## Chapter 3: Series

- **3.1 Definitions and Examples**

Given a sequence $(a_n)_{n=1}^{\infty}$, the symbol

$$\boxed{\sum_{n=1}^{\infty} a_n := a_1 + a_2 + a_3 + \cdots + a_n + \cdots}$$

is called a **series** (of real numbers).

**Definition 3.1** Let $\sum_{n=1}^{\infty} a_n$ be a series.

1. The sequence $(s_n)_{n=1}^{\infty}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ \boxed{s_n} &\boxed{= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k} \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number $s_n$ being the $n$**-th partial sum**.

2. $\sum_{n=1}^{\infty} a_n$ is said to **converge** if $(s_n)_{n=1}^{\infty}$ converges.

In this case, the number $\boxed{s = \lim_{n\to\infty} s_n}$ is called the **sum of the series** and we write

$$\boxed{\sum_{n=1}^{\infty} a_n = s}$$

A series which does not converge is said to **diverge**.

- **Theorem 3.1**

Consider the **geometric** series

$$\boxed{\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots ,}$$

where $a \neq 0,\ r \in \mathbb{R}$ . It converges if $|r| < 1$ with

$$\boxed{\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}}$$

and diverges if $|r| \geq 1$.

**Proof.** See Example 3.1 of the study guide. $\square$

- **Example 1**

Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent ?

**Solution.** Rewrite the $n$-th term of the series as $ar^n$:

$$a_n = 2^{2n}3^{1-n} = (2^2)^n \times 3^1 \times 3^{-n} = \frac{4^n \times 3}{3^n} = 3\left(\frac{4}{3}\right)^n$$

Thus, $r = \frac{4}{3}$. So, $|r| = \left|\frac{4}{3}\right| = \frac{4}{3} > 1$. $\Rightarrow |r| \geq 1$

YES!

Hence, the series diverges.

- **Example 2**

$r = \frac{\frac{5}{16}}{-\frac{5}{4}} = \frac{-\frac{5}{64}}{\frac{5}{16}} = -\frac{1}{4}$

The series

a

$$\sum_{n=0}^{\infty} \frac{(-1)^n 5}{4^n} = \boxed{5} - \frac{5}{4} + \frac{5}{16} - \frac{5}{64} + \cdots$$

is a geometric series with $a = 5$ and $r = -1/4$. It converges to

$$s = \frac{a}{1-r} = \frac{5}{1+\frac{1}{4}} = 4$$

- **Theorem 3.2**

  $\Rightarrow$

  If the series $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$ .

  $\Leftarrow$ No!

  **Proof.** See the study guide. $\square$

- The contrapositive of Theorem 3.2 is very useful:

- **Theorem 3.3 (Test for Divergence)**

  $\Leftarrow$

  $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$ or fails to exist.

  $\Rightarrow$ No!

- **Example 3**

  $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges since $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0.$

  (see Chapter 2, Lecture 1).

- **Example 4**

  $\sum_{n=1}^{\infty}(-1)^n$ diverges since $\lim_{n\to\infty}(-1)^n$ does not exist

  (see Chapter 2, Lecture 1).

- From Theorem 2.2, we immediately infer

- **Theorem 3.4 (Sum Laws)**

  If $\sum a_n = A \in \mathbb{R}$ and $\sum b_n = B \in \mathbb{R}$, then

  1. $\sum(ca_n) = c\sum a_n = cA$, (for any number $c$)

  2. $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$

  3. $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$

- **Example 5**

  Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{5}{n(n+1)} + \frac{1}{5^n}\right)$.

**Solution.** Recall from Calculus I that

$$\sum_{n=1}^{m} \frac{1}{n(n+1)} = \sum_{n=1}^{m} \left( \frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{m+1}.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{n(n+1)} = \lim_{m\to\infty} \left(1 - \frac{1}{m+1}\right) = 1. \quad (1)$$

And

$a \rightarrow$ $\quad r = \frac{1/25}{1/5} = \frac{1/125}{1/25} = \frac{1}{5}$

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \frac{1}{625} + \cdots$$

is a geometric series with $a = 1/5$ and $r = 1/5$. Hence,

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{a}{1-r} = \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{1}{4}. \quad (2)$$

Therefore,

$(1) \qquad (2)$

$$\sum_{n=1}^{\infty} \left( \frac{5}{n(n+1)} + \frac{1}{5^n} \right) = 5\sum_{n=1}^{\infty} \frac{1}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{5^n} = 5(1) + \frac{1}{4} = \frac{21}{4}.$$

- **Example 6**

  Find the sum of the series $\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}}$.

  **Solution.**

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{3^{n-1} - 1}{6^{n-1}} &= \sum_{n=1}^{\infty} \left( \frac{3^{n-1}}{6^{n-1}} - \frac{1}{6^{n-1}} \right) \\
&= \sum_{n=1}^{\infty} \left( \frac{3}{6} \right)^{n-1} - \sum_{n=1}^{\infty} \left( \frac{1}{6} \right)^{n-1} \\
&= \sum_{n=1}^{\infty} \left( \frac{1}{2} \right)^{n-1} - \sum_{n=1}^{\infty} \left( \frac{1}{6} \right)^{n-1} \\
&= \frac{1}{1 - \frac{1}{2}} - \frac{1}{1 - \frac{1}{6}} \\
&= 2 - \frac{6}{5} = \frac{4}{5}
\end{aligned}$$

$a=1 \quad r = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-1} = 1 + \frac{1}{2} + \frac{1}{4} +$

$\sum_{n=1}^{\infty} \left(\frac{1}{6}\right)^{n-1} = 1 + \frac{1}{6} + \frac{1}{36} +$

$a=1 \quad r = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{6}$

$\leftarrow \boxed{S = \frac{a}{1-r}}$

- **Theorem 3.5.**

  $\sum_{n=1}^{\infty} a_n$ converges if and only if for each $\epsilon > 0$ there is $K \in \mathbb{N}$ such that for all $m \geq k \geq K$

  $$\left| \sum_{n=k}^{m} a_n \right| < \epsilon .$$

  **Proof.** See the study guide □

- **Tutorial 3.1.1.**

  1. Test each of the following series for convergence or divergence:

  (a) $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)$, (b) $\sum_{n=1}^{\infty} n \sin\left(\frac{1}{n}\right)$, (c) $\sum_{n=1}^{\infty} \frac{n^2 - 1}{n - 50n^2}$.

  2. Which of the following is valid? Justify your conclusions.

  (a) If $a_n \to 0$ as $n \to \infty$, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $a_n \not\to 0$ as $n \to \infty$, then $\sum_{n=1}^{\infty} a_n$ is divergent.

(c) If $\sum_{n=1}^{\infty} a_n$ is divergent, then $a_n \not\to 0$ as $n \to \infty$.