ematical Foundations of Data Science (COMS4055A)
Class Test 2

19 May 2022, 14h00–16h00, RSEH

Name: [redacted] Row: ______ Seat: 13 Signature: ______

Student Number: [redacted] ID Number: [redacted]

For marking purposes only

Question 1	
Question 2	
Total	

Instructions

- Answer all questions in pen. **Do not write in pencil.**
- This test consists of 3 pages. Ensure that you are not missing any pages.
- This is a **closed-book** test: you may not consult any written material or notes.
- You are allocated 2 hours to complete this test.
- There are 2 questions and 60 marks available.
- Ensure your cellphone is switched off. [redacted]
- You may use a calculator during the test.
- Round off to 2 decimal places and simplify your answers fully.

Question 1 Linear Algebra [30 Marks]

1. Compute the determinant of A where. [6] 5

$$A = \begin{bmatrix} 2 & 0 & 1 & 2 & 7 \\ 2 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2. Let $B \in \mathbb{R}^{2\times 3}$ with a singular value decomposition of $B = U\tilde{B}V^T$. Let $x \in \mathbb{R}^3$ be a **coordinate** vector in terms of the canonical basis. Specifically:

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \quad V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & \frac{-2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{-1}{3} \end{bmatrix} \quad x = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}\frac{1}{3}$$

   (a) Determine the matrix B. [6]

   (b) Compute $\hat{x}$ where $\hat{x} = Bx$. [3]

   (c) Project x onto the basis defined by the right singular vectors (V). Call this new vector $\tilde{x}$. [4]

   (d) Compute $\hat{\tilde{x}} = \tilde{B}\tilde{x}$. Note $\hat{\tilde{x}}$ is a vector in terms of the basis defined by the left singular vectors. [3]

   (e) Project $\hat{\tilde{x}}$ onto the canonical basis **from** the basis defined by the left singular vectors (U). **Hint:** use your answer from question 2(d) [3]

3. Determine the [redacted] SVD of A where: [5]

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Question 2 Vector Calculus [30 Marks]

1. Compute the derivative $f'(x)$ for $f(x)$ shown below (note log refers to the natural log). [8]

$$f(x) = log(x^4)sin(x^3)$$

2. Compute the third order Taylor polynomial T_3 of $f(x) = sin(x) + cos(x)$ at $x_0 = \frac{\pi}{2}$ (you don't have to foil out all the powers of $x - x_0$). [7]

   Remember that the Taylor polynomial of degree n of $f : \mathbb{R} \longrightarrow \mathbb{R}$ at x_0 is defined as

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

   where $f^{(k)}(x_0)$ is the kth derivative of f at x_0 (which we assume exists) and $\frac{f^{(k)}(x_0)}{k!}$ are the coefficients of the polynomial, according to **Definition 5.3** of the textbook.

3. What will be the dimensionality of the Jacobian for the following functions differentiated with respect to x (x is a column vector, x^T is a row vector and standard matrix multiplication is used):

   (a) $f(x) = x^2 + y$ for $x \in \mathbb{R}$ and $y \in \mathbb{R}$. [1]

   (b) $f(x) = x^T Ax$ for $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. [2]

   (c) $F(x) = xx^T$ for $x \in \mathbb{R}^n$. [3]

4. Consider the function $f(x) = \sqrt{(x^3 + \cos(x^3))} - (x^3 + \cos(x^3))^2 + \cos(x^3)$. [9]
   Depict $f(x)$ as a data flow graph. Make sure to define all intermediate variables.