Chapter 2 Lecture 4

2.2 Bounded and Monotonic Sequences

Definition 2.3.
1. (a_n) is **bounded above** if $\exists M \in \mathbb{R}$ such that $a_n \leq M \; \forall n \in \mathbb{N}$.
2. (a_n) is **bounded below** if $\exists m \in \mathbb{R}$ such that $a_n \geq m \; \forall n \in \mathbb{N}$.
3. (a_n) is **bounded** if it is bounded above and bounded below.

Note. $(a_n)_{n=n_0}^{\infty}$ is bounded (above, below) if and only if the set $\{a_n \; : \; n \in \mathbb{Z}, n \geq n_0\}$ is bounded (above, below).

Definition 2.4. A sequence $(a_n)_{n=1}^{\infty}$ is called
(1) **increasing** if $a_n \leq a_{n+1}$ for all indices n,
(2) **strictly increasing** if $a_n < a_{n+1}$ for all indices n,
(3) **decreasing** if $a_n \geq a_{n+1}$ for all indices n,
(4) **increasing** if $a_n > a_{n+1}$ for all indices n,
(5) **monotonic** if it is either increasing or decreasing,
(6) **strictly monotonic** if it is either strictly increasing or strictly decreasing.

Theorem 2.7. Every convergent sequence is bounded.

Proof.

Since (a_n) converges, $\exists\ L$ and k such that $|\,a_n - L\,| < 1\ \forall\ n \geq k$.

Hence $\{a_n\ :\ n < k\}$ and $\{a_n\ :\ n \geq k\} \subset (L-1, L+1)$ are bounded

So (a_n) is bounded as the union of two bounded sets. $\square$

Every unbounded sequence diverges.

Definition 2.5.

1. $a_n \to \infty$ as $n \to \infty$ if $\forall\ A \in \mathbb{R}\ \exists K \in \mathbb{R}$ s.t. $a_n > A\ \forall n \geq K$.
2. $a_n \to -\infty$ as $n \to \infty$ if $\forall\ A \in \mathbb{R}\ \exists K \in \mathbb{R}$ s.t. $a_n < A\ \forall n \geq K$.

Example 2.2.1.

1. Let $a_n = (-1)^n$. Then (a_n) is bounded but not convergent.
2. Let $a_n = n$. Then (a_n) is unbounded (by the Archimedean principle) so diverges.

Example 2.2.2. Prove that $n^2 - n^3 + 10 \to -\infty$ as $n \to \infty$.
Solution. Let $A < 0$ and put $K = -A + 11$. Then $K > 11$, and for $n \geq K$, we have

$$\begin{aligned} n^2 - n^3 + 10 &= -n^2(n-1) + 10 \\ &\leq -10n^2 + 10 \leq -10K^2 + 10 \\ &= -10(K-1)(K+1) \leq -100(K+1) \\ &\leq -K = A - 11 \leq A. \end{aligned}$$

Theorem 2.8.
1. Let (a_n) be a sequence with $a_n > 0\ \forall n$. Then

$$\lim_{n\to\infty} a_n = \infty \Leftrightarrow \lim_{n\to\infty} \frac{1}{a_n} = 0\,.$$

2. Let (a_n) be a sequence with $a_n < 0\ \forall n$. Then

$$\lim_{n\to\infty} a_n = -\infty \Leftrightarrow \lim_{n\to\infty} \frac{1}{a_n} = 0\,.$$

Proof. 1. Assume that $\lim\limits_{n\to\infty} a_n = \infty$. Let $\epsilon > 0$.
Put $\frac{1}{\epsilon}$. Then $\exists K$ s.t. $0 < A < a_n\ \forall n \geq K$. Then

$$0 < \frac{1}{a_n} < \frac{1}{A} = \epsilon$$

for these n, which shows that $\lim\limits_{n\to\infty} \frac{1}{a_n} = 0$.

Conversely, assume that $\lim_{n\to\infty} \frac{1}{a_n} = 0$. Let $A \in \mathbb{R}$ and put $A_0 = \max\{A, 1\}$. Then $A_0 > 0$ and put $\epsilon = \frac{1}{A_0} > 0$.
Hence $\exists K \in \mathbb{R}$ s.t. $\frac{1}{a_n} = \left|\frac{1}{a_n}\right| < \epsilon \quad \forall n \geq K$.
This gives $a_n > \frac{1}{\epsilon} = A_0 \geq A \quad \forall n \geq K$.

2. The proof is similar. □

Theorem 2.9.
Refer to your study guide for more rules for infinite limits.

Definition 2.6. Let $A \subseteq \mathbb{R}$ be nonempty.
If A is not bounded above, we write $\sup A = \infty$, and if A is not bounded below, we write $\inf A = -\infty$.

Theorem 2.10.
1. Let (a_n) be an increasing sequence.
If (a_n) is bounded, then (a_n) converges, and $\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$.
If (a_n) is not bounded, then (a_n) diverges to ∞ .

2. Let (a_n) be a decreasing sequence.
If (a_n) is bounded, then (a_n) converges, and
$\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$.
If (a_n) is not bounded, then (a_n) diverges to $-\infty$.

Proof.
1. Let (a_n) be bounded. Put $L = \sup\{a_n : n \in \mathbb{N}\}$ and let $\epsilon > 0$.
Let's show that $\exists K \in \mathbb{N}$ s.t. $n \geq K$ gives $L-\epsilon < a_n < L+\epsilon$.
By definition of the supremum, $a_n \leq L \leq L + \epsilon$ for all n
and by Theorem 1.6(b), there is K such that $L - \epsilon < a_K$.
Then for all $n \geq K$, $L - \epsilon < a_K \leq a_n$.
Hence $L - \epsilon < a_n < L + \epsilon$ for all $n \geq K$, which proves
$a_n \to L$ as $n \to \infty$.
Now assume that (a_n) is not bounded. Since $a_0 \leq a_n$ for all n,
(a_n) is bounded below. Hence (a_n) is not bounded above.
Let $A \in \mathbb{R}$. Then there is an index K such that $a_K > A$,
and thus $a_n \geq a_K > A$ for all $n \geq K$.
2. The proof is similar. □

Example 2.2.3. Let $a_n = \left(1 + \frac{1}{n}\right)^n$ and $b_n = \left(1 + \frac{1}{n}\right)^{n+1}$.

Then $a_n < b_n \ \forall \ n \in \mathbb{N}$, (a_n) is an increasing sequence, (b_n) is a decreasing sequence, both sequences converge to the same number, and the limit is denoted e, Euler's number.

Proof. $a_n < b_n$ since $b_n = a_n\left(1 + \frac{1}{n}\right)$.

Using Bernoulli's inequality, see Example 1.3.2, we have

$$\frac{a_{n+1}}{a_n} = \frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^n} = \left(\frac{n(n+2)}{(n+1)^2}\right)^{n+1}\frac{n+1}{n} = \left(1 - \frac{1}{(n+1)^2}\right)^{n+1}\frac{n+1}{n}$$

$$\geq \left(1 - \frac{n+1}{(n+1)^2}\right)\frac{n+1}{n} = \frac{n}{n+1}\frac{n+1}{n} = 1 \text{ and}$$

$$\frac{b_n}{b_{n+1}} = \frac{\left(\frac{n+1}{n}\right)^{n+1}}{\left(\frac{n+2}{n+1}\right)^{n+2}} = \left(\frac{(n+1)^2}{n(n+2)^2}\right)^{n+2}\frac{n}{n+1} = \left(1 + \frac{1}{n(n+2)}\right)^{n+2}\frac{n}{n+1}$$

$$\geq \left(1 + \frac{n+2}{n(n+2)}\right)\frac{n}{n+1} = \frac{n+1}{n}\frac{n}{n+1} = 1$$

Hence (a_n) is increasing and bounded (above by b_1).

By Theorem 2.10, (a_n) has a limit α. Similarly, (b_n) has a limit β. Finally

$$\beta = \lim_{n\to\infty} b_n = \lim_{n\to\infty}\left(\left(1 + \frac{1}{n}\right)a_n\right) = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)\lim_{n\to\infty} a_n = \alpha. \ \square$$