University of the Witwatersrand
School of Mathematics

MATH2007 (Multivariable Calculus)
Tutorial solutions

Note: there may be more than one way to solve a problem, and there may be many equivalent solutions. If you are not sure whether your own attempt is correct, please consult a tutor or lecturer!

Section 1.1: Derivatives and Differentials

1. This question revolves around "lifting" partial derivatives to ordinary derivatives in order to show that some basic properties of ordinary derivatives also hold for partial derivatives (i.e. linearity and the product rule):

$$\begin{tikzcd} \dfrac{\partial}{\partial x_1} fg \arrow[r, "\text{product rule}"] \arrow[d] & g\dfrac{\partial f}{\partial x_1} + f\dfrac{\partial g}{\partial x_1} \\ \dfrac{d}{dt}\varphi\psi \arrow[r, "\text{product rule}"] & \psi\dfrac{d\varphi}{dt} + \varphi\dfrac{d\psi}{dt} \arrow[u] \end{tikzcd}$$

(a) Since $\varphi(t) = f(t, x_2, \ldots, x_n) = f(\mathbf{x})\big|_{x_1=t}$, differentiation of φ with respect to t (where x_2, $\ldots$, x_n are treated as constant) is the same as partial differentiation of f with respect to x_1:

$$\begin{aligned} \varphi' &= \lim_{h\to 0} \frac{\varphi(t+h) - \varphi(t)}{h} \\ &= \lim_{h\to 0} \frac{f(t+h, x_2, \ldots, x_n) - f(t, x_2, \ldots, x_n)}{h} \\ &= \left[\lim_{h\to 0} \frac{f(x_1+h, x_2, \ldots, x_n) - f(x_1, x_2, \ldots, x_n)}{h}\right]_{x_1=t} \\ &= \left.\frac{\partial f}{\partial x_1}\right|_{x_1=t}. \end{aligned}$$

The equality shown above is independent of the choice of f. So, for example, we alse have

$$\psi' = \left.\frac{\partial g}{\partial x_1}\right|_{x_1=t}.$$

We will use this fact in the following answers.

(b) In the same way as part (a), we have

$$(a\varphi + b\psi)' = \left.\frac{\partial}{\partial x_1}(af + bg)\right|_{x_1=t} \tag{1}$$

where $(af+bg)(\mathbf{x}) := af(\mathbf{x})+bg(\mathbf{x})$ is the usual point-wise addition and scalar multiplication of functions. On the other hand

$$(a\varphi + b\psi)' = a\varphi' + b\psi' = a\left.\frac{\partial f}{\partial x_1}\right|_{x_1=t} + b\left.\frac{\partial g}{\partial x_1}\right|_{x_1=t} \tag{2}$$

from part (a). The left hand sides of (1) and (2) are identical, and equating the right hand sides yields

$$\left.\frac{\partial}{\partial x_1}(af + bg)\right|_{x_1=t} = a\left.\frac{\partial f}{\partial x_1}\right|_{x_1=t} + b\left.\frac{\partial g}{\partial x_1}\right|_{x_1=t}.$$

Since t is arbitrary, we may take $t = x_1$ so that

$$\frac{\partial}{\partial x_1}(af + bg) = a\frac{\partial f}{\partial x_1} + b\frac{\partial g}{\partial x_1}.$$

(c) Similar to part (b), we have

$$(\varphi\psi)' = \left.\frac{\partial}{\partial x_1} fg\right|_{x_1=t},$$
$$(\varphi\psi)' = \psi\varphi' + \varphi\psi' = g\left.\frac{\partial f}{\partial x_1}\right|_{x_1=t} + f\left.\frac{\partial g}{\partial x_1}\right|_{x_1=t}.$$

Equating right hand sides of the above two equations and taking $t = x_1$ yields

$$\frac{\partial}{\partial x_1} fg = g\frac{\partial f}{\partial x_1} + f\frac{\partial g}{\partial x_1}.$$

2. (a) We have two curves, $\mathbf{r}_1(s) = (s, s^2)^T$ and $\mathbf{r}_2(t) = (t^2, t)^T$. The curves intersect when $(s, s^2) = (t^2, t)$, in other words $t = s^2 = (t^2)^2 = t^4$. Thus we have the equation $t - t^4 = t(1 - t^3) = 0$ which has the solution $t = 0$ or $t = 1$ (we consider only the real valued solutions). When $t = 0$, we have $s = t^2 = 0$ and $s = 0$. When $t = 1$, we have $s = 1$. Thus the two curves intersect at $(0, 0)^T$ (i.e. when $t = s = 0$) and at $(1, 1)^T$ (i.e. when $t = s = 1$). If the curves have perpendicular tangents at the points of intersection, then the curves are orthogonal at the points of intersection. The tangent directions are:

$$\mathbf{r}_1'(s) = \begin{pmatrix} 1 \\ 2s \end{pmatrix}, \qquad \mathbf{r}_2'(t) = \begin{pmatrix} 2t \\ 1 \end{pmatrix}.$$

At the intersection point $(0, 0)^T$ we have $s = t = 0$ and the dot product

$$\mathbf{r}_1'(0) \cdot \mathbf{r}_2'(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$$

is zero, i.e. the two curves are orthogonal at $(0, 0)^T$. At the intersection point $(1, 1)^T$ we have $s = t = 1$ and the dot product

$$\mathbf{r}_1'(1) \cdot \mathbf{r}_2'(1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 4 \neq 0$$

is non-zero, i.e. the two curves are *not* orthogonal at $(1, 1)^T$.

(b) We have two curves, $\mathbf{r}_1(s) = (s, s^2)^T$ and $\mathbf{r}_2(t) = (t, 1/2 - t^2)^T$. The curves intersect when $(s, s^2) = (t, 1/2 - t^2)$, in other words $s = t$ and $s^2 = t^2 = \frac{1}{2} - t^2$. Thus $s = t = 1/2$ or $s = t = -1/2$. Thus the two curves intersect at $(1/2, 1/4)^T$ (i.e. when $t = s = 1/2$) and at $(-1/2, 1/4)^T$ (i.e. when $t = s = -1/2$). If the curves have perpendicular tangents at the points of intersection, then the curves are orthogonal at the points of intersection. The tangent directions are:

$$\mathbf{r}_1'(s) = \begin{pmatrix} 1 \\ 2s \end{pmatrix}, \qquad \mathbf{r}_2'(t) = \begin{pmatrix} 1 \\ -2t \end{pmatrix}.$$

At the intersection point $(1/2, 1/4)^T$ we have $s = t = 1/2$ and the dot product

$$\mathbf{r}_1'(1/2) \cdot \mathbf{r}_2'(1/2) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

is zero, i.e. the two curves are orthogonal at $(1/2, 1/4)^T$. At the intersection point $(-1/2, -1/2)^T$ we have $s = t = -1/2$ and the dot product

$$\mathbf{r}_1'(-1/2) \cdot \mathbf{r}_2'(-1/2) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

is zero, i.e. the two curves are orthogonal at $(-1/2, 1/4)^T$.

3. A circle with radius 5 centered at the origin has the parameterization $(5\cos t, 5\sin t)^T$. The same circle, but with center $(8,-3)$ is obtained by adding $(8,-3)$ to every point on the circle (a translation). Thus the parameterized circle with radius 5 and center $(8,-3)$ is

$$\mathbf{r}(t) = \begin{pmatrix} 5\cos t + 8 \\ 5\sin t - 3 \end{pmatrix}.$$

The point $(5,1)$ is on the circle, since

$$(5-8)^2 + (1-(-3))^2 = 5^2.$$

Let the point $(5,1)$ be obtained at $t = t_*$. Our parameterization yields $5 = 5\cos t_* + 8$ and $1 = 5\sin t_* - 3$, or $\cos t_* = -3/5$ and $\sin t_* = 4/5$. A tangent vector at $(5,1)$ is

$$\mathbf{r}'(t_*) = \begin{pmatrix} -5\sin t_* \\ 5\cos t_* \end{pmatrix} = \begin{pmatrix} -4 \\ -3 \end{pmatrix}.$$

The tangent line at $(5,1)$ is (where s is a real parameter)

$$T_{(5,1)}(s) = \mathbf{r}(t_*) + s\mathbf{r}'(t_*) = \begin{pmatrix} 5 \\ 1 \end{pmatrix} + s\begin{pmatrix} -4 \\ -3 \end{pmatrix}.$$

4. We have

$$\begin{aligned} f'(x_1,x_2) &= \begin{pmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial x}{\partial x_2} \end{pmatrix} = \begin{pmatrix} -x_2\sin x_1 & \cos x_1 \end{pmatrix} \\ f'(2,7) &= \begin{pmatrix} -7\sin 2 & \cos 2 \end{pmatrix} \\ df[(2,7);\mathbf{h}] &= f'(2,7)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} -7\sin 2 & \cos 2 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = -(7\sin 2)h_1 + (\cos 2)h_2 \end{aligned}$$

where $\mathbf{h} = (h_1,h_2)^T \in \mathbb{R}^2$.

5. We have

$$\begin{aligned} f'(x,y,z) &= \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial x}{\partial y} & \dfrac{\partial x}{\partial z} \end{pmatrix} = \begin{pmatrix} e^y & (x-2z)e^y & -2e^y \end{pmatrix} \\ df[(x,y,z);\mathbf{h}] &= f'(x,y,z)\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} e^y & (x-2z)e^y & -2e^y \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = e^y h_1 + (x-2z)e^y h_2 - 2e^y h_3 \end{aligned}$$

where $\mathbf{h} = (h_1,h_2,h_3)^T \in \mathbb{R}^3$.

6. (a) First, note that $\mathbf{v}$ is a tangent vector to $\mathbf{r}(t)$ (at $\mathbf{r}(t)$) if and only if $\mathbf{v}$ is non-zero and parallel to $\mathbf{r}'(t)$. The derivative is

$$\mathbf{F}'(x_1,x_2) = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial x_2} \\ \dfrac{\partial F_2}{\partial x_1} & \dfrac{\partial F_2}{\partial x_2} \\ \dfrac{\partial F_3}{\partial x_1} & \dfrac{\partial F_3}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{x_1} & 0 \\ 2x_2e^{2x_1} & e^{2x_1} \\ -x_2^2\sin x_1 & 2x_2\cos x_1 \end{pmatrix}$$

so that

$$d\mathbf{F}\left[\begin{pmatrix} a_1 \\ a_2 \end{pmatrix};\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\right] = \begin{pmatrix} \frac{1}{a_1} & 0 \\ 2a_2e^{2a_1} & e^{2a_1} \\ -a_2^2\sin a_1 & 2a_2\cos a_1 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} \frac{h_1}{a_1} \\ 2a_2h_1e^{2a_1} + e^{2a_1}h_2 \\ -a_2^2h_1\sin a_1 + 2a_2h_2\cos a_1 \end{pmatrix}.$$

If $\mathbf{v}$ is a tangent vector for curve $\mathbf{r}(t)$ at $\mathbf{r}(t)$, then $\mathbf{v}$ is parallel to $\mathbf{r}'(t)$, i.e. $\mathbf{v} = \alpha\mathbf{r}'(t) = (0, \alpha)^T$ for some $\alpha \in \mathbb{R}$. A tangent vector for the curve

$$\mathbf{F}(\mathbf{r}(t)) = \mathbf{F}\begin{pmatrix}1\\t\end{pmatrix} = \begin{pmatrix}0\\te^2\\t^2\cos 1\end{pmatrix}$$

at $\mathbf{F}(\mathbf{r}(t))$ is

$$(\mathbf{F}(\mathbf{r}(t)))' = \begin{pmatrix}0\\e^2\\2t\cos 1\end{pmatrix}.$$

Now,

$$d\mathbf{F}\left[(\mathbf{r}(t); \mathbf{v}\right] = d\mathbf{F}\left[\begin{pmatrix}1\\t\end{pmatrix}; \begin{pmatrix}0\\\alpha\end{pmatrix}\right] = \begin{pmatrix}0\\\alpha e^2\\2\alpha t\cos 1\end{pmatrix} = \alpha\begin{pmatrix}0\\e^2\\2t\cos 1\end{pmatrix}$$

which is a tangent vector (parallel to $(\mathbf{F}(\mathbf{r}(t)))'$) to the curve $\mathbf{F}(\mathbf{r}(t))$ at $\mathbf{F}(\mathbf{r}(t))$.

(b) The derivative is

$$\mathbf{F}'(x,y,z) = \begin{pmatrix}\dfrac{\partial F_1}{\partial x} & \dfrac{\partial F_1}{\partial y} & \dfrac{\partial F_1}{\partial z}\\ \dfrac{\partial F_2}{\partial x} & \dfrac{\partial F_2}{\partial y} & \dfrac{\partial F_1}{\partial z}\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ \dfrac{y}{1+z^2} & \dfrac{x}{1+z^2} & -\dfrac{2xyz}{(1+z^2)^2}\end{pmatrix}$$

so that

$$d\mathbf{F}\left[\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}; \begin{pmatrix}h_1\\h_2\\h_3\end{pmatrix}\right] = \begin{pmatrix}1 & 0 & 0\\ \dfrac{a_2}{1+a_3^2} & \dfrac{a_1}{1+a_3^2} & -\dfrac{2a_1a_2a_3}{(1+a_3^2)^2}\end{pmatrix}\begin{pmatrix}h_1\\h_2\\h_3\end{pmatrix}$$
$$= \begin{pmatrix}h_1\\ \dfrac{(a_2h_1 + a_1h_2)(1+a_3^2) - 2a_1a_2a_3h_3}{(1+a_3^2)^2}\end{pmatrix}.$$

If $\mathbf{v}$ is a tangent vector for curve $\mathbf{r}(t)$ at $\mathbf{r}(t)$, then $\mathbf{v}$ is parallel to $\mathbf{r}'(t)$, i.e. $\mathbf{v} = \alpha\mathbf{r}'(t) = (0, 2\alpha t, 0)^T$ for some $\alpha \in \mathbb{R}$. A tangent vector for the curve

$$\mathbf{F}(\mathbf{r}(t)) = \mathbf{F}\begin{pmatrix}2\\t^2\\1\end{pmatrix} = \begin{pmatrix}1\\t^2\end{pmatrix}$$

at $\mathbf{F}(\mathbf{r}(t))$ is

$$(\mathbf{F}(\mathbf{r}(t)))' = \begin{pmatrix}0\\2t\end{pmatrix}.$$

Now,

$$d\mathbf{F}\left[(\mathbf{r}(t); \mathbf{v}\right] = d\mathbf{F}\left[\begin{pmatrix}2\\t^2\\1\end{pmatrix}; \begin{pmatrix}0\\2\alpha t\\0\end{pmatrix}\right] = \begin{pmatrix}0\\2\alpha t\end{pmatrix} = \alpha\begin{pmatrix}0\\2t\end{pmatrix}$$

which is a tangent vector (parallel to $(\mathbf{F}(\mathbf{r}(t)))'$) to the curve $\mathbf{F}(\mathbf{r}(t))$ at $\mathbf{F}(\mathbf{r}(t))$.

7. We write $\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), \ldots, F_m(\mathbf{x}))^T$ and $\mathbf{G}(\mathbf{x}) = (G_1(\mathbf{x}), \ldots, G_m(\mathbf{x}))^T$.

(a) By linearity of the partial derivatives (see question 1)

$$
\begin{aligned}
(\alpha\mathbf{F}+\beta\mathbf{G})' &= \begin{pmatrix} \frac{\partial}{\partial x_1}(\alpha F_1+\beta G_1) & \cdots & \frac{\partial}{\partial x_n}(\alpha F_1+\beta G_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1}(\alpha F_m+\beta G_m) & \cdots & \frac{\partial}{\partial x_n}(\alpha F_m+\beta G_m) \end{pmatrix} \\
&= \begin{pmatrix} \alpha\frac{\partial F_1}{\partial x_1}+\beta\frac{\partial G_1}{\partial x_1} & \cdots & \alpha\frac{\partial F_1}{\partial x_n}+\beta\frac{\partial G_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \alpha\frac{\partial F_m}{\partial x_1}+\beta\frac{\partial G_m}{\partial x_1} & \cdots & \alpha\frac{\partial F_m}{\partial x_n}+\beta\frac{\partial G_m}{\partial x_n} \end{pmatrix} \\
&= \alpha\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{pmatrix} + \beta\begin{pmatrix} \frac{\partial G_1}{\partial x_1} & \cdots & \frac{\partial G_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial G_m}{\partial x_1} & \cdots & \frac{\partial G_m}{\partial x_n} \end{pmatrix} \\
&= \alpha\mathbf{F}'+\beta\mathbf{G}'.
\end{aligned}
$$

(b) By the product rule for partial derivatives (see question 1)

$$
\begin{aligned}
(g\mathbf{F})' &= (gF_1, gF_2, \dots, gF_m)' \\
&= \begin{pmatrix} \frac{\partial}{\partial x_1}(gF_1) & \cdots & \frac{\partial}{\partial x_n}(gF_1) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x_1}(gF_m) & \cdots & \frac{\partial}{\partial x_n}(gF_m) \end{pmatrix} \\
&= \begin{pmatrix} F_1\frac{\partial g}{\partial x_1}+g\frac{\partial F_1}{\partial x_1} & \cdots & F_1\frac{\partial g}{\partial x_n}+g\frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ F_m\frac{\partial g}{\partial x_1}+g\frac{\partial F_m}{\partial x_1} & \cdots & F_m\frac{\partial g}{\partial x_n}+g\frac{\partial F_m}{\partial x_n} \end{pmatrix}
\end{aligned}
$$

8. (a) We have

$$
\begin{aligned}
\varphi' &= \begin{pmatrix} \frac{\partial\varphi}{\partial x_1} & \frac{\partial\varphi}{\partial x_2} & \frac{\partial\varphi}{\partial x_3} \end{pmatrix} = \begin{pmatrix} -2 & 2x_2x_3 & x_2^2 \end{pmatrix}, \\
\mathbf{F}' &= \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} \end{pmatrix} = \begin{pmatrix} -2e^{-2x_1} & 5x_3^5 & 25x_2x_3^4 \\ -2 & 1 & \cos x_3 \end{pmatrix}.
\end{aligned}
$$

(b) Theorem 1.1.4 provides

$$
\begin{aligned}
(\varphi\mathbf{F})' &= \varphi\mathbf{F}'+\mathbf{F}\varphi' \\
&= (x_3x_2^2-2x_1)\begin{pmatrix} -2e^{-2x_1} & 5x_3^5 & 25x_2x_3^4 \\ -2 & 1 & \cos x_3 \end{pmatrix} + \begin{pmatrix} e^{-2x_1}+5x_2x_3^5 \\ x_2-2x_1+\sin x_3 \end{pmatrix}\begin{pmatrix} -2 & 2x_2x_3 & x_2^2 \end{pmatrix}.
\end{aligned}
$$

Now try calculate $(\varphi\mathbf{F})'$ directly and compare your answer with this result!

9. Here we consider linearity in the second argument (i.e. $\mathbf{h}$):

$$\begin{aligned} d\mathbf{F}[\mathbf{a}; \alpha\mathbf{h} + \beta\mathbf{g}] &= \mathbf{F}'(\mathbf{a})(\alpha\mathbf{h} + \beta\mathbf{g}) \\ &= \alpha\mathbf{F}'\mathbf{h} + \beta\mathbf{F}'\mathbf{g} && \text{(matrix multiplication is linear)} \\ &= \alpha d\mathbf{F}[\mathbf{a}; \mathbf{h}] + \beta d\mathbf{F}[\mathbf{a}; \mathbf{g}] \end{aligned}$$

for all $\alpha, \beta \in \mathbb{R}$ and $\mathbf{h}, \mathbf{g} \in \mathbb{R}^n$.

10. The results follows immediately from Theorem 1.1.4:

$$\begin{aligned} d(g\mathbf{F})[\mathbf{a}; \mathbf{h}] &= (g\mathbf{F})'(\mathbf{a})\mathbf{h} \\ &= (g(\mathbf{a})\mathbf{F}'(\mathbf{a}) + \mathbf{F}(\mathbf{a})g'(\mathbf{a}))\mathbf{h} \\ &= g(\mathbf{a})\mathbf{F}'(\mathbf{a})\mathbf{h} + \mathbf{F}(\mathbf{a})g'(\mathbf{a})\mathbf{h} \\ &= g(\mathbf{a})d\mathbf{F}[\mathbf{a}; \mathbf{h}] + \mathbf{F}(\mathbf{a})dg[\mathbf{a}; \mathbf{h}]. \end{aligned}$$

11. (a) Noting that $\mathbf{u} \cdot \mathbf{v} = \sum_{j=1}^{n} u_j v_j$, we obtain

$$(\mathbf{u}\cdot\mathbf{v})' = \left(\sum_{j=1}^{n} u_j v_j\right)' = \sum_{j=1}^{n}(u_j v_j)' = \sum_{j=1}^{n}(u_j' v_j + u_j v_j') = \sum_{j=1}^{n} u_j' v_j + \sum_{j=1}^{n} u_j v_j' = \mathbf{u}'\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v}'.$$

(b) We wish to make a statement about $\mathbf{v} \cdot \mathbf{p}(t)$ for all $t \in \mathbb{R}$. It is given that $\mathbf{v} \cdot \mathbf{p}'(t) = 0$ for all $t \in \mathbb{R}$. Since $\mathbf{v}$ is constant

$$(\mathbf{v} \cdot \mathbf{p}(t))' = \mathbf{v}' \cdot \mathbf{p}(t) + \mathbf{v} \cdot \mathbf{p}'(t) = \mathbf{v} \cdot \mathbf{p}'(t) = 0,$$

it follows that $\mathbf{v} \cdot \mathbf{p}(t)$ is constant (i.e. the same value for all $t \in \mathbb{R}$). It is also given that $\mathbf{v} \cdot \mathbf{p}(0) = 0$, which is the (constant) value $\mathbf{v} \cdot \mathbf{p}(t) = 0$ for all $t \in \mathbb{R}$. Thus $\mathbf{v}$ and $\mathbf{p}(t)$ are orthogonal for all $t \in \mathbb{R}$.

(c) Since $\mathbf{u}(t)$ is a unit vector, $\|\mathbf{u}(t)\| = \sqrt{\mathbf{u}(t) \cdot \mathbf{u}(t)} = 1$. It suffices to consider $\mathbf{u}(t) \cdot \mathbf{u}(t) = 1$ for all t (can you explain why?). Then we have, for all t,

$$(\mathbf{u}(t) \cdot \mathbf{u}(t))' = 1'$$

if and only if

$$\mathbf{u}'(t) \cdot \mathbf{u}(t) + \mathbf{u}(t) \cdot \mathbf{u}'(t) = 0$$

if and only if

$$2\mathbf{u}'(t) \cdot \mathbf{u}(t) = 0.$$

Since $\mathbf{u}'(t) \cdot \mathbf{u}(t) = 0$ for all t, $\mathbf{u}(t)$ and $\mathbf{u}'(t)$ are orthogonal for all t.

(d) Notice that $\mathbf{u}(t)$ is a unit vector for all t, hence part (c) applies. Straightforward calculation yields

$$\begin{aligned} \varphi' &= e^{-t^2} - 2t^2 e^{-t^2} \\ \mathbf{u}' &= \begin{pmatrix} -2\sin 2t \\ 2\cos 2t \end{pmatrix} \\ (\varphi\mathbf{u})' &= \mathbf{u}\varphi' + \varphi\mathbf{u}' = (e^{-t^2} - 2t^2 e^{-t^2})\mathbf{u} + te^t\mathbf{u}' \end{aligned}$$

and the last derivative (by part (c)) is a sum of two orthogonal vectors.

12. (a) We have $\mathbf{F}' = \begin{pmatrix} z\cos\theta & -rz\sin\theta & r\cos\theta \\ z\sin\theta & rz\cos\theta & r\sin\theta \\ 0 & 0 & 1 \end{pmatrix}$.

(b) It follows that $\det(\mathbf{F}') = rz^2\cos^2\theta + rz^2\sin^2\theta = rz^2$.

13. (a) The velocity is $\mathbf{r}'(t) = \begin{pmatrix} 4t \\ 2t-4 \\ 3 \end{pmatrix}$ and the acceleration is $\mathbf{r}''(t) = (\mathbf{r}'(t))' = \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$.

(b) The component of acceleration in the direction of the velocity is given by a projection

$$\frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \cdot \mathbf{r}''(t) = \frac{1}{\sqrt{16t^2 + (2t-4)^2 + 9}} \begin{pmatrix} 4t \\ 2t-4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix} = \frac{20t - 8}{\sqrt{20t^2 - 16t + 25}}.$$

14. We note that $\mathbf{r}(t) = \rho(t)\mathbf{u}(t)$, where $\mathbf{u}(t) = (\cos t, \sin t)^T$ is a unit vector (so that question 11 (c) applies).

(a) We apply the product rule, $\mathbf{r}'(t) = \mathbf{u}(t)\rho'(t) + \rho(t)\mathbf{u}'(t) = \rho'(t)\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + \rho(t)\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$.

(b) Linearity of differentiation yields

$$\mathbf{r}''(t) = (\mathbf{r}'(t))' = \rho''(t)\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + \rho'(t)\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + \rho'(t)\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + \rho(t)\begin{pmatrix} -\cos t \\ -\sin t \end{pmatrix}.$$

Grouping the $\mathbf{u}$ and $\mathbf{u}'$ terms yields the desired result

$$\mathbf{r}''(t) = (\rho''(t) - \rho(t))\begin{pmatrix} \cos t \\ \sin t \end{pmatrix} + 2\rho'(t)\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}.$$

(c) For all $a, b \in \mathbb{R}$, we have that

$$\|a\mathbf{u} + b\mathbf{u}'\| = \sqrt{(a\mathbf{u} + b\mathbf{u}') \cdot (a\mathbf{u} + b\mathbf{u}')} = \sqrt{a^2 + b^2}$$

since $\mathbf{u}$ and $\mathbf{u}'$ are unit vectors and orthogonal to each other. It follows from $\mathbf{r}'(t) = \rho'(t)\mathbf{u}(t) + \rho(t)\mathbf{u}'(t)$, that

$$\|\mathbf{r}'(t)\| = \sqrt{(\rho'(t))^2 + (\rho(t))^2}.$$

15. If $\mathbf{r}(t)$ lies on the circle of radius R with center 0 for all $t \in \mathbb{R}$, then $\|\mathbf{r}(t)\| = R$ is constant. Differentiating $\mathbf{r}(t) \cdot \mathbf{r}(t) = R^2$ twice yields

$$(\mathbf{r}(t) \cdot \mathbf{r}(t))' = 2\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$$
$$(\mathbf{r}(t) \cdot \mathbf{r}(t))'' = 2\mathbf{r}''(t) \cdot \mathbf{r}(t) + 2\mathbf{r}'(t) \cdot \mathbf{r}'(t) = 0$$

Solving the last equation for $\|\mathbf{r}'(t)\|^2 = \mathbf{r}'(t) \cdot \mathbf{r}'(t)$ yields

$$\|\mathbf{r}'(t)\| = \sqrt{-\mathbf{r}''(t) \cdot \mathbf{r}(t)}.$$

16. Applying Theorem 1.1.5, we find

$$\boldsymbol{\omega}' = (\mathbf{r} \times \mathbf{r}')' = \mathbf{r}' \times \mathbf{r}' + \mathbf{r} \times \mathbf{r}'' = \mathbf{r} \times \mathbf{r}''$$

where we used that $\mathbf{r}' \times \mathbf{r}' = \mathbf{0}$ (the cross product of any vector with itself is the zero vector).

17. (a) In terms of limits,

$$\begin{aligned} h'(t) &= \lim_{s \to 0} \frac{h(t+s) - h(t)}{s} \\ &= \lim_{s \to 0} \frac{f(x_1 + t + s, x_2) - f(x_1 + t, x_2)}{s} \\ &= \left.\frac{\partial f}{\partial x_1}\right|_{x_1 \to x_1 + t} = \frac{\partial f}{\partial x_1}(x_1 + t, x_2). \end{aligned}$$

(b) If $\dfrac{\partial f}{\partial x_1} = 0$ for all $\mathbf{x} \in \mathbb{R}^2$, then $h'(t) = 0$ for all $t \in \mathbb{R}$. It follows that

$$h(b) - h(0) = \int_0^b h'(t)\,dt = 0.$$

Clearly, $h(b) = h(0)$. In other words, $f(x_1 + b, x_2) = f(x_1, x_2)$ for all $b \in \mathbb{R}$ including $b = -x_1$ which provides $f(0, x_2) = f(x_1, x_2)$.

18. Since $\mathbf{F} \cdot \mathbf{F} = 1$ for all $\mathbf{x} \in \mathbb{R}^n$, we have (Theorem 1.1.4)

$$(\mathbf{F} \cdot \mathbf{F})' = 2\mathbf{F}^T\mathbf{F}' = \mathbf{0}.$$

Now assume that $\det(\mathbf{F}') \neq 0$ so that $(\mathbf{F}')^{-1}$ exists. Then,

$$2\mathbf{F}^T\mathbf{F}'(\mathbf{F}')^{-1} = \mathbf{0}\,(\mathbf{F}')^{-1} = \mathbf{0}.$$

It follows that $\mathbf{F}^T = \mathbf{0}$ and $\|\mathbf{F}\| = 0$. But $\|\mathbf{F}\| = 1$, so our assumption that $\det(\mathbf{F}') \neq 0$ was false; i.e. $\det(\mathbf{F}') = 0$.

Section 1.2: Vector analysis

1. (a) $\nabla f = \begin{pmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \dfrac{2x_1}{x_1^2 + x_2^2} \\ \dfrac{2x_2}{x_1^2 + x_2^2} \end{pmatrix}.$

   (b) $\nabla f = \begin{pmatrix} \dfrac{\partial f}{\partial x_1} \\ \dfrac{\partial f}{\partial x_2} \\ \dfrac{\partial f}{\partial x_3} \end{pmatrix} = \begin{pmatrix} -x_2 e^{x_1 x_2} + \cos(x_1 - x_2) \\ -x_1 e^{x_1 x_2} - \cos(x_1 - x_2) \\ 2x_3 \end{pmatrix}.$

2. (a) $\mathbf{G} = \operatorname{curl} \mathbf{F} = \begin{pmatrix} x_1 x_3 - x_2 \cos x_3 \\ -x_2 x_3 \\ -x_1 e^{x_2} \end{pmatrix}$ and $\operatorname{div} \mathbf{F} = e^{x_2} + \sin x_3 + x_1 x_2$.

   (b) Theorem 1.2.7 provides $\nabla \cdot G = \nabla \cdot (\nabla \times \mathbf{F}) = 0$. Check this fact by direct calculation! (See also Section 1.2, question 7).

3.
$$\begin{aligned}
&\operatorname{curl} \mathbf{F} = \begin{pmatrix} 1 \\ x_1 - 1 \\ 0 \end{pmatrix}, \\
&\operatorname{div} \mathbf{F} = x_3 + e^{x_2}, \\
&\nabla \cdot (\nabla \times \mathbf{F}) = \operatorname{div}(\operatorname{curl} \mathbf{F}) = 0, \\
&\mathbf{F} \cdot ((\nabla \times \mathbf{F}) = \begin{pmatrix} x_1 x_3 \\ e^{x_2} \\ x_1 + x_2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ x_1 - 1 \\ 0 \end{pmatrix} = x_1 x_3 + (x_1 - 1)e^{x_2}, \\
&\nabla \times (\nabla \times \mathbf{F}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\end{aligned}$$

4. (i) Since $\nabla^2 u = \dfrac{\partial^2 u}{\partial x_1} + \dfrac{\partial^2 u}{\partial x_2} + \dfrac{\partial^2 u}{\partial x_3} + \dfrac{\partial^2 u}{\partial x_4} = 0 - e^{x_3} \sin x_2 + e^{x_3} \sin x_2 + 0 = 0$, u is harmonic.

(ii) Since $\nabla^2 u = \dfrac{\partial^2 u}{\partial x_1} + \dfrac{\partial^2 u}{\partial x_2} + \dfrac{\partial^2 u}{\partial x_3} = 6x_3 + 0 - 6x_3 = 0$, u is harmonic.

(iii) Since $\nabla^2 u = \dfrac{\partial^2 u}{\partial x_1} + \dfrac{\partial^2 u}{\partial x_2} = 2 + 2 \neq 0$, u is *not* harmonic.

5. By straightforward application of linearity and the product rule for partial derivatives.

6. (a) We have

$$f_h(y+k) - f_h(y) = \psi(x+h, y+k) - \psi(x, y+k) - \psi(x+h, y) + \psi(x, y)$$

and

$$g_k(x+h) - g_k(x) = \psi(x+h, y+k) - \psi(x+h, y) - \psi(x, y+k) + \psi(x, y).$$

Clearly $f_h(y+k) - f_h(y) = g_k(x+h) - g_k(x)$.

(b) The first mean value theorem states that there exists a $K_1 \in [0, k]$ such that

$$\frac{f_h(y+k) - f_h(y)}{(y+k) - y} = f_h'(y + K_1)$$

or equivalently

$$f_h(y+k) - f_h(y) = k f_h'(y + K_1).$$

Similarly, there exists $H_1 \in [0, h]$ such that

$$g_k(x+h) - g_k(x) = h g_k'(x + H_1).$$

(c) Since (part (a)) $f_h(y+k) - f_h(y) = g_k(x+h) - g_k(x)$, we have from part (b)

$$f_h(y+k) - f_h(y) = k f_h'(y + K_1) = h g_k'(x + H_1) = g_k(x+h) - g_k(x).$$

(d) Now (see question 1 of Section 1.1),

$$k f_h'(y + K_1) = k \left(\left.\frac{\partial \psi}{\partial y}\right|_{(x,y)\to(x+h, y+K_1)} - \left.\frac{\partial \psi}{\partial y}\right|_{(x,y)\to(x, y+K_1)} \right)$$

and

$$h g_k'(x + H_1) = h \left(\left.\frac{\partial \psi}{\partial x}\right|_{(x,y)\to(x+H_1, y+k)} - \left.\frac{\partial \psi}{\partial x}\right|_{(x,y)\to(x+H_1, y)} \right).$$

Equating the right hand sides of these two equations (which are equal by part (c)) yields that

$$k\left(\frac{\partial \psi}{\partial y}(x+h, y+K_1) - \frac{\partial \psi}{\partial y}(x, y+K_1)\right) = h\left(\frac{\partial \psi}{\partial x}(x+H_1, y+k) - \frac{\partial \psi}{\partial x}(x+H_1, y)\right). \tag{3}$$

(e) Similar to part (b), we apply the first mean value theorem to each side of (3):

$$k\left(\frac{\partial \psi}{\partial y}(x+h, y+K_1) - \frac{\partial \psi}{\partial y}(x, y+K_1)\right) = kh\frac{\partial}{\partial x}\frac{\partial \psi}{\partial y}(x+H_2, y+K_1)$$

for some $H_2 \in [0, h]$, and

$$h\left(\frac{\partial \psi}{\partial x}(x+H_1, y+k) - \frac{\partial \psi}{\partial x}(x+H_1, y)\right) = hk\frac{\partial}{\partial y}\frac{\partial \psi}{\partial x}(x+H_1, y+K_2)$$

for some $K_2 \in [0, k]$. Thus

$$kh\frac{\partial}{\partial x}\frac{\partial \psi}{\partial y}(x+H_2, y+K_1) = hk\frac{\partial}{\partial y}\frac{\partial \psi}{\partial x}(x+H_1, y+K_2).$$

(f) Dividing by hk in the previous equation yields As $h \to 0$ and $k \to 0$, we must have $H_1, H_2, K_1, K_2 \to 0$.

$$\frac{\partial}{\partial x}\frac{\partial \psi}{\partial y}(x + H_2, y + K_1) = \frac{\partial}{\partial y}\frac{\partial \psi}{\partial x}(x + H_1, y + K_2)$$

and taking the limits $h \to 0$ and $k \to 0$ (and $H_1, H_2, K_1, K_2 \to 0$) yields

$$\frac{\partial}{\partial x}\frac{\partial \psi}{\partial y}(x, y) = \frac{\partial}{\partial y}\frac{\partial \psi}{\partial x}(x, y).$$

Here we have assumed that all of the partial derivatives exist, and are continuous so that we can take the limits and equate the limits to the function values.

7. We need to prove that $\nabla \cdot (\nabla \times \mathbf{F}) \equiv 0$. We have

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{F}) &= \nabla \cdot \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{pmatrix} \\
&= \frac{\partial}{\partial x_1}\left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) \\
&= \frac{\partial^2 F_3}{\partial x_1 \partial x_2} - \frac{\partial^2 F_3}{\partial x_2 \partial x_1} + \frac{\partial^2 F_2}{\partial x_3 \partial x_1} - \frac{\partial^2 F_3}{\partial x_1 \partial x_3} + \frac{\partial^2 F_1}{\partial x_2 \partial x_3} - \frac{\partial^2 F_3}{\partial x_3 \partial x_2}
\end{aligned}$$

and applying Theorem 1.2.6 yields $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

8. Applying Theorem 1.2.5(b) and Theorem 1.2.5(d) yields

$$\nabla^2(\phi\psi) = \nabla \cdot (\nabla(\phi\psi)) = \nabla \cdot (\psi\nabla\phi + \phi\nabla\psi) = (\nabla\psi) \cdot (\nabla\phi) + \psi\nabla^2\phi + (\nabla\phi) \cdot (\nabla\psi) + \phi\nabla^2\psi.$$

Since $\nabla^2\phi = \nabla^2\psi = 0$, we find

$$\nabla^2(\phi\psi) = 2(\nabla\psi) \cdot (\nabla\phi).$$

9. If f is harmonic, then $\nabla^2 f = 0$. We have by Theorem 1.2.5(b)

$$\nabla^2 f^2 = \nabla \cdot (\nabla f^2) = \nabla \cdot (2f\nabla f) = (\nabla(2f)) \cdot (\nabla f) + 2f(\nabla \cdot \nabla f) = 2\|\nabla f\|^2$$

where we used Theorem 1.2.5(d) and (a).

10. Theorem 1.2.5(f) and Theorem 1.2.7, provides

$$\nabla \times (f\nabla g) = (\nabla f) \times (\nabla g) + f(\nabla \times (\nabla g)) = (\nabla f) \times (\nabla g).$$

11. Since

$$\nabla \times (g\nabla f) = (\nabla g) \times (\nabla f) + g(\nabla \times (\nabla f)) = (\nabla g) \times (\nabla f)$$

by Theorem 1.2.5(f) and Theorem 1.2.7, and

$$\nabla \times (f\nabla g) = (\nabla f) \times (\nabla g) + f(\nabla \times (\nabla g)) = -(\nabla g) \times (\nabla f)$$

we have $\nabla \times (g\nabla f) = \nabla \times (f\nabla g)$ if and only if

$$(\nabla g) \times (\nabla f) = -(\nabla g) \times (\nabla f)$$

i.e. $(\nabla g) \times (\nabla f) = 0$ (which means that ∇g and ∇f are parallel vectors for all $\mathbf{x} \in \mathbb{R}^3$).

12. Setting $\nabla^2 f = 4 - c\cos x + 2b = 0$ yields $b = \frac{c}{2}\cos x - 2$. Since b must be constant, we have $c = 0$ and $b = -2$. This is the only constraint, i.e. $f(x, y) = ax + 2x^2 - 2y^2$ is harmonic for all $a \in \mathbb{R}$.

13. Theorem 1.2.5(d) provides

$$\nabla \cdot (f \nabla f) = (\nabla f) \cdot (\nabla f) + f \nabla \cdot (\nabla f) = \|\nabla f\|^2 + f(\nabla \cdot (\nabla f)).$$

Thus $\nabla \cdot (f \nabla f) = \|\nabla f\|^2$ if and only if $f(\nabla \cdot (\nabla f)) = 0$. If $f = 0$ on some region of $\mathbb{R}^2$, then $\nabla \cdot (\nabla f) = 0$ identically in that region. If $f \neq 0$ on some region of $\mathbb{R}^2$, then $\nabla \cdot (\nabla f) = 0$. It remains to discuss the boundary between regions where $f = 0$ and $\nabla \cdot (\nabla f) = 0$.

Section 1.3: Chain Rule

1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous and have continuous derivatives.

   (a) From the first mean value theorem

   $$g(t+h) - g(t) = hg'(t+\delta)$$

   for some δ which lies between 0 and h,

   (b) and similarly from the first mean value theorem

   $$f(a+s) - f(a) = sf'(a+\epsilon)$$

   for some ϵ which lies between 0 and s.

   (c) By setting $a = g(t)$ and $s = g(t+h) - g(t) = hg'(t+\delta)$ we see that

   $$f(g(t+h)) - f(g(t)) = sf'(g(t)+\epsilon) = hg'(t+\delta)f'(g(t)+\epsilon)$$

   where ϵ lies between 0 and s.

   (d) Observing that as $h \to 0$ we have $s \to 0$ (and hence $\delta \to 0$ and $\epsilon \to 0$), we may conclude that

   $$\begin{aligned} \lim_{h\to 0} \frac{f(g(t+h)) - f(g(t))}{h} &= \lim_{h\to 0} \frac{hg'(t+\delta)f'(g(t)+\epsilon)}{h} \\ &= \lim_{h\to 0} g'(t+\delta)f'(g(t)+\epsilon) \\ &= g'(t)f'(g(t)). \end{aligned}$$

2. (a) Theorem 1.3.2 provides $[f(\mathbf{G}(t))]' = f'(\mathbf{G}(t))\mathbf{G}'(t)$.

   (b) Theorem 1.3.2 with $n = 3$ yields

   $$\begin{aligned} [f(\mathbf{G}(t))]' &= \sum_{i=1}^{3} \left.\frac{\partial f}{\partial x_i}\right|_{\mathbf{x}=\mathbf{G}(t)} \frac{dG_i}{dt} \\ &= (2x_1)\Big|_{(x_1,x_2,x_3)=(G_1(t),G_2(t),G_3(t))} \frac{dG_1}{dt} + (2x_2)\Big|_{(x_1,x_2,x_3)=(G_1(t),G_2(t),G_3(t))} \frac{dG_2}{dt} \\ &\quad - 1\Big|_{(x_1,x_2,x_3)=(G_1(t),G_2(t),G_3(t))} \frac{dG_3}{dt} \\ &= 2G_1(t)G_1'(t) + 2G_2(t)G_2'(t) - G_3'(t). \end{aligned}$$

   (c) Since $\mathbf{G}(t) \in A$ for all $t \in \mathbb{R}$, we have $f(\mathbf{G}(t)) = k$ for all $t \in \mathbb{R}$. It follows that

   $$0 = \frac{d}{dt} f(\mathbf{G}(t)) = (\nabla f)(\mathbf{G}(t)) \cdot \mathbf{G}'(t)$$

   where we used Theorem 1.3.2 again, and since the tangent vector $\mathbf{G}'(t)$ is orthogonal to $(\nabla f)(\mathbf{G}(t))$ (i.e. their dot product is zero), $(\nabla f)(\mathbf{G}(t))$ is orthogonal to the curve given by $\mathbf{G}(t)$.

3. (a)
$$\nabla f = \begin{pmatrix} e^x \\ -2y \end{pmatrix}.$$

(b) Since $f(\mathbf{r}(t)) = 0$, we have
$$0 = \frac{d}{dt} f(\mathbf{r}(t)) = (\nabla f)(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = \begin{pmatrix} e^x \\ -2y \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} = e^x x' - 2yy'.$$
Thus, $y' = \dfrac{e^x x'}{2y}$.

(c) Since $y' = \dfrac{e^x x'}{2y}$, we have $\dfrac{dy}{dx} = \dfrac{y'}{x'} = \dfrac{e^x}{2y}$.

4. (a) Theorem 1.3.2 provides $(f \circ \mathbf{G}(t))' = f'(\mathbf{G}(t))\mathbf{G}'(t)$.

(b) Let $\mathbf{G}(t) = \begin{pmatrix} x(t) & y(t) & t \end{pmatrix}^T$ so that $f(x(t), y(t), t) = (f \circ \mathbf{G})(t)$. It follows that
$$\begin{aligned} \frac{d}{dt} f(x(t), y(t), t) = (f \circ \mathbf{G}(t))' &= \left.\frac{\partial f}{\partial x}\right|_{(x,y,t)} \frac{dx}{dt} + \left.\frac{\partial f}{\partial y}\right|_{(x,y,t)} \frac{dy}{dt} + \left.\frac{\partial f}{\partial t}\right|_{(x,y,t)} \frac{dt}{dt} \\ &= x' \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + \frac{\partial f}{\partial t}. \end{aligned}$$

(c) If $f(x(t), y(t), t)$ is constant for $x(t) = e^{-t}$ and $y(t) = t^2$, then
$$0 = \frac{d}{dt} f(e^{-t}, t^2, t) = -e^{-t} \frac{\partial f}{\partial x} + 2t \frac{\partial f}{\partial y} + \frac{\partial f}{\partial t}.$$
Thus, the partial derivatives of f must obey
$$-e^{-t} \frac{\partial f}{\partial x} + 2t \frac{\partial f}{\partial y} + \frac{\partial f}{\partial t} = 0.$$
Since $x(t) = e^{-t}$ we may replace any t with $t = -\ln x$. On the other hand, $y(t) = t^2$ we may replace any t with $t = \sqrt{y}$. This provides 8 more correct answers, although even more are possible.

5. (a)
$$\nabla f = \begin{pmatrix} -2x \sin(x^2 - 3y) \\ 3 \sin(x^2 - 3y) \end{pmatrix}$$

(b) Since $f(\mathbf{r}(t)) = 0$ for all t, we have $\dfrac{d}{dt} f(\mathbf{r}(t)) = 0$. It follows that
$$\begin{aligned} 0 &= \frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \\ &= \begin{pmatrix} -2x(t) \sin(x^2(t) - 3y(t)) \\ 3 \sin(x^2(t) - 3y(t)) \end{pmatrix} \cdot \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} \\ &= -2x(t) \sin(x^2(t) - 3y(t)) x'(t) + 3 \sin(x^2(t) - 3y(t)) y'(t) \\ &= (3y'(t) - 2x(t)x'(t)) \sin(x^2(t) - 3y(t)). \end{aligned}$$
and consequently, $\sin(x^2(t) - 3y(t)) = 0$ or
$$\sin(x^2(t) - 3y(t)) = 0 \qquad \text{or} \qquad y'(t) = \frac{2}{3} x(t) x'(t).$$
When $\sin(x^2(t) - 3y(t)) = 0$ we have $y(t) = \frac{1}{3} x^2(t) + k\pi$ for some $k \in \mathbb{Z}$. In *both cases* we find
$$y'(t) = \frac{2}{3} x(t) x'(t).$$

(c)

$$\frac{dy}{dx} = \frac{\dfrac{dy}{dt}}{\dfrac{dx}{dt}} = \frac{\dfrac{2}{3}xx'}{x'} = \frac{2}{3}x.$$

6. (a)

$$\frac{\partial}{\partial x_j}(F_i \circ \mathbf{G})(\mathbf{x}) = \sum_{k=1}^{p} \frac{\partial F_i}{\partial x_k}\Big|_{\mathbf{x}\to\mathbf{G}(\mathbf{x})} \frac{\partial G_k}{\partial x_j}\Big|_{\mathbf{x}}.$$

(b)

$$\frac{\partial}{\partial x_j}(F_i \circ \mathbf{G})(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial F_i}{\partial x_1} & \cdots & \dfrac{\partial F_i}{\partial x_p} \end{pmatrix}\Big|_{\mathbf{x}\to\mathbf{G}(\mathbf{x})} \begin{pmatrix} \dfrac{\partial G_1}{\partial x_j} \\ \vdots \\ \dfrac{\partial G_p}{\partial x_j} \end{pmatrix}\Bigg|_{\mathbf{x}}.$$

(c)

$$(\mathbf{F}\circ\mathbf{G})'(\mathbf{x}) = \begin{pmatrix} (F_1\circ\mathbf{G})(\mathbf{x}) \\ \vdots \\ (F_m\circ\mathbf{G})(\mathbf{x}) \end{pmatrix}' = \begin{pmatrix} \dfrac{\partial}{\partial x_1}(F_1\circ\mathbf{G})(\mathbf{x}) & \cdots & \dfrac{\partial}{\partial x_n}(F_1\circ\mathbf{G})(\mathbf{x}) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial}{\partial x_1}(F_m\circ\mathbf{G})(\mathbf{x}) & \cdots & \dfrac{\partial}{\partial x_n}(F_m\circ\mathbf{G})(\mathbf{x}) \end{pmatrix}.$$

(d)

$$(\mathbf{F}\circ\mathbf{G})'(\mathbf{x}) = \begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \cdots & \dfrac{\partial F_1}{\partial x_p} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial F_m}{\partial x_1} & \cdots & \dfrac{\partial F_m}{\partial x_p} \end{pmatrix}\Bigg|_{\mathbf{x}\to\mathbf{G}(\mathbf{x})} \begin{pmatrix} \dfrac{\partial G_1}{\partial x_1} & \cdots & \dfrac{\partial G_1}{\partial x_p} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial G_p}{\partial x_1} & \cdots & \dfrac{\partial G_p}{\partial x_p} \end{pmatrix}\Bigg|_{\mathbf{x}} = \mathbf{F}'(\mathbf{G}(\mathbf{x}))\mathbf{G}'(\mathbf{x}).$$

7. We have

$$\begin{aligned}
(\varphi\circ\mathbf{F})'(r,\theta) &= \varphi'(\mathbf{F}(r,\theta)\mathbf{F}'(r,\theta) \\
&= \begin{pmatrix} 2x-6y & 2y-6x \end{pmatrix}\Big|_{(x,y)=(r\cos\theta, r\sin\theta)} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} 2r\cos\theta - 6r\sin\theta & 2r\sin\theta - 6r\cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\
&= \begin{pmatrix} 2r - 12r\cos\theta\sin\theta & 6r^2(\sin^2\theta - \cos^2\theta) \end{pmatrix} \\
&= \begin{pmatrix} 2r - 6r\sin 2\theta & -6r^2\cos 2\theta \end{pmatrix}.
\end{aligned}$$

(This is easy to check using $(\varphi\circ\mathbf{F})(r,\theta) = r^2 - 3r^2\sin 2\theta$.)

8. (a) Straightforward calculation yields

$$\begin{aligned}
d(\mathbf{F}\circ\mathbf{G})[\mathbf{a};\mathbf{h}] &= (\mathbf{F}\circ\mathbf{G})'(\mathbf{a})\mathbf{h} \\
&= \mathbf{F}'(\mathbf{G}(\mathbf{a}))\underbrace{\mathbf{G}'(\mathbf{a})\mathbf{h}} && \text{(chain rule)} \\
&= \mathbf{F}'(\mathbf{G}(\mathbf{a}))d\mathbf{G}[\mathbf{a};\mathbf{h}] \\
&= d\mathbf{F}[\mathbf{G}(\mathbf{a}); d\mathbf{G}[\mathbf{a};\mathbf{h}]].
\end{aligned}$$

(b) The differential

$$d\mathbf{F}[\mathbf{r}(t);\mathbf{r}'(t)] = \mathbf{F}'(\mathbf{r}(t))\mathbf{r}'(t) = (\mathbf{F}\circ\mathbf{r})'(t)$$

is a derivative of the composite function $\mathbf{F}\circ\mathbf{r}$. Every tangent vector $\mathbf{h}$ at the point $\mathbf{r}(t)$ to the curve parametrised by $\mathbf{r}(t)$ is a scalar multiple $\mathbf{h} = k\mathbf{r}'(t)$ of $\mathbf{r}'(t)$. Linearity of the differential (in $\mathbf{h}$) yields that

$$d\mathbf{F}[\mathbf{r}(t);\mathbf{h}] = d\mathbf{F}[\mathbf{r}(t);k\mathbf{r}'(t)] = kd\mathbf{F}[\mathbf{r}(t);\mathbf{r}'(t)] = k(\mathbf{F}\circ\mathbf{r})'(t)$$

which is a tangent vector to the curve parametrised by $(\mathbf{F}\circ\mathbf{r})(t)$ at the point $(\mathbf{F}\circ\mathbf{r})(t)$.

9. The area of the parallelogram P is given by $\operatorname{Area}(P) = |\det\begin{pmatrix}\mathbf{h} & \mathbf{k}\end{pmatrix}|$ (or equivalently, the magnitude of the cross product of $\mathbf{h}$ and $\mathbf{k}$ in $\mathbb{R}^3$). Similarly,

$$\begin{aligned}
\operatorname{Area}(Q) &= |\det\begin{pmatrix}d\mathbf{F}[\mathbf{a};\mathbf{h}] & d\mathbf{F}[\mathbf{a};\mathbf{k}]\end{pmatrix}| \\
&= |\det\begin{pmatrix}\mathbf{F}'(\mathbf{a})\mathbf{h} & \mathbf{F}'(\mathbf{a})\mathbf{k}\end{pmatrix}| \\
&= |\det((\mathbf{F}'(\mathbf{a}))\begin{pmatrix}\mathbf{h} & \mathbf{k}\end{pmatrix}| \\
&= |\det(\mathbf{F}'(\mathbf{a}))\det(\begin{pmatrix}\mathbf{h} & \mathbf{k}\end{pmatrix})| \\
&= |\det(\mathbf{F}'(\mathbf{a}))|\,|\det(\begin{pmatrix}\mathbf{h} & \mathbf{k}\end{pmatrix})| \\
&= |\det(\mathbf{F}'(\mathbf{a}))|\operatorname{Area}(P).
\end{aligned}$$

For a parallelopiped P in $\mathbb{R}^3$ with sides $\mathbf{h}$, $\mathbf{k}$ and $\mathbf{u}$ we have $\operatorname{Area}(P) = |\det\begin{pmatrix}\mathbf{h} & \mathbf{k} & \mathbf{u}\end{pmatrix}|$ (or equivalently, the absolute value of the scalar triple product of $\mathbf{h}$, $\mathbf{k}$ and $\mathbf{u}$). As above

$$\begin{aligned}
\operatorname{Area}(Q) &= |\det\begin{pmatrix}d\mathbf{F}[\mathbf{a};\mathbf{h}] & d\mathbf{F}[\mathbf{a};\mathbf{k}] & d\mathbf{F}[\mathbf{a};\mathbf{u}]\end{pmatrix}| \\
&\;\vdots \\
&= |\det(\mathbf{F}'(\mathbf{a}))|\operatorname{Area}(P).
\end{aligned}$$

10. (a) $(\mathbf{F}\circ\mathbf{G})'(\mathbf{x}) = \mathbf{F}'(\mathbf{G}(\mathbf{x}))\mathbf{G}'(\mathbf{x})$.

(b) Since $(\mathbf{F}\circ\mathbf{G})(\mathbf{x})) = \mathbf{x}$, we have

$$\frac{\partial}{\partial x_i}(F_j\circ\mathbf{G})(\mathbf{x})) = \frac{\partial}{\partial x_i}x_j = \begin{cases}1 & i=j,\\ 0 & i\neq j.\end{cases}$$

It follows that

$$\frac{\partial}{\partial\mathbf{x}}(\mathbf{F}\circ\mathbf{G}) = \det((\mathbf{F}\circ\mathbf{G})') = \det I_n = 1$$

where I_n is the $n\times n$ identity matrix. Applying the chain rule yields

$$\begin{aligned}
\det((\mathbf{F}\circ\mathbf{G})') &= \det(\mathbf{F}'(\mathbf{G}(\mathbf{x}))\mathbf{G}'(\mathbf{x})) \\
&= \det(\mathbf{F}'(\mathbf{G}(\mathbf{x})))\det(\mathbf{G}'(\mathbf{x})) \\
&= \det(\mathbf{F}'(\mathbf{y}))\det(\mathbf{G}'(\mathbf{x})) \\
&= \left.\frac{\partial\mathbf{F}}{\partial\mathbf{x}}\right|_{\mathbf{x}\to\mathbf{y}}\frac{\partial\mathbf{G}}{\partial\mathbf{x}} \\
&= 1.
\end{aligned}$$

Clearly

$$\frac{\partial\mathbf{G}}{\partial\mathbf{x}} = \left(\left.\frac{\partial\mathbf{F}}{\partial\mathbf{x}}\right|_{\mathbf{x}\to\mathbf{y}}\right)^{-1}.$$

(c)

$$\begin{aligned}(\mathbf{F}\circ\mathbf{G})'(\mathbf{x}) &= \mathbf{F}'(\mathbf{G}(\mathbf{x}))\mathbf{G}'(\mathbf{x})\\ &= \begin{pmatrix} D_1F_1 & D_2F_1 & D_3F_1\\ D_1F_2 & D_2F_2 & D_3F_2\end{pmatrix}\Big|_{\mathbf{x}\to\mathbf{G}(\mathbf{x})}\mathbf{G}'(\mathbf{x})\\ &= \begin{pmatrix} D_1F_1 & D_2F_1 & D_3F_1\\ D_1F_2 & D_2F_2 & D_3F_2\end{pmatrix}\Big|_{\mathbf{x}\to\mathbf{G}(\mathbf{x})}\begin{pmatrix} x_2 & x_1\\ 2x_1 & -2x_2\\ 1 & 1\end{pmatrix}\end{aligned}$$

and the last matrix is inconveniently expanded as

$$\begin{pmatrix} x_2(D_1F_1)(\mathbf{G}(\mathbf{x}))+2x_1(D_2F_1)(\mathbf{G}(\mathbf{x}))+(D_3F_1)(\mathbf{G}(\mathbf{x})) & x_1(D_1F_1)(\mathbf{G}(\mathbf{x}))-2x_2(D_2F_1)(\mathbf{G}(\mathbf{x}))+(D_3F_1)(\mathbf{G}(\mathbf{x}))\\ x_2(D_1F_2)(\mathbf{G}(\mathbf{x}))+2x_1(D_2F_2)(\mathbf{G}(\mathbf{x}))+(D_3F_2)(\mathbf{G}(\mathbf{x})) & x_1(D_1F_2)(\mathbf{G}(\mathbf{x}))-2x_2(D_2F_2)(\mathbf{G}(\mathbf{x}))+(D_3F_2)(\mathbf{G}(\mathbf{x}))\end{pmatrix}.$$

(d) Since

$$\mathbf{F}(\mathbf{G}(\mathbf{x})) = \begin{pmatrix} G_1(\mathbf{x})G_2(\mathbf{x})\\ 2G_1(\mathbf{x})+G_3^2(\mathbf{x})\end{pmatrix} = \begin{pmatrix} x_1\\ x_3\end{pmatrix}$$

we find

$$(\mathbf{F}\circ\mathbf{G})'(\mathbf{x}) = \begin{pmatrix} y_2 & y_1 & 0\\ 2 & 0 & 2y_3\end{pmatrix}\Big|_{\mathbf{y}=\mathbf{G}(\mathbf{x})}\mathbf{G}'(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}.$$

On the left hand side we applied the chain rule, on the right hand side we differentiated $\mathbf{F}(\mathbf{G}(\mathbf{x}))$ in the usual way. It follows that

$$\begin{pmatrix} G_2(\mathbf{x}) & G_1(\mathbf{x}) & 0\\ 2 & 0 & 2G_3(\mathbf{x})\end{pmatrix}\begin{pmatrix} D_1G_1 & D_2G_1 & D_3G_1 & D_4G_1\\ D_1G_2 & D_2G_2 & D_3G_2 & D_4G_2\\ D_1G_3 & D_2G_3 & D_3G_3 & D_4G_3\end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}.$$

The first row and first column yields

$$G_2(\mathbf{x})(D_1G_1)(\mathbf{x}) + G_1(\mathbf{x})(D_1G_2)(\mathbf{x}) = 1$$

and similarly for the second row and third column

$$2(D_3G_1)(\mathbf{x}) + 2G_3(\mathbf{x})(D_3G_3)(\mathbf{x}) = 1.$$

Section 1.4: Directional Derivatives

1. The given direction vector is a unit vector. The gradient of φ is

$$\nabla\varphi = \begin{pmatrix} e^{x_2}\\ x_1e^{x_2}\\ -x_4\sin x_3\\ \cos(x_3)\end{pmatrix}.$$

Using Theorem 1.4.2, we find

$$D_{\frac{1}{6}\begin{pmatrix}5\\3\\1\\1\end{pmatrix}}\varphi\begin{pmatrix}2\\3\\\frac{\pi}{4}\\5\end{pmatrix} = \nabla\varphi\begin{pmatrix}2\\3\\\frac{\pi}{4}\\5\end{pmatrix}\cdot\frac{1}{6}\begin{pmatrix}5\\3\\1\\1\end{pmatrix} = \begin{pmatrix} e^3\\ 2e^3\\ -\frac{5}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}\cdot\frac{1}{6}\begin{pmatrix}5\\3\\1\\1\end{pmatrix} = \frac{11}{6}e^3 - \frac{4}{6\sqrt{2}}.$$

2. The given direction vector is not a unit vector. The unit vector in the given direction is $1/\sqrt{6}(2,-1,1)^T$. The gradient of φ is

$$\nabla\varphi = \begin{pmatrix} 2x_1e^{3x_3} \\ -2x_2e^{3x_3} \\ 3(x_1^2-x_2^2)e^{3x_3} \end{pmatrix}.$$

Using Theorem 1.4.2, we find

$$D_{\frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}}\varphi\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \nabla\varphi\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}\cdot\frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6e^3 \\ -4e^3 \\ 15e^3 \end{pmatrix}\cdot\frac{1}{\sqrt{6}}\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \frac{31}{\sqrt{6}}e^3.$$

3. (a)

$$\nabla\phi = \begin{pmatrix} e^y \\ xe^y \\ -2 \end{pmatrix}.$$

(b) The unit vector in the direction $\mathbf{v}$ is $1/\sqrt{14}(2,-3,1)^T$. The rate of increase of ϕ at $(1,0,3)$ in this direction is

$$\nabla\phi(1,0,3)\cdot\frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\cdot\frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = -\frac{3}{\sqrt{14}}.$$

(c) The direction of fastest increase of ϕ at $(1,0,3)$ is in the direction of $\nabla\phi(1,0,3) = (1,1,-2)^T$. Suppose that $\mathbf{v}$ is in the same direction, then $\mathbf{v} = k\nabla\phi(1,0,3)$ for some real k, i.e. $k=2$, $k=-3$ and $-2k=1$ which is obviously impossible! Thus the direction of $\mathbf{v}$ is not the direction of fastest increase. In the direction $\nabla\phi(1,0,3)$, ϕ increases at a rate of $\|\nabla\phi(1,0,3)\| = \|(1,1,-2)^T\| = \sqrt{6}$.

4. (a) Let $\mathbf{u}$ be a unit vector. The directional derivative of f at $\mathbf{x}$ is given by (Theorem 1.4.2)

$$\nabla f(\mathbf{x})\cdot\mathbf{u} = \|\nabla f(\mathbf{x})\|\,\|\mathbf{u}\|\,\cos\theta = \|\nabla f(\mathbf{x})\|\,\cos\theta$$

where θ is the angle between the vectors $\nabla f(\mathbf{x})$ and $\mathbf{u}$. The directional derivative of f at $\mathbf{x}$ gives the rate of increase of f in the direction $\mathbf{u}$. The fastest increase is obviously obtained when $\cos\theta = 1$, i.e. when $\mathbf{u}$ has the same direction as $\nabla f(\mathbf{x})$.

(b) The direction of fastest increase is

$$\nabla f(2,\ln 3,-2) = \left.\begin{pmatrix} 2x_1e^{x_2} \\ (x_1^2-x_3)e^{x_2} \\ -e^{x_2} \end{pmatrix}\right|_{x_1=2,x_2=\ln 3,x_3=-1} = \begin{pmatrix} 12 \\ 15 \\ -3 \end{pmatrix}.$$

The unit vector in the direction of $\mathbf{v}$ is $1/\sqrt{30}(-1,2,-5)$. The directional derivative of f at $(2,\ln 3,-1)$ in the direction of $\mathbf{v}$ is

$$\nabla f(2,\ln 3,-2)\cdot\frac{1}{\sqrt{30}}\begin{pmatrix} -1 \\ 2 \\ -5 \end{pmatrix} = \begin{pmatrix} 12 \\ 15 \\ -3 \end{pmatrix}\cdot\frac{1}{\sqrt{30}}\begin{pmatrix} -1 \\ 2 \\ -5 \end{pmatrix} = \frac{33}{\sqrt{30}}.$$

5. (a) If f is constant on $\mathbf{r}(t) = t\mathbf{u}+\mathbf{x}_0$, then $f(\mathbf{r}(t)) = c$ for all t, where c is some constant. It follows that

$$\frac{d}{dt}f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t))\cdot\mathbf{r}'(t) = \nabla f(\mathbf{r}(t))\cdot\mathbf{u} = 0 \qquad (4)$$

for all t. At $t = 0$, we have $\mathbf{r}(0) = \mathbf{x}_0$. By Theorem 1.4.2, we have

$$D_{\mathbf{u}} f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \cdot \mathbf{u} = \nabla f(\mathbf{r}(0)) \cdot \mathbf{u} = 0$$

where we used (4) at $t = 0$.

(b) Let $\mathbf{r}(t) = t\mathbf{u} + \mathbf{x}_0$ parametrise the line through $\mathbf{x}_0$ in the direction $\mathbf{u}$. Note that we are *not assuming* that f is constant on this line (i.e. we are not continuing part (a)). Let $h(t) = (f \circ \mathbf{r})(t)$. Then $h(t)$ has a minimum (respectively maximum) at $t = 0$ if f has a minimum (respectively maximum) at $\mathbf{x}_0$.
(Note: the previous sentence is not "if and only if"!).

If f has a minimum (maximum) at $\mathbf{x}_0$ then h has a minimum (maximum) at $t = 0$, i.e. $h'(0) = 0$. In other words,

$$0 = h'(0) = (f \circ \mathbf{r})'(0) = \nabla f(\mathbf{r}(0)) \cdot \mathbf{r}'(0) = \nabla f(\mathbf{x}_0) \cdot \mathbf{u} = D_{\mathbf{u}} f(\mathbf{x}_0).$$

(c) Let $\mathbf{v}$ be any non-zero vector, $\mathbf{r}(t) = t\mathbf{v} + \mathbf{x}$ and $h(t) = (f \circ \mathbf{r})(t)$. Then

$$h'(t) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(0) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{v} = 0$$

for all t. Thus $h(t)$ is constant, and $f(\mathbf{x})$ is constant on the line $\mathbf{r}(t)$. However, the choice of $\mathbf{v}$ and $\mathbf{x}$ was arbitrary. Let $\mathbf{v} = \mathbf{y} - \mathbf{x}$ where $\mathbf{y} \neq \mathbf{x}$. Then $h(0) = h(1)$ so that

$$f(\mathbf{x}) = f(\mathbf{r}(0)) = h(0) = h(1) = f(\mathbf{r}(1)) = f(\mathbf{y}).$$

Section 1.5: Tangents and Normals

1. (a) A tangent vector is $G'(t)$, or any non-zero scalar multiple thereof.

   (b) If all tangent vectors $k\mathbf{G}'(t)$ (k a non-zero scalar) at $\mathbf{G}(t)$ are orthogonal to $\nabla f(\mathbf{G}(t))$, then for $k = 1$

   $$(f \circ \mathbf{G})(t) = \nabla f(\mathbf{G}(t)) \cdot \mathbf{G}'(t) = 0$$

   for all t. Thus $f \circ \mathbf{G}$ is constant. Since all points on the curve C are given by $\mathbf{G}(t)$ for some t, f is constant on the curve C parametrised by $\mathbf{G}(t)$.

2. (a) Since $\varphi(\mathbf{b}) = \varphi\begin{pmatrix}3\\5\\2\end{pmatrix} = 3^2 \cdot 5 - 6 \cdot 2^3 = 45 - 48 = -3$, it follows that $\mathbf{p} \in S$.

   (b) A normal vector to S at $\mathbf{p}$ is

   $$\nabla\varphi(\mathbf{p}) = \begin{pmatrix}2x_1x_2\\x_1^2\\-18x_3^2\end{pmatrix}\Bigg|_{\mathbf{p}} = \begin{pmatrix}30\\9\\-72\end{pmatrix}.$$

   (c) The tangent vectors of S at $\mathbf{p}$ are

   $$\begin{aligned}
   T_{\mathbf{p}} &= \left\{ \begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3 \,:\, \begin{pmatrix}a\\b\\c\end{pmatrix} \cdot \begin{pmatrix}30\\9\\-72\end{pmatrix} = 0 \right\} \\
   &= \left\{ \begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3 \,:\, 30a + 9b - 72c = 0 \right\} \\
   &= \left\{ \begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3 \,:\, b = 8c - \frac{10}{3}a \right\} \\
   &= \left\{ \begin{pmatrix}a\\8c - \frac{10}{3}a\\c\end{pmatrix} \,:\, a, c \in \mathbb{R}, \right\}.
   \end{aligned}$$

Note that there are many different ways to describe this set!

(d) The tangent plane to S at $\mathbf{p}$ is given by

$$\mathbf{p} + T_{\mathbf{p}} = \{\, \mathbf{p} + \mathbf{t} \,:\, \mathbf{t} \in T_{\mathbf{p}} \,\} = \left\{ \begin{pmatrix} 3+a \\ 5+8c-\frac{10}{3}a \\ 2+c \end{pmatrix} \;:\; a, c \in \mathbb{R}, \right\}.$$

We could also write

$$\mathbf{p} + T_{\mathbf{p}} = \left\{ \begin{pmatrix} 3 \\ 5 \\ 2 \end{pmatrix} + a \begin{pmatrix} 1 \\ -\frac{10}{3} \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 8 \\ 1 \end{pmatrix} \;:\; a, c \in \mathbb{R}, \right\}$$

which explicitly gives two spanning vectors for the plane. Again, there are (infinitely) many pairs of spanning vectors for the plane and infinitely many ways to describe the plane.

(e) If $\mathbf{q}$ lies in the tangent plane to S at $\mathbf{p}$, then

$$\begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 3+a \\ 5+8c-\frac{10}{3}a \\ 2+c \end{pmatrix}$$

for some a and c. Consequently

$$6 = 3 + a \qquad 3 = 5 + 8c - \frac{10}{3}a \qquad 3 = 2 + c.$$

Obviously $a = 3$ and $c = 1$ and the middle equation is satisfied: $5 + 8 \cdot 1 - \frac{10}{3} \cdot 3 = 3$. Thus $\mathbf{q}$ does lie in the tangent plane to S at $\mathbf{p}$.

Alternatively, $\mathbf{q}$ lies in the tangent plane to S at $\mathbf{p}$ if $(\mathbf{q} - \mathbf{p}) \cdot \nabla\varphi(\mathbf{p}) = 0$, and indeed

$$(\mathbf{q} - \mathbf{p}) \cdot \nabla\varphi(\mathbf{p}) = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 30 \\ 9 \\ -72 \end{pmatrix} = 90 - 18 - 72 = 0.$$

(f) Since $\mathbf{q}$ is in the tangent plane to S at $\mathbf{p}$, and every tangent to S is a vector (not a point) which lies in that plane, $\mathbf{q} - \mathbf{p}$ is tangent to S at $\mathbf{p}$. However,

$$\mathbf{q} \cdot \nabla\varphi(\mathbf{p}) = \begin{pmatrix} 6 \\ 3 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 30 \\ 9 \\ -72 \end{pmatrix} = -9 \neq 0$$

so that $\mathbf{q}$ is not tangent to S at $\mathbf{p}$.

3. (a) Since $\varphi(1, 2, 1) = 1 + 2 - 4 = -1$, it follows that $(1, 2, 1) \in S$.

(b) A normal to S at $(1, 2, 1)$ is

$$\nabla\varphi(1, 2, 1) = \begin{pmatrix} 4x \\ -2y \\ 1 \end{pmatrix}\Bigg|_{(x,y,z)=(1,2,1)} = \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}.$$

(c) The tangent vectors to S at $(1,2,1)$ are given by

$$T_{(1,2,1)} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \, : \, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \nabla\varphi(1,2,1) = 0 \right\}$$
$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \, : \, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix} = 0 \right\}$$
$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \, : \, 4a - 4b + c = 0 \right\}$$
$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \, : \, c = 4b - 4a \right\}$$
$$= \left\{ \begin{pmatrix} a \\ b \\ 4b - 4a \end{pmatrix} \, : \, a, b \in \mathbb{R} \right\}.$$

(d) The tangent plane to S at $(1,2,1)$ is given by

$$P = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + T_{(1,2,1)} = \left\{ \begin{pmatrix} 1+a \\ 2+b \\ 1 + 4b - 4a \end{pmatrix} \, : \, a, c \in \mathbb{R}, \right\}.$$

(e) For each point $\mathbf{r}(t)$, $t \in \mathbb{R}$, on the line, we have

$$\left(\mathbf{r}(t) - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right) \cdot \nabla\varphi(1,2,1) = \begin{pmatrix} t+1 \\ 2t+1 \\ 4t \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix} = 4t + 4 - 8t - 4 + 4t = 0$$

so that the points $\mathbf{r}(t)$ lie in P for all $t \in \mathbb{R}$.

4. (a) Since $\varphi(3/2, \sqrt{3}, 0) = 9 + 27 + 0 = 36$, $\mathbf{x}_0$ lies on S.

(b) A normal to S at $\mathbf{x}_0$ is

$$\nabla\varphi(3/2, \sqrt{3}, 0) = \begin{pmatrix} 8x \\ 18y \\ 2z \end{pmatrix} \Bigg|_{(x,y,z)=(3/2,\sqrt{3},0)} = \begin{pmatrix} 12 \\ 18\sqrt{3} \\ 0 \end{pmatrix}.$$

(c) The tangent vectors to S at $\mathbf{x}_0$ are given by

$$T_{\mathbf{x}_0} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \, : \, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \nabla\varphi(3/2, \sqrt{3}, 0) \right\}$$
$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \, : \, \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 18\sqrt{3} \\ 0 \end{pmatrix} \right\}$$
$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \, : \, 12a + 18\sqrt{3}b = 0 \right\}$$
$$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \, : \, a = -\frac{3}{2}\sqrt{3}b \right\}$$

$$= \left\{ \begin{pmatrix} -\frac{3}{2}\sqrt{3}b \\ b \\ c \end{pmatrix} \ : \ b, c \in \mathbb{R} \right\}.$$

(d)

$$\begin{aligned}
\varphi(\mathbf{r}(u,v)) &= 36\cos^2 u \cos^2 v + 36\sin^2 u \cos^2 v + 36\sin^2 v \\
&= 36(\cos^2 u + \sin^2 u)\cos^2 v + 36\sin^2 v \\
&= 36\cos^2 v + 36\sin^2 v = 36 \\
\mathbf{r}(\pi/3, 0) &= \begin{pmatrix} 3\cos\frac{\pi}{3}\cos 0 \\ 2\sin\frac{\pi}{3}\cos 0 \\ 6\sin 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \sqrt{3} \\ 0 \end{pmatrix} = \mathbf{x}_0.
\end{aligned}$$

(e) The partial derivatives are

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial u}(\pi/3, 0) &= \left.\begin{pmatrix} -3\sin u \cos v \\ 2\cos u \cos v \\ 0 \end{pmatrix}\right|_{(u,v)=(\pi/3,0)} = \begin{pmatrix} -\frac{3\sqrt{3}}{2} \\ 1 \\ 0 \end{pmatrix} \\
\frac{\partial \mathbf{r}}{\partial v}(\pi/3, 0) &= \left.\begin{pmatrix} -3\cos u \sin v \\ -2\cos u \sin v \\ 6\cos v \end{pmatrix}\right|_{(u,v)=(\pi/3,0)} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}
\end{aligned}$$

Now

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial u}(\pi/3, 0) \cdot \nabla\varphi(\mathbf{x}_0) &= \begin{pmatrix} -\frac{3\sqrt{3}}{2} \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 18\sqrt{3} \\ 0 \end{pmatrix} = -18\sqrt{3} + 18\sqrt{3} = 0 \\
\frac{\partial \mathbf{r}}{\partial v}(\pi/3, 0) \cdot \nabla\varphi(\mathbf{x}_0) &= \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 18\sqrt{3} \\ 0 \end{pmatrix} = 0
\end{aligned}$$

so that $\dfrac{\partial \mathbf{r}}{\partial u}(\pi/3, 0)$, $\dfrac{\partial \mathbf{r}}{\partial v}(\pi/3, 0) \in T_{\mathbf{x}_0}$. Finally,

$$\frac{\partial \mathbf{r}}{\partial u}(\pi/3, 0) \cdot \nabla\varphi(\mathbf{x}_0) \times \frac{\partial \mathbf{r}}{\partial v}(\pi/3, 0) \cdot \nabla\varphi(\mathbf{x}_0) = \begin{pmatrix} -\frac{3\sqrt{3}}{2} \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 9\sqrt{3} \\ 0 \end{pmatrix} = \frac{1}{2}\nabla\varphi(\mathbf{x}_0).$$

Since the cross product $\frac{\partial \mathbf{r}}{\partial u}(\pi/3, 0) \cdot \nabla\varphi(\mathbf{x}_0) \times \frac{\partial \mathbf{r}}{\partial v}(\pi/3, 0) \cdot \nabla\varphi(\mathbf{x}_0)$ is a multiple of the gradient $\nabla\varphi(\mathbf{x}_0)$ at $\mathbf{x}_0$, it is normal to S at $\mathbf{x}_0$.

5. (a) Obviously $\mathbf{r}(u,v) \in S$ (since it is the parametric form for S), i.e. $\varphi(\mathbf{r}(u,v)) = 0$ for all $u, v \in \mathbb{R}$. The chain rule provides

$$\begin{aligned}
\begin{pmatrix} 0 & 0 \end{pmatrix} &= (\varphi \circ \mathbf{r})'(u,v) \\
&= \varphi'(\mathbf{r}(u,v))\mathbf{r}'(u,v) \\
&= (\nabla\varphi(\mathbf{r}(u,v)))^T \begin{pmatrix} \frac{\partial \mathbf{r}}{\partial u}(u,v) & \frac{\partial \mathbf{r}}{\partial v}(u,v) \end{pmatrix} \\
&= \begin{pmatrix} \nabla\varphi(\mathbf{r}(u,v)) \cdot \frac{\partial \mathbf{r}}{\partial u}(u,v) & \nabla\varphi(\mathbf{r}(u,v)) \cdot \frac{\partial \mathbf{r}}{\partial v}(u,v) \end{pmatrix}
\end{aligned}$$

and since $\nabla\varphi(\mathbf{r}(u,v)) \cdot \dfrac{\partial \mathbf{r}}{\partial u}(u,v) = 0$ we have $\dfrac{\partial \mathbf{r}}{\partial u}(u,v) \in T_{\mathbf{r}(u,v)}$. Similarly, $\dfrac{\partial \mathbf{r}}{\partial v}(u,v) \in T_{\mathbf{r}(u,v)}$.

(b) First, suppose that $\dfrac{\partial \mathbf{r}}{\partial u}(u,v)$ and $\dfrac{\partial \mathbf{r}}{\partial v}(u,v)$ are linearly independent. Since the cross product

$$\frac{\partial \mathbf{r}}{\partial u}(u,v) \times \frac{\partial \mathbf{r}}{\partial v}(u,v)$$

is perpendicular to the plane which contains $\dfrac{\partial \mathbf{r}}{\partial u}(u,v)$ and $\dfrac{\partial \mathbf{r}}{\partial v}(u,v)$, the cross product is perpendicular to $\mathbf{r}(u,v) + T_{\mathbf{r}(u,v)}$ (the tangent plane to S at $\mathbf{r}(u,v)$) and hence also normal to S at $\mathbf{r}(u,v)$.

However, if $\dfrac{\partial \mathbf{r}}{\partial u}(u,v)$ and $\dfrac{\partial \mathbf{r}}{\partial v}(u,v)$ are linearly dependent, then

$$\frac{\partial \mathbf{r}}{\partial u}(u,v) \times \frac{\partial \mathbf{r}}{\partial v}(u,v) = \mathbf{0}$$

which is trivially normal to S at $\mathbf{r}(u,v)$.

6. Since $\mathbf{x}_0$ is a regular point of S, we have $\nabla\varphi(\mathbf{x}_0) \neq \mathbf{0}$. There are different approaches to prove the given statement. For example, let A be the matrix

$$A = \nabla\varphi(\mathbf{x}_0)(\nabla\varphi(\mathbf{x}_0))^T.$$

We can prove that $\operatorname{rank}(A) = 1$ (can you show this?), it follows immediately that the null space of A is $T_{\mathbf{x}_0}$ which must have dimension $n - \operatorname{rank}(A) = n - 1$. Another (equivalent) approach is the following. Since $\nabla\varphi(\mathbf{x}_0) \neq \mathbf{0}$, we can find an orthogonal basis

$$B = \{\, \mathbf{b}_1 = \nabla\varphi(\mathbf{x}_0),\, \mathbf{b}_2,\, \dots,\, \mathbf{b}_n \,\}$$

for $\mathbb{R}^n$ (for example, by applying the Gram-Schmidt process). Then

$$\begin{aligned}
T_{\mathbf{x}_0} &= \{\, \mathbf{v} = \beta_1\mathbf{b}_1 + \cdots + \beta_n\mathbf{b}_n \,:\, \nabla\varphi(\mathbf{x}_0)\cdot\mathbf{v} = 0 \,\} \\
&= \{\, \mathbf{v} = \beta_1\mathbf{b}_1 + \cdots + \beta_n\mathbf{b}_n \,:\, \mathbf{b}_1\cdot\mathbf{v} = 0 \,\} \\
&= \{\, \mathbf{v} = \beta_1\mathbf{b}_1 + \cdots + \beta_n\mathbf{b}_n \,:\, \beta_1 = 0 \,\} \\
&= \{\, \beta_2\mathbf{b}_2 + \cdots + \beta_n\mathbf{b}_n \,\} \\
&= \operatorname{span}\{\, \mathbf{b}_2,\, \dots,\, \mathbf{b}_n \,\}
\end{aligned}$$

which has dimension $n-1$.

7. First note that $(\mathbf{F}^{-1}\circ\mathbf{F})(\mathbf{x}) = \mathbf{x}$ so that

$$(\mathbf{F}^{-1}\circ\mathbf{F})'(\mathbf{x}) = (\mathbf{F}^{-1})'(\mathbf{F}(\mathbf{x}))\mathbf{F}'(\mathbf{x}) = I_n$$

where I_n is the $n\times n$ identity matrix. Thus $(\mathbf{F}^{-1})'(\mathbf{F}(\mathbf{x}))$ is invertible for all $\mathbf{x}\in\mathbb{R}^n$. Secondly, $\nabla f(\mathbf{x})\cdot\mathbf{y} = f'(\mathbf{x})\mathbf{y}$ and $\nabla f(\mathbf{x}) = [f'(\mathbf{x})]^T$ fot all $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$.

(a) We have

$$\begin{aligned}
\mathbf{F}(S) &= \{\, \mathbf{F}(\mathbf{x}) \,:\, \mathbf{x}\in S \,\} \\
&= \{\, \mathbf{F}(\mathbf{x}) \,:\, \mathbf{F}^{-1}(\mathbf{F}(\mathbf{x}))\in S \,\} \\
&= \{\, \mathbf{y} \,:\, \mathbf{F}^{-1}(\mathbf{y})\in S \,\} \\
&= \{\, \mathbf{y} \,:\, \varphi(\mathbf{F}^{-1}(\mathbf{y})) = 0 \,\} \\
&= \{\, \mathbf{y} \,:\, \psi(\mathbf{y}) = 0 \,\}
\end{aligned}$$

where $\psi(\mathbf{y}) := \varphi(\mathbf{F}^{-1}(\mathbf{y}))$.

(b) Since $\mathbf{x}_0$ is a regular point of S, $\nabla\varphi(\mathbf{x}_0) \neq 0$ and

$$\begin{aligned}
\nabla\psi(\mathbf{F}(\mathbf{x}_0)) &= \nabla(\varphi \circ \mathbf{F}^{-1})(\mathbf{F}(\mathbf{x}_0)) \\
&= [(\varphi \circ \mathbf{F}^{-1})'(\mathbf{F}(\mathbf{x}_0))]^T \\
&= [\varphi'(\mathbf{F}^{-1}(\mathbf{F}(\mathbf{x}_0)))(\mathbf{F}^{-1})'(\mathbf{F}(\mathbf{x}_0))]^T \\
&= [\varphi'(\mathbf{x}_0)(\mathbf{F}^{-1})'(\mathbf{F}(\mathbf{x}_0))]^T \\
&= [(\mathbf{F}^{-1})'(\mathbf{F}(\mathbf{x}_0))]^T \nabla\varphi(\mathbf{x}_0) \neq \mathbf{0},
\end{aligned}$$

where $\nabla\varphi(\mathbf{x}_0) \neq 0$ and $(\mathbf{F}^{-1})'(\mathbf{F}(\mathbf{x}_0))$ is invertible (and hence maps non-zero vectors to non-zero vectors), we have that $\mathbf{F}(\mathbf{x}_0)$ is a regular point of $\mathbf{F}(S)$.

(c) Let $\mathbf{v} \in T_{\mathbf{x}_0}(S)$ be a tangent vector to S at $\mathbf{x}_0$. In other words $\nabla\varphi(\mathbf{x}_0) \cdot \mathbf{v} = 0$. Then

$$\begin{aligned}
\nabla\psi(\mathbf{F}(\mathbf{x}_0)) \cdot (\mathbf{F}'(\mathbf{x}_0)\mathbf{v}) &= \psi'(\mathbf{F}(\mathbf{x}_0))\mathbf{F}'(\mathbf{x}_0)\mathbf{v} \\
&= (\psi \circ \mathbf{F})'(\mathbf{x}_0)\mathbf{v} \\
&= (\varphi \circ \mathbf{F}^{-1} \circ \mathbf{F})'(\mathbf{x}_0)\mathbf{v} \\
&= \varphi'(\mathbf{x}_0)\mathbf{v} \\
&= \nabla\varphi(\mathbf{x}_0) \cdot \mathbf{v} = 0.
\end{aligned}$$

Consequently, every $\mathbf{v} \in T_{\mathbf{x}_0}(S)$ satisfies $\mathbf{F}'(\mathbf{x}_0)\mathbf{v} \in T_{\mathbf{F}(\mathbf{x}_0)}(\mathbf{F}(S))$. Thus we have shown that $\mathbf{F}'(\mathbf{x}_0)T_{\mathbf{x}_0}(S) \subset T_{\mathbf{F}(\mathbf{x}_0)}(\mathbf{F}(S))$.

(d) Since $\det \mathbf{F}'(\mathbf{x}_0) \neq 0$, the matrix $\mathbf{F}'(\mathbf{x}_0)$ describes a one-to-one map from $T_{\mathbf{x}_0}(S)$ to $T_{\mathbf{F}(\mathbf{x}_0)}(\mathbf{F}(S))$. Furthermore, both tangent spaces have dimension $n-1$ (see question 6) so we must have $\mathbf{F}'(\mathbf{x}_0)T_{\mathbf{x}_0}(S) = T_{\mathbf{F}(\mathbf{x}_0)}(\mathbf{F}(S))$. Finally,

$$T_{\mathbf{F}(\mathbf{x}_0)}(\mathbf{F}(S)) = \mathbf{F}'(\mathbf{x}_0)T_{\mathbf{x}_0}(S) = \{\, \mathbf{F}'(\mathbf{x}_0)\mathbf{v} \,:\, \mathbf{v} \in T_{\mathbf{x}_0}(S) \,\} = \{\, d\mathbf{F}[\mathbf{x}_0; \mathbf{v}] \,:\, \mathbf{v} \in T_{\mathbf{x}_0}(S) \,\}.$$

Section 1.6: Maxima and Minima

1. (a) The critical points are given by

$$\nabla f(x_1, x_2) = \begin{pmatrix} 2x_1 - 2x_2 \cos x_1 \\ -\sin x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

i.e. we solve the simultaneous equations

$$\begin{cases} 2x_1 - 2x_2 \cos x_1 = 0, \\ -2\sin x_1 = 0 \end{cases}$$

The second equation, $-2\sin x_1 = 0$, yields $x_1 = k\pi$ for $k \in \mathbb{Z}$. The first first equation, for $x_1 = k\pi$, becomes $2k\pi - 2x_2(-1)^k = 0$. In other words,

$$\begin{cases} 2x_1 - 2x_2 \cos x_1 = 0, \\ -2\sin x_1 = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} x_1 = k\pi, \\ x_2 = (-1)^k k\pi, \end{cases} \qquad k \in \mathbb{Z}.$$

There are infinitely many critical points. For example, $k \in \{-1, 0, 1, 2\}$ yields the critical points $(-\pi, \pi)$, $(0, 0)$, $(\pi, -\pi)$ and $(2\pi, 2\pi)$. The infinitely many critical points are

$$\begin{pmatrix} k\pi \\ (-1)^k k\pi \end{pmatrix}, \qquad k \in \mathbb{Z}.$$

We have

$$\frac{\partial^2 f}{\partial x_1^2} = 2 + 2x_2 \sin x_1, \qquad \frac{\partial^2 f}{\partial x_2^2} = 0, \qquad \frac{\partial^2 f}{\partial x_1 x_2} = -2\cos x_1.$$

and the discriminant is given by

$$\Delta = \frac{\partial^2 f}{\partial x_1^2}\frac{\partial^2 f}{\partial x_2^2} - \left(\frac{\partial^2 f}{\partial x_1 \partial x_2}\right)^2 = -4\cos^2 x_1 \leq 0.$$

At the critical point given by $x_1 = k\pi$, we have $\Delta = -4$, i.e. every critical point is a saddle point.

(b) The critical points are given by

$$\nabla f(x,y) = \begin{pmatrix} 3x^2 - 4xy + y^2 \\ -2x^2 + 2xy + 3y^2 - 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The first equation, $3x^2 - 4xy + y^2 = 0$, is a quadratic form so let's try to solve it first (the second equation is also quadratic, but not a quadratic form). The first equation may be rewritten as

$$3x^2 - 3xy - xy + y^2 = 3x(x-y) - y(x-y) = (3x-y)(x-y) = 0$$

which provides the solution $y = 3x$, i.e.

$$\begin{cases} y = 3x \\ -2x^2 + 2xy + 3y^2 - 3 = 0 \end{cases} \Rightarrow \begin{cases} y = 3x \\ 31x^2 = 3 \end{cases} \tag{5}$$

and the solution $y = x$, i.e.

$$\begin{cases} y = x \\ -2x^2 + 2xy + 3y^2 - 3 = 0 \end{cases} \Rightarrow \begin{cases} y = x \\ 3x^2 = 3 \end{cases} \tag{6}$$

Now equation (5) yields two critical points, namely $(\sqrt{3/31},\, 3\sqrt{3/31})$ and $(-\sqrt{3/31},\, -3\sqrt{3/31})$. Similarly equation (6) yields the two critical points $(1,1)$ and $(1,-1)$. We have

$$\frac{\partial^2 f}{\partial x^2} = 6x - 4y, \qquad \frac{\partial^2 f}{\partial y^2} = 2x + 6y, \qquad \frac{\partial^2 f}{\partial xy} = -4x + 2y.$$

and the discriminant is given by

$$\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = (6x-4y)(2x+6y) - (-4x+2y)^2 = -4x^2 + 44xy - 28y^2.$$

We tabulate and classify the critical points:

(x,y)	$\Delta(x,y)$		$f_{xx}(x,y)$		
$(1,1)$	12	>0	2	>0	strict local min.
$(-1,-1)$	12	>0	-2	<0	strict local max.
$(\sqrt{3/31},\, 3\sqrt{3/31})$	-4	<0			saddle point
$(-\sqrt{3/31},\, -3\sqrt{3/31})$	-4	<0			saddle point

(c) The critical points are given by

$$\nabla f(x,y) = \begin{pmatrix} 3y - 3x^2 \\ 3x + 3y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From $3x + 3y = 0$ we find $y = -x$ which substituted into $3y - 3x^2 = 0$ yields $x(x+1) = 0$. Thus we have two critical points, $(x,y) = (0,0)$ and $(x,y) = (-1,1)$. We have

$$\frac{\partial^2 f}{\partial x^2} = -6x, \qquad \frac{\partial^2 f}{\partial y^2} = 3, \qquad \frac{\partial^2 f}{\partial x \partial y} = 3.$$

and the discriminant is given by

$$\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = -18x - 9.$$

We tabulate and classify the critical points:

(x,y)	$\Delta(x,y)$		$f_{xx}(x,y)$		
$(0,0)$	-9	<0			saddle point
$(-1,1)$	9	>0	6	>0	strict local min.

(d) The critical points are given by

$$\nabla f(x,y) = \begin{pmatrix} 4y - 4x^3 \\ 4x - 4y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Clearly, $x = y$ and $x(x^2 - 1) = 0$ which yields the three critical points $(0,0)$, $(1,1)$ and $(-1,-1)$. We have

$$\frac{\partial^2 f}{\partial x^2} = -12x^2, \qquad \frac{\partial^2 f}{\partial y^2} = -4, \qquad \frac{\partial^2 f}{\partial x \partial y} = 4.$$

and the discriminant is given by

$$\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 48x^2 - 16.$$

We tabulate and classify the critical points:

(x,y)	$\Delta(x,y)$		$f_{xx}(x,y)$		
$(0,0)$	-16	<0			saddle point
$(1,1)$	32	>0	-12	<0	strict local max.
$(-1,-1)$	32	>0	-12	<0	strict local max.

(e) The critical points are given by

$$\nabla f(x,y) = \begin{pmatrix} 2x(2 + \cos(x+2y)) - (1+x^2)\sin(x+2y) \\ -2(1+x^2)\sin(x+2y) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Since $1 + x^2 \neq 0$ (for real solutions), the second equation yields $\sin(x + 2y) = 0$, i.e. $x = k\pi - 2y$ where $k \in \mathbb{Z}$. The first equation reduces to

$$2(k\pi - 2y)(2 + (-1)^k) = 0$$

and since $2 + (-1)^k \neq 0$ for all $k \in \mathbb{Z}$ we must have $y = k\pi/2$ (and hence $x = 0$). We have

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= 4 + (1 - x^2)\cos(x+2y) - 4x\sin(x+2y), \\
\frac{\partial^2 f}{\partial y^2} &= -4(1+x^2)\cos(x+2y), \\
\frac{\partial^2 f}{\partial x \partial y} &= -4x\sin(x+2y) - 2(1+x^2)\cos(x+2y).
\end{aligned}$$

and the discriminant is given by

$$\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2.$$

At the critical point $(0, k\pi/2)$ (we have infinitely many critical points) we find

$$\Delta = 4(4 + (-1)^k)(-1)^{k+1} - 4$$

so that $\Delta > 0$ when k is odd, and $\Delta < 0$ when k is even. Here we used the fact that $4 + (-1)^k > 1$ for all $k \in \mathbb{Z}$. We have $f_{yy}(0, k\pi/2) = 4(-1)^{k+1}$. We tabulate and classify the (infinitely many) critical points:

(x,y)		$\Delta(x,y)$		$f_{yy}(x,y)$		
$(0,k\pi/2)$	k even	-24	<0			saddle point
$(0,k\pi/2)$	k odd	8	>0	4	>0	strict local min.

(f) The critical points are given by

$$\nabla f(x,y) = \begin{pmatrix} 9x^2+3y-2x \\ 3x+6y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The second equation provides $x=-2y$, and so the first equation becomes $36y^2+7y=0$, i.e. we have the two critical points $(0,0)$ and $(7/18,-7/36)$. We have

$$\frac{\partial^2 f}{\partial x^2} = 18x-2, \qquad \frac{\partial^2 f}{\partial y^2} = 6, \qquad \frac{\partial^2 f}{\partial x \partial y} = 3.$$

and the discriminant is given by

$$\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 108x-21.$$

We tabulate and classify the critical points:

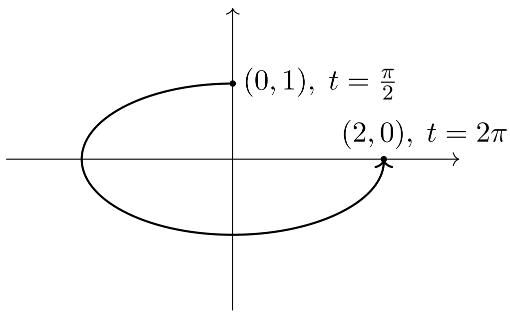
(x,y)	$\Delta(x,y)$		$f_{xx}(x,y)$		
$(0,0)$	-21	<0			saddle point
$(7/18,-7/36)$	21	>0	5	>0	strict local min.

Section 2.2: Scalar Path Integrals

1. (a) $\displaystyle\int_\Gamma f\,ds = \int_a^b f(\mathbf{r}(t)) \left\| \frac{d\mathbf{r}}{dt} \right\| dt.$

   (b) The parametrisation

$$\left\{ \mathbf{r}(t) \,:\, \mathbf{r}(t) = \begin{pmatrix} 2\cos t \\ \sin t \end{pmatrix},\ t \in \left[\frac{\pi}{2}, 2\pi\right] \right\}$$



yields

$$f(\mathbf{r}(t)) = f(2\cos t, \sin t) = \sqrt{4\cos^2 t + 16\sin^2 t},$$
$$\frac{d\mathbf{r}}{dt} = \begin{pmatrix} -2\sin t \\ \cos t \end{pmatrix}, \qquad \left\| \frac{d\mathbf{r}}{dt} \right\| = \sqrt{4\sin^2 t + \cos^2 t}$$

so that

$$
\begin{aligned}
\int_\Gamma f\,ds &= \int_{\frac{\pi}{2}}^{2\pi} \sqrt{4\cos^2 t + 16\sin^2 t}\,\sqrt{4\sin^2 t + \cos^2 t}\,dt \\
&= 2\int_{\frac{\pi}{2}}^{2\pi} (4\sin^2 t + \cos^2 t)\,dt \\
&= 2\int_{\frac{\pi}{2}}^{2\pi} (3\sin^2 t + 1)\,dt \\
&= 2\int_{\frac{\pi}{2}}^{2\pi} \left(\frac{3}{2}(1-\cos(2t)) + 1\right)\,dt \qquad\qquad (2\sin^2 t = 1 - \cos t) \\
&= 2\left[\frac{5}{2}t - \frac{3}{2}\sin(2t)\right]_{\frac{\pi}{2}}^{2\pi} \\
&= 10\pi - \frac{5}{2}\pi = \frac{15}{2}\pi.
\end{aligned}
$$

Of course, other parametrisations are possible and yield the same result.

2. Since cube roots are unique in $\mathbb{R}$, $x = (y^2)^{1/3}$ which yields the parametrisation

$$
\left\{ \mathbf{r}(t) \,:\, \mathbf{r}(t) = \begin{pmatrix} t^{\frac{2}{3}} \\ t \end{pmatrix},\ t \in [-1,1] \right\}.
$$

The length of the curve Γ is

$$
\begin{aligned}
\int_\Gamma 1\,ds &= \int_{-1}^{1} \|\mathbf{r}'(t)\|\,dt = \int_{-1}^{1} \left\| \begin{pmatrix} \frac{2}{3}t^{-\frac{1}{3}} \\ 1 \end{pmatrix} \right\|\,dt \\
&= \int_{-1}^{1} \sqrt{\frac{4}{9}t^{-\frac{2}{3}} + 1}\,dt = \int_{-1}^{1} \left|t^{-\frac{1}{3}}\right| \sqrt{\frac{4}{9} + t^{\frac{2}{3}}}\,dt \\
&= \int_{-1}^{0} (-t^{-\frac{1}{3}})\sqrt{\frac{4}{9} + t^{\frac{2}{3}}}\,dt + \int_{0}^{1} (t^{-\frac{1}{3}})\sqrt{\frac{4}{9} + t^{\frac{2}{3}}}\,dt \\
&= \left[-\left(\frac{4}{9} + t^{\frac{2}{3}}\right)^{\frac{3}{2}}\right]_{-1}^{0} + \left[\left(\frac{4}{9} + t^{\frac{2}{3}}\right)^{\frac{3}{2}}\right]_{0}^{1} \\
&= \left(-\left(\frac{4}{9}\right)^{\frac{3}{2}} + \left(\frac{13}{9}\right)^{\frac{3}{2}} + \left(\frac{13}{9}\right)^{\frac{3}{2}} - \left(\frac{4}{9}\right)^{\frac{3}{2}}\right) \\
&= 2\left(\frac{13}{9}\right)^{\frac{3}{2}} - \frac{16}{27}.
\end{aligned}
$$

Alternatively, since $x > 0$, let $x = t^2$ so that $y = t^3$ where $t \in [-1,1]$, i.e.

$$
\left\{ \mathbf{r}(t) \,:\, \mathbf{r}(t) = \begin{pmatrix} t^2 \\ t^3 \end{pmatrix},\ t \in [-1,1] \right\}.
$$

The length of the curve Γ is

$$
\begin{aligned}
\int_\Gamma 1\,ds &= \int_{-1}^{1} \|\mathbf{r}'(t)\|\,dt = \int_{-1}^{1} \left\| \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix} \right\|\,dt \\
&= \int_{-1}^{1} \sqrt{4t^2 + 9t^4}\,dt = \int_{-1}^{1} |t|\sqrt{4 + 9t^2}\,dt \\
&= \int_{-1}^{0} (-t)\sqrt{4 + 9t^2}\,dt + \int_{0}^{1} t\sqrt{4 + 9t^2}\,dt
\end{aligned}
$$

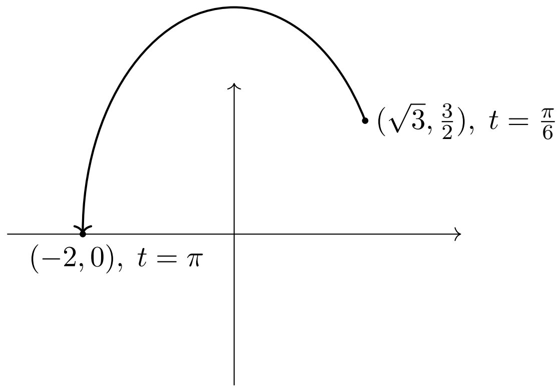
$$= -\left[\frac{1}{27}(4+9t^2)^{\frac{3}{2}}\right]_{-1}^{0} + \left[\frac{1}{27}(4+9t^2)^{\frac{3}{2}}\right]_{0}^{1}$$

$$= \frac{1}{27}(-(4^{\frac{3}{2}} - (4+9)^{\frac{3}{2}}) + (4+9)^{\frac{3}{2}} - 4^{\frac{3}{2}}) = \frac{2}{27}\left(13^{\frac{3}{2}} - 8\right)$$

which is the same answer as found above.

3. We use the parametrisation

$$\left\{ \mathbf{r}(t)\ :\ \mathbf{r}(t) = \begin{pmatrix} 2\cos t \\ 3\sin t \end{pmatrix},\ t \in \left[\frac{\pi}{6}, \pi\right] \right\}$$



The interval $[\frac{\pi}{6}, \pi]$ is found as follows. We have $(\sqrt{3}, \frac{3}{2}) = (2\cos t_1, 3\sin t_1)$ which yields $t_1 = \frac{\pi}{6}$, while $(-2, 0) = (2\cos t_2, 3\sin t_2)$ yields $t_2 = \pi$ with $t_2 > t_1$ so that we have the correct orientation. Thus

$$f(\mathbf{r}(t)) = f(2\cos t, 3\sin t) = \frac{4\cos^2 t}{\sqrt{13 - 4\cos^2 t - 9\sin^2 t}},$$

$$\frac{d\mathbf{r}}{dt} = \begin{pmatrix} -2\sin t \\ 3\cos t \end{pmatrix}, \qquad \left\|\frac{d\mathbf{r}}{dt}\right\| = \sqrt{4\sin^2 t + 9\cos^2 t}$$

so that, using $13 = 13\cos^2 t + 13\sin^2 t$,

$$\begin{aligned}
\int_\Gamma f\,ds &= \int_{\frac{\pi}{6}}^{\pi} f(2\cos t, \sin t)\left\|\frac{d\mathbf{r}}{dt}\right\| dt \\
&= \int_{\frac{\pi}{6}}^{\pi} 4\cos^2 t \frac{\sqrt{4\sin^2 t + 9\cos^2 t}}{\sqrt{9\cos^2 t + 4\sin^2 t}}\,dt \\
&= \int_{\frac{\pi}{6}}^{\pi} 4\cos^2 t\,dt \\
&= [2t + \sin(2t)]_{\frac{\pi}{6}}^{\pi} \\
&= 2\pi + 0 - \frac{\pi}{3} - \sin(\pi/3) = \frac{5\pi}{3} - \frac{\sqrt{3}}{2}.
\end{aligned}$$

4. (a) The length of the path Γ is

$$\begin{aligned}
\int_\Gamma 1\,ds &= \int_{-1}^{1}\left\|\begin{pmatrix} 6t^2 \\ 6t \end{pmatrix}\right\| dt = \int_{-1}^{1}\sqrt{36t^4 + 36t^2}\,dt \\
&= \int_{-1}^{1} 6|t|\sqrt{t^2+1}\,dt = -\int_{-1}^{0} 6t\sqrt{t^2+1}\,dt + \int_{0}^{1} 6t\sqrt{t^2+1}\,dt \\
&= -\left[2(t^2+1)^{\frac{3}{2}}\right]_{-1}^{0} + \left[2(t^2+1)^{\frac{3}{2}}\right]_{0}^{1} \\
&= -4 + 2^{\frac{7}{2}} = 4(\sqrt{8} - 1).
\end{aligned}$$

(b) We find

$$\begin{aligned}
\int_\Gamma f\,ds &= \int_{-1}^{1} f(2t^3,3t^2)\left\|\begin{pmatrix}6t^2\\6t\end{pmatrix}\right\| dt\\
&= \int_{-1}^{1} |2t^3|\sqrt{3t^2+3}\sqrt{36t^4+36t^2}\,dt\\
&= \int_{-1}^{1} 12\sqrt{3}|t^4|\sqrt{t^2+1}\sqrt{t^2+1}\,dt\\
&= \int_{-1}^{1} 12\sqrt{3}t^4(t^2+1)\,dt = \int_{-1}^{1} 12\sqrt{3}(t^6+t^4)\,dt\\
&= \left[\frac{12}{7}\sqrt{3}t^7+\frac{12}{5}\sqrt{3}t^5\right]_{-1}^{1}\\
&= \sqrt{3}\frac{288}{35}.
\end{aligned}$$

(c) The average of f over Γ is

$$\frac{\displaystyle\int_\Gamma f\,ds}{\displaystyle\int_\Gamma 1\,ds} = \frac{72\sqrt{3}}{35(\sqrt{8}-1)}.$$

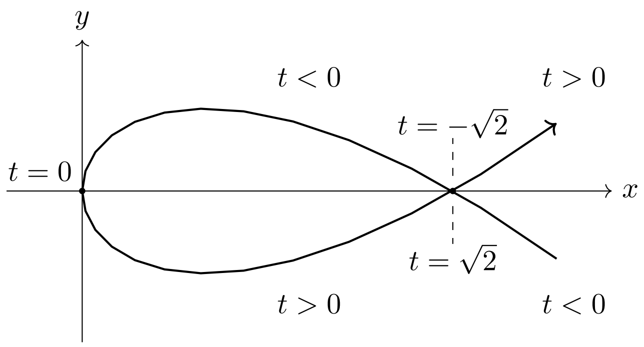
5. First we must describe the loop. A loop is either a closed curve which does not intersect itself (such as a circle) or a part of a curve which is closed because it intersects itself. In other words, the intersection is given by as two distinct parameter values $t_1 \neq t_2$ which satisfy

$$\mathbf{r}(t_1) = \begin{pmatrix}\sqrt{6}t_1^2\\ t_1(t_1^2-2)\end{pmatrix} = \begin{pmatrix}\sqrt{6}t_2^2\\ t_2(t_2^2-2)\end{pmatrix} = \mathbf{r}(t_2).$$

Thus $\sqrt{6}t_1^2 = \sqrt{6}t_2^2$, i.e. $t_1 = t_2$ (which we has assumed is false – $t_1 \neq t_2$) or $t_1 = -t_2$. The second coordinate gives

$$t_1(t_1^2-2) = t_2(t_2^2-2) = -t_1(t_1^2-2)$$

since $t_2 = -t_1$. Obviously $2t_1(t_1^2-2) = 0$ which yields $t_1 = 0$ (but then $t_2 = -t_1 = 0 = t_1$ which does not satisfy $t_1 \neq t_2$), or $t_1 = \sqrt{2}$ and $t_2 = -\sqrt{2}$. (Of course $t_2 = \sqrt{2}$ and $t_1 = -\sqrt{2}$ is also a solution, but it yields the same two values of t for the intersection!). The point of intersection is $(\sqrt{6}t_1^2, t_1(t_1^2-2)) = (2\sqrt{6}, 0)$.



The sign table for $(x,y) = (\sqrt{6}t^2, t(t^2-2))$ completes the picture:

t	$\cdots$	$-\sqrt{2}$	$\cdots$	0	$\cdots$	$\sqrt{2}$	$\cdots$
x	$+$	$2\sqrt{6}$	$+$	0	$+$	$2\sqrt{6}$	$+$
y	$-$	0	$+$	0	$-$	0	$+$

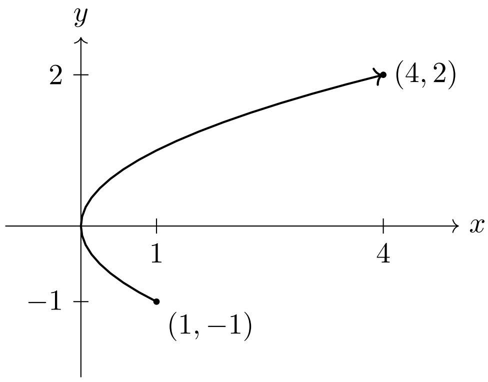
The length of the loop is $\int_\Gamma 1\,ds$ where

$$\Gamma = \left\{ \begin{pmatrix} \sqrt{6}t^2 \\ t(t^2-2) \end{pmatrix} \,:\, t \in [-\sqrt{2}, \sqrt{2}] \right\}$$

In other words, the length is

$$\begin{aligned} \int_\Gamma 1\,ds &= \int_{-\sqrt{2}}^{\sqrt{2}} \|\mathbf{r}'(t)\|\,dt = \int_{-\sqrt{2}}^{\sqrt{2}} \left\| \begin{pmatrix} 2\sqrt{6}t \\ 3t^2-2 \end{pmatrix} \right\| dt \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{9t^4+12t^2+4}\,dt = \int_{-\sqrt{2}}^{\sqrt{2}} (3t^2+2)\,dt \\ &= \left[t^3+2t\right]_{-\sqrt{2}}^{\sqrt{2}} = 8\sqrt{2}. \end{aligned}$$

6. The curve is parabolic in y, i.e.



A parametrisation of Γ is

$$\Gamma = \left\{ \begin{pmatrix} t^2 \\ t \end{pmatrix} \,:\, t \in [-1, 2] \right\}$$

so that

$$\begin{aligned} \int_\Gamma \frac{x}{\sqrt{1+3x+y^2}}\,ds &= \int_{-1}^{2} \frac{t^2}{\sqrt{1+3t^2+t^2}} \left\| \begin{pmatrix} 2t \\ 1 \end{pmatrix} \right\| dt \\ &= \int_{-1}^{2} \frac{t^2}{\sqrt{1+4t^2}} \sqrt{1+4t^2}\,dt \\ &= \int_{-1}^{2} t^2\,dt = 3. \end{aligned}$$

Section 2.3: Vector Path Integrals

1. (a) $\displaystyle\int_\Gamma \mathbf{F}\cdot d\mathbf{r} = \int_a^b \left(\mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} \right) dt$ where $\cdot$ denotes the dot product.

   (b) We have the parametrisation given by $\mathbf{r}(t) = \begin{pmatrix} t \\ t^2+3 \end{pmatrix}$, i.e.

$$\Gamma = \left\{ \begin{pmatrix} t \\ t^2+3 \end{pmatrix} \,:\, t \in [1, 3] \right\}.$$

It follows that

$$\begin{aligned}
\int_\Gamma \mathbf{F}\cdot d\mathbf{r} &= \int_1^3 \left(\mathbf{F}(t,t^2+3)\cdot \frac{d}{dt}\begin{pmatrix} t \\ t^2+3\end{pmatrix}\right)\, dt \\
&= \int_1^3 \begin{pmatrix} t \\ \sin(t^2+3)\end{pmatrix}\cdot\begin{pmatrix} 1 \\ 2t\end{pmatrix}\, dt \\
&= \int_1^3 (t+2t\sin(t^2+3))\, dt \\
&= \left[\frac{t^2}{2} - \cos(t^2+3)\right]_1^3 = 4-\cos(12)+\cos(4).
\end{aligned}$$

(c) We have the parametrisation given by $\mathbf{r}(t) = \begin{pmatrix} t \\ \ln t\end{pmatrix}$, i.e.

$$\Gamma = \left\{ \begin{pmatrix} t \\ \ln t\end{pmatrix} \ : \ t\in[e,e^3] \right\}.$$

It follows that

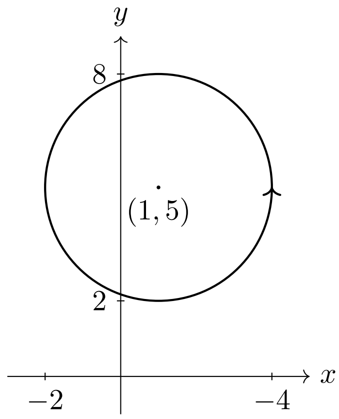
$$\begin{aligned}
\int_\Gamma \mathbf{F}\cdot d\mathbf{r} &= \int_e^{e^3} \left(\mathbf{F}(t,\ln t)\cdot \frac{d}{dt}\begin{pmatrix} t \\ \ln t\end{pmatrix}\right)\, dt \\
&= \int_e^{e^3} \begin{pmatrix} \ln t \\ t^2\end{pmatrix}\cdot\begin{pmatrix} 1 \\ \frac{1}{t}\end{pmatrix}\, dt \\
&= \int_e^{e^3} (\ln t + t)\, dt \\
&= \left[t\ln t - t + \frac{t^2}{2}\right]_e^{e^3} = \frac{e^6}{2} + 2e^3 - \frac{e^2}{2}.
\end{aligned}$$

2. We have the parametrisation given by $\mathbf{r}(t) = \begin{pmatrix} t^2-t+1 \\ t\end{pmatrix}$ with $t\in[-1,2]$. The parametrisation has the required orientation since $\mathbf{r}(-1)=(3,-1)$ is before $\mathbf{r}(2)=(3,2)$ on the oriented curve and $-1<2$, i.e. -1 is before 2 for the parameter t. Thus

$$\begin{aligned}
\int_\Gamma \mathbf{v}\cdot d\mathbf{r} &= \int_{-1}^2 \left(\mathbf{v}(t^2-t+1,t)\cdot \frac{d}{dt}\begin{pmatrix} t^2-t+1 \\ t\end{pmatrix}\right)\, dt \\
&= \int_{-1}^2 \begin{pmatrix} t \\ -2t^2+2t-2\end{pmatrix}\cdot\begin{pmatrix} 2t-1 \\ 1\end{pmatrix}\, dt \\
&= \int_{-1}^2 (t-2)\, dt \\
&= \left[\frac{t^2}{2} - 2t\right]_{-1}^2 = -\frac{9}{2}.
\end{aligned}$$

3.

$$\Gamma = \left\{ \begin{pmatrix} 3\cos t - 1 \\ 3\sin t - 5 \end{pmatrix} \,:\, t \in [0, 2\pi] \right\}$$



$$\begin{aligned}
\int_\Gamma \mathbf{v} \cdot d\mathbf{r} &= \int_0^{2\pi} \left(\mathbf{v}(3\cos t - 1, 3\sin t - 5) \cdot \frac{d}{dt} \begin{pmatrix} 3\cos t - 1 \\ 3\sin t - 5 \end{pmatrix} \right) \, dt \\
&= \int_0^{2\pi} \begin{pmatrix} 15\cos t - 3\sin t \\ 3\cos t - 1 \end{pmatrix} \cdot \begin{pmatrix} -3\sin t \\ 3\cos t \end{pmatrix} \, dt \\
&= \int_0^{2\pi} (-45\sin t\cos t - 3\cos t + 9) \, dt \\
&= [45\cos t - 3\sin t + 9t]_0^{2\pi} = 18\pi.
\end{aligned}$$

4. (a) We have the parametrisation

$$\Gamma = \left\{ \begin{pmatrix} a\cos t \\ b\sin t \end{pmatrix} \,:\, t \in [0, 2\pi] \right\}.$$

(b) It follows that

$$\begin{aligned}
\int_\Gamma \mathbf{v} \cdot d\mathbf{r} &= \int_0^{2\pi} \left(\mathbf{v}(a\cos t, b\sin t) \cdot \frac{d}{dt} \begin{pmatrix} a\cos t \\ b\sin t \end{pmatrix} \right) \, dt \\
&= \int_0^{2\pi} \begin{pmatrix} a\cos t - b\sin t \\ a\cos t + b\sin t \end{pmatrix} \cdot \begin{pmatrix} -a\sin t \\ b\cos t \end{pmatrix} \, dt \\
&= \int_0^{2\pi} \left(\frac{b^2 - a^2}{2} \sin(2t) + ab \right) \, dt \\
&= \left[\frac{a^2 - b^2}{4} \cos(2t) + abt \right]_0^{2\pi} = 2ab\pi.
\end{aligned}$$

5. (a) The line segment from $(1,1)$ to (N,N) is given by $\mathbf{r}_1(t) = (1,1) + t((N,N) - (1,1))$ for $t \in [0,1]$, i.e.

$$\Gamma_1(N) = \left\{ \begin{pmatrix} 1 + t_1(N-1) \\ 1 + t_1(N-1) \end{pmatrix} \,:\, t_1 \in [0,1] \right\}.$$

(b) The circle Γ_2 is given by $\mathbf{r}_2(t) = (2\cos t, 2\sin t)$,

$$\Gamma_2 = \left\{ \begin{pmatrix} 2\cos t_2 \\ 2\sin t_2 \end{pmatrix} \,:\, t_2 \in [0, 2\pi] \right\}.$$

(c) The tangent to Γ_2 at $\mathbf{x} = \mathbf{r}_2(t) = (2\cos t, 2\sin t)$ (i.e. an arbitrary point on Γ_2 given at $t_2 = t$), is

$$\mathbf{r}_2'(t) = \frac{d}{dt_2} \begin{pmatrix} 2\cos t_2 \\ 2\sin t_2 \end{pmatrix} \Bigg|_{\mathbf{x}} = \begin{pmatrix} -2\sin t_2 \\ 2\cos t_2 \end{pmatrix} \Bigg|_{t_2 = t} = \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix}.$$

It follows that $\mathbf{x}\cdot\mathbf{r}_2'(t) = -4\cos t\sin t + 4\cos t\sin t = 0$ and

$$\mathbf{F}(\mathbf{x})\cdot\mathbf{r}_2'(t) = \frac{1}{\|\mathbf{x}\|^3}\mathbf{x}\cdot\mathbf{r}_2'(t) = 0.$$

Thus $\displaystyle\int_{\Gamma_2}\mathbf{F}\cdot d\mathbf{r}_2 = \int_{\Gamma_2}\mathbf{F}(\mathbf{r}(t_2))\cdot\mathbf{r}_2'(t_2)\,dt_2 = 0.$

(d) Since $\|(1+t_1(N-1),1+t_1(N-1))\| = \sqrt{2}(1+t_1(N-1))$ when $t_1>0$ and $N>0$,

$$\begin{aligned}
\int_{\Gamma_1(N)}\mathbf{F}\cdot d\mathbf{r}_1 &= \int_0^1\left(\mathbf{F}(1+t_1(N-1),1+t_1(N-1))\cdot\frac{d}{dt}\begin{pmatrix}1+t_1(N-1)\\1+t_1(N-1)\end{pmatrix}\right)\,dt_1\\
&= \int_0^1\frac{1}{\sqrt{8}(1+t_1(N-1))^2}\begin{pmatrix}1\\1\end{pmatrix}\cdot\begin{pmatrix}N-1\\N-1\end{pmatrix}\,dt_1\\
&= \int_0^1\frac{1}{\sqrt{2}}\left(\frac{N-1}{(1+t_1(N-1))^2}\right)\,dt_1\\
&= \frac{1}{\sqrt{2}}\left[\frac{1}{(1+t_1(N-1))}\right]_0^1 = \frac{1}{\sqrt{2}N}-\frac{1}{\sqrt{2}}.
\end{aligned}$$

Thus, $\displaystyle\lim_{N\to\infty}\int_{\Gamma_1(N)}\mathbf{F}\cdot d\mathbf{r}_1 = -\frac{1}{\sqrt{2}}.$

6. This question is identical to question 3, using different notation.

7. We have

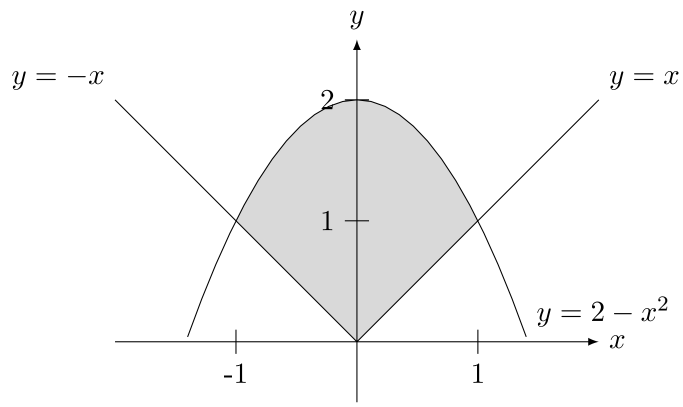
$$\begin{aligned}
\int_\Gamma\sqrt{x^2+y^2+4z}\,ds &= \int_{-2\pi}^{2\pi}\sqrt{\cos^2 t+\sin^2 t+4t^t}\left\|\begin{pmatrix}-\sin t\\\cos t\\2t\end{pmatrix}\right\|\,dt\\
&= \int_{-2\pi}^{2\pi}(1+4t^2)\,dt = 4\pi+\frac{64}{3}\pi^3.
\end{aligned}$$

and

$$\begin{aligned}
\int_\Gamma(x-zy)\,dx+(zx+y)\,dy+4z\,dz &= \int_\Gamma\begin{pmatrix}x-zy\\zx+y\\4z\end{pmatrix}\cdot d\mathbf{r}\\
&= \int_{-2\pi}^{2\pi}\begin{pmatrix}\cos t-t^2\sin t\\t^2\cos t+\sin t\\4t^2\end{pmatrix}\cdot\begin{pmatrix}-\sin t\\\cos t\\2t\end{pmatrix}\,dt\\
&= \int_{-2\pi}^{2\pi}(t^2+8t^3)\,dt = \frac{16}{3}\pi^3.
\end{aligned}$$

Section 2.4: Double Integrals and Fubini's Theorem

1. (a)



(b) The outer integral, in x, is taken between the smallest x value – -1 – and the largest x value – 1 – in the region, while the domain for y (for the inner integral) depends on x. Clearly, $|x| \le y \le 2 - x^2$, so that

$$\iint_D f(x,y)\,dx\,dy = \int_{-1}^{1} \int_{|x|}^{2-x^2} (x+y)\,dy\,dx.$$

Alternatively: The outer integral, in y, is taken between the smallest y value – 0 – and the largest y value – 2 – in the region, while the domain for x (for the inner integral) depends on y. The two curves intersect for $x < 0$ at $-x = |x| = y = 2 - x^2$ (i.e. $x^2 - x - 2 = 0$ so that $x = -1$ and $y = 1$) and for $x \ge 0$ at $x = |x| = y = 2 - x^2$ (i.e. $x^2 + x - 2 = 0$ so that $x = 1$ and $y = 1$). We have to consider two regions, $0 \le y \le 1$ which is bounded by the absolute value $y \ge |x|$ ($-y \le x \le y$), and $1 \le y \le 2$ bounded by the parabola $y \le 2 - x^2$ ($-\sqrt{2-y} \le x \le \sqrt{2-y}$):

$$\iint_D f(x,y)\,dx\,dy = \int_0^1 \int_{-y}^{y} (x+y)\,dx\,dy + \int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} (x+y)\,dx\,dy$$

(c)

$$\begin{aligned}
&\iint_D f(x,y)\,dx\,dy \\
&\quad= \int_{-1}^{1} \int_{|x|}^{2-x^2} (x+y)\,dy\,dx \\
&\quad= \int_{-1}^{1} \left[xy + \frac{y^2}{2} \right]_{y=|x|}^{y=2-x^2} dx \\
&\quad= \int_{-1}^{1} \left(x(2-x^2) + \frac{(2-x^2)^2}{2} - x|x| - \frac{x^2}{2} \right) dx \\
&\quad= \int_{-1}^{1} \left(\frac{x^4}{2} - \frac{5}{2}x^2 + 2 \right) dx \qquad \begin{pmatrix}\text{odd functions integrate} \\ \text{to zero on } [-1,1]\end{pmatrix} \\
&\quad= \left[\frac{x^5}{10} - \frac{5}{6}x^3 + 2x \right]_{-1}^{1} \\
&\quad= \frac{1}{5} - \frac{5}{3} + 4 = \frac{38}{15}.
\end{aligned}$$

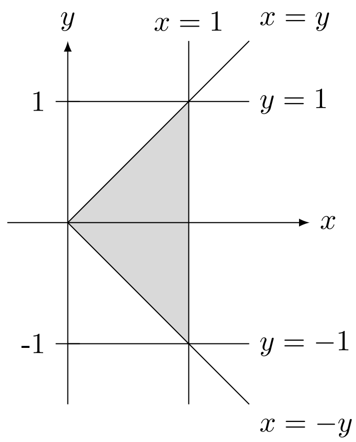
Always check that this makes sense! The answer is a number, independent of x and y, so after integrating with respect to y, we should only see x. After the final integration, neither x nor y should appear in the answer.

Alternatively:

$$\begin{aligned}
&\iint_D f(x,y)\,dx\,dy \\
&\quad= \int_0^1 \int_{-y}^{y} (x+y)\,dx\,dy + \int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} (x+y)\,dx\,dy \\
&\quad= \int_0^1 \int_{-y}^{y} y\,dx\,dy + \int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} y\,dx\,dy \qquad (x \text{ is an odd function}) \\
&\quad= \int_0^1 \Big[xy \Big]_{x=-y}^{x=y} dy + \int_1^2 \Big[xy \Big]_{x=-\sqrt{2-y}}^{x=\sqrt{2-y}} dy \\
&\quad= \int_0^1 2y^2\,dy + \int_1^2 2y\sqrt{2-y}\,dy
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{3} + \int_1^2 \left(2(y-2)\sqrt{2-y} + 4\sqrt{2-y}\right)\, dy \\
&= \frac{2}{3} + \int_1^2 \left(-2(2-y)^{3/2} + 4\sqrt{2-y}\right)\, dy \\
&= \frac{2}{3} + \left[\frac{4}{5}(2-y)^{5/2} - \frac{8}{3}(2-y)^{3/2}\right]_1^2 \\
&= \frac{2}{3} - \frac{4}{5} + \frac{8}{3} = \frac{38}{15}.
\end{aligned}
$$

2. (a) We have $-1 \leq y \leq 1$ and $|y| \leq x \leq 1$. Thus, the region is:



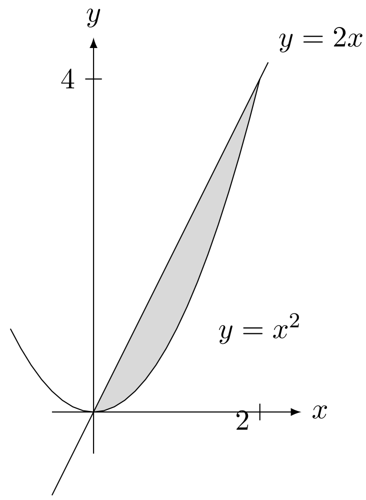
(b) From the sketch we see that $0 \leq x \leq 1$ and $-x \leq y \leq x$, so that

$$
\int_{-1}^{1} \int_{|y|}^{1} (1+2y)e^{x^2}\, dx\, dy = \int_0^1 \int_{-x}^{x} (1+2y)e^{x^2}\, dy\, dx.
$$

(c)

$$
\begin{aligned}
\int_0^1 \int_{-x}^{x} (1+2y)e^{x^2}\, dy\, dx &= \int_0^1 \left[(y+y^2)e^{x^2}\right]_{y=-x}^{y=x}\, dx \\
&= \int_0^1 2xe^{x^2}\, dx \\
&= e^{x^2}\Big|_0^1 = e-1.
\end{aligned}
$$

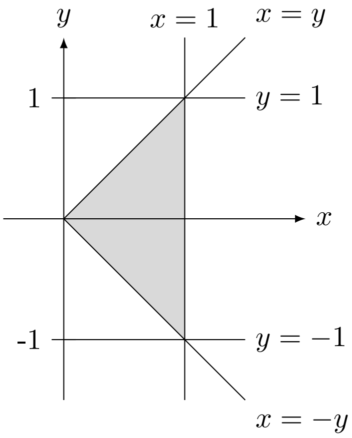
3. The two curves $y = 2x$ and $y = x^2$ intersect when $2x = x^2$, i.e. $x = 0$ or $x = 2$.



Since D is bounded above by $y = 2x$, $y \leq 2x$ in D, and since D is bounded below by $y = x^2$, $y \geq x^2$. Thus, $x^2 \leq y \leq 2x$ — but this is only true for $x \in [0, 2]$. Thus, $D = \{(x, y) : x \in [0, 2],\ y \in [x^2, 2x]\}$. Then

$$\begin{aligned}
\iint_D f\, dx\, dy &= \int_0^2 \int_{x^2}^{2x} \sqrt{2x + y + 1}\, dy\, dx \\
&= \int_0^2 \left[\frac{2}{3}(2x + y + 1)^{3/2} \right]_{y=x^2}^{y=2x} dx \\
&= \int_0^2 \left(\frac{2}{3}(4x + 1)^{3/2} - \frac{2}{3}(x + 1)^3 \right) dx \\
&= \left[\frac{1}{15}(4x + 1)^{5/2} - \frac{1}{6}(x + 1)^4 \right]_0^2 \\
&= \frac{1}{15}3^5 - \frac{1}{6}3^4 - \frac{1}{15} + \frac{1}{6} = \frac{14}{5}.
\end{aligned}$$

4. (a) Since $|y| \leq x \leq 1$ and $y \in [-1, 1]$, the region is:



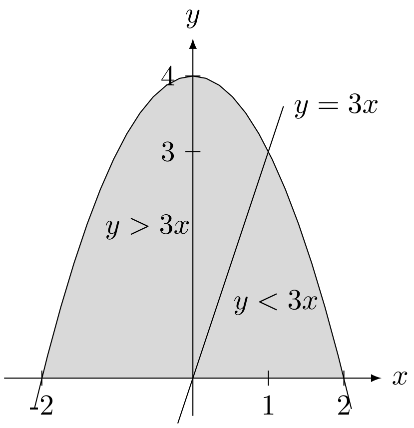
(b) Since $|y| \leq x$ if and only if $-x \leq y \leq x$, and $0 \leq |y| \leq x \leq 1$ (and $y = 0$ is included the domain), we have

$$D = \{(x, y) : y \in [-1, 1],\ x \in [\,|y|, 1]\} = \{(x, y) : x \in [0, 1],\ y \in [-x, x]\}.$$

(c)

$$\begin{aligned}
\int_{-1}^1 \int_{|y|}^1 e^{-x^2}\, dx\, dy &= \int_0^1 \int_{-x}^{x} e^{-x^2}\, dy\, dx \\
&= \int_0^1 \left[ye^{-x^2} \right]_{y=-x}^{y=x} dx \\
&= \int_0^1 2xe^{-x^2} dx \\
&= \left[-e^{-x^2} \right]_0^1 = 1 - e^{-1}.
\end{aligned}$$

5. Since D is bounded above by $y = 4 - x^2$, $y \leq 4 - x^2$ in D, and since D is bounded below by $y = 0$, $y \geq 0$. Thus, $0 \leq y \leq 4 - x^2$ — but this is only true for $x \in [-2, 2]$. Thus, $D = \{(x, y) : x \in [-2, 2],\ y \in [0, 4 - x^2]\}$. The region is



We note that $|3x-y| = 3x-y$ if $3x \geq y$, and $|3x-y| = y-3x$ otherwise. The line $y=3x$ intersects $y = 4-x^2$ at $x=1$ and $y=3$. From the figure, we see 3 subregions,

$$
\begin{aligned}
D_1 &= \{(x,y) \in D \,:\, y \leq 3x\} = \{(x,y) \,:\, y \in [0,3],\, x \in [y/3, \sqrt{4-y}]\},\\
D_2 &= \{(x,y) \in D \,:\, y \geq 3x,\, x \leq 0\} = \{(x,y) \,:\, x \in [-2,0],\, y \in [0, 4-x^2]\},\\
D_2 &= \{(x,y) \in D \,:\, y \geq 3x,\, x \geq 0\} = \{(x,y) \,:\, x \in [0,1],\, y \in [3x, 4-x^2]\}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
\iint_{D_1} |3x-y|\,dx\,dy &= \int_0^3 \int_{y/3}^{\sqrt{4-y}} (3x-y)\,dx\,dy\\
&= \int_0^3 \left[\frac{3}{2}x^2 - xy\right]_{x=y/3}^{x=\sqrt{4-y}} dy\\
&= \int_0^3 \frac{3}{2}(4-y) - y\sqrt{4-y} + \frac{1}{6}y^2\,dy\\
&= \int_0^3 6 - \frac{3}{2}y + (4-y)\sqrt{4-y} - 4\sqrt{4-y} + \frac{1}{6}y^2\,dy\\
&= \left[6y - \frac{3}{4}y^2 - \frac{2}{5}(4-y)^{5/2} + \frac{8}{3}(4-y)^{3/2} + \frac{1}{18}y^3\right]_0^3\\
&= 18 - \frac{27}{4} - \frac{2}{5} + \frac{8}{3} + \frac{27}{18} + \frac{64}{5} - \frac{64}{3} = \frac{389}{60},
\end{aligned}
$$

and

$$
\begin{aligned}
\iint_{D_2} |3x-y|\,dx\,dy &= \int_{-2}^0 \int_0^{4-x^2} (y-3x)\,dy\,dx\\
&= \int_{-2}^0 \left[\frac{y^2}{2} - 3xy\right]_{y=0}^{y=4-x^2} dx\\
&= \int_{-2}^0 \frac{x^4}{2} - 4x^2 + 8 - 12x + 3x^3\,dx\\
&= \left[\frac{x^5}{10} - \frac{4x^3}{3} + 8x - 6x^2 + \frac{3}{4}x^4\right]_{-2}^0\\
&= \frac{16}{5} - \frac{32}{3} + 16 + 24 - 12 = \frac{308}{15}.
\end{aligned}
$$

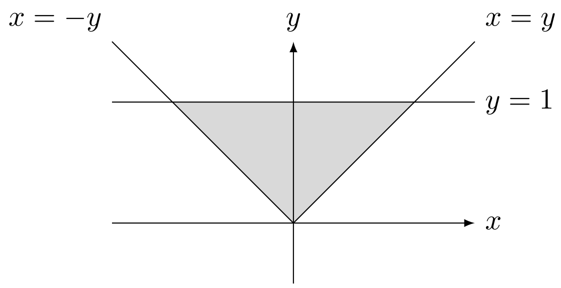
Finally,

$$\begin{aligned}
\iint_{D_3} |3x-y|\,dx\,dy &= \int_0^1 \int_{3x}^{4-x^2} (y-3x)\,dy\,dx \\
&= \int_0^1 \left[\frac{y^2}{2} - 3xy\right]_{3x}^{4-x^2} dx \\
&= \int_0^1 \frac{x^4}{2} - 4x^2 + 8 - 12x + 3x^3 + \frac{9}{2}x^2\,dx \\
&= \left[\frac{x^5}{10} - \frac{4x^2}{3} + 8x - 6x^2 + \frac{3}{4}x^4 + \frac{3}{2}x^3\right]_0^1 \\
&= \frac{1}{10} - \frac{4}{3} + 8 - 6 + \frac{3}{4} + \frac{3}{2} = \frac{181}{60}.
\end{aligned}$$

Thus,

$$\iint_D |3x-y|\,dx\,dy = \iint_{D_1} |3x-y|\,dx\,dy + \iint_{D_2} |3x-y|\,dx\,dy + \iint_{D_3} |3x-y|\,dx\,dy = \frac{901}{30}.$$

6. (a) We have $y \le 1$ and $y \ge |x|$.

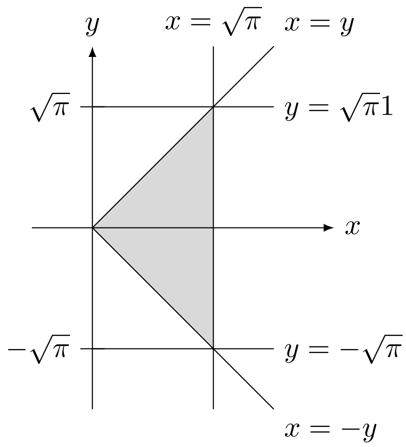


(b) $D = \{(x,y) \,:\, 1 \le y \le |x|,\ -1 \le x \le 1\} = \{(x,y) \,:\, -y \le x \le y,\ 0 \le y \le 1\}$.

(c) The second representation of D yields

$$\begin{aligned}
\iint_D ye^{-x}\,dx\,dy &= \int_0^1 \int_{-y}^{y} ye^{-x}\,dx\,dy \\
&= \int_0^1 -y(e^{-y} - e^{y})\,dy \\
&= \left[y(e^{-y} + e^{y}) + e^{-y} - e^{y}\right]_0^1 \\
&= 2e^{-1}.
\end{aligned}$$

7. (a) We have $x \le \sqrt{\pi}$, $x \ge |y|$ and $y \in [-\sqrt{\pi}, \sqrt{\pi}]$.



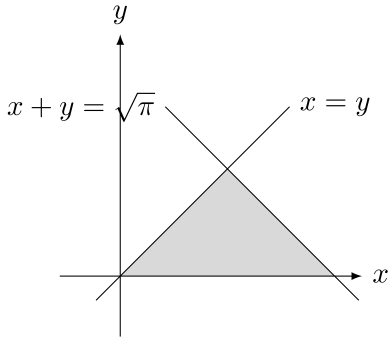
(b) $D = \{(x,y) \,:\, y \in [-\sqrt{\pi}, \sqrt{\pi}],\, x \in [\,|y|, \sqrt{\pi}]\} = \{(x,y) \,:\, x \in [0, \sqrt{\pi}],\, y \in [-x, x]\}$.

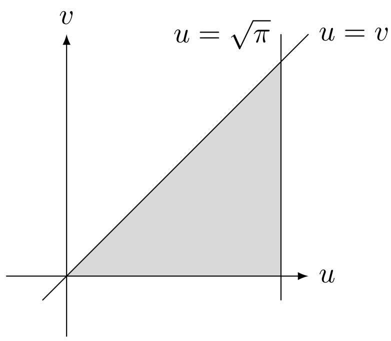
(c)

$$\begin{aligned}
\int_{-\sqrt{\pi}}^{\sqrt{\pi}} \int_{|y|}^{\sqrt{\pi}} \sin(x^2)\, dx\, dy &= \int_0^{\sqrt{\pi}} \int_{-x}^{x} \sin(x^2)\, dy\, dx \\
&= \int_0^{\sqrt{\pi}} 2x \sin(x^2)\, dx \\
&= \Big[-\cos(x^2)\Big]_0^{\sqrt{\pi}} \\
&= 2.
\end{aligned}$$

Section 2.5: Change of Variables

1. The region is as follows. The line $x = y$ is equivalent to $v = 0$. The line $x + y = \sqrt{\pi}$ is equivalent to $u = \sqrt{\pi}$. Finally, $y = 0$ is equivalent to $\frac{1}{2}(u - v) = 0$.





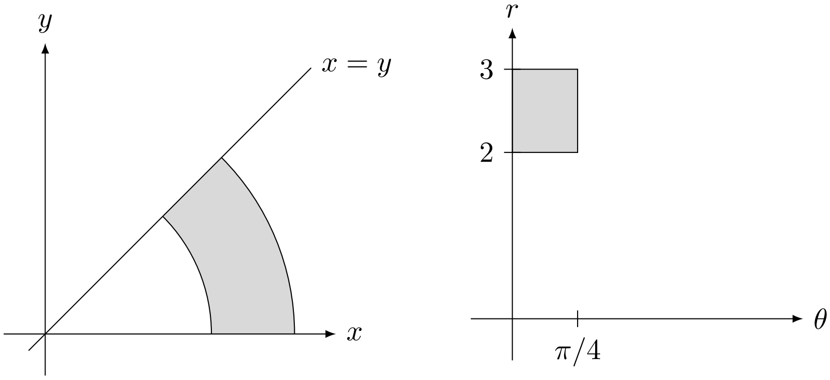
Since $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$, the Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} = -\frac{1}{2}.$$

Thus,

$$\begin{aligned}
\iint_D e^{\frac{x-y}{x+y}} \sin((x+y)^2)\, dx\, dy &= \int_0^{\sqrt{\pi}} \int_0^{u} e^{v/u} \sin(u^2) \left| -\frac{1}{2} \right| dv\, du \\
&= \frac{1}{2} \int_0^{\sqrt{\pi}} u(e-1) \sin(u^2)\, du \\
&= \frac{e-1}{4} \cos(u^2) \Big|_0^{\sqrt{\pi}} \\
&= \frac{1-e}{2}.
\end{aligned}$$

2. The annulus is given by $4 \le r^2 \le 9$, i.e. $2 \le r \le 3$ (we assume $r \ge 0$ and $0 \le \theta < 2\pi$ to ensure that $(r,\theta) \to (x,y)$ is one to one on the annulus). The first quadrant is given by $0 \le \theta \le \pi/2$ and $y = x$ yields $\theta = \pi/4$.



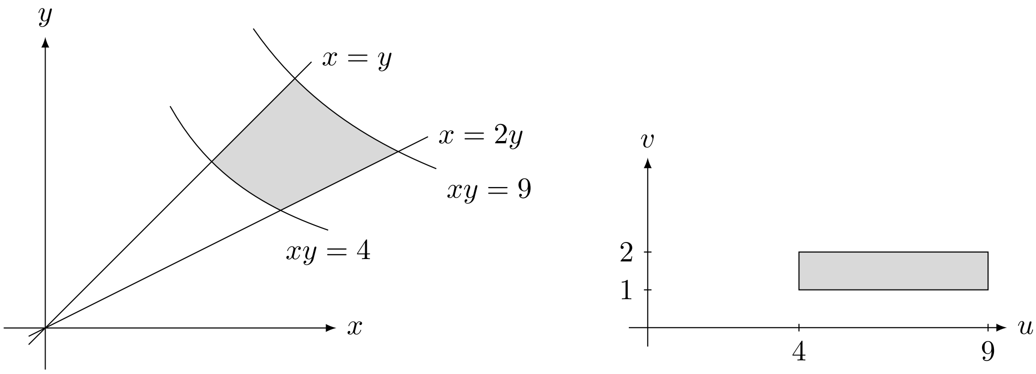
The Jacobian is given by

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{pmatrix} = \det\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} = r.$$

Thus,

$$\begin{aligned}
&\iint_D \left(x^2 + 2y^2 + \frac{y^4}{x^2}\right) e^{\left(xy + \frac{y^3}{x}\right)}\, dx\, dy \\
&= \iint_D \left(\frac{(x^2+y^2)^2}{x^2}\right) e^{\left((x^2+y^2)\frac{y}{x}\right)}\, dx\, dy \\
&= \int_2^3 \int_0^{\pi/4} \left(r^2 \sec^2\theta\right) e^{r^2 \tan\theta} |r|\, d\theta\, dr \\
&= \int_2^3 r \left(\int_0^{r^2} e^u\, du\right) dr && (u = r^2\tan\theta,\ du = r^2\sec^2\theta\, d\theta) \\
&= \int_2^3 \left(re^{r^2} - r\right) dr \\
&= \frac{1}{2}(e^9 - e^4 - 5).
\end{aligned}$$

3. The region D is given by $y \le x$ (so $x/y \ge 1$), $x \le 2y$ (so $x/y \le 2$), $y \ge 4/x$ (so $xy \ge 4$) and $y \le 9/x$ (so $xy \le 9$). Hence, $1 \le v \le 2$ and $4 \le u \le 9$,



We have $uv = x^2$ so that, in the first quadrant, $x = \sqrt{uv}$. Similarly, $y = \sqrt{u/v}$. We note that $|u| = u$ and $|v| = v$ in the region of integration. The Jacobian is given by

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} = \det\begin{pmatrix} \dfrac{v}{2\sqrt{uv}} & \dfrac{u}{2\sqrt{uv}} \\ \dfrac{1}{2v\sqrt{u/v}} & -\dfrac{u}{2v^2\sqrt{u/v}} \end{pmatrix} = -\frac{1}{2v}.$$

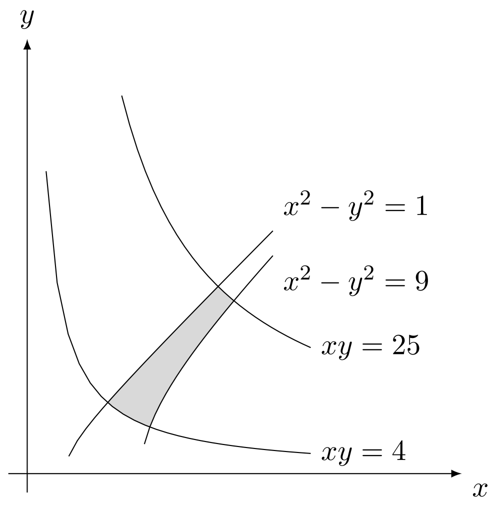
Equivalently,

$$\det \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix} = \det \begin{pmatrix} y & x \\ 1/y & -x/y^2 \end{pmatrix} = -2\frac{x}{y} = -2v$$

so that the Jacobian is $-\frac{1}{2v}$. Thus,

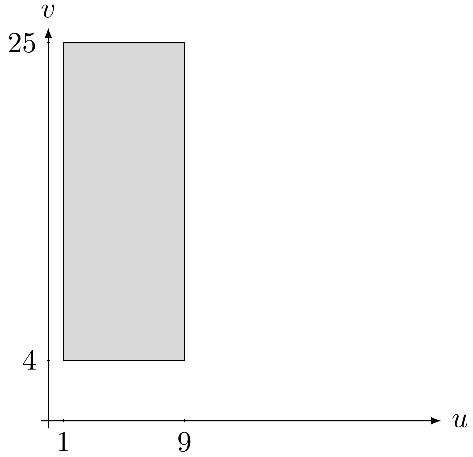
$$\begin{aligned}
\iint_D (x^2+y^2)\,dx\,dy &= \int_1^2 \int_4^9 \left(uv + \frac{u}{v}\right) \left|-\frac{1}{2v}\right| du\,dv \\
&= \frac{1}{2}\int_1^2 \int_4^9 \left(u + \frac{u}{v^2}\right) du\,dv \\
&= \frac{1}{4}\int_1^2 \left[u^2 + \frac{u^2}{v^2}\right]_4^9 dv \\
&= \frac{65}{4}\int_1^2 \left(1 + \frac{1}{v^2}\right) dv \\
&= \frac{65}{4}\left[v - \frac{1}{v}\right]_1^2 \\
&= \frac{195}{8}.
\end{aligned}$$

4. (a)



(b) We have

$$D^* = \{(u,v) : 1 \le u \le 9,\ 4 \le v \le 25\}.$$



(c)

$$\frac{\partial \mathbf{T}(u,v)}{\partial(u,v)} = \frac{\partial(u,v)}{\partial(x,y)}^{-1} = \det\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}^{-1} = \det\begin{pmatrix} 2x & -2y \\ y & x \end{pmatrix}^{-1} = \frac{1}{2(x^2+y^2)}.$$

(d)

$$f(\mathbf{T}(u,v))\left|\frac{\partial \mathbf{T}(u,v)}{\partial(u,v)}\right| = (x^2+y^2)\frac{1}{2(x^2+y^2)} = \frac{1}{2}.$$

(e)

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(\mathbf{T}(u,v))\left|\frac{\partial \mathbf{T}(u,v)}{\partial(u,v)}\right| = \int_1^9 \int_4^{25} \frac{1}{2}\,dv\,du = \frac{1}{2}\cdot 8\cdot 21 = 84.$$

5. Since

$$9 \le 10x^2 - 16xy + 10y^2 = 36r^2$$

we have $r \ge 1/2$. From

$$36r^2 = 10x^2 - 16xy + 10y^2 \le 36$$

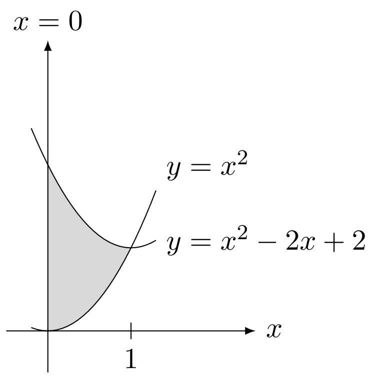
we obtain $r \le 1$. The Jacobian is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{pmatrix} = \det\begin{pmatrix} 3\cos\theta + \sin\theta & r(-3\sin\theta + \cos\theta) \\ 3\cos\theta - \sin\theta & r(-3\sin\theta - \cos\theta) \end{pmatrix} = -6r.$$

Thus,

$$\begin{aligned} \iint_D \frac{1}{5x^2 - 8xy + 5y^2}\,dx\,dy &= \int_0^{2\pi}\int_{1/2}^1 \frac{1}{18r^2}|-6r|\,dr\,d\theta \\ &= \int_0^{2\pi}\int_{1/2}^1 \frac{1}{3r}\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{1}{3}\ln 2\,d\theta \\ &= \frac{2}{3}\pi\ln 2. \end{aligned}$$

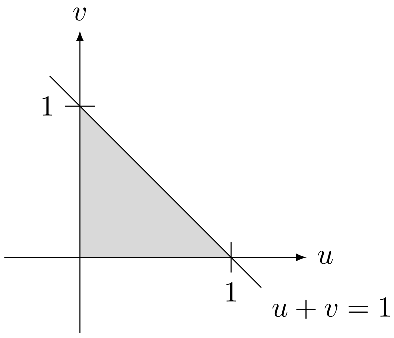
6. (a) The curves $y = x^2 - 2x + 2$ and $y = x^2$ intersect at $x = y = 1$.



(b) The line $x = 0$ (the y-axis), is the line $v = 0$. The curve $y = x^2 - 2x + 2$ is the curve $v^2 + 2u = v^2 - 2v + 2$, i.e. the straight line $u + v = 1$. Finally, $y = x^2$ is the curve $v^2 + 2u = v^2$, i.e. $u = 0$. From the sketch above, we see $x^2 \le y \le x^2 - 2x + 2$, i.e.

$v^2 \le v^2 + 2u$ and $v^2 + 2u \le v^2 - 2v + 2$ so that $0 \le u$ and $u \le 1 - v$. We also have $0 \le x \le 1$ so that $0 \le v \le 1$. Hence

$$D^* = \{(u, v) \,:\, 0 \le u \le 1 - v,\, 0 \le v \le 1\}.$$



(c)

$$\frac{\partial \mathbf{T}(u,v)}{\partial(u,v)} = \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} 0 & 1 \\ 2 & 2v \end{pmatrix} = -2.$$

(d)

$$f(\mathbf{T}(u,v)) = e^{v^2+2u-v^2} + v = e^{2u} + v.$$

(e)

$$\begin{aligned}
\iint_D f(x,y)\,dx\,dy &= \int_0^1 \int_0^{1-v} (e^{2u} + v)|-2|\,du\,dv \\
&= 2\int_0^1 \left[\frac{1}{2}e^{2u} + uv\right]_{u=0}^{u=1-v} dv \\
&= 2\int_0^1 \left(\frac{1}{2}e^{2-2v} - \frac{1}{2} + v - v^2\right) dv \\
&= 2\left[-\frac{1}{4}e^{2-2v} - \frac{v}{2} + \frac{v^2}{2} - \frac{v^3}{3}\right]_0^1 \\
&= -\frac{1}{2} - 1 + 1 - \frac{2}{3} + \frac{e^2}{2} \\
&= \frac{e^2}{2} - \frac{7}{6}.
\end{aligned}$$

Section 2.6: Green's Theorem

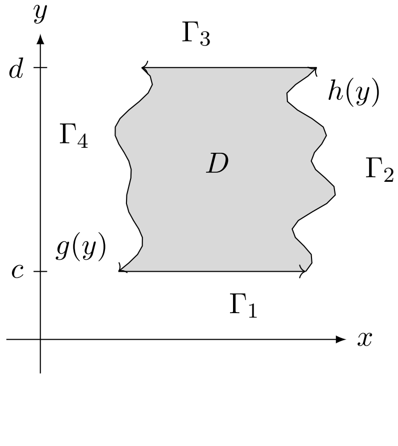
1. Let

$$\Gamma_1 = \{\mathbf{r}_1(t) = (t, c) \,:\, g(c) \le t \le h(c)\},$$

$$\Gamma_2 = \{\mathbf{r}_2(t) = (h(t), t) \,:\, c \le t \le d\},$$

$$\Gamma_3 = \{\mathbf{r}_3(t) = (g(d) + h(d) - t, d) \,:\, g(d) \le t \le h(d)\},$$

$$\Gamma_4 = \{\mathbf{r}_4(t) = (g(c+d-t), c+d-t) \,:\, c \le t \le d\}.$$



Then $\partial D = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4$. Hence,

$$\int_{\partial D} \begin{pmatrix} 0 \\ f \end{pmatrix} \cdot d\mathbf{r}$$

$$
\begin{aligned}
&= \int_{\Gamma_1} \begin{pmatrix} 0 \\ f \end{pmatrix} \cdot d\mathbf{r}_1 + \int_{\Gamma_2} \begin{pmatrix} 0 \\ f \end{pmatrix} \cdot d\mathbf{r}_2 + \int_{\Gamma_3} \begin{pmatrix} 0 \\ f \end{pmatrix} \cdot d\mathbf{r}_3 + \int_{\Gamma_4} \begin{pmatrix} 0 \\ f \end{pmatrix} \cdot d\mathbf{r}_4 \\
&= \int_{g(c)}^{h(c)} \begin{pmatrix} 0 \\ f(t,c) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt + \int_c^d \begin{pmatrix} 0 \\ f(h(t),t) \end{pmatrix} \cdot \begin{pmatrix} h'(t) \\ 1 \end{pmatrix} dt \\
&\quad + \int_{g(d)}^{h(d)} \begin{pmatrix} 0 \\ f(g(d)+h(d)-t,d) \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 0 \end{pmatrix} dt \\
&\quad + \int_c^d \begin{pmatrix} 0 \\ f(g(c+d-t),c+d-t) \end{pmatrix} \cdot \begin{pmatrix} g'(c+d-t) \\ -1 \end{pmatrix} dt \\
&= 0 + \int_c^d f(h(t),t)\,dt + 0 - \int_c^d f(g(c+d-t),c+d-t)\,dt \\
&= \int_c^d f(h(t),t)\,dt + \int_c^d f(g(u),u)\,dt. \qquad \begin{pmatrix} u = c+d-t,\ du = -dt, \\ t = c \Rightarrow u = d, \\ t = d \Rightarrow u = c \end{pmatrix}
\end{aligned}
$$

Also,

$$
\begin{aligned}
\iint_D \frac{\partial f}{\partial x}\,dx\,dy &= \int_c^d \int_{g(y)}^{h(y)} \frac{\partial f}{\partial x}\,dx\,dy \\
&= \int_c^d [f(x,y)]_{x=g(y)}^{x=h(y)}\,dy \\
&= \int_c^d (f(g(y),y) - f(h(y),y))\,dy \\
&= \int_c^d f(g(y),y\,dy - \int_c^d f(h(y),y))\,dy.
\end{aligned}
$$

Thus,

$$
\int_{\partial D} \begin{pmatrix} 0 \\ f \end{pmatrix} \cdot d\mathbf{r} = \iint_D \frac{\partial f}{\partial x}\,dx\,dy.
$$

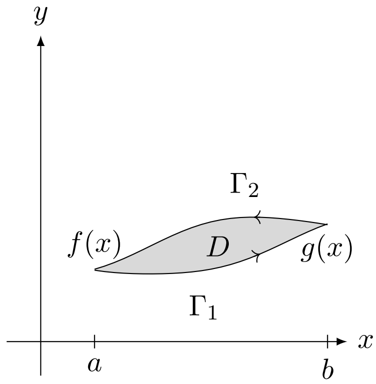
2. (a) Let

$$
\Gamma_1 = \{\mathbf{r}_1(t) = (t, g(t)) \,:\, a \le t \le b\},
$$

$$
\Gamma_2 = \{\mathbf{r}_2(t) = (a+b-t, f(a+b-t)) \,:\, a \le t \le b\}.
$$

Then $\partial D = \Gamma_1 + \Gamma_2$. One way to check the orientation is:

$$
\begin{aligned}
&\mathbf{r}_1'(t) = (1, g'(t)) && \text{(points right)}, \\
&\mathbf{r}_2'(t) = (-1, -f'(a+b-t)) && \text{(points left)}.
\end{aligned}
$$



(b)

$$
\begin{aligned}
I &= \int_{\partial D} \begin{pmatrix} P \\ 0 \end{pmatrix} \cdot d\mathbf{r} \\
&= \int_{\Gamma_1} \begin{pmatrix} P \\ 0 \end{pmatrix} \cdot d\mathbf{r}_1 + \int_{\Gamma_2} \begin{pmatrix} P \\ 0 \end{pmatrix} \cdot d\mathbf{r}_2 \\
&= \int_a^b \begin{pmatrix} P(t, g(t)) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ g'(t) \end{pmatrix} dt \\
&\quad + \int_a^b \begin{pmatrix} P(a+b-t, f(a+b-t)) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -f'(a+b-t) \end{pmatrix} dt
\end{aligned}
$$

$$= \int_a^b P(t, g(t))\, dt - \int_a^b P(a+b-t, f(a+b-t))\, dt$$

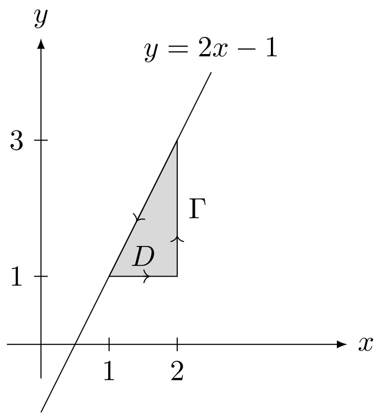
$$= \int_a^b P(t, g(t))\, dt - \int_a^b P(u, f(u))\, du. \qquad \begin{pmatrix} u = a+b-t,\ du = -dt, \\ t = a \Rightarrow u = b, \\ t = b \Rightarrow u = a \end{pmatrix}$$

(c)

$$\begin{aligned} J = \iint_D \frac{\partial P}{\partial y}\, dx\, dy &= \int_a^b \int_{g(x)}^{f(x)} \frac{\partial P}{\partial y}\, dy\, dx \\ &= \int_a^b P(x, f(x)) - P(x, g(x))\, dx \\ &= \int_a^b P(x, f(x))\, dx - \int_a^b P(x, g(x))\, dx. \end{aligned}$$

(d) Clearly, $J = -I$.

3. The region for the path integral is



$$\begin{aligned} \int_\Gamma \begin{pmatrix} e^{x+2y} + xy \\ 2e^{x+2y} + \sin y \end{pmatrix} \cdot d\mathbf{r} &= \iint_D \frac{\partial}{\partial x}(2e^{x+2y} + \sin y) - \frac{\partial}{\partial y}(e^{x+2y} + xy)\, dx\, dy \\ &= \int_1^2 \int_1^{2x-1} 2e^{x+2y} - 2e^{x+2y} - x\, dy\, dx \\ &= \int_1^2 (-xy)\Big|_{y=1}^{y=2x-1}\, dx \\ &= \int_1^2 2x - 2x^2\, dx \\ &= 3 - \frac{14}{3} = -\frac{5}{3}. \end{aligned}$$

This can be verified by calculating the path integral along Γ, which yields the same answer.

4. We have

$$\begin{aligned} \int_{\partial D} \underbrace{ax}_{Q(x,y)}\, dy + \underbrace{(a-1)y}_{P(x,y)}\, dx &= \iint_D \frac{\partial}{\partial x} ax - \frac{\partial}{\partial y}(a-1)y\, dx\, dy \\ &= \iint_D a - (a-1)\, dx\, dy = \iint_D 1\, dx\, dx = \text{Area}(D) \end{aligned}$$

for all a.

5. To make a loop, the curve must intersect itself, i.e. $\mathbf{r}(t_1) = \mathbf{r}(t_2)$ for some $t_1 \neq t_2$. Hence, we must find $t_1 \neq t_2$ from

$$\begin{pmatrix} t_1 - t_1^3 \\ 1 + t_1^2 \end{pmatrix} = \begin{pmatrix} t_2 - t_2^3 \\ 1 + t_2^2 \end{pmatrix}$$

which yields, for the x-coordinate, $t_1 - t_2 = t_1^3 - t_2^3$ and for the y-coordinate: $t_1^2 - t_2^2 = (t_1 - t_2)(t_1 + t_2) = 0$. The latter equation yields $t_2 = -t_1$, since $t_1 \neq t_2$. Inserting $t_1 = -t_2$ into the first equation yields $2t_1 = 2t_1^3$ that $t_1 = 0$ (which we exclude since then $t_2 = -t_1 = 0 = t_1$ but $t_1 \neq t_2$) or $t_1^2 = 1$. Thus $t_1 = -1$ and $t_2 = 1$ (or vice-versa). Since a is arbitrary in question 4, we may set $a = 1$. Let D be the region with boundary

$$\partial D = \{\, \mathbf{r}(t) \;:\; t \in [-1, 1] \,\},$$

then

$$\begin{aligned}
\text{Area}(D) &= \iint_D 1 \, dx \, dy \\
&= \int_{\partial D} ax \, dy + (a - y)y \, dx && \text{(Question 4)} \\
&= \int_{\partial D} x \, dy && \text{(with } a = 1) \\
&= \int_{-1}^{1} (t - t^3) \, 2t \, dt (x = t - t^3, \; y = 1 + t^2) \\
&= \left[\frac{2}{3} t^3 - \frac{2}{5} t^5 \right]_{-1}^{1} \\
&= \frac{8}{15}.
\end{aligned}$$

6. (a) Points on the line between $\mathbf{a}$ and $\mathbf{b}$ have the form $\mathbf{a} + t(\mathbf{b} - \mathbf{a})$. Thus,

$$\Gamma_i = \{\, (x_{i-1} + t(x_i - x_{i-1}), y_{i-1} + t(y_i - y_{i-1}) \;:\; t \in [0, 1] \,\}.$$

(Other parametrisations are possible).

(b)

$$\begin{aligned}
\int_{\Gamma_i} \begin{pmatrix} -y \\ 0 \end{pmatrix} \cdot d\mathbf{r} &= \int_0^1 \begin{pmatrix} -y_{i-1} - t(y_i - y_{i-1}) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x_i - x_{i-1} \\ y_i - y_{i-1} \end{pmatrix} dt \\
&= \int_0^1 y_{i-1}(x_{i-1} - x_i) + t(y_i - y_{i-1})(x_i - x_{i-1}) \, dt \\
&= \left[y_{i-1}(x_{i-1} - x_i)t + \frac{1}{2}(y_i - y_{i-1})(x_i - x_{i-1})t^2 \right]_0^1 \\
&= y_{i-1}(x_{i-1} - x_i) + \frac{1}{2}(y_i - y_{i-1})(x_i - x_{i-1}) \\
&= \frac{1}{2}(y_i + y_{i-1})(x_i - x_{i-1}).
\end{aligned}$$

(c) Since a is arbitrary in question 4, we may set $a = 0$ to show that

$$\int_{\partial D} \begin{pmatrix} -y \\ 0 \end{pmatrix} \cdot d\mathbf{r} = \sum_{i=1}^{N} \int_{\Gamma_i} \begin{pmatrix} -y \\ 0 \end{pmatrix} \cdot d\mathbf{r} = \text{Area})D.$$

Thus, $\text{Area}(D) = \displaystyle\sum_{i=1}^{N} \frac{1}{2}(y_i + y_{i-1})(x_i - x_{i-1})$.

7. (a) $\mathbf{w}(\mathbf{x}) = \mathbf{x} \times \mathbf{v}(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} v_1(x_1, x_2) \\ v_1(x_1, x_2) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ x_1 v_2(x_1, x_2) - x_2 v_1(x_1, x_2) \\ 0 \end{pmatrix}$ and hence we identify $W(x_1, x_2) = x_1 v_2(x_1, x_2) - x_2 v_1(x_1, x_2)$.

(b) Here we write $x_1(t) = \rho \cos(t)$, $x_2 = \rho \sin(t)$ and hence $x_1' = -x_2$ and $x_2' = x_1$. Note also that $\rho = \sqrt{(x_1')^2 + (x_2')^2}$. Hence,

$$
\begin{aligned}
\rho \int_{C_\rho} \mathbf{v} \cdot d\mathbf{r} &= \rho \int_0^{2\pi} \begin{pmatrix} v_1(x_1, x_2) \\ v_1(x_1, x_2) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} x_1' \\ x_2' \\ 0 \end{pmatrix} dt \\
&= \rho \int_0^{2\pi} \begin{pmatrix} v_1(x_1, x_2) \\ v_1(x_1, x_2) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix} dt \\
&= \int_0^{2\pi} (v_1(x_1, x_2) x_1 - x_2 v_1(x_1, x_2)) \rho \, dt \\
&= \int_0^{2\pi} (v_1(x_1, x_2) x_1 - x_2 v_1(x_1, x_2)) \left\| \begin{pmatrix} x_1' \\ x_2' \\ 0 \end{pmatrix} \right\| dt \\
&= \int_{C_\rho} W \, ds.
\end{aligned}
$$

(c) By Green's theorem,

$$
\begin{aligned}
\int_{C_\rho} W \, ds &= \rho \int_{C_\rho} \mathbf{v} \cdot d\mathbf{r} \\
&= \rho \iint_{D_\rho} \left(\frac{\partial}{\partial x_1} v_2 - \frac{\partial}{\partial x_2} v_1 \right) dx_1 \, dx_2 \\
&= \rho \iint_{D_\rho} (\nabla \times \mathbf{v}) \cdot \mathbf{e}_3 \, dx_1 \, dx_2
\end{aligned}
$$

where D_ρ is the disc with centre $(0, 0)$ and radius ρ.

(d) The length of C_ρ is $2\pi\rho$ and the area of D_ρ is $\pi\rho^2$. Thus,

$$
\begin{aligned}
\frac{\int_{C_\rho} W \, ds}{2\pi\rho} &= \frac{1}{2\pi} \iint_{D_\rho} (\nabla \times \mathbf{v}) \cdot \mathbf{e}_3 \, dx_1 \, dx_2 \\
&= \frac{\rho^2}{2} \frac{\iint_{D_\rho} (\nabla \times \mathbf{v}) \cdot \mathbf{e}_3 \, dx_1 \, dx_2}{\pi \rho^2}.
\end{aligned}
$$

So, the average of W is $\dfrac{\rho^2}{2}$ times the average of $(\nabla \times \mathbf{v}) \cdot \mathbf{e}_3$.

8.

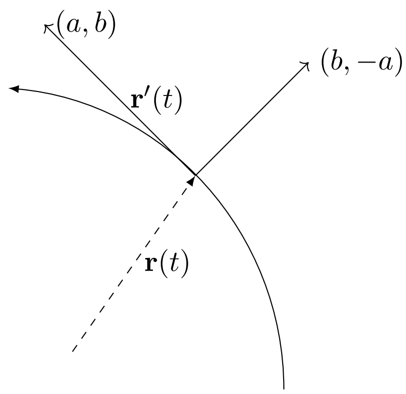
$$
\begin{aligned}
\int_\Gamma \nabla\varphi \cdot d\mathbf{r} &= \int_a^b \nabla\varphi(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} \, dt \\
&= \int_a^b \frac{d}{dt} \varphi(\mathbf{r}(t)) \, dt \\
&= \varphi(\mathbf{r}(b)) - \varphi(\mathbf{r}(a)).
\end{aligned}
$$

9. (a) Since $G_1 = -F_2$ and $G_2 = F_1$,

$$\begin{aligned}
\iint_D \nabla\cdot\mathbf{F}\,dx\,dy &= \iint_D \left(\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}\right)dx\,dy \\
&= \iint_D \left(\frac{\partial G_2}{\partial x}-\frac{\partial G_1}{\partial y}\right)dx\,dy \\
&= \int_{\partial D} \mathbf{G}\cdot d\mathbf{r} \qquad\qquad \text{(by Green's theorem)} \\
&= \int_{\partial D} G_1\,dx + G_2\,dy \\
&= \int_{\partial D} F_1\,dy - F_2\,dx.
\end{aligned}$$

(b) The outward pointing normal vector to ∂D at $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}\right)$, i.e. in the direction (rotated $-90°$)

$$\begin{pmatrix}\cos(-\frac{\pi}{2}) & -\sin(-\frac{\pi}{2})\\ \sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2})\end{pmatrix}\begin{pmatrix}\frac{dx}{dt}\\ \frac{dy}{dt}\end{pmatrix} = \begin{pmatrix}\frac{dy}{dt}\\ -\frac{dx}{dt}\end{pmatrix}.$$



Thus, the unit outward pointing normal vector is

$$\mathbf{n} = \frac{1}{\left\|\left(\frac{dx}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2\right\|}\begin{pmatrix}\frac{dy}{dt}\\ -\frac{dx}{dt}\end{pmatrix}.$$

(c) Straightforward calculation yields

$$\begin{aligned}
\int_{\partial D}\mathbf{F}\cdot\mathbf{n}\,ds &= \int_{\partial D}\begin{pmatrix}F_1\\F_2\end{pmatrix}\cdot\frac{1}{\left\|\left(\frac{dx}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2\right\|}\begin{pmatrix}\frac{dy}{dt}\\ -\frac{dx}{dt}\end{pmatrix}\left\|\left(\frac{dx}{dt}\right)^2+\left(\frac{dx}{dt}\right)^2\right\|dt \\
&= \int_{\partial D} F_1\frac{dy}{dt}\,dt - F_2\frac{dx}{dt}dt \\
&= \int_{\partial D} F_1\,dy - F_2\,dx \\
&= \iint \nabla\cdot\mathbf{F}\,dx\,dy,
\end{aligned}$$

where the last equality follows from (a).

10. (a) We have

$$\frac{\partial H}{\partial z} = \int_x^y \frac{\partial f}{\partial z}(z,p)\,dp,$$

and, by the fundamental theorem of calculus,

$$\frac{\partial H}{\partial x} = -f(z,x), \qquad \frac{\partial H}{\partial y} = f(z,y).$$

Thus,

$$H'(x,y,z) = \begin{pmatrix}\frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z}\end{pmatrix} = \begin{pmatrix}-f(z,x) & f(z,y) & \int_x^y \frac{\partial f}{\partial z}(z,p)\,dp\end{pmatrix}.$$

(b)

$$
\begin{aligned}
(H \circ \mathbf{r})'(t) &= H'(\mathbf{r}(t))\mathbf{r}'(t) \\
&= \begin{pmatrix} -f(z,x) & f(z,y) & \int_x^y \frac{\partial f}{\partial z}(z,p)\,dp \end{pmatrix} \begin{pmatrix} \varphi_1('t) \\ \varphi_2('t) \\ 1 \end{pmatrix} \\
&= -f(z,x)\varphi_1'(t) + f(z,y)\varphi_2'(t) + \int_x^y \frac{\partial f}{\partial z}(z,p).
\end{aligned}
$$

(c) Identifying $x = \varphi_1(t)$, $y = \varphi_2(t)$ and $z = t$, we have

$$
\begin{aligned}
(H \circ \mathbf{r})'(t) &= \frac{d}{dt}\int_{\varphi_1(t)}^{\varphi_2(t)} f(t,p)\,dp \\
&= -f(t,\varphi_1(t))\varphi_1'(t) + f(t,\varphi_2(t))\varphi_2'(t) + \int_{\varphi_1(t)}^{\varphi_2(t)} \frac{\partial f}{\partial t}(t,p).
\end{aligned}
$$