iReduct: Differential Privacy with Reduced Relative Errors

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ABSTRACT

Prior work in differential privacy has produced techniques for answering aggregate queries over sensitive data in a privacy-preserving way. These techniques achieve privacy by adding noise to the query answers. Their objective is typically to minimize *absolute* errors while satisfying differential privacy. Thus, query answers are injected with noise whose scale is *independent of whether the answers are large or small*. The noisy results for queries whose true answers are small therefore tend to be dominated by noise, which leads to inferior data utility.

This paper introduces *iReduct*, a differentially private algorithm for computing answers with reduced *relative* errors. The basic idea of *iReduct* is to inject different amounts of noise to different query results, so that smaller (larger) values are more likely to be injected with less (more) noise. The algorithm is based on a novel resampling technique that employs *correlated* noise to improve data utility. Performance is evaluated on an instantiation of *iReduct* that generates *marginals*, i.e., projections of multi-dimensional histograms onto subsets of their attributes. Experiments on real data demonstrate the effectiveness of our solution.

Categories and Subject Descriptors

H.2.0 [**DATABASE MANAGEMENT**]: Security, integrity, and protection

General Terms

Algorithms, Security

Keywords

Privacy, Differential Privacy

1. INTRODUCTION

Disseminating aggregate statistics of private data has much benefit to the public. For example, census bureaus already publish aggregate information about census data to improve decisions about the allocation of funding; hospitals could release aggregate information about their patients for medical research. However, since those datasets contain private information, it is important to ensure that the published statistics do not leak sensitive information about any individual who contributed data. Methods for protecting against such information leakage have been the subject of research for decades.

The current state-of-the-art paradigm for privacy-preserving data publishing is *differential privacy*. Differential privacy requires that the aggregate statistics reported by a data publisher should be perturbed by a randomized algorithm $\mathcal{G}$, so that the output of $\mathcal{G}$ remains roughly the same even if any single tuple in the input data is arbitrarily modified. This ensures that given the output of $\mathcal{G}$, an adversary will not be able to infer much about any single tuple in the input, and thus privacy is protected.

In this paper, we consider the setting in which the goal is to publish answers to a batch set of queries, each of which maps the input dataset to a real number [4,8,16,17,21,23,32]. The pioneering work by Dwork et al. [8] showed that it is possible to achieve differential privacy by adding i.i.d. random noise to each query answer if the noise is sampled from a Laplace distribution where the scale of the noise is appropriately calibrated to the query set. In recent years, there has been much research on developing new differentially private methods with improved accuracy [4,16,17,21,32]. Such work has generally focused on reducing the *absolute* error of the queries, and thus the amount of noise injected into a query is *independent of the actual magnitude of the query answer*.

In many applications, however, the error that can be tolerated is *relative* to the magnitude of the query answer: larger answers can tolerate more noise. For example, suppose that a hospital performs queries q_1 and q_2 on a patient record database in order to obtain counts of two different but potentially overlapping medical conditions. Suppose the first condition is relatively rare, so that the exact answer for q_1 is 10, while the second is common, so that the exact answer for q_2 is 10000. In that case, the noisy answer to q_1 might be dominated by noise, even though the result of q_2 would be quite accurate. Other applications where *relative* errors are important include learning classification models [12], selectivity estimation [13], and approximate query processing [29].

Our Contributions. This paper introduces several techniques that automatically adjust the scale of the noise to reduce *relative* errors while still ensuring differential privacy. Our main contribution is the *iReduct* algorithm (Section 4). In a nutshell, *iReduct* initially obtains very rough estimates of the query answers and subsequently uses this information to iteratively refine its estimates with the goal of minimizing relative errors. Ordinarily, iterative resampling would incur a considerable privacy "cost" because each noisy answer to the same query leaks additional information. We avoid this cost by using an innovative resampling function we call *NoiseDown*. The key to *NoiseDown* is that it resamples conditioned on

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previous samples, generating a sequence of correlated estimates of successively reduced noise magnitude. We prove that the privacy cost of the sequence is equivalent to the cost of sampling a *single* Laplace random variable having the same noise magnitude as the last estimate in the sequence produced by *NoiseDown*.

To demonstrate the application of *iReduct*, we present an instantiation of *iReduct* for generating *marginals*, i.e., the projections of a multi-dimensional histogram onto various subsets of its attributes (Section 5). With extensive experiments on real data (Section 6), we show that *iReduct* has lower relative errors than existing solutions and other baseline techniques introduced in this paper. In addition, we demonstrate the practical utility of optimizing for relative errors, showing that it yields more accurate machine learning classifiers.

2. PRELIMINARIES

2.1 Problem Formulation

Let T be a dataset, and $Q = [q_1, \ldots, q_m]$ be a sequence of m queries on T, each of which maps T to a real number. We aim to publish the results of all queries in Q on T using an algorithm that satisfies ϵ*-differential privacy*, a notion of privacy defined based on the concept of *neighboring datasets*.

DEFINITION 1 (NEIGHBORING DATASETS). *Two datasets T_1 and T_2 are* ***neighboring datasets*** *if they have the same cardinality but differ in one tuple.* □

DEFINITION 2 (ϵ-DIFFERENTIAL PRIVACY [9]). *A randomized algorithm $\mathcal{G}$ satisfies* ϵ***-differential privacy***, *if for any output O of $\mathcal{G}$ and any neighboring datasets T_1 and T_2,*

$$Pr\left[\mathcal{G}(T_1) = O\right] \leq e^{\epsilon} \cdot Pr\left[\mathcal{G}(T_2) = O\right]. \;\square$$

We measure the quality of the published query results by their *relative errors*. In particular, the relative error of a published numerical answer r^* with respect to the original answer r is defined as

$$err(r^*) = \frac{|r^* - r|}{\max\{r, \delta\}}, \tag{1}$$

where δ is a user-specified constant (referred to as the *sanity bound*) used to mitigate the effect of excessively small query results. This definition of relative error follows from previous work [13,29]. For ease of exposition, we assume that all queries in Q have the same sanity bound, but our techniques can be easily extended to the case when the sanity bound varies from query to query. Table 1 summarizes the notations that will be frequently used in this paper.

2.2 Differential Privacy via Laplace Noise

Dwork et al. [9] show that ϵ-differential privacy can be achieved by adding i.i.d. noise to the result of each query in Q. Specifically, the noise is sampled from the *Laplace distribution* which has the following probability density function (p.d.f.)

$$Pr[\eta = x] = \frac{1}{2\lambda} e^{-|x-\mu|/\lambda}, \tag{2}$$

where μ is the mean of the distribution (we assume $\mu = 0$ unless stated otherwise) and λ (referred to as the *noise scale*) is a parameter that controls the degree of privacy protection. A random variable that follows the Laplace distribution has a standard deviation of $\sqrt{2} \cdot \lambda$ and an expected absolute deviation of λ (ie. $E|\eta - \mu| = \lambda$).

Symbol	Description
$q(T)$	the result of a query q on a dataset T
Q	a given sequence of m queries $[q_1, \ldots, q_m]$
Λ	a sequence of m noise scales $[\lambda_1, \ldots, \lambda_m]$
δ	the sanity bound for the queries in Q
$S(Q)$	the sensitivity of Q
$GS(Q, \Lambda)$	the generalized sensitivity of Q with respect to Λ
x	an arbitrary real number
y_i	a random variable such that $y_i - x$ follows a Laplace distribution
ξ	$\xi = \min\{x, y-1\}$
f	the conditional probability density function of y' given x, y, λ, and λ', as defined in Equation 6

Table 1: Frequently Used Notations

The resulting privacy depends both on the scale of the noise and the *sensitivity* of Q. Informally, sensitivity measures the maximum change in query answers due to changing a single tuple in the database.

DEFINITION 3 (SENSITIVITY [9]). *The* ***sensitivity*** *of a sequence Q of queries is defined as*

$$S(Q) = \max_{T_1, T_2} \sum_{i=1}^{m} |q_i(T_1) - q_i(T_2)| \tag{3}$$

where T_1 and T_2 are any neighboring datasets. □

Dwork et al. prove that adding Laplace noise of scale λ leads to $(S(Q)/\lambda)$-differential privacy.

PROPOSITION 1 (PRIVACY FROM LAPLACE NOISE [9]). *Let Q be a sequence of queries and $\mathcal{G}$ be an algorithm that adds i.i.d. noise to the result of each query in Q, such that the noise follows a Laplace distribution of scale λ. Then, $\mathcal{G}$ satisfies $(S(Q)/\lambda)$-differential privacy.* □

3. FIRST-CUT SOLUTIONS

Dwork et al.'s method, described above, may have high relative errors because it adds noise of fixed scale to every query answer *regardless of whether the answer is large or small*. Thus, queries with small answers have much higher expected relative errors than queries with large answers. In this paper, we propose several techniques that calibrate the noise scale to reduce relative errors. This section describes two approaches. The first is a simple idea that achieves uniform expected relative errors but fails to satisfy differential privacy, while the second is a first attempt at a differentially private solution.

Before presenting the strategies, we introduce an extension to Dwork et al.'s method that adds unequal noise to query answers (adapted from [32]). Let $\Lambda = [\lambda_1, \ldots, \lambda_m]$ be a sequence of positive real numbers such that q_i will be answered with noise scale λ_i for $i \in [1, m]$. We extend the notion of sensitivity to a sequence of queries Q and a corresponding sequence of noise scales Λ.

DEFINITION 4 (GENERALIZED SENSITIVITY [32]). *Let Q be a sequence of m queries $[q_1, \ldots, q_m]$, and Λ be a sequence of m positive constants $[\lambda_1, \ldots, \lambda_m]$. The* ***generalized sensitivity*** *of Q with respect to Λ is defined as*

$$GS(Q, \Lambda) = \max_{T_1, T_2} \sum_{i=1}^{m} \left(\frac{1}{\lambda_i} \cdot |q_i(T_1) - q_i(T_2)| \right), \tag{4}$$

where T_1 and T_2 are any neighboring datasets. □

The following proposition states that adding Laplace noise with unequal scale leads to $GS(Q, \Lambda)$-differential privacy.

PROPOSITION 2 (PRIVACY FROM UNEQUAL NOISE [32]). *Let $Q = [q_1, \ldots, q_m]$ be a sequence of m queries and $\Lambda = [\lambda_1, \ldots, \lambda_m]$ be a sequence of m positive real numbers. Let $\mathcal{G}$ be an algorithm that adds independent noise to the result of each query q_i in Q ($i \in [1, m]$), such that the noise follows a Laplace distribution of scale λ_i. Then, $\mathcal{G}$ satisfies $GS(Q, \Lambda)$-differential privacy.* □

For convenience, we use *LaplaceNoise*(T, Q, Λ) to denote the above algorithm. On input T, Q, Λ, it returns a sequence of noisy answers $Y = [y_1, \ldots, y_m]$, where $y_i = q_i(T) + \eta_i$ and η_i is a sample from a Laplace distribution of scale λ_i for $i \in [1, m]$.

3.1 Proportional Laplace Noise

To reduce relative errors, a natural but ultimately flawed approach is to set the scale of the noise to be proportional to the actual query answer. (Extremely small answers are problematic, so we can set the noise scale to be the maximum of $q_i(T)$ and the sanity bound δ.) We refer to this strategy as *Proportional*. The expected relative error for query q_i is $\lambda_i / \max\{q_i(T), \delta\}$, and so setting $\lambda_i = c \max\{q_i(T), \delta\}$ for some constant c ensures that all query answers have equal expected relative errors, thereby minimizing the *worst-case relative error*.

Unfortunately *Proportional* is flawed because it violates differential privacy. This is because the scale of the noise depends on the input. As a consequence, two neighboring datasets may have different noise scales for the same query, and hence, some outputs may become considerably more likely on one dataset than the other. The following example illustrates the privacy defect of *Proportional*.

EXAMPLE 1. Let $\delta = 1$ and $\epsilon = 1$. Suppose that we have a census dataset that reports the ages of a population consisting of 5 individuals: $T_1 = \{42, 17, 35, 19, 55\}$. We have two queries: q_1 asks for the number of teenagers (ages $13 - 19$), and q_2 asks for the number of people under the age of retirement (ages under 65). Since we have $q_1(T_1) = 2$ and $q_2(T_2) = 5$, the *Proportional* algorithm would set $\Lambda = [\lambda_1, \lambda_2]$ such that $\lambda_1 = 2c$ and $\lambda_2 = 5c$ for some constant c. It can be verified that $c = 7/10$ achieves $GS(Q, \Lambda) = \epsilon$. Thus, we set $\lambda_1 = 2c = 1.4$ and $\lambda_2 = 5c = 3.5$.

Now consider a neighboring dataset T_2 in which one of the teenagers has been replaced with a 20-year old: $T_2 = \{42, 17, 35, \underline{20}, 55\}$. Compared to T_1, the answer on T_2 for q_1 is smaller by 1 and q_2 is unchanged, and accordingly *Proportional* sets the scales so that $\lambda_1 = c$ and $\lambda_2 = 5c$. It can be verified that $c = 6/5$ achieves $GS(Q, \Lambda) = \epsilon$ and so $\lambda_1 = 1.2$ and $\lambda_2 = 6$. Thus, the expected error of q_1 is *higher* on T_1 than on T_2.

If we consider the probability of an output that gives a highly inaccurate answer for the first query, such as $y_1 = 102$ and $y_2 = 5$, we can see it is much more likely on T_1 than T_2:

$$
\begin{aligned}
&\frac{Pr[\textit{Proportional} \text{ outputs } y_1, y_2 \text{ given } T_1]}{Pr[\textit{Proportional} \text{ outputs } y_1, y_2 \text{ given } T_2]} \\
&= \frac{\exp(-|y_1 - q_1(T_1)|/1.4) \cdot \exp(-|y_2 - q_2(T_1)|/3.5)}{\exp(-|y_1 - q_1(T_2)|/1.2) \cdot \exp(-|y_2 - q_2(T_2)|/6)} \\
&= \exp(-100/1.4 + 101/1.2) \\
&= \exp(535/42) > \exp(\epsilon).
\end{aligned}
$$

This demonstrates that *Proportional* violates differential privacy. □

Algorithm *TwoPhase* (T, Q, δ, ϵ_1, ϵ_2)
1. let $m = |Q|$ and q_i be the i-th ($i \in [1, m]$) query in Q
2. initialize $\Lambda = [\lambda_1, \ldots, \lambda_m]$ such that $\lambda_i = S(Q)/\epsilon_1$
3. $Y =$ *LaplaceNoise*(T, Q, Λ)
4. $\Lambda' =$ *Rescale*$(Y, \Lambda, \delta, \epsilon_2)$
5. if $GS(Q, \Lambda') \leq \epsilon_2$
6. $\quad Y' =$ *LaplaceNoise*(T, Q, Λ')
7. $\quad$ for any $i \in [1, m]$
8. $\quad\quad y_i = (\lambda_i'^2 \cdot y_i + \lambda_i^2 \cdot y_i')/(\lambda_i^2 + \lambda_i'^2)$
9. return Y

Figure 1: The *TwoPhase* Algorithm

3.2 Two-Phase Noise Injection

To overcome the drawback of *Proportional*, a natural idea is to ensure that the scale of the noise is decided in a differentially private manner. Towards this end, we may first apply *LaplaceNoise* to compute a noisy answer for each query in Q, and then use the noisy answers (instead of the exact answers) to determine the noise scale. This motivates the *TwoPhase* algorithm illustrated in Figure 1.

TwoPhase takes as input five parameters: a dataset T, a sequence Q of queries, a sanity bound δ, and two positive real numbers ϵ_1 and ϵ_2. Its output is a sequence $Y = [y_1, \ldots, y_m]$ such that y_i is a noisy answer for each query q_i. The algorithm is called *TwoPhase* because it interacts with the private data twice. In the first phase (Lines 1-3), it uses *LaplaceNoise* to compute noisy query answers, such that the scale of noise for all queries is equal to $S(Q)/\epsilon_1$.

In the second phase (Lines 4-8), the noise scales are adjusted (based on the output of the first phase) and a second sequence of noisy answers is obtained. Intuitively, we would like to rescale the noise so that it is reduced for queries that had small noisy answers in the first phase. This could be done in a manner similar to *Proportional* where the noisy answer y_i is used instead of the private answer $q_i(T)$—i.e., set λ_i' proportional to $\max\{y_i, \delta\}$. However, it is possible to get even lower relative errors by exploiting the structure of the query sequence Q. For now, we will treat this as a "black box" (referred to as *Rescale* on Line 4) and we will instantiate it when we consider specific applications in Section 5. Using the adjusted noise scales Λ', *TwoPhase* regenerates a noisy result y_i' for each query q_i (Line 6), and then estimates the final answer as a weighted average between the two noisy answers for each query (Line 7). Specifically, the final answer for q_i equals $(\lambda_i'^2 \cdot y_i + \lambda_i^2 \cdot y_i')/(\lambda_i^2 + \lambda_i'^2)$, which is an estimate of $q_i(T)$ with the minimum variance among all unbiased estimators of $q_i(T)$ given y_i and y_i' (this can be proved using a Lagrange multiplier and the fact that y_i and y_i' have variance $2\lambda_i^2$ and $2\lambda_i'^2$, respectively).

The following proposition states that *TwoPhase* ensures ϵ-differential privacy.

PROPOSITION 3 (PRIVACY OF *TwoPhase*). *TwoPhase ensures ϵ-differential privacy when its input parameters ϵ_1 and ϵ_2 satisfy $\epsilon_1 + \epsilon_2 \leq \epsilon$.* □

PROOF. *TwoPhase* only interacts with the private data T through two invocations of the *LaplaceNoise* mechanism. The first invocation satisfies ϵ_1-differential privacy, which follows from Proposition 1 and the fact that $\lambda_i = S(Q)/\epsilon_1$ for $i \in [1, m]$. Before the second invocation, the algorithm checks that the generalized sensitivity is at most ϵ_2, therefore by Proposition 2 it satisfies ϵ_2-differential privacy. Finally, differentially private algorithms compose: the sequential application of algorithms $\{G_i\}$, each satisfying ϵ_i-differential privacy, yields $(\sum_i \epsilon_i)$-differential privacy [24]. Therefore *TwoPhase* satisfies $(\epsilon_1 + \epsilon_2)$-differential privacy. □

TwoPhase differs from *Proportional* in that the noise scale is decided based on noisy answers rather than the correct (but private!) answers. While this is an improvement over *Proportional* from a privacy perspective, *TwoPhase* has two principal limitations in terms of utility. The first obvious issue is that errors in the first phase can lead to mis-calibrated noise in the second phase. For example, if we have two queries q_1 and q_2 with $q_1(T) < q_2(T)$, the first phase may generate y_1 and y_2 such that $y_1 > y_2$. In that case, the second phase of *TwoPhase* would incorrectly reduce the noise for q_2.

The second, more subtle issue is that given the requirement that $\epsilon_1 + \epsilon_2 \leq \epsilon$ for a fixed ϵ, it is unclear how to set ϵ_1 and ϵ_2 so that the expected relative error is minimized. There is a tradeoff: If ϵ_1 is too small, the answers in the first phase will be inaccurate and the noise will be mis-calibrated in the second phase, possibly leading to high relative errors for some queries. Although increasing ϵ_1 makes it more likely that the noise scale in the second phase will be appropriately calibrated, the answers in the second phase will be less accurate overall as the noise scale of all queries increases with decreasing ϵ_2. In general, ϵ_1 and ϵ_2 must be chosen to strive a balance between the effectiveness of the first and second phases, which may be challenging without prior knowledge of the data distribution. In the next section, we will remedy this deficiency with a method that does not require user inputs on the allocation of privacy budgets.

4. ITERATIVE NOISE REDUCTION

This section presents *iReduct* (iterative noise reduction), an improvement over the *TwoPhase* algorithm discussed in Section 3. The core of *iReduct* is an iterative process that adaptively adjusts the amount of noise injected into each query answer. *iReduct* begins by producing a noisy answer y_i for each query $q_i \in Q$ by adding Laplace noise of relatively large scale. In each subsequent iteration, *iReduct* first identifies a set $Q_\triangle$ of queries whose noisy answers are small and may therefore have high relative error. Next, *iReduct* resamples a noisy answer for each query in $Q_\triangle$, reducing the noise scale of each query by a constant. This iterative process is repeated until *iReduct* cannot decrease the noise scale of any answer without violating differential privacy. Intuitively, *iReduct* optimizes the relative errors of the queries because it gives queries with smaller answers more opportunities for noise reduction.

The aforementioned iterative process is built upon an algorithm called *NoiseDown* which takes as input a noisy result y_i and outputs a new version of y_i with reduced noise scale. We will introduce *NoiseDown* in Sections 4.1 and 4.2 below and present the details of *iReduct* in Section 4.3.

4.1 Rationale of *NoiseDown*

At a high level, given a query q and a dataset T, *iReduct* estimates $q(T)$ by iteratively invoking *NoiseDown* to generate noisy answers to q with reduced noise scales. The properties of the *NoiseDown* function ensure that an adversary who sees the entire sequence of noisy answers can infer no more about the dataset T than an adversary who sees only the last answer in the sequence. For ease of exposition, we focus on executing a single invocation of *NoiseDown*. Let Y (Y') be a Laplace random variable with mean value $q(T)$ and scale λ (λ'), such that $\lambda' < \lambda$. Intuitively, Y represents the noisy answer to q obtained in one iteration, and Y' corresponds to the noisy estimate generated in the next iteration.

Given Y, the simplest approach to generating Y' is to add to the true answer $q(T)$ fresh Laplace noise that is independent of Y and has scale λ'. This approach, however, incurs considerable privacy cost. For example, let q be a count query, such that for any two neighboring datasets T_1 and T_2, we have $q(T_1) - q(T_2) \in \{-1, 0, 1\}$. We will show that an algorithm that publishes independent samples Y' and Y in this scenario can satisfy ϵ-differential privacy only if $\epsilon \geq \frac{1}{\lambda'} + \frac{1}{\lambda}$. In other words, the privacy "cost" of publishing independent estimates of Y' and Y is $\frac{1}{\lambda'} + \frac{1}{\lambda}$.

For any neighboring datasets T_1 and T_2, let q be a count query such that $q(T_1) = c$ and $q(T_2) = c + 1$ for some integer constant c. Then for any noisy answers $y, y' < c$, we have

$$\begin{aligned}
&\frac{Pr\left[Y' = y', Y = y \mid T = T_1\right]}{Pr\left[Y' = y', Y = y \mid T = T_2\right]} \\
&= \frac{Pr\left[Y' = y', Y = y \mid q(T) = c\right]}{Pr\left[Y' = y', Y = y \mid q(T) = c + 1\right]} \\
&= \frac{\frac{1}{2\lambda'} \exp\left(-|c - y'|/\lambda'\right) \cdot \frac{1}{2\lambda} \exp\left(-|c - y|/\lambda\right)}{\frac{1}{2\lambda'} \exp\left(-|c + 1 - y'|/\lambda'\right) \cdot \frac{1}{2\lambda} \exp\left(-|c + 1 - y|/\lambda\right)} \\
&= \exp\left(\frac{1}{\lambda'}\left(|c + 1 - y'| - |c - y'|\right) + \frac{1}{\lambda}\left(|c + 1 - y| - |c - y|\right)\right) \\
&= \exp\left(1/\lambda' + 1/\lambda\right).
\end{aligned}$$

In contrast, the privacy cost of publishing Y' alone is $1/\lambda'$. That is, we pay an extra cost of $1/\lambda$ for sampling Y, even though the sample is discarded once we generate Y'[1].

Intuitively, the reason for this excess privacy cost is that both Y and Y' leak information about the state of the dataset T. Even though Y is less accurate than Y', an adversary who knows Y in addition to Y' has more information than an adversary that knows only Y', because the two samples are *independent*. The problem, then, is that the sampling process for Y' does not depend on the previously sampled value of Y. Therefore, instead of sampling Y' from a fresh Laplace distribution, we want to sample Y' from a distribution that depends on Y. Our aim is to do so in a way such that Y provides no new information about the dataset once the value of Y' is known. We can formalize this objective as follows. From an adversary's perspective, the dataset T is unknown and can be modeled as a random variable. The adversary tries to infer the contents of the dataset by looking at Y' and Y. For any possible dataset T_1, we want the following property to hold:

$$Pr\left[T = T_1 \mid Y = y, Y' = y'\right] = Pr[T = T_1 \mid Y' = y'] \quad (5)$$

To see how this restriction allows us to use the privacy budget more efficiently, let us again consider any two neighboring datasets T_1 and T_2. Observe that when Equation 5 is satisfied, we can apply Bayes' rule (twice) to show that:

$$\begin{aligned}
&Pr[Y' = y', Y = y \mid T = T_i] \\
&= Pr[T = T_i \mid Y' = y', Y = y] \cdot \frac{Pr[Y' = y', Y = y]}{Pr[T = T_i]} \\
&= Pr[T = T_i \mid Y' = y'] \cdot \frac{Pr[Y' = y', Y = y]}{Pr[T = T_i]} \\
&= \frac{Pr[Y' = y' \mid T = T_i] Pr[T = T_i]}{Pr[Y' = y']} \cdot \frac{Pr[Y' = y', Y = y]}{Pr[T = T_i]} \\
&= Pr[Y' = y' \mid T = T_i] \cdot Pr[Y = y \mid Y' = y']
\end{aligned}$$

where the term $Pr[Y = y \mid Y' = y']$ does not depend on the value of T.

[1]Rather than discarding Y, one could combine Y and Y' to derive an estimate of $q(T)$, as done in the *TwoPhase* algorithm (see Line 8 in Figure 1). This does reduce the expected error, but it still has excess privacy cost compared to *NoiseDown*, as explained in Appendix A.

This allows us to derive an upper bound on the privacy cost of an algorithm that outputs Y followed by Y'. Let us again consider a count query q such that $q(T_1) - q(T_2) \in \{-1, 0, 1\}$. Let Y' be a random variable that (i) follows a Laplace distribution with mean $q(T)$ and scale λ' but (ii) is generated in a way that *depends* on the observed value for Y. If these criteria are satisfied and Equation 5 holds, it follows that:

$$\begin{aligned}&\frac{Pr[Y'=y', Y=y \mid T=T_1]}{Pr[Y'=y', Y=y \mid T=T_2]}\\ =&\frac{Pr[Y'=y' \mid T=T_1]\cdot Pr[Y=y \mid Y'=y']}{Pr[Y'=y' \mid T=T_2]\cdot Pr[Y=y \mid Y'=y']}\\ =&\frac{Pr[Y'=y' \mid T=T_1]}{Pr[Y'=y' \mid T=T_2]} \le \exp(1/\lambda')\end{aligned}$$

The last step works because of our assumption that Y' follows a Laplace distribution with scale λ'. The above inequality implies that obtaining *correlated* samples Y' and Y now incurs a total privacy cost of just $1/\lambda'$, i.e., no privacy budget is wasted on Y.

In summary, if Y' follows a Laplace distribution but is sampled from a distribution that depends on Y, and if Equation 5 is satisfied, then we can perform the desired resampling procedure without incurring any loss in the privacy budget. We now define the conditional probability distribution of Y'.

DEFINITION 5 (NOISE DOWN DISTRIBUTION). *The* Noise Down distribution *is a conditional probability distribution on Y' given that $Y = y$. It is defined by the following conditional probability distribution function (p.d.f.). Let μ, λ, λ' be fixed parameters. The conditional p.d.f. of Y' given that $Y = y$ is defined as:*

$$f_{\mu,\lambda,\lambda'}(y'|Y=y) = \frac{\lambda}{\lambda'}\cdot\frac{\exp\left(-\frac{|y'-\mu|}{\lambda'}\right)}{\exp\left(-\frac{|y-\mu|}{\lambda}\right)}\cdot\gamma(\lambda',\lambda,y',y) \quad (6)$$

where $\gamma(\lambda', \lambda, y', y)$

$$\begin{aligned}=&\frac{1}{4\lambda}\cdot\frac{1}{\cosh\left(\frac{1}{\lambda'}\right)-1}\cdot\Bigg(2\cosh\left(\tfrac{1}{\lambda'}\right)\cdot\exp\left(-\tfrac{|y-y'|}{\lambda}\right)\\ &-\exp\left(-\tfrac{|y-y'-1|}{\lambda}\right)-\exp\left(-\tfrac{|y-y'+1|}{\lambda}\right)\Bigg). \quad (7)\end{aligned}$$

and $\cosh(\cdot)$ *denotes the hyperbolic cosine function, i.e.,* $\cosh(z) = (e^z + e^{-z})/2$ *for any z.*

Theorem 1 shows that this conditional distribution has the desired properties, namely, (i) Y' follows a Laplace distribution and (ii) releasing Y in addition to Y' leaks no additional information.

THEOREM 1 (PROPERTIES OF NOISE DOWN). *Let Y be a random variable that follows a Laplace distribution with mean $q(T)$ and scale λ. Let Y' be a random variable drawn from a Noise Down distribution conditioned on the value of Y with parameters μ, λ, λ' such that $\mu = q(T)$ and $\lambda' < \lambda$. Then, Y' follows a Laplace distribution with mean $q(T)$ and scale λ'. Further, Y' and Y satisfy Equation 5 for any values of y', y, and T_1.*

Theorem 1 provides the theoretical basis for the iterative resampling that is key to *iReduct*. The next section describes an algorithm for sampling from the Noise Down distribution.

4.2 Implementing *NoiseDown*

To describe an algorithm for sampling from the Noise Down distribution (Equation 6), it is sufficient to fix $Y = y$ for some real constant y and fix parameters μ, λ, and λ'. Let $f : \mathbb{R} \to [0, 1]$ be

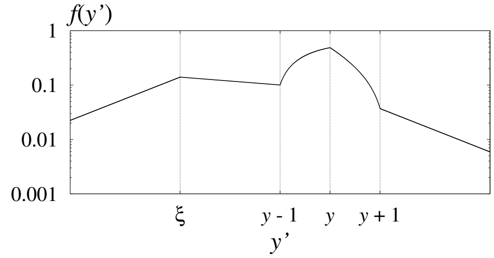


Figure 2: Illustration of f

defined by $f(y') = f_{\mu,\lambda,\lambda'}(y' \mid Y = y)$, i.e., the conditional p.d.f. from Equation 6. We now describe an algorithm for sampling from the probability distribution defined by f. In the following derivation, we focus on the case where $\mu \le y$; we will describe later the modifications that are necessary to handle the case where $\mu > y$.

Let $\xi = \min\{\mu, y - 1\}$. By Equation 6, for any $y' \le \xi$,

$$f(y') = e^{y'/\lambda'}\cdot\gamma(\lambda',\lambda,y',y)\cdot\frac{\lambda}{\lambda'}\cdot\exp\left(\frac{-\mu}{\lambda'}+\frac{y-\mu}{\lambda}\right),$$

where

$$\begin{aligned}\gamma(\lambda',\lambda,y',y) =& e^{y'/\lambda}\cdot\frac{1}{4\lambda}\cdot\frac{1}{\cosh\left(\frac{1}{\lambda'}\right)-1}\\ &\cdot\left(2\cosh\left(\tfrac{1}{\lambda'}\right)\cdot\exp\left(\tfrac{-y}{\lambda}\right)-\exp\left(\tfrac{1-y}{\lambda}\right)-\exp\left(\tfrac{-1-y}{\lambda}\right)\right).\end{aligned}$$

(Note that the function γ above is as defined in Equation 7 but takes a simplified form given $y' < \xi$). As μ, y, λ, and λ' are all given, $f(y') \propto \exp(y'/\lambda' + y'/\lambda)$ holds. Similarly, it can be verified that

1. $\forall y' \in (\xi, y-1]$, $f(y') \propto \exp(y'/\lambda - y'/\lambda')$;
2. $\forall y' \in [y+1, +\infty)$, $f(y') \propto \exp(-y'/\lambda' - y'/\lambda)$.

For example, Figure 2 illustrates f with the y-axis in log-scale.

In summary, f conforms to an exponential distribution on each of the following intervals: $(-\infty, \xi]$, $(\xi, y-1]$, and $[y+1, +\infty)$. When the probability mass of f on each of the three intervals is known, it is straightforward to generate random variables that follow f on those intervals. Let $\theta_1 = \int_{-\infty}^{\xi} f(y')dy'$, $\theta_2 = \int_{\xi}^{y-1} f(y')dy'$, and $\theta_3 = \int_{y+1}^{+\infty} f(y')dy'$. It can be shown that

$$\theta_1 = \frac{\lambda\cdot\left(\cosh(\frac{1}{\lambda'})-\cosh(\frac{1}{\lambda})\right)\cdot\exp\left((\frac{1}{\lambda'}+\frac{1}{\lambda})\cdot(\xi-\mu)\right)}{2(\lambda'+\lambda)\cdot\left(\cosh(\frac{1}{\lambda'})-1\right)}, \quad (8)$$

$$\begin{aligned}\theta_2 =& \frac{\lambda\cdot\cosh(\frac{1}{\lambda'})\cdot\left(e^{1/\lambda'}-e^{1/\lambda}\right)\cdot(1-e^{-1/\lambda'-1/\lambda})}{4(\lambda-\lambda')\cdot\left(\cosh(\frac{1}{\lambda'})-1\right)}\\ &\cdot\left(1-\exp\left((\frac{1}{\lambda'}-\frac{1}{\lambda})\cdot(\xi-y+1)\right)\right), \quad (9)\end{aligned}$$

$$\theta_3 = \frac{\lambda\cdot\left(\cosh(\frac{1}{\lambda'})-\cosh(\frac{1}{\lambda})\right)\cdot\exp\left(\frac{\mu-y-1}{\lambda'}-\frac{\mu-y+1}{\lambda}\right)}{2(\lambda'+\lambda)\cdot\left(\cosh(\frac{1}{\lambda'})-1\right)}. \quad (10)$$

For the remaining interval $(y-1, y+1)$ on which f has a complex form, we resort to a standard importance sampling approach. Specifically, we first generate a random sample y' that is uniformly distributed in $(y-1, y+1)$. After that, we toss a coin that comes

Algorithm *NoiseDown* (μ, y, λ, λ')

initialize a boolean variable *invert* = true
if $\mu > y$
  set $\mu = -\mu$, $y = -y$, and *invert* = false
$\xi = \min\{\mu, y - 1\}$
let $\theta_1, \theta_2, \theta_3$, and φ be as defined in Eqn. 8, 9, 10, and 11
generate a random variable u uniformly distributed in $[0, 1]$
if $u \in [0, \theta_1)$
  generate a random variable $y' \in (-\infty, \xi]$ such that $Pr[y' = y'] \propto \exp(y'/\lambda' + y'/\lambda)$
else if $u \in [\theta_1, \theta_1 + \theta_2]$
  generate a random variable $y' \in (\xi, y - 1]$ such that $Pr[y' = y'] \propto \exp(y'/\lambda - y'/\lambda')$
else if $u \in [1 - \theta_3, 1]$
  generate a random variable $y' \in [y + 1, +\infty)$ such that $Pr[y' = y'] \propto \exp(-y'/\lambda' - y'/\lambda)$
else
  while true
    generate a random variable y' uniformly distributed in $(y - 1, y + 1)$
    generate a random variable u' uniformly distributed in $[0, 1]$
    if $u' \leq f(y')/\varphi$ then break
if *invert* = true then return y'; otherwise, return $-y'$

Figure 3: The *NoiseDown* Algorithm

up heads with a probability $f(y')/\varphi$, where

$$\varphi = \frac{1}{2\lambda'} \cdot \frac{\cosh\left(\frac{1}{\lambda'}\right) - \exp\left(-\frac{1}{\lambda}\right)}{\cosh\left(\frac{1}{\lambda'}\right) - 1} \cdot \exp\left(\frac{y - \mu}{\lambda} - \frac{\max\{0, y - \mu - 1\}}{\lambda'}\right). \tag{11}$$

If the coin shows a tail, we resample y' from a uniform distribution on $(y - 1, y + 1)$, and we toss the coin again. This process is repeated until the coin shows a head, at which time we return y' as a sample from f. The following proposition proves the correctness of our sampling approach.

PROPOSITION 4 (NOISE DOWN SAMPLING). *Given $\mu \leq y$, we have $f(y') < \varphi$ for any $y' \in (y - 1, y + 1)$.* □

So far, we have focused on the case where $\mu \leq y$. For the case when $\mu > y$, we first set $\mu = -\mu$, $y = -y$, and then generate y' using the method described above. After that, we set $y' = -y'$ before returning it as a sample from f. The correctness of this method follows from the following property of the Noise Down distribution $f_{\mu,\lambda,\lambda'}(y'|Y = y)$ (as defined Equation 6):

$$f_{\mu,\lambda,\lambda'}(y'|Y = y) \;=\; f_{-\mu,\lambda,\lambda'}(-y'|Y = -y)$$

for any $\mu, \lambda, \lambda', y', y$. As a summary, Figure 3 shows the pseudo-code of the *NoiseDown* function that takes as input $\mu, y, \lambda, \lambda'$ and returns a sample from the distribution defined in Equation 6.

4.3 The *iReduct* Algorithm

We are now ready to present the *iReduct* algorithm, as illustrated in Figure 4. It takes as input a dataset T, a sequence Q of queries on T, a sanity bound δ, and three positive real numbers ϵ, λ_{max}, and λ_Δ. *iReduct* starts by initializing a variable λ_i for each query $q_i \in Q$ ($i \in [1, m]$), setting $\lambda_i = \lambda_{max}$ (Lines 1-2). The user-specified parameter λ_{max} is a large constant that corresponds to the greatest amount of Laplace noise that a user is willing to accept in any query answer returned by *iReduct*. For example, if Q is a sequence of count queries on T then a user may set λ_{max} to 10% of the number of tuples in the dataset T.

Algorithm *iReduct* (T, Q, δ, ϵ, λ_{max}, λ_Δ)

1. let $m = |Q|$ and q_i be the i-th ($i \in [1, m]$) query in Q
2. initialize $\Lambda = [\lambda_1, \ldots, \lambda_m]$, such that $\lambda_i = \lambda_{max}$
3. if $GS(Q, \Lambda) > \epsilon$ then return $\emptyset$
4. Y = *LaplaceNoise*(T, Q, Λ)
5. let $Q' = Q$
6. while $Q' \neq \emptyset$
7. $\quad Q_\Delta$ = *PickQueries*(Q', Y, Λ, δ)
8. $\quad$ for each $i \in [1, m]$
9. $\quad\quad$ if $q_i \in Q_\Delta$ then $\lambda_i = \lambda_i - \lambda_\Delta$
10. $\quad$ if $GS(Q, \Lambda) \leq \epsilon$
11. $\quad\quad$ for each $i \in [1, m]$
12. $\quad\quad\quad$ if $q_i \in Q_\Delta$ then y_i = *NoiseDown*$\big(q_i(T), y_i, \lambda_i + \lambda_\Delta, \lambda_i\big)$
13. $\quad$ else
14. $\quad\quad$ for each $i \in [1, m]$
15. $\quad\quad\quad$ if $q_i \in Q_\Delta$ then $\lambda_i = \lambda_i + \lambda_\Delta$
16. $\quad Q' = Q' \setminus Q_\Delta$
17. return Y

Figure 4: The *iReduct* Algorithm

As a second step, *iReduct* checks whether the conservative setting of the noise scale guarantees ϵ-differential privacy (Line 3). This is done by measuring the generalized sensitivity of the query sequence given noise scales Λ. If the generalized sensitivity exceeds ϵ, *iReduct* returns an empty set to indicate that the results of Q cannot be released without adding excessive amounts of noise to the queries. Otherwise, *iReduct* generates a noisy result y_i for each query $q_i \in Q$ by applying Laplace noise of scale λ_i (Line 4).

Given Y, *iReduct* iteratively applies *NoiseDown* to adjust the noise in y_i so that noise magnitude is reduced for queries that appear to have high relative errors (Lines 5-16). In each iteration, *iReduct* first identifies a set Q_Δ of queries (Line 7) and then tries to decrease the noise scales of the queries in Q_Δ by a constant λ_Δ (Lines 8-16). The selection of Q_Δ is performed by an application-specific function that we refer to as *PickQueries*. An instantiation of *PickQueries* is given in Section 5.3, but in general, any algorithm can be applied, so long as the algorithm utilizes only the sanity bound δ, the noisy queries answers seen so far, and the queries' noise scales, and does not rely on the true answer $q_i(T)$ of any query q_i. This requirement ensures that the selection of Q_Δ does not reveal any private information beyond what has been disclosed by the noisy results generated so far. For example, if we aim to minimize the maximum relative error of the query results, we may implement *PickQueries* as a function that returns the query q_i that maximizes $\lambda_i / \max\{y_i, \delta\}$, i.e., the query whose noise scale is likely to be large with respect to its actual result.

Once Q_Δ is selected and the scales λ_i have been adjusted accordingly, *iReduct* checks whether the revised scales are permissible given the privacy budget. This is done by measuring the generalized sensitivity given the revised scales (Line 10). If the revised scales are permissible, *iReduct* applies *NoiseDown* to reduce the noise scale of each q_i in Q_Δ by λ_Δ (Lines 11-12). Otherwise, *iReduct* reverts the changes in Λ and removes the queries in Q_Δ from its working set (Line 13-16). This iterative process is repeated until no more queries remain in the working set, which indicates that *iReduct* cannot find a subset of queries whose noise can be reduced without violating the privacy constraint. At this point, it outputs the noisy answers Y.

THEOREM 2 (PRIVACY OF *iReduct*). *iReduct ensures ϵ'-differential privacy whenever its input parameter ϵ satisfies $\epsilon \leq \epsilon'$.* □

5. CASE STUDY: PRIVATE MARGINALS

In this section, we present an instantiation of the *TwoPhase* and *iReduct* algorithms for generating privacy-preserving *marginals*. Section 5.1 describes the problem and discusses existing solutions and Sections 5.2 and 5.3 describe the instantiations of *TwoPhase* and *iReduct*.

5.1 Motivation and Existing Solution

A marginal is a table of counts that summarize a dataset along some dimensions. Marginals are widely used by the U.S. Census Bureau and other federal agencies to release statistics about the U.S. population. More formally, a marginal M of a dataset T is a table of counts that corresponds to a subset $\mathcal{A}$ of the attributes in T. If $\mathcal{A}$ contains k attributes $A_1, A_2, \ldots, A_k$, then M comprises $\prod_{i=1}^{k} |A_i|$ counts, where $|A_i|$ denotes the number of values in the domain of A_i. Each count in M pertains to a point $\langle v_1, v_2, \ldots, v_k\rangle$ in the k-dimensional space $A_1 \times A_2 \times \cdots \times A_k$, and the count equals the number of tuples whose value on A_i is v_i ($i \in [1, k]$). For example, Table 2 illustrates a dataset T that contains three attributes: Age, (Marital) Status, and Gender. Table 3 shows a marginal of T on $\{\text{Status}, \text{Gender}\}$. In general, a marginal defined over a set of k attributes is referred to as a *k-dimensional marginal*.

Age	Status	Gender
23	Single	M
25	Single	F
35	Married	F
37	Married	F
85	Widowed	F

Table 2: A dataset T

	Gender	
Status	M	F
Single	1	1
Married	0	2
Divorced	0	0
Widowed	0	1

Table 3: A marginal of T

While a dataset with k attributes can be equivalently represented as a single k-dimensional marginal, such a marginal is likely very sparse (i.e., all counts near zero), and thus will not be able to tolerate the random noise added for privacy. Instead, we therefore publish a set $\mathcal{M}$ of low dimensional marginals, each of which is a projection onto j dimensions for some small $j < k$. This is common practice at statistical agencies, and is consistent with prior work on differentially private marginal release [1].

We can publish $\mathcal{M}$ in a ϵ-differentially private manner as long as we add Laplace noise of scale $2 \cdot |\mathcal{M}|/\epsilon$ to each count in every marginal. This is because changing a record affects only two counts in each marginal (each count would be offset by one), thus the sensitivity of the marginals is $2 \cdot |\mathcal{M}|$, i.e., Laplace noise of scale $2 \cdot |\mathcal{M}|/\epsilon$ suffices for privacy, as shown in Proposition 1. However, adding an equal amount of noise to each marginal in $\mathcal{M}$ may lead to sub-optimal results. For example, suppose that $\mathcal{M}$ contains two marginals M_1 and M_2, such that M_1 (M_2) has a large (small) number of counts, all of which are small (large). If identical amount of noise is injected to M_1 and M_2, then the noisy counts in M_1 would have much higher relative errors than the noisy counts in M_2. Intuitively, a better solution is to apply smaller (larger) amount of noise to M_1 (M_2), so as to balance the quality of M_1 and M_2 without degrading their overall privacy guarantee.

We measure the utility of a set of noisy marginals in terms of minimizing *overall error*, as defined next. Each marginal $M_i \in \mathcal{M}$ is represented as a sequence of queries $[q_{i1}, \ldots, q_{i|M_i|}]$ where q_{ij} denotes the j^{th} query in M_i for $i \in [1, |\mathcal{M}|]$ and $j \in [1, |M_i|]$. Let M_i^* denote a noisy version of marginal M_i and let y_{ij} denote to the noisy answer to q_{ij} for $i \in [1, |\mathcal{M}|]$ and $j \in [1, |M_i|]$.

DEFINITION 6 (OVERALL ERROR OF MARGINALS). *The* overall error *of a set of noisy marginals* $\{M_1^*, \ldots, M_{|\mathcal{M}|}^*\}$ *is defined as*

$$\frac{1}{|\mathcal{M}|} \cdot \sum_{i=1}^{|\mathcal{M}|} \frac{1}{|M_i^*|} \cdot \sum_{j=1}^{|M_i^*|} \frac{|y_{ij} - q_{ij}(T)|}{\max\{\delta, q_{ij}(T)\}}$$

□

We next describe how we instantiate *TwoPhase* and *iReduct* with this utility goal in mind. The query sequence input to both algorithms is simply the concatenation of the individual query sequences for each marginal; i.e., $Q = [q_{11}, \ldots, q_{1|M_1|}, \ldots, q_{|\mathcal{M}|1}, \ldots, q_{|\mathcal{M}||M_{|\mathcal{M}|}|}]$.

5.2 Instantiation of *TwoPhase*

To instantiate *TwoPhase*, we must describe the implementation of the *Rescale* subroutine that was treated as a "black box" in Section 3.2.

Let us first consider an ideal scenario where the first phase of *TwoPhase* returns the exact count of every query q_{ij}. In this case, we know that if we added Laplace noise of scale λ_i to every count in a marginal M_i, then each count $q_{ij}(T)$ in M_i would have an expected relative error of $\lambda_i / \max\{\delta, q_{ij}(T)\}$. (Recall that the expected absolute deviation of a Laplace variable equals its scale.) The expected average relative error in M_i would therefore be $\lambda_i/|M_i| \cdot \left(\sum_{j=1}^{|M_i|} 1/\max\{\delta, q_{ij}(T)\}\right)$. Consequently, to minimize the expected overall error of the marginals, it suffices to ensure that $\sum_{M_i} \left(\lambda_i/|M_i| \cdot \sum_{j=1}^{|M_i|} 1/\max\{\delta, q_{ij}(T)\}\right)$ is minimized, subject to the privacy constraint that the marginals should ensure ϵ-differential privacy, i.e., $\sum_{i=1}^{|\mathcal{M}|} 2/\lambda_i \le \epsilon$. Using a Lagrange multiplier, it can be shown that λ_i should be proportional to $\sqrt{|M_i|/(\sum_{j=1}^{|M_i|} 1/\max\{\delta, q_{ij}(T)\})}$. We refer to this optimal strategy for deciding noise scale as the *Oracle* method. *Oracle* is similar to the *Proportional* algorithm described in Section 3.1, but sets its λ_i values to minimize average rather than worst-case relative error.

Rescale sets the noise scale similarly to how it is set by *Oracle* except that it uses the noisy counts to approximate the exact counts. To be precise, λ_i is proportional to $\sqrt{|M_i|/\sum_{j=1}^{|M_i|} \max\{\delta, y_{ij}\}}$, where y_{ij} denotes the noisy answer produced by the first phase of *TwoPhase*.

5.3 Instantiation of *iReduct*

Before we can use *iReduct* to publish marginals, we need to instantiate two of its subroutines: (i) a method for computing the generalized sensitivity of a set of weighted marginal queries and (ii) an implementation of the *PickQueries* black box (discussed in Section 4.3) that chooses a set Q_Δ of marginal queries for noise reduction. In practice, the sensitivity of a set of marginals depends only on the smallest noise scale in each marginal, and so we gain the best tradeoff between low sensitivity and high utility by always selecting the same noise scale for every count in a given marginal. We maintain the invariant by (i) initially setting all marginal counts to have the same noise scale λ_{max} and (ii) having *PickQueries* always pass to *NoiseDown* the queries corresponding to a single marginal M_i^* in which the noise scale for each cell in M_i^* is reduced by the same constant λ_Δ.

As the counts in the same marginal always have identical noise scale, we can easily derive the generalized sensitivity of the marginal queries as follows. Assume that, in the beginning of a certain iteration, the counts in M_i^* ($i \in [1, |\mathcal{M}|]$) have noise scale λ_i.

	Age	Gender	Marital Status	State	Birth Place	Race	Education	Occupation	Class of Worker
Brazil	101	2	4	26	29	5	5	512	4
US	92	2	4	51	52	14	5	477	4

Table 4: Sizes of Attribute Domains

It can be verified that the generalized sensitivity of the marginals is

$$g_\alpha = \sum_{i \in [1, |\mathcal{M}|]} \frac{2}{\lambda_i} \tag{12}$$

Consider that we apply *NoiseDown* on a noisy marginal M_j^* ($j \in [1, |\mathcal{M}|]$). The noise scale of the queries in M_j^* would become $\lambda_j - \lambda_\Delta$, in which case the generalized sensitivity would become:

$$g_\beta = \frac{2}{\lambda_j - \lambda_\Delta} + \sum_{i \in [1, |\mathcal{M}|] \wedge i \neq j} \frac{2}{\lambda_i}. \tag{13}$$

Recall that, in each iteration of *iReduct*, we aim to ensure that the generalized sensitivity of the queries does not exceed ϵ. Therefore, we can apply *NoiseDown* on a marginal M_j^* only if $g_\beta \leq \epsilon$.

We now discuss our implementation of the *PickQueries* function. Notice that when we apply *NoiseDown* to a noisy marginal M_j^* ($j \in [1, |\mathcal{M}|]$), the expected errors of the estimated counts for that marginal decrease (due to the decrease in the noise scale of M_j^*) but the privacy guarantee degrades. Ideally, running *NoiseDown* on the selected marginal should lead to a large drop in the overall error and a small increase in the privacy overhead. To identify good candidates, we adopt the following methods to quantify the changes in the overall error and privacy overhead that are incurred by invoking *NoiseDown* on a marginal M_j^*.

First, recall that when we employ *NoiseDown* to reduce the noise scale λ_j of M_j^* by a constant λ_Δ, the generalized sensitivity of the noisy marginals increases from g_α to g_β, where g_α and g_β are as defined in Equations 12 and 13. In light of this, we quantify the cost of privacy entailed by applying *NoiseDown* on M_j^* as

$$g_\beta - g_\alpha = \frac{2}{\lambda_j - \lambda_\Delta} - \frac{2}{\lambda_j}. \tag{14}$$

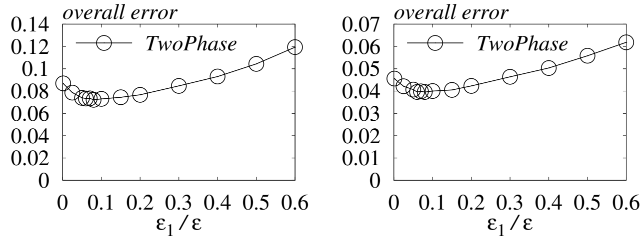
Second, for each noisy count y in M_j^*, we estimate its relative error as $\lambda_j / \max\{y, \delta\}$, where δ is the sanity bound. In other words, we use y to approximate the true count, and we estimate the absolute error of y as the expected absolute deviation λ_j of the Laplace noise added in y. Accordingly, the average relative error of M_j^* is estimated as $\lambda_j \cdot \sum_{y \in M_j^*} 1/\max\{y, \delta\}$. Following the same rationale, we estimate the average relative error of M_j^* after the application of *NoiseDown* as $(\lambda_j - \lambda_\Delta) \cdot \sum_{y \in M_j^*} 1/\max\{y, \delta\}$. Naturally, the decrease in the overall error of the marginals (resulted from invoking *NoiseDown* on M_j^*) is quantified as

$$\frac{1}{|\mathcal{M}|} \cdot \Big(\lambda_j \cdot \sum_{y \in M_j^*} \frac{1}{\max\{y, \delta\}} - (\lambda_j - \lambda_\Delta) \cdot \sum_{y \in M_j^*} \frac{1}{\max\{y, \delta\}}\Big)$$
$$= \frac{\lambda_\Delta}{|\mathcal{M}|} \cdot \sum_{y \in M_j^*} \frac{1}{\max\{y, \delta\}} \tag{15}$$

Given Equations 14 and 15, in each iteration of *iReduct*, we heuristically choose the marginal M_j^* that maximizes

$$\Big(\frac{\lambda_\Delta}{|\mathcal{M}|} \cdot \sum_{y \in M_j^*} \frac{1}{\max\{y, \delta\}}\Big) \Big/ \Big(\frac{2}{\lambda_j - \lambda_\Delta} - \frac{2}{\lambda_j}\Big).$$

That is, we select the marginal that maximizes the ratio between the estimated decrease in overall error and the estimated increase in privacy cost.



(a) Brazil (b) USA

Figure 5: Overall Error vs. ϵ_1/ϵ (1D Marginals)

6. EXPERIMENTS

We evaluate the accuracy of the proposed algorithms on three tasks: estimating all one-way marginals (Section 6.3), estimating all two-way marginals (Section 6.4), and learning a Naive Bayes classifier (Section 6.5).

6.1 Experimental Settings

We use two datasets[2] that are composed of census records collected from Brazil and the US respectively. Each dataset contains nine attributes, whose domain sizes are as illustrated in Table 4. The Brazil dataset consists of nearly 10 million tuples, while the US dataset has around 14 million records.

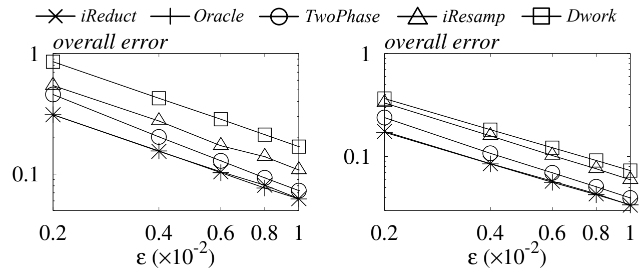
We use *iReduct* to generate noisy marginals from each dataset, and we compare the performance of *iReduct* against four alternate methods. The first method is the *Oracle* algorithm presented in Section 5.2, which utilizes the exact counts in the marginals to decide the noise scale of each marginal in a manner that minimizes the expected overall error. Although *Oracle* does not conform to ϵ-differential privacy, it provides a lower bound on the error incurred by any member of a large class of ϵ-differential privacy algorithms. The second and third methods are the *TwoPhase* and *iResamp* algorithms presented in Section 3 and Appendix A respectively. The final technique, henceforth referred to as *Dwork*, is Dwork et al.'s method (Section 2.2), which adds an equal amount of noise to each marginal.

We measure the quality of noisy marginals by their overall errors, as defined in Section 5.1. In every experiment, we run each algorithm 10 times, and we report the mean of the overall errors incurred by the algorithm. Among the input parameters of *iReduct*, we fix $\lambda_{\max} = |T|/10$ and $\lambda_\Delta = |T|/10^6$, where $|T|$ denotes the number of tuples in the dataset. That is, *iReduct* starts by adding Laplace noise of scale $|T|/10$ to each marginal and, in each iteration, it reduces the noise scale of a selected marginal by $|T|/10^6$. All of our experiments are run on a computer with a 2.66GHz CPU and 24GB memory.

6.2 Calibrating *TwoPhase*

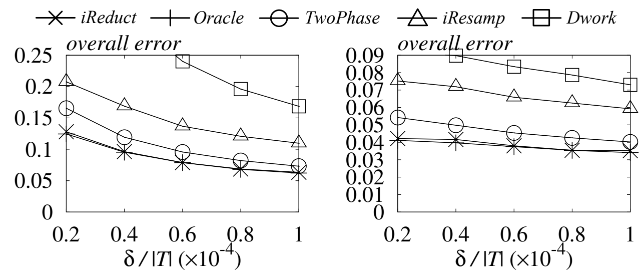
As was discussed in Section 3.2, the performance of the *TwoPhase* algorithm depends on how the fixed privacy budget is allocated across its two phases. This means that before we can use

[2]Both datasets are available online as part of the *Integrated Public Use Microdata Series* [25].



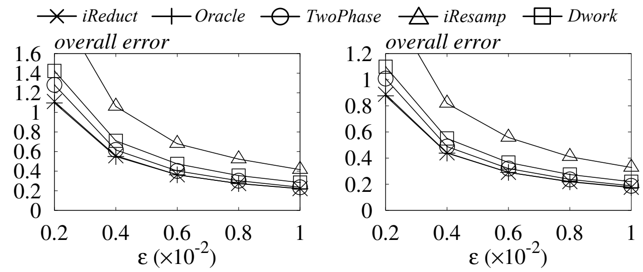
(a) Brazil (b) USA

Figure 6: Overall Error vs. ϵ (1D Marginals)



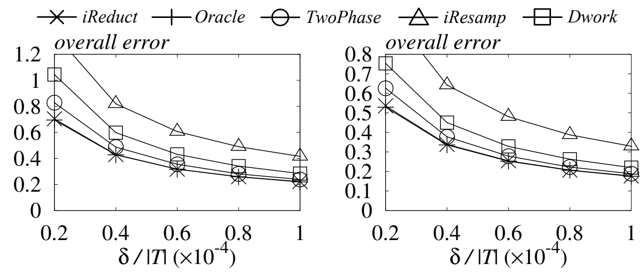
(a) Brazil (b) USA

Figure 7: Overall Error vs. δ (1D Marginals)



(a) Brazil (b) USA

Figure 8: Overall Error vs. ϵ (2D Marginals)



(a) Brazil (b) USA

Figure 9: Overall Error vs. δ (2D Marginals)

the *TwoPhase* algorithm, we must decide how to set the values of ϵ_1 and ϵ_2 to optimize the quality of the noisy marginals.

To determine the appropriate setting, we fix $\epsilon = 0.01$ and measure the overall error of *TwoPhase* when ϵ_1 varies and $\epsilon_2 = \epsilon - \epsilon_1$. Figure 5 illustrates the overall error on the set of one-dimensional marginals as a function of ϵ_1/ϵ. As ϵ_1/ϵ increases, the overall error of *TwoPhase* decreases until it reaches a minimum and then increases monotonically. When ϵ_1 is too small, the first step of *TwoPhase* is unable to generate accurate estimates of the marginal counts. This renders the second step of *TwoPhase* less effective, since it relies on the estimates from the first step to determine the noise scale for each marginal. On the other hand, making ϵ_1 too large causes ϵ_2 to becomes small and forces the second step to inject large amounts of noise into all of the marginal counts. The utility of the noisy marginals is optimized only when ϵ_1/ϵ strikes a good balance between the effectiveness of the first and second steps. As shown in Figure 5, the overall error of *TwoPhase* hits a sweet spot when $\epsilon_1 \in [0.06\epsilon, 0.08\epsilon]$. We set $\epsilon_1 = 0.07\epsilon$ in the experiments that follow.

6.3 Results on 1D Marginals

We compare the overall error incurred when each of the five algorithms described above is used to produce differentially private estimates of the one-dimensional marginals of the Brazil and USA census data. We vary both ϵ and δ. Figure 6 shows the overall error of each algorithm as a function of ϵ, with δ fixed to $10^{-4} \cdot |T|$. *iReduct* yields error values that are almost equal to those of *Oracle*, which indicates that its performance is close to optimal. *TwoPhase* performs worse than *iReduct* in all cases, but it still consistently outperforms the other methods. The overall error of *iResamp* is comparable to that of *Dwork*.

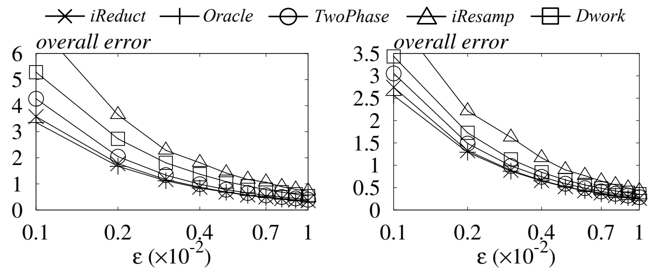
Figure 7 illustrates the overall error of each method as a function of the sanity bound δ when $\epsilon = 0.01$. The relative performance of each method is the same as in Figure 6. The overall errors of all algorithms decrease as δ increases, since a larger δ leads to lower relative errors for marginal counts smaller than δ (see Equation 1).

In the aforementioned experiments, every method but *iReduct* needs only a few linear scans of the marginal counts to generate its output, and therefore incurs negligible computation cost. In contrast, the inner loop of *iReduct* is executed $O(\lambda_{max}/\lambda_{\Delta})$ times, and the *iReduct* takes around 5 seconds to output a set of marginal counts. The relative high computational overhead of *iReduct* is justified by the fact that it provides improved data utility over the other methods.

6.4 Results on 2D Marginals

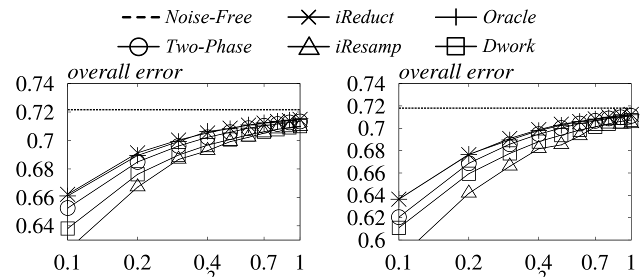
In the second sets of experiments, we consider the task of generating all two-dimensional marginals of the datasets. For the input parameters of *TwoPhase*, we set $\epsilon_1/\epsilon = 0.025$ based on an experiment similar to that described in Section 6.2. Figure 8 shows the overall error of each method when $\delta = 10^{-4} \cdot |T|$ and ϵ varies. The overall error of *iReduct* is almost the same as that of *Oracle*. *TwoPhase* is outperformed by *iReduct*, but its overall error is consistently lower than that of *iResamp* or *Dwork*. Interestingly, the performance gaps among *iReduct*, *TwoPhase*, and *Dwork* are not as substantial as the case for one-dimensional marginals. The reason is that a large fraction of the two-dimensional marginals are sparse. As a consequence, *iReduct* and *TwoPhase* apply roughly equal amounts of noise to those marginals in order to reduce overall error. The noise scale of the marginals is therefore relatively uniform: the allocation of noise scale selected by *iReduct* or *TwoPhase* does not differ too much from the allocation used by *Dwork*. This explains why the improvement of *iReduct* and *TwoPhase* over *Dwork* is less significant. Figure 9 illustrates the overall error of each algorithm as a function of δ, with $\epsilon = 0.1$. The relative performance of each technique remains the same as in Figure 8.

Regarding computation cost, *iReduct* requires around 15 minutes to generate each set of two-dimensional marginals. This running time is considerably longer than for one-dimensional marginals because *iReduct* must handle more marginal counts in the two-dimensional case.



(a) Brazil (b) USA

Figure 10: Overall Error vs. ϵ (Marginals for Classifier)



(a) Brazil (b) USA

Figure 11: Accuracy of Classifier vs. ϵ

6.5 Results on Naive Bayes Classifier

Our last set of experiments demonstrates that relative error is an important metric for real-world analytic tasks. In these experiments, we consider the task of constructing a Naive Bayes classifier from each dataset. We use *Education* as the class variable and the remaining attributes as feature variables. The construction of such a classifier requires 9 marginals: a one-dimensional marginal on *Education* and 8 two-dimensional marginals, each of which contains *Education* along with another attribute. We apply each algorithm to generate noisy versions of the 9 marginals, and we measure the accuracy of the classifier built from the noisy data[3].

For robustness of measurements, we adopt 10-fold cross-validation in evaluating the accuracy of classification. That is, we first randomly divide the dataset into 10 equal-size subsets. Next, we take the union of 9 subsets as the training set and use the remaining subset for validation. This process is repeated 10 times in total, using each subset for validation exactly once. We report (i) the average overall error of the 10 sets of noisy marginals generated from the 10 training datasets and (ii) the average accuracy of the classifiers built from the 10 sets of noisy marginals.

Figure 10 illustrates the overall error incurred by each algorithm when $\delta = 10^{-4} \cdot |T|$ and ϵ varies. (The input parameter ϵ_1 of *TwoPhase* is set to 0.03 based on an experiment similar to that described in Section 6.2.) Again, *iReduct* and *Oracle* entail the smallest errors in all cases, followed by *TwoPhase*. In addition, *iResamp* consistently incurs higher error than *Dwork* does. Figure 11 shows the accuracy of the classifiers built upon the output of each algorithm. The dashed line in the figure illustrates the accuracy of a classifier constructed from a set of marginals without any injected noise. Observe that methods that incur lower relative errors lead to more accurate classifiers.

7. RELATED WORK

There is a plethora of techniques [1–5, 5, 9–12, 14, 15, 17–20, 22, 26–28, 32] for enforcing ϵ-differential privacy in the publication of various types of data, such as relational tables [9, 17, 21, 32], search logs [15, 20], data mining results [12, 23], and so on. None of these techniques optimizes the relative errors of the published data; instead, they optimize either (i) the variance of the noisy results or (ii) some application-specific metric such as the accuracy of classification [12]. Below, we will discuss several pieces of work that are closely related to (but different from) ours.

Barak et al. [1] devise a technique for publishing marginals of a given dataset. Their objective, however, is not to improve the accuracy of the released marginal counts. Instead, their technique is designed to make the noisy marginals more user-friendly by ensuring that (i) every marginal count is non-negative and (ii) all marginals are *consistent*, i.e. the sum of the counts in one marginal should equal the sum in any other marginal. Kasiviswanathan et al. [19] present a theoretical study on marginal publishing, and they prove several lower bounds on the absolute errors of the noisy marginal counts. Nevertheless, they do not propose any concrete algorithm for releasing marginals.

Blum et al. [4] present a method for releasing one-dimensional data that provides a worst-case guarantee on the absolute errors of range count query results. Hay et al. [17] propose a algorithm that improves the performance bound of Blum et al.'s method, while Xiao et al. [32] devise an approach for multi-dimensional data that achieves a bound similar to Hay et al.'s. Li et al. [21] generalize both Hay et al.'s and Xiao et al.'s approaches, and propose an optimal technique that minimizes the absolute errors of any given set of range count queries. In principle, the techniques above can also be adapted for marginal publishing, since each marginal count can be regarded as the answer to a range count query. However, since those techniques optimize only the absolute errors of the marginal counts, they would incur large relative errors for small counts in the marginals, as is the case for Dwork et al's method.

Besides the aforementioned work on ϵ-differential privacy, there is also a technique by Xiao et al. [31] that is worth mentioning. Given a dataset T, Xiao et al.'s technique generates a collection of noisy datasets $\{T_1^*, T_2^*, \ldots, T_m^*\}$ that satisfies two conditions. First, for any $j < i$, T_j^* contains more noise than T_i^*. Second, $Pr[T \mid T_1^*, \ldots, T_m^*] = Pr[T \mid T_m^*]$, i.e., the combination of all noisy data reveals no more information than the least noisy version T_m^* of T. Our *NoiseDown* algorithm provides a very similar guarantee. Nevertheless, Xiao et al.'s method cannot be used to implement *NoiseDown*, since their method is applicable only when the noisy data is computed using an approach called *randomized response* [30]. In contrast, *NoiseDown* requires that the the noise injected into query outputs follow a Laplace distribution.

8. CONCLUSIONS

Existing techniques for differentially private query publishing optimize only the absolute error of the published query results, which may lead to large relative errors for queries with small answers. This paper remedies the problem with *iReduct*, a novel algorithm for generating differentially private results with reduced relative errors. We present an instantiation of *iReduct* for generating marginals, and we demonstrate its effectiveness through extensive experiments with real data. *iReduct* outperforms several other candidate approaches and, unlike *TwoPhase*, it does not require parameter tuning.

Our technique for generating marginals could be used as a ba-

[3]Following previous work [6], we postprocess each noisy marginal cell y by setting $y = \max\{y + 1, 1\}$ before constructing the classifier from the marginals.

sis for releasing a table of "synthetic" records. Since our computations are differentially private, the table would satisfy a much stronger guarantee than approaches based on anonymity. Furthermore, many counting queries executed on the synthetic records would have low relative errors relative to the original data. There is a tight connection between marginals and log-linear statistical models, as well as algorithms for sampling [7]. However, a challenge remains: the noise introduced for privacy may produce marginals that are *infeasible*, meaning that it is impossible to construct a table that is consistent with the marginals.

Finally, note that the *iReduct* algorithm introduces subtle correlations between the answers that it outputs. This is because the magnitude of the noise injected into each query answer depends not only on the true answer for that query but also on noisy answers to the other queries under consideration. We believe that such correlations may be necessary to minimize relative error. In fact, differentially private techniques that produce correlated answers are not uncommon [1, 4, 15, 17, 21, 29]. Further investigation into the implications of such correlations is left as future work.

9. ACKNOWLEDGMENTS

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APPENDIX

A. ALTERNATE METHOD FOR NOISE REDUCTION

Let q_i be a query, and let y_i' (resp. y_i) be a noisy version of $q_i(T)$ injected with Laplace noise of scale λ_i' (resp. λ_i), such that (i) $\lambda_i' < \lambda_i$, and (ii) y_i' and y_i are generated independently. As discussed in Section 3, we can combine y_i' and y_i to obtain an estimate of $q_i(T)$ that is more accurate than either y_i' or y_i alone. In particular, let $y_i^* = (\lambda_i^2 \cdot y_i' + \lambda_i'^2 \cdot y_i)/(\lambda_i'^2 + \lambda_i^2)$. It can be verified that y_i^* has a variance $2\lambda_i'^2\lambda_i^2/(\lambda_i'^2 + \lambda_i^2)$, which is smaller than the variances of both y_i' and y_i. Furthermore, the variance of y_i^* is the minimum among all unbiased estimators of $q(T)$ given y_i' and y_i.

Nevertheless, generating y_i^* is not cost-effective since, intuitively, it incurs a privacy overhead of $1/\lambda_i' + 1/\lambda_i$ (i.e., the combined cost of producing y_i' and y_i). In contrast, with the same privacy cost, we could have first generated y_i and then applied *NoiseDown* on y_i to obtain a noisy version of $q_i(T)$ with a variance $2\lambda_i'^2\lambda_i^2/(\lambda_i' + \lambda_i)^2$, which is smaller than the variance of y_i^*. This indicates that generating and combining independent noisy versions leads to lower data utility than *NoiseDown* does. To further illustrate this point, we present *iResamp* (iterative resampling), an alternative to *iReduct* that iteratively samples independent noisy query answers instead of using *NoiseDown*. We compare the performance of *iResamp* with that of *iReduct* in the experiments in Section 6.

Figure 12 shows the pseudo-code of *iResamp*. Like *iReduct*, *iResamp* generates initial estimates of query values by adding a large amount of Laplace noise to the true values (Lines 1-5 in Figure 12) and then iteratively refines the noisy query results (Lines 6-21). In each iteration, it also invokes the *PickQueries* black box function to select a set Q_Δ queries for noise reduction. The reduction process, however, is not performed by applying *NoiseDown*, but by generating new noisy answers that are independent of the noisy estimates obtained in previous iterations (Line 15). Specifically, if the most recent noisy answer generated for a query $q_i \in Q_\Delta$ has noise scale λ_i then the new noisy result will have noise scale $\lambda_i/2$, i.e. the amount of noise is reduced by half each time a value is resampled (Lines 8-9). After the new noisy estimate is generated, *iResamp* takes a weighted average of all the noisy results that have been obtained for q_i so far (Line 16) in order to derive a variance-minimized estimate of $q_i(T)$. In particular, if there are k independent noisy samples $y_i^{(j)}$ ($j \in [1, n]$) of $q_i(T)$ such that $y_i^{(j)}$ has noise scale $\lambda_i^{(j)}$, then the variance-minimized unbiased estimate of $q_i(T)$ is:

$$y_i^* = \frac{\sum_{j=1}^{k} y_i^{(j)} / \left(\lambda_i^{(j)} \cdot \lambda_i^{(j)}\right)}{\sum_{j=1}^{k} 1 / \left(\lambda_i^{(j)} \cdot \lambda_i^{(j)}\right)}. \tag{16}$$

The new estimates are then used by the *PickQueries* function for query selections in the next iteration (Line 17).

Algorithm *iResamp* $(T, Q, \delta, \epsilon, \lambda_{max})$

1. let $m = |Q|$ and q_i be the i-th ($i \in [1, m]$) query in Q
2. initialize $\Lambda = [\lambda_1, \ldots, \lambda_m]$, such that $\lambda_i = \lambda_{max}$
3. if $GS(Q, \Lambda) > \epsilon$ then return $\emptyset$
4. Y = *LaplaceNoise*(T, Q, Λ)
5. let $Q' = Q$
6. while $Q' \neq \emptyset$
7. Q_Δ = *PickQueries*(Q', Y, Λ, δ)
8.   for each $i \in [1, m]$
9.   if $q_i \in Q_\Delta$ then $\lambda_i = \lambda_i/2$
10.   let $\Lambda' = [\lambda_1', \ldots, \lambda_m']$, such that $\lambda_i' = 1/(2/\lambda_i - 1/\lambda_{max})$
11.   if $GS(Q, \Lambda') \leq \epsilon$
12.   for each $i \in [1, m]$
13.   if $q_i \in Q_\Delta$
14.   let $y_i^{(1)}, \ldots y_i^{(k-1)}$ be the noisy versions of $q_i(T)$ that have been generated in the previous iterations
15. $y_i^{(k)}$ = *LaplaceNoise*(T, q_i, λ_i)
16.   calculate y_i^* based on $y_i^{(1)}, \ldots y_i^{(k)}$ according to Eqn. 16
17. $y_i = y_i^*$
18.   else
19.   for each $i \in [1, m]$
20.   if $q_i \in Q_\Delta$ then $\lambda_i = 2\lambda_i$
21. $Q' = Q' \setminus Q_\Delta$
22. return Y

Figure 12: The *iResamp* Algorithm

To show that the algorithm guarantees ϵ-differential privacy, it suffices to show that the set of all noisy query estimates (generated in all iterations) of all the marginals is ϵ-differentially private. Notice that for any query q_i, the k-th noisy answer $y_i^{(k)}$ for q_i (generated by *iResamp*) has noise scale $\lambda_{max}/2^{k-1}$. This is true because the first noisy answer $y_i^{(1)}$ has noise scale λ_{max} and the noise scale of $y_i^{(j)}$ equals half of the noise scale of $y_i^{(j-1)}$ ($j > 1$). By summing the terms of the resulting geometric series, it can be verified that the privacy cost of producing $y_i^{(1)}, \ldots, y_i^{(k)}$ is bounded above by the cost of generating a noisy answer for q_i with Laplace noise of scale $2/\lambda_i - 1/\lambda_{max}$, where λ_i denotes the noise scale of $y_i^{(k)}$. Therefore, to ensure that all noisy results produced by *iResamp* are ϵ-differentially private, it suffices to guarantee that the sequence Q of input queries has generalized sensitivity at most ϵ with respect to $\Lambda' = [\lambda_1', \ldots, \lambda_m']$, where $\lambda_i' = 1/(2/\lambda_i - 1/\lambda_{max})$. This explains the rationale of Lines 10 and 11 in Figure 12. The above argument concludes our proof of the following theorem:

THEOREM 3 (PRIVACY OF *iResamp*). *iResamp ensures ϵ'-differential privacy whenever its input parameter ϵ satisfies $\epsilon \leq \epsilon'$.* □