MATH2001–Basic Analysis Final Examination June 2011

Time: 90 minutes Total marks: 90 marks

SECTION A Multiple choice

Answer the multiple choice questions on the computer card provided. There is ONLY ONE correct answer to each question. Please ensure that your student number is entered on the card, by pencilling in the requisite digit for each block.

Question 1 .... [3 marks]

Let (a_n) be a bounded sequence. Which of the following statements is true for all such sequences?

A. (a_n) converges if and only if $a_n - \liminf_{k\to\infty} a_k \to 0$ as $n \to \infty$.

B. $\limsup_{n\to\infty} a_n \in \mathbb{R}$ and $\liminf_{n\to\infty} a_n \in \mathbb{R}$ and

$$2 \lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n + \liminf_{n\to\infty} a_n.$$

C. (a_n) converges if and only if $\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

D. For all $n \in \mathbb{N}$, $a_n \leq \limsup_{k\to\infty} a_k \in \mathbb{N}$.

E. None of the above.

Question 2 .... [3 marks]

Consider the series $\sum_{n=1}^{\infty} a_n$. Which of the following statements is false?

A. If $\limsup_{n\to\infty} |a_n| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

B. If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

C. If $a_n \neq 0$ for all n and if $\limsup_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

D. If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

E. None of the above.

Questions 3–7 below refer to the following theorem and its proof.

Theorem. Suppose that f is continuous on the closed interval $[a, b]$ with $f(a) \neq f(b)$. Then for any number k between $f(a)$ and $f(b)$ there exists a number c in the open interval (a, b) such that $f(c) = k$.

Question 3 .... [1 mark]

How is this theorem called?

A. Fermat's Theorem

B. Intermediate Value Theorem

C. Mean Value Theorem

D. Monotone Continuity Theorem

E. Rolle's Theorem

Question 4 .... [3 marks]

Assume that $f(a) < f(b)$ and define $S = \{x \in [a, b] : f(x) < k\}$. Then

A. $S \neq \emptyset$ since $b \notin S$ and S is bounded above by k;

B. $S \neq \emptyset$ since $a \in S$ and S is bounded above by k;

C. $S \neq \emptyset$ since $b \in S$ and S is bounded above by k;

D. $S \neq \emptyset$ since $b \in S$ and S is bounded above by b;

E. $S \neq \emptyset$ since $a \in S$ and S is bounded above by b.

Question 5 .... [2 marks]

The set S defined in Question 3 has a supremum

A. since $S \neq \emptyset$;

B. by Dedekind completeness;

C. since $f(a) < f(b)$;

D. since f is continuous;

E. since $a < b$.

Question 6 .... [3 marks]

Let $c = \sup S$ and assume that $f(c) < k$. Since $k - f(c) > 0$ and f is continuous from the right at c,

A. there is $\delta > 0$ such that $c \leq x < c + \delta$ implies $|f(x) - f(c)| < k$;

B. there is $\delta > 0$ such that $c \leq x < c + \delta$ implies $|f(x) - f(c)| < f(c) - k$;

C. there is $\delta > 0$ such that $c \leq x < c + \delta$ implies $|f(x) - f(c)| < f(x) - k$;

D. there is $\delta > 0$ such that $c - \delta \leq x < c$ implies $f(x) < k$;

E. there is $\delta > 0$ such that $c \leq x < c + \delta$ implies $|f(x) - f(c)| < k - f(c)$.

Question 7 .... [3 marks]

The correct answer in Question 6 contradicts the assumptiom that $f(c) < k$ since the answer would imply that

A. $f(x) \in S$ for all $c \leq x < c + \delta$;

B. S is unbounded;

C. $S = \emptyset$;

D. $\sup S < c$;

E. S is not closed.

Question 8 .... [3 marks]

The function $f(x) = \dfrac{\sin x(e^x - 1)}{x^2}$ is

A. is differentiable at $x = 0$ since $\lim\limits_{x\to 0} \dfrac{f(x)}{x}$ exists;

B. is differentiable at $x = 0$ since $\lim\limits_{x\to 0} \dfrac{f(x) - 1}{x}$ exists;

C. is differentiable at $x = 0$ since $\sin x$, $e^x - 1$ and x^2 are differentiable at $x = 0$;

D. is not differentiable at $x = 0$ since $\lim\limits_{x\to 0} \dfrac{f(x)}{x}$ does not exist;

E. is not differentiable at $x = 0$ since f is not defined at $x = 0$;

Question 9 .... [3 marks]

Let I be an interval and $f : I \to \mathbb{R}$. Which of the following statements is not always true?

A. If f is continuous and $I = [a, b]$, $a < b$, then f is bounded.

B. If f is continuous and unbounded, then I is not a bounded interval.

C. If f is differentiable, then f is continuous.

D. If f is differentiable and $I = [a, b]$, $a < b$, then
$|f(b) - f(a)| \leq (b - a) \sup\limits_{x\in(a,b)} = |f'(x)|$.

E. If f is differentiable and $I = [a, b]$, $a < b$, then
$f(b) - f(a) \leq (b - a) \sup\limits_{x\in(a,b)} = f'(x)$.

SECTION B

Answer this section in the answer book provided.

Question 1 .... [9 marks]

Write down the definitions of the following limits of functions where $a, L \in \mathbb{R}$ and f is a real-valued functions. Also write down the assumptions for the domain of f.

(a) $\lim_{x \to a^-} f(x) = L.$ (3)

(b) $\lim_{x \to a^+} f(x) = -\infty.$ (3)

(c) $\lim_{x \to \infty} f(x) = L.$ (3)

Question 2 .... [14 marks]

Prove from the definitions that

(a) $\lim_{x \to -1^-} \frac{1}{x+1} = -\infty$ (6)

(b) $\lim_{x \to -\infty} \frac{x^2+1}{x^2-1} = 1$ (8)

Question 3 .... [10 marks]

Let $a \in \mathbb{R}$ and let f and g be continuous at a. Prove that the product fg is also continuous at a.

Question 4 .... [17 marks]

Recall that $e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$.

(a) Show that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$. (6)
Hint. You may use Bernoulli's inequality.

(b) Using part (a) or otherwise show that $e^x \leq \frac{1}{1-x}$ for $x > -1$. (3)

(c) Show that $\lim_{h \to 0} \frac{e^h - 1}{h}$ exists, and find this limit. (5)

(d) Show that e^x is differentiable and find its derivative. (3)

Question 5 .... [6 marks]

Let $f : [0, 2] \to \mathbb{R}$ be continuous with $f(0) = f(2)$. Show that there is $c \in [0, 1]$ such that $f(c) = f(c + 1)$.

Question 6 .... [10 marks]

(a) Let (a_n) be a sequence of real numbers with $a_n \neq 0$ for all n. Prove that (7)

$$\limsup_{n \to \infty} \sqrt[n]{|a_n|} \leq \limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|.$$

(b) Give an example for a sequence (a_n) where the inequality in (a) is strict. (3)
Write down the values of for $\limsup_{n \to \infty} \sqrt[n]{|a_n|}$ and $\limsup_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|$ for your example.
No proof is required.