Multivariable Calculus

MATH2007

2.1 Vector Analysis and Parametrization

WITS UNIVERSITY

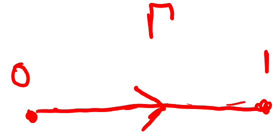
School of Mathematics

Definition (2.1.1)**.** Let $\underline{r} : [a, b] \to \mathbb{R}^n$ and where $a < b$, then a set of the form

$$\Gamma = \{\underline{r}(t) \mid t \in [a, b]\}$$

is called a **curve** and the function $\underline{r}(t), t \in [a, b]$, is called a **parametrisation** of Γ. If Γ has a direction or orientation (usually indicated by an arrow along the curve), then Γ is called a **path** or **oriented curve**.

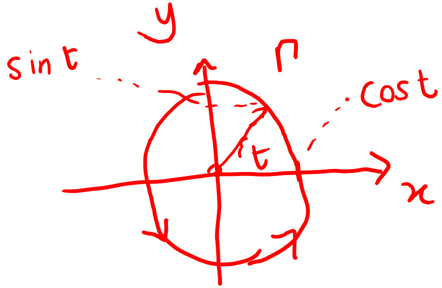
Example. Straight line in $\mathbb{R}$ with orientation from 0 to 1:



$r(t) = t \qquad t \in [0,1]$

$\Gamma = \{ t : \quad t \in [0,1] \}$

Example. Unit circle oriented in the anticlockwise direction:



in $\mathbb{R}^2$ $\quad \underline{r}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} \qquad t \in [0, 2\pi]$

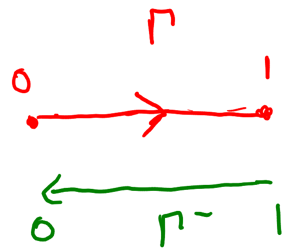
$\Gamma = \left\{ \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} : \quad t \in [0, 2\pi] \right\}$

Definition (2.1.1)**.** Let Γ be a Oriented path parametrised by $\underline{r}(t), t \in [a, b]$, if the direction $\underline{r}(t)$ moves as t increases is the same as the direction associated with Γ, then $\underline{r}(t), t \in [a, b]$, is said to be an **orientation preserving** parametrisation of Γ. Otherwise $\underline{r}(t), t \in [a, b]$, is an **orientation reversing** parametrisation of Γ. If Γ is an oriented curve, we denote by Γ^{-}, the curve Γ but with reversed orientation.

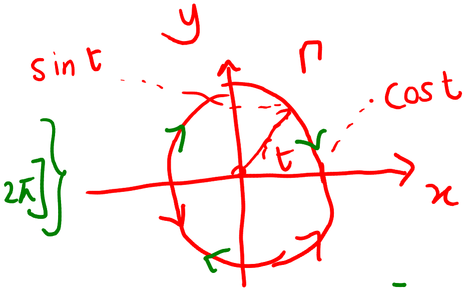
Example. Straight line in $\mathbb{R}$ with orientation from 0 to 1:

$r(t) = t \quad t \in [0,1] \qquad r(t) = 1 - t \quad t \in [0,1]$

$\Gamma = \{ t : t \in [0,1] \} \qquad \Gamma^{-} = \{ 1-t : t \in [0,1] \}$

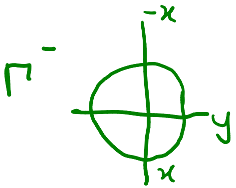


Example. Unit circle oriented in the anticlockwise direction:



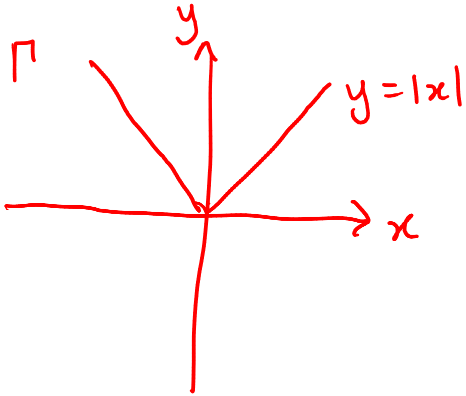
$\underline{r}(t) = \begin{pmatrix} \cos(-t) \\ \sin(-t) \end{pmatrix} \quad t \in [0, 2\pi] \qquad \Gamma^{-} = \left\{ \begin{pmatrix} \cos(-t) \\ \sin(-t) \end{pmatrix} : t \in [0, 2\pi] \right\}$

or $\underline{r}(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} \quad t \in [0, 2\pi] \qquad \Gamma^{-} = \{ \dots \}$



Definition (2.1.1)**.** A curve Γ is said to be **piecewise smooth**, if the parametrisation $\{\, \underline{r}(t) \,|\, t \in [a,b] \,\}$ of Γ, where $\underline{r}'(t)$ exists and is non-zero except for at most finitely many points of Γ. The values of $\underline{r}(t)$ where either $\underline{r}'(t)$ does not exist or is zero are called **singular** points of Γ.

Example. $y = |x|$, $x \in [-1, 1]$: in $\mathbb{R}^2$



$$\underline{r}(t) = \begin{pmatrix} t \\ |t| \end{pmatrix} \quad t \in [-1, 1]$$

$$\underline{r}'(t) = \begin{cases} \begin{pmatrix} 1 \\ 1 \end{pmatrix} & t \in (0, 1] \\ \begin{pmatrix} 1 \\ -1 \end{pmatrix} & t \in [-1, 0) \end{cases}$$

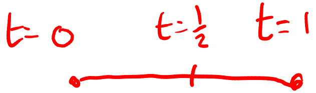
Singular point at $t = 0$

$$\Gamma = \left\{ \begin{pmatrix} t \\ |t| \end{pmatrix} \,\middle|\, t \in [-1, 1] \right\}$$

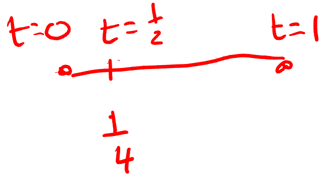
Are parametrizations unique?

Example. Straight line in $\mathbb{R}$ with orientation from 0 to 1:

$$\underline{r}(t) = \{ t : t \in [0,1] \}$$



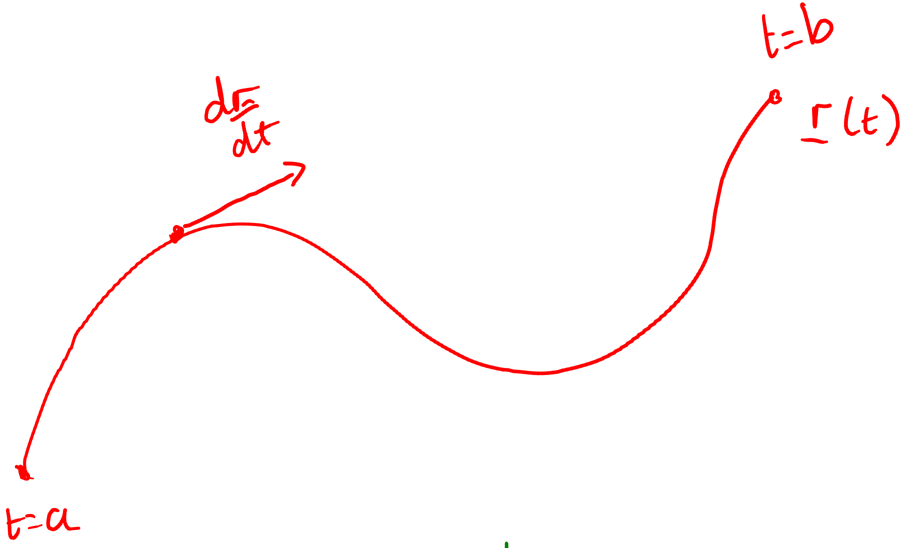
$$\underline{r}(t) = \{ t^2 : t \in [0,1] \}$$



$$\underline{r}(t) = \{ \sin(t) : t \in [0,\tfrac{\pi}{2}] \}$$

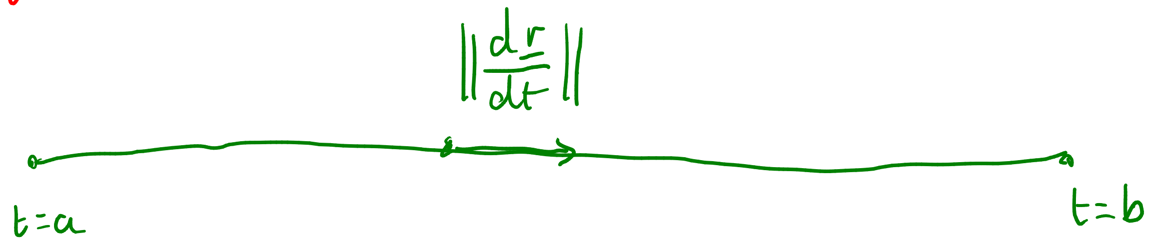
A physical interpretation:

If $\underline{r}(t)$ is a path, then $\dfrac{d\underline{r}(t)}{dt}$ is the velocity and $\left\|\dfrac{d\underline{r}(t)}{dt}\right\|$ is the speed of an object moving along the path with position $\underline{r}(t)$ at time t.



speed: how fast

velocity: how fast and in what direction

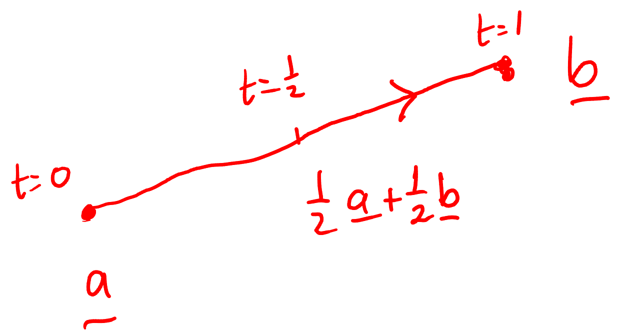


Some standard parametrisations are as follows:

1. The line segment from $\underline{a}$ to $\underline{b}$: in $\mathbb{R}^n$

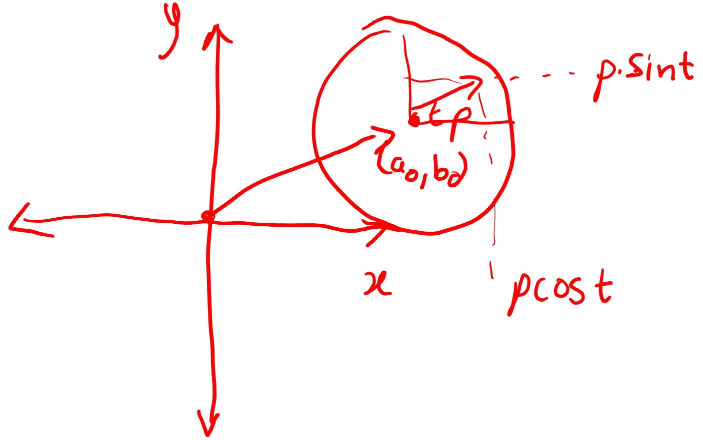
$$\underline{r}(t) = (1-t)\underline{a} + t\underline{b} \qquad t \in [0,1]$$

$$= \underline{a} + t(\underline{b} - \underline{a})$$



2. The circle centre (a_0, b_0) with radius ρ:

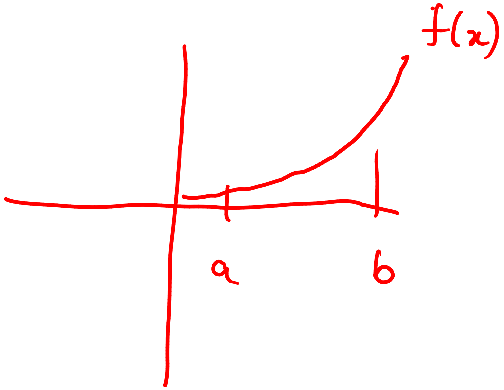
$$\underline{r}(t) = \begin{pmatrix} \rho\cos t \\ \rho\sin t \end{pmatrix} + \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \qquad t \in [0, 2\pi]$$



Some standard parametrisations are as follows:

3. The explicit curve $y = f(x)$ where x goes from a to b and $a < b$:

$$\underline{r}(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix} \qquad t \in [a,b]$$



4. The explicit curve $x = f(y)$ where y goes from a to b and $a < b$:

$$\underline{r}(t) = \begin{pmatrix} f(t) \\ t \end{pmatrix} \qquad t \in [a,b]$$