博弈论：作业一

问题一 Let $\{a_n\}$ be a sequence of positive real number. Denote by $S_n = \sum_{i=1}^{n} a_i$. If $S_{n+1} \geq 2S_n$, then there exists a constant $c > 0$, such that $a_n \geq 2^n c$ for every positive n.

解. 由 $S_{n+1} \geq 2S_n$ 可得 $S_n \geq 2^{n-1}S_1$，从而可得 $S_{n+1} - S_n \geq S_n \geq 2^{n-1}S_1 = 2^{n-1}a_1$，即 $a_{n+1} \geq 2^{n-1}a_1$，即 $a_n \geq 2^{n-2}a_1$。因此，取 $c = a_1/4$，则对于任意正整数 n，$a_n \geq 2^n c$。

问题二 Suppose that $(1, 1, -1)$ is an eigenvector of matrix

$$\begin{bmatrix} 2 & -1 & b \\ 5 & a & 3 \\ -1 & 2 & -1 \end{bmatrix}$$

Solve a, b, and the corresponding eigenvalue.

解. 令特征向量 $(1, 1, -1)$ 对应的特征值为 λ，则可得方程

$$\begin{bmatrix} 2 & -1 & b \\ 5 & a & 3 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda \\ -\lambda \end{bmatrix}$$

求解方程，得到 $a = -4$，$b = 3$，对应的特征值 $\lambda = -2$。

问题三 For $\epsilon \in [0, 1]$, prove that

$$\frac{1}{2}\left(1 + \sqrt{1 + 4\epsilon^2}\right) e^{1 - \sqrt{1+4\epsilon^2}} \leq e^{-(\epsilon^2 - \epsilon^3)/2}$$

解. 不等式两边取对数，则原问题转化为证明

$$1 - \sqrt{1 + 4\epsilon^2} + \log\left(1 + \sqrt{1 + 4\epsilon^2}\right) - \log 2 \leq -\frac{\epsilon^2 - \epsilon^3}{2}$$

令 $f(\epsilon) = 1 - \sqrt{1 + 4\epsilon^2} + \log\left(1 + \sqrt{1 + 4\epsilon^2}\right) - \log 2 + \frac{\epsilon^2 - \epsilon^3}{2}$，$\epsilon \in [0, 1]$，则 $f(\epsilon) = 0$，且

$$\begin{aligned} f'(\epsilon) &= \epsilon - \frac{3}{2}\epsilon^2 - \frac{4\epsilon}{\sqrt{1 + 4\epsilon^2}} + \frac{4\epsilon}{1 + 4\epsilon^2 + \sqrt{1 + 4\epsilon^2}} \\ &= \frac{\epsilon\left(-3\epsilon - 3\epsilon\sqrt{1 + 4\epsilon^2} + 2\sqrt{1 + 4\epsilon^2} - 6\right)}{2\left(1 + \sqrt{1 + 4\epsilon^2}\right)} \end{aligned}$$

注: $-\frac{4\epsilon}{\sqrt{1+4\epsilon^2}} + \frac{4\epsilon}{1+4\epsilon^2+\sqrt{1+4\epsilon^2}} = \frac{-4\epsilon(1+4\epsilon^2)}{\sqrt{1+4\epsilon^2}\left(1+4\epsilon^2+\sqrt{1+4\epsilon^2}\right)} = \frac{-4\epsilon\sqrt{1+4\epsilon^2}}{\left(1+\sqrt{1+4\epsilon^2}\right)\sqrt{1+4\epsilon^2}} = \frac{-4\epsilon}{1+\sqrt{1+4\epsilon^2}}$

当 $\epsilon \in [0, 1]$ 时，$-3\epsilon - 3\epsilon\sqrt{1 + 4\epsilon^2} + 2\sqrt{1 + 4\epsilon^2} - 6 \leq -3\epsilon - 3\epsilon\sqrt{1 + 4\epsilon^2} + 2\sqrt{5} - 6 \leq 0$，即，$f'(\epsilon) \leq 0$。因此，当 $\epsilon \in [0, 1]$ 时，$f(\epsilon) \leq 0$，则原式得证。