Learning Guide Module

Subject Code	Math 3	Mathematics 3
Module Code	7.0	*Inverse Functions*
Lesson Code	7.1.1	*Functions That Have Inverses 1*
Time Limit		30 minutes

TARGET

Time Allocation: 1 minute
Actual Time Allocation: _____ minutes

By the end of this learning guide, the students will have been able to:

1. Determine the relationship of a function and its inverse;
2. Determine whether a function is one-to-one; and
3. Define the inverse of a function.

HOOK

Time Allocation: 2 minutes
Actual Time Allocation: _____ minutes

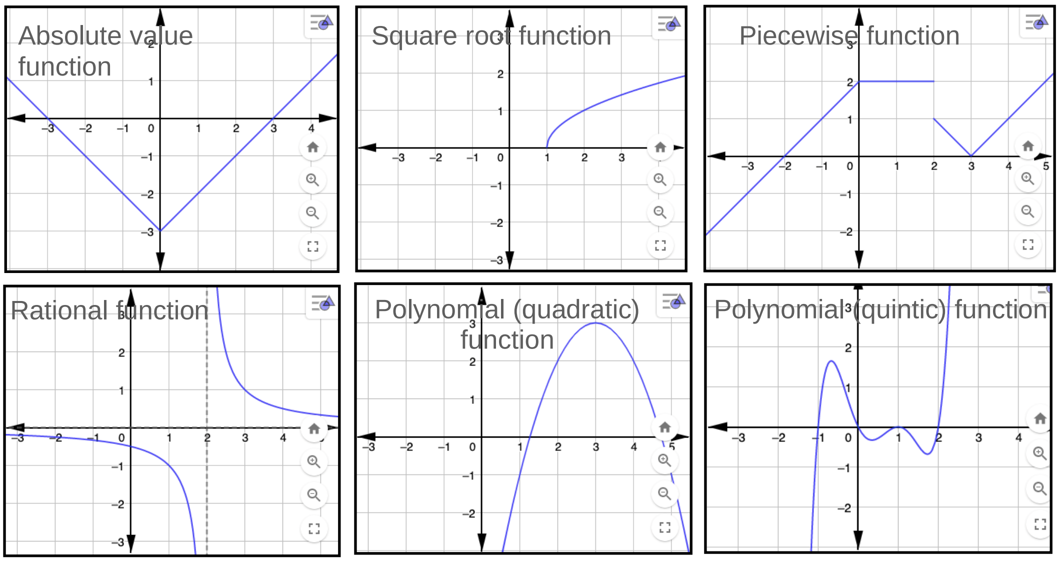


Figure 1. Graphs of different types of functions

In the previous learning guides, we were introduced to different types of functions. We learned about the properties of polynomial functions, rational functions, square-root functions, absolute-value functions, and piecewise-defined functions. Figure 1 shows the graphs of different functions.

We know that a function is a *rule* that assigns every element in the domain to exactly one element in the range. This relationship forms an *ordered pair*. The element from the domain is the *input* of the function while its corresponding element in the range is the *output*. When the rule is represented by an equation, we can plug the input into the equation in order to get the output. There are instances however when the output is given and we want to know the input. What strategies can we use in this case?

This is similar to *working backwards.* It is a problem-solving strategy where you start with the final step and work your way back to the starting point. In this strategy, we try to *undo* or *reverse* the process. For instance, if the operation is addition, we perform subtraction. If it is multiplication, we perform division. If it's squaring a number, we take the square root, and so on.

This is how inverse of functions work.

What are inverses of functions? Do all functions have inverses? If a function has an inverse, is the inverse a function too? If an inverse exists, is the inverse of a function unique?

IGNITE

Time Allocation: 15 minutes
Actual Time Allocation: _____ minutes

Consider the square-root function described by the equation: $y = \sqrt{2x-7}$.

What is the value of y when $x = 5$? By substitution of $x = 5$, we get $y = \sqrt{3}$. (Note that we only take the positive root, i.e. the *principal square root.*)

What is the value of x when $y = 10$?

The given function tells us that the rule to get the output y is to multiply the input x by 2, subtract 7, then take the square root of the result. Doing this process in reverse, we will square the value of y (in this case, 10), add 7, then divide the result by 2. Hence, we get $x = \frac{107}{2}$.

What we actually did is the process that takes place when we use the inverse of a given function. The inverse of a function enables us to solve for the input given the output.

For notations, we use f, g, h, etc. for functions and we use f^{-1}, g^{-1}, h^{-1}, etc. to denote their inverses, respectively.

CAUTION:

1. The -1 does not act as an exponent. f^{-1} (read as "f *inverse*" or "*inverse of* f") is not the same as $\frac{1}{f}$.
2. If you have some familiarity with Calculus, f' is NOT the same as f^{-1}. The first refers to the first derivative of f (read as "f prime"), while the second refers to the inverse of f.

Do all functions have inverses? Let's take a look at the next examples.

Example 1: A function f is defined by the set of ordered pairs $\{(2,3),(3,5),(4,7),(5,9),(6,11)\}$. What is f^{-1}?

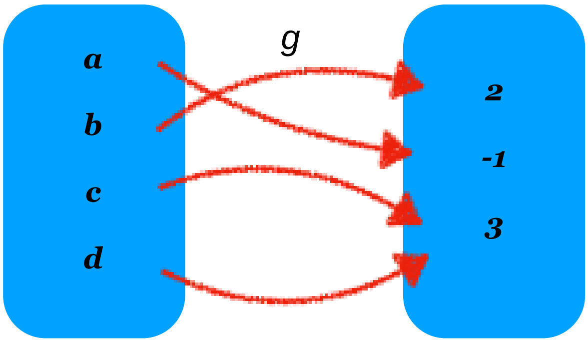
For inverse of functions, there will be a switching of roles. The output of the function becomes the input of the inverse, and the input of the function becomes the output of the inverse. Therefore, we have f^{-1} described by the set of ordered pairs below.

$$\{(3,2),(5,3),(7,4),(9,5),(11,6)\}$$

NOTE:
The ordered pair (x, y) of f will become (y, x) for its inverse. As a result, the domain of f becomes the range of f^{-1} and the range of f becomes the domain of f^{-1}.

In Example 1, the domain of f is $\{2, 3, 4, 5, 6\}$ and its range is $\{3, 5, 7, 9, 11\}$. The domain of f^{-1} is $\{3, 5, 7, 9, 11\}$ and its range is $\{2, 3, 4, 5, 6\}$.

Example 2:
Now consider the function g described by the mapping below and determine g^{-1}.



The function g has the ordered pairs (a, -1), (b, 2), (c, 3), and (d, 3).

Observe that g has a *many-to-one* correspondence. Its inverse, g^{-1}, will have the ordered pairs (-1, a), (2, b), (3, c), and (3, d).

Notice that this becomes problematic. Due to the switching of roles of x and y, the mapping now becomes a *one-to-many* correspondence, which is NOT a function. (Recall: Definition of a function)

If the inverse is not a function, then we say the inverse does not exist. So, when does a function have an inverse? What type of correspondence or mapping will guarantee that the inverse of a function is a function too?

The answer is, it must be a *one-to-one* correspondence or mapping.

REMARK:
There are cases though when we can restrict the domain of the function such that its resulting inverse will still be a function. This will be discussed further in the next learning guide.

How do we know if a function is one-to-one?

If we know the graph of the function, we can use the *Horizontal Line Test*.

The Horizontal Line Test
On the Cartesian Plane, if no horizontal line intersects the graph of the function more than once, then we say that the function is one-to-one.

Try this!
Using the horizontal line test, identify which of the following is/are one-to-one. (Note: You may use graphing software (GeoGebra or Desmos) for functions whose graphs are difficult to visualize.)

1. $f(x) = 5x + 18$
2. $g(x) = (x+4)^2 + 1$
3. $p(x) = x^3 - 5$
4. $q(x) = 2|x+3| - 4$
5. $r(x) = \frac{x-2}{x+1}$

Answer:
The functions $f, p,$ and r are all one-to-one functions.

Observe that the function f is a linear function. In Math 2, we learned that the graph of a linear function is a straight line. Since the slope of f is 5, the graph is a slanted line. It satisfies the horizontal line test; therefore, it is one-to-one.

The function p is a cubic function. Using transformation of functions, we can tell that the graph of p is the same as the graph of $y = x^3$ but translated 5 units down. The graph satisfies the horizontal line test; hence, it is also one-to-one.

The rational function r is one-to-one. Its graph is shown in Figure 2.

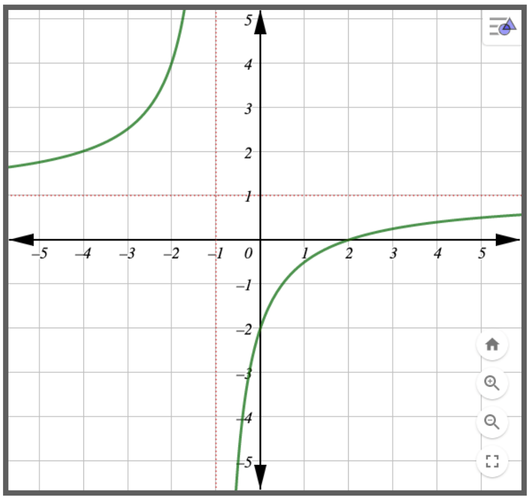


Figure 2: Graph of rational function $r(x)$ and its asymptotes $x = -1$ and $y = 1$

On the other hand, the graph of function g, which is a quadratic function, is a parabola. Its graph does not satisfy the horizontal line test. Hence, it is a many-to-one function.

Likewise, absolute value functions such as q has a V-shaped graph. This type of graph fails the horizontal line test. Therefore, this function is also not one-to-one.

Alternatively, we can show algebraically if a function is one-to-one using its definition.

> ***Definition of One-to-One Functions***
> A function f is one-to-one if for any x_1 and x_2 in the domain of f and $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

Equivalently, it can be stated as

$$\text{If } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

Example 3:
Show by definition that $p(x) = x^3 - 5$ and $r(x) = \dfrac{x-2}{x+1}$ are one-to-one functions.

Solution:
Let's start with $p(x) = x^3 - 5$. Let x_1 and x_2 be in the domain of p. (Hint: The plan is to assume that the outputs $p(x_1) = p(x_2)$, then we will show that it should follow that $x_1 = x_2$.)

$$p(x_1) = p(x_2)$$
$$(x_1)^3 - 5 = (x_2)^3 - 5$$
$$(x_1)^3 - 5 + 5 = (x_2)^3 - 5 + 5$$
$$(x_1)^3 = (x_2)^3$$
$$\sqrt[3]{(x_1)^3} = \sqrt[3]{(x_2)^3}$$
$$\therefore x_1 = x_2$$

Now, we use the same technique to show that $r(x) = \frac{x-2}{x+1}$ is also one-to-one. Let x_1 and x_2 be in the domain of r. (Note that -1 is not in the domain of r.)

$$r(x_1) = r(x_2)$$
$$\frac{x_1-2}{x_1+1} = \frac{x_2-2}{x_2+1} \qquad \text{(Since } x_1, x_2 \neq -1\text{, we can cross multiply.)}$$
$$(x_1 - 2)(x_2 + 1) = (x_2 - 2)(x_1 + 1) \qquad \text{(Use the FOIL Method.)}$$
$$x_1x_2 + x_1 - 2x_2 - 2 = x_2x_1 + x_2 - 2x_1 - 2 \qquad \text{(Cancel some terms to simplify the equation.)}$$
$$x_1 - 2x_2 = x_2 - 2x_1$$
$$x_1 + 2x_1 = x_2 + 2x_2 \qquad \text{(Simplify.)}$$
$$3x_1 = 3x_2$$
$$\therefore x_1 = x_2$$

Example 4:
Show that $g(x) = (x + 4)^2 + 1$ is NOT one-to-one.

Solution:
Let us use the same method in Example 3. We let x_1 and x_2 be in the domain of g and suppose that $g(x_1) = g(x_2)$. Can we show that $x_1 = x_2$?

$$g(x_1) = g(x_2)$$
$$(x_1 + 4)^2 + 1 = (x_2 + 4)^2 + 1 \qquad \text{(Subtract 1 from both sides.)}$$
$$(x_1 + 4)^2 = (x_2 + 4)^2 \qquad (*)$$

CAUTION: Can we take the square root of both sides? Why or why not? Remember that a square has a positive square root and a negative square root. Hence, from equation (*), we can have

$$|x_1 + 4| = |x_2 + 4| \qquad (**)$$

With our knowledge of absolute value of numbers, we can tell that x_1 and x_2 are NOT necessarily equal. For instance, $x_1 = 0$ and $x_2 = -8$. The equation (**) is true but $x_1 \neq x_2$.

Therefore, given that $g(x_1) = g(x_2)$, we cannot conclude that $x_1 = x_2$ for any x_1 and x_2 in the domain of g. Therefore, $g(x)$ is not one-to-one.

Alternative Solution:
We can approach Example 4 by identifying a counter-example. By definition of one-to-one functions, whenever $x_1 \neq x_2$, then $g(x_1) \neq g(x_2)$.

A *counter-example* is an example that satisfies the given conditions but leads you to a contradiction. (HINT: The idea is to find two numbers $(x_1 \neq x_2)$ that make $g(x_1) = g(x_2)$.)

An example would be $x_1 = -5$ and $x_2 = -3$. The corresponding function values will be as follows.

$$g(-5) = (-5+4)^2 + 1 = (-1)^2 + 1 = 2$$
$$g(-3) = (-3+4)^2 + 1 = (1)^2 + 1 = 2$$

Since we are able to show that $g(-5) = g(-3)$ given that $-5 \neq -3$, this contradicts the definition of one-to-one functions. Hence, g is not one-to-one.

The case when $x_1 = 0$ and $x_2 = -8$ in the previous solution is also a counter-example.

Now that we know how to check if a function is one-to-one, we can now define mathematically what the inverse of a function is.

> ***Definition of Inverse of a Function***
> Let f be a one-to-one function. The inverse of a function f, denoted by f^{-1}, is the set of all ordered pairs (y, x) where (x, y) belongs to f.

In the next learning guide, we will learn how to get the equation of the inverse of a function.

NAVIGATE

Time Allocation: 10 minutes
Actual Time Allocation: _____ minutes

Check your understanding by answering the following items. Items marked with an asterisk (*) will be graded.

A. Given the indicated value of x or y, determine the value of the other variable.

1. $y = 3x - 10$; $x = 10$

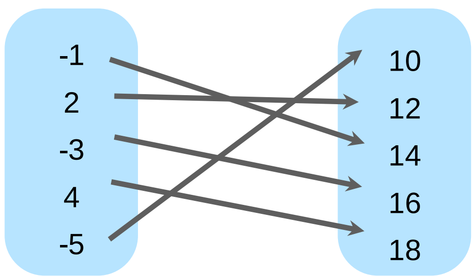
* 2. $y = 5 - 4x^2$; $x = -6$

3. $y = \frac{1+3x}{5-2x}$; $y = 3$

* 4. $y = \sqrt{x-5}$; $y = 0$

B. Which functions are one to one?

1.



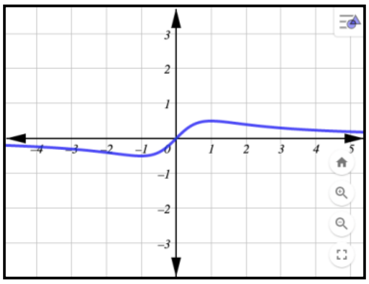
*2. $\{(-2,1),(-1,4),(0,5),(1,4),(2,1)\}$

3. $g(x) = |x+2|, x \geq -2$

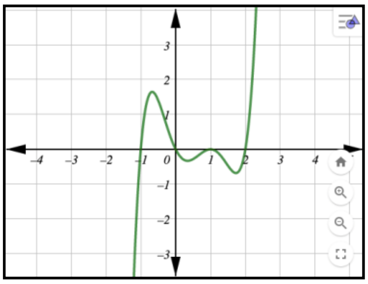
*4. $f(x) = \begin{cases} x, & if\ x \leq 0 \\ \sqrt{x}, & if\ x > 0 \end{cases}$

5. $g(x) = \dfrac{|x|}{x} + 6$

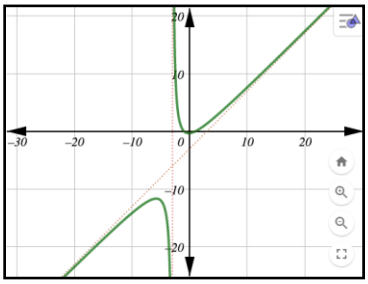
C. Which of the graphs of functions below are one-to-one?



(1)



(2)



(3)

D. Show algebraically that $f(x) = \sqrt[3]{x+9}$ is a one-to-one function. *

E. Identify which of the following statements is/are always true (A), sometimes true (S), or never true (N).

1. If a function is a rational function, then it is one-to-one.
2. The constant function, $y = c$, where $c \in \mathbb{R}$, has an inverse. *
3. A polynomial function in the form $f(x) = ax^n$, where $a \neq 0$ and n is odd, is one-to-one.

KNOT

Time Allocation: 2 minutes
Actual Time Allocation: _____ minutes

Some points to remember:

1. If a function has a one-to-one correspondence, then its inverse is a function too.
2. The horizontal line test can be used to identify if a function is one-to-one.
3. A function f is one-to-one if for any x_1 and x_2 in the domain of f, whenever $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$. Equivalently, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.
4. All linear functions (with nonzero slope) are one-to-one. Quadratic functions and absolute value functions are many-to-one.
5. Some rational functions are one-to-one. The same can be said about polynomial functions.
6. The inverse of a one-to-one function f, denoted by f^{-1}, is the set of all ordered pairs (y, x) where (x, y) belongs to f.

References:

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Stewart, J., Redlin, L., & Watson, S. (2012). *Algebra and trigonometry* (3rd ed.). Brooks/Cole, Cengage Learning. http://en.bookfi.net/book/1373416

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Answer Key:

A. 1. 20

3. $\frac{14}{9}$

B. 1. One-to-one; 3. One-to-one; 5. Not one-to-one

C. None

E. 1. S; 3. A