Chapter 5: IDENTITY, INVERSE & WELL DEFINED MAPPINGS

Mphako-Banda

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG
SCHOOL OF MATHEMATICS

LEARNING OUTCOMES FOR THE LECTURE

By the end of this lecture, students will be able to:

- ♣ define a function or mapping
- ♣ determine whether a given rule defines a function or mapping
- ♣ define the domain, codomain and range of a function
- ♣ determine the domain, codomain and range of a given function
- ♣ determine whether a given rule is a well defined function or mapping

DOMAIN, CODOMAIN RANGE AND GRAPHS OF MAPPINGS

Definition (5.1.1 (1))

Let A and B be non empty sets. $\alpha : A \to B$ *is a function or mapping if it is a rule that assigns each element of A with exactly one element of B. i.e.*
$\forall a \in A \quad \exists b \in B \quad | \quad \alpha(a) = b.$

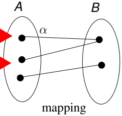
Examples:

$\alpha : \mathbb{R} \to \mathbb{R} \quad ; \quad \alpha(x) = x^2$ is a mapping.

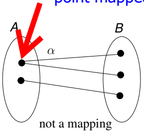
$\alpha : \mathbb{R} \to \mathbb{R} \quad ; \quad \alpha(x) = \pm\sqrt{x}$ is not a mapping.

eg. x=4; α(4)=2 and α(4)=-2. x=4 has been mapped to two values in B.

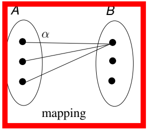
each of these points
mapped to
one value in B

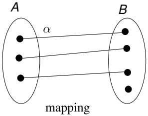


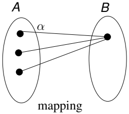
point mapped to two different values in B

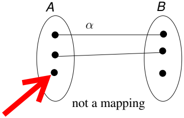


each point in A has to be
mapped to one point B.
it does not matter if a
point in B does not have
a corresponding point(s)
in A









point not mapped to any point in B

Definition (5.1.1 (2))

A is called the Domain of α denoted by $D(\alpha)$.

Definition (5.1.1 (3))

B is called the codomain of α denoted by $CoD(\alpha)$.

Definition (5.1.1 (4))

$\alpha(A) = \{\alpha(a) \mid a \in A\}$ is called the range or image of α denoted by $Im(\alpha)$.

$\alpha(a)$ is the image of a under α.

ordered pairs, a value in an A and its image in B

Definition (5.1.1 (5))

Graph of α, $G(\alpha) = \{(a, \alpha(a)) \mid a \in A\}$.

Defining a well defined mapping

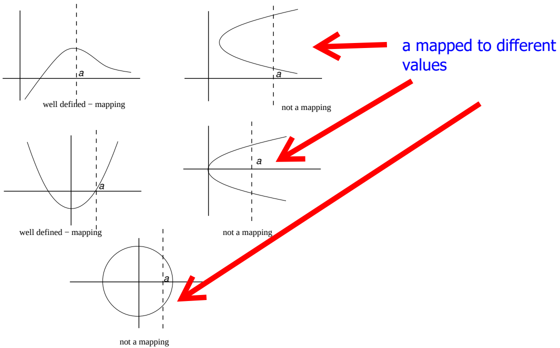
(i) specify domain and codomain

(ii) specify the rule that assigns exactly one element $\alpha(a)$ for each $a \in A$

That is, we must specify the domain, the co-domain and the well defined action. In algebra we usually use Greek lower case letters, to represent mappings (functions).

5.1.2.Test for a mapping or a function

Test for well defined i.e Let $\alpha : A \to B$. Show that $a = b \quad \Rightarrow \quad \alpha(a) = \alpha(b)$. This is equivalent to the vertical line test in $\mathbb{R}^2$.



Example (5.1.3 (a))

$\alpha : \mathbb{Z} \to \mathbb{Z} \quad ; \quad \alpha(x) = 2x$; *x integer.*
$D(\alpha) = \mathbb{Z}, \qquad CoD(\alpha) = \mathbb{Z}$
Range $\alpha = Im(\alpha) = \alpha(\mathbb{Z}) = 2\mathbb{Z}$ *even integers.*

Testing for well defined:
Show that $x_1 = x_2 \quad \Rightarrow \quad \alpha(x_1) = \alpha(x_2)$.
Let $x_1, x_2 \in \mathbb{Z} \quad | \quad x_1 = x_2$
$\Rightarrow \quad 2x_1 = 2x_2$
$\Rightarrow \quad \alpha(x_1) = \alpha(x_2)$
Thus α is well defined.

Example (5.1.3 (b))

$\alpha : \mathbb{R} \to \mathbb{R} \quad ; \quad \alpha(x) = x^2.$
$D(\alpha) = \mathbb{R}, \qquad CoD(\alpha) = \mathbb{R}$
$Range\ \alpha = Im(\alpha) = \alpha(\mathbb{R}) = \mathbb{R}^+ \cup \{0\}.$
Is α well defined?

not always the case that range=codomain

$x_1 = x_2 \quad \Rightarrow \quad x_1^2 = x_2^2 \quad \Rightarrow \quad \alpha(x_1) = \alpha(x_2)$. Thus α is well defined.

Definition (5.1.1 (6))

*If $A = S \times S$, $B = S$ then a mapping $\alpha : S \times S \to S$ or $\alpha : A \to B$ **is a binary operation** on the set S.*

rule combining two values in A
to create a new value in B

Example

$\alpha : T \times T \rightarrow T \quad ; \quad \boxed{\alpha(t_1, t_2) = t_1.}$

rule how to combine
two elements in the domain

$D(\alpha) = T \times T \qquad CoD(\alpha) = T$

$Range\, \alpha = Im(\alpha) = \alpha(T \times T) = T.$

TEST for well defined

Let $(t_1, t_2), (s_1, s_2) \in T \times T$.

$(t_1, t_2) = (s_1, s_2) \quad \Rightarrow \quad t_1 = s_1, \quad t_2 = s_2.$

Thus $\alpha(t_1, t_2) = \alpha(s_1, s_2)$ since $t_1 = s_1$. Thus α is well defined.

α is called a projection of $T \times T$ onto T.

α is a binary operation.

Example

Let $S = T \times T$.
$\alpha_1 : S \to T \quad ; \quad \alpha_1(t_1, t_2) = t_1 + t_2$.
$\alpha_2 : S \to T \quad ; \quad \alpha_2(t_1, t_2) = t_1 t_2$.
α_1 *and* α_2 *are binary operations.* $D(\alpha_1) = D(\alpha_2) = S$ *and* $CoD(\alpha_1) = CoD(\alpha_2) = T$
Range $\alpha_1 = Im(\alpha_1) = T =$ *Range* $\alpha_2 = Im(\alpha_2)$.

Testing for well defined:

$(t_1, t_2) = (s_1, s_2) \quad \Rightarrow \quad t_1 = s_1, \quad t_2 = s_2.$
$\Rightarrow \quad t_1 + t_2 = s_1 + s_2 \quad \Rightarrow \quad \alpha_1(t_1, t_2) = \alpha_1(s_1, s_2).$
Similarly, $(t_1, t_2) = (s_1, s_2) \quad \Rightarrow \quad t_1 = s_1, \quad t_2 = s_2.$
$\Rightarrow \quad t_1 t_2 = s_1 s_2 \quad \Rightarrow \quad \alpha_2(t_1, t_2) = \alpha_2(s_1, s_2).$
Thus α_1 and α_2 are well defined.

Example

$$\alpha : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \quad ; \quad \alpha(t_1, t_2) = \frac{t_1}{t_2}.$$

α *is not well defined, since* $\dfrac{t_1}{t_2} \notin \mathbb{Z}$. also, points like (t,0) do not have an image in $\mathbb{Z}$.

Example

$$\alpha : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q} \quad ; \quad \alpha(t_1, t_2) = \frac{t_1}{t_2}$$

α *is not well defined, since* $\dfrac{t_1}{t_2}$ *not defined if* $t_2 = 0$.

Example

on $\mathbb{Z}_n$, *define* $\oplus : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n \quad ; \quad \oplus(\overline{a}, \overline{b}) = \overline{a+b}$.
$\odot : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n \quad ; \quad \odot(\overline{a}, \overline{b}) = \overline{ab}$.

Is $\odot$ well defined? If $\overline{a} = \overline{b}$ and $\overline{c} = \overline{d}$ then

(***We need to show that*** $\overline{ac} = \overline{bd}$.)

$a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$

$$\Rightarrow \quad a - b = t_1 n \quad \text{and} \quad c - d = t_2 n \quad \text{where} \quad t_1, t_2 \in \mathbb{Z}$$

$$\Rightarrow \quad a = t_1 n + b \quad \text{and} \quad c = t_2 n + d$$

$$\Rightarrow \quad ac = (t_1 n + b)(t_2 n + d) = bd + (bt_2 + dt_1 + t_1 t_2 n)n$$

$$\Rightarrow \quad ac - bd = (bt_2 + dt_1 + t_1 t_2 n)n$$

$$\Rightarrow \quad ac \equiv bd \pmod{n}$$

$$\Rightarrow \quad \overline{ac} = \overline{bd}.$$

Is $\oplus$ well defined? Exercise for students

Please do this exercise.