Stereometry

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This paper contains basic definitions, facts and formulas related to the 3d geometry useful for programming competitions. Almost no algorithms nor geomtrical data structures included.

1 Basic definitions

A 3d *point* is defined by its coordinates (x, y, z).

A 3d *vector* is also defined by its coordinates $\vec{v} = (x, y, z)$. We will identify points and vectors, that allows us to add vector to a point without writing extra classes and methods.

Let $\vec{v_1} = (x_1, y_1, z_1)$ and $\vec{v_2} = (x_2, y_2, z_2)$. Then:

- $\vec{v_1} + \vec{v_2} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ is the sum of vectors;
- $\vec{v_1} - \vec{v_2} = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$ is the difference of vectors;
- $k\vec{v_1} = \vec{v_1}k = (kx_1, ky_1, kz_1)$ is the vector multiplied by the real value k;
- $\vec{v_1} \cdot \vec{v_2} = (v_1, v_2) = x_1x_2 + y_1y_2 + z_1z_2$ is the *scalar* product of vectors;

As well as in the 2d case, the operations above have intuitive geometrical meaning. Sum of vectors may be find by triangle or parallelogram rule, the difference of vectors is a vector connecting endpoints of two vectors.

For vector $\vec{v} = (x, y, z)$ its Euclidean norm is defined as $\sqrt{x^2 + y^2 + z^2}$. If $\vec{v} \neq \vec{0}$, $\vec{v}/|v|$ always has a unit norm (i.e. 1).

Scalar product $(\vec{u}, \vec{v})$ has the following geometrical meaning: if $u \neq \vec{0}$, $(\vec{u}, \vec{v}) = \operatorname{proj}_{\vec{u}} \vec{v} \times |u|$ where $\operatorname{proj}_{\vec{u}} \vec{v}$ is the signed projection length of vector $\vec{v}$ on the vector $\vec{u}$. In particular, when $\vec{u}$ has unit norm, $(\vec{u}, \vec{v})$ is a length of a component of vector $\vec{v}$ that is parallel to $\vec{u}$.

The geometrical meaning of $\operatorname{sgn}(\vec{u}, \vec{v})$ for non-zero vectors: if it is zero, $\vec{u}$ and $\vec{v}$ are orthogonal, if it is positive, the angle between them is acute, otherwise it is obtuse. If one of vectors is a zero-vector, scalar product is always zero.

The following equation is held:

$$(\vec{u}, \vec{v}) = |v||u| \cos \angle(\vec{u}, \vec{v})$$

2 Vector and mixed products

A vector product of $\vec{v_1}$ and $\vec{v_2}$ is

$$[\vec{v_1}, \vec{v_2}] = \begin{vmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = (y_1z_2 - y_2z_1, z_1x_2 - z_2x_1, x_1y_2 - x_2y_1)$$

Note that in 3d case the vector product is a vector!

$|[\vec{v_1}, \vec{v_2}]|$ is an area of a parallelogram built by vector $\vec{v_1}$ and $\vec{v_2}$. The following equation is held:

$$|[\vec{v_1}, \vec{v_2}]| = |v||u| \sin \angle(\vec{u}, \vec{v})|$$

Vector product is always orthogonal to both $\vec{v_1}$ and $\vec{v_2}$ (it may be proven by taking scalar product with any of $\vec{v_1}$ and $\vec{v_2}$).

A mixed product $(\vec{v_1}, \vec{v_2}, \vec{v_3})$ is $(\vec{v_1}, [\vec{v_2}, \vec{v_3}])$. Also, by definition:

$$(\vec{v_1}, \vec{v_2}, \vec{v_3}) = (x_1, y_1, z_1) \cdot \begin{vmatrix} \vec{e_x} & \vec{e_y} & \vec{e_z} \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Geometric meaning of a mixed product is a signed volume of a parallelepiped built by v_1, v_2, v_3 as a edges. A tetrahedron volume is 6 times smaller than the parallelepiped volume.

The $\operatorname{sgn}(\vec{u}, \vec{v}, \vec{w})$ has several geometric meanings:

- if $\vec{u}$ is a fixed vector, then $\operatorname{sgn}(\vec{u}, \vec{v}, \vec{w})$ is zero if $\vec{v}, \vec{w}$ and $\vec{u}$ lie in the same plane, is positive if $\vec{w}$ is counter-clockwise from $\vec{v}$ when watching straight at the end of $\vec{u}$ (in direction of $-\vec{u}$), and negative otherwise;

- $\operatorname{sgn}(\vec{u}, \vec{v}, \vec{w})$ depends on which half-space w belongs in respect to the plane containing $\vec{u}$ and $\vec{v}$.

Let $\operatorname{vol}(A, B, C, D) = \operatorname{vol} ABCD = (\vec{AB}, \vec{AC}, \vec{AD})$.

An angle between vectors $\vec{u}$ and $\vec{v}$ may be found using the formula:

$$\angle(\vec{u}, \vec{v}) = atan2(|[\vec{u}, \vec{v}]|, (\vec{u}, \vec{v}))$$

Note that, contrary to 2d case, this angle is always between 0 and π and may not be treated as a signed angle (in particular, there is no such notion as clockwise or counter-clockwise when we talk about arbitrary angle in the plane).

Exercise: given a line AB and n points $p_1, \ldots, p_n$ that belong to one half-space not containing the line AB, sort these points by their angle around the line AB (use the sign of mixed product as a comparison predicate, its similar to using vector product sign in 2d case).

3 Planes

The plane equation: $Ax + By + Cz = D$ where at least one of A, B and C is not zero. This is equivalent to the following: $(A, B, C) \cdot (x, y, z) = D$ that shows that (A, B, C) is a *normal* vector to the plane (i.e. vector orthogonal to the plane). Indeed, for any two points p_1 and p_2 belonging to the plane $(A, B, C)(\vec{p_1} - \vec{p_2}) = D - D = 0$.

The *signed distance* from the point (x_0, y_0, z_0) to the plane $Ax+By+Cz = D$ is $\frac{Ax_0+By_0+Cz_0-D}{\sqrt{A^2+B^2+C^2}}$; it is zero if (x_0, y_0, z_0) belongs to the plane, it is positive if (x_0, y_0, z_0) lies in the half-space in the direction of vector (A, B, C), and it is negative otherwise.

Plane may also be defined by three non-collinear points on it. If A, B and C belong to the plane, then normal vector may be restored as $\vec{n} = [\vec{AB}, \vec{AC}]$.

4 Lines, line division rule

The easiest way to define the line in 3d space is the choose two points on it.

The barycentric line division rule: if X belongs to the line AB, then $\vec{X} = \frac{\vec{AX}}{\vec{AB}}\vec{B} + \frac{\vec{XB}}{\vec{AB}}\vec{A}$. Here $\frac{\vec{u}}{\vec{v}}$ is the proportion between two vectors belonging to the same line: it is $\frac{|u|}{|v|}$ if they have the same direction and $-\frac{|u|}{|v|}$ otherwise. Note that $\frac{\vec{AX}}{\vec{AB}} + \frac{\vec{XB}}{\vec{AB}} = \frac{\vec{AX}+\vec{XB}}{\vec{AB}} = \frac{\vec{AB}}{\vec{AB}} = 1$.

Equivalent statement: if λ is a real number, then $\lambda\vec{B} + (1-\lambda)\vec{A} = \vec{A} + \lambda(\vec{B} - \vec{A})$ is always a point of the line AB that divides segment AB in proportion of $\lambda : (1-\lambda)$.

Barycentric line division rule is a useful thing. We will illustrate how to apply it to intersect line with a plane.

Suppose you have a line AB and plane CDE. Then their intersection point O may be found using the following formula:

$$O = \frac{\operatorname{vol} CDEB \cdot \vec{A} + \operatorname{vol} ACDE \cdot \vec{B}}{\operatorname{vol} CDEB + \operatorname{vol} ACDE}$$

This is true because $\operatorname{vol} CDEB : \operatorname{vol} ACDE$ is equal to the ratio of signed distances between B and CDE and between A and CDE, which is equal to the ratio in which plane CDE divides the segment (line) AB.

If the plane is defined by a normal vector $\vec{n}$ (and the equation $(\vec{n}, (x, y, z)) = D$), then the intersection of a line AB and plane may also be found by using a barycentric rule:

$$O = \frac{((\vec{n}, \vec{B}) - D)\vec{A} + (D - (\vec{n}, \vec{A}))\vec{B}}{(\vec{n}, \vec{AB})}$$

This is true because $(\vec{n}, \vec{B}) - D$ is also proportional to the signed distance between B and CDE (with a coefficient of $|\vec{n}|$) and similar for the second vector.

Exercise: write down similar formula for the 2d case and the line intersection.

Now we will discuss the following problem: you are given two lines in 3d space, find the distance between them. The full solution is now missing in this text, but it may eventually get here.

5 Polytopes

Polytope may be define by specifying its faces. Each face is a polygone located in some plane. Vertices are connected with edges, each edge is adjacent to exactly two faces.

Euler Formula: for a polytope that has at least three non-collinear points, $V - E + F = 2$ where V is the number of vertices, E is the number of edges and F is the number of faces.

Lemma: $E \le 3V - 6$. Proof is left as an exercise.

These facts are enough to show that for any two numbers x and y among V, E, F one may bound $x = \mathcal{O}(y)$, for example, number of edges and number of faces are both linear in terms of number of vertices.

In order to calculate the volume of a polytope, one should express it as a sum of several pyramids, some of which will be included with a negative sign (as well as in triangle formula for calculating polygon area in 2d case). Suppose that all faces are oriented counter-clockwise. Then $\operatorname{vol} P = \sum\limits_{\text{face } f} \operatorname{vol}(O, f)$ where $\operatorname{vol}(O, f)$ is the area of a pyramid with its vertex at O and its base in f. Volume of pyramid may also be found as a sum of volumes of several tetrahedrons forming the pyramid. All volumes are considered to be signed.

If not all faces are oriented counter-clockwise, you may orient each face f such that the signed volume $\mathrm{vol}(p_i, f)$ is positive where p_i is any vertex of the polygon non-belonging to f.

Polytope may also be defined as a convex hull of a given set. A naive approach to find all faces of a polytope defined as a convex hull is to try each of the possible $\Theta(n^3)$ triples of points, find a plane passing through them and then check that all other points lie by one side from it, resulting in a $\Theta(n^4)$ algorithm.

More efficient approach is to perform a gift wrapping: suppose you have an edge v_iv_j of a convex hull. Sort all points around the line v_iv_j, then find two outermost points p_a and p_b in this order. They yield two faces $v_iv_jv_a$ and $v_iv_jv_b$ and four new edges. Continue conisdering edges in a DFS or BFS like procedure. Such an algorithm works in $\mathcal{O}(n^2)$.

There are much more complicated algorithms for finding convex hull in $\mathcal{O}(n \log n)$ or even $\mathcal{O}(n \log h)$ where h is the number of points in a convex hull, but we will not discuss them.

6 Geometry of masses

Center of mass of a set of a finite set of points p_i associated with the masses w_i may be found using the formula $(\sum\limits_i p_i w_i)/(\sum\limits_i w_i)$. Center of mass of a solid tetrahedron $ABCD$ is located in a point $(A+B+C+D)/4$ (as well as if we had only 4 points A, B, C, D).

When dealing with points with masses, you may replace a system of points (or a solid body like tetrahedron) with their center of mass, associating a total mass of replaced points with it. For example, you may replace a tetrahedron $ABCD$ with a point $(A+B+C+D)/4$ of mass $\mathrm{vol}\, ABCD$.

7 Spherical geometry

The main idea: keep spehere radius $r = 1$ and use Euclidean coordinate system, that makes everything simpler.

Suppose $\vec{v} = (x, y, z)$, $|\vec{v}| = 1$. The spherical coordinates are: latitude ψ and longitude ϕ.

Spherical coordinates $\to$ Euclidean coordinates:

$$z = \sin\psi$$

$$x = \cos\psi\cos\phi$$

$$y = \cos\psi\sin\phi$$

Euclidean coordinates $\to$ spherical coordinates:

$$\psi = \operatorname{atan2}(z, \sqrt{x^2+y^2})$$

$$\phi = \operatorname{atan2}(y, x)$$

Distance on sphere between points with radius-vectors $\vec{u}$ and $\vec{v}$ is $d(\vec{u}, \vec{v}) = \angle(\vec{u}, \vec{v}) = \operatorname{atan2}(|[\vec{u}, \vec{v}]|, \vec{u} \cdot \vec{v})$.

Arc intersection on sphere: if A_1B_1 and A_2B_2 are two arcs on the sphere:

$$\vec{n_1} = [\vec{A_1}, \vec{B_1}], \quad \vec{n_2} = [\vec{A_2}, \vec{B_2}], \quad \vec{X} = [\vec{n_1}, \vec{n_2}]/|\vec{n_1}, \vec{n_2}|$$