MATH 104, SUMMER 2008, REVIEW SHEET FOR MIDTERM 2

To do well on this exam you should be able to do at least each of the following[1]:

Section 11: Subsequences

(1) Statement of the Bolzano-Weierstrass Theorem and its applications.
(2) Define subsequential limits.
(3) List all the subsequential limits of a given sequence. (For example, $\left(\sin \frac{n\pi}{4}\right)_{n\in\mathbb{N}}$).
(4) State the relationship between $\limsup s_n$, $\liminf s_n$, and the set of all subsequential limits of (s_n).

Section 12: $\limsup$'s and $\liminf$'s

(1) Prove a simple fact requiring the use of the definition of $\limsup s_n$ or $\liminf s_n$. (For example, if (s_{n_k}) is a subsequence of (s_n), then $\limsup s_{n_k} \leq \limsup s_n$).
(2) Use Theorem 12.2 and Corollary 12.3 to calculate limits of certain sequences.

Section 14: Series

(1) Identify and compute the limit of a geometric series.
(2) Prove that if $\sum a_n$ converges, then $\lim a_n = 0$. Use this fact to show certain series are divergent.
(3) Give a counterexample to show falsity of the converse of (2) above.
(4) Use the comparison test to determine whether a given series converges.
(5) State the root test and the ratio test, and use these tests to determine whether a given series converges.
(6) State the Cauchy Criterion for series, and use it to determine whether a given series converges.

Section 15: Alternating Series and Integral Tests

(1) State the alternating series theorem and Integral Test (see lecture notes).
(2) Use the integral test or alternating series test to determine whether a given series converges. [e.g. for alternating series, $\sum \frac{(-1)^n}{\ln n}$]

Section 17: Continuous Functions

(1) Define continuous at x_0.
(2) State the ϵ-δ property for continuity at x_0.
(3) Use the definition of continuity to prove that a given function is continuous at a real number x_0.
(4) Use the ϵ-δ condition for continuity to prove that a given function is continuous at a real number x_0.
(5) Prove that a given function is not continuous at a given real number.
(6) Use the definition of continuity and the limit laws for sequences to prove the basic results about continuity of the sum, product, absolute value, composition, etc ... of continuous functions.
(7) Determine, with proof, whether a given function is continuous at a given point.

Section 18: Properties of Continuous Functions

(1) State the Intermediate Value Theorem.

[1]This review guide was summarized by Benjamin Johnson, a Berkeley math GSI.

(2) Use the intermediate value theorem to prove, for example, that every positive real number has a cube root.

Section 19: Uniform Continuity

(1) Define uniformly continuous on S.
(2) Give an example of a function f and a set S such that f is continuous on S but not uniformly continuous on S.
(3) Use the definition of uniform continuity to prove that a given function is uniformly continuous on a set S.

If you have mastered the above, you can expect to do very well on at least 80% of the exam. While there may be one or two more creatively designed problems, everything on the exam will be related to something covered in the lecture; and most of the exam will be a direct modification of a previous quiz question or homework assignment.

Additional resources:

(1) Your quiz questions are one of your best resources. You ought to be able to answer any of the previous quiz questions quickly and effectively. Solutions to all quizzes given so far this semester are posted on the course website.
(2) Your homework problems are another resource. You should attempt to re-work any homework problem that you did not answer correctly the first time. Use the homework solutions on the course website to help you accomplish this.
(3) Your textbook is a good resource. The explanations in the textbook are pretty good most of the time, and there are several more exercises in the textbook than I have assigned in the homework, providing more opportunity for you to practice working examples, and structuring proofs effectively.
(4) The practice problems below will help you check if you have mastered the required materials. You can expect to have a real exam with similar difficulty level.

GOOD LUCK!!

Practice problems

(1) (a) State the Bolzano-Weierstrass Theorem.
(b) State the definition of a bounded function.
(c) Prove that if (x_n) is a bounded sequence in $\mathrm{dom}(f)$ and f is a bounded function, then there exists a *convergent* subsequence (y_k) of (x_n) such that $(f(y_k))$ is also a convergent sequence.

(2) Find the (i) subsequential limits, (ii) liminf's and limsup's, and (iii) limits (if the limit exists) of the following sequence.
(a) $\left(\cos \frac{n\pi}{4}\right)_{n\in\mathbb{N}}$ (b) $\left(\sin \frac{n\pi}{7}\right)_{n\in\mathbb{N}}$ (c) $\left(\left(\sin \frac{n\pi}{7}\right)^n\right)_{n\in\mathbb{N}}$

(3) (a) Prove that for any subsequences (s_{n_k}) of (s_n),

$$\limsup_{k\to\infty} s_{n_k} \le \limsup_{n\to\infty} s_n,$$

(b) Suppose (s_n) is a decreasing sequence. Then for any subsequence (s_{n_k}),

$$\lim_{k\to\infty} s_{n_k} = \lim_{n\to\infty} s_n.$$

(4) Find the limit of the following sequence

$$a_n = \frac{1}{n}\left[(2n)!!\right]^{1/n}.$$

Note $(2n)!! = (2n)(2n-2)(2n-4)\cdots 4\cdot 2$ is the product of all positive even numbers not exceeding $2n$.

(5) Are the following series convergent or divergent? Justify your answers.
(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$
(b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2+n+1}$
(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1+2^n}{3^n}$
(d) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$
(e) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \left(\frac{n}{n+1}\right)^n$
(f) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^{1/2}}$
(g) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(2n)!}{n^{2n}}$

(6) Prove that if $\sum a_n$ converges and $a_n > 0$ for all n, then $\sum \frac{a_n^2}{1+a_n}$ also converges.

(7) (a) State the Cauchy Criterion for testing convergence of series.
(b) Suppose $\sum a_n$ is an absolutely convergent series and (b_n) is a bounded sequence. Prove that $\sum a_n b_n$ converges.

(8) Use Cauchy Criterion to prove that $\sum a_n$ converges implies $\lim a_n = 0$. Then give an counter example to show the converse is not true.

(9) (a) State the definition of a function f being continuous at $x_0 \in \mathrm{dom}(f)$.
 (b) State the ϵ-δ property for continuity at $x_0 \in \mathrm{dom}(f)$.

(10) Use limit laws to prove that if f and g are both continuous at x_0 and $g(x_0) \neq 0$, then f/g is continuous at x_0.

(11) (a) Let

$$f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Prove that f is continuous at 0.

 (b) Let

$$g(x) = \begin{cases} \cos\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

Prove that g is discontinuous at 0.

(12) Thomae's function[2], named after Johannes Karl Thomae, also known as the popcorn function, the raindrop function, the ruler function, the Riemann function or the Stars over Babylon (by John Horton Conway) is defined as follows:

$$f(x) = \begin{cases} \frac{1}{q} \text{ if } x = \frac{p}{q} \in \mathbb{Q}, \text{ where } q > 0, \gcd(p,q) = 1 \\ 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Note that if $x = 0$ we take $q = 1$.

Prove that f is (a) continuous at all irrational numbers and (b) discontinuous at all rational numbers.

Hint: Part (a) is hard. Note that the sets $\{\frac{p}{q} \in (x_0, x_0+1) : \gcd(p,q) = 1, \frac{1}{q} \geq \epsilon\}$ *and* $\{\frac{p}{q} \in (x_0 - 1, x_0) : \gcd(p,q) = 1, \frac{1}{q} \geq \epsilon\}$ *are finite. If you think this is too hard, skip it.*

(13) Use ϵ-δ property to show that $f(x) = x^2 + \sqrt{x} + 1$ is continuous on $\mathbb{R}$.

(14) (a) State the Intermediate Value Theorem.
 (b) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous, and $f(a)f(b) < 0$ for some pair of real numbers $a < b$. Show that there is an $x \in \mathbb{R}$ with $f(x) = 0$.

(15) (a) State the definition of a function f being uniformly continuous on $S \subset \mathrm{dom}(f)$.
 (b) Prove that the function $\tan x$ is uniformly continuous on $[0, a]$ for any positive $a \in (0, \pi/2)$. You may use the fact that $|\sin\alpha| \leq |\alpha|$ for $\alpha \in \mathbb{R}$.
 (c) Prove that the function $h(x) = \sin\left(\frac{1}{x}\right)$ is not uniformly continuous on $(0, 1)$.

[2] See http://en.wikipedia.org/wiki/Thomae's_function