ROBOTICS

KINEMATICS USING THE JACOBIAN – PART 2

MORE COMPLEX ARMS

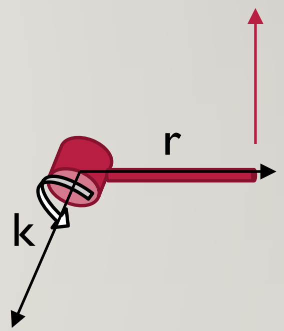
- We saw what to do with a 2 jointed, 2D arm in the last lecture
- What happens if we have many joints operating in a 3D space?
- Sounds a lot harder…

A JOINT IN 3D

- Let's look at what happens with a single joint
- This joint can be either prismatic:
  - No effect on angular velocity of the end effector
  - The end effector moves in the z direction of the joint extension
  - Let k be a unit vector in the z direction
  - $v = \dot{q}k$
  - $\omega = 0$
- Or revolute

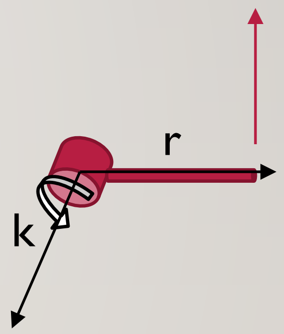
A JOINT IN 3D

- Let's look at what happens with a single joint
- This joint can be either prismatic:
  - $v \ = \dot{q}k$
  - $\omega = \ 0$
- Or revolute
  - The end effector rotates around the z-axis
  - It moves in a direction that is perpendicular to the z axis and the vector joining the joint to the end effector



A JOINT IN 3D

- Let’s look at what happens with a single joint
- This joint can be either prismatic:
  - $v = \dot{q}k$
  - $\omega = 0$
- Or revolute
  - The end effector rotates around the z-axis
  - It moves in a direction that is perpendicular to the z axis and the vector joining the joint to the end effector
  - $\omega = \dot{q}k$
  - $v = \dot{q}k \times r$



A JOINT IN 3D

- But what happens if we have many joints?
- How do we figure out the effect of the velocity of joint i on the end effector velocity?
  - Transformation matrix!
- We need to transform the unit vector k to the world frame
- All that means is that we need to apply the relevant transformation matrix T_{i-1}^{0} to k
- $\begin{bmatrix} x_x & y_x & z_x & o_x \\ x_y & y_y & z_y & o_y \\ x_z & y_z & z_z & o_x \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} z_x \\ z_y \\ z_z \\ 0 \end{bmatrix}$

ANGULAR VELOCITY JACOBIAN

- Remember the angular velocity Jacobian should accomplish the following:
- $\omega = \begin{bmatrix} J_{\omega,1} & J_{\omega,2} \ldots J_{\omega,n} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix}$, so you can see that each $J_{\omega,i}$ denotes the effect $\dot{q}_i$ has on ω
- What is the effect of joint i? If it's revolute, it's $\dot{q}k$, so the Jacobian is k, but transformed to the world reference frame as in the previous slide
- So $J_{\omega,i} = \begin{bmatrix} z_x \\ z_y \\ z_z \end{bmatrix}$ = The z component of the transformation matrix T_{i-1}^{0}

ANGULAR VELOCITY JACOBIAN EXAMPLE

- Our arm is RRP (the first two joints are revolute and the third prismatic)
- The transformation matrices are presented below
- $T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} T_2^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & 0 \\ s_1 c_2 & -c_1 & -s_1 s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} T_3^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & -q_3 c_1 s_2 \\ s_1 c_2 & -c_1 & -s_1 s_2 & -q_3 s_1 s_2 \\ -s_2 & 0 & -c_2 & 3 - q_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- What does the angular velocity Jacobian look like?

ANGULAR VELOCITY JACOBIAN EXAMPLE

- Our arm is RRP (the first two joints are revolute and the third prismatic)
- $T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $T_2^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & 0 \\ s_1 c_2 & -c_1 & -s_1 s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $T_3^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & -q_3 c_1 s_2 \\ s_1 c_2 & -c_1 & -s_1 s_2 & -q_3 s_1 s_2 \\ -s_2 & 0 & -c_2 & 3 - q_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- What does the angular velocity Jacobian look like?
- $J_\omega = \begin{bmatrix} \quad \end{bmatrix}$

ANGULAR VELOCITY JACOBIAN EXAMPLE

- Our arm is RRP (the first two joints are revolute and the third prismatic)
- $T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} T_2^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & 0 \\ s_1 c_2 & -c_1 & -s_1 s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} T_3^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & -q_3 c_1 s_2 \\ s_1 c_2 & -c_1 & -s_1 s_2 & -q_3 s_1 s_2 \\ -s_2 & 0 & -c_2 & 3 - q_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- What does the angular velocity Jacobian look like?
- $J_\omega = \begin{bmatrix} 0 & & \\ 0 & & \\ 1 & & \end{bmatrix}$

ANGULAR VELOCITY JACOBIAN EXAMPLE

- Our arm is RRP (the first two joints are revolute and the third prismatic)
- $T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $T_2^0 = \begin{bmatrix} c_1c_2 & s_1 & -c_1s_2 & 0 \\ s_1c_2 & -c_1 & -s_1s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $T_3^0 = \begin{bmatrix} c_1c_2 & s_1 & -c_1s_2 & -q_3c_1s_2 \\ s_1c_2 & -c_1 & -s_1s_2 & -q_3s_1s_2 \\ -s_2 & 0 & -c_2 & 3-q_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- What does the angular velocity Jacobian look like?
- $J_\omega = \begin{bmatrix} 0 & -s_1 & \\ 0 & c_1 & \\ 1 & 0 & \end{bmatrix}$

ANGULAR VELOCITY JACOBIAN EXAMPLE

- Our arm is RRP (the first two joints are revolute and the third prismatic)
- $T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} T_2^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & 0 \\ s_1 c_2 & -c_1 & -s_1 s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} T_3^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & -q_3 c_1 s_2 \\ s_1 c_2 & -c_1 & -s_1 s_2 & -q_3 s_1 s_2 \\ -s_2 & 0 & -c_2 & 3 - q_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- What does the angular velocity Jacobian look like?
- $J_\omega = \begin{bmatrix} 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

ANGULAR VELOCITY JACOBIAN EXAMPLE

- $J_\omega = \begin{bmatrix} 0 & -s_1 & 0 \\ 0 & c_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$
- Joint 1 is rotated at 0 and is rotating at 2 r/s, joint 2 is rotated at $-\frac{\pi}{2}$ and rotating at 1r/s, joint 3 is fixed at 5m.
- $\omega = J_\omega \dot{q}$
- $J_\omega = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $, \dot{\mathrm{q}} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \omega = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$

LINEAR VELOCITY JACOBIAN

- If joint n is prismatic, $v = \dot{q}k$, so the Jacobian will just be k, transformed to the world reference frame, so will be the z component of the transformation matrix T_{i-1}^{0} (Same as the angular velocity Jacobian for a revolute joint)
- If joint n is revolute, $v = \dot{q}k \times r$ where r is the distance between the joint axis and the end effector
- $r = \left(o_n^0 - o_{i-1}^0\right)$
- $v = \dot{q}_i z_{i-1}^0 \times \left(o_n^0 - o_{i-1}^0\right)$
- $J_{v_i} = z_{i-1}^0 \times \left(o_n^0 - o_{i-1}^0\right)$

LINEAR VELOCITY JACOBIAN EXAMPLE

- Same RRP arm as before
- $T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $T_2^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & 0 \\ s_1 c_2 & -c_1 & -s_1 s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $T_3^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & -q_3 c_1 s_2 \\ s_1 c_2 & -c_1 & -s_1 s_2 & -q_3 s_1 s_2 \\ -s_2 & 0 & -c_2 & 3 - q_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- What does the linear velocity Jacobian look like?
- $J_v = \begin{bmatrix} \quad \end{bmatrix}$

LINEAR VELOCITY JACOBIAN EXAMPLE

- $T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} T_2^0 = \begin{bmatrix} c_1c_2 & s_1 & -c_1s_2 & 0 \\ s_1c_2 & -c_1 & -s_1s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} T_3^0 = \begin{bmatrix} c_1c_2 & s_1 & -c_1s_2 & -q_3c_1s_2 \\ s_1c_2 & -c_1 & -s_1s_2 & -q_3s_1s_2 \\ -s_2 & 0 & -c_2 & 3 - q_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- $J_v = \begin{bmatrix} \quad \end{bmatrix}$

- $J_{v_1} = z_{i-1}^0 \times \left(o_n^0 - o_{i-1}^0\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -q_3c_1s_2 \\ -q_3s_1s_2 \\ 3 - q_3c_2 \end{bmatrix} = \begin{bmatrix} q_3s_1s_2 \\ -q_3c_1s_2 \\ 0 \end{bmatrix}$

LINEAR VELOCITY JACOBIAN EXAMPLE

- $T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $T_2^0 = \begin{bmatrix} c_1c_2 & s_1 & -c_1s_2 & 0 \\ s_1c_2 & -c_1 & -s_1s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $T_3^0 = \begin{bmatrix} c_1c_2 & s_1 & -c_1s_2 & -q_3c_1s_2 \\ s_1c_2 & -c_1 & -s_1s_2 & -q_3s_1s_2 \\ -s_2 & 0 & -c_2 & 3-q_3c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- $J_v = \begin{bmatrix} q_3s_1s_2 & & \\ -q_3c_1s_2 & & \\ 0 & & \end{bmatrix}$
- $J_{v_2} = z_{i-1}^0 \times \left(o_n^0 - o_{i-1}^0\right) = \begin{bmatrix} -s_1 \\ c_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} -q_3c_1s_2 - 0 \\ -q_3s_1s_2 - 0 \\ 3 - q_3c_2 - 3 \end{bmatrix} = \begin{bmatrix} -q_3c_1c_2 \\ -q_3s_1c_2 \\ q_3s_1^2s_2 + q_3c_1^2s_2 \end{bmatrix} = \begin{bmatrix} -q_3c_1c_2 \\ -q_3s_1c_2 \\ q_3s_2 \end{bmatrix}$

LINEAR VELOCITY JACOBIAN EXAMPLE

- $T_1^0 = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} T_2^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & 0 \\ s_1 c_2 & -c_1 & -s_1 s_2 & 0 \\ -s_2 & 0 & -c_2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} T_3^0 = \begin{bmatrix} c_1 c_2 & s_1 & -c_1 s_2 & -q_3 c_1 s_2 \\ s_1 c_2 & -c_1 & -s_1 s_2 & -q_3 s_1 s_2 \\ -s_2 & 0 & -c_2 & 3 - q_3 c_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- $J_v = \begin{bmatrix} q_3 s_1 s_2 & -q_3 c_1 c_2 & \\ -q_3 c_1 s_2 & -q_3 s_1 c_2 & \\ 0 & q_3 s_2 & \end{bmatrix}$
- $J_{v_3} = z_{i-1}^0 = \begin{bmatrix} -c_1 s_2 \\ -s_1 s_2 \\ -c_2 \end{bmatrix}$

LINEAR VELOCITY JACOBIAN EXAMPLE

- $J_v = \begin{bmatrix} q_3 s_1 s_2 & -q_3 c_1 c_2 & -c_1 s_2 \\ -q_3 c_1 s_2 & -q_3 s_1 c_2 & -s_1 s_2 \\ 0 & q_3 s_2 & -c_2 \end{bmatrix}$
- Find the linear velocity of the endpoint when joint 1 is at 0 degrees and is rotating at 2r/s, joint 2 is at $-\frac{\pi}{2}$ degrees and is rotating at -1 r/s and joint 3 is extended to 2m and is not moving

STATIC FORCES

- We've been relating the joint state and the joint velocity to the end effector state and velocity
- Can we relate the forces acting on the joints to the forces acting on the endpoint?
- Picture an arm carrying a weight at the end effector
- If the system is at rest (not moving) we can say that the forces acting on the end effector are balanced out by the joint forces

STATIC FORCES

- If the system is at rest (not moving) we can say that the work applied to the end effector is balanced out by the work done by the joints
- $F.\delta p = \tau.\delta\theta$
- $F^T.\delta p = \tau^T.\delta\theta$
- But here’s the cool bit: $\delta p = J\delta\theta$
- So $F^T J\delta\theta = \tau^T\delta\theta$
- So $\tau \ = \ J^T F$

STATIC FORCES

- So $\tau = J^T F$
- This is great because we already know how to get the Jacobian
- Let's take the example of the RRP arm we used earlier:
- $J_v = \begin{bmatrix} q_3 s_1 s_2 & -q_3 c_1 c_2 & -c_1 s_2 \\ -q_3 c_1 s_2 & -q_3 s_1 c_2 & -s_1 s_2 \\ 0 & q_3 s_2 & -c_2 \end{bmatrix}$
- With Joint 1 at 0 radians, Joint 2 at $-\frac{\pi}{2}$ radians and Joint 3 at 2m, the Jacobian becomes
- $J_v = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$

STATIC FORCES

- $J_v = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$
- Now let's say the arm is carrying a 10kg weight (100N). This would be a downward force in the z direction
- $\tau = J^T F$
- $\tau = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -2 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -100 \end{bmatrix} = \begin{bmatrix} 0 \\ 200 \\ 0 \end{bmatrix}$
- So joint 2 applies a torque of 200Nm to keep the system static