Basic Analysis 2015 — Solutions of Tutorials

Section 4.2

Tutorial 4.2.1

1. Find the derivatives of
(a) $\arctan x$, (b) $\arccos x$, (c) $\ln x$ and $\ln|x|$,
(d) $\ln|f(x)|$, where $f : I \to \mathbb{R} \setminus \{0\}$ is differentiable on I.

Solution. From (a) $\frac{d}{dx}\tan x = \sec^2(x) > 0$, $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$, (b) $\frac{d}{dx}\cos x = -\sin x < 0$, $x \in (0,\pi)$, (c) $\frac{d}{dx}e^x = e^x > 0$, $x \in \mathbb{R}$, it follows in view of the First Mean Value Theorem that these three functions are strictly monotonic on $(-\frac{\pi}{2}, \frac{\pi}{2})$, $[0,\pi]$ and $\mathbb{R}$, respectively. Hence it follows from Theorem 4.8 that their inverses are differentiable on $\mathbb{R}$, $(-1,1)$, and $(0,\infty)$, respectively, and are given by

$$\frac{d}{dx}\arctan x = \frac{1}{\sec^2(\arctan x)} = \frac{1}{1+\tan^2(\arctan x)} = \frac{1}{1+x^2}, \quad x \in \mathbb{R},$$

$$\frac{d}{dx}\arccos x = \frac{1}{-\sin(\arccos x)} = -\frac{1}{\sqrt{1-\cos^2(\arccos x)}} = \frac{1}{\sqrt{1-x^2}}, \quad x \in (-1,1),$$

$$\frac{d}{dx}\ln x = \frac{1}{e^{\ln x}} = \frac{1}{x}, \quad x \in (0,\infty).$$

Since $\ln|x| = \ln x$ for $x > 0$ and $\ln|x| = \ln(-x)$ for $x < 0$, it follows from the chain rule that

$$\frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}, \quad x < 0,$$

so that

$$\frac{d}{dx}\ln|x| = \frac{1}{x}, \quad x \in \mathbb{R} \setminus \{0\}.$$

(d) follows now from the chain rule:

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}, \quad x \in I.$$

2. Let $x > 0$ and let $r = \frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}^*$. Define $x^r = (x^p)^{\frac{1}{q}}$.
Show that $x^r = \exp(r \ln x)$.

Proof. Let $g(y) = y^q$ for $y \in [0,\infty)$. We are first going to show that g is a bijection from $[0,\infty)$ onto $[0,\infty)$. Indeed, g is continuous on $[0,\infty)$, differentiable on $(0,\infty)$, $g(y) \ge 0$ for all $y \in [0,\infty)$, $g'(y) > 0$ for all $y > 0$, and $g(y) \ge y$ for all $y \ge 1$. From the intermediate value theorem we get $g([0,\infty)) = [0,\infty)$, and the mean value theorem gives that g is injective.
Hence $z^{\frac{1}{q}}$ is well defined for $z \in [0,\infty)$.
Next we are going to show that $(\exp(t))^s = \exp(st)$ for all $t \in \mathbb{R}$ and $s \in \mathbb{Z}$.
Indeed, if $p = 0$, then

$$(\exp(t))^0 = 1 = \exp(0) = \exp(0 \cdot t).$$

If $s > 0$, then it follows from Tutorial 2.2.1,8(c), that

$$(\exp(t))^s = \exp(st).$$

If $s < 0$, then it follows from above and Tutorial 2.2.1,8(c), that

$$(\exp(t))^s = \frac{1}{(\exp(t))^{-s}} = \frac{1}{\exp(-st)} = \exp(st).$$

Therefore it follows with $t = \ln x$ and $s = p$ as well as $t = r\ln x$ and $s = q$ that

$$(x^r)^q = x^p = (\exp(\ln x))^p = \exp(p\ln x) = \exp(qr\ln x) = (\exp(r\ln x))^q,$$

and the work at the beginning of this proof shows that $x^r = \exp(r\ln x)$. $\square$

3. In view of tutorial problem 2 above, we define $x^r = \exp(r \ln x)$ for all $x > 0$ and $r \in \mathbb{R}$.
(a) Show that $\lim_{x \to -\infty} \exp(x) = 0$.

Proof. From $\exp(x) \geq 1 + x$ for all $x \in \mathbb{R}$, see Theorem 4.9, part 1, we have concluded in the proof of Theorem 4.9 that $e^x \leq \dfrac{1}{1-x}$ for $x < 1$. On the other hand, by Tutorial 2.2.1,8(c), $\exp(x) = \left(\exp\left(\frac{x}{2}\right)\right)^2 \geq 0$, and the Sandwich Theorem gives $\lim_{x \to -\infty} \exp(x) = 0$. □

(b) Show that $\lim_{x \to \infty} \dfrac{1}{x^r}$ for all $r > 0$.
Solution. First observe that $\exp' = \exp > 0$ shows that exp is strictly increasing and bijective from $[0, \infty)$ onto $[1, \infty)$. Hence the inverse ln is strictly increasing and bijective from $[1, \infty)$ onto $[0, \infty)$. In particular, $x \to \infty$ implies $\ln x \to \infty$. From parts 2 and 3(a) we find

$$\lim_{x \to \infty} \frac{1}{x^r} = \lim_{x \to \infty} \frac{1}{\exp(r \ln x)} = \lim_{x \to \infty} \exp(-r \ln x) = 0$$

since $-r \ln x \to -\infty$ as $x \to \infty$.