ROBOTICS

DYNAMICS

DYNAMICS

- At the end of the last lecture we talked about systems that are at rest
- In this lecture we'll look at the modelling some dynamical systems

FORCES

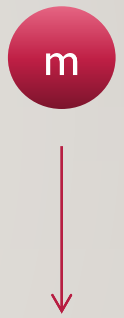
- Newton's Second Law
  - $F = ma$
  - $F = m\ddot{x}$
- Drag force
  - $F = -c\dot{x}$
- Spring force
  - $F = -kx$

THE LAGRANGIAN

- Lagrangian = Kinetic Energy – Potential Energy
- $\mathcal{L}(\mathrm{q}, \dot{q}) = T(q, \dot{q}) - V(q, \dot{q})$
- The joint forces are then calculated as
- $\tau_j = \frac{d}{dt}\left(\frac{\delta \mathcal{L}}{\delta \dot{q}_j}\right) - \frac{\delta \mathcal{L}}{\delta q_j}$

EXAMPLE : FALLING MASS

- A falling mass
- Kinetic Energy $T = \frac{1}{2}m\dot{x}^2$
- Gravitational Potential Energy $V = mgx$
- $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - mgx$
- Joint forces? 0
- So $\frac{d}{dt}\left(\frac{\delta\mathcal{L}}{\delta\dot{q}_j}\right) - \frac{\delta\mathcal{L}}{\delta q_j} = 0$



EXAMPLE : FALLING MASS

- $\frac{\delta \mathcal{L}}{\delta \dot{q}_j} = m\dot{x}\,, \frac{\delta \mathcal{L}}{\delta q_j} = -mg$
- $\frac{d}{dt}(m\dot{x}) - (-mg) = 0$
- $m\ddot{x} + mg = 0$
- $\ddot{x} = -g$
- That's a good confirmation that we're doing something right

From previous slide:

- $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - mgx$
- $\frac{d}{dt}\left(\frac{\delta \mathcal{L}}{\delta \dot{q}_j}\right) - \frac{\delta \mathcal{L}}{\delta q_j} = 0$

EXAMPLE: MASS ON A SPRING

- Kinetic Energy $T = \frac{1}{2}m\dot{x}^2$
- Potential Energy $V = \frac{1}{2}kx^2$
- $\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$
- $\frac{\delta\mathcal{L}}{\delta\dot{q}_j} = m\dot{x}, \frac{\delta\mathcal{L}}{\delta q_j} = -kx$
- $m\ddot{x} - (-kx) = 0$
- $m\ddot{x} + kx = 0$
- Requires you to solve the DE

From previous slides:

- $\mathcal{L} = T - V$
- $\frac{d}{dt}\left(\frac{\delta\mathcal{L}}{\delta\dot{q}_j}\right) - \frac{\delta\mathcal{L}}{\delta q_j} = 0$

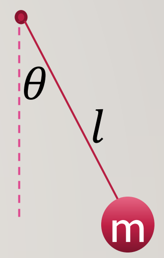
EXAMPLE: MASS ON A SPRING

- In this case we're lucky that our DE has a closed form solution
- For a DE of the form $\ddot{x} + \omega_0^2 x = 0$, the solution is:
- $x(t) = x_0 \cos(\omega_0 t) + \frac{\dot{x}_0}{\omega_0} \sin(\omega_0 t)$
- $\ddot{x} + \frac{k}{m} x = 0$
- $\omega_0 = \sqrt{\frac{k}{m}}$

From previous slides:

- $m\ddot{x} + kx = 0$

EXAMPLE: PENDULUM



- Mass m, Length l, Angle from vertical θ
- Position = $(l\sin(\theta), -l\cos\theta)$
- Velocity = $(l\dot{\theta}\cos\theta, l\dot{\theta}\sin\theta)$
- Kinetic Energy $T = \frac{1}{2}mv^2$
- $v^2 = (l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2 = l^2\dot{\theta}^2$
- Potential Energy $V = -mgl\cos\theta$
- $\mathcal{L} = T - V = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$

EXAMPLE: PENDULUM

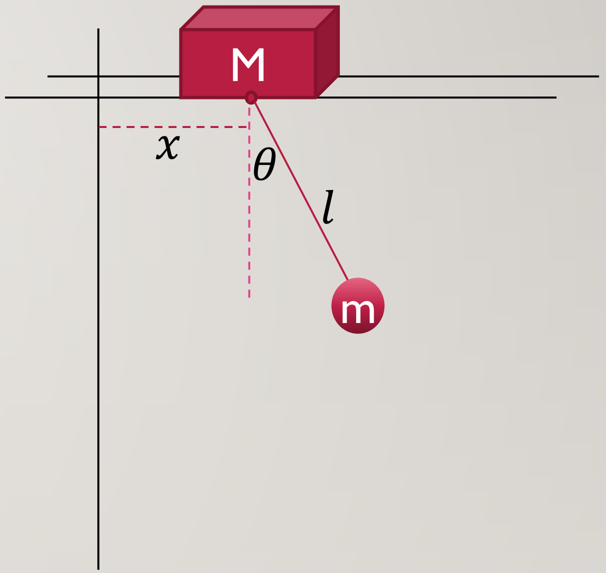
- $\frac{\delta \mathcal{L}}{\delta \dot{q}_j} = ml^2\dot{\theta}, \frac{\delta \mathcal{L}}{\delta q_j} = -mgl\sin\theta$
- $ml^2\ddot{\theta} + mgl\sin\theta = 0$
- $l\ddot{\theta} + g\sin\theta = 0$
- $\ddot{\theta} = -\frac{g}{l}\sin\theta$
- For small oscillations, we can use small angle approximation that $\sin\theta \approx \theta$
- $\ddot{\theta} + \frac{g}{l}\theta = 0$ and use the closed form solution we had earlier

From previous slide:

- $\mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$
- $\frac{d}{dt}\left(\frac{\delta \mathcal{L}}{\delta \dot{q}_j}\right) - \frac{\delta \mathcal{L}}{\delta q_j} = 0$

EXAMPLE: PENDULUM ON A CART

- Modelled using two variables, x and θ
- Position = $(x + l\sin\theta, -l\cos\theta)$
- Velocity = $(\dot{x} + l\dot{\theta}\cos\theta, l\dot{\theta}\sin\theta)$
- $T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[(\dot{x} + l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2]$
- $V = -mgl\cos\theta$
- $\mathcal{L} = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m[(\dot{x} + l\dot{\theta}\cos\theta)^2 + (l\dot{\theta}\sin\theta)^2] + mgl\cos\theta$



EXAMPLE: PENDULUM ON A CART

- Now we have $\frac{d}{dt}\left(\frac{\delta\mathcal{L}}{\delta\dot{x}}\right)-\frac{\delta\mathcal{L}}{\delta x}=0$
    - And $\frac{d}{dt}\left(\frac{\delta\mathcal{L}}{\delta\dot{\theta}}\right)-\frac{\delta\mathcal{L}}{\delta\theta}=0$
- $(M+m)\ddot{x}+ml\ddot{\theta}\cos\theta-ml\dot{\theta}^2\sin\theta=0$
- $\ddot{\theta}+\frac{\ddot{x}}{l}\cos\theta+\frac{g}{l}\sin\theta=0$

From previous slide:

- $\mathcal{L}=\frac{1}{2}M\dot{x}^2+\frac{1}{2}m\left[\left(\dot{x}+l\dot{\theta}\cos\theta\right)^2+(l\dot{\theta}\sin\theta)^2\right]+mgl\cos\theta$
- $\frac{d}{dt}\left(\frac{\delta\mathcal{L}}{\delta\dot{q}_j}\right)-\frac{\delta\mathcal{L}}{\delta q_j}=0$