Tutorial Solutions of Chapter 2 Lecture 1

Tutorial 2.1.1.(1)

(a) Prove, using the definition of convergence, that the sequence $(a_n) = \left(\dfrac{n}{n+1}\right)$ does not converge to 2.

Proof : We first estimate

$$\frac{n}{n+1} - 2 = \frac{-n-2}{n+1} = -1 - \frac{1}{n+1} < -1$$

Let $\epsilon = 1$. Then

$$|\, a_n - 2\,| = \left| \frac{n}{n+1} - 2 \right| > 1 = \epsilon$$

for all $n \in \mathbb{N}$. Hence there is no K such that

$$\left| \frac{n}{n+1} - 2 \right| < \epsilon$$

for all $n > K$. Therefore the sequence (a_n) does not converge to 2. $\square$

(b) Prove, using the definition of convergence, that the sequence $\big(a_n\big) = \big((-1)^n\big)$ does not converge to any L.

Proof : Assume, by proof of contradiction, that the sequence (a_n) converges to some $L \in \mathbb{R}$.

Let $\epsilon > 0$. Then there would exists K such that $|\,(-1)^n - L\,| < \epsilon$ for all $n > K$. In particular,

$$\big|\,(-1)^{2n} - L\,\big| < \epsilon \quad \text{and} \quad \big|\,(-1)^{2n+1} - L\,\big| < \epsilon\,,$$

But then,

$$2 = |\,1 - (-1)\,| = |\,1 - L - (-1 - L)\,| \le |\,1 - L\,| + |\,-1 - L\,| = \big|\,(-1)^{2n} - L\,\big| + |\,(-1)^{2n+1} - L\,| < 2\epsilon$$

which is clearly false if $\epsilon \le 1$. $\square$