14:00 hrs	07 /6/2007	EX. HALL	Exams Office Use Only

University of the Witwatersrand, Johannesburg

Course or topic No(s)	MATH2001
Course or topic name(s) Paper Number & title	BASIC ANALYSIS
Examination/Test* to be held during month(s) of (*delete as applicable)	June 2007
Year of Study (Art & Sciences leave blank)	
Degrees/Diplomas for which this course is prescribed (BSc (Eng) should indicate which branch)	BSc, BCom, BA
Faculty/ies presenting candidates	
Internal examiners and telephone number(s)	Mr. A. Blecher - Ext. 76202 Prof. C.C. Labuschagne - Ext. 76215
External examiner(s)	Dr. F.E.S. Bullock (UNISA)
Calculator policy	Calculators allowed
Time allowance	Course Nos: MATH2001; Hours: 1.5 Hours
Instructions to candidates (Examiners may wish to use this space to indicate, inter alia, the contribution made by this examination or test towards the year mark, if appropriate)	Answer section A on the computer card provided and Section B in the exam book provided. Total : 90

Internal Examiners or Heads of Department are requested to sign the declaration overleaf

Section A Multiple Choice

Answer the multiple choice questions on the computer sheet provided.
There is only **one correct answer** to each question.
Each question is worth 2 marks.

Question 1

Let $|x+2| < \max\{2, \varepsilon\}$. Then:

(A) $\max\{2, \varepsilon\} < 2$

(B) $\max\{2, \varepsilon\} < \varepsilon$

(C) $|x+2| < 2$ or $|x+2| < \varepsilon$

(D) $|x+2| < 2$ and $|x+2| < \varepsilon$

(E) None of the above are true.

Question 2

Let $\delta = \min\{1, \frac{\varepsilon}{2}\}$. Then:

(A) $|x+3| < 1 \Rightarrow |x+3| < \delta$

(B) $|x+3| < 1 \Rightarrow |x+3| < \frac{\varepsilon}{2}$

(C) $|x+3| < \delta \Rightarrow |x+4| < \frac{\varepsilon}{2}$

(D) $|x+3| < \delta \Rightarrow |x+4| < 2$

(E) None of the above are true.

Question 3

Let $\varepsilon > 0$, $\delta_1 > 0$, $\delta_2 > 0$ and $f(x), g(x), L$ and M be such that

$$\begin{aligned} & 0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \tfrac{\varepsilon}{2} \\ \text{and} \quad & 0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \tfrac{\varepsilon}{2}. \end{aligned}$$

Then:

(A) $0 < |x-a| < \max\{\delta_1, \delta_2\} \Rightarrow |f(x) - L| + |g(x) - M| < \varepsilon$

(B) $0 < |x-a| < \max\{\delta_1, \delta_2\} \Rightarrow |f(x) + g(x) - (L+M)| < 2\varepsilon$

(C) $0 < |x-a| < \min\{\delta_1, \delta_2\} \Rightarrow \frac{|f(x)-L|}{|g(x)-M|} < 1$

(D) $0 < |x-a| < \min\{\delta_1, \delta_2\} \Rightarrow |f(x) + g(x) - (L+M)| < \varepsilon$

(E) None of the above are true.

Question 4

The series $\sum(-1)^n \dfrac{n}{n+3}$ can be shown to

(A) converge, by the alternating series test.

(B) diverge, by the alternating series test or by the divergence test.

(C) converge, by the ratio test.

(D) diverge, by the ratio test.

(E) None of the above.

Question 5

Find the radius of convergence of the series $\sum \dfrac{2^n\, n^3}{3^n\,(n^2+1)}\, x^n$.

(A) ∞

(B) 0

(C) 1

(D) $\frac{2}{3}$

(E) $\frac{3}{2}$

EACH OF THE REMAINING QUESTIONS IN THIS SECTION PERTAINS TO A RELEVANT PART OF THE PROOF OF THE FOLLOWING THEOREM:

Theorem. Let f is a real function which is continuous on the closed interval $[a, b]$ and $f(a) < f(b)$. If $f(a) < 0 < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = 0$.

Proof. We must prove that there exists $c \in (a, b)$ such that $f(c) = 0$. Define

$$S = \{x \in [a, b] : f(x) < 0\}$$

and let $c = \sup S$. To prove that $f(c) = 0$, we use an argument by contradiction.

Case (i): Assume that $f(c) < 0$. Then $c \neq b$. Thus, $a \leq c < b$. Since $f(x)$ tends to $f(c)$, as x tends to c from the right, it follows that there exists $\delta > 0$ such that

$$x \in [c, c + \delta] \Rightarrow f(c) < 0.$$

In particular, $f(c + \delta) < 0$. Hence, $c + \delta \in S$. This is a contradiction. Hence, it is false that $f(c) < 0$.

Case (ii): Assume that $f(c) > 0$. Then $c \neq a$. Hence $a < c \leq b$. Since $f(x)$ tends to $f(c)$ as x tends to c from the left, there exists $\delta_1 > 0$ such that

$$x \in [c - \delta_1, c] \Rightarrow f(c) > 0.$$

Then there exists $s \in S$ such that $s \in [c - \delta_1, c]$. This leads to a contradiction. Hence, it is false that $f(c) > 0$.

We conclude from Cases (i) and (ii) that $f(x) = 0$, which completes the proof of the theorem. □

AFTER READING THE ABOVE THEOREM AND THE PROOF THEREOF, ANSWER THE FOLLOWING QUESTIONS ABOUT THE PROOF OF THE THEOREM:

Question 6

The set S is not empty, since

(A) f is continuous on the right at a.

(B) f is continuous on the left at b.

(C) there exists $c \in (a, b)$ such that $f(c) = 0$.

(D) $a \in S$.

(E) none of the above are true.

Question 7

The set S is bounded above, since

(A) $b \in S$.

(B) b is an upper bound of S.

(C) f is continuous on the left at b.

(D) $b \in \mathbb{R}$.

(E) none of the above are true.

Question 8

The supremum c of the set S exists, because

(A) $\mathbb{R}$ satisfies the Archimedean property.

(B) $\mathbb{R}$ satisfies the well-ordering property.

(C) $\mathbb{R}$ satisfies the completness property.

(D) $\mathbb{R}$ satisfies the projection property.

(E) none of the above are true.

Question 9

In **Case (i)**, $c \neq b$, because

(A) f is bounded on $[a, b]$.

(B) $f(b) > 0$.

(C) S is not empty.

(D) f is right continuous at c.

(E) none of the above are true.

Question 10

In **Case (i)**, because $c \neq b$,

(A) f is right continuous at c.

(B) f is left continuous at c.

(C) f is not necessarily right continuous at c.

(D) f is continuous at c.

(E) none of the above are true.

Question 11

In **Case (i)**, the fact that $c + \delta \in S$ leads to a contradiction, because

(A) $c + \frac{\delta}{2} \in S$.

(B) S is bounded below by a.

(C) S is not empty.

(D) $c + \delta > c$ and c is the least upper bound of S.

(E) none of the above are true.

Question 12

In **Case (ii)**, $f(x)$ tends to $f(c)$, as x tends to c from the left, because

(A) f is bounded on $[c, b]$.

(B) $f(b) > 0$.

(C) f is left continuous at c

(D) f is right continuous at c.

(E) none of the above are true.

Question 13

In **Case (ii)**, there exists $s \in S$ such that $s \in [c - \delta_1, c]$, because

(A) $c \in S$.

(B) $\delta_1 > 0$ and $c = \sup S$.

(C) $\delta_1 > 0$ and f is continuous on $[a, b]$.

(D) there exists $c \in (a, b)$ such that $f(c) = 0$.

(E) none of the above are true.

Question 14

In **Case (ii)**, the existence of $s \in S$ such that $s \in [c - \delta_1, c]$ leads to a contradiction, because

(A) $f(s) < 0$ and $f(s) > 0$ for such s,

(B) the sets S and $[c - \delta_1, c]$ are non intersecting,

(C) $c = \sup S$,

(D) there exists $c \in (a, b)$ such that $f(c) = 0$,

(E) none of the above are true.

Question 15

We may conclude from Cases (i) and (ii) that $f(c) = 0$, because

(A) exactly one of $r > 0$, $r = 0$, $r < 0$ holds for any real number r,

(B) the proof is complete,

(C) f is continuous on $[a, b]$,

(D) $\mathbb{R}$ is complete,

(E) none of the above are true.

Section A Total marks: [30]

Section B.

Question 1

(a) Use the ε-definition for convergence of a sequence to prove that

$$\lim_{n\to\infty} \frac{n^2+n+1}{2n^2+3} = \frac{1}{2}.$$

(6)

(b) Suppose $(a_n), (b_n)$ and (c_n) are sequence of real numbers such that $a_n \le b_n \le c_n$ for all $n \in \mathbb{N}$. If $L \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} c_n = L$, prove that

$$\lim_{n\to\infty} b_n = L.$$

(6)

[12]

Question 2

(a) Let f be a real function which is defined on $\mathbb{R}$ and $a, L \in \mathbb{R}$. State the definition of $\lim_{x\to a} f(x) = L$. (2)

(b) Show that for all $x \in \mathbb{R}$

$$|x-1| < \frac{1}{4} \Longrightarrow \frac{1}{|2x-1|} < 2.$$

(3)

(c) Use the definition in (a) to show that

$$\lim_{x\to 1} \left(\frac{1-3x}{2x-1}\right) = -2.$$

(The implication in (b) is useful if you use it appropriately in (c).) (7)

[12]

Question 3

(a) Let g be a real function defined for all of $\mathbb{R}$, $a, M \in \mathbb{R}$ and $\lim_{x\to a} g(x) = M \neq 0$. Use an ε, δ-argument to show that $\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{M}$.
(You may assume that if $g(x) \to M$, then there exists δ_1 such that

$$0 < |x-a| < \delta_1 \Rightarrow |g(x)| > \frac{|M|}{2}.)$$

(6)

(b) Show that $f(x) = \sqrt{1-x}$ is continuous on the left at $x = 1$. (3)

(c) Use the intermediate value theorem to show that there exists $x \in (0,1)$ such that $e^x = 3\sin(\frac{\pi x}{2})$.
(You may assume that $\sin(\frac{\pi x}{2})$ is continuous for all x but otherwise give full justification for your answer.) (3)

[12]

Question 4

(a) Let $a_n \geq 0$ for each $n \in \mathbb{R}$ and let $K \in \mathbb{R}$ be such that $s_n = \sum_{i=1}^{n} a_i \leq K$ for all $n \in \mathbb{N}$. Show that (s_n) is a convergent sequence. (4)

(b) You may assume that if $\sum a_n$ is convergent and $\sum b_n$ is convergent, then $\sum[a_n + b_n]$ is convergent. Use this fact to prove that if $\sum x_n$ is convergent and $\sum y_n$ is divergent, then $\sum[x_n + y_n]$ is divergent. (4)

(c) Decide if the series

$$\sum\left[\frac{1}{n} - \frac{1}{3^n}\right]$$

is convergent or divergent. You are required to justify your answer. (4)

[12]

Question 5

(a) In each of the following cases decide if the series converges or diverges. Then prove your answer.

(i) $\sum\left(\frac{1}{n} - \frac{2}{2n+1}\right)$.

(ii) $\sum\left(\frac{1}{n} + \frac{2}{2n+1}\right)$.

(7)

(b) Use the ratio test to test the series

$$\sum \frac{2^{3n}(n!)^2}{(2n)!}$$

for convergence. (5)

[12]

Section B Total marks: [60]

Total marks: [90]