UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG
Computer Science and Applied Mathematics

APPM2007
Lagrangian Mechanics

Tutorial 1

Question 1 (10 Points)

Methane is a chemical compound with the chemical formula CH_4 (one atom of carbon and four atoms of hydrogen). Methane is a tetrahedral molecule with four equivalent $C-H$ bonds. Show that the angle subtended at the central Carbon atom by any two Hydrogen atoms in a methane molecule is 109.5°. (Hint: use the symmetry of the tetrahedron structure of the CH_4 molecule to aid your computations.)

Solution 1 (10 Points)

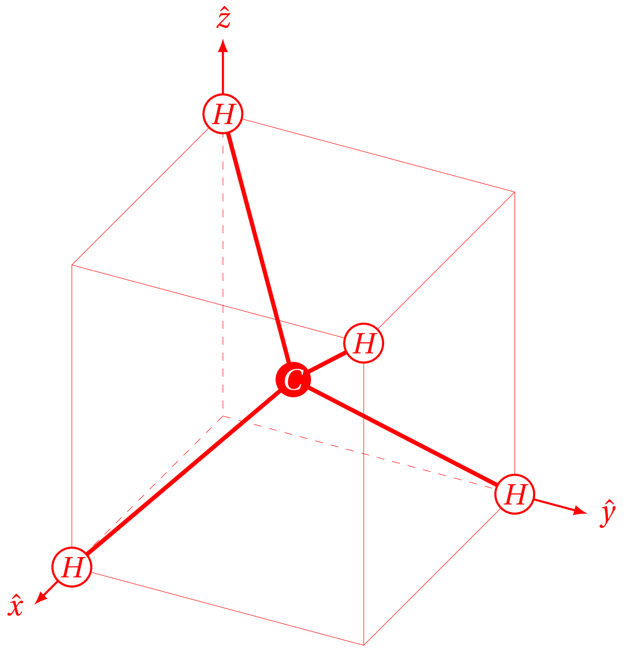
Consider the space $\mathbb{R}^3$ and place the atoms in the methane molecule in the following positions,

$$\begin{aligned} H &\to p = (2,0,0) \\ H &\to q = (0,2,0) \\ H &\to r = (0,0,2) \\ H &\to s = (2,2,2) \\ C &\to t = (1,1,1). \end{aligned}$$

This positioning reproduces the tetrahedron shape of the methane molecule in $\mathbb{R}^3$. The displacement vectors specifying the relative positions of each H atom relative to the central C atom are then

$$\begin{aligned} \vec{tp} &= (1,-1,-1) \\ \vec{tq} &= (-1,1,-1) \\ \vec{tr} &= (-1,-1,1) \\ \vec{ts} &= (1,1,1). \end{aligned}$$

By symmetry under relabelling of the H atom placements relative to the position of the central C atom, we may consider any two vectors for consideration in the calculation to



follow, without loss of generality. Consider the vectors $\vec{tp}$ and $\vec{tp}$. Clearly,

$$\|\vec{tp}\| = \|\vec{tq}\| = (-1, 1, -1) = \sqrt{3}$$

and

$$\vec{tp} \cdot \vec{tq} = -1.$$

Recall that

$$\vec{tp} \cdot \vec{tq} = \|\vec{tp}\| \, \|\vec{tq}\| \cos(\theta)$$

such that

$$\cos(\theta) = \frac{\vec{tp} \cdot \vec{tq}}{\vec{tp} \cdot \vec{tq}} = -\frac{1}{3}.$$

Therefore,

$$\theta = \arccos\left(-\frac{1}{3}\right) = 109.47° \approx 109.5°.$$

$\square$

Question 2 (5 Points)

Consider the space $\mathbb{R}^2$. Compare the area element of the Rectilinear and Polar Co-ordinate systems on $\mathbb{R}^2$. Do the Rectilinear and Polar Co-ordinate area elements co-incide?

Solution 2 (5 Points)

Consider the infinitesimal area element in each case for simplicity. Note that the magnitude of the area element of the rectilinear co-ordinate system is

$$\|\mathrm{d}\vec{a}\| = \mathrm{d}\|\vec{a}\| = \mathrm{d}a = \mathrm{d}x\,\mathrm{d}y\,,$$

where $\mathrm{d}x$ and $\mathrm{d}y$ are the extents of the infinitesimal area element in the $\hat{x}$- and $\hat{y}$-directions, respectively. Note that this area element has fixed magnitude everywhere in $\mathbb{R}^2$, as such it is position independent. In polar co-ordinates, this becomes

$$\|\mathrm{d}\vec{a}\| = \mathrm{d}\,\|\vec{a}\| = \mathrm{d}a = \mathrm{d}r\,r\mathrm{d}\theta\,,$$

which is more commonly written as $\mathrm{d}a = r\mathrm{d}r\,\mathrm{d}\theta$, where $\mathrm{d}r$ is the extent of the infinitesimal area element in the $\hat{r}$-direction and $r\mathrm{d}\theta$ is the extent of the same area element in the $\hat{\theta}$-direction. Notice that this places a position dependence r on the magnitude of the area element in the polar co-ordinate system. Therefore, these area elements do not co-incide. $\square$

Question 3 (10 Points)

Consider the plane $\mathbb{R}^2$ and the unit Sphere S^2 with north pole marked p. Construct the following co-ordinate mappings

1. $f : U \to S^2 \setminus \{p\}$ with $U \subset \mathbb{R}^2$ is finite. (Hint: use Riemann Normal Co-ordinates.)
2. $f : \mathbb{R}^2 \to S^2 \setminus \{p\}$. (Hint: use a co-ordinate projection where points in $c = \{z \in \mathbb{R}^2 : \|z\| \leq 1\}$ are mapped to the southern hemisphere of S^2 and points outside c are mapped to the northern hemisphere of S^2.)

Compare the qualitative differences and similarities of these two co-ordinatisation maps.

Solution 3 (10 Points)

To perform each of the following co-ordinatisations, it will be usefull to first embed S^2 in $\mathbb{R}^3$ to form a geometric description of the problem. In each construction it will be instructive to denote the radius of S^2 by R.

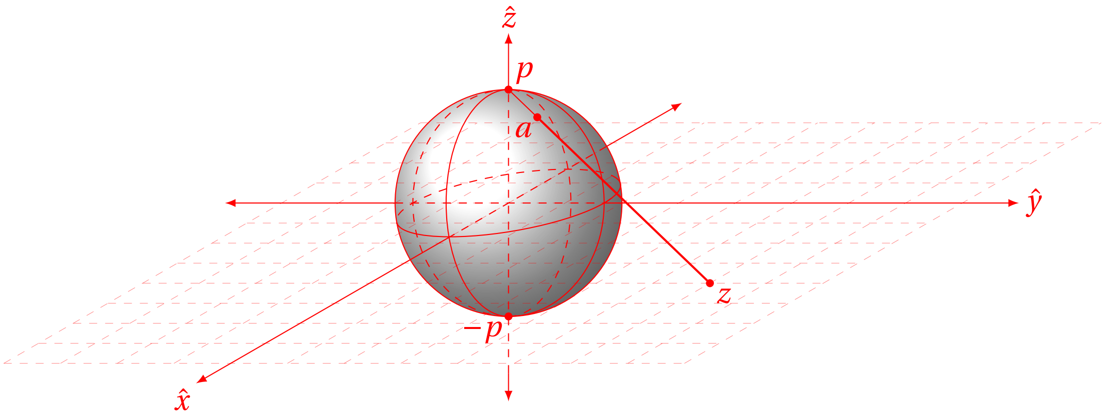
1. Introduce a system of Riemann Normal Co-ordinates (RNC) to S^2 by starting at point $-p$ and then moving radially outward from $-p$ towards the anti-podal point p. The radial distance l along the surface S^2 between $-p$ and p is equal to half the diameter of S^2,

$$l = \pi R.$$

   A point z on any radial line connecting p and $-p$ has radial position on the interval $[0, \pi R]$. Let $t \in [0, 1]$ denote the fraction of the distance between $-p$ and p occupied by z. Then the radial component of z is $Rt\pi$. Since this measures a fractional angular displacement, it is convenient to define $\theta = tR\pi$. The angular position of z corresponds to the angular displacement of z at fixed distance $0 < c < \pi R$ from $-p$, from the initial radius line connecting $-p$ to p. This measures an angle ϕ that is uniquely defined up to a multiple of 2π. Combining these radial and angular positions, with the assignment of $R = 1$, we denote z by the co-ordinate pair (θ, ϕ) with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$. This is a finite co-ordinitisation of S^2. Additionally, the points $-p$ and p have ambiguous descriptions with respect to ϕ. Each such point is called an *exceptional point*.

2. Intersect S^2 with $\mathbb{R}^2$ such that the equator lies on the unit circle in$\mathbb{R}^2$. In this configuration, $p = (0, 0, R)$ and the line connecting p and $-p$ intersects the plane at $(0, 0, 0)$ and $\mathbb{R}^2$ corresponds to all points in $\mathbb{R}^3$ with z-co-ordinate equal to zero. Every line throught p meeting S^2 meets exactly one point in $\mathbb{R}^2$, with the exception of lines tangent to S^2 at p. Choose a point $z = (r, s) \in \mathbb{R}^2$. This corresponds to a point

$z = (r, s, 0) \in \mathbb{R}^3$. The line passing through p and z intersects S^2 at some point a. This associates to each point $z \in \mathbb{R}^2$ a unique point $a \in S^2$. We call this association a *stereographic projection* (SP). Consider the figure below.



The line passing through z and a colinear with the vector

$$\vec{v} = (r, s, 0) - (0, 0, R) = (r, s, -R)$$

connecting p and z. Therefore the vector $\vec{w}$ connecting p and a must originate at p and extend some distance along $\vec{v}$, so

$$\vec{w} = \vec{p} + \lambda\vec{w} = (\lambda r, \lambda s, R - \lambda).$$

The point a corresponds to the intersection of the line $(\lambda r, \lambda s, R - \lambda)$ with S^2

$$x^2 + y^2 + z^2 = R^2.$$

This gives an equation for λ

$$\lambda^2 \left(r^2 + s^2 + R^2\right)^2 - 2\lambda = 0.$$

So,

$$\lambda = 0 \quad \text{or} \quad \lambda = \frac{2}{r^2 + s^2 + R^2}.$$

Note that $\lambda = 0$ corresponds to the points $p \in S^2$ and the origin in $\mathbb{R}^2$. For $\lambda \neq 0$, we have

$$a(r, s) = \left(\frac{2r}{r^2 + s^2 + R^2}, \frac{2s}{r^2 + s^2 + R^2}, \frac{r^2 + s^2 - R^2}{r^2 + s^2 + R^2}\right).$$

Recall that $R = 1$ on S^2, so

$$a(r, s) = \left(\frac{2r}{r^2 + s^2 + 1}, \frac{2s}{r^2 + s^2 + 1}, \frac{r^2 + s^2 - 1}{r^2 + s^2 + 1}\right).$$

The number of parameters needed to identify a point $(r, s) \in \mathbb{R}^2$ now corresponds with the number of parameters needed to specify $a(r, s)$. Note that this co-ordinatisation also works at p, but corresponds to $r, s \to \infty$, which leads to an ambiguity in the description. Therefore, p is an exceptional point in this co-ordinatisation.

Each co-ordinate mapping contains at least one exceptional point, where the SP contains one fewer exceptional point than that of RNC at a cost of adding one extra dimension to the description. An interesting consequence of each mapping is that the RNC maintains a fixed scale over the whole sphere, while SP distorts the co-ordinate scale because the unit disc is mapped to the southern hemisphere of S^2 and the remainder of $\mathbb{R}^2$ is mapped to the northern hemisphere. $\square$

Question 4 (10 Points)

Consider a marked point in $p \in \mathbb{R}^3$. Let α, β and γ denote the angles subtended at the origin by the vector $\vec{p}$ and each of the co-ordinate axes $\hat{x}$, $\hat{y}$ and $\hat{z}$. Show that

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = 1.$$

Solution 4 (10 Points)

Define the displacement vector starting at the origine of $\mathbb{R}^3$ and terminating at the point $p = (a, b, c)$ as $\vec{r}$, and define $\vec{x}$, $\vec{y}$ and $\vec{z}$ as arbitrary length vectors directed along the positive co-ordinate directions. By these definitions, it is clear that,

$$\begin{aligned}
\vec{r} &= (a, b, c) \\
\vec{x} &= (s, 0, 0) \\
\vec{y} &= (0, t, 0) \\
\vec{z} &= (0, 0, u).
\end{aligned}$$

Note that

$$\vec{r} \cdot \vec{r} = \|\vec{r}\|^2 = a^2 + b^2 + c^2.$$

For notational simplicity, denote $\|\vec{r}\| = r$. Similarly, it is clear that $\|\vec{x}\| = s$, $\|\vec{y}\| = t$ and $\|\vec{z}\| = u$. Then, from the dot product of $\vec{r}$ with $\vec{x}$, $\vec{y}$ and $\vec{z}$ we recover the following relation among the projections of $\vec{r}$ along each vector $\vec{x}$, $\vec{y}$ and $\vec{z}$ in terms of the internal co-ordinate representations of each vector,

$$\begin{aligned}
\cos(\alpha) &= \frac{\vec{r} \cdot \vec{x}}{\|r\| \, \|x\|} = \frac{as}{rs} = \frac{a}{r} \\
\cos(\beta) &= \frac{\vec{r} \cdot \vec{y}}{\|r\| \, \|y\|} = \frac{bt}{rt} = \frac{b}{r} \\
\cos(\gamma) &= \frac{\vec{r} \cdot \vec{z}}{\|r\| \, \|z\|} = \frac{cu}{ru} = \frac{c}{r}.
\end{aligned}$$

Notice that the angular cosines are independent of the scaling of the direction vectors $\vec{x}$, $\vec{y}$ or $\vec{z}$. Clearly,

$$\cos^2(\alpha) + \cos^2(\beta) + \cos^2(\gamma) = \frac{a^2 + b^2 + c^2}{r^2} = \frac{r^2}{r^2} = 1.$$

Notice that this formulation made no prior assumptions of the specific direction or magnitude of $\vec{r}$ or the magnitudes of the directional vectors $\vec{x}$, $\vec{y}$ or $\vec{z}$ and is therefore a more general statement that that implied by the original question. $\square$