Spacing





Numerical Solutions

Equal Spacing



$$\left.\frac{\partial\varphi}{\partial x}\right|_A = \frac{\varphi_1 - \varphi_0}{h}, \qquad \left.\frac{\partial\varphi}{\partial x}\right|_B = \frac{\varphi_0 - \varphi_3}{h}$$

$$\Rightarrow \left.\frac{\partial^2\varphi}{\partial x^2}\right|_0 = \frac{\left.\frac{\partial\varphi}{\partial x}\right|_A - \left.\frac{\partial\varphi}{\partial x}\right|_B}{h}$$

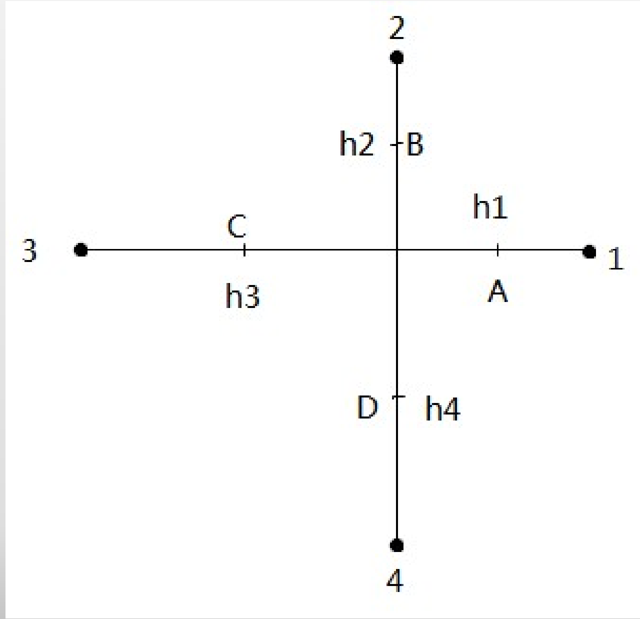
$$= \frac{1}{h^2}\left[(\varphi_1 - \varphi_0) + (\varphi_3 - \varphi_0)\right]$$

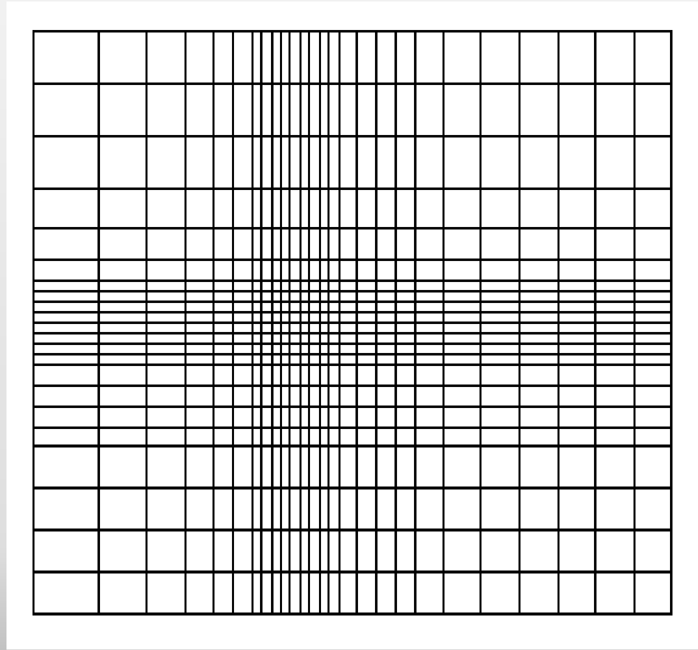
Similarly $\left.\dfrac{\partial^2\varphi}{\partial y^2}\right|_0 = \dfrac{1}{h^2}\left[(\varphi_2 - \varphi_0) + (\varphi_4 - \varphi_0)\right]$

$$\boxed{\Rightarrow \varphi_0 = \frac{1}{4}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4)}$$

Numerical Solutions

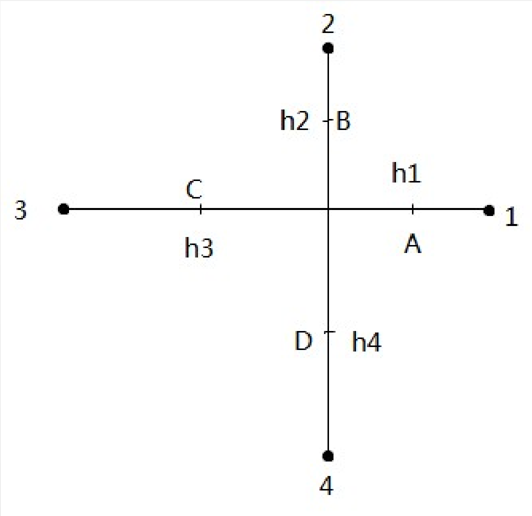
Unequal Spacings





Numerical Solutions

Unequal Spacings



$$\boxed{\varphi_0 = \frac{1}{\beta}\left(m\varphi_1 + n\varphi_2 + p\varphi_3 + q\varphi_4\right)}$$

$$where \quad \begin{cases} \beta = \dfrac{1}{h_1 h_3} + \dfrac{1}{h_2 h_4} \\ m = \dfrac{1}{h_1(h_1 + h_3)} \\ n = \dfrac{1}{h_2(h_2 + h_4)} \\ p = \dfrac{1}{h_3(h_3 + h_1)} \\ q = \dfrac{1}{h_4(h_2 + h_4)} \end{cases}$$

Numerical Solutions

Iteration & Convergence

Synchronization Iterative Method(SIM):

$$\varphi_{i,j}^{(1)} = \frac{1}{4}\left(\varphi_{i,j-1}^{(0)} + \varphi_{i,j+1}^{(0)} + \varphi_{i-1,j}^{(0)} + \varphi_{i+1,j}^{(0)}\right)$$

$$\text{Iterate...}$$

$$\varphi_{i,j}^{(k+1)} = \frac{1}{4}\left(\varphi_{i,j-1}^{(k)} + \varphi_{i,j+1}^{(k)} + \varphi_{i-1,j}^{(k)} + \varphi_{i+1,j}^{(k)}\right)$$

$$\text{Until}$$

$$\left|\varphi_{i,j}^{(k+1)} - \varphi_{i,j}^{(k)}\right| < w$$

Minimum error

Numerical Solutions

Iteration & Convergence

Asynchronous(or Gauss-Seidel) Iterative Methods(AIM):

$$\varphi_{i,j}^{(k+1)} = \frac{1}{4}\left(\varphi_{i,j-1}^{(k+1)} + \varphi_{i,j+1}^{(k)} + \varphi_{i-1,j}^{(k+1)} + \varphi_{i+1,j}^{(k)}\right)$$

Numerical Solutions

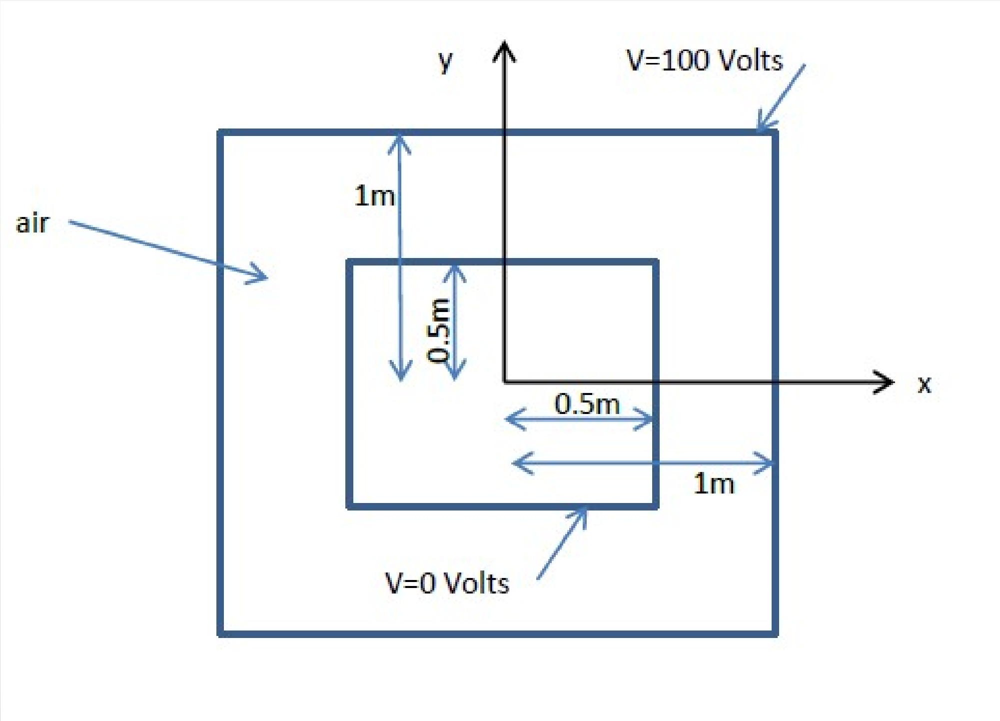
Iteration & Convergence

Successive Over Relaxation Iteration Method(SOR Method):

$$\varphi_{(i,j)}^{(k+1)} = \varphi_{(i,j)}^{(k)} + \frac{\alpha}{4}\left(\varphi_{i,j-1}^{(k+1)} + \varphi_{i,j+1}^{(k)} + \varphi_{i-1,j}^{(k+1)} + \varphi_{i+1,j}^{(k)} - 4\varphi_{(i,j)}^{(k)}\right)$$

$$\alpha \in [1, 2)$$

Program Realization



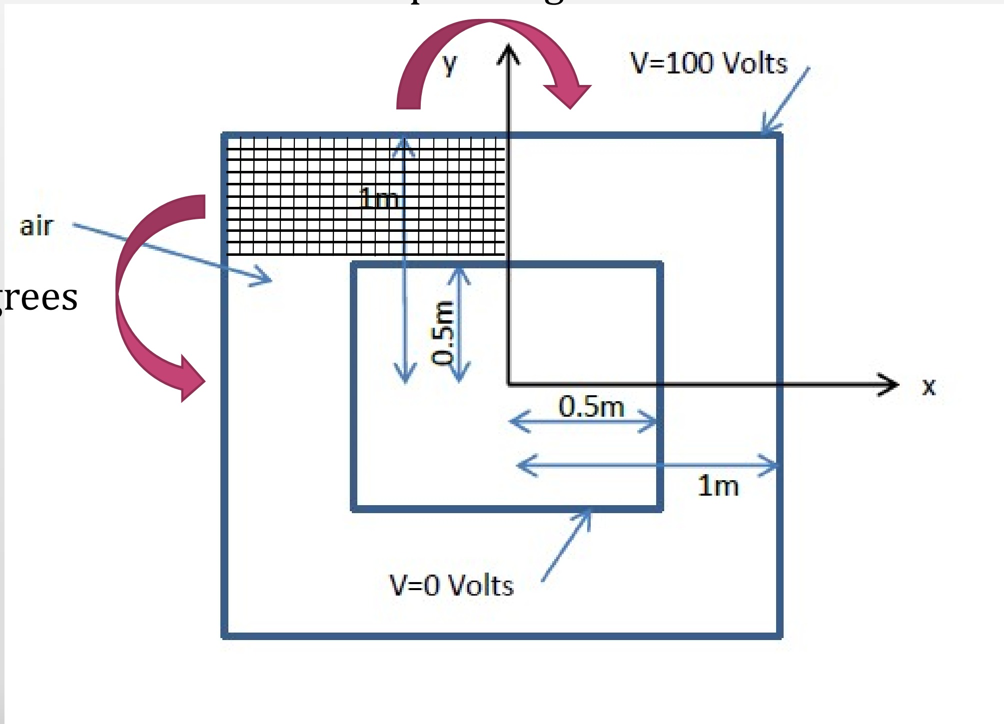
- Potential φ
- Electric field intensity $\overrightarrow{E}$
- Capacitance per meter C

Program Realization

Potential Calculating

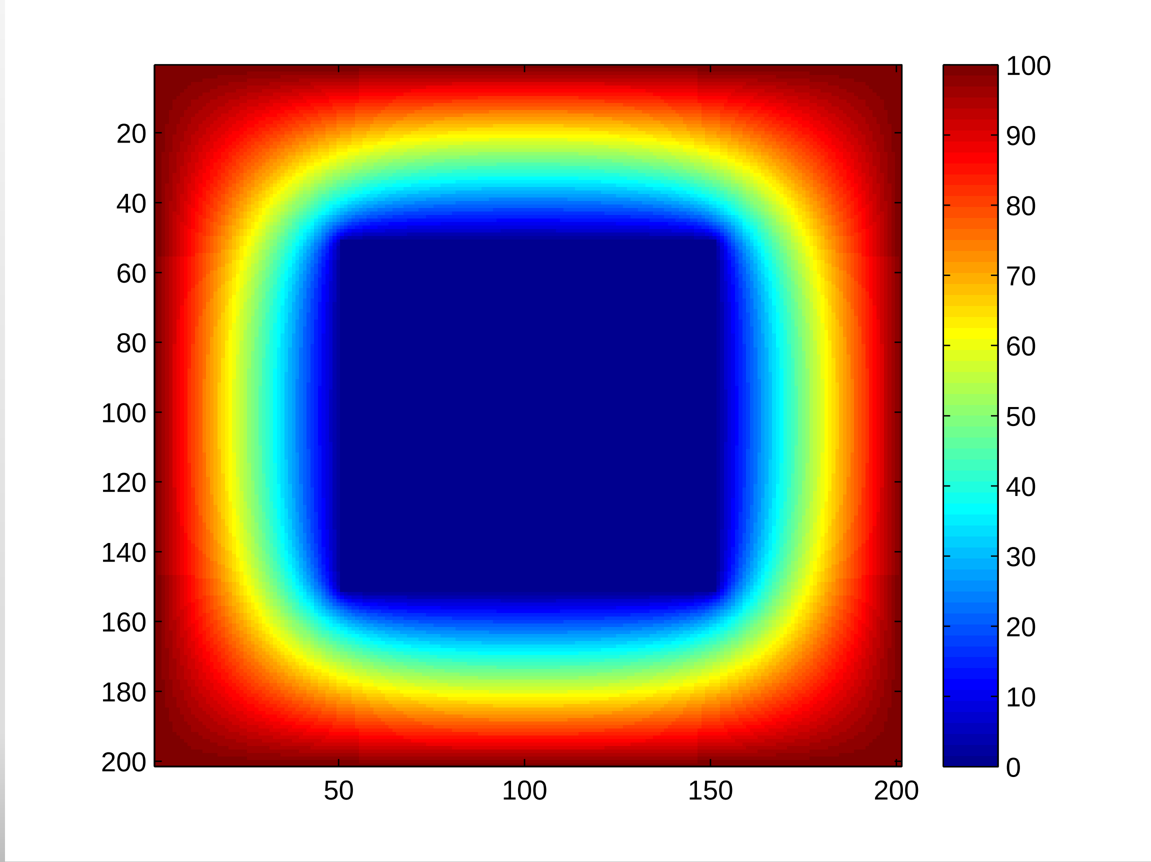
Flip left-right

Rotate 90 degrees



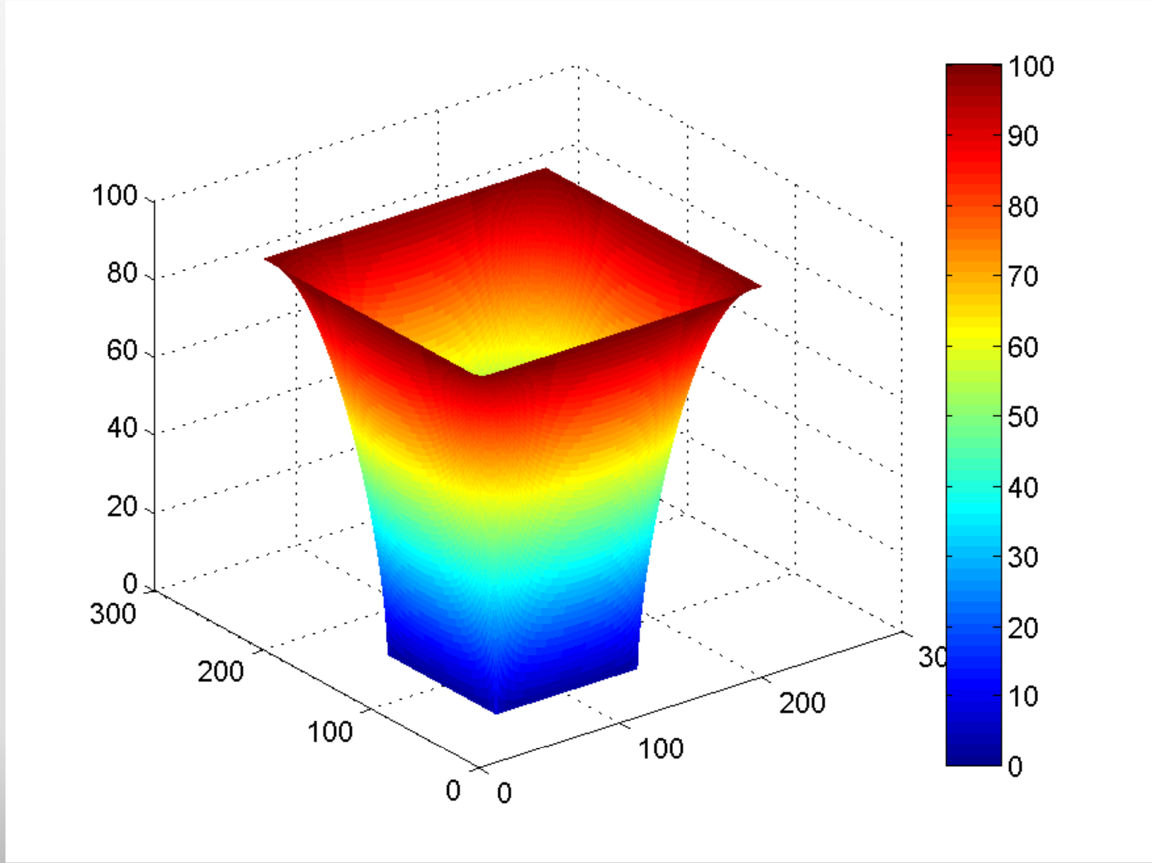
Program Realization

Potential Calculating



Program Realization

Potential Calculating



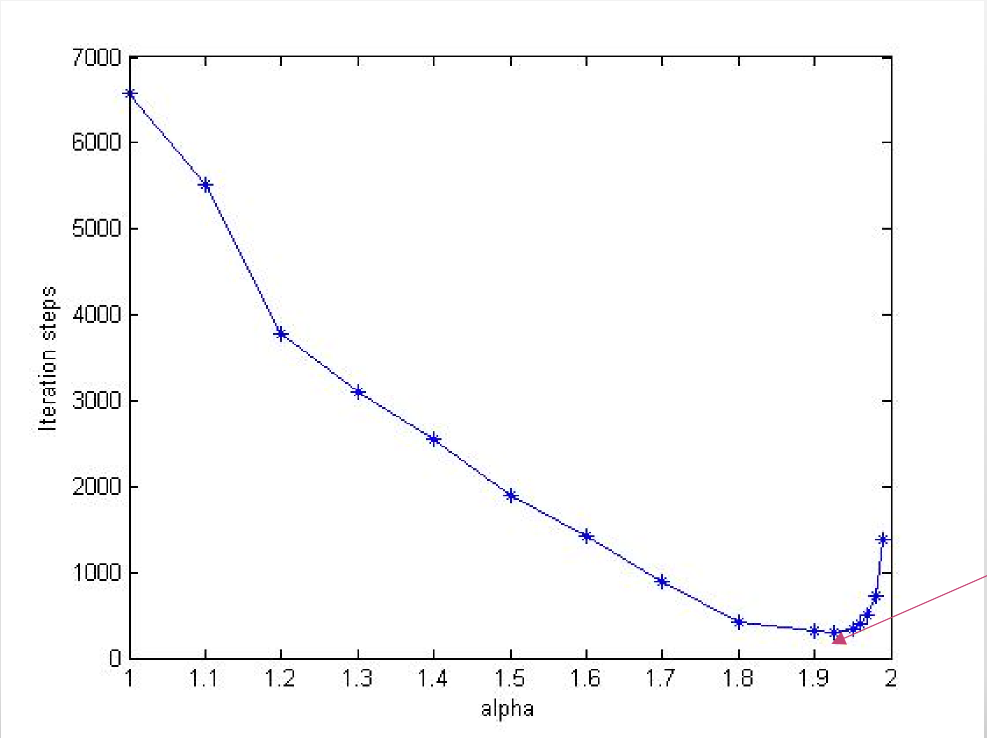
Program Realization

Potential Calculating

Asynchronous Iterative Method(AIM):	6579 steps
Successive Over Relaxation Iteration Method(SOR Method):	296 steps(Min)

Program Realization

Potential Calculating

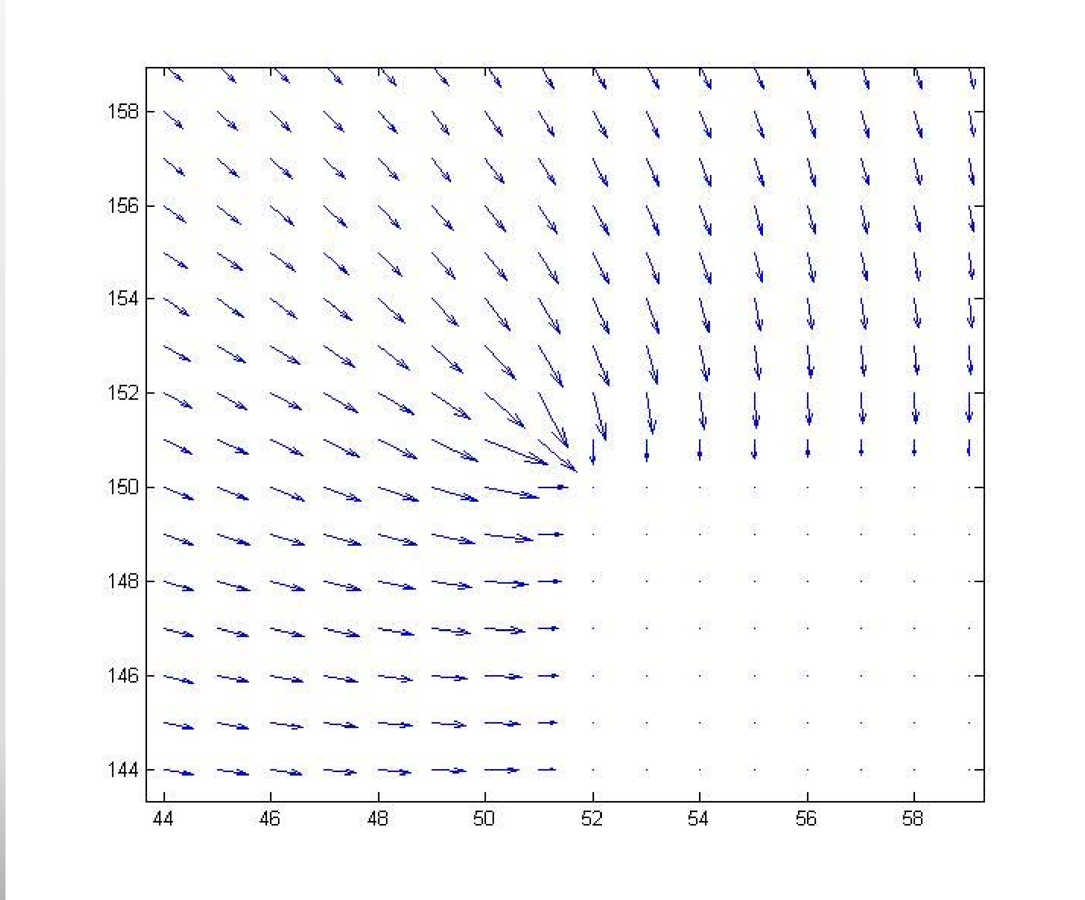


$\alpha = 1.925$

Program Realization

Electric Field Intensity Calculating

$$\overrightarrow{E} = -\nabla\varphi \qquad \Rightarrow$$

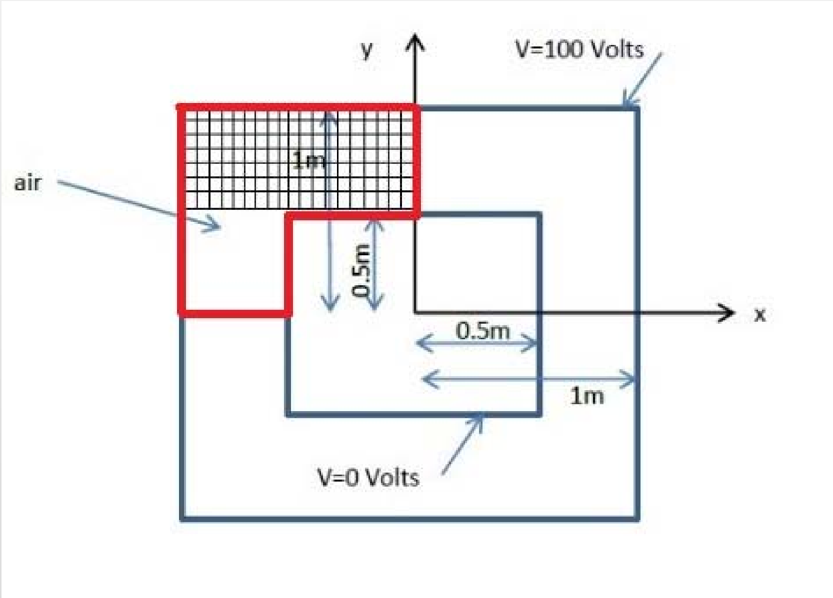


Program Realization

Capacitance Calculating

$$q = \oint \epsilon_0 \overrightarrow{E} ds$$

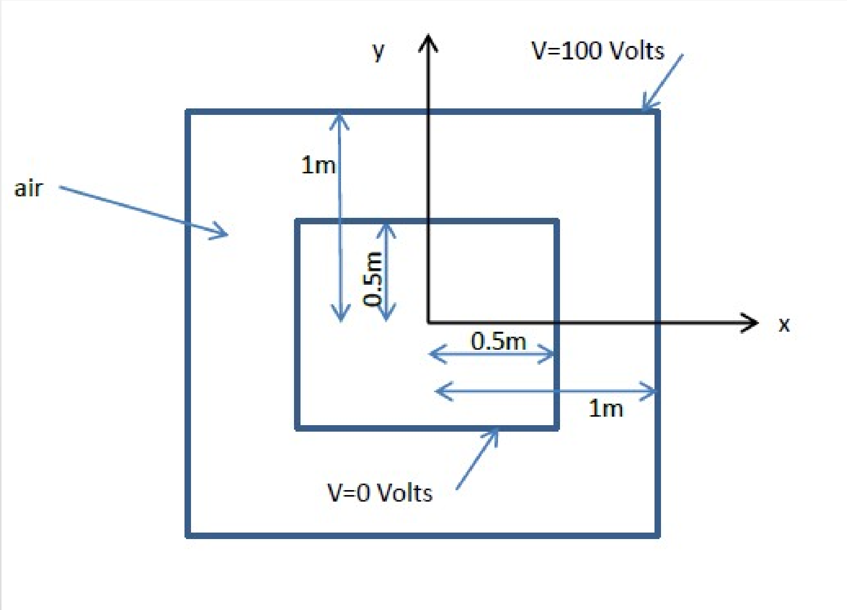
$$= \epsilon_0 \left(\oint E_x dl + E_y dl \right)$$



Program Realization

Capacitance Calculating

$$C = \frac{Q}{U} = \frac{4q}{U} = 0.73819 \qquad pF/m$$



Summary

- Derivation & Range (Albert & Dean)
- Three examples of analytical solutions (Ruth)
- Numerical solutions: equal spacing & unequal spacings; (Sullen & Angela)

  iteration & convergence (Dean)
- Program realization: φ $\overrightarrow{E}$ C (Dean)

Question Time