CEE362G Fall 2015
Project 1
Due: October 5, 2015

One particular class of inverse problems is image deblurring. In this case, we may have a lot of data but each observation is affected by blurring or smearing caused by the optical or other sensor used. The sensor does not measure the "true value at a point" but a weighted average around it. In this case, the matrix $\boldsymbol{H}$, although square, is very poorly conditioned.

We will see to what extent we can, using the linear inversion method we saw in the first few lectures, restore the true image from the blurred image we get from the raw data following an example from *Hansen* (1997). The physical significance of the problem is described in *Shaw* (1972). The example is 1-D because 2-D would be much more computationally expensive to run and thus would require more sophisticated tools.

The relation between the observation y and the unknown s is given through a Fredholm integral equation of the first kind:

$$y\left(t\right)=\int_{-\pi/2}^{\pi/2}K\left(t,u\right)s\left(u\right)du+v \tag{1}$$

where the kernel K is a known function and v is small random noise. The kernel is

$$K\left(t,u\right)=\left(\cos\left(t\right)+\cos\left(u\right)\right)^{2}\left(\frac{\sin\left(\zeta\right)}{\zeta}\right)^{2} \tag{2}$$

where $\zeta=\pi\left(\sin\left(t\right)+\sin\left(u\right)\right)$.

We will assume that it is sufficient to use a simple quadrature scheme to approximate the integral equation for the matrix-vector form $\boldsymbol{y}=\boldsymbol{H}\boldsymbol{s}+\boldsymbol{v}$. It is straightforward to program. You can verify your result using Hansen's MATLAB function `shaw.m`, to be found at here. This assumes $m=n$, an even number and the matrix $\boldsymbol{A}$ this function gives as output is the matrix $\boldsymbol{H}$ we are looking for.

1. To see the effect of blurring caused by this kernel, consider a few examples of known $s\left(t\right)$ where you evaluate $y\left(t\right)$.

2. Investigate the distribution of the spectrum of $\boldsymbol{H}$. Keeping the interval $[-\pi/2,\pi/2]$, use $n=m=N$ and try for the cases $N=20$, then 200, and then 2000.

3. For $m=n=200$, plot the "most important" vectors that span the rowspace of $\boldsymbol{H}$.

4. For $m=n=200$. Assume that

$$s\left(t\right)=2\exp\left(-6\left(t-0.8\right)^{2}\right)+\exp\left(-2\left(t+0.5\right)^{2}\right) \tag{3}$$

compute y and then see whether using `pinv` will yield the right result.

5. Repeat 4 with noise with standard deviation 1.E-12 added to the y data. Are the results affected? Try higher and lower noise levels.

6. Now, let us look at some data. Use the data y in file 2015Assign1_1.txt (first column is data coordinate t, second column is data value y). In this case, $n = m = 400$. Find the posterior mean and standard deviation of s. Use the linear inversion method we covered in the first few lectures (try `GenLinInv.m`). Try several different $\boldsymbol{Q}$ and $\boldsymbol{R}$ till you are satisfied you have a "reasonable" result.

7. Generate 10 conditional realizations.

8. During post-processing, one member of the team suggests what the solution may be. Can you provide evidence supporting or refuting this claim?

References

Hansen, P. C. (1997), *Rank-Deficient and Discrete Ill-Posed Problems: Numerical Aspects of Linear Inversion*, SIAM Monographs on Mathematical Modeling and Computation 4, SIAM.

Shaw, C. B. (1972), Improvement of the resolution of an instrument by numerical solution of an integral equation, *J. Math. Anal. Appl.*, *37*, 83–112.