Question 1 The linear operator $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$ is given by the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & -2 \\ 3 & -2 & 0 \end{pmatrix}$$

in the standard basis. Find the matrix B of $\mathcal{A}$ in the basis $\{(1,0,1),(1,1,0),(0,1,1)\}$.

[10]

$$(T \mid A) = \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 & 1 & -2 \\ 1 & 0 & 1 & 3 & -2 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 2 & 1 & -2 \\ 0 & -1 & 1 & 2 & -1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & -1 & -2 & -2 \\ 0 & 1 & 1 & 2 & 1 & -2 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & -3 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right)$$

$$T^{-1}A = \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & -1 \\ 2 & 0 & -1 \end{pmatrix}$$

Therefore $B = T^{-1}A \,.\, T$

$$= \begin{pmatrix} 1 & -2 & -3 \\ 0 & 1 & -1 \\ 2 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -1 & -5 \\ -1 & 1 & 0 \\ 1 & 2 & -1 \end{pmatrix}$$

Question 2 Prove that the characteristic polynomial of a linear operator does not depend on the choice of a basis.

[10]

Let A : V → V be a linear operator
Then let e = $\{e_1, \ldots, e_n\}$ & f = $\{f_1, \ldots, f_n\}$ be a bases for V

Let $[A]_e$ &$[A]_f$ be matrices of perpendicular linear operator A in the respective basis.

Then there exist an invertible matrix C such that $[A]_f = C^{-1}[A]_e C$

Let $P_A(\lambda)$ represent perpendicular characteristic polynomial.

Then : $P_{[A]_f}(\lambda) = \det\big([A]_f - \lambda I\big)$

$$= \det(C^{-1}[A]_e C - \lambda I)$$
$$= \det(C^{-1}[A]_e C - \lambda C^{-1} C)$$
$$= \det((C^{-1}.C)([A]_e - \lambda I))$$
$$= \det(C^{-1}) * \ det([A]_e - \lambda I) \ * \det(C)$$
$$= \det(C^{-1}) \ * \det(C) * \ P_{[A]_e}(\lambda)$$
$$= \ P_{[A]_e}(\lambda)$$

Therefore $P_{[A]_f}(\lambda) \ = P_{[A]_e}(\lambda)$

Question 3 Determine whether the matrix

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & -1 \\ 2 & 0 & 0 \end{pmatrix}$$

is diagonalizable, and if yes, find a diagonal matrix D and a matrix T such that $D = T^{-1}AT$.

[10]

$$\det\left(\begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & -1 \\ 2 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right) = 0$$

$$\det\left(\begin{pmatrix} -1-\lambda & 0 & 1 \\ 2 & 1-\lambda & -1 \\ 2 & 0 & -\lambda \end{pmatrix}\right) = 0$$

Therefore

[$(-1-\lambda)$*($1-\lambda$)*($-\lambda$) + 0*(-1)*2 + 1*2*0 – 1*$(1-\lambda)$*2 – $(-1-\lambda)$*(-1)*0 – 0*2*$(-\lambda)$] = 0

$-\lambda^3 + \lambda - 2 + 2\lambda = 0$

$-\lambda^3 + 3\lambda - 2 = 0$

$(\lambda - 1)^2(x + 2) = 0$

$\lambda = 1 \; or \; \lambda = \; -2$

For $\lambda = 1$

$$\left(\begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & -1 \\ 2 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 0 & 1 \\ 2 & 0 & -1 \\ 2 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then do RREF

$$\begin{bmatrix} -2 & 0 & 1 & 0 \\ 2 & 0 & -1 & 0 \\ 2 & 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 2 & 0 & -1 & 0 \\ 2 & 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore

$$\mathrm{x} = \frac{1}{2}\, z$$
$$\mathrm{y} = \mathrm{y}$$
$$\mathrm{z} = \mathrm{z}$$

$$\vec{v} = y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$

<u>For $\boldsymbol{\lambda = -2}$</u>

$$\left(\begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & -1 \\ 2 & 0 & 0 \end{pmatrix} - \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & -1 \\ 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Then do RREF

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 3 & -1 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 3 & -3 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore

$x = z$
$y = -z$
$z = z$

$$\vec{v} = z\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Therefore since there is 3 eigen vectors it diagonalizable

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \& \quad T = \begin{pmatrix} 0 & \frac{1}{2} & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$

Question 4 From the Cauchy-Bunyakowski inequality deduce that for any vectors x, y of an inner product space, $||x + y|| \leq ||x|| + ||y||$.

[10]

$\|x + y\| \leq \|x\| + \|y\|$

Since

(x +y , x +y)

= (x + y)(x + y)

$= x^2 + 2xy + y^2$

= (x , x) + 2 (x , y) + (y , y)

And we know

| (x + y) | $\leq \|x\| * \|y\|$ (from Cauchy- Bunyakowski)

It follows that

$\|x + y\|^2 \leq \|x\|^2 + 2 * \|x\| * \|y\| + \|y\|^2$

$\leq (\|x\| + \|y\|)^2$

Therefore $\|x + y\| \leq \|x\| + \|y\|$

Question 5 Using the Gram-Schmidt process, transform the basis $\{(0,1,1),(1,0,1),(1,1,0)\}$ of $\mathbb{R}^3$ into an orthonormal basis.

[10]

Let $a_1 = (0,1,1)$, $a_2 = (1,0,1)$ & $a_3 = (1,1,0)$

$b_1 = a_1 = (0,1,1)$

$\|b_1\| = \sqrt{0^2 + 1^2 + 1^2} = \sqrt{2}$

$$b_2 = a_2 - \frac{(a_2\,,\ b_1)}{(b_1\,,\ b_1)}\, b_1$$

$$= (1,0,1) - \frac{1*0+0*1+1*1}{0*0+1*1+1*1}\ (0,1,1)$$

$$= (1,0,1) - \frac{1}{2}\ (0,1,1)$$

$$= (1,0,1) - \left(0, \frac{1}{2}, \frac{1}{2}\right)$$

$$= \left(1, -\frac{1}{2}, \frac{1}{2}\right)$$

$$\|b_2\| = \sqrt{1^2 + (-\tfrac{1}{2})^2 + (\tfrac{1}{2})^2} = \frac{\sqrt{6}}{2}$$

$$b_3 = a_3 - \frac{(a_3\,,\ b_1)}{(b_1\,,\ b_1)}\, b_1 - \frac{(a_3\,,\ b_2)}{(b_2\,,\ b_2)}\, b_2$$

$$= (1,1,0) - \frac{1*0+1*1+0*1}{0*0+1*1+1*1}\ (0,1,1) - \frac{1*1+1*\left(-\frac{1}{2}\right)+0*\left(\frac{1}{2}\right)}{1*1+\left(-\frac{1}{2}\right)*\left(-\frac{1}{2}\right)+\left(\frac{1}{2}\right)*\left(\frac{1}{2}\right)}\left(1, -\frac{1}{2}, \frac{1}{2}\right)$$

$$= (1,1,0) - \frac{1}{2}\ (0,1,1) - \frac{1}{3}\left(1, -\frac{1}{2}, \frac{1}{2}\right)$$

$$= (1,1,0) - \left(0, \frac{1}{2}, \frac{1}{2}\right) - \left(\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}\right)$$

$$= \left(\frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\right)$$

$$\|b_3\| = \sqrt{(\tfrac{2}{3})^2 + (\tfrac{2}{3})^2 + (-\tfrac{2}{3})^2} = \frac{2\sqrt{3}}{3}$$

Therefore

$$C_1 = \frac{1}{\|b_1\|}\ b_1 \ = \frac{1}{\sqrt{2}}\ (0,1,1) = \left(0, \frac{1}{\sqrt{2}}\ , \frac{1}{\sqrt{2}}\ \right)$$

$$C_2 = \frac{1}{\|b_2\|}\ b_2 \ = \frac{\sqrt{6}}{3}\ \left(1, -\frac{1}{2}\ , \frac{1}{2}\right) \ = \left(\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}\ , \frac{\sqrt{6}}{6}\right)$$

$$C_3 = \frac{1}{\|b_3\|}\ b_3 \ = \frac{\sqrt{3}}{2}\ \left(\frac{2}{3}\ , \frac{2}{3}, -\ \frac{2}{3}\right) \ = \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\ , -\ \frac{\sqrt{3}}{3}\right)$$

Therefore

$$\{\left(0, \frac{1}{\sqrt{2}}\ , \frac{1}{\sqrt{2}}\ \right)\ , \left(\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{6}\ , \frac{\sqrt{6}}{6}\right)\ , \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\ , -\ \frac{\sqrt{3}}{3}\right)\}$$

Question 6 Find a system of linear equations whose solution space is the subspace $\langle a_1, a_2, a_{3,} a_4 \subseteq \mathbb{R}^5$, where

$$a_1 = (2, 1, -1, 0, 1), a_2 = (-1, 1, -2, -1, 0), a_3 = (2, 0, 1, 0, -1)$$
$$a4 = (3, 2, -2, -1, 0)$$

[10]

1

$$(a_1, x) \Leftrightarrow 0 = (2,1,-1,0,1) * (x_1, x_2, x_3, x_4, x_5)$$
$$0 = 2x_1 + x_2 - x_3 + x_5$$

$$(a_2, x) \Leftrightarrow 0 = (-1,1,-2,-1,0) * (x_1, x_2, x_3, x_4, x_5)$$
$$0 = -x_1 + x_2 - 2x_3 - x_4$$

$$(a_3, x) \Leftrightarrow 0 = (2,0,1,0,-1) * (x_1, x_2, x_3, x_4, x_5)$$
$$0 = 2x_1 + x_3 - x_5$$

$$(a_4, x) \Leftrightarrow 0 = (3,2,-2,-1,0) * (x_1, x_2, x_3, x_4, x_5)$$
$$0 = 3x_1 + 2x_2 - 2x_3 - x_4$$

Therefore

$$\begin{bmatrix} 2 & 1 & -1 & 0 & 1 & 0 \\ -1 & 1 & -2 & -1 & 0 & 0 \\ 2 & 0 & 1 & 0 & -1 & 0 \\ 3 & 2 & -2 & -1 & 0 & 0 \end{bmatrix}$$

Calc too long

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & -4 & -8 & 0 \\ 0 & 0 & 1 & -2 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore

$x_1 = -x_4 - 2x_5$
$x_2 = 4x_4 + 8x_5$
$x_3 = 2x_4 + 5x_5$
$x_4 = x_4$
$x_5 = x_5$

$$\vec{v} = x_4 \begin{pmatrix} -1 \\ 4 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -2 \\ 8 \\ 5 \\ 0 \\ 1 \end{pmatrix}$$

Therefore basis is

$b_1 = (-1,4,2,1,0)$ & $b_2 = (-2,8,5,0,1)$

Therefore required linear system of eqn is

$-x_1 + 4x_2 + 2x_3 + x_4 = 0$
$-2x_1 + 8x_2 + 5x_3 + x_5 = 0$