Homework 3

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Problem 1:

1. 求

即 ~~计算~~ $b^T x + c\sum x_i \ln x_i$ 取min值时的 x

令 $b=(b_1, \cdots b_n)$

原式 $= \sum(c x_i \ln x_i + b_i x_i)$ 约束条件为 $1^T x = 1$

~~故要求~~ $\cancel{f_0(x) = \sum(c x_i \ln x_i + b_i x_i)}$

记 ~~minimize~~

subject to $\cancel{1^T x = 1}$ ~~的最优解~~

~~时的最优解为~~ $\cancel{\hat{x}}$

令 $\Phi(x, \lambda) = f_0(x) + \lambda(\sum x_i - 1) = \sum(c x_i \ln x_i + b_i x_i + \lambda x_i) - \lambda$

$$\begin{cases} \frac{\partial \Phi}{\partial x_i} = 0 \\ \frac{\partial \Phi}{\partial \lambda} = 0 \end{cases} \rightarrow \begin{cases} x_i = e^{-\frac{b_i+\lambda+c}{c}} \\ \sum x_i = 1. \end{cases} \rightarrow x_i = e^{-\frac{b_i + c\ln(\sum e^{-\frac{b_i}{c}})}{c}}$$

~~故~~ $\cancel{x = (e^{-\frac{b_i+\lambda+c}{c}}}$

故 $x = \left(e^{-\frac{b_1 + c\ln(\sum e^{-\frac{b_i}{c}})}{c}}, e^{-\frac{b_2 + c\ln(\sum e^{-\frac{b_i}{c}})}{c}} \cdots\cdots\right)$

Problem 2:

2.

(1). $\lambda=0$ 时 $g(\lambda)=\inf c^T x=-\infty$

$\lambda>0$ 时 $g(\lambda)=\inf\left(c^T x+\lambda f(x)\right)=\lambda \inf\left(\left(\frac{c}{\lambda}\right)^T x+f(x)\right)$

$=-\lambda f_1^*\left(-\frac{c}{\lambda}\right)$

故对偶问题为 minimize $-\lambda f_1^*\left(\frac{-c}{\lambda}\right)$

subject to $\lambda \geq 0$

(2). ~~∵ g(λ) 关于 x 无下界.~~

~~故对偶函数~~

(2). ∵ g 是凹函数 故 -g 为凸函数.

故对偶问题是凸的

Problem 3:

3.

(1). $L(x,\lambda) = x_1^2 + x_2^2 + \lambda_1\left[(x_1-1)^2 + (x_2-1)^2\right] + \lambda_2\left[(x_1-1)^2 + (x_2+1)^2\right]$

(2). 由 SCQ 知. 若 $\exists x$ 使 $\begin{cases}(x_1-1)^2 + (x_2-1)^2 \le 2 \\ (x_1-1)^2 + (x_2+1)^2 \le 2\end{cases}$ 成立即满足强对偶性

令 $x = (1,0)^T$ 显然都满足. 故此问题满足强对偶性

(3) KKT Condition:

$(x_1-1)^2 + (x_2-1)^2 \le 2$

$(x_1-1)^2 + (x_2+1)^2 \le 2$

$\lambda_1 \ge 0, \quad \lambda_2 \ge 0.$

$\lambda_1\left[(x_1-1)^2 + (x_2-1)^2 - 2\right] = 0, \quad \lambda_2\left[(x_1-1)^2 + (x_2+1)^2 - 2\right] = 0$

$\nabla(x_1^2 + x_2^2) + \lambda_1 \nabla\left[(x_1-1)^2 + (x_2-1)^2 - 2\right] + \lambda_2 \nabla\left[(x_1-1)^2 + (x_2+1)^2 - 2\right] = 0$

(即 $2x_1 + 2\lambda_1(x_1-1) + 2\lambda_2(x_1-1) = 0, \quad 2x_2 + 2\lambda_1(x_2-1) + 2\lambda_2(x_2+1) = 0$)

Problem 4:

4. $trX = \sum \lambda_i(X) = r$ $\quad\lambda_i$ 为X的第i个特征值

$tr(AX) = \sum \lambda_j(AX)$ $\quad\lambda_j$ 为AX的第j个特征值

取X为对角阵 此时 $X=\begin{bmatrix}1 & & & \\ & 1 & & \\ & & 0 & \\ & & & \ddots \\ & & & & 1\end{bmatrix}$ (r个1).

此时AX变为A中n-r列置为0.

此时AX中特征值至多有r个非零不同的

这实际上也是A的r个特征值.

f(A)是求A最大的r个特征值之和.

而 $\max(AX)$ 也是 tr. 相同

Problem 5:

5. 原问题为

$$\min \ \log\det X^{-1}$$

$$\text{s.t.} \ A_i^T X A_i \preceq B_i \quad i=1,\dots,m.$$

对偶问题:

$$\text{maximize} \ \log\det\left(\sum_{i=1}^{m} \lambda_i A_i A_i^T\right) - 1^T\lambda + n.$$

$$\text{subject to} \ \lambda \succeq 0.$$