Tutorial 1.2 - Questions

1. In each of the following cases, state if the given set is bounded above or not. If a set is bounded above, give two different upper bounds for the set, give the supremum of the set and state if the set has a maximum or not.

   a. $(-3,2)$
   b. $(1,\infty)$
   c. $[10,11]$
   d. $\{5,4\}$
   e. $\{10,9,8,7,6,5,4,3,2,1,0,-1,-2,-5\}$
   f. $(-\infty,2]$
   g. $\{x \in \mathbb{R} : x^2 < 3\}$
   h. $\{x \in \mathbb{R} : x^2 \leq 3\}$

2. For each of the sets in Q.1, state if the given set is bounded below or not. If a set is bounded below, give two different lower bounds for the set, give the infimum of the set and state if the set has a minimum or not.

3. Let S be a nonempty subset of $\mathbb{R}$.

   a. If a is the greatest element of S, then what is $\sup S$?
   b. If $\sup S = a$, then what are the upper bounds of S?
   c. If $\sup S = a$, does S have a maximum?
   d. If $\sup S = a$ and $a \in S$, does S have a maximum?

4. Prove Proposition 1.4.

5. Prove Theorem 1.7.

6. Prove Theorem 1.8.

7. Let S and T be non-empty subsets of $\mathbb{R}$ which are bounded above. Use Theorem 1.6 to prove that $\sup(S+T) = \sup S + \sup T$.

8. Decide which of the following statements are **True** and which are **False**.

   a. $\frac{1}{2} \in \{0,1\}$
   b. $3 \in (0,3)$
   c. $17 \in [0,17]$

d. $17 \in (-3,18)$
e. $17 \in [16,18]$
f. $2 \in \{1,3,5,7\}$
g. $2.5 \in \{x \in \mathbb{R} : x^2 \geq 4\}$
h. $-1 \in \{x \in \mathbb{R} : 2x + 7 < 5\}$

9. Assume that the Dedekind cut property, Theorem 1.9, as well as the ordered field axioms are satisfied. Show that the Dedekind completeness holds.

10. Show that if $S \subset \mathbb{Z}$, $S \neq \emptyset$, and S is bounded below, then S has a minimum.

11. Show that $\sqrt{2}$ is irrational.

12. Show that the rational numbers satisfy the axioms (A1)-(A4), (M1)-(M4), (D) and (O1)-(O3).

13. Let $a, b \in \mathbb{Q}$ with $b \neq 0$ and $r \in \mathbb{R} \setminus \mathbb{Q}$. Show that $a + br \in \mathbb{R} \setminus \mathbb{Q}$.

14. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, x^n is defined inductively by

    a. $x^0 = 1$,
    b. $x^{n+1} = xx^n$ for $n \in \mathbb{N}$.

    Show that

    i. $x^n x^m = x^{n+m}$,
    ii. $(x^n)^m = x^{nm}$,
    iii. $x^n y^n = (xy)^n$.

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