Tutorial 2.2.1.

1. Prove Theorem 2.9.

Consider $k \in \mathbb{R}$ and sequences with the following properties as $n \to \infty$:
$a_n \to \infty$, $\quad b_n \to \infty$, $\quad c_n \to c \in \mathbb{R}$, $d_n \to -\infty$.
Then as $n \to \infty$,

(a) $ka_n = \begin{cases} \infty & \text{if} \quad k > 0, \\ -\infty & \text{if} \quad k < 0, \\ 0 & \text{if} \quad k = 0. \end{cases}$

(b) $a_n + b_n \to \infty$,

(c) $a_n + c_n \to \infty$,

(d) $-d_n \to \infty$,

(e) $a_n c_n \to \begin{cases} \infty & \text{if} \quad c > 0, \\ -\infty & \text{if} \quad c < 0. \end{cases}$

(f) $a_n b_n \to \infty$.

Proof.

(a) If $k = 0$, then $ka_n = 0$ for all $n \in \mathbb{N}$, and therefore $ka_n \to 0$ as $n \to \infty$ by Theorem 2.3(a).
Now let $k \neq 0$. Let $A \in \mathbb{R}$. Since $a_n \to \infty$ as $n \to \infty$, there is $K \in \mathbb{R}$ such that $a_n > \dfrac{A}{k}$ for $n > K$.
For $k > 0$, we conclude $ka_n > A$ for $n > K$, and therefore $ka_n \to \infty$ as $n \to \infty$.
For $k < 0$, we conclude $ka_n < A$ for $n > K$, and therefore $ka_n \to -\infty$ as $n \to \infty$.

(b) Let $A \in \mathbb{R}$. There are $K_1, K_2 \in \mathbb{R}$ such that $a_n > A$ for $n \geq K_1$ and $b_n > 0$ for $n \geq K_2$.
For $n \geq K = \max\{K_1, K_2\}$, we conclude $a_n + b_n > A + 0 = A$. Therefore $a_n + b_n \to \infty$ as $n \to \infty$.

(c) Let $A \in \mathbb{R}$. There are $K_1, K_2 \in \mathbb{R}$ such that $a_n > A - c + 1$ for $n \geq K_1$ and $c_n > c - 1$ for $n \geq K_2$.
For $n \geq K = \max\{K_1, K_2\}$, we conclude $a_n + c_n > A - c + 1 + c - 1 = A$. Therefore $a_n + c_n \to \infty$ as $n \to \infty$.

(d) is similar to (a) with $k = -1$.

(e) First let $c > 0$. Let $A \in \mathbb{R}$. There are $K_1, K_2 \in \mathbb{R}$ such that $a_n > \dfrac{2A}{c}$ and $c_n > \dfrac{c}{2}$ for $n \geq K_2$.
For $n \geq K = \max\{K_1, K_2\}$, we conclude $a_n c_n = \dfrac{2A}{c}\dfrac{c}{2} = A$. Therefore $a_n c_n \to \infty$ as $n \to \infty$.
If $c < 0$, then $a_n(-c_n) \to \infty$ as $n \to \infty$ since $-c > 0$, and then $a_n c_n \to -\infty$ by part (a).

(f) Let $A \in \mathbb{R}$. There are $K_1, K_2 \in \mathbb{R}$ such that $a_n > A$ for $n \geq K_1$ and $b_n > 1$ for $n \geq K_2$.
For $n \geq K = \max\{K_1, K_2\}$, we conclude $a_n b_n > A \cdot 1 = A$. Therefore $a_n b_n \to \infty$ as $n \to \infty$.

2. Use suitable rules or first principles to find
(a) $\displaystyle\lim_{n\to\infty} \left(n^2 + 2n - 10\right)$ $\quad$ (b) $\displaystyle\lim_{n\to\infty} \left(n - \frac{1}{n}\right)$ $\quad$ (c) $\displaystyle\lim_{n\to\infty} \frac{n^3 - 3n^2}{n+1}$

Solution. We use the trivial statement that $n \to \infty$ as $n \to \infty$.

(a) Since $n^2 + 2n - 10 = n\left(n + 2 + \dfrac{1}{n}\right)$, it follows from Theorems 2.3, 2.6 and 2.9 that $\displaystyle\lim_{n\to\infty} \left(n^2 + 2n - 10\right) = \infty$.

(b) In view of Theorems 2.6 and 2.9, $\lim_{n\to\infty}\left(n-\frac{1}{n}\right)=\infty$.

(c) Since

$$\frac{n^3-3n^2}{n+1}=n^2\frac{n-3}{n+1}=n^2\left(1-\frac{4}{n+1}\right),$$

it follows from Theorems 2.3, 2.6 and 2.9 that $\lim_{n\to\infty}\frac{n^3-3n^2}{n+1}=\infty$.

3. Prove that if $\lim_{n\to\infty}|a_n|=\infty$, then (a_n) diverges.

Proof. Assume that (a_n) converges. By Theorem 2.7, (a_n) is bounded, i.e. there are $m, M\in\mathbb{R}$ such that

$$m\le a_n\le M \text{ for all } n\in\mathbb{N},$$

and hence $-a_n\le -m$. It follows that

$$|a_n|\le \max\{M,-m\}:=L \text{ for all } n\in\mathbb{N},$$

which contradicts $\lim_{n\to\infty}|a_n|=\infty$, since the latter implies that there are $n\in\mathbb{N}$ with $|a_n|>L$.

4. Prove that if $p\in\mathbb{N}$, $p>0$, then $n^p\to\infty$ as $n\to\infty$.

Proof. The result follows from Theorems 2.6 and 2.8.

5. Define a sequence as follows:

$$a_0=0,\ a_1=\frac{1}{2},\ a_{n+1}=\frac{1}{3}\left(1+a_n+a_{n-1}^3\right)\quad\text{for } n\ge 2\,.$$

(a) Use induction to show that $0\le a_n\le\frac{2}{3}$ for all $n\in\mathbb{N}$.
(b) Use induction to show that $a_n\le a_{n+1}$ for all $n\in\mathbb{N}$.
(c) Explain why we may conclude that (a_n) converges.
(d) Using the fact that $\lim_{n\to\infty}a_n=\lim_{n\to\infty}a_{n-1}=\lim_{n\to\infty}a_{n+1}$, find $\lim_{n\to\infty}a_n$.

Solution.
(a) The statement is true for $n=0$ and $n=1$ since $a_0=0$ and $a_1=\frac{1}{2}$.
Now let $n\ge 1$ and assume that the estimate is true for all $k\in\mathbb{N}$ with $k\le n$. Clearly, $0\le a_{n+1}$.
Furthermore,

$$\begin{aligned} a_{n+1} &= \frac{1}{3}\left(1+a_n+a_{n-1}^3\right)\\ &\le \frac{1}{3}\left(1+\frac{2}{3}+\frac{8}{27}\right)\\ &\le \frac{1}{3}\frac{53}{27}<\frac{1}{3}\frac{54}{27}=\frac{2}{3}. \end{aligned}$$

(b) A straight forward calculation gives

$$a_2=\frac{1}{3}\left(1+a_1+a_0^3\right)=\frac{1}{2}.$$

Hence, we have $a_0\le a_1\le a_2$.
Now let $n\ge 2$ and assume that $a_k\le a_{k+1}$ for $k\in\mathbb{N}$ such that $k<n$. Then

$$a_n=\frac{1}{3}\left(1+a_{n-1}+a_{n-2}^3\right)\le a_{n+1}=\frac{1}{3}\left(1+a_n+a_{n-1}^3\right)=a_{n+1}.$$

(c) In (a), we have seeen that (a_n) is bounded, and in (b), we have seen that the sequence is increasing. By Theorem 2.10 (1), the sequence converges.
(d) Let $a=\lim_{n\to\infty}a_n$. Taking limits as $n\to\infty$ in $a_{n+1}=\frac{1}{3}\left(1+a_n+a_{n-1}^3\right)$ we arrive at

$$a=\frac{1}{3}\left(1+a+a^3\right),$$

which can be written as

$$a^3 - 2a + 1 = 0.$$

Hence a is a zero of the polynomial p is given by

$$p(x) = x^3 - 2x + 1.$$

We have $p(0) = 1 > 0$ and $p(1) = 0$, so that p has one zero at 1 and one zero on the negative real axis. Hence a must be the remaining third zero of p, i.e. a is the unique zero of p in the interval $(0, 1)$.

6. Let $\lim\limits_{n\to\infty} a_n = \infty$, $\lim\limits_{n\to\infty} b_n = \infty$, $\lim\limits_{n\to\infty} c_n = 0$. Show, by giving examples, that no general conclusion can be made about the behaviour of the following sequences:
(a) $a_n - b_n$, (b) $a_n c_n$, (c) $\dfrac{a_n}{b_n}$, (d) $\dfrac{a_n}{c_n}$.

Solution.
(a) We show that any sequence can be obtained in this way. So let (d_n) be a sequence and define

$$a_n = n + |\,d_n\,| + d_n, \qquad b_n = n + |\,d_n\,|.$$

Then $a_n \geq n$ and $b_n \geq n$ show that both $\lim\limits_{n\to\infty} a_n = \infty$ and $\lim\limits_{n\to\infty} b_n = \infty$, whereas $a_n - b_n = d_n$.
(b) Again, let (d_n) be sequence and define now

$$a_n = n(1 + |\,d_n|), \qquad c_n = \frac{d_n}{n(1 + |\,d_n|)}.$$

Then $a_n \geq n$, so that $\lim\limits_{n\to\infty} a_n = \infty$, $|\,c_n\,| \leq \dfrac{1}{n}$, so that $\lim\limits_{n\to\infty} c_n = 0$, and $a_n c_n = d_n$.
(c) Here let (d_n) be sequence with $d_n > 0$ for all $n \in, \mathbb{N}$ and define

$$a_n = \begin{cases} n & \text{if} \quad d_n \leq 1, \\ n d_n & \text{if} \quad d_n > 1, \end{cases} \qquad b_n = \begin{cases} \dfrac{n}{d_n} & \text{if} \quad d_n \leq 1, \\ n & \text{if} \quad d_n > 1. \end{cases}$$

The $a_n, b_n \geq n$, so that $\lim\limits_{n\to\infty} a_n = \infty$ and $\lim\limits_{n\to\infty} b_n = \infty$. Furthermore, $\dfrac{a_n}{b_n} = d_n$.
(d) Here, of course, $c_n \neq 0$ for all $n \in \mathbb{N}$. This case is different since we can conclude that

$$\left|\frac{a_n}{c_n}\right| \to \infty \text{ as } n \to \infty.$$

We can therefore make the following statements:
(i) If there are atmost finitely many $n \in \mathbb{N}$ such that $c_n < 0$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{c_n} = \infty$.
(ii) If there are atmost finitely many $n \in \mathbb{N}$ such that $c_n > 0$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{c_n} = -\infty$.
(iii) If there are infinitely many $n \in \mathbb{N}$ such that $c_n < 0$ and infinitely many $n \in \mathbb{N}$ such that $c_n > 0$, then $\left(\dfrac{a_n}{c_n}\right)$ is neither bounded above nor bounded below.

7. Let (a_n) and (b_n) be sequences such that $a_n \leq b_n$ for all $n \in \mathbb{N}$. Show that

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n \text{ and } \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} b_n\,.$$

Proof. We are proving the statement for lim inf. The proof for lim sup is similar. If (a_n) is not bounded below, then $\liminf\limits_{n\to\infty} a_n = -\infty$, and nothing has to be proved.
Now let (a_n) be bounded below and define $\alpha_n = \inf\{a_k \,:\, k \geq n\}$, $n \in \mathbb{N}$. Since $a_k \leq b_k$ for all $k \in \mathbb{N}$, $\alpha_n \leq a_k \leq b_k$ for $k \geq n$, which shows that α_n is a lower bound of $\{b_k \,:\, k \geq n\}$. Hence (b_k) is bounded below and $\beta_n := \inf\{b_k \,:\, k \geq n\} \geq \alpha_n$.
If $\liminf\limits_{n\to\infty} b_n = \infty$, nothing has to be shown. So let $\beta = \liminf\limits_{n\to\infty} b_n \in \mathbb{R}$. Then

$$\alpha_n \leq \beta_n \leq \beta,$$

and hence (why?)

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \alpha_n \le \beta = \liminf_{n\to\infty} b_n.$$

8. (a) Show that $\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$ exists for all $x \in \mathbb{R}$ and that $\exp(1) = e$.
Hint. Adapt the proof of Example 2.2.3.
(b) Use Bernoulli's inequality to prove that

$$\lim_{n\to\infty}\left(\frac{1+\frac{x+y}{n}}{1+\frac{x+y}{n}+\frac{xy}{n^2}}\right)^n = 1$$

for all $x, y \in \mathbb{R}$.
(c) Use (b) to show that $\exp(x+y) = \exp(x)\exp(y)$ for all $x, y \in \mathbb{R}$.
(d) Show that $\exp(x) \ge 1 + x$ for all $x > 0$.
(e) Show that $\exp(x) > 0$ for all $x \in \mathbb{R}$.
(f) Show that exp is strictly increasing.
(g) Show that $\exp(n) = e^n$ for all $n \in \mathbb{Z}$.

Proof.

(a) $\exp(1) = e$ follows from Example 2.2.3. Also, $\exp(0) = \lim_{n\to\infty} 1 = 1$.
Now let $x \in \mathbb{R}$ and put $a_n = \left(1+\frac{x}{n}\right)^n$. For the limit, we may assume that $n \ge n_0$, where $n_0 > -x$. Hence $n + x > 0$ and $x \le n + x$, which gives

$$-\frac{x}{(n+1)(n+x)} > -1.$$

Using Bernoulli's inequality, we calculate

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\left(\frac{n+1+x}{n+1}\right)^{n+1}}{\left(\frac{n+x}{n}\right)^n} = \left(\frac{n(n+1+x)}{(n+1)(n+x)}\right)^{n+1}\frac{n+x}{n} \\
&= \left(1-\frac{1}{(n+1)(n+x)}\right)^{n+1}\frac{n+x}{n} \\
&\ge \left(1-\frac{x}{n+x}\right)\frac{n+x}{n} = \frac{n}{n+x}\,\frac{n+x}{n} = 1.
\end{aligned}$$

By the Archimedean principle, we can choose $k \in \mathbb{N}\setminus\{0\}$ such that $x \le k$. For $n \ge n_0$, we have

$$0 < 1+\frac{x}{n} \le 1+\frac{k}{n} \le \left(1+\frac{1}{n}\right)^k .$$

Therefore,

$$\left(1+\frac{x}{n}\right)^n < \left(\left(1+\frac{1}{n}\right)^n\right)^k < e^k,$$

where the inequality follows from Example 2.2.3. Hence (a_n) is bounded, and it follows by Theorem 2.10 that (a_n) converges. $\square$

(b) For sufficiently large n, here $n > -2(x+y)$ such that $n^2 > -2xy$, we have

$$\left(\frac{1+\frac{x+y}{n}}{1+\frac{x+y}{n}+\frac{xy}{n^2}}\right)^n = \left(1-\frac{\frac{xy}{n^2}}{1+\frac{x+y}{n}+\frac{xy}{n^2}}\right)^n \ge 1-\frac{\frac{xy}{n^2}}{1+\frac{x+y}{n}+\frac{xy}{n^2}},$$

and the right hand side tends to 1 as $n \to \infty$. Hence

$$\liminf_{n\to\infty}\left(\frac{1+\frac{x+y}{n}}{1+\frac{x+y}{n}+\frac{xy}{n^2}}\right)^n \ge 1.$$

Similarly,

$$\left(\frac{1+\frac{x+y}{n}}{1+\frac{x+y}{n}+\frac{xy}{n^2}}\right)^n = \left(1+\frac{\frac{xy}{n^2}}{1+\frac{x+y}{n}+\frac{xy}{n^2}}\right)^n \geq 1+\frac{\frac{xy}{n}}{1+\frac{x+y}{n}},$$

and the right hand side tends to 1 as $n \to \infty$. Hence

$$\liminf_{n\to\infty}\left(\frac{1+\frac{x+y}{n}}{1+\frac{x+y}{n}+\frac{xy}{n^2}}\right)^n \geq 1.$$

which implies

$$\limsup_{n\to\infty}\left(\frac{1+\frac{x+y}{n}}{1+\frac{x+y}{n}+\frac{xy}{n^2}}\right)^n = \frac{1}{\liminf_{n\to\infty}\left(\frac{1+\frac{x+y}{n}}{1+\frac{x+y}{n}+\frac{xy}{n^2}}\right)^n} \leq 1.$$

Hence lim sup and lim inf of the sequence in question are both 1, and the sequence converges to 1 by Theorem 2.11. $\square$

(c) Putting $a_n(x) = \left(1+\dfrac{x}{n}\right)^n$, the sequence considered in (b) is $\left(\dfrac{a_n(x+y)}{a_n(x)a_n(y)}\right)$. Therefore

$$\exp(x+y) = \lim_{n\to\infty} a_n(x+y) = \lim_{n\to\infty}\frac{a_n(x+y)}{a_n(x)a_n(y)}\lim_{n\to\infty} a_n(x)\lim_{n\to\infty} a_n(y) = (1)\exp(x)\exp(y).$$

$\square$

(d) With the notation from (c) and the Bernoulli's inequality, we have

$$a_n(x) = \left(1+\frac{x}{n}\right)^n \geq 1+x.$$

Now take the limit . $\square$

(e) Since $a_n(x) > 0$ for sufficiently large n, we have $\exp(x) \geq 0$. Also, $a_n(0) = 1$ shows $\exp(0) = 1$. From (c), we have $\exp(x)\exp(-x) = \exp(0) = 1$, so that $\exp(x) \neq 0$. Hence, $\exp(x) > 0$. $\square$

(f) Let $x, y \in \mathbb{R}$ with $x < y$. In view of $y - x > 0$, it follows from (c) and (d) that

$$\exp(x) < (1+(y-x))\exp(x) \leq \exp(y-x)\exp(x) = \exp(y).$$

$\square$

(g) We already know this for $n = 0$ and $n = 1$. For integers $n \geq 2$ the statement follows by induction on n and by (c), and for negative integers it then follows immediately from (c) since

$$\exp(-x) = \frac{\exp(-x)\exp(x)}{\exp(x)} = \frac{\exp(0)}{\exp(x)} = \frac{1}{\exp(x)}.$$

$\square$