Math 104, Midterm Examination 2 – Solution

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1. (10 points) Find the following limit and justify your answer.

$$\lim_{n\to\infty} \frac{1}{n^2}\left[(2n)!\right]^{1/n}$$

Solution: Let $s_n = \dfrac{(2n)!}{n^{2n}}$, then we need to find the limit $\lim_{n\to\infty}(s_n)^{1/n}$. Since

$$\begin{aligned}\left|\frac{s_{n+1}}{s_n}\right| &= \frac{(2n+2)!}{(n+1)^{2n+2}}\cdot\frac{n^{2n}}{(2n)!}\\ &= \frac{(2n+2)(2n+1)}{(n+1)^2}\cdot\left(\frac{n}{n+1}\right)^{2n}\\ &= \frac{4+2/n}{1+1/n}\cdot\frac{1}{\left[\left(1+\frac{1}{n}\right)^n\right]^2}\\ &\to 4/e^2, \quad n\to\infty,\end{aligned}$$

by limits laws. Then we apply the theorem in textbook to conclude that

$$\lim_{n\to\infty}|s_n|^{1/n} = \lim_{n\to\infty}\left|\frac{s_{n+1}}{s_n}\right| = \frac{4}{e^2}.$$

2. (10 points) Is the following series convergent? Justify your answer.

$$\sum_{n=1}^{\infty}(-1)^n\cos(\frac{\pi}{n})$$

Solution: The series diverges. Denote by $a_n = (-1)^n\cos\left(\dfrac{\pi}{n}\right)$. Consider the subsequences (a_{2m}) and (a_{2m+1}).

$$\begin{aligned}\lim_{m\to\infty} a_{2m} &= \lim_{m\to\infty}\cos\left(\frac{\pi}{2m}\right) = 1,\\ \lim_{m\to\infty} a_{2m+1} &= \lim_{m\to\infty}\left[-\cos\left(\frac{\pi}{2m+1}\right)\right] = -1.\end{aligned}$$

So the sequence (a_n) diverges. Hence $\sum a_n$ diverges.

3. (10 points) Is the following series convergent? Justify your answers.

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{-n^2}$$

Solution: Let $a_n = \left(1+\frac{1}{n}\right)^{-n^2}$. Then

$$\lim_{n\to\infty}|a_n|^{1/n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-n} = \frac{1}{e} < 1.$$

By the Ratio Test, the series $\sum a_n$ converges.

4. (15 points) Prove that the function

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

is discontinuous at any $x_0 \in \mathbb{R}$.

Proof. (i) Fix any $x_0 \notin \mathbb{Q}$. Then $f(x_0) = 1$. By denseness of rationals, for any $n \in \mathbb{N}$ there exists $r_n \in \mathbb{Q}$ such that

$$x_0 < r_n < x_0 + \frac{1}{n}.$$

Thus $\lim r_n = x_0$ by squeeze theorem. But

$$\lim_{n\to\infty} f(r_n) = \lim 0 = 0 \neq f(x_0) = 1.$$

We conclude that f is discontinuous at any $x_0 \in \mathbb{R}\setminus\mathbb{Q}$.

(ii) Fix any $x_0 \in \mathbb{Q}$. Then $f(x_0) = 0$. By denseness of rationals, for any $n \in \mathbb{N}$ there exists $r_n' \in \mathbb{Q}$ such that

$$x_0 - \sqrt{2} < r_n' < x_0 - \sqrt{2} + \frac{1}{n}.$$

Thus $\lim r_n' = x_0 - \sqrt{2}$ by squeeze theorem, or equivalently

$$\lim_{n\to\infty}(r_n' + \sqrt{2}) = x_0.$$

Since $r_n' + \sqrt{2}$ is irrational,

$$\lim_{n\to\infty} f(r_n' + \sqrt{2}) = \lim_{n\to\infty} 1 = 1 \neq f(x_0) = 0.$$

We conclude that f is discontinuous at any $x_0 \in \mathbb{Q}$.

Finally, we see that f is discontinuous at any $x \in \mathbb{R}$. $\square$

5. (a) (5 points) State the Intermediate Value Theorem (If you cite the intermediate value property, you need to state that property precisely).

   Solution: See textbook.

   (b) (10 points) Prove that the following equation has at least *TWO* real solutions:

$$\sin x = x^2 - 1.$$

   You may assume (without proof) that the functions x^2 and $\sin x$ are continuous on $\mathbb{R}$.

   Proof. Let $f(x) = \sin x - x^2 + 1$. Then $f(x)$ is continuous on $\mathbb{R}$ since $\sin x$ and x^2 are continuous.

$$\begin{aligned} f(0) &= 1 > 0; \\ f(\pi) &= -\pi^2 + 1 < 0; \\ f(-\pi) &= -\pi - \pi^2 + 1 < 0. \end{aligned}$$

   By the Intermediate Value Theorem, there exist $x_1 \in (0, \pi)$ and $x_2 \in (-\pi, 0)$ such that

$$f(x_1) = 0, \quad f(x_2) = 0.$$

   So x_1 and x_2 are two real solutions of the equation $\sin x = x^2 - 1$. □

6. (a) (5 points) State the definition of a uniformly continuous function f on $S \subset \mathrm{dom}(f)$.

   Solution: See textbook.

   (b) (15 points) Use the above definition to prove that $f(x) = \frac{1}{\sqrt{x}}$ is uniformly continuous on $[a, \infty)$ for any given real number $a > 0$.

   Proof. For any $\epsilon > 0$, let $\delta = 2a\sqrt{a}\epsilon > 0$. Then $x, y \in [a, \infty)$

and $|x - y| < \delta$ imply that

$$\begin{aligned}
|f(x) - f(y)| &= \left|\frac{1}{\sqrt{x}} - \frac{1}{\sqrt{y}}\right| = \frac{|\sqrt{y} - \sqrt{x}|}{\sqrt{xy}} \\
&= \frac{|(\sqrt{y} - \sqrt{x})(\sqrt{y} + \sqrt{x})|}{\sqrt{xy}(\sqrt{y} + \sqrt{x})} \\
&= \frac{|y - x|}{\sqrt{xy}(\sqrt{y} + \sqrt{x})} \\
&< \frac{\delta}{\sqrt{a^2}(\sqrt{a} + \sqrt{a})} = \frac{\delta}{2a\sqrt{a}} = \epsilon.
\end{aligned}$$

By definition, f is uniformly continuous on $[a, \infty)$. □

7. (15 points) Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. Prove that if $\sum b_n$ converges and

$$\limsup_{n\to\infty} \frac{a_n}{b_n} < 1,$$

then $\sum a_n$ also converges.

Proof. Since

$$\limsup_{n\to\infty} \frac{a_n}{b_n} = \lim_{N\to\infty} \sup\left\{\frac{a_n}{b_n} : n > N\right\} = L < 1,$$

there exists $N_0 > 0$ such that

$$\left|\sup\left\{\frac{a_n}{b_n} : n > N_0\right\} - L\right| < \epsilon_0 = \frac{1 - L}{2} > 0.$$

So

$$\sup\left\{\frac{a_n}{b_n} : n > N_0\right\} < \frac{1 - L}{2} + L = \frac{1 + L}{2} < 1.$$

Therefore, for any $n > N_0$,

$$\frac{a_n}{b_n} < \frac{1 + L}{2} < 1 \Rightarrow a_n < b_n.$$

Because $\sum b_n$ converges, by the comparison test, $\sum a_n$ also converges. □