Chapter 3: EQUIVALENCE RELATIONS

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LEARNING OUTCOMES FOR THE LECTURE

By the end of this lecture, students will be able to:

- determine properties of a partition of a set A under an equivalence relation
- determine a partition of a set A under a given equivalence relation
- define a transversal of an equivalence relation
- determine a transversal of a set A under a given equivalence relation
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PARTITIONS

Theorem (3.2.1)

Let $A \neq \emptyset$ and $\approx$ be an equivalence relation on a set A, and $a, b \in A$. Then

"a is related to a"

(i) $a \in [a], \quad \forall a \in A.$

Proof : $a \approx a \quad \forall a \in A$ by reflexive prop

$\Rightarrow \quad a \in [a], \quad \forall a \in A.$

$\Rightarrow [a] \neq \emptyset.$

$\Rightarrow$) [a]=[b] $\Rightarrow$ a $\approx$ b

$\Longleftarrow$ a $\approx$ b $\Rightarrow$ [a]=[b]

(ii) $[a] = [b]$ if and only if $a \approx b$.

Proof : $\Rightarrow$

$[a] = [b] \Rightarrow b \in [a] \Rightarrow b \approx a \Rightarrow a \approx b$ by symmetric prop

$\Leftarrow$

$a \approx b \Rightarrow b \approx a$ symmetric prop $\Rightarrow a \in [b]$ and $b \in [a]$.
Now $c \in [a] \Rightarrow c \approx a$, but $a \approx b$ therefore $c \approx b$ by transitive prop $\Rightarrow c \in [b] \Rightarrow [a] \subseteq [b]$.
Similarly, $[b] \subseteq [a]$ so $[a] = [b]$.

(iii) If $[a] \neq [b]$ then $[a] \cap [b] = \emptyset$.

Proof : ***(proving contrapositive)***

$c \in [a] \cap [b] \Rightarrow c \in [a]$ and $c \in [b]$

$\Rightarrow c \approx a$ and $c \approx b$

$\Rightarrow a \approx b$ by transitivity prop

$\Rightarrow [a] = [b]$ by part(ii).

$\therefore [a] \neq [b] \Rightarrow [a] \cap [b] = \emptyset$.

assume the contrary to the given condition, i.e. assume that the intersection is non empty and reach a contradiction.

(iv) If $a \in C$ where C is any equivalence class then $C = [a]$.
Let C be an equivalence class in A such that $C = [b]$. If $a \in C \Rightarrow a \in [b] \Rightarrow a \approx b \Rightarrow [a] = [b] = C$ by part(ii).
e.g. On $\mathbb{Z}$, $a \approx b$ iff $a - b = 2k$, $k \in \mathbb{Z}$
$[0] = [2] = [-8]$
$[1] = [-1] = [-99]$

Recall definition of a partition from chapter 1
If A is a non-empty set, a family or collection Σ of subsets of A is a partition of A, with the elements in Σ called cells, if

(i) no cell $A_i \in \Sigma$ is empty. That is $A_i \neq \emptyset$ for all $A_i \in \Sigma$. [Theorem 0.1 part(i)]

- the cells are pair-wise disjoint. That is : $A_i \cap A_j = \emptyset$ for all A_i and A_j in the partition Σ. [Theorem 0.1 part(iii)]
- every element of A belongs to some cell. That is: If $a \in A_i$ for some $A_i \in \Sigma$. By (ii) above a will belong to exactly one cell in the partition. We can write A as the union of the cells in the partition as follows: $A \bigcup_{A_i \in \Sigma} A_i$. [Theorem 0.1 part(i)]

Hence, if $A \neq \emptyset$ and $\approx$ is an equivalence relation on A, then
$\Sigma = A_{\approx} = \{C| \quad C \text{ is an equivalence class of } A\}$
is a partition of A.

Example:Let $A = \mathbb{Z}$, $\quad a \approx b$ iff $a - b = 2k$, $k \in \mathbb{Z}$

$\Sigma = \{[0], [1]\}$.

$\Sigma = [0] \cup [1]$

$[0] \cap [1] = \emptyset$

Definition (3.2.2 TRANSVERSAL)

Let Σ be the set of disjoint, distinct equivalence classes of A under $\approx$. Let τ be a set consisting of exactly one element from each equivalence class. Then the set τ is called a Transversal to A under $\approx$

e.g. On $\mathbb{Z}$, $a \approx b$ iff $a - b = 2k$, $k \in \mathbb{Z}$
$\tau = \{0, 1\} = \{-110, 99\}$.
$[0] \cup [1] = \mathbb{Z}$ and $[0] \cap [1] = \emptyset$.