14:00 hrs	02/ 06 /2017	Examinations and Graduation Office Central Block Exams Hall	Exams Office Use Only

University of the Witwatersrand, Johannesburg

Course or topic No(s)	Math 2021
Course or topic name(s) Paper number & title	Multivariable Calculus
Examination/Test* to be held during month(s) of (*delete as applicable)	June 2017
Year of study (Art & Sciences leave blank)	2nd year
Degrees/Diplomas for which this course is prescribed (BSc (Eng) should indicate which branch)	B Sc ; B Com; B A.
Faculty/ies presenting candidates	Science / Commerce/ Arts
Internal examinar(s) and telephone number(s)	Porf S Currie Dr M Archibald
External examiner(s)	
Calculator policy	Calculators NOT allowed

Time allowance

Course Nos	Math 2021	Time	2 hours

Instruction to candidates (Examiners may wish to use this space to indicate, inter alia, the contribution made by this examination or test towards the year mark, if appropriate)	Answer all questions

Internal Examiners or Heads of Department are requested to sign the Declaration overleaf

Question 1: **9 marks**

Let Γ be a portion of the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$ taken clockwise from $(\sqrt{3},\frac{3}{2})$ to $(0,-3)$.

Find $\displaystyle\int_\Gamma \frac{x^2}{\sqrt{13-x^2-y^2}}\,ds.$ (Hint: $13 = 13\cos^2 t + 13\sin^2 t$)

Question 2: **18 marks**

Using the change of variables $u = xy$ and $v = \frac{x}{y}$, evaluate the integral

$$\iint_D (x^2+y^2)\ dxdy,$$

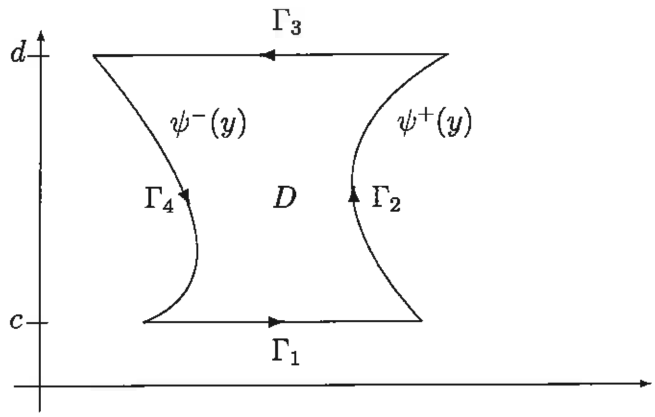
where D is the region in the first quadrant bounded by the curves $x = y$, $y = \frac{x}{2}$, $xy = 1$ and $x = 2$. Sketch the region D and the transformed region D'.

Question 3: **11 marks**

Let D be a region in $\mathbb{R}^2$ of the form $D = \{(x,y)|\,\psi^-(y) \leqslant x \leqslant \psi^+(y), y \in [c,d]\}$ with the boundary $\partial D = \Gamma_1+\Gamma_2+\Gamma_3+\Gamma_4$ (see diagram at the bottom of the page) and let $P : D \to \mathbb{R}$. Prove that

(a) $\displaystyle\iint_D D_1 P\,da, = \int_c^d P(\psi^+(t),t) - P(\psi^-(t),t)\,dt.$

(b) $\displaystyle\int_{\Gamma_1} P\,dy = 0$ and $\displaystyle\int_{\Gamma_2} P\,dy = \int_c^d P(\psi^+(t),t)dt.$

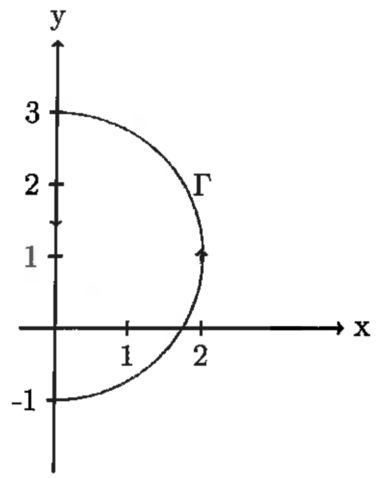


Question 4: **19 marks**

(a) Using Green's Theorem evaluate the path integral

$$\int_\Gamma \begin{pmatrix} xy \\ x^2 + \sin y \end{pmatrix} \cdot d\mathbf{r},$$

where Γ is the semi-circle centred at $(0, 1)$ with radius 2 in the right-half plane as indicated in the diagram below.



(b) **Without** using Green's Theorem calculate $\int_\Gamma \begin{pmatrix} x \\ y \end{pmatrix} \cdot d\mathbf{r}$, where Γ is as given in the diagram above in Question 4(a).

(Hint: remember that you may split Γ into Γ_1 and Γ_2)

Question 5: **10 marks**

Evaluate

$$\iint_S \mathbf{F} \cdot d\mathbf{a},$$

where $\mathbf{F} = (x, y, z)$ and S is the portion of the surface $z = 1 - x^2 - y^2$ lying in the first octant with normal taken so as to have a positive z component.

Question 6: **8 marks**

Let S be the portion of the cylinder $x^2 + y^2 = 4$ lying between $z = 0$ and $z = 3$. Use Stoke's Theorem to evaluate

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{a}$$

where $\mathbf{F} = \begin{pmatrix} (z-2)y \\ x(2-z) \\ \frac{z}{3} \end{pmatrix}$. Here S is oriented having normal away from the z-axis.

Question 7: **9 marks**

Let $\mathbf{F}$ be a vector field on a domain D. Prove that the following are equivalent:

(i) $\int_\Gamma \mathbf{F} \cdot d\gamma = 0$ for every closed path Γ in D;

(ii) $\mathbf{F}$ is a conservative vector field.

Question 8: **15 marks**

Prove that if $\mathbf{F}$ is a conservative vector field on a domain D then $\mathbf{F}$ is a gradient vector field.

Question 9: **10 marks**

Let Γ be a path that begins at the origin of $\mathbb{R}^3$ and ends at the point $(2, -1, \pi)$. Evaluate

$$\int_\Gamma y^2 \sin z \, dx + 2xy \sin z \, dy + (z + xy^2 \cos z) \, dz$$

by

(i) checking that a potential function for this vector field exists;

(ii) finding the potential function;

(iii) using the potential function to do the integration.

Question 10: **11 marks**

Use Gauss' divergence theorem to calculate

$$\iint_S \mathbf{F} \cdot d\mathbf{a},$$

where $\mathbf{F}(x, y, z) = (x^2, y^2, z - 3)$ and V is the region lying above the plane $z = 1$ and below the cone $z = 3 - (x^2 + y^2)^{\frac{1}{2}}$.

Total : 120 marks