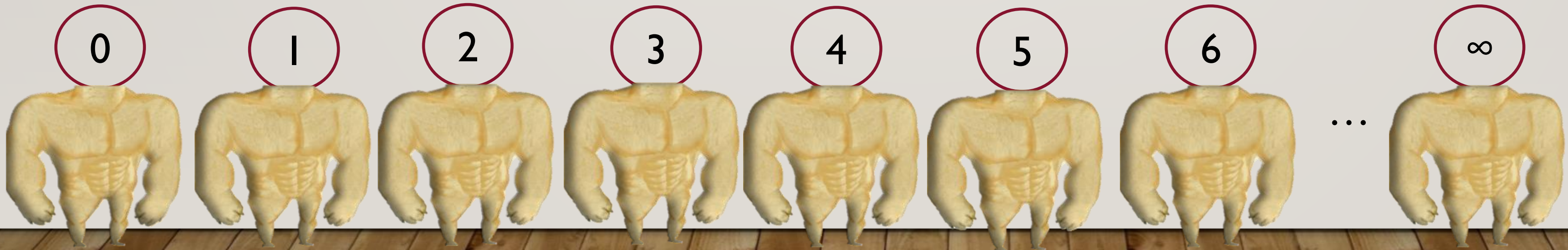
ANALYSIS OF ALGORITHMS

LECTURE 7 : MATHEMATICAL INDUCTION

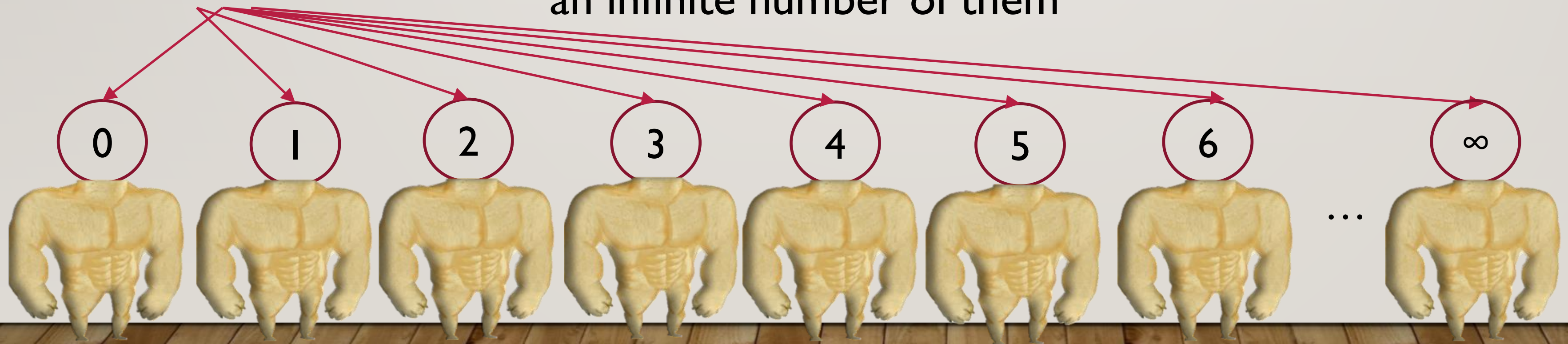
LORD OF THE MEMES

- I would like to steal Steve's title of MemeLord
- Now, let's say there's an infinitely long line of people
- I want one of my memes to go to the entire set of people
- Basically, I want the statement "Person *n* has my meme" to be true for every natural number *n*



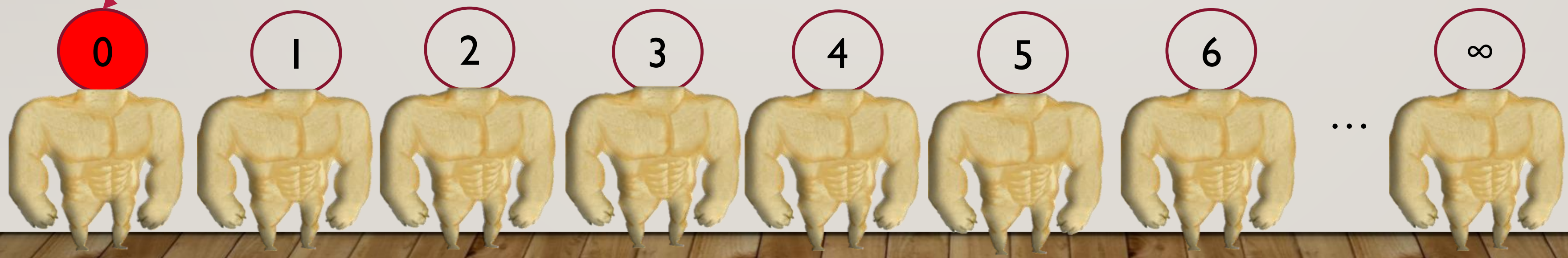
THAT'S TOO MUCH, MAN

- Now, I would like to steal Steve's title of MemeLord
- So I want one of my memes to go to the entire set of people
- Basically I want the statement "Person *n* has my meme" to be true for all *n*
- I could exhaustively send it to all of them, but… man, there's an infinite number of them



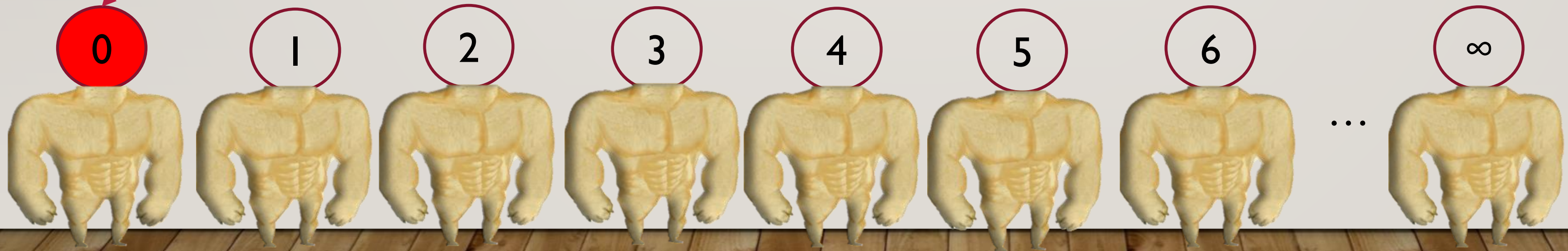
THERE CAN BE ONLY ONE

- So what I really want to do is just send it to the first person
- Now the statement "Person *n* has my meme" is true for n=0
- But this meme is missing something, so no one is gonna pass it on. It just dies at person 0. So the statement won't even be true for n=1



BE BETTER

- What if the meme is so good that any person who receives it will be compelled to pass it on?



RESISTANCE IS FUTILE

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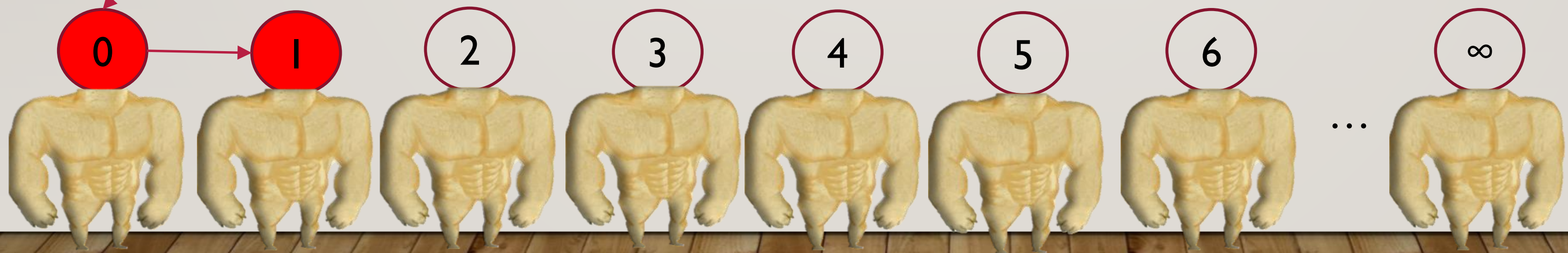
- Person 0 is compelled by the power of memes to send it to Person 1



RESISTANCE IS FUTILE

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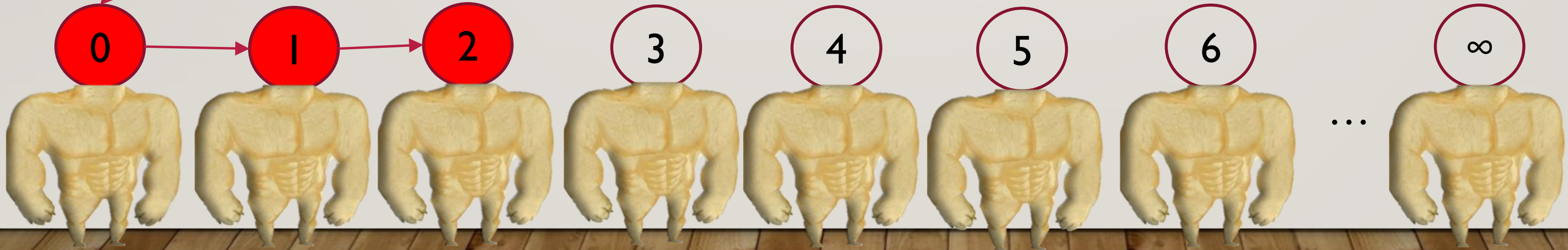
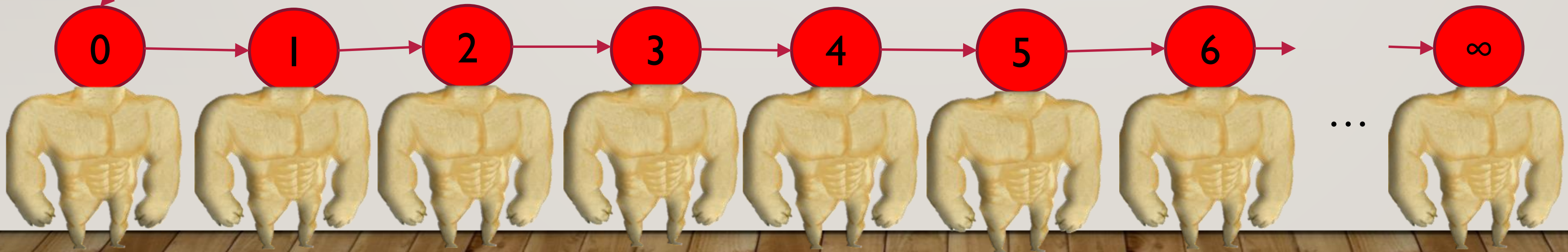
- But Person 1 is powerless to resist the power of memes and has to send it to person 2



TO INFINITY AND BEYOND

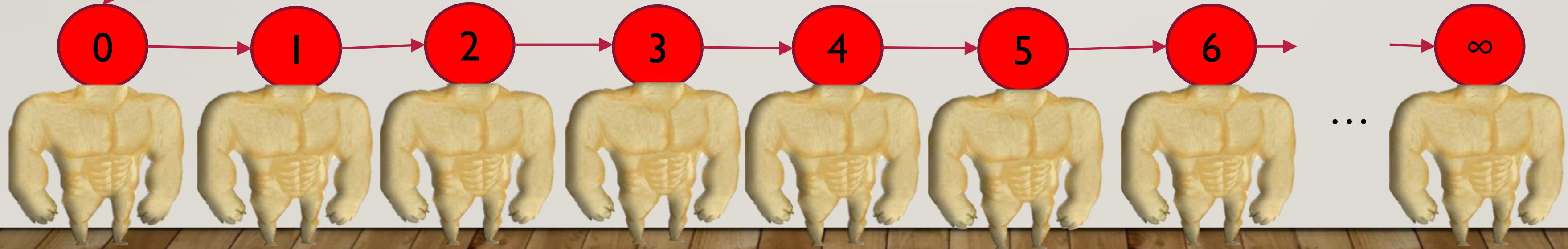
- Oh now we're talking
- Person 0 is compelled by the power of memes to send it to Person 1
- But Person 1 is powerless to resist the power of memes and has to send it to person 2
- More generally, person k is compelled to send it to person k+1, for all k



WHAT JUST HAPPENED?

To prove the statement "Person n has my meme" true for all *n*, I needed to do two things

1. Ensure that the statement was true for *n*=0 (If I didn't send the meme to anyone, no one would have it) [base case]
2. Ensure that the meme would definitely be passed – So if "Person *k* has my meme" is true than "Person *k*+1" has my meme is also true [induction step]

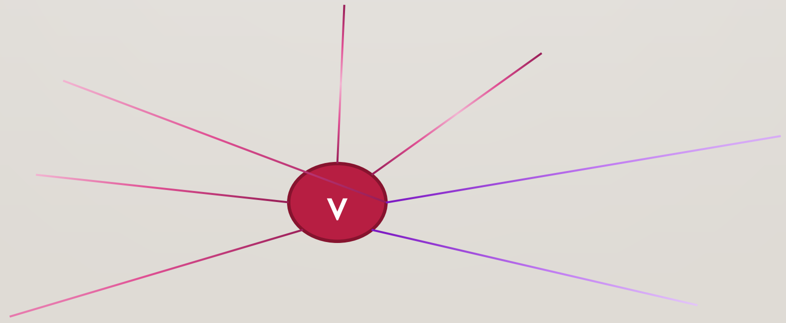


IS THIS USEFUL WITH GRAPHS?

- Consider this statement:
  - A graph G whose vertex of maximum degree v has degree r will always be $r+1$ colourable

IS THIS USEFUL WITH GRAPHS?

- Consider this statement:
  - A graph *G* whose vertex of maximum degree *v* has degree *r* will always be *r*+1 colourable
- That's kind of a big statement. Coz there are an infinite number of possible graphs, and we're saying that no matter what graph it is, if the vertex with the biggest degree has a degree of say… 7, then the graph can be coloured using 8 colours. No matter what the rest of the graph looks like.



GRAPH INDUCTION EXAMPLE

- Consider this statement:
  - A graph G whose vertex of maximum degree v has degree r will always be $r+1$ colourable
- How do we use induction? Well, remember with our example we said we were trying to prove the statement "Person n has the meme" is true for every natural number n
- So we kind of have to decide what statement we can make that we will want to prove true for all n
- With graphs, we will very often use the number of vertices in the graph as our n:

GRAPH INDUCTION EXAMPLE

- Consider this statement:
  - A graph *G* whose vertex of maximum degree *v* has degree *r* will always be *r*+1 colourable
- How do we use induction? Well, remember with our example we said we were trying to prove the statement "Person *n* has the meme" is true for every natural number *n*
- So we kind of have to decide what statement we can make that we will want to prove true for all *n*
- With graphs, we will very often use the number of vertices in the graph as our *n*:
  - All graphs with n vertices whose vertex of maximum degree *v* has degree *r* will always be *r*+1 colourable

GRAPH INDUCTION EXAMPLE

- We must then prove the statement "All graphs with n vertices whose vertex of maximum degree *v* has degree *r* will always be *r*+1 colourable" is true for every natural number *n*
- What do we do to make it true for every natural number *n*…
- First, we make sure that it is true for the first *n* (*n*=0)
- Well, if there are no vertices, the maximum degree is 0. Can we colour a graph with no vertices using at most 1 colour? Yeah, sure. We can use 0 colours.
- That might seem unconvincing to you. Let's confirm with *n*=1 (We don't have to do this, but it's illustrative

GRAPH INDUCTION EXAMPLE

- Must check if "All graphs with n vertices whose vertex of maximum degree *v* has degree *r* will always be *r*+1 colourable" is true for *n*=1
- In this case "All graphs" is a bit grandiose, because there's only one graph of size 1. It looks like this: 

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- In this case, the vertex with maximum degree is vertex 0, which has degree 0. So *r*=0.
- Now, is the graph *r*+1 colourable? Can I colour it using 1 colour?
- Yes, so "All graphs with 1 vertex whose vertex of maximum degree *v* has degree *r* will always be *r*+1 colourable"

GRAPH INDUCTION EXAMPLE

- Okay, so we've got our base case (n=0 is fine, but n=1 if you really want to)
- Now we want to show that if the statement
  - "All graphs with ***k*** vertices whose vertex of maximum degree v has degree r will always be r+1 colourable" is true

  then
  - "All graphs with ***k+1*** vertices whose vertex of maximum degree v has degree r will always be r+1 colourable" is true
- Basically we showed that the first guy has the meme, now we want to show that anyone that has the meme is compelled to pass it on, because then everyone will have it

GRAPH INDUCTION EXAMPLE

- So now we're allowed to assume
  - "All graphs with ***k*** vertices whose vertex of maximum degree *v* has degree *r* will always be *r*+1 colourable" is true

  And we must prove that

  - "All graphs with ***k+1*** vertices whose vertex of maximum degree *v* has degree *r* will always be *r*+1 colourable" is true

GRAPH INDUCTION EXAMPLE

- Consider a graph G with ***k*+1** vertices, whose vertex of maximum degree *v* has degree *r*
- Construct a graph G’ by removing v and its edges from G.
- G’ has **k** vertices. G’ has a vertex of maximum degree *u.* Now, since we haven’t added any edges, *u* must have a degree that’s less than or equal to r
- So G’ must be r+1 colourable, by our induction hypothesis
- Now, let’s construct G by adding *v* and its edges back to G’. But we’ll keep the colours from G’
- The only vertex we then have to colour is *v*
- *v* has *r* neighbours, so at most *r* colours can be used. So there’s one left over for *v*
- So G is *r*+1-colourable