- **Tutorial 3.2.1.**

1. Test $\displaystyle\sum_{n=1}^{\infty}\left\{\left(\frac{1}{2}\right)^n + 5\left(\frac{3}{4}\right)^n\right\} n\sin\left(\frac{1}{n}\right)$ for convergence.

Solution.

Let $a_n = \left(\frac{1}{2}\right)^n n$ and $b_n = \left(\frac{3}{4}\right)^n n$. Since

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{1}{2}\right)^{n+1}(n+1)}{\left(\frac{1}{2}\right)^n n} = \frac{1}{2}\left(1 + \frac{1}{n}\right) \to \frac{1}{2} < 1 \text{ as } n \to \infty$$

it follows from the Ratio Test that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

Similarly, it follows from

$$\frac{b_{n+1}}{b_n} = \frac{\left(\frac{3}{4}\right)^{n+1}(n+1)}{\left(\frac{3}{4}\right)^n n} \to \frac{3}{4} < 1 \text{ as } n \to \infty$$

that $\sum_{n=1}^{\infty} b_n$ converges. Then it follows from

$|\sin \frac{1}{n}| < 1$

absolute conv $\Downarrow$ conv

$$\left|\left(\frac{1}{2}\right)^n n \sin\left(\frac{1}{n}\right)\right| \leq a_n, \quad \left|\left(\frac{3}{4}\right)^n n \sin\left(\frac{1}{n}\right)\right| \leq b_n,$$

the Comparison Test and Theorem 3.4 that also $\sum_{n=1}^{\infty} \left\{\left(\frac{1}{2}\right)^n + 5\left(\frac{3}{4}\right)^n\right\} n \sin\left(\frac{1}{n}\right)$ converges.

2. Prove that the sequence $(a_n)_{n=1}^{\infty}$ converges if and only if
(i) $(a_{2n})_{n=1}^{\infty}$ converges,
(ii) $(a_{2n-1})_{n=1}^{\infty}$ converges,
(i) $(a_n - a_{n-1}) \to 0$ as $n \to \infty$.

Proof. Assume that the sequence $(a_n)_{n=1}^{\infty}$ converges. By definition, there is $L \in \mathbb{R}$ and for all $\epsilon > 0$ there is $K \in \mathbb{R}$ such that for all $n \geq K$: $|a_n - L| < \frac{\epsilon}{2}$. Since $n \leq 2n - 1 < 2n$, it follows for all $n \geq K + 1$ that

$$|a_{2n} - L| < \frac{\epsilon}{2} < \epsilon,$$

$$|a_{2n-1} - L| < \frac{\epsilon}{2} < \epsilon,$$

$$|\, a_n - a_{n-1} \,| \leq |\, a_n - L \,| + |\, a_{n-1} - L \,| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which proves (i), (ii), and (iii).

Conversely, let (i), (ii) and (iii) hold. Let

$$L_1 = \lim_{n\to\infty} a_{2n-1} \text{ and } L_2 = \lim_{n\to\infty} a_{2n}.$$

Let $\epsilon > 0$. Then there are $K_0 \in \mathbb{R}$ such that

$$|\, a_n - a_{n-1} \,| < \frac{\epsilon}{3}$$

for all $n \geq K_0$, $K_1 \in \mathbb{R}$ such that

$$|\, a_{2n-1} - L_1 \,| < \frac{\epsilon}{3}$$

for all $n \geq K_1$, and $K_2 \in \mathbb{R}$ such that

$$|\, a_{2n} - L_2 \,| < \frac{\epsilon}{3}$$

for all $n \geq K_2$. Put $K = \max\{K_0, K_1, K_2 \cancel{-2}\}$ and choose some $n \geq K$. Then $2n > 2n - 1 \geq K$ and

$$\begin{aligned}
|\, L_2 - L_1 \,| &= |\, (L_2 - a_{2n}) + (a_{2n} - a_{2n-1}) + (a_{2n-1} - L_1) \,| \\
&\leq |\, L_2 - a_{2n} \,| + |\, a_{2n} - a_{2n-1} \,| + |\, a_{2n-1} - L_1 \,| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Hence $L_2 = L_1$ by Lemma 2.1. For $n \geq 2K + 1$ we now conclude: If n is even, then $n = 2k$ with $k \geq K \geq K_2$, and therefore

$$|a_n - L_2| = |a_{2k} - L_2| < \frac{\epsilon}{3} < \epsilon;$$

If n is odd, then $n = 2k - 1$ with $k \geq K \geq K_1$, and therefore

$$|a_n - L_1| = |a_{2k-1} - L_1| < \frac{\epsilon}{3} < \epsilon.$$

This shows that the sequence $(a_n)_{n=1}^{\infty}$ converges. □

3. Use Tut 2 to prove that the Alternating Series.
Hint. Show that $(s_{2n})_{n=1}^{\infty}$ and $(s_{2n-1})_{n=1}^{\infty}$ are monotonic sequences.

Proof. Without loss of generality we may assume that the alternating series is of the form $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$.

We calculate

$$\begin{aligned} s_{2n} &= \sum_{k=1}^{2n}(-1)^{k-1}b_k = \sum_{k=1}^{n} b_{2k-1} - \sum_{k=1}^{n} b_{2k} \\ &= \sum_{k=1}^{n}(b_{2k-1} - b_{2k}). \end{aligned}$$

Since the sequence (b_n) is decreasing, it follows that each term of the series on the right hand side, $b_{2k-1} - b_{2k}$, is nonnegative, and hence the sequence s_{2n} is increasing. We further calculate

$$\begin{aligned} s_{2n-1} &= \sum_{k=1}^{2n-1} (-1)^{k-1} b_k = b_1 + \sum_{k=1}^{n-1} b_{2k+1} - \sum_{k=1}^{n-1} b_{2k} \\ &= b_1 - \boxed{\sum_{k=1}^{n-1} (b_{2k} - b_{2k+1}).} \quad \geq 0 \end{aligned}$$

Hence the sequence (s_{2n-1}) is decreasing. We further calculate

$$s_{2n} = s_{2n-1} + (-1)^{2n-1} b_{2n} \leq s_1 - b_{2n} \leq s_1 = b_1,$$

which shows that (s_{2n}) is bounded and increasing. By Theorem 2.10, (s_{2n}) converges.
Furthermore, $s_{2n-1} = s_{2n} - (-1)^{2n-1} b_{2n}$, so that s_{2n-1} is the difference of two convergent sequences and thus convergent. Finally, $s_n - s_{n-1} = (-1)^{n-1} b_n \to 0$ as $n \to \infty$ shows that the sequence (s_n) satisfies the assumptions (i), (ii), (iii) of part 2. Hence the alternating series converges. $\square$

4. Use the alternating series test, ratio test or root test to test for convergence:

(a) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{(-1)^n n}{2n+1}\right)^{2n}$, (b) $\displaystyle\sum_{n=1}^{\infty}\frac{n!2^n}{(2n)!}$,

(c) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(e-\left(1+\frac{1}{n}\right)^n\right)$, (d) $\displaystyle\sum_{n=1}^{\infty}\frac{n^n}{(n!)^2}$,

(e) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n}$.

Solution.

(a)

$\frac{1}{4} < 1$

$$\sqrt[n]{\left|\left(\frac{(-1)^n n}{2n+1}\right)^{2n}\right|}=\left(\frac{n}{2n+1}\right)^2=\left(\frac{1}{2+\frac{1}{n}}\right)^2\to\frac{1}{4}\text{ as } n\to\infty$$

shows that the series converges in view of the root test.

(b)

$$\frac{\dfrac{(n+1)!2^{n+1}}{(2(n+1))!}}{\dfrac{n!2^n}{(2n)!}}=\frac{2(n+1)}{(2n+2)(2n+1)}=\frac{1}{2n+1}\to 0\text{ as } n\to\infty$$

$0 < 1$

shows that the series converges in view of the ratio test.

(c) We know from Example 2.2.3 that $\left(1+\frac{1}{n}\right)^n$ increases and has limit e. Hence this alternating series converges by the alternating series test.

$e-\left(1+\frac{1}{n}\right)^n \longrightarrow e-e=0$

(d)

$$\frac{\dfrac{(n+1)^{n+1}}{((n+1)!)^2}}{\dfrac{n^n}{(n!)^2}} = \frac{\left(\dfrac{n+1}{n}\right)}{n+1} = \frac{\left(1+\dfrac{1}{n}\right)}{n+1} \to 0 \text{ as } n \to \infty$$

<1

(e)

$$\frac{\dfrac{2^{n+1}}{n+1}}{\dfrac{2^n}{n}} = \frac{2n}{n+1} \to 2 \text{ as } n \to \infty$$

$2>1$

shows that the series diverges in view of the ratio test.