Solution.

最大化收益

$$\max_{q_i \geq 0} u_i(q_1, \dots, q_n) = \max_{q_i \geq 0}(a - b(q_1 + \dots + q_n) - c)q_i$$

令 $\dfrac{\partial u_i}{\partial q_i} = 0$, 得

$$a - b\sum_{j \neq i} q_j - 2bq_i - c = 0$$

$$q_i = \frac{a - c - b\sum_{j \neq i} q_j}{2b}$$

类似地，可以得到 Best Response Correspondence: $B_i(q_{-i}) = \max\left(0, \frac{a-c-b\sum_{j \neq i} q_j}{2b}\right)$

假设$\{q_1^*, \dots, q_n^*\}$达到纳什均衡, 根据轮换对称性可知, $q_1^* = q_2^* = \dots = q_n^*$

若$q_i^* = 0$, 则不满足纳什均衡的条件. 故$q_i^* > 0$

于是由

$$q_i^* = \frac{a - c - b\sum_{j \neq i} q_j^*}{2b}$$

得到

$$q_i^* = \frac{a - c}{(n+1)b}, \quad i = 1, 2, \dots, n$$

相应地

$$u_i = \frac{(a-c)^2}{(n+1)^2 b}$$

- Best Response Correspondence: $B_i(q_{-i}) = \max(0, (a-c-bq_{-i})/2b)$

$\because$ 若 $q_{-i} \geq (a-c)/b$，则任意 $q_i>0$ 都有 $u_i(q_i, q_{-i}) \leq 0$，则 $q_i=0$

若 $q_{-i} < (a-c)/b$，则：

$$u_i(q_i, q_{-i}) = (a-c-b(q_i+q_{-i}))q_i$$

$$\frac{\partial u_i}{\partial q_i} = a-c-bq_{-i}-2bq_i=0$$

$$\Rightarrow q_i = (a-c-bq_{-i})/2b$$

- The Nash equilibria: $\{(\frac{a-c}{(n+1)b}, \frac{a-c}{(n+1)b}, \cdots, \frac{a-c}{(n+1)b})\}$

$\because$ $q_i^* = B_i(q_{-i}^*) = (a-c-bq_{-i}^*)/2b$

$\therefore$ 写作：

$$q_1+\frac{1}{2}q_2+\frac{1}{2}q_3+\cdots+\frac{1}{2}q_n=\frac{a-c}{2b} \quad \cdots ①$$

$$\frac{1}{2}q_1+q_2+\frac{1}{2}q_3+\cdots+\frac{1}{2}q_n=\frac{a-c}{2b} \quad \cdots ②$$

$$\vdots$$

$$\frac{1}{2}q_1+\frac{1}{2}q_2+\frac{1}{2}q_3+\cdots+q_n=\frac{a-c}{2b} \quad \cdots ⓝ$$

可得：

$$\left.\begin{array}{l} ①-② \Rightarrow q_1^*=q_2^* \\ ②-③ \Rightarrow q_2^*=q_3^* \\ \vdots \\ (n-1)-(n) \Rightarrow q_{n-1}^*=q_n^* \end{array}\right\} q_1^*=q_2^*=q_3^*=\cdots=q_n^*$$

解得：

$$q_1^*=q_2^*=q_3^*=\cdots=q_n^*=\frac{a-c}{(n+1)b}$$

- 证明 $q_i^*>0$：不妨设 $q_1^*=0$，则 $\sum_{i=2}^{n} q_i \geq (a-c)/b$

① 若 $\sum_{i=2}^{n} q_i > (a-c)/b$，则任意 $i\,(i\neq 1)$ 有 $u_i(q_i, q_{-i})<0$，将减小 q_i，总会调整至 $\sum_{i=2}^{n} q_i=\frac{a-c}{b}$

② 若 $\sum_{i=2}^{n} q_i = (a-c)/b$，同上有 $u_i(q_i, q_{-i})=0$，也会减小 q_i 使收益增大，终将导致 $\sum_{i=2}^{n} q_i < \frac{a-c}{b}$

$\Rightarrow q_i^* \neq 0$，则 $q_i^*>0$ 得证