Chapter 2: Sequence (Cont...)

Theorem 2.4 (Sandwich Theorem).
If $a_n \le b_n \le c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then $\lim_{n\to\infty} b_n = L$.

Proof.
Let $\epsilon > 0$ and choose k_1 and k_2 such that
$|a_n - L| < \epsilon$ if $n \ge k_1$ and $|c_n - L| < \epsilon$ if $n \ge k_2$.
In particular, for $n \ge K = \max\{k_1, k_2\}$,
$L - \epsilon < a_n < L + \epsilon$ and $L - \epsilon < c_n < L + \epsilon$
gives $L-\epsilon < a_n \le b_n \le c_n < L+\epsilon$. Thus, $L-\epsilon < b_n < L+\epsilon$.
Hence, $|b_n - L| < \epsilon$ if $n \ge K$. $\square$

Theorem 2.5. Let $x \in \mathbb{R}$. Then x^n converges if and only if $-1 < x \leq 1$.

Proof.

(1) For $x = 1$, $\lim_{n \to \infty} x^n = \lim_{n \to \infty} 1^n = 1$, by Theorem 2.3(a).

(2)Let $0 < x < 1$ and let $\epsilon > 0$. Then $0 < 1 < \dfrac{1}{x}$ and therefore $y := \dfrac{1}{x} - 1 > 0$.

Then $\dfrac{1}{x^n} = \left(\dfrac{1}{x}\right)^n = (1 + y)^n \geq 1 + ny$, by Bernoulli's ineq .

Put $K = \dfrac{1 - \epsilon}{y\epsilon}$.

Then, for $n > K$, $0 < x^n \leq \dfrac{1}{1 + ny} < \dfrac{1}{1 + \frac{1-\epsilon}{\epsilon}} = \epsilon$.

That is, $0 < x^n < \epsilon$. Hence $x^n \to 0$ as $n \to \infty$.

(3) Now let $x > 1$.

Then $0 < \dfrac{1}{x} < 1$. Hence $\left(\dfrac{1}{x}\right)^n \to 0$ as $n \to \infty$.

Assume that x^n converges to some L.

Then $1 = \lim_{n \to \infty} 1^n = \lim_{n \to \infty} \left(\dfrac{1}{x}\right)^n \cdot \lim_{n \to \infty} x^n = 0 \cdot L = 0$.

That is, 1 = 0 which is impossible.

(4) The cases $-1 < x < 0$ are left as an exercise. $\square$

Theorem 2.6. If $r > 0$, then $\lim_{n\to\infty} \frac{1}{n^r} = 0$.

Proof. By induction on positive integer r.

(1) Let $r = 1$ and choose $\epsilon > 0$. Let $K = \frac{2}{\epsilon}$.

Then, for $n \geq K$, $0 < \frac{1}{n} \leq \frac{1}{K} = \frac{\epsilon}{2} < \epsilon$,

Thus, $0 < \frac{1}{n} < \epsilon$. Hence, $\frac{1}{n} \to 0$ as $n \to \infty$.

(2) Now assume the statement holds for an integer $r > 0$.

That is, $\lim_{n\to\infty} \frac{1}{n^r} = 0$.

(3) Then $\lim_{n\to\infty} \frac{1}{n^{r+1}} = \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n^r} = 0 \cdot 0 = 0$.

So the result follows for all positive integers r by induction.

□

Tutorial 2.1.1. Cont ...

3. **Contractive maps.** Suppose that for some $c \in \mathbb{R}$ with $0 < c < 1$, we have $|a_{n+1} - L| \leq c|a_n - L|$ for all $n \in \mathbb{N}$.
(a) Use induction on n to prove that $|a_n - L| \leq c^n |a_0 - L|$.
(b) Use the Sandwich Theorem and the fact that $\lim_{n\to\infty} c^n = 0$ to prove that $\lim_{n\to\infty} a_n = L$.

4. **Recursive algorithm for finding $\sqrt{a}$.** Let $a > 1$ and define

$$a_0 = a \text{ and } a_n = \frac{1}{2}\left(a_{n-1} + \frac{a}{a_{n-1}}\right) \text{ for } n \geq 1 .$$

(a) Prove that $0 < a_n - \sqrt{a} = \frac{1}{2a_{n-1}}\left(a_{n-1} - \sqrt{a}\right)^2$ for $n \geq 1$.
(b) Use (a) to prove that $0 \leq a_n - \sqrt{a} \leq \frac{1}{2}\left(a_{n-1} - \sqrt{a}\right)$ for $n \geq 1$.
(c) Deduce that $\lim_{n\to\infty} a_n = \sqrt{a}$.
(d) Apply four steps of the recursive algorithm with $a = 3$ to approximate $\sqrt{3}$.