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MATH 6367: Optimization and Variational Methods II, Spring 2015
Extra Assignment

Problem: Apply the L-shaped method to find the solution of the following deterministic equivalent program (DEP)

$$\min\ 6x + 10\mathcal{Q}(x)$$
$$\text{subject to}\ \ 0 \leq x \leq 2$$

of a two-stage stochastic linear program with recourse function below.

$$\mathcal{Q}(x) = E_{\omega}Q\big(x, \xi(\omega)\big)$$
$$Q(x, \xi) \quad \begin{cases} \xi \quad x, & x \leq \xi \\ x \quad \xi, & x \geq \xi \end{cases}$$

The random variable ξ takes the values $\xi_1 = 0$ and $\xi_2 = 0$ with probability 0.5 each.

$$Q_1(x, \xi_1) = \begin{cases} \quad x, & x \leq 0 \\ x, & x \geq 0 \end{cases}$$
$$Q_2(x, \xi_2) = \begin{cases} 1 \quad x, & x \leq 1 \\ x \quad 1, & x \geq 1 \end{cases}$$
$$\mathcal{Q}(x) = E_{\omega}Q\big(x, \xi(\omega)\big) = 0.5Q_1(x, \xi_1) + 0.5Q_2(x, \xi_2)$$

$$\mathcal{Q}(x) = E_{\omega}Q\big(x, \xi(\omega)\big) = \begin{cases} 0.5 \quad x, & x \leq 0 \\ 0.5, & 0 \leq x \leq 1 \\ x \quad 0.5, & x \geq 1 \end{cases}$$

Since the recourse function and objective function are of low dimensionality, they are plotted in Figure 1 respectively. The graphs will enable verification of the L-shaped method's solution.

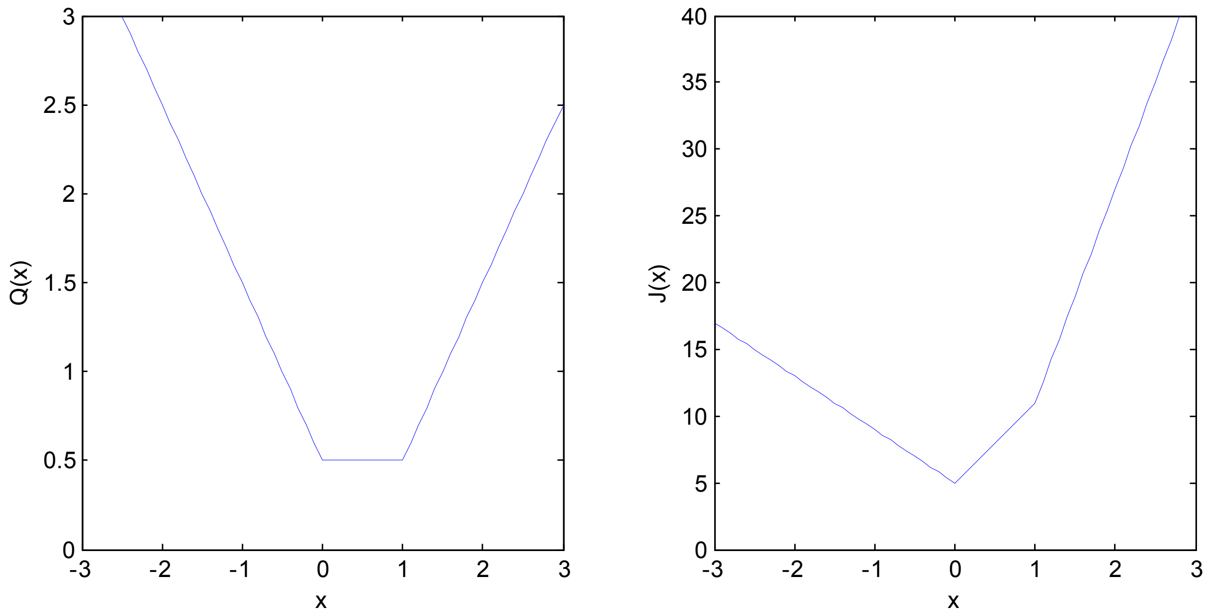

Figure 1. The recourse function $Q(x)$ is plotted on the left. The objective function *J(x)* is plotted on the right.

L-shaped method

Preliminary setup for the problem:

From $Q(x, \xi(\omega))$, we obtain q, W, (ω), and $T(\omega)$.

$$q = 1$$

$$W = 1$$

$$_k(\omega) = \begin{cases} \xi_k, & x \leq \xi_k \\ \xi_k, & x > \xi_k \end{cases}$$

$$T_k(\omega) = \begin{cases} 1, & x \leq \xi_k \\ 1, & x > \xi_k \end{cases}$$

Iteration 1:

Step 0: set $r = s = \nu = 0, D_r = 0, d_r = 0, E_s = 0, e_s = 0$

Step 1: set $\nu = \nu + 1 = 1,\ \theta^1 = \quad \infty$

$\min J(x, \theta) = c^T x + \theta = 6x$

subject to $Ax = b$

$0 \leq x \leq 2$

by inspection, $x^1 = 0$

Step 2:

$$\min \tilde{J}\,(y, v^+, v^-) = e^T v^+ + e^T v^-$$

subject to $Wy + Iv^+ \quad Iv^- = \quad_k \quad T_k x^v$

$$y \geq 0,\ v^+ \geq 0,\ v^- \geq 0$$

$\tilde{J}(y, v^+, v^-) = 0\ \ for\ 1 \leq k \leq 2$ since $x^1 = 0$ is feasible.

Step 3:

$$\min \hat{J}(y) = q^T y = y$$

subject to $Wy = \quad_k \quad T_k x^v$

Solve the associated dual LP as follows

$$\max \quad_k \pi_k^1$$

subject to $\pi_k^1 + s = 1$

$$s \geq 0$$

For $k = 1$, $\quad_1 = 0$ and $T_1 = 1$.

Then, $\pi_1^1 = 1$.

For $k = 2$, $\quad_2 = 1$ and $T_2 = 1$.

Then, $\pi_2^1 = 1$.

$$E_1 = \sum_{k=1}^{2} p_k (\pi_k^1) T_k = 0.5(1)(1) + 0.5(1)(1) = 1$$

$$e_1 = \sum_{k=1}^{2} p_k (\pi_k^1) \quad_k = 0.5(1)(0) + 0.5(1)(1) = 0.5$$

$$J^1 = e_1 \quad E_1 x^1 = 0.5 \quad (1)(0) = 0.5$$

check $\theta^1 \geq J^1 \rightarrow False$

set $s = s + 1 = 1$ and return to step 1

Iteration 2:

Step 1: set $\nu = \nu + 1 = 2$

Add the optimality cut: $E_1 x + \theta \geq e_1 \ \rightarrow \ \theta \geq 0.5 \quad x$

$\min J(x, \theta) = c^T x + \theta = 6x$

subject to $\theta \geq 0.5 \quad x$
$0 \leq x \leq 2$

by inspection, $x^2 = 0$ and $\theta^2 = 0.5$

Step 2:

$\min \tilde{J}\,(y, v^+, v^-) = e^T v^+ + e^T v^-$

subject to $Wy + Iv^+ \quad Iv^- = \quad_k \quad T_k x^v$
$y \geq 0,\ v^+ \geq 0,\ v^- \geq 0$

$\tilde{J}(y, v^+, v^-) = 0\ \ for\ 1 \leq k \leq 2$ since $x^2 = 0$ is feasible.

Step 3:

$\min \hat{J}(y) = q^T y = y$

subject to $Wy = \quad_k \quad T_k x^v$

Solve the associated dual LP as follows

max $\quad_k \pi_k^2$

Subject to $\pi_k^2 + s = 1$
$s \geq 0$

For $k = 1$, $\quad_1 = 0$ and $T_1 = 1$.

Then, $\pi_1^2 = 1$.

For $k = 2$, $\quad_2 = 1$ and $T_2 = 1$.

Then, $\pi_2^2 = 1$.

$$E_1 = \sum_{k=1}^{2} p_k(\pi_k^1) T_k = 0.5(1)(1) + 0.5(1)(1) = 1$$

$$e_1 = \sum_{k=1}^{2} p_k(\pi_k^1) \quad_k = 0.5(1)(0) + 0.5(1)(1) = 0.5$$

$J^2 = e_2 \quad E_2 x^2 = 0.5 \quad (1)(0) = 0.5$

check $\theta^2 \geq J^2 \rightarrow True$

Therefore, terminate the algorithm.

Then the optimal solutions for x and θ are $x = 0$ and $\theta = 0.5$.

For the objective function, the optimal solution is

$$J(x = 0) = 6x + 10Q(x) = 6(0) + 10(0.5)$$

$$J(x = 0) = 5$$

We refer back to Figure 1 to verify the solution. The recourse function $Q(x)$ is optimal on the interval $x \in [0,1]$, but the objective function $J(x)$ has a global minimum at $x = 0$.

Appendix

The Matlab code used to make Figure 1 is included below.

Function *Q.m*

```
function y = Q(x,xi)
% Recourse/value function

y1 = (xi - x).*(x <= xi);
y2 = (x - xi).*(x >= xi);
y = y1 + y2;
```

Main script *main.m*

```
clear; clc;
% Alexander Hebert
% Math 6367
% Extra assignment
% main script

% Compute the recourse function Q(x)

x = [-3:0.1:3];
xi1 = 0;
Q1 = Q(x,xi1);
xi2 = 1;
Q2 = Q(x,xi2);
Q_expected = 0.5*Q1 + 0.5*Q2;

figure(1)
plot(x,Q_expected)
xlabel('x')
ylabel('Q(x)')

% Compute the objective function J(x)

J = 6*x + 10*Q_expected;
```

```
% Plot both functions side by side
figure(1)

subplot(1,2,1)
plot(x,Q_expected)
xlabel('x')
ylabel('Q(x)')
axis([-3 3 0 3])

subplot(1,2,2)
plot(x,J)
xlabel('x')
ylabel('J(x)')
axis([-3 3 0 40])
```