Solutions for Selected Problems from Aluffi's *Algebra: Chapter 0*

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Contents

Chapter I

Preliminaries: Set theory and categories

1 Naive set theory

1.1 Locate a discussion of Russell's paradox, and understand it.

1.2 Prove that if $\sim$ is an equivalence relation on a set S, then the corresponding family $\mathcal{P}_\sim$ defined in §1.5 is indeed a partition of S: that is, its elements are nonempty, disjoint, and their union is S. [§1.5]

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small test 3 $\square$

1.3 Given a partition $\mathcal{P}$ on a set S, show how to define a relation $\sim$ on S such that

$\mathcal{P}$ is the corresponding partition. [§1.5]

1.4 How many different equivalence relations may be defined on the set $\{1, 2, 3\}$?

1.5 Give an example of a relation that is reflexive and symmetric but not transitive. What happens if you attempt to use this relation to define a partition on the set? (Hint: Thinking about the second question will help you answer the first one.)

1.6 Define a relation $\sim$ on the set $\mathbb{R}$ of real numbers by setting $a \sim b \iff b - a \in \mathbb{Z}$. Prove that this is an equivalence relation, and find a 'compelling' description for $\mathbb{R}/\sim$. Do the same for the relation $\approx$ on the plane $\mathbb{R} \times \mathbb{R}$ defined by declaring $(a_1, a_2) \approx (b_1, b_2) \iff b_1 - a_1 \in \mathbb{Z}$ and $b_2 - a_2 \in \mathbb{Z}$. [§II.8.1, II.8.10]

2 Functions between sets

2.1 How many different bijections are there between a set S with n elements and itself? [§II.2.1]

2.2 Prove statement (2) in Proposition 2.1. You may assume that given a family of disjoint nonempty subsets of a set, there is a way to choose one element in each member of the family[13]. [§2.5, V.3.3]

2.3 Prove that the inverse of a bijection is a bijection and that the composition of two bijections is a bijection.

2.4 Prove that 'isomorphism' is an equivalence relation (on any set of sets). [§4.1]

2.5 Formulate a notion of *epimorphism*, in the style of the notion of *monomorphism* seen in §2.6, and prove a result analogous to Proposition 2.3, for epimorphisms and surjections. [§2.6, §4.2]

Solution: The dual notion is as follows: an *epimorphism* is a function $f : A \to B$ such that for any two functions $\alpha', \alpha'' : B \to Z$, if $\alpha' \circ f = \alpha'' \circ f$, then $\alpha' = \alpha''$. The result analogous to Proposition 2.3 is that $f : A \to B$ is an epimorphism if and only if it is surjective.

First assume that f is surjective and $\alpha' \circ f = \alpha'' \circ f$. We must show that for any $b \in B$, $\alpha'(b) = \alpha''(b)$. By surjectivity there exists $a \in A$ such that $f(a) = b$. Then $\alpha'(b) = \alpha'(f(a)) = \alpha''(f(a)) = \alpha''(b)$.

Conversely, assume that f is *not* surjective, so that $\operatorname{im} f$ is not the whole of B. Then we can define functions $\alpha', \alpha'' : B \to \{0,1\}$ that agree on $\operatorname{im} f$ but differ on some $b \in B \setminus \operatorname{im} f$. Specifically, let $\alpha'(b) = 1$ and $\alpha''(b) = 0$, while α' and α'' agree on all other points in $\operatorname{im} f$. Then $\alpha' \circ f = \alpha'' \circ f$, but $\alpha' \neq \alpha''$, showing that f is not an epimorphism. □

2.6 With notation as in Example 2.4, explain how any function $f : A \to B$ determines a section of π_A.

2.7 Let $f : A \to B$ be any function. Prove that the graph Γ_f of f is isomorphic to A.

2.8 Describe as explicitly as you can all terms in the canonical decomposition (cf. §2.8) of the function $\mathbb{R} \to \mathbb{C}$ defined by $r \mapsto e^{2\pi i r}$. (This exercise matches one assigned previously. Which one?)

2.9 Show that if $A' \cong A''$ and $B' \cong B''$, and further $A' \cap B' = \emptyset$ and $A'' \cap B'' = \emptyset$, then $A' \cup B' \cong A'' \cup B''$. Conclude that the operation $\amalg B$ (as described in §1.4) is well-defined up to isomorphism (cf. §2.9). [§2.9, 5.7]

2.10 Show that if A and B are finite sets, then $|B^A| = |B|^{|A|}$. [§2.1, 2.11, §II.4.1]

2.11 In view of Exercise 2.10, it is not unreasonable to use 2^A to denote the set of functions from an arbitrary set A to a set with 2 elements (say $\{0,1\}$). Prove that there is a bijection between 2^A and the *power set* of A (cf. §1.2). [§1.2, III.2.3]

3 Categories

3.1 Let C be a category. Consider a structure C^{op} with

- $\mathrm{Obj}(\mathsf{C}^{\mathrm{op}}) := \mathrm{Obj}(\mathsf{C})$;
- for A, B objects of C^{op} (hence objects of C), $\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, B) := \mathrm{Hom}_{\mathsf{C}}(B, A)$.

Show how to make this into a category (that is, define composition of morphisms in C^{op} and verify the properties listed in §3.1).

Intuitively, the 'opposite' category C^{op} is simply obtained by 'reversing all the arrows' in C. [§5.1, §VIII.1.1, §IX.1.2, IX.1.10]

3.2 If A is a finite set, how large is $\mathrm{End}_{\mathsf{Set}}(A)$?

3.3 Formulate precisely what it means to say that 1_A is an identity with respect to composition in Example 3.3, and prove this assertion. [§3.2]

3.4 Can we define a category in the style of Example 3.3 using the relation $<$ on the set $\mathbb{Z}$?

3.5 Explain in what sense Example 3.4 is an instance of the categories considered in Example 3.3. [§3.2]

3.6 (Assuming some familiarity with linear algebra.) Define a category V by taking $\mathrm{Obj}(\mathsf{V}) = \mathbb{N}$ and letting $\mathrm{Hom}_{\mathsf{V}}(n, m) =$ the set of $m \times n$ matrices with real entries, for all $n, m \in \mathbb{N}$. (I will leave the reader the task of making sense of a matrix with 0 rows or columns.) Use product of matrices to define composition. Does this category 'feel' familiar? [§VI.2.1, §VIII.1.3]

3.7 Define carefully objects and morphisms in Example 3.7, and draw the diagram corresponding to composition. [§3.2]

3.8 A *subcategory* C' of a category C consists of a collection of objects of C with sets of morphisms $\mathrm{Hom}_{\mathsf{C}'}(A, B) \subseteq \mathrm{Hom}_{\mathsf{C}}(A, B)$ for all objects A, B in $\mathrm{Obj}(\mathsf{C}')$, such that identities and compositions in C make C' into a category. A subcategory C' is full if $\mathrm{Hom}_{\mathsf{C}'}(A, B) = \mathrm{Hom}_{\mathsf{C}}(A, B)$ for all A, B in $\mathrm{Obj}(\mathsf{C}')$. Construct a category of *infinite sets* and explain how it may be viewed as a full subcategory of Set. [4.4, §VI.1.1, §VIII.1.3]

3.9 An alternative to the notion of *multiset* introduced in §2.2 is obtained by considering sets endowed with equivalence relations; equivalent elements are taken to be multiple instances of elements 'of the same kind'. Define a notion of morphism between such enhanced sets, obtaining a category MSet containing (a 'copy' of) Set as a full subcategory. (There may be more than one reasonable way to do this! This is intentionally an open-ended exercise.) Which objects in MSet determine ordinary multisets as defined in §2.2 and how? Spell out what a morphism of multisets would be from this point of view. (There are several natural notions of morphisms of multisets. Try to define morphisms in MSet so that the notion you obtain for ordinary multisets captures your intuitive understanding of these objects.) [§2.2, §3.2, 4.5]

3.10 Since the objects of a category C are not (necessarily interpreted as) sets, it is not clear how to make sense of a notion of 'subobject' in general. In some situations it *does* make sense to talk about subobjects, and the subobjects of any given object A in C are in one-to-one correspondence with the morphisms $A \to \Omega$ for a fixed, special object Ω of C, called a *subobject classifier*. Show that Set has a subobject classifier.

3.11 Draw the relevant diagrams and define composition and identities for the category $\mathsf{C}^{A,B}$ mentioned in Example 3.9. Do the same for the category $\mathsf{C}^{\alpha,\beta}$ mentioned in Example 3.10. [§5.5, 5.12]

4 Morphisms

4.1 Composition is defined for *two* morphisms. If more than two morphisms are given, e.g.,

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{i} E,$$

then one may compose them in several ways, for example:

$$(ih)(gf), \quad (i(hg))f, \quad i((hg)f), \quad \text{etc.}$$

so that at every step one is only composing two morphisms. Prove that the result of any such nested composition is independent of the placement of the parentheses. (Hint: Use induction on n to show that any such choice for $f_n f_{n-1} \cdots f_1$ equals

$$((\cdots((f_n f_{n-1}) f_{n-2}) \cdots) f_1).$$

Carefully working out the case $n = 5$ is helpful.) [§4.1, §II.1.3]

4.2 In Example 3.3 we have seen how to construct a category from a set endowed with a relation, provided this latter is reflexive and transitive. For what types of relations is the corresponding category a groupoid (cf. Example 4.6)? [§4.1]

4.3 Let A, B be objects of a category C, and let $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ be a morphism.

- Prove that if f has a right-inverse, then f is an epimorphism.
- Show that the converse does not hold, by giving an explicit example of a category and an epimorphism without a right-inverse.

4.4 Prove that the composition of two monomorphisms is a monomorphism. Deduce that one can define a subcategory C_{mono} of a category C by taking the objects as in C and defining $\mathrm{Hom}_{\mathsf{C}_{\text{mono}}}(A, B)$ to be the subset of $\mathrm{Hom}_{\mathsf{C}}(A, B)$

consisting of monomorphisms, for all objects A, B. (Cf. Exercise 3.8; of course, in general C_{mono} is not full in C.) Do the same for epimorphisms. Can you define a subcategory $\mathsf{C}_{\text{nonmono}}$ of C by restricting to morphisms that are *not* monomorphisms?

4.5 Give a concrete description of monomorphisms and epimorphisms in the category MSet you constructed in Exercise 3.9. (Your answer will depend on the notion of morphism you defined in that exercise!)

5 Universal properties

5.1 Prove that a final object in a category C is initial in the opposite category C^{op} (cf. Exercise 3.1).

5.2 Prove that $\emptyset$ is the unique initial object in Set. [§5.1]

5.3 Prove that final objects are unique up to isomorphism. [§5.1]

5.4 What are initial and final objects in the category of 'pointed sets' (Example 3.8)? Are they unique?

5.5 What are the final objects in the category considered in §5.3? [§5.3]

5.6 Consider the category corresponding to endowing (as in Example 3.3) the set $\mathbb{Z}^+$ of positive integers with the divisibility relation. Thus there is exactly one morphism $d \to m$ in this category if and only if d divides m without remainder; there is no morphism between d and m otherwise. Show that this category has products and coproducts. What are their 'conventional' names? [§VII.5.1]

5.7 Redo Exercise 2.9, this time using Proposition 5.4.

5.8 Show that in every category C the products $A \times B$ and $B \times A$ are isomorphic, if they exist. (Hint: Observe that they both satisfy the universal property for the product of A and B; then use Proposition 5.4.)

5.9 Let C be a category with products. Find a reasonable candidate for the universal property that the product $A \times B \times C$ of three objects of C ought to satisfy, and prove that both $(A \times B) \times C$ and $A \times (B \times C)$ satisfy this universal property. Deduce that $(A \times B) \times C$ and $A \times (B \times C)$ are necessarily isomorphic.

5.10 Push the envelope a little further still, and define products and coproducts for *families* (i.e., indexed sets) of objects of a category.

Do these exist in Set?

It is common to denote the product $\underbrace{A \times \cdots \times A}_{n \text{ times}}$ by A^n.

5.11 Let A, resp. B be a set, endowed with an equivalence relation $\sim_A$, resp. $\sim_B$. Define a relation $\sim$ on $A \times B$ by setting

$$(a_1, b_1) \sim (a_2, b_2) \iff a_1 \sim_A a_2 \text{ and } b_1 \sim_B b_2.$$

(This is immediately seen to be an equivalence relation.)

- Use the universal property for quotients (§5.3) to establish that there are canonical quotient maps $q_A : A \to A/\sim_A$, $q_B : B \to B/\sim_B$, and $q : A \times B \to (A \times B)/\sim_{A\times B}$, and that these induce functions $(A \times B)/\sim_{A\times B} \to A/\sim_A$ and $(A \times B)/\sim_{A\times B} \to B/\sim_B$.
- Prove that $(A \times B)/\sim_{A\times B}$, together with these induced functions, satisfies the universal property for the product of $A/\sim_A$ and $B/\sim_B$.
- Conclude (without further work) that $(A \times B)/\sim_{A\times B} \cong (A/\sim_A) \times (B/\sim_B)$.

5.12 Define the notions of *fibered products* and *fibered coproducts*, as terminal objects of the categories $\mathsf{C}_{\alpha,\beta}$, $\mathsf{C}^{\alpha,\beta}$ considered in Example 3.10 (cf. also Exercise 3.11), by stating carefully the corresponding universal properties.

As it happens, Set has both fibered products and coproducts. Define these objects 'concretely', in terms of naive set theory. [II.3.9, III.6.10, III.6.11]

Chapter II

Groups, first encounter

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Chapter III

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Groups, second encounter

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