Chapter 2: Sequence (Cont...)

Lemma 2.1. If $L, M \in \mathbb{R}$ such that $|L - M| < \epsilon$ for all $\epsilon > 0$, then $L = M$.

Proof. Assume $L \neq M$. Then either $L < M$ or $L > M$. But

$$L < M \implies L - M < 0 \implies |L - M| = M - L > 0$$

$$L > M \implies L - M > 0 \implies |L - M| = L - M > 0$$

so that $|L - M| > 0$. Then

$$0 < \frac{|L - M|}{2} < |L - M|$$

which contradicts $|L - M| < \epsilon$ for $\epsilon = \dfrac{|L - M|}{2}$.

Hence the assumption $L \neq M$ must be false, and $L = M$ follows. □

Note. The proof above uses the fact that,

$$A \implies B \quad \text{is equivalent to} \quad \neg B \implies \neg A\,,$$

where '¬' is the symbole for the negation.

Theorem 2.2. If the sequence (a_n) converges, then its limit is unique.

Proof. Assume that (a_n) converges to L and M.

Then there are numbers k_L and k_M such that for all $n \in \mathbb{N}$

$$|a_n - L| < \frac{\epsilon}{2} \text{ if } n \geq k_L, \quad \text{and} \quad |a_n - M| < \frac{\epsilon}{2} \text{ if } n \geq k_M .$$

Take $K = \max\{k_L, k_M\}$.

For positive integers $n \geq K$, we have $n \geq k_L$ and $n \geq k_M$.

Therefore,

$$|L-M| = |(L-a_n)+(a_n-M)| \leq |a_n-L|+|a_n-M| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon$$

Thus, $|L - M| < \epsilon$ for any $\epsilon > 0$. By Lemma 2.1, $L = M$.
□

Theorem 2.3 (Limit Laws). Let $c \in \mathbb{R}$ and suppose that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$ both exist. Then

(a) $\lim\limits_{n\to\infty} c = c$.

Proof. For all $\epsilon > 0$ and all $n \in \mathbb{N}$, with $a_n = c$,

$|\, a_n - c \,| = |\, c - c \,| = 0 < \epsilon$.

(b) $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = L + M$.

Proof. Let $\epsilon > 0$. There are numbers k_L and k_M

such that for all $n \in \mathbb{N}$, $|\, a_n - L \,| < \dfrac{\epsilon}{2}$ if $n \geq k_L$,

and $|\, a_n - M \,| < \dfrac{\epsilon}{2}$ if $n \geq k_M$.

Take $K = \max\{k_L, k_M\}$. For positive integers $n \geq K$,

we have $n \geq k_L$ and $n \geq k_M$.

Therefore, $|\,(a_n + b_n) - (L + M)\,| = |\,(a_n - L) + (b_n - M)\,|$

$\leq |\, a_n - L \,| + |\, b_n - M \,| < \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon$.

Theorem 2.3 (Limit Laws) Cont...

(c) $\lim\limits_{n\to\infty}(ca_n) = c\lim\limits_{n\to\infty} a_n = cL$.

Proof. Let $\epsilon > 0$.Then there is a number K such that

$$|a_n - L| < \frac{\epsilon}{1+|c|} \text{ if } n \geq K \ .$$

For positive integers $n \geq K$, we have

$$|ca_n - cL| = |c(a_n - L)| = |c||a_n - L| \leq |c|\frac{\epsilon}{1+|c|} < \epsilon$$

(d) $\lim\limits_{n\to\infty}(a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right) = LM$.

Proof. We consider 2 cases:

Case 1: $L = M = 0$. Let $\epsilon > 0$.

Then there are numbers k_L and k_M such that

$$|a_n - 0| < \epsilon \text{ if } n \geq k_L, \text{ and } |b_n - 0| < 1 \text{ if } n \geq k_M \ .$$

Put $K = \max\{k_L, k_M\}$. For positive integer $n \geq K$, we have $n \geq k_L$ and $n \geq k_M$ and therefore

$|a_n b_n| = |a_n||b_n| < \epsilon \cdot 1 = \epsilon$.

Case 2: L and M are arbitrary. Then

$a_n b_n = (a_n - L)(b_n - M) + L(b_n - M) + a_n M$.

By (a) and (b), $(a_n - L) \to 0$ and $(b_n - M) \to 0$

as $n \to \infty$, and by (b), (c), and Case 1, it follows that

$(a_n b_n)$ converges with $a_n b_n = 0 + L \cdot 0 + LM = LM$.

(e) If $M \neq 0$, $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \dfrac{L}{M}$. (The proof is long.)

(f) If $L \neq 0$ and $M = 0$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$ does not exist.

Proof. Assume that $P = \lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$ exists. Then by (d)

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{a_n}{b_n} b_n\right) = \lim_{n\to\infty} \frac{a_n}{b_n} \cdot \lim_{n\to\infty} b_n$$

$= P \cdot M = P \cdot 0 = 0$ which contradicts $L \neq 0$.

(g) If $k \in \mathbb{Z}^+$, $\lim_{n\to\infty} a_n^k = \left(\lim_{n\to\infty} a_n\right)^k = L^k$.

Proof: Follows from (d) by induction.

(h) If $k \in \mathbb{Z}^+$, $\lim_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{\lim_{n\to\infty} a_n} = \sqrt[k]{L}$.

If k is even, we assume $a_n \geq 0$ and $L \geq 0$.

Proof: For $k = 2$ only.

If $L = 0$, let $\epsilon > 0$ and choose K such that $a_n < \epsilon^2$

for $n \geq K$. Then $\sqrt{a_n} < \epsilon$ for these n, and

$\lim_{n\to\infty} \sqrt{a_n} = 0 = \sqrt{L}$.

If $L > 0$, then

$$|\sqrt{a_n} - L| = \left| \frac{a_n + L}{\sqrt{a_n} - L} \right| = \frac{|a_n - L|}{\sqrt{a_n} - \sqrt{L}} \leq \frac{1}{\sqrt{L}} |a_n - L|,$$

and choosing K such that $|a_n - L| < \epsilon\sqrt{L}$ for $n \geq K$,

it follows that $\left|\sqrt{a_n} - \sqrt{L}\right| < \epsilon$ for $n \geq K$.

(i) If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$. **Proof:** Assignment.

Tutorial 2.1.1(2)

(a) Prove that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} |a_n| = |L|$.

Hint: Use (and prove) the inequality

$$||x| - |y|| \leq |x - y|.$$

Use Tutorial 1.1.2,2(d) to prove this inequality.

(b) Give an example to show that the converse to part (a) is not true. **Hint:** Use $a_n = (-1^n)$.