Basic Analysis 2015 — Solutions of Tutorials

Section 4.1

Tutorial 4.1.1 Let $a < b$ be real numbers, $f, g : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b).
1. Show that g is injective on $[a, b]$ if $g'(x) \neq 0$ for all $x \in (a, b)$.

Proof. Let $x, y \in [a, b]$ with $x < y$. By the First Mean Value Theorem there is a $c \in (x, y)$ such that

$$g'(c) = \frac{g(y) - g(x)}{y - x}.$$

Since $y - x \neq 0$ and since $g'(c) \neq 0$ by assumption, it follows that

$$g(y) - g(x) = g'(c)(y - x) \neq 0.$$

This shows that g is injective. □

2. Prove the Second Mean Value Theorem: If $g'(x) \neq 0$ for all $x \in (a, b)$, then there is $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Hint. Consider the function

$$h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - (g(a)).$$

Proof. In view of part 1 of this tutorial, $g(b) - g(a) \neq 0$, and h is therefore well defined, continuous on $[a, b]$ and differentiable on (a, b). By the First Mean Value Theorem, there is $c \in (a, b)$ such that

$$h'(c) = \frac{h(b) - h(a)}{b - a}. \tag{1}$$

But

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) \tag{2}$$

and

$$h(a) = f(a), \quad h(b) = f(b) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(b) - (g(a)) = f(b) - (f(b) - f(a)) = f(a).$$

Hence $h(b) = h(a)$, and (1) gives $h'(c) = 0$. The statement of the Second Mean Value Theorem follows now from (2). □

3. Show that the statement of the Second Mean Value Theorem remains correct if one replaces the condition that $g'(x) \neq 0$ for all $x \in (a, b)$ with the weaker condition that $g(a) \neq g(b)$ and that there is no $x \in (a, b)$ with $g'(x) = f'(x) = 0$.

Solution. Since $g(a) \neq g(b)$, the function h defined in part 2 is again well defined, and again there is $c \in (a, b)$ such that (1) holds with $h'(c) = 0$. Since $g'(c) = 0$ would then imply $f'(c) = 0$ by (2), which is impossible by assumption, it follows that $g'(c) \neq 0$, and the statement of the Second Mean Value Theorem follows.

4. Prove the following one-sided version of l'Hôpital's Rule: If $f(a) = g(a) = 0$, $g'(x) \neq 0$ for x near a and $\lim_{x\to a^+} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

Proof. Let $L = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that $a+\delta \leq b$, $g'(x) \neq 0$ and $\left|\frac{f'(x)}{g'(x)} - L\right| < \varepsilon$ for $x \in (a, a+\delta)$. The Second Mean Value Theorem shows that for each $x \in (a, a+\delta)$ there is $c \in (a, a+x) \subset (a, a+\delta)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)}$$

and therefore

$$\left|\frac{f(x)}{g(x)} - L\right| = \left|\frac{f'(c)}{g'(c)} - L\right| < \varepsilon.$$

Hence

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$ □

***Tutorial 4.1.2** Let $-\infty \leq a < b \leq \infty$ and let $f, g : (a, b) \to \mathbb{R}$ be differentiable on (a, b) such that $g'(x) \neq 0$ for all $x \in (a, b)$. Prove the following one-sided versions of l'Hôpital's Rule.

1. If $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$ and $\lim_{x\to a^+} \frac{f'(x)}{g'(x)}$ exists as a proper or improper limit, then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

Proof. It is clear that the statement of part 1 of Tutorial 4.4.1 also holds in this case. Hence there is at most one $d \in (a, b)$ such that $g(d) = 0$, and replacing b with d, if necessary, we may assume that $g(x) \neq 0$ for all $x \in (a, b)$. Assume first that $\lim_{x\to a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$. Let $\varepsilon > 0$. Then there is $b_\varepsilon \in (a, b)$ such that $\left|\frac{f'(x)}{g'(x)} - L\right| < \frac{\varepsilon}{2}$ for all $x \in (a, b_\varepsilon)$. Let $x \in (a, b_\varepsilon)$. Since $\lim_{y\to a^+}(|f(y)| + |g(y)|) = 0$ we can find $y \in (a, b_\varepsilon)$ such that

$$|f(y)| + |g(y)| < \frac{1}{4} \min\left\{ \frac{\varepsilon g(x)^2}{|f(x)| + |g(x)|}, |g(x)| \right\}. \tag{1}$$

Then

$$\left|\frac{f(x)}{g(x)} - L\right| \leq \left|\frac{f(x)}{g(x)} - \frac{f(x) - f(y)}{g(x) - g(y)}\right| + \left|\frac{f(x) - f(y)}{g(x) - g(y)} - L\right|. \tag{2}$$

But by the Second Mean Value Theorem there is $c \in (y, x) \subset (a, b_\varepsilon)$ such that

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(c)}{g'(c)}$$

and therefore

$$\left|\frac{f(x) - f(y)}{g(x) - g(y)} - L\right| = \left|\frac{f'(c)}{g'(c)} - L\right| < \frac{\varepsilon}{2}. \tag{3}$$

Furthermore,

$$\begin{aligned}\frac{f(x)}{g(x)} - \frac{f(x) - f(y)}{g(x) - g(y)} &= \frac{f(x)(g(x) - g(y)) - g(x)(f(x) - f(y))}{g(x)(g(x) - g(y))} \\ &= \frac{g(x)f(y) - f(x)g(y)}{g(x)(g(x) - g(y))}\end{aligned}$$

and, in view of (1),

$$|g(x) - g(y)| \geq |g(x)| - |g(y)| > \frac{1}{2}|g(x)|$$

shows that

$$\left|\frac{f(x)}{g(x)} - \frac{f(x)-f(y)}{g(x)-g(y)}\right| \le 2\frac{(|f(x)|+|g(x)|)(|f(y)|+|g(y)|)}{g(x)^2} < \frac{\varepsilon}{2}. \tag{3}$$

Substituting (2) and (3) into (1) gives

$$\left|\frac{f(x)}{g(x)} - L\right| \le \varepsilon \text{ for all } x \in (a, b_\varepsilon),$$

which proves

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = L = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$

if $L \in \mathbb{R}$.

Now let $L = \infty$. Let $A \in \mathbb{R}$. Then there is $b_A \in (a, b)$ such that $\dfrac{f'(x)}{g'(x)} > A + 1$ for all $x \in (a, b_A)$. As above, with $\varepsilon = 2$, for every $x \in (a, b_A)$ we can find $y \in (a, x)$ such that (3) holds. Hence, for every $x \in (a, b_A)$ there is $c \in (a, b_A)$ such that

$$\frac{f(x)}{g(x)} \ge \frac{f(x)-f(y)}{g(x)-g(y)} - 1 = \frac{f'(c)}{g'(c)} - 1 < A + 1 - 1 = A,$$

which proves

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \infty = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$

if $L = \infty$. Alternatively, one could interchange f and g and use the previous case with $L = 0$; but then additional justification is needed.

In the case $L = \infty$, we may replace f with $-f$.

Note that the case $a \in \mathbb{R}$ and $L \in \mathbb{R}$ is exactly the special case dealt with in Tutorial 4.1.1 part 4. □

2. If $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = \infty$ and $\lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)}$ exists as a proper or improper limit, then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

Proof. As we have seen in the proof of part 1, we may assume that $g(x) \neq 0$ for all $x \in (a, b)$. Assume first that $\lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)} = L \in \mathbb{R}$. Let $\varepsilon > 0$. Then there is $b_\varepsilon \in (a, b)$ such that $\left|\dfrac{f'(x)}{g'(x)} - L\right| < \dfrac{\varepsilon}{4}$ for all $x \in (a, b_\varepsilon)$. We may also assume that $f(x) > 0$ and $g(x) > 0$ for all $x \in (a, b_\varepsilon)$. Choose some $y \in (a, b_\varepsilon)$. Then there is $d \in (a, y)$ such that for all $x \in (a, d)$,

$$\min\{f(x), g(x)\} > \frac{16(|L|+1+\varepsilon)}{\varepsilon} \max\{f(y), g(y)\}. \tag{4}$$

As in part 1 we have for all $x \in (a, d)$ that

$$\left|\frac{f(x)}{g(x)} - L\right| \le \left|\frac{f(x)}{g(x)} - \frac{f(x)-f(y)}{g(x)-g(y)}\right| + \left|\frac{f(x)-f(y)}{g(x)-g(y)} - L\right|. \tag{5}$$

But by the Second Mean Value Theorem there is $c \in (x, y) \subset (a, b_\varepsilon)$ such that

$$\frac{f(y)-f(x)}{g(y)-g(x)} = \frac{f'(c)}{g'(c)}$$

and therefore

$$\left|\frac{f(x)-f(y)}{g(x)-g(y)} - L\right| = \left|\frac{f'(c)}{g'(c)} - L\right| < \frac{\varepsilon}{4}. \tag{6}$$

Furthermore, as in part 1,

$$\begin{aligned}\frac{f(x)}{g(x)} - \frac{f(x)-f(y)}{g(x)-g(y)} &= \frac{f(x)(g(x)-g(y)) - g(x)(f(x)-f(y))}{g(x)(g(x)-g(y))}\\ &= \frac{g(x)f(y) - f(x)g(y)}{g(x)(g(x)-g(y))}\end{aligned}$$

and, in view of (4),

$$g(x) - g(y) > \frac{1}{2} g(x).$$

Therefore, again in view of (4),

$$\begin{aligned}
\left| \frac{f(x)}{g(x)} - \frac{f(x) - f(y)}{g(x) - g(y)} \right| &\le 2 \frac{(f(x) + g(x))(f(y) + g(y))}{g(x)^2} \\
&= 2 \left(\frac{f(x)}{g(x)} + 1 \right) \left(\frac{f(y)}{g(x)} + \frac{g(y)}{g(x)} \right) \\
&\le \frac{\varepsilon}{4(L + 1 + \varepsilon)} \left(\frac{f(x)}{g(x)} + 1 \right) \\
&\le \frac{\varepsilon}{4(L + 1 + \varepsilon)} \left(\left| \frac{f(x)}{g(x)} - L \right| + L + 1 \right) \\
&< \frac{1}{4} \left| \frac{f(x)}{g(x)} - L \right| + \frac{\varepsilon}{4}. \qquad (7)
\end{aligned}$$

Substituting (6) and (7) into (5) gives

$$\left| \frac{f(x)}{g(x)} - L \right| < \frac{\varepsilon}{2} + \frac{1}{4} \left| \frac{f(x)}{g(x)} - L \right| \text{ for all } x \in (a, b_\varepsilon),$$

which gives

$$\left| \frac{f(x)}{g(x)} - L \right| < \varepsilon \text{ for all } x \in (a, b_\varepsilon).$$

This proves

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = L = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

if $L \in \mathbb{R}$.

Now let $L = \infty$. From $\lim_{x \to a^+} \frac{f'(x)}{g'(x)} = \infty$ it easily follows that $\lim_{x \to a^+} \frac{g'(x)}{f'(x)} = 0 \in \mathbb{R}$ and therefore $\lim_{x \to a^+} \frac{g(x)}{f(x)} = 0$ by what we already have shown. But since $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = \infty$ we can find $d \in (a, b)$ such that $f(x) > 0$ and $g(x) > 0$ for all $x \in (a, d)$. Then it follows that $\lim_{x \to a^+} \frac{f(x)}{g(x)} = \infty$. □