Solution of Laplace's Equation

Dean Chan

***Abstract*—This paper considers the derivation and range of Laplace's Equation in electromagnetism, three examples of analytical solutions and numerical schemes. Then programs are written to realize the numerical schemes and further calculate the capacitance per-meter of a square coaxial cable. Also, ways to accelerate the convergence rate are introduced and tested.**

***Keywords*—Solution, Laplace's Equation, numerical schemes, MATLAB, electromagnetism**

I. Introduction

In mathematics, Laplace's Equation is a second-order partial differential equation named after Pierre-Simon Laplace who first studied its properties. This is often written as:

$$\Delta\varphi = 0 \qquad or \qquad \nabla^2\varphi = 0 \tag{1}$$

where $\Delta = \nabla^2$ is the Laplace operator and φ is a scalar function. In this paper, φ indicates the potential. Laplace's Equation is important in many fields of science, notably the fields of electromagnetism, which is the background of this paper.

In this paper, we first derivate Laplace's Equation from Maxwell's Equations using the knowledge of differential geometry. What follows is the analytical solutions of Laplace's Equation. Solutions of Laplace's equation are called harmonic functions; they are all analytic within the domain where the equation is satisfied. Here we choose three boundray conditions to illustrate the practical method of solution. Many complex industrial and engineering problems need solutions of Laplace's Equation. What is frustrating is that these solutions cannot often be obtained analytically because of the occurrence of irregular boundaries for which systems of separable coordinates are unknown. Therefore we develop numerical schemes to overcome this barrier. Numerical schemes usually consist of massive calculation, which is difficult to human but favorable to computers. Therefore, the last section of this paper tells the procedure of programming with MATLAB to oprate a numeriacl scheme. In this way we greatly reduce the amount of labor required.

II. Derivation and range of Laplace's Equation

Gauss's Law says that the divergence of electric flux density is equal to the charge density.

$$\nabla \bullet \overrightarrow{D} = \rho_V \tag{2}$$

where

$$\overrightarrow{D} = \epsilon\overrightarrow{E} \tag{3}$$

As we all know, the intensity of electric field is equal to the opposite of the gradient of potential.

$$\overrightarrow{E} = -\nabla\varphi = -\left(\frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y} + \frac{\partial\varphi}{\partial z}\right) \tag{4}$$

Therefore

$$\nabla \bullet \left(\epsilon\overrightarrow{E}\right) = \rho_V \tag{5}$$

$$\nabla \bullet (-\epsilon\nabla\varphi) = \rho_V \tag{6}$$

$$\nabla^2\varphi = -\frac{\rho_V}{\epsilon} \tag{7}$$

In free space, $\rho_V = 0$, so

$$\nabla^2\varphi = 0 \tag{8}$$

This is the well-known Laplace's Equation.

Laplace's Equation can be concluded as the solution of a function φ defined in an area D where $\rho_V = 0$. The Newmann(or second-type) boundary condition does not give the function φ itself directly. Instead, the normal derivative along the boundary of area D is given directly. Therefore, in order to be available, the Laplace's Equation must be analytical in the area D which is provided. Of course, all of these must be in free space that is $\rho_V = 0$. The function φ is called harmonic function. With all of these satisfied, the Laplace's Equation can be effective.

III. Examples of analytical solutions

Finding the solution of Laplace's Equation is a progress of integration. There must be some unknown constants. This means Laplace's Equation cannot describe the static electric field uniquely. We need information of boundary condition to find the very solution. Three representational examples of analytical solutions corresponding to different boundary conditions are as follows.

A. *Infinite parallel conductive plates*

Two infinite parallel conductive plates with a distance of l are shown as the first sketch of Fig.1. The potential difference between the two plates is V. The Laplace's Equation and boundary conditions can be discribed as follows:

$$\begin{cases} \nabla^2\varphi = 0 \\ \varphi(z=0) = 0 \\ \varphi(z=l) = V \end{cases}$$

Any points, regardless of its (x, y) coordinate, share the same potential as long as they stand at the same hight. Therefore the equations are easy to solve.

$$\varphi = \frac{V}{l}z \tag{9}$$

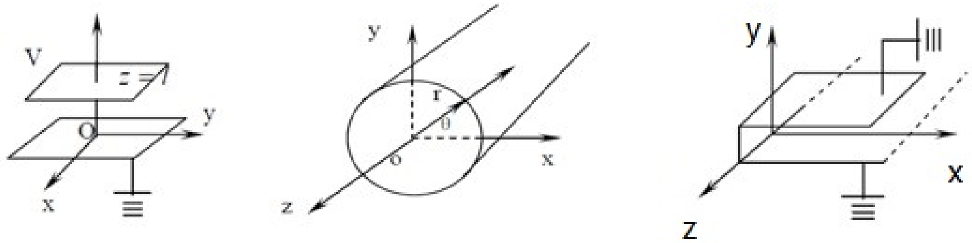

Fig. 1. Examples of different boundary conditions

B. Infinitely long cylindrical conductor

The figure of this example can be seen at the second position of Fig.1. The infinitely long cylindrial conductor, whose radius is a, has a uniform charge distribution of σ. Our work is to calculate the potential of the outer space of the conductor. Here we use the cylindrical coordinate so that φ has no relationship with θ. For the convenience of calculation, we assume the potential of the surface is 0.

$$\begin{cases} \nabla^2\varphi = 0 \\ \varphi(r=a) = 0 \end{cases}$$

Then we utilize the common routine: solving the equation with some unknown constants and figuring them out with boundary conditions afterwards. The solution is

$$\varphi(r) = -\frac{a\sigma}{\epsilon_0}\ln\frac{r}{a} \tag{10}$$

where ϵ_0 is the permittivity of free space, equal to $8.85\times 10^{-12}F/m$.

C. Semi-infinite plates

The one lies at the third position is another common condition. The distance between the two grounded semi-infinite plates is b. The left plate has a potential of V. It is easy to see that φ is not related to z. Here we use a little trick

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \tag{11}$$

$$Let \qquad \varphi = X(x)Y(y) \tag{12}$$

$$where \qquad \begin{cases} X(x) = Ae^{kx} + Be^{-kx} \\ Y(y) = C\sin ky + D\cos ky \end{cases} \tag{13}$$

According to the boundary condition,

$$y = 0, \varphi = 0 \Rightarrow D = 0 \qquad (AB \neq 0)$$

$$y = b, \varphi = 0 \Rightarrow \sin kb = 0 \qquad kb = n\pi$$

$\therefore \varphi$ is related to n. So it can be written as

$$\varphi_n(x,y) = \left(A_n e^{kx} + B_n e^{-kx}\right)\left(C_n' \sin\frac{n\pi}{b}y\right) \tag{14}$$

where $n = 1,2,3...$ and $B_n C_n' = C_n$. Further

$$x \to \infty, \varphi = 0 \Rightarrow A_n = 0$$

$$x = 0, \varphi = V \Rightarrow C_n = \begin{cases} \dfrac{4V}{m\pi} & m = oddnum. \\ 0 & m = evennum. \end{cases}$$

Let $m = 2n+1, n = 0,1,2...$ Finally, we got the soution as

$$\varphi(x,y) = \frac{4V}{\pi}\sum_{m=0}^{\infty}\frac{1}{2n+1}\sin\frac{(m+1)\pi y}{b}e^{-\dfrac{(2n+1)\pi x}{b}} \tag{15}$$

IV. Numerical schemes for the solution

Many differential equations, especially these with complex(or not smooth) boundray conditions, cannot be solved analytically, such as the square coaxial cable shown in Fig.2. However, in science and engineering, a numeric approximation to the solution is often good enough to solve a problem. The basic idea of numerical solution is replacing the continuous fixed solution area by the grid composed of a finite number of discrete points, which are called nodes in the grid; use discrete variable defined in the grid to approximate the continuous variable in region with continuous quantity; use difference quotient to approximate the derivative of original equations and boundary conditions, so the original differential equations and boundary conditions are approximated by the algebraic equations. The solution of this equation can be approximated in the discrete points on the original problem. According to the different shape of the grid(square or oblong), the numerical schemes can be divided into two categories: equal spacing scheme and unequal spacings scheme.

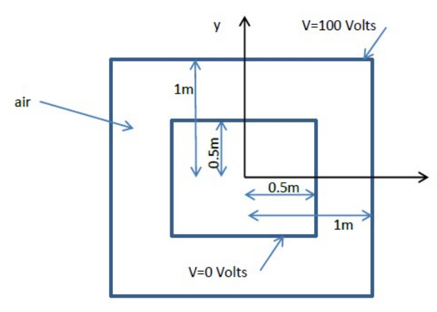

Fig. 2. Square coaxial cable

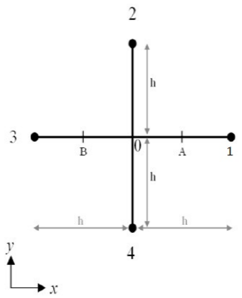

Fig. 3. Equal spacing

A. Cartesian grid with equal spacing

From Fig.3, we apply

$$\frac{\partial \varphi}{\partial x}\Big|_A = \frac{\varphi_1 - \varphi_0}{h}, \qquad \frac{\partial \varphi}{\partial x}\Big|_B = \frac{\varphi_0 - \varphi_3}{h} \tag{16}$$

$$\Rightarrow \frac{\partial^2 \varphi}{\partial x^2}\Big|_0 = \frac{\frac{\partial \varphi}{\partial x}\Big|_A - \frac{\partial \varphi}{\partial x}\Big|_B}{h} \tag{17}$$

$$= \frac{1}{h^2}\left[(\varphi_1 - \varphi_0) + (\varphi_3 - \varphi_0)\right] \tag{18}$$

Similarly

$$\frac{\partial^2 \varphi}{\partial y^2}\Big|_0 = \frac{1}{h^2}\left[(\varphi_2 - \varphi_0) + (\varphi_4 - \varphi_0)\right] \tag{19}$$

$$\Rightarrow \varphi_0 = \frac{1}{4}\left(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4\right) \tag{20}$$

B. Cartesian grid with unequal spacings

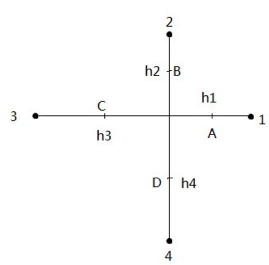

Fig. 4. Unequal spacings

When making the grid, there will not always be enough luck to lie the boundary on your nodes. Therefore, we have to make meshgrid with unequal spacings to fit, for which Fig.4 demonstrates well.

To obtain the expression of φ_0, we utilize the same derivation process when dealing with Cartesian grid with equal spacing. The solution is

$$\varphi_0 = \frac{1}{\beta}\left(m\varphi_1 + n\varphi_2 + p\varphi_3 + q\varphi_4\right) \tag{21}$$

$$where \quad \begin{cases} \beta = \dfrac{1}{h_1 h_3} + \dfrac{1}{h_2 h_4} \\ m = \dfrac{1}{h_1(h_1 + h_3)} \\ n = \dfrac{1}{h_2(h_2 + h_4)} \\ p = \dfrac{1}{h_3(h_3 + h_1)} \\ q = \dfrac{1}{h_4(h_2 + h_4)} \end{cases} \tag{22}$$

C. Iteration & Convergence

After the gridding, Laplace's Equation is repalced with a set of difference equations, making it easier to calculate.

In the area D, every node has a corresponding equation. To solve a practical issue, we usually adopt large quantities of nodes for the sake of accuracy. Therefore, the rank is usually too high for hand computation, giving computers a good chance to show their talents. The most common algorithm is Synchronization Iterative Method(SIM), also known as Simple Iterative Method. First, assign a original value $\varphi_{i,j}^{(0)}$ to every node of area D. Then substitute these values into equation (20)or (21).

$$\varphi_{i,j}^{(1)} = \frac{1}{4}\left(\varphi_{i,j-1}^{(0)} + \varphi_{i,j+1}^{(0)} + \varphi_{i-1,j}^{(0)} + \varphi_{i+1,j}^{(0)}\right) \tag{23}$$

Use $\varphi_{i,j}^{(1)}$ as the first approximation value. When the 4 values at the right side of equation (23) locate at the boundray, use the boundray condition. Substituting $\varphi_{i,j}^{(1)}$ to the right side can get the second approximation value. Generally, from the k-th approximation value, utilizing equation (23) will lead to the (k+1)-th approximation value.

$$\varphi_{i,j}^{(k+1)} = \frac{1}{4}\left(\varphi_{i,j-1}^{(k)} + \varphi_{i,j+1}^{(k)} + \varphi_{i-1,j}^{(k)} + \varphi_{i+1,j}^{(k)}\right) \tag{24}$$

Continue the iteration until

$$\left|\varphi_{i,j}^{(k+1)} - \varphi_{i,j}^{(k)}\right| < w \tag{25}$$

here w is the minimum error decided on the basis of practical issues.

To operate Synchronization Iterative Method on a computer, two set of arrays(or matrices) are needed to store the old values and new values. Asynchronous(or Gauss-Seidel) Iterative Methods(AIM), described by equation (26), can overcome this shortage.

$$\varphi_{i,j}^{(k+1)} = \frac{1}{4}\left(\varphi_{i,j-1}^{(k+1)} + \varphi_{i,j+1}^{(k)} + \varphi_{i-1,j}^{(k+1)} + \varphi_{i+1,j}^{(k)}\right) \tag{26}$$

In equation (26), half of the old values are replaced by new ones, making the convergence rate twice as fast as that of SIM.

It is proved by actual computation that the convergence rate is still too slow when the number of nodes goes too large. To satisfy the need of calculation speed, Successive

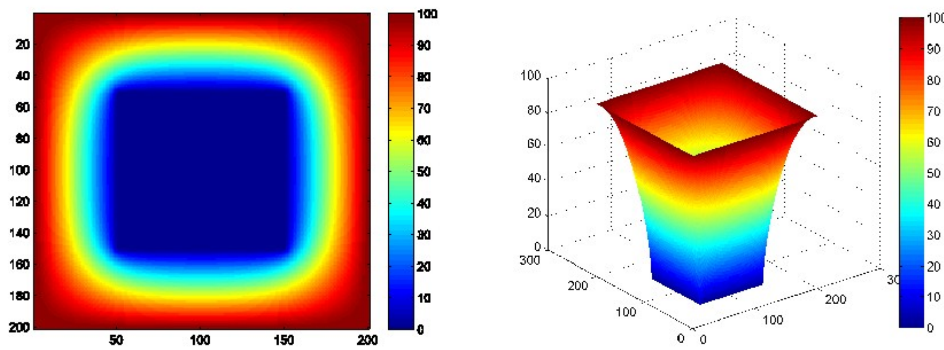

Fig. 5. Potential distribution

Over Relaxation Iteration Method(SOR Method) is made on the basis of AIM. Equation (27) discribes the basic idea of SOR Method.

$$\varphi_{(i,j)}^{(k+1)} = \varphi_{(i,j)}^{(k)} + \frac{\alpha}{4}\left(\varphi_{i,j-1}^{(k+1)} + \varphi_{i,j+1}^{(k)} + \varphi_{i-1,j}^{(k+1)} + \varphi_{i+1,j}^{(k)} - 4\varphi_{(i,j)}^{(k)}\right) \quad (27)$$

Here α is an accelerative convergence factor in the range of $[1, 2)$. It is uaually decided by practical experience.

V. Program realization

A. Numerical solution of Laplace's Equation

There's still a problem remaining unsolved in Fig.3. This section illustrates the program realization of numerical scheme derivated above. We divide the cross section of square coaxial cable into a 200*200 meshgrid, and then utilize Asynchronous Iterative Method to compute the potential of every point between the inner and outer surfaces. Thanks to the symmetry, we only have to do the calculation in the shaded area of Fig.6. and then mirror it to other areas. w is set as 0.000001. The detailed algorithm is shown in TABLE I.

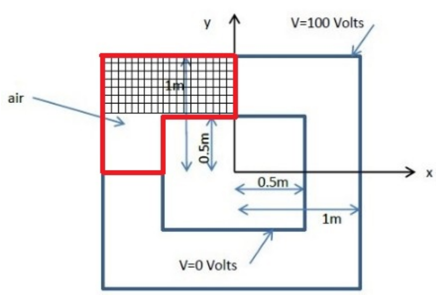

Fig. 6. Shaded area for computing

After 6579 steps of iteration, we get the potential distribution maps in 2 dimensions and 3 dimentions shown in Fig.5.

SOR Method gives the more fast performance. When $\alpha = 1$, it has the same iteration steps as that of AIM. The steps goes to minimum as 296 when $\alpha = 1.925$. Fig.7 shows the relationship between iteration steps and α.

TABLE I. Algorithm to compute the potential

Algorithm 1
1. Grid the cross section into 200*200, getting a matrix of 201*201 nodes
2. Assign boundray potential to the matrix
3. Compute the potential of every node in the shaded area
4. Mirror the shaded area to others
5. Keep doing step 3 and step 4 until $\lvert\varphi_{i,j}^{(k+1)} - \varphi_{i,j}^{(k)}\rvert < 0.000001$
6. Draw the diagrammatic map

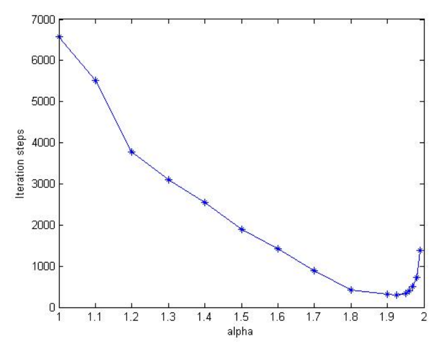

Fig. 7. Steps-alpha curve

B. Capacitance calculating

The basic idea to calculate the capacitance is shown by Algorithm 2 in TABLE II. A glimpse of electric field intensity E is shown in Fig.8.

TABLE II. Algorithm to compute the capacitance

Algorithm 2
1. Find the gradient of φ in x and y directions
2. Creating matrices as EX and EY to store the opposite value of the gradient of φ.
3. Do the loop integral to E on the red line in Fig.6, getting hypothetical charge Q'
4. $C = \frac{4Q'}{U}, \qquad U = 100V$

The capacitance of this coaxial cable per-meter is $0.73819pF$.

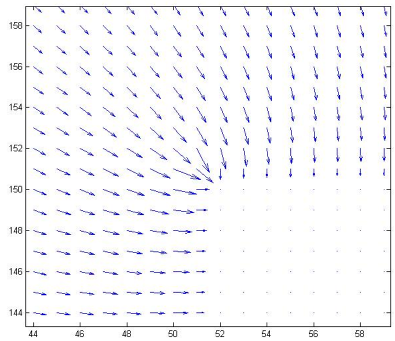

Fig. 8. Partial view of electric field intensity

VI. Conclusion

Both analytical and numerical solutions have been disscussed above. Analytical solutions are precise but difficult to calculate sometimes. It is important to theoretical study. Numerical schemes, talked about 2 gridding methods and 3 iteration methods, are useful in engineering application. It is suggested to judge the condition before using any of them to obtain efficiency of calculation.

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