Tutorial 2.1.1 Solutions of Chapter 2 Lecture 3

(3). Contractive maps. Suppose that for some $c \in \mathbb{R}$ with $0 < c < 1$, we have
$|\, a_{n+1} - L \,| \leq c|\, a_n - L \,|$ for all $n \in \mathbb{N}$.
(a) Use induction on n to prove that $|\, a_n - L \,| \leq c^n |\, a_0 - L \,|$.
(b) Use the Sandwich Theorem and the fact that $\lim_{n\to\infty} c^n = 0$ to prove that $\lim_{n\to\infty} a_n = L$.

Proof.
(a) Induction base: For $n = 0$, we have $|\, a_0 - L \,| = c^0 |\, a_0 - L \,|$.
Induction step: Assume the estimate is true for n. Then

$$\begin{aligned} |\, a_{n+1} - L \,| &\leq c|\, a_n - L \,| \\ &\leq cc^n |\, a_0 - L \,| \qquad \text{(by induction hypothesis)} \\ &= c^{n+1} |\, a_0 - L \,| . \end{aligned}$$

Hence the estimate is true for $n+1$. By the principle of mathematical induction, the estimate is true for all $n \in \mathbb{N}$.
(b) By (a),

$$0 \leq |\, a_n - L \,| \leq c^n |\, a_0 - L \,|$$

for all $n \in \mathbb{N}$. By the proof of Theorem 2.5, $\lim_{n\to\infty} c^n = 0$. Hence the limits on the left and the right in the estimate above are both 0, and the Sandwich Theorem gives $\lim_{n\to\infty} |\, a_n - L \,| = 0$, which means that $\lim_{n\to\infty} a_n = L$ (by Theorem 2.3(i)) .

(4). Recursive algorithm for finding $\sqrt{a}$. Let $a > 1$ and define

$$a_0 = a \text{ and } a_n = \frac{1}{2}\left(a_{n-1} + \frac{a}{a_{n-1}}\right) \text{ for } n \geq 1 .$$

(a) Prove that $0 < a_n - \sqrt{a} = \dfrac{1}{2a_{n-1}} \left(a_{n-1} - \sqrt{a}\right)^2$ for $n \geq 1$.
(b) Use (a) to prove that $0 \leq a_n - \sqrt{a} \leq \dfrac{1}{2} \left(a_{n-1} - \sqrt{a}\right)$ for $n \geq 1$.
(c) Deduce that $\lim_{n\to\infty} a_n = \sqrt{a}$.
(d) Apply four steps of the recursive algorithm with $a = 3$ to approximate $\sqrt{3}$.

Proof.
(a) Assume $a_{n-1} \neq 0$, we get

$$\begin{aligned} a_n - \sqrt{a} &= \frac{1}{2}\left(a_{n-1} + \frac{a}{a_{n-1}}\right) - \sqrt{a} \\ &= \frac{1}{2a_{n-1}} \left(a_{n-1}^2 + a\right) - \sqrt{a} \\ &= \frac{1}{2a_{n-1}} \left(a_{n-1} + \sqrt{a}\right)^2 . \end{aligned}$$

For $n = 0$, $a_0 \neq \sqrt{a}$.
By induction and the above identity, it follows that $a_n - \sqrt{a} > 0$ for $n \geq 1$ and particularly $a_n > 0$.
(b) We have

$$0 \leq \frac{1}{a_{n-1}} \left(a_{n-1} - \sqrt{a}\right) < 1 ,$$

and the identity in part (a) gives

$$a_n - \sqrt{a} = \frac{1}{2a_{n-1}} \left(a_{n-1} + \sqrt{a}\right) \left(a_{n-1} + \sqrt{a}\right) \leq \frac{1}{2} \left(a_{n-1} + \sqrt{a}\right) .$$

(c) $\lim_{n\to\infty} a_n = \sqrt{a}$. This immediately follows from (a), (b) and tutorial question 3.

(d) The values and their approximation are

$$a_0 = 3, \quad a_1 = 2, \quad a_2 = \frac{7}{4} = 1.75, \quad a_3 = \frac{97}{56} \approx 1.73214, \quad a_4 = \frac{18817}{10864} \approx 1.73205\,.$$

$\square$