Chapter 7: Groups of Symmetry

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LEARNING OUTCOMES FOR THE LECTURE

By the end of this lecture, students will be able to:

- ♣ find a composition of two permutations
- ♣ find the inverse of a permutation
- ♣ identify fixed and moved elements in a permutation
- ♣ find the orbit of any element in a permutation
- ♣ use cycle notation to represent a permutation

(NOTE 7.2.3 (2))

Let in S_4,

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}; \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}; \alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$$

Note that $\beta\alpha \neq \alpha\beta$.

(NOTE 7.2.3 (3))

Find Inverses in S_n.

$$\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix} \text{ then } \delta^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 6 & 1 & 3 & 4 & 5 \end{pmatrix}$$

Read from 2nd row of array to 1st row of array.

(NOTE 7.2.3 (4))

S_n is called symmetric group of n elements.
$|S_n| = n!$ *so* $|S_4| = 4! = 24$.

FIXED AND MOVED ELEMENTS

Note:

In Note 7.2.3 (2), $\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$.

We see that 2 and 3 are fixed and 1 and 4 are moved.

Where as $\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$

has 1 and 4 fixed and 2 and 3 moved.

Definition (7.3.1)

Let $X = \{1, 2, \cdots, n\}$ *and* $\alpha \in S_n$.

1. k *is* *fixed* *by* α *if* $\alpha(k) = k, k \in X$.
2. k *is* *moved* *by* α *if* $\alpha(k) \neq k, k \in X$.
3. α *and* β *are* *disjoint* *if no element of* X *is moved by both* α *and* β.

The set of elements moved by α is disjoint from the set of elements moved by β. For example, in Note 7.2.3 (2), $\alpha\beta$ moves $\{1, 4\}$ and $\beta\alpha$ moves $\{2, 3\}$ $\therefore$ $\alpha\beta$ and $\beta\alpha$ are disjoint.

Example (7.3.2 (1))

$e \in S_n$ moves no elements and fixes all elements.

$\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$ *fixes no elements.*

$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$ *fixes 1 and 4, moves 2 and 3.*

Example (7.3.2 (2))

Let $\sigma \in S_n$ where $X = \{1, 2, \cdots, n\}$. We say

$a \equiv b \quad$ *iff* $\quad b = \sigma^r(a) \quad a, b \in X$, *for some $r \in \cancel{\mathbb{N}}$.* $\mathbb{Z}$

please note correction

EXCERCISE

Show that $a \equiv b \quad \text{iff} \quad b = \sigma^r(a) \quad a, b \in X$ defines an equivalence relation on S_n.

(i) $a \equiv a$ since $\sigma^0 = e$ and $e(a) = a \quad \forall a \in X$.

(ii)

$$\begin{aligned} a \equiv b \quad &\Rightarrow \quad b = \sigma^r(a) \\ &\Rightarrow \quad (\sigma^r)^{-1}(b) = (\sigma^r)^{-1}\sigma^r(a) \\ &\Rightarrow \quad \sigma^{-r}(b) = a \\ &\Rightarrow \quad b \equiv a \quad \text{as} \quad -r \in \mathbb{Z} \end{aligned}$$

(iii $a \equiv b \Rightarrow b = \sigma^r(a)$ and $b \equiv c \Rightarrow c = \sigma^k(b)$
$\Rightarrow \quad c = \sigma^k(b) = \sigma^k(\sigma^r(a)) = \sigma^{k+r}(a)$
$\Rightarrow \quad a \equiv c$ as $k + r \in \mathbb{Z}$.

Equivalence Classes:

$$\begin{aligned} [a] &= \{b \in X | b \equiv a\} \\ &= \{\cdots, \sigma^{-2}(a), \sigma^{-1}(a), a, \sigma(a), \sigma^2(a), \cdots\} \\ &= \text{Orbit of } a \text{ under } \sigma. \end{aligned}$$

e.g.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 4 & 5 & 6 & 2 \end{pmatrix}$$

$$[1] = \{\sigma^r(1) | r \in \mathbb{Z}\} = \{1, 3, 4, 5, 6, 2\} = X$$

Exactly one orbit or equivalence class under σ.

$$\delta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 2 & 4 & 6 \end{pmatrix}$$

$[1] = \{1\}$ under δ
$[2] = \{2, 3, 5, 4\}$ under δ
$[6] = \{6\}$ under δ

EXERCISE

Find the equivalence classes of τ. $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix}$

Let $\approx$ be equiv relation $a \approx b$ iff $b = \tau^r(a)$, $r \in \mathbb{Z}$
$[1] = \{1, 3, 5\} = [3] = [5]$; $[2] = \{2\}$; $[4] = \{4\}$.
$[1] \cup [2] \cup [4] = X$.

Cycle Decomposition

Definition (7.4.1 (1))

Let $X = \{1, 2, \cdots, n\}$ *and let* $k_1, k_2, \cdots, k_r$ *be* r *distinct elements of* X. *Cycle* $\sigma = (k_1 k_2 \cdots k_r)$ *is a permutation in* S_n *is defined by*
$\sigma(k_1) = k_2, \sigma(k_2) = k_3, \cdots, \sigma(k_r) = k_1$ *and* $\sigma(k) = (k)$ *if* $k \notin \{k_1, k_2, \cdots, k_r\}$.
That is: $\sigma(k_i) = k_{i+1}$ *for* $i = 1, 2, \cdots, r-1$ *and* $\sigma(k_r) = k_1$ *and* $\sigma(k) = k$ *if* $k \notin \{k_1, k_2, \cdots, k_r\}$.

Definition (7.4.1 (2)) $(k_1\ k_2\ k_3 \ldots k_r)$

Let $X = \{1, 2, \cdots, n\}$ *and let* $k_1, k_2, \cdots, k_r$ *be r distinct elements of* X. σ *has length r and is an r-cycle.*
e.g. $\tau = (1 \quad 3 \quad 5) = (3 \quad 5 \quad 1) = (5 \quad 1 \quad 3)$ *is a 3-cycle*
$\sigma = (1 \quad 3 \quad 4 \quad 5 \quad 6 \quad 2)$ *is a 6-cycle which is equal to* $(3 \quad 4 \quad 5 \quad 6 \quad 2 \quad 1) = \cdots = (2 \quad 1 \quad 3 \quad 4 \quad 5 \quad 6)$.

Definition (7.4.1 (3))

Let $X = \{1, 2, \cdots, n\}$ *and let* $k_1, k_2, \cdots, k_r$ *be r distinct elements of* X. *Representation is not unique but has a cyclic character.*