Question 1

1. First, to help with explaining, let's let the fixed coin be described as the "inner coin" and the coin rolling around the other to be the "outer coin".

From the start to finish of the motion, the center of the outer coin moves in a circular path as it moves around the inner coin. Thus this path constructs another circle around the outer edge of the inner coin when a full revolution is completed. This circle's radius will then be the sum of the coins' radii, i.e. it will have a radius of $r + r = 2r$. Hence the circumference of this circle is double either coin's circumference.

Thus we see the center of the moving coin travels twice the coin's circumference resulting in the outer coin making two complete revolutions around its own center while completing one revolution around the inner coin.

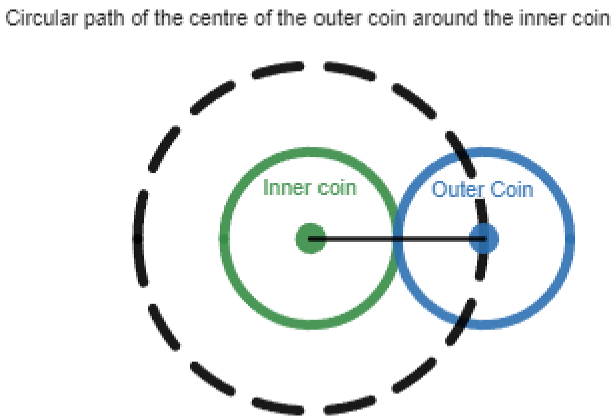

Figure 1: Construction of the circular path of the center of the outer coin around the inner coin

This problem sheds light on the idea of simultaneous motion. Both transnational and rotational motion.

2. In the general case of two coins with different radii, say R and r, in which the coin with radius r rolls around the coin of radius R, we see that the coin of radius r will have a radius of $R + r$ and hence has a circumference of $2\pi(R + r)$. Dividing the distance covered of the rolling coin ($2\pi(R + r)$) by the circumference of the rolling coin ($2\pi r$) we have:

$$\frac{2\pi(R + r)}{2\pi r} = \frac{R + r}{r} = \frac{R}{r} + 1$$

Hence in general, the rolling coin will make $\frac{R}{r}+1$ rotations around its own axis during one revolution around the larger coin.

Thus in the case of a coin of radius $\frac{r}{2}$ rolled around a coin of radius r we see that it will make

$$\frac{r}{\frac{r}{2}}+1=3$$

Thus the coin of radius $\frac{r}{2}$ will rotate 3 times around its own axis during one revolution around the coin of radius r.

Question 2

First lets construct our $\vec{p}$ vector:

$$\vec{p}=\begin{pmatrix} ar\cos(\phi)+r\cos(\theta)\cos(\phi) \\ ar\sin(\phi)+r\cos(\theta)\sin(\phi) \\ r\sin(\theta) \end{pmatrix}=\begin{pmatrix} r\cos(\phi)(a+\cos(\theta)) \\ r\sin(\phi)(a+\cos(\theta)) \\ r\sin(\theta) \end{pmatrix}$$

1. Constructing our tangent vectors:

$$\begin{aligned}\vec{e}_r=\frac{\partial\vec{p}}{\partial r}&=\frac{\partial}{\partial r}\begin{pmatrix} r\cos(\phi)(a+\cos(\theta)) \\ r\sin(\phi)(a+\cos(\theta)) \\ r\sin(\theta) \end{pmatrix}\\ &=\underrightarrow{\begin{pmatrix} \cos(\phi)(a+\cos(\theta)) \\ \sin(\phi)(a+\cos(\theta)) \\ \sin(\theta) \end{pmatrix}}\end{aligned}$$

$$\begin{aligned}\vec{e}_\theta=\frac{\partial\vec{p}}{\partial\theta}&=\frac{\partial}{\partial\theta}\begin{pmatrix} r\cos(\phi)(a+\cos(\theta)) \\ r\sin(\phi)(a+\cos(\theta)) \\ r\sin(\theta) \end{pmatrix}\\ &=\frac{\partial}{\partial\theta}\begin{pmatrix} ar\cos(\phi)+r\cos(\theta)\cos(\phi) \\ ar\sin(\phi)+r\cos(\theta)\sin(\phi) \\ r\sin(\theta) \end{pmatrix}\\ &=\begin{pmatrix} -r\sin(\theta)\cos(\phi) \\ -r\sin(\theta)\sin(\phi) \\ r\cos(\theta) \end{pmatrix}\\ &=\underrightarrow{r\begin{pmatrix} -\sin(\theta)\cos(\phi) \\ -\sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}}\end{aligned}$$

$$
\begin{aligned}
\vec{e}_\phi = \frac{\partial \vec{p}}{\partial \phi} = \frac{\partial}{\partial \phi} &\begin{pmatrix} r\cos(\phi)(a+\cos(\theta)) \\ r\sin(\phi)(a+\cos(\theta)) \\ r\sin(\theta) \end{pmatrix} \\
&= \begin{pmatrix} -r\sin(\phi)(a+\cos(\theta)) \\ r\cos(\phi)(a+\cos(\theta)) \\ 0 \end{pmatrix} \\
&= r\begin{pmatrix} -\sin(\phi)(a+\cos(\theta)) \\ \cos(\phi)(a+\cos(\theta)) \\ 0 \end{pmatrix}
\end{aligned}
$$

2. To find the area element of the outer surface of the torus we have that:

$$dA = r\|\vec{e}_\theta \times \vec{e}_\phi\| d\theta d\phi$$

Substituting in we have that:

$$dA = r\left\| r\begin{pmatrix} -\sin(\theta)\cos(\phi) \\ -\sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} \times r\begin{pmatrix} -\sin(\phi)(a+\cos(\theta)) \\ \cos(\phi)(a+\cos(\theta)) \\ 0 \end{pmatrix} \right\| d\theta d\phi$$

Here we can immediately take out r as common factor to get that:

$$
\begin{aligned}
dA &= r\,\|r\| \left\| \begin{pmatrix} -\sin(\theta)\cos(\phi) \\ -\sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} \times \begin{pmatrix} -\sin(\phi)(a+\cos(\theta)) \\ \cos(\phi)(a+\cos(\theta)) \\ 0 \end{pmatrix} \right\| d\theta d\phi \\
&= r^2 \left\| \begin{pmatrix} -\sin(\theta)\cos(\phi) \\ -\sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} \times \begin{pmatrix} -\sin(\phi)(a+\cos(\theta)) \\ \cos(\phi)(a+\cos(\theta)) \\ 0 \end{pmatrix} \right\| d\theta d\phi
\end{aligned}
$$

Now just solving for:

$$\begin{pmatrix} -\sin(\theta)\cos(\phi) \\ -\sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} \times \begin{pmatrix} -\sin(\phi)(a+\cos(\theta)) \\ \cos(\phi)(a+\cos(\theta)) \\ 0 \end{pmatrix}$$

Which is equivalent to:

$$\left| \begin{pmatrix} \hat{e}_\theta & \hat{e}_\phi & \hat{e}_r \\ -\sin(\theta)\cos(\phi) & -\sin(\theta)\sin(\phi) & \cos(\theta) \\ -\sin(\phi)(a+\cos(\theta)) & \cos(\phi)(a+\cos(\theta)) & 0 \end{pmatrix} \right|$$

Solving we have that:

$$= \hat{e}_\theta\left(-\cos(\theta)\cos(\phi)(a+\cos(\theta))\right) - \hat{e}_\phi\left(\cos(\theta)\sin(\phi)(a+\cos(\theta)\right) + \hat{e}_r\left(-\sin(\theta)\cos^2(\phi)(a+\cos(\theta)) - \sin(\theta)\sin^2(\phi)(a+\cos(\theta)\right)$$

$$= \hat{e}_\theta\left(-\cos(\theta)\cos(\phi)(a+\cos(\theta))\right) - \hat{e}_\phi\left(\cos(\theta)\sin(\phi)(a+\cos(\theta)\right) + \hat{e}_r\left(-\sin(\theta)(a+\cos(\theta))\right)$$

Taking the Magnitude of the above equation we have that:

$$\begin{aligned}
&= \sqrt{\left(-\cos(\theta)\cos(\phi)(a+\cos(\theta))\right)^2 + \left(-\cos(\theta)\sin(\phi)(a+\cos(\theta))\right)^2 + \left(-\sin(\theta)(a+\cos(\theta))\right)^2} \\
&= \sqrt{(\cos^2(\theta)\cos^2(\phi)(a+\cos(\theta))^2 + (\cos^2(\theta)\sin^2(\phi)(a+\cos(\theta))^2 + \sin^2(\theta)(a+\cos(\theta))^2)} \\
&= \sqrt{(a+\cos(\theta))^2\left(\cos^2(\theta)\cos^2(\phi) + \cos^2(\theta)\sin^2(\phi) + \sin^2(\theta)\right)} \\
&= (a+\cos(\theta))\sqrt{\cos^2(\theta)(\cos^2(\phi)+\sin^2(\phi)) + \sin^2(\theta)} \\
&= (a+\cos(\theta))\sqrt{\cos^2(\theta)(1) + \sin^2(\theta)} \\
&= (a+\cos(\theta))\sqrt{1} \\
&= (a+\cos(\theta))
\end{aligned}$$

Substituting into our original equation

$$\begin{aligned}
dA &= r^2 \left\| \begin{pmatrix} -\sin(\theta)\cos(\phi) \\ -\sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} \times \begin{pmatrix} -\sin(\phi)(a+\cos(\theta)) \\ \cos(\phi)(a+\cos(\theta)) \\ 0 \end{pmatrix} \right\| d\theta d\phi \\
&= \underrightarrow{r^2\big(a+\cos(\theta)\big)d\theta d\phi}
\end{aligned}$$

3. For the volume element we have that:

$$dV = \left|\left(\vec{e}_r \cdot \vec{e}_\theta \times \vec{e}_\phi\right)\right| d\theta d\phi dr$$

From our solution in the previous question we have that:

$$\begin{aligned}
dV &= \left| \left(\begin{pmatrix} \cos(\phi)(a+\cos(\theta)) \\ \sin(\phi)(a+\cos(\theta)) \\ \sin(\theta) \end{pmatrix} \cdot r^2 \begin{pmatrix} -\cos(\theta)\cos(\phi)(a+\cos(\theta)) \\ -\cos(\theta)\sin(\phi)(a+\cos(\theta)) \\ -\sin(\theta)(a+\cos(\theta)) \end{pmatrix} \right) \right| d\theta d\phi dr \\
&= r^2 \left| \left(\begin{pmatrix} \cos(\phi)(a+\cos(\theta)) \\ \sin(\phi)(a+\cos(\theta)) \\ \sin(\theta) \end{pmatrix} \cdot \begin{pmatrix} -\cos(\theta)\cos(\phi)(a+\cos(\theta)) \\ -\cos(\theta)\sin(\phi)(a+\cos(\theta)) \\ -\sin(\theta)(a+\cos(\theta)) \end{pmatrix} \right) \right| d\theta d\phi dr
\end{aligned}$$

Solving for:

$$\left(\begin{pmatrix} \cos(\phi)(a+\cos(\theta)) \\ \sin(\phi)(a+\cos(\theta)) \\ \sin(\theta) \end{pmatrix} \cdot \begin{pmatrix} -\cos(\theta)\cos(\phi)(a+\cos(\theta)) \\ -\cos(\theta)\sin(\phi)(a+\cos(\theta)) \\ -\sin(\theta)(a+\cos(\theta)) \end{pmatrix} \right)$$

We have that:

$$
\begin{aligned}
&= -\cos^2(\phi)\cos(\theta)(a+\cos(\theta))^2 - \cos(\theta)\sin^2(\phi)(a+\cos(\theta))^2 - \sin^2(\theta)(a+\cos(\theta))\\
&= -\cos(\theta)(a+\cos(\theta))^2(\cos^2(\phi)+\sin^2(\phi)) - \sin^2(\theta)(a+\cos(\theta)\\
&= -\cos(\theta)(a+\cos(\theta))^2 - \sin^2(\theta)(a+\cos(\theta)\\
&= -(a+\cos(\theta))(\cos(\theta)(a+\cos(\theta)) + \sin^2(\theta))\\
&= -(a+\cos(\theta))(a\cos(\theta)+\cos^2(\theta)+\sin^2(\theta))\\
&= -(a+\cos(\theta))(a\cos(\theta)+1)\\
&= -(a+\cos(\theta))(1+a\cos(\theta))
\end{aligned}
$$

We now take the absolute value of this as seen in the original equation:

$$
\begin{aligned}
&= \Big| -(a+\cos(\theta))(1+a\cos(\theta))\Big|\\
&= (a+\cos(\theta))(1+a\cos(\theta))
\end{aligned}
$$

now substituting into our original equation:

$$
\begin{aligned}
dV &= r^2\left|\left(\begin{pmatrix}\cos(\phi)(a+\cos(\theta))\\ \sin(\phi)(a+\cos(\theta))\\ \sin(\theta)\end{pmatrix}\cdot\begin{pmatrix}-\cos(\theta)\cos(\phi)(a+\cos(\theta))\\ -\cos(\theta)\sin(\phi)(a+\cos(\theta))\\ -\sin(\theta)(a+\cos(\theta))\end{pmatrix}\right)\right| d\theta d\phi dr\\
&= \underrightarrow{r^2(a+\cos(\theta))(1+a\cos(\theta))d\theta d\phi dr}
\end{aligned}
$$

4. Solving for the surface area we have that:

$$
\begin{aligned}
dA &= r^2\big(a+\cos(\theta)\big)d\theta d\phi\\
A &= \int_0^{2\pi}\int_0^{2\pi} r^2\big(a+\cos(\theta)\big)\,d\theta\,d\phi\\
&= \int_0^{2\pi}\left[\int_0^{2\pi} r^2\big(a+\cos(\theta)\big)\,d\theta\right]\,d\phi\\
&= \int_0^{2\pi}\left[a\theta r^2+\sin(\theta)r^2\right]\Bigg|_0^{2\pi}\,d\phi\\
&= \int_0^{2\pi}\Big(a(2\pi)r^2+\sin(2\pi)r^2-a(0)r^2-\sin(0)r^2\Big)\,d\phi\\
&= \int_0^{2\pi} 2\pi a r^2\,d\phi\\
&= 2\pi a r^2\int_0^{2\pi} d\phi\\
&= 2\pi a r^2\phi\Bigg|_0^{2\pi}\\
&= \underrightarrow{4\pi^2 a r^2}
\end{aligned}
$$

5. Solving for the volume we have that:

$$\begin{aligned}
dV &= r^2(a+\cos(\theta))(1+a\cos(\theta))d\theta d\phi dr \\
V &= \int_0^{2\pi}\int_0^{2\pi}\int_0^{r} r^2(a+\cos(\theta))(1+a\cos(\theta))drd\theta d\phi \\
&= \int_0^{2\pi}\int_0^{2\pi}\left[\frac{r^3}{3}(a+\cos(\theta))(1+a\cos(\theta))\right]\Bigg|_0^{r} d\theta d\phi \\
&= \int_0^{2\pi}\left(\int_0^{2\pi}\frac{r^3}{3}(a+\cos(\theta))(1+a\cos(\theta))d\theta\right)d\phi \\
&= \int_0^{2\pi}\left(\frac{r^3}{3}\int_0^{2\pi}(a+\cos(\theta))(1+a\cos(\theta))d\theta\right)d\phi \\
&= \int_0^{2\pi}\left(\frac{r^3}{3}\left[\int_0^{2\pi}(a+a\cos(\theta)+\cos(\theta)+a\cos^2(\theta))d\theta\right]\right)d\phi \\
&= \int_0^{2\pi}\left(\frac{r^3}{3}\left[a\theta+a\sin(\theta)+\sin(\theta)+a\left(\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\right)\right]\Bigg|_0^{2\pi}\right)d\phi \\
&= \int_0^{2\pi}\left(\frac{r^3}{3}\left[a(2\pi)+a\sin(2\pi)+\sin(2\pi)+a\left(\pi+\frac{\sin(4\pi)}{4}\right)-0\right]\right)d\phi \\
&= \int_0^{2\pi}\left(\frac{r^3}{3}\left[2a\pi+0+0+a\pi+0\right]\right)d\phi \\
&= \int_0^{2\pi}\left(\frac{r^3}{3}\left[3a\pi\right]\right)d\phi \\
&= \int_0^{2\pi}\left(\pi a r^3\right)d\phi \\
&= \left(\pi a r^3\right)\phi\Bigg|_0^{2\pi} \\
&= \pi a r^3\left(2\pi-0\right) \\
&= \underrightarrow{2\pi^2 a r^3}
\end{aligned}$$

6. To find a equation of a rectangle with the same area as the given torus we should first understand the different components of the parameterization of the torus and what exactly they mean.

The parameter r of the torus denotes the radius of the tube of the torus itself, thus meaning the circumference of a circular cross section of the torus will have a perimeter of $2\pi r$.

The parameter a is the distance from the the center of the tube to the inner edge of the torus.

The parameter θ the angle of rotation around the tube.

Finally, the parameter ϕ represents the rotation around the torus' axis of Revolution.

Now, when the torus is cut open and flattened, transforming into a rectangle we can say this rectangle has a length (L) and width (W). To relate these dimensions to that of the torus we have that:

The length (L) of the rectangle relates to the circumference of the circular cross-section of the torus as described above. This circumference is $2\pi r$ and thus the length (L) of the rectangle is $2\pi r$

The width (W) of the rectangle relates to the circumference of the tube itself. The tube has a radius ar with a circumference of $2\pi ar$ and thus the width (W) of the rectangle is $2\pi ar$

Verifying that this corresponds to our surface area of the torus

$$\begin{aligned} L \times W &= (2\pi r) \times 2\pi ar \\ &= 4\pi^2 ar^2 \end{aligned}$$

Which is identical to our surface area of the torus.
Thus we see that the Length of the rectangle is related to the parameter r of the torus and the width of the rectangle is related to both the parameters r and a of the torus, equivalently:

$$L(r) = 2\pi r$$

And

$$W(a, r) = 2\pi ar$$

Question 3

1. g is what we call a metric tensor. Now, the metric tensor fundamentally contains all the dot product information (specifically between our tangent vectors) in the space in which we are working.

Thus we find that: $g_{\phi t}$ actually means the dot product between our tangent vectors $\vec{e}_\phi$ and $\vec{e}_t$:

$$g_{\phi t} = \vec{e}_\phi \cdot \vec{e}_t$$

This idea holds true for all elements in our matrix g:

$$g_{ij} = \vec{e}_i \cdot \vec{e}_j$$

Where $i, j \in \{t, r, \theta, \phi\}$

This is significant because this means that the metric tensor g contains information about the curvature and geometry of the space in which we are working. Specifically it stores information which provides a way to measure distances, angles, and other geometric properties within the space we are working. As a result, the metric tensor is said to so-called encode the information regarding the length and angle measurements in the given space.

To illustrate this point lets consider a 2 Dimensional surface which has been mapped into a 3 Dimensional sphere:

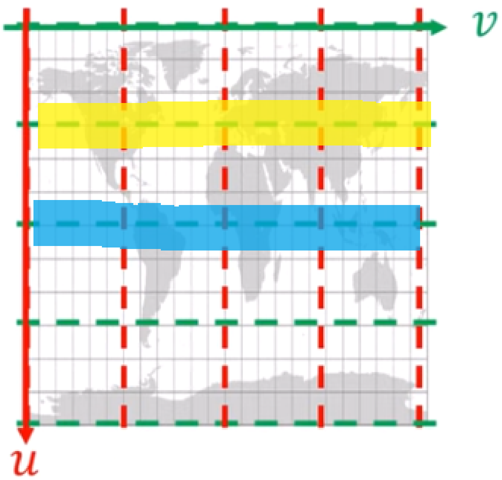

Figure 2: Lines of longitude on our 2 Dimensional Map

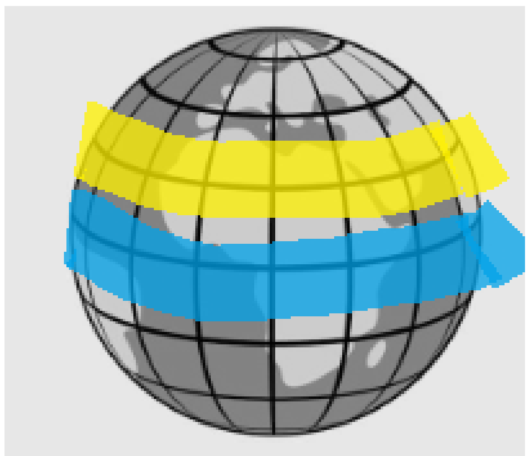

Figure 3: The corresponding lines of longitude on our 3 Dimensional Globe

Clearly here we see that in figure 2, it would seem that the yellow and blue line are of equal length. But when we translate it now to the globe they are different lengths. This distortion of space in this example from a 2 Dimensional "map" to a 3 Dimensional Globe is contained within what we call a metric tensor. It contains information pertaining to angles and distances which are otherwise no easily seen by the human eye.

2. Proving that $g_{\phi t} = g_{t\phi}$ is essentially asking us to prove that the dot product is commutative, i.e.

$$g_{\phi t} = g_{t\phi} \Rightarrow \vec{e}_\phi \cdot \vec{e}_t = \vec{e}_t \cdot \vec{e}_\phi$$

Proving the above statement we have for the general case of two arbitrary vectors $\vec{a}$ and $\vec{b}$:

$$\begin{aligned}
\vec{a} \cdot \vec{b} &= \sum_{i=1}^{n} \vec{a}_i \vec{b}_i && \text{Definition of the dot product} \\
&= \sum_{i=1}^{n} \vec{b}_i \vec{a}_i && \text{Real multiplication is commutative} \\
&= \vec{b} \cdot \vec{a} && \text{Definition of dot product}
\end{aligned}$$

Therefore we have shown that:

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

Thus since $\vec{a}$ and $\vec{b}$ were arbitrary it is true that:

$$\vec{e}_\phi \cdot \vec{e}_t = \vec{e}_t \cdot \vec{e}_\phi$$

Following from this we have:

$$g_{\phi t} = \vec{e}_\phi \cdot \vec{e}_t = \vec{e}_t \cdot \vec{e}_\phi = g_{t\phi}$$

Thus we have proven that:

$$g_{\phi t} = g_{t\phi}$$

3. The metric tensor g is a useful quantity in Geometry since as said before, it contains all the length and angle information a given space.

Evidently, this is important in geometry since fundamentally geometry is the study of the lengths and shapes of different things in a given space, and the metric tensor contains and defines all this information which is needed. Thus proving to be useful in the context of Geometry.

4. Constructing the arclength for a free particle of mass m moving in this black-hole geometry we have that:

$$\text{arc-length} = \int_{\lambda_i}^{\lambda_f} \left\| \frac{d\vec{P}}{d\lambda} \right\| d\lambda$$

Where $\vec{P}$ is our function our function describing our black hole space and λ_i is our initial time and λ_f is our final time.

Solving for $\left\|\frac{d\vec{P}}{d\lambda}\right\|$ we have that:

$$\left\| \frac{d\vec{P}}{d\lambda} \right\|^2 = \begin{pmatrix} \frac{dt}{d\lambda} & \frac{dr}{d\lambda} & \frac{d\theta}{d\lambda} & \frac{d\phi}{d\lambda} \end{pmatrix} g \begin{pmatrix} \frac{dt}{d\lambda} \\ \frac{dr}{d\lambda} \\ \frac{d\theta}{d\lambda} \\ \frac{d\phi}{d\lambda} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{dt}{d\lambda} & \frac{dr}{d\lambda} & \frac{d\theta}{d\lambda} & \frac{d\phi}{d\lambda} \end{pmatrix} \begin{pmatrix} g_{tt} & 0 & 0 & g_{t\phi} \\ 0 & g_{rr} & 0 & 0 \\ 0 & 0 & g_{\theta\theta} & 0 \\ g_{\phi t} & 0 & 0 & g_{\phi\phi} \end{pmatrix} \begin{pmatrix} \frac{dt}{d\lambda} \\ \frac{dr}{d\lambda} \\ \frac{d\theta}{d\lambda} \\ \frac{d\phi}{d\lambda} \end{pmatrix}$$

Multiplying out our first 2 matricies we get that:

$$\begin{aligned}
&= \begin{pmatrix} \frac{dt}{d\lambda} g_{tt} + \frac{d\phi}{d\lambda} g_{\phi t} & \frac{dr}{d\lambda} g_{rr} & \frac{d\theta}{d\lambda} g_{\theta\theta} & \frac{dt}{d\lambda} g_{\phi t} + \frac{d\phi}{d\lambda} g_{\phi\phi} \end{pmatrix} \begin{pmatrix} \frac{dt}{d\lambda} \\ \frac{dr}{d\lambda} \\ \frac{d\theta}{d\lambda} \\ \frac{d\phi}{d\lambda} \end{pmatrix} \\
&= \left(\frac{dt}{d\lambda} g_{tt} + \frac{d\phi}{d\lambda} g_{\phi t} \right) \frac{dt}{d\lambda} + \left(\frac{dr}{d\lambda} \frac{dr}{d\lambda} g_{rr} \right) + \left(\frac{d\theta}{d\lambda} \frac{d\theta}{d\lambda} g_{\theta\theta} \right) + \left(\frac{dt}{d\lambda} g_{\phi t} + \frac{d\phi}{d\lambda} g_{\phi\phi} \right) \frac{d\phi}{d\lambda} \\
&= \left(\frac{dt}{d\lambda} g_{tt} + \frac{d\phi}{d\lambda} g_{\phi t} \right) \frac{dt}{d\lambda} + \left(\frac{dr}{d\lambda} \frac{dr}{d\lambda} g_{rr} \right) + \left(\frac{d\theta}{d\lambda} \frac{d\theta}{d\lambda} g_{\theta\theta} \right) + \left(\frac{dt}{d\lambda} g_{\phi t} + \frac{d\phi}{d\lambda} g_{\phi\phi} \right) \frac{d\phi}{d\lambda} \\
&= \left(\frac{dt}{d\lambda} g_{tt} \frac{dt}{d\lambda} + \frac{d\phi}{d\lambda} g_{\phi t} \frac{dt}{d\lambda} + \frac{dr}{d\lambda} g_{rr} \frac{dr}{d\lambda} + \frac{d\theta}{d\lambda} g_{\theta\theta} \frac{d\theta}{d\lambda} + \frac{dt}{d\lambda} g_{\phi t} \frac{d\phi}{d\lambda} + \frac{d\phi}{d\lambda} g_{\phi\phi} \frac{d\phi}{d\lambda} \right) \\
&= \left(g_{tt} \left(\frac{dt}{d\lambda} \right)^2 + 2 g_{\phi t} \frac{dt}{d\lambda} \frac{d\phi}{d\lambda} + g_{rr} \left(\frac{dr}{d\lambda} \right)^2 + g_{\theta\theta} \left(\frac{d\theta}{d\lambda} \right)^2 + g_{\phi\phi} \left(\frac{d\phi}{d\lambda} \right)^2 \right)
\end{aligned}$$

Therefore we have that:

$$\left\| \frac{d\vec{P}}{d\lambda} \right\| = \sqrt{\left(g_{tt} \left(\frac{dt}{d\lambda} \right)^2 + 2 g_{\phi t} \frac{dt}{d\lambda} \frac{d\phi}{d\lambda} + g_{rr} \left(\frac{dr}{d\lambda} \right)^2 + g_{\theta\theta} \left(\frac{d\theta}{d\lambda} \right)^2 + g_{\phi\phi} \left(\frac{d\phi}{d\lambda} \right)^2 \right)}$$

Substituting back into our integral:

$$\text{arc-length} = \int_{\lambda_i}^{\lambda_f} \sqrt{\left(g_{tt} \left(\frac{dt}{d\lambda} \right)^2 + 2 g_{\phi t} \frac{dt}{d\lambda} \frac{d\phi}{d\lambda} + g_{rr} \left(\frac{dr}{d\lambda} \right)^2 + g_{\theta\theta} \left(\frac{d\theta}{d\lambda} \right)^2 + g_{\phi\phi} \left(\frac{d\phi}{d\lambda} \right)^2 \right)} d\lambda$$

$$\begin{aligned}
= \int_{\lambda_i}^{\lambda_f} &\Bigg(-(1 - \frac{Rr}{r^2 + a^2 \cos^2(\theta)}) \left(\frac{dt}{d\lambda} \right)^2 + 2 \left(-a \sin^2(\theta) \frac{Rr}{r^2 + a^2 \cos^2(\theta)} \right) \frac{dt}{d\lambda} \frac{d\phi}{d\lambda} + \\
&+ \frac{r^2 + a^2 \cos^2(\theta)}{r^2 + a^2 - Rr} \left(\frac{dr}{d\lambda} \right)^2 + \left(r^2 + a^2 \cos^2(\theta) \right) \left(\frac{d\theta}{d\lambda} \right)^2 \\
&+ \sin^2(\theta) \left(r^2 + a^2 \sin^2(\theta) \frac{Rr}{r^2 + a^2 \cos^2(\theta)} \right) \left(\frac{d\phi}{d\lambda} \right)^2 \Bigg)^{\frac{1}{2}} d\lambda
\end{aligned}$$

5. Note, as given in the prompt we will convert to the line-element notation to simplify the following set of equations.

In the case that $a = 0$ we have that many terms in the above integral will cancel:

$$\begin{aligned}ds^2 = \Bigg(&-(1 - \frac{Rr}{r^2 + 0^2\cos^2(\theta)})dt^2 + 2\left(-0\sin^2(\theta)\frac{Rr}{r^2 + 0^2\cos^2(\theta)}\right)dtd\phi + \\ &+ \frac{r^2 + 0^2\cos^2(\theta)}{r^2 + 0^2 - Rr}dr)^2 + \left(r^2 + 0^2\cos^2(\theta)\right)d\theta^2 \\ &+ \sin^2(\theta)\left(r^2 + 0^2\sin^2(\theta)\frac{Rr}{r^2 + 0^2\cos^2(\theta)}\right)d\phi^2\Bigg)\end{aligned}$$

cancelling terms we have:

$$= -\left(1 - \frac{R}{r}\right)dt^2 + \frac{r^2}{r^2 - Rr}dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2$$

since $R \ll r$, we have $\frac{R}{r} \to 0$ and $\frac{r^2}{r^2 - Rr} \to 1$. Thus:

$$= -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2$$

The inclusion of a negative term $(-dt^2)$ in our line element shows us that the line element in this particular space differs from the line element in Euclidean space.

This negative term gives rise to peculiar behavior within the space, as it implies the possibility of an arc-length of zero or even negatively valued distances. Consequently, it suggests that two unique points within this space could have zero or negative distance between them.