UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG
School of Computer Science and Applied Mathematics

APPM2023
Mechanics II
2023

Assignment-03

State Machines and Phase-Space

Issued: 15 August 2023 **Total:** 46 points

Due: 17:00, 23 August 2023

Instructions

- Read all the instructions and questions carefully.
- Typeset the solution document using 'Assignment.cls' LaTeX document template. Submissions that have not used this template shall receive a zero grade.
- Use plain written English where necessary.
- Students are free to use whatever resources at their disposal to answer this assignment, including Computer Algebra and Graphing software. However, all necessary calculation steps and details should be include to obtain full credit.
- Students may use the `Mathematica` and LaTeX supplementary resources posted on the course Moodle page to complete this assignment. In particular, students should consult the example script files on the course Moodle site to help answer this assignment.
- Students are encouraged to work in groups. However, this is to be individual work and each student must submit their own report.
- Plagiarised submissions shall receive a zero grade.
- No late submissions shall be considered.
- Do not submit any `Mathematica` code for this assignment.

Introduction

The objective of this assignment is to develop an understanding of Classical Mechanics as a state machine by conducting a numerical study of Newton's Laws of motion. Students shall perform computations by hand and use the computer algebra system `Mathematica` to verify their manual computations, perform numerical evaluations of the resulting equations, and construct graphical representations of the solutions.

Consider a particle of mass m moving freely with position $\vec{p}$ on the surface of the unit sphere S^2. Suppose the surface of S^2 in $\mathbb{R}^3$ is parameterize by

$$\vec{p}(\theta,\phi) = \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}.$$

The motion of the particle on the surface is given by a pair of functions $\theta(t)$ and $\phi(t)$, each parameterized by a time parameter t. Note that we can also write this as

$$\vec{p}(\theta,\phi) = \sin(\theta)\cos(\phi)\,\hat{x} + \sin(\theta)\sin(\phi)\,\hat{y} + \cos(\theta)\,\hat{z}$$

where θ and ϕ are implicitly functions of t. To help setup the coordinate system in the calculations that will follow, consider the following `Mathematica` code to define a set of coordinate unit vectors

```
xhat = {1, 0, 0};
yhat = {0, 1, 0};
zhat = {0, 0, 1};
```

and then use the following `Mathematica` code to define $\vec{p}(t)$,

```
p[t_] :=  Sin[\[Theta][t]] Cos[\[Phi][t]] xhat
        + Sin[\[Theta][t]] Sin[\[Phi][t]] yhat
        + Cos[\[Theta][t]] zhat ;
```

We construct a parametric path on S^2 by substituting the parametric functions $\theta(t)$ and $\phi(t)$ into $\vec{p}$ in place of the free parameters θ and ϕ, respectively. The velocity of a particle moving along this parametric curve is then simply

$$\vec{v}(t) = \left(\frac{\mathrm{d}\vec{p}}{\mathrm{d}t}\right).$$

Use the following `Mathematica` code to compute $\vec{v}(t)$

```
v[t_] := D[p[s], s] /. {s -> t};
```

where `D` computes the derivative with respect to the parameter s and we then parameter substitute s with t to evaluate the velocity function at the point t. To aid in parameter substitution, define a parameter substitution list

```
PARAMS = {m -> 1, g -> 9.81, ...};
```

where m is the mass of the particle and g is the gravitational acceleration and the ... should be replaced with any remaining elements in the parameter list. Students may find the functions Dot and FullSimplify useful, and are encouraged to consult the Mathematica documentation for more information on these and other functions.

We can use Mathematica to solve coupled sets of differential equations. Suppose we have a pair of coupled differential equations for the dependent variables θ and ϕ that are each functions of an independent variable t. We use Mathematica to solve these differential equations numerically by first compiling a list of dependent coordinates as

```
COORDS = {\[Theta], \[Phi]};
```

and then specifying the corresponding differential equations in the form

```
EqnTheta[t_] := ... == 0; (* first equation *)
  EqnPhi[t_] := ... == 0; (* second equation *)
```

where ... should be replaced with the appropriate mathematical expressions for the dependent variables \[Theta][t] and \[Phi][t] and independent variable t. These equations should be collected in a list

```
EQNS = {EqnTheta[t], EqnPhi[t]}; (* keep list in order
   above *)
```

Initial conditions for the numerical solver can be placed in a list

```
INITS = {
    \[Theta][0] == ...,
    \[Theta]'[0] == ...,
    \[Phi][0] == ...,
    \[Phi]'[0] == ...
    };
```

corresponding to the inital values of $\theta(0)$, $\phi(0)$ and so on, where ... should be replaced with the appropriate initial values. We then solve the equations numerically using Mathematica and the following code,

```
Soln[eqns_, coords_, inits_, params_, tf_] := Module[{vars},
    vars = NDSolveValue[
        Flatten[
            {
            eqns //. params, (* parameter subsitition here
              *)
            inits
            }
            ],
        coords,
        {t, 0, tf}];
```

```
Return[vars];
];
```

where NDSolveValue is the numerical solver and the prime notation corresponds to differentiation with respect to the independent variable t. The numerical solver will comput the values of the dependent varaibles with respect to the independent variable t ranging from 0 to T. The system can now be solved for the given choice of initial conditions using

```
solved = Soln[EQNS, COORDS, INITS, PARAMS, T];
```

The output from the solver is in the form of a tuple that needs separating. We do this using

```
thetaSol[t_] := solved[[1]][t]; (* first component *)
  phiSol[t_] := solved[[2]][t]; (* second component *)
```

The functions thetaSol[t] and phiSol[t] can now be used as regular functions in subsequent code, meaning that they can be plotted on the interval [0, T]. Similarly, the numerically computed first derivatives of these functions are evaluated using thetaSol'[t] and phiSol'[t] at any point on the interval [0, T].

Students should also consult the example example script files on the course Moodle site to help answer these questions. You may use Mathematica to check your computations in the questions to follow.

Question 1 (5 Points)

Show that Newton's Second Law gives rise to a deterministic state machine. Argue that this state-machine is also reversible.

Question 2 (5 Points)

Show that a particle moving with constant motion in the Cartesian plane with position $(x(t), y(t))$ will move a long the line

$$y(x) = mx + c.$$

Your solution must not rely on computing $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$. No credit shall be awarded to solutions that compute $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$.

Question 3 (6 Points)

Draw the state/phase space of a particle that moves with the parametric motion

$$x(t) = -e^{-at^2}t.$$

where $1 < a$ is a positive constant. Use `Mathematica` to generate this plot. Include all necessary labels. Add your student number as text to the background of each plot graph. Describe the long term ($t \to \infty$ limit) behaviour of the system.

Question 4 (30 Points)

Consider the spherical coordinate embedding

$$\begin{aligned}
x(\rho, \theta, \phi) &= \rho \sin(\theta)\cos(\phi) \\
y(\rho, \theta, \phi) &= \rho \sin(\theta)\sin(\phi) \\
z(\rho, \theta, \phi) &= \rho \cos(\theta)
\end{aligned}$$

where ρ is the radial distance from the origin, θ is the polar angle measured from the positive z-axis and ϕ is the azimuth measured in the $x-y$-plane starting on the positive x-axis, and define the position vector

$$\vec{p}(\rho, \theta, \phi) = x(\rho, \theta, \phi)\hat{x} + y(\rho, \theta, \phi)\hat{y} + z(\rho, \theta, \phi)\hat{z}.$$

Suppose we construct a rigid spherical pendulum comprising a small bead that is free to slide along the surface of a sphere of fixed radius ρ in this coordinate system and is subject only to gravity that is directed vertically in the negative z-direction. Answer the following questions.

1. Invert the embedding functions by computing the expressions for $\rho(x, y, z)$, $\theta(x, y, z)$ and $\phi(x, y, z)$. Show all working. (3 points)

2. Show that the embedding functions $x(\rho,\theta,\phi)$, $y(\rho,\theta,\phi)$ and $z(\rho,\theta,\phi)$ are not invertable at some ρ, θ and ϕ values. Explain why these functions are not invertable at these points. (3 points)

3. Show that the motion a bead on the sphere is subject to a holonomic constraint in the $x-y-z$-coordinates. Then show that, subject to this holonomic constraint, fixing any two coordinates in (x,y,z) fixes that value of the third, but this is not true in the case of the coordinates (ρ,θ,ϕ). (3 points)

4. Show that

$$\ddot{\vec{p}} = \left(\ddot{\rho} - \rho\dot{\theta}^2 - \rho\sin^2(\theta)\dot{\phi}^2\right)\hat{\rho} + \left(\rho\ddot{\theta} + 2\dot{\theta}\dot{\rho} - \rho\sin(\theta)\cos(\theta)\dot{\phi}^2\right)\hat{\theta} \\ + \left(\rho\sin(\theta)\ddot{\phi} + 2\rho\cos(\theta)\dot{\theta}\dot{\phi} + 2\sin(\theta)\dot{\rho}\dot{\phi}\right)\hat{\phi}.$$

Show all working. (Hint: compute $\sin(\theta)\hat{\rho} + \cos(\theta)\hat{\theta}$.) (6 points)

5. Compute the generalized components of force for the bead in the (ρ,θ,ϕ)-coordinate system and determine the equations of motion of for the bead. (7 points)

6. Solve the equations of motion of the bead using moving on a unit sphere `mathematica`, subject to the initial conditions

$$\theta(0) = \frac{\pi}{6}, \quad \dot{\theta}(0) = -\frac{1}{2}, \quad \phi(0) = 0 \quad \text{and} \quad \dot{\phi}(0) = \frac{3}{2}$$

and generate the following correctly labeled plots, modulo 2π,

(a) the phase-portrait for $\theta(t)$. (4 points)

(b) the phase portrait for $\phi(t)$. (4 points)

Add your student number as text to the background of each plot graph.