Multivariable Calculus

MATH2007

2.6 Green's Theorem
(Part 1)

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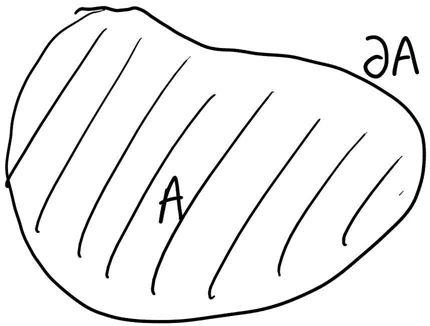
$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a)$$



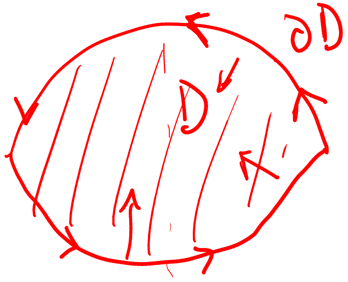
$$\boxed{\iint_A ? = \int_{\partial A} ? \ \ldots}$$

$$= \int_a^b ? \ \ldots$$

$$= F(b) - F(a)$$

Definition. Let $D \subset \mathbb{R}^2$ be of Type III. We denote the boundary of D by ∂D. We say that D is positively oriented if ∂D is oriented such that D lies on one's left when facing in the direction of orientation of ∂D.

Note. The first co-ordinate is the horizontal axis and the second co-ordinate is the vertical axis. This convention must be adhered to.



Theorem (2.6.1 Green's Theorem)**.** Let D be a region in $\mathbb{R}^2$ with boundary ∂D oriented anticlockwise (i.e., with positive orientation), then for $\underline{F} : \mathbb{R}^2 \to \mathbb{R}^2$ we have

$$\iint_D \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx_1\, dx_2 = \int_{\partial D} \underline{F} \cdot d\underline{r}.$$

$x \quad y$

$\partial x \quad \partial y$

Note. Some other forms of Green's Theorem are

$$\iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy = \int_{\partial D} F_1\, dx + F_2\, dy$$

and if we identify $\underline{F}(\underline{x})$ with the vector in $\mathbb{R}^3$ given by $\begin{bmatrix} F_1 \\ F_2 \\ 0 \end{bmatrix}$ then Green's Theorem can be written as

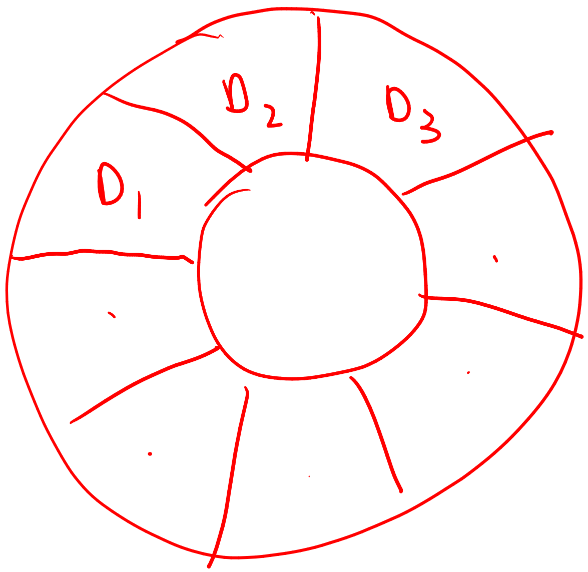
($\underline{x} \in \mathbb{R}^2$

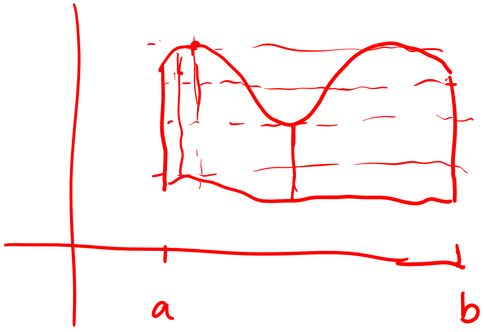
$$\iint_D (\nabla \times \underline{F}) \cdot \underline{e}_3\, dx_1\, dx_2 = \int_{\partial D} \underline{F} \cdot d\underline{r}.$$

We only prove Green's Theorem for the case of a region D in $\mathbb{R}^2$ of the form Type III

$$D = \{(x, y) \mid \underline{p(x)} \le y \le \underline{q(x)}, x \in [a, b]\} = \{(x, y) \mid \underline{g(y)} \le x \le \underline{h(y)}, y \in [c, d]\}.$$

$$\iint_D \cdots = \iint_{D_1} \cdot + \iint_{D_2} + \cdots$$





Lemma (2.6.3)**.** Let D be a region in $\mathbb{R}^2$ of TYPE I, i.e of the form

$$D = \{(x, y) \,|\, p(x) \leq y \leq q(x), x \in [a, b]\},$$

then

$$\underline{\int_{\partial D} f \; dx} = - \underline{\iint_D \frac{\partial f}{\partial y}} \, dy \, dx.$$

(or alternatively)

$$\int_{\partial D} \begin{bmatrix} f \\ 0 \end{bmatrix} \cdot d\underline{r} = - \iint_D \frac{\partial f}{\partial y} \, dy \, dx.$$

$$d\underline{r} = \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Lemma (2.6.4)**.** Let D be a region in $\mathbb{R}^2$ of TYPE II, i.e. of the form

$$D = \{(x, y) \,|\, g(y) \leq x \leq h(y), y \in [c, d]\}$$

then

$$\int_{\partial D} \begin{bmatrix} 0 \\ f \end{bmatrix} \cdot d\underline{r} = \iint_D \frac{\partial f}{\partial x} \, dx \, dy.$$

$$\int_{\partial D} f \; dy = \iint_D \frac{\partial f}{\partial x} \, dx \, dy$$

Proof of Green's Theorem. □

$$\int_{\partial D} \underline{F} \cdot d\underline{r} = \int_{\partial D} F_1 \, dx + \int_{\partial D} F_2 \, dy$$

$$= -\iint_D \frac{\partial F_1}{\partial y} \, dx \, dy + \iint_D \frac{\partial F_2}{\partial x} \, dx \, dy$$

$$= \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx \, dy \, .$$

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2.6 Green's Theorem (Part 2)

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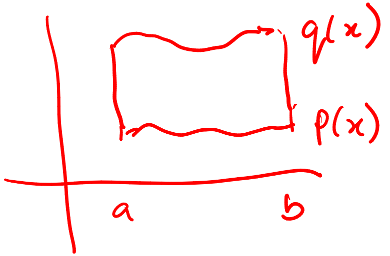
Lemma (2.6.3)**.** Let D be a region in $\mathbb{R}^2$ of TYPE I, i.e of the form

$$D = \{(x, y) \mid \underline{p(x) \leq y \leq q(x), x \in [a, b]}\},$$



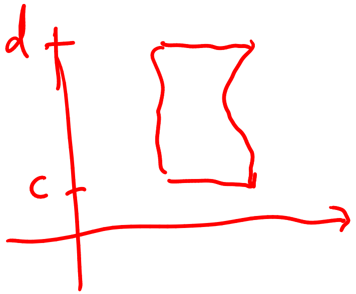
then

$$\int_{\partial D} f \, dx = -\iint_D \frac{\partial f}{\partial y} \, dy \, dx.$$

(or alternatively)

$$\int_{\partial D} \begin{bmatrix} f \\ 0 \end{bmatrix} \cdot d\underline{r} = -\iint_D \frac{\partial f}{\partial y} \, dy \, dx.$$

Lemma (2.6.4)**.** Let D be a region in $\mathbb{R}^2$ of TYPE II, i.e. of the form



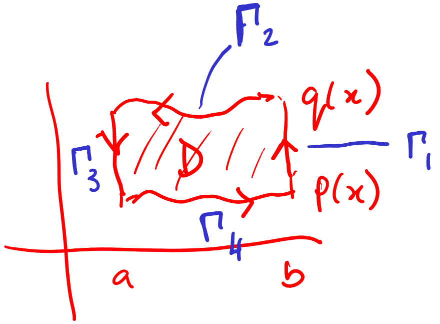
$$D = \{(x, y) \mid g(y) \leq x \leq h(y), y \in [c, d]\}$$

then

$$\int_{\partial D} \begin{bmatrix} 0 \\ f \end{bmatrix} \cdot d\underline{r} = \iint_D \frac{\partial f}{\partial x} \, dx \, dy.$$

Exercise.

Proof of Lemma 2.6.3. □



$$\int_{\partial D} f\,dx = \int_{\Gamma_1} f\,dx + \int_{\Gamma_2} f\,dx + \int_{\Gamma_3} f\,dx + \int_{\Gamma_4} f\,dx$$

$$= \int_{\Gamma_1} f\,dx - \int_{\Gamma_2^-} f\,dx + \int_{\Gamma_3} f\,dx + \int_{\Gamma_4} f\,dx$$

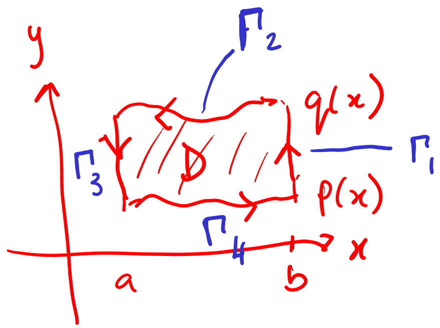
$$\iint_D \left(-\frac{\partial f}{\partial y}\right) dx\,dy = -\int_a^b \left(\int_{p(x)}^{q(x)} \frac{\partial f}{\partial y}\,dy\right) dx$$

$$= -\int_a^b \Big[f(x,y)\Big]_{y=p(x)}^{y=q(x)}\,dx$$

$$= -\int_a^b f(x,q(x))\,dx + \int_a^b f(x,p(x))\,dx$$

$$\Gamma_1 = \left\{ \underline{r}_1(t) : \underline{r}_1(t) = \begin{pmatrix} b \\ (1-t)p(b) + t\,q(b) \end{pmatrix},\ t \in [0,1] \right\}$$

$$\underline{r}_1(t) = (1-t)\begin{pmatrix} b \\ p(b) \end{pmatrix} + t \begin{pmatrix} b \\ q(b) \end{pmatrix}$$ Straight line



$$\Gamma_2^- = \left\{ \underline{r}_2(t) : \underline{r}_2(t) = \begin{pmatrix} t \\ q(t) \end{pmatrix},\ t \in [a,b] \right\}$$

$$\Gamma_3 = \left\{ \underline{r}_3(t) : \underline{r}_3(t) = \begin{pmatrix} a \\ (1-t)q(a) + t\,p(a) \end{pmatrix};\ t \in [0,1] \right\}$$

$$\Gamma_4 = \left\{ \underline{r}_4(t) : \underline{r}_4(t) = \begin{pmatrix} t \\ p(t) \end{pmatrix},\ t \in [a,b] \right\}$$

$$\int_{\Gamma_1} f\,dx = \int_{\Gamma_1} \begin{pmatrix} f(\underline{r}_1(t)) \\ 0 \end{pmatrix} \cdot d\underline{r}_1 = \int_0^1 \begin{pmatrix} f(\underline{r}_1(t)) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ q(b)-p(b) \end{pmatrix} dt = 0.$$

$$\int_{\Gamma_3} f\,dx = \int_0^1 \begin{pmatrix} f(\underline{r}_3(t)) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ p(a)-q(a) \end{pmatrix} dt = 0.$$

$$\Gamma_2^- = \left\{ \underline{r}_2(t) : \ \underline{r}_2(t) = \begin{pmatrix} t \\ q(t) \end{pmatrix}, \quad t \in [a,b] \right\}$$

$$\Gamma_4 = \left\{ \underline{r}_4(t) : \ \underline{r}_4(t) = \begin{pmatrix} t \\ p(t) \end{pmatrix}, \quad t \in [a,b] \right\}$$

$$-\int_{\Gamma_2^-} f \, dx = -\int_a^b \begin{pmatrix} f(\underline{r}_2(t)) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ q'(t) \end{pmatrix} dt = -\int_a^b f(t, q(t)) \, dt$$

$$\int_{\Gamma_4} f \, dx = \int_a^b \begin{pmatrix} f(\underline{r}_4(t)) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ p'(t) \end{pmatrix} dt = \int_a^b f(t, p(t)) \, dt$$

$$\int_{\partial D} f \, dx = \cancel{\int_{\Gamma_1} f \, dx} - \int_{\Gamma_2^-} f \, dx + \cancel{\int_{\Gamma_3} f \, dx} + \int_{\Gamma_4} f \, dx = -\int_a^b f(t, q(t)) \, dt + \int_a^b f(t, p(t)) \, dt$$

$$\iint_D \left(-\frac{\partial f}{\partial y} \right) dx \, dy = -\int_a^b f(x, q(x)) \, dx + \int_a^b f(x, p(x)) \, dx = \int_{\partial D} f \, dx .$$

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2.6 Green's Theorem
(Part 3)

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Example. Use Green's Theorem to calculate $\int_\Gamma \begin{pmatrix} x \\ yx \end{pmatrix} \cdot d\underline{r}$ where Γ is the unit circle.

F_1

F_2

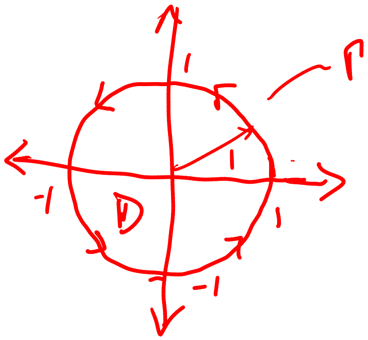
$$\int_\Gamma \begin{pmatrix} x \\ yx \end{pmatrix} \cdot d\underline{r} = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dx\, dy$$

$$= \iint_D (y - 0)\, dx\, dy$$

$$= \int_{-1}^{1} \left(\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y\, dx \right) dy$$

$$= \int_{-1}^{1} xy \Big|_{x=-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} dy = \int_{-1}^{1} 2y\sqrt{1-y^2}\, dy$$

$$= \left[\frac{2}{3}(1-y^2)^{3/2} \right]_{-1}^{1} = 0 - 0 = 0.$$



$D: x^2 + y^2 \le 1$

$x^2 \le 1 - y^2$

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2.6 Green's Theorem
(Part 4)

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Example. If D is a positive oriented region of area A in $\mathbb{R}^2$. Show that

$$\underbrace{A = \int_{\partial D} x\, dy}_{} = \underbrace{- \int_{\partial D} y\, dx}_{A =} = \underbrace{\frac{1}{2} \int_{\partial D} (-y, x) \cdot d\underline{r}}_{A =}.$$

$$A = \iint_D 1\, dx\, dy = \int_{\partial D} \underline{F}(x,y) \cdot d\underline{r}$$

$\underline{F}(x,y)$: $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$

1) $F_2 = x \quad F_1 = 0$

2) $F_2 = 0 \quad F_1 = -y$

3) $F_2 = \frac{1}{2}x \quad F_1 = -\frac{1}{2}y$

$$= \int_{\partial D} \begin{pmatrix} 0 \\ x \end{pmatrix} \cdot d\underline{r} = \int_{\partial D} x\, dy \qquad \text{(Option 1)}$$

$$= \int_{\partial D} \begin{pmatrix} -y \\ 0 \end{pmatrix} \cdot d\underline{r} = \int_{\partial D} (-y)\, dx \qquad \text{(Option 2)}$$

$$= \int_{\partial D} \begin{pmatrix} -\frac{1}{2}y \\ \frac{1}{2}x \end{pmatrix} \cdot d\underline{r} = \int_{\partial D} \frac{1}{2}(-y, x) \cdot d\underline{r} \qquad \text{(Option 3)}$$

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2.6 Green's Theorem
(Part 5)

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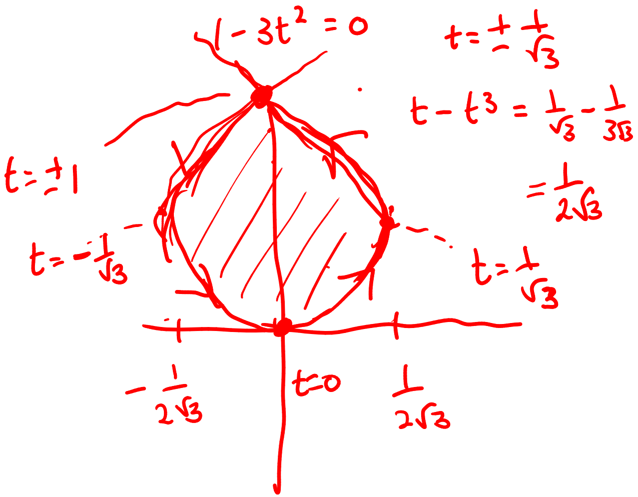
Example. Calculate the area bounded by the curve $\underline{r}(t) = \begin{pmatrix} t - t^3 \\ 1 + t^2 \end{pmatrix}$.



$$\begin{pmatrix} t_1 - t_1^3 \\ 1 + t_1^2 \end{pmatrix} = \begin{pmatrix} t_2 - t_2^3 \\ 1 + t_2^2 \end{pmatrix}$$

$1 + t_1^2 = 1 + t_2^2 \Rightarrow t_1 = \pm t_2$ (take $t_1 \neq t_2$)

i.e $t_1 = -t_2$

$t_1 - t_1^3 = t_2 - t_2^3 \Rightarrow t_1 - t_1^3 = -t_1 + t_1^3$

$\Rightarrow 2(t_1 - t_1^3) = 0$

$\Rightarrow 2t_1(1 - t_1^2) = 0 \Rightarrow \cancel{t_1 = 0}$, $t_1 = 1$ or $t_1 = -1$

$t_1 \neq t_2$

$A = \iint_D 1\, dx\, dy = \frac{1}{2} \int_{\partial D} \begin{pmatrix} -y \\ x \end{pmatrix} d\underline{r}$ (see previous example)

$$A = \iint_D 1 \, dx\,dy = \frac{1}{2}\int_{\partial D} \begin{pmatrix} -y \\ x \end{pmatrix} d\underline{r} \qquad \underline{r}(t) = \begin{pmatrix} t - t^3 \\ 1 + t^2 \end{pmatrix}$$

$$= \frac{1}{2}\int_{-1}^{1} \begin{pmatrix} -(1+t^2) \\ t - t^3 \end{pmatrix} \cdot \begin{pmatrix} 1 - 3t^2 \\ 2t \end{pmatrix} dt$$

$$= \frac{1}{2}\int_{-1}^{1} (3t^2 - 1)(1+t^2) + 2t(t - t^3) \, dt$$

$$= \frac{1}{2}\int_{-1}^{1} t^4 + 4t^2 - 1 \, dt = \frac{1}{2}\left[\frac{t^5}{5} + \frac{4}{3}t^3 - t \right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{2}{5} + \frac{8}{3} - 2 \right] = \frac{1}{5} + \frac{4}{3} - 1 = \frac{8}{15}.$$