Tutorial Solutions Ch1

Multivariable Calculus (University of the Witwatersrand, Johannesburg)

Chapter 1, Part 4: Directional Derivatives

1.

$$\nabla\phi = \begin{pmatrix} e^{x_1} \\ x_1 e^{x_1} \\ -x_4 \sin x_3 \\ \cos x_3 \end{pmatrix}$$

so the required derivative is given by

$$\begin{pmatrix} \frac{5}{6} \\ \frac{3}{6} \\ \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} \cdot \begin{pmatrix} e^3 \\ 2e^3 \\ \frac{-5}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{6}\left(11e^3 - \frac{4}{\sqrt{2}}\right)$$

2.

$$\nabla\phi = \begin{pmatrix} 2x_2 e^{3x_3} \\ -2x_2 e^{3x_3} \\ 3(x_1^2 - x_2^2)e^{3x_3} \end{pmatrix},$$

so the required directional derivative is given by

$$\frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 6e^3 \\ -4e^3 \\ 15e^3 \end{pmatrix} = \frac{1}{\sqrt{6}}\,(12 + 4 + 15)\, e^3.$$

3. (a) $\nabla\phi = \begin{pmatrix} e^y \\ xe^y \\ -2 \end{pmatrix}$

   (b) Required rate of increase is

$$\frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \frac{-3}{\sqrt{14}}.$$

   (c) The direction of fastest increase at $(1, 0, 3)$ is $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ and there does *not* exist $k > 0$ such that $k\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} = \frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$. So $\mathbf{v}$ is not the direction of fastest increase of ϕ at $(1, 0, 3)$; the direction of fastest (maximum) increase is the direction of $\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$, and the rate of increase in that direction is $\left\|\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}\right\| = \sqrt{6}$.

4. (a) See notes.

(b) $\nabla f = \begin{pmatrix} 2x_1 e^{x_2} \\ (x_1^2 - x_3)e^{x_2} \\ -e^{x_3} \end{pmatrix}$, so the direction of maximum increase at $(2, \ln 3, -1)$ is

$$\nabla f(2, \ln 3, -1) = \begin{pmatrix} 12 \\ 9 \\ -3 \end{pmatrix}.$$

The directional derivative of f at $\mathbf{x}_0$ in the direction of $\mathbf{v}$ is

$$\frac{1}{\sqrt{30}} \begin{pmatrix} -1 \\ 2 \\ -5 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 9 \\ -3 \end{pmatrix} = \frac{21}{\sqrt{30}}.$$

5. (a) From the definition of the directional derivative we have

$$\mathrm{D}_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{t \to 0} \frac{f(\mathbf{x}_0 + t\mathbf{u}) - f(\mathbf{x}_0)}{t} = \lim_{t \to 0} \frac{f(\mathbf{r}(t+0)) - f(\mathbf{r}(0))}{t} = [f \circ \mathbf{r}]'(0).$$

But f is constant along $\mathbf{r}(t)$ so $f(\mathbf{r}(t)) = k \ \forall t$, giving $[f \circ \mathbf{r}]'(t) = 0$, and thus $\mathrm{D}_{\mathbf{u}} f(\mathbf{x}_0) = 0$.

(b) Let $h(t) = f \circ \mathbf{r}(t)$ then $h(t)$ has a maximum or minimum at $t = 0$ if f has a max or min at $\mathbf{x}_0$. Thus $h'(0) = 0$ so

$$h'(0) = f'(\mathbf{r}(0))\mathbf{r}'(0) = f'(\mathbf{x}_0)\mathbf{u} = \nabla f(\mathbf{x}_0) \cdot \mathbf{u} = \mathrm{D}_{\mathbf{u}} f(\mathbf{x}_0).$$

(c) Let h be as above, then if $\nabla f(\mathbf{x}) = 0 \ \forall \mathbf{x} \in \mathbb{R}^n$, we see that $h'(t) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{u} = 0 \ \forall t$. Thus $h(t)$ is a constant. In particular, $h(t) = h(0)$ giving $f(\mathbf{x}_0 + t\mathbf{u}) = f(\mathbf{x}_0)$.

Now let $\mathbf{y} \in \mathbb{R}^n$. If $\mathbf{y} \neq \mathbf{x}_0$, let $\mathbf{u} = \frac{\mathbf{y}-\mathbf{x}_0}{\|\mathbf{y}-\mathbf{x}_0\|}$ and $t = \|\mathbf{y} - \mathbf{x}_0\|$. Then

$$\mathbf{x}_0 + t\mathbf{u} = \mathbf{x}_0 + \|\mathbf{y} - \mathbf{x}_0\| \frac{\mathbf{y} - \mathbf{x}_0}{\|\mathbf{y} - \mathbf{x}_0\|} = \mathbf{y},$$

so $f(\mathbf{y}) = f(\mathbf{x}_0)$, i.e. f is constant in $\mathbb{R}^n$.