Learning Guide Module

Subject Code	Math 3	Mathematics 3
Module Code	7.0	*Inverse Functions*
Lesson Code	7.1.2	*Functions That Have Inverses 2*
Time Limit		30 minutes

TARGET

Time Allocation: 1 minute
Actual Time Allocation: _____ minutes

By the end of this learning guide, the students will have been able to:

1. Find the inverse of a one-to-one function;
2. Find the inverse of functions with domain restrictions; and
3. Identify the properties of inverse functions.

HOOK

Time Allocation: 2 minutes
Actual Time Allocation: _____ minutes

Anchor Task:
A girl tossed a stress ball vertically upward from a height of 4 feet according to the equation $y = -16x^2 + 16x + 4$, where y is the height in feet of the stress ball and x is the time in seconds. Determine a function that gives the time of the stress ball on its way down. After how many seconds will the ball hit the ground?

Figure 1. Photo showing a young girl playing a game of catch in the garden.

Note: From FreeImages by mtreasure, https://www.freeimages.com/premium/image-of-girl-playing-catch-in-garden-throwing-catching-ball-824208

Inverse of functions are useful because they allow us to reverse the rule determined by a given function. This enables us to solve a variety of problems like the anchor task above.

We learned from the previous learning guide that not all functions have inverses. One-to-one correspondence guarantees that a function will have an inverse.

The function and its inverse are related such that their domain and range are interchanged. This means that the domain and range of a given function is the range and domain, respectively, of its inverse.

How do we get the equation of the inverse of the function? Is the inverse unique? How do we restrict the domain of some many-to-one functions such that its inverse may exist? What other properties does a function have in relation to its inverse?

IGNITE

Time Allocation: 15 minutes
Actual Time Allocation: _____ minutes

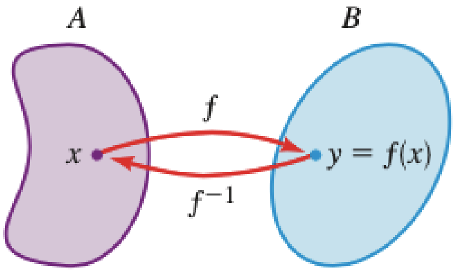

Figure 2. Mapping of f and f^{-1}
Note: From *Algebra and trigonometry* (p. 243) by Stewart, J., Redlin, L., & Watson, S. Copyright 2012 by Brooks/Cole, Cengage Learning.

Figure 2 illustrates the mapping of f and its inverse, f^{-1}. Observe that f^{-1} maps the output back to the input.

How do we determine the equation of the inverse of a function? The idea is to reverse the rule of the function and switch the roles of the input and the output.

Let's try some examples.

Example 1: Find the inverse function of $f(x) = \frac{1}{2}x + \frac{21}{3}$. Give the domain and range of f^{-1}.

Solution:

Since f is a linear function (with nonzero slope), then it is one-to-one. Hence, its inverse is a function.

Firstly, we switch the roles of x and y in $y = \frac{1}{2}x + \frac{21}{3}$ as shown below, then solve for y.

$$x = \frac{1}{2}y + \frac{21}{3} \quad \Rightarrow \quad x - \frac{21}{3} = \frac{1}{2}y \quad \Rightarrow \quad 2\left(x - \frac{21}{3}\right) = y$$

Therefore, the inverse function of f is given by $f^{-1}(x) = 2x - \frac{42}{3}$.

Note that the inverse of a linear function is also linear. In Example 1, the domain and the range are the set of real numbers for both the function and its inverse.

Observe that by manipulating the equation of the function to isolate y, we can arrive at its inverse.

Example 2: Give the domain and range of $g(x) = \sqrt[3]{x+5} - 2$ and find its inverse, g^{-1}. Determine the domain and range of the inverse function.

Solution:
We can visualize the graph of g using transformation of functions. The graph of g is the same as the graph of $y = \sqrt[3]{x}$ but shifted 5 units to the left and 2 units down. The graph satisfies the horizontal line test and therefore, is one-to-one. (You may use a graphing software to verify the graph.)

Both the domain and the range of g is the set of all real numbers.

We determine the equation for g^{-1} as follows.

Switch the roles of x and y in $y = \sqrt[3]{x+5} - 2$, then solve for y.

$$y = \sqrt[3]{x+5} - 2 \Rightarrow \qquad x = \sqrt[3]{y+5} - 2 \Rightarrow \qquad x + 2 = \sqrt[3]{y+5}$$

$$\Rightarrow (x+2)^3 = y + 5 \qquad \Rightarrow (x+2)^3 - 5 = y$$

Therefore, $g^{-1}(x) = (x+2)^3 - 5$. Since the inverse reverses the effect of g, then it makes sense that the inverse of a cube root function is a cubic function.

The domain of g^{-1} and its range are both the set of real numbers.

In summary, below shows the steps in writing the equation of the inverse of a function.

HOW TO FIND THE INVERSE OF A ONE-TO-ONE FUNCTION
1. Given a one to one function $f(x) = y$, interchange x and y.
2. Solve the resulting equation for y in terms of x.
3. Write the equation as $f^{-1}(x) = y$.

Inverse Function Property
If f and f^{-1} are inverses of each other, then $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

This property tells us of the cancelation of a function and its inverse. This is similar to additive inverses for addition and multiplicative inverses for multiplication. Figure 3 illustrates what happens during the composition of the function and its inverse, i.e. $f \circ f^{-1}$.

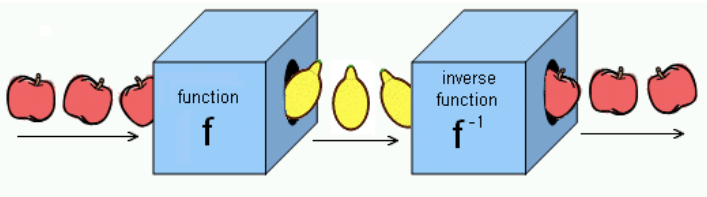

Figure 3. Apples to lemons to illustrate a function and its inverse.
Note: From expii by Pan, J. (n.d.), https://www.expii.com/t/inverse-of-nonconstant-linear-functions-4376

Try this!
Verify that $g(x) = \sqrt[3]{x+5} - 2$ and $g^{-1}(x) = (x+2)^3 - 5$ are inverses of each other using the inverse function property.

Solution:
If the two functions are inverses of each other, then this must hold true: $g(g^{-1}(x)) = g^{-1}(g(x)) = x$.

Using composition of functions, we will show that $(g \circ g^{-1})(x) = (g^{-1} \circ g)(x) = x$.

$$(g(g^{-1}(x)) = \sqrt[3]{[(x+2)^3 - 5] + 5} - 2 = \sqrt[3]{(x+2)^3} - 2 = x + 2 - 2 = x$$

$$g^{-1}(g(x)) = ([\sqrt[3]{x+5} - 2] + 2)^3 - 5 = (\sqrt[3]{x+5})^3 - 5 = x + 5 - 5 = x$$

Hence, by the inverse function property, $y = \sqrt[3]{x+5} - 2$ and $y = (x+2)^3 - 5$ are inverses of each other.

Example 3:
Verify that the inverse of $h(x) = \frac{3x-2}{5x-3}$ is itself. Determine the domain and range of h.

Solution:
The given rational function has asymptotes $x = \frac{3}{5}$ and $y = \frac{3}{5}$. Note that $h(x)$ is a one-to-one function. (You may graph the function manually or verify the graph using a graphing software.) Therefore, its inverse exists.

Applying the property of inverse, we have:

$$(h \circ h)(x) = \frac{3\left(\frac{3x-2}{5x-3}\right)-2}{5\left(\frac{3x-2}{5x-3}\right)-3} = \frac{\frac{9x-6}{5x-3}-2}{\frac{15x-10}{5x-3}-3} = \frac{9x-6-2(5x-3)}{5x-3} \cdot \frac{5x-3}{15x-10-3(5x-3)} = \frac{-x}{-1}$$

We are able to show that $(h \circ h)(x) = x$, hence $h(x)$ is the inverse of itself.

Alternatively, we can also determine the inverse of h using the steps described above and see that we will arrive at the same function, h.

Lastly, the domain and the range of h and h^{-1} will be exactly the same. That is, the domain and range are $\{x | x \in \mathbb{R}, x \neq \frac{3}{5}\}$ and $\{y | y \in \mathbb{R}, y \neq \frac{3}{5}\}$.

In this example, the asymptotes of the function and its inverse remains the same. In the next learning guide, we will learn some properties on the graphs of inverse functions.

The next examples will involve many-to-one functions.

Example 4:
Restrict the domain of $y = x^2$ such that the function will be one-to-one and obtain its corresponding inverse function. Identify the domain and range of the inverse.

Since the graph of $y = x^2$ is a parabola (i.e. many-to-one), we can split its domain into two intervals starting at the vertex. Given that the vertex is at the origin, one interval will be $x > 0$ and the other interval is $x < 0$. (You may also use the inequality $\geq$ or $\leq$.)

Observe that by doing so, $y = x^2$ when $x > 0$ becomes a one-to-one function, in the same way as $y = x^2$ when $x < 0$.

Now we can obtain the inverse of each.

$$y = x^2, x > 0 \Rightarrow x = y^2 \Rightarrow \pm\sqrt{x} = y$$

Since we restricted the domain of the function to $(0, +\infty)$, this becomes the range of its inverse. Hence, we take the positive square root, i.e. $y = \sqrt{x}$.

Likewise, for $y = x^2, x < 0$, the domain is $(-\infty, 0)$ which is also going to be the range of the inverse. Hence, we take the negative square root, i.e. $y = -\sqrt{x}$.

Now we are ready to answer the anchor task posted at the start of this module (see: Hook).

Example 5. Anchor Task
A girl tossed a stress ball vertically upward from a height of 4 feet according to the equation $y = -16x^2 + 16x + 4$, where y is the height in feet of the stress ball and x is the time in seconds. Determine a function that gives the time of the stress ball on its way down. After how many seconds will the ball hit the ground?

Solution:
In Math 2, we know that a trajectory of a ball thrown upward may be represented using a quadratic function. The inverse of the function $y = -16x^2 + 16x + 4$ will give the time it takes for the ball to reach a given height.

The function is not one-to-one, hence, we will have to restrict the domain of the function in order to get its inverse.

We start by writing the function in vertex-form: $y = -16(x - \frac{1}{2})^2 + 8$. Hence, this tells us that we can split the domain into intervals $(-\infty, \frac{1}{2})$ and $(\frac{1}{2}, \infty)$.

We obtain the inverse of the function as follows.

$$y = -16(x - \tfrac{1}{2})^2 + 8 \quad \Rightarrow x - 8 = -16(y - \tfrac{1}{2})^2 \Rightarrow -\frac{x-8}{16} = (y - \tfrac{1}{2})^2$$

$$\Rightarrow \pm\sqrt{\frac{8-x}{16}} = y - \frac{1}{2} \Rightarrow y = \frac{1}{2} \pm \frac{\sqrt{8-x}}{4}$$

Observe that due to the switching of roles, for $y = \frac{1}{2} \pm \frac{\sqrt{8-x}}{4}$ the input x is now the height of the ball and output y is the time it takes to reach that height.

Which of the two represents the time the ball is *on its way down*?

Based on $y = -16(x - \frac{1}{2})^2 + 8$, the maximum height of the ball is 8 feet and the time the ball will reach this height is after 0.5 seconds the ball was thrown upward. Hence, the time *on its way up* is represented by the interval $[0, \frac{1}{2})$ while the time it is *on its way down* is $\left(\frac{1}{2}, \infty\right)$.

Since the domain and range of the function and its inverse will switch, the range for the inverse (the time the ball is on its way down) will be the interval $\left(\frac{1}{2}, \infty\right)$ and this is achieved when $y = \frac{1}{2} + \frac{\sqrt{8-x}}{4}$ (i.e. $y > \frac{1}{2}$ when $x < 8$). The ball will hit the ground (i.e. when $x = 0$) after 1.21 seconds.

REMARK:
The range of $y = -16(x - \frac{1}{2})^2 + 8$ is $(-\infty, 8)$. This is also the domain of $y = \frac{1}{2} + \frac{\sqrt{8-x}}{4}$.

NAVIGATE

Time Allocation: 10 minutes
Actual Time Allocation: _____ minutes

Check your understanding by answering the items below. Do as indicated. Items marked with an asterisk (*) will be graded.

A. Complete the table below with the missing information. Use knowledge on transformation of functions or a graphing software to verify that the functions are one-to-one.

Given	Domain of f	Range of f	Inverse of f, i.e. f^{-1}	Domain of f^{-1}	Range of f^{-1}
1. $f(x) = \sqrt{9-x}$					
*2. $f(x) = (x-1)^5$					
3. $f(x) = \frac{2x-1}{x+5}$					

B. Tell whether the given pairs of functions are inverses of each other.

1. $y = 100x + 5$ and $y = \frac{x-5}{100}$

* 2. $y = \frac{x}{x+4}$ and $y = \frac{x-4}{x}$

3. $y = \sqrt[3]{2x-1}$ and $y = (2x-1)^3$

* C. Restrict the domain of $f(x) = |x-2| - 7$ and determine its inverse.

* D. Give a function (other than the identity function y=x) that is the inverse of itself.

KNOT

Time Allocation: 2 minutes
Actual Time Allocation: _____ minutes

Some points to remember:

1. To find the inverse of a one-to-one function, we perform the following steps:
   a. Given a one-to-one function $f(x) = y$, interchange *x* and *y*.
   b. Solve the resulting equation for *y* in terms of *x*.
   c. Write the equation as $f^{-1}(x) = y$.
2. If f and f^{-1} are inverses of each other, then $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.
3. The domain of a many-to-one function may be restricted such that it becomes one-to-one in that interval, and thus obtain its inverse.

References:

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Answer Key:

A.

Given	Domain of f	Range of f	Inverse of f, i.e. f^{-1}	Domain of f^{-1}	Range of f^{-1}
1. $f(x) = \sqrt{9-x}$	$(-\infty, 9)$	$[0, +\infty)$	$f^{-1}(x) = -x^2 + 9$	$[0, +\infty)$	$(-\infty, 9)$
3. $f(x) = \frac{2x-1}{x+5}$	$(-\infty, -5) \cup (-5, +\infty)$	$(-\infty, 2) \cup (2, +\infty)$	$f^{-1}(x) = \frac{5x+1}{2-x}$	$(-\infty, 2) \cup (2, +\infty)$	$(-\infty, -5) \cup (-5, +\infty)$

B. (1) and (3) are not inverses of each other.