ROBOTICS

KINEMATICS USING THE JACOBIAN

KINEMATICS IN MOTION

- Now we know how to get the position of the end effector given the joint angles
- When doing control, we often care about velocities
- Can we formulate a relationship between the motor velocities and the end effector velocity?
- Necessary for control of end effector velocities

KINEMATICS IN MOTION

- We use v to denote the linear velocity and ω to represent the angular velocity of the end effector
- For every joint i, we use q_i to represent its state
- We use $\dot{q}_i$ to represent the velocity of joint i

KINEMATICS IN MOTION

- Lets take the example of a 2D two joint arm
  - We want to find some mapping to convert between joint velocities and end effector velocity
  - Now this mapping will not be global because the velocity will have a different effect depending on the state
  - We are thus looking for some $f(q)$ such that
  - $\begin{bmatrix} v \\ \omega \end{bmatrix} = f(q) \begin{bmatrix} \dot{q_1} \\ \dot{q_2} \end{bmatrix}$

THE JACOBIAN

- This relationship is called the Jacobian
  - $\begin{bmatrix} v \\ \omega \end{bmatrix}$ = $\mathrm{J}(q_1, q_2) \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$
- Lets ignore the angular velocity part for now
  - $v = J_v(q_1, q_2) \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$
  - $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$ = $\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$

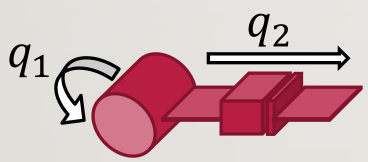
HOW DO WE GET THE JACOBIAN?

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$$

- How do we get the J values
- We want to figure out how x changes relative to q_1 and
- We are thus interested in the partial derivatives
- $J_{11} = \frac{\delta x}{\delta q_1}, J_{12} = \frac{\delta x}{\delta q_2}, J_{21} = \frac{\delta y}{\delta q_1}, J_{22} = \frac{\delta y}{\delta q_2}$

EXAMPLE

- Simple system with a revolute and prismatic joint



- $x = q_2 cos(q_1)$
- $y = q_2 sin(q_1)$

$$J_{11} = \frac{\delta x}{\delta q_1} = \ ...$$

$$J_{12} = \frac{\delta x}{\delta q_2} = \ ...$$

$$J_{21} = \frac{\delta y}{\delta q_1} = \cdots$$

$$J_{22} = \frac{\delta y}{\delta q_2} = \ ...$$

EXAMPLE

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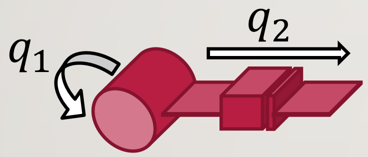


- $x \ = q_2 cos(q_1)$
- $y \ = q_2 sin(q_1)$

$$J_{11} = \frac{\delta x}{\delta q_1} = -q_2 \sin(q_1)$$

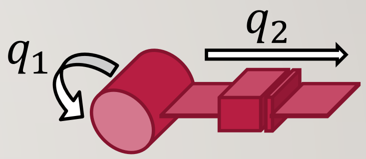
$$J_{12} = \frac{\delta x}{\delta q_2} = \cos(\mathrm{q}_1)$$

$$J_{21} = \frac{\delta y}{\delta q_1} = q_2 \cos(q_1)$$

$$J_{22} = \frac{\delta y}{\delta q_2} = \sin(q_1)$$

ANGULAR VELOCITY

- Now to find the angular velocity jacobian J_ω
- $\omega = \begin{bmatrix} J_1 & J_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$
- What effect does the actuation of each joint have on the angle of the end effector?
- $J_1 = 1, J_2 = 0$
- We can now combine the linear velocity and angular velocity Jacobians



FULL JACOBIAN

- $\begin{bmatrix} v \\ \omega \end{bmatrix}$ = $\mathrm{J}(q_1, q_2)\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$
- $\begin{bmatrix} \dot{x} \\ \dot{y} \\ \omega \end{bmatrix} = \begin{bmatrix} -q_2 \sin(q_1) & \cos(q_1) \\ q_2 \cos(q_1) & \sin(q_1) \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$

WAIT, HOW DOES THIS WORK AGAIN?

- $\begin{bmatrix} v \\ \omega \end{bmatrix} = \mathrm{J}(\boldsymbol{q})\,\dot{\boldsymbol{q}}$
- This means that we can plug in the joint velocities and get the end effector velocities
- But we often want to do the inverse – for instance, if we want to draw a line
- $\dot{\boldsymbol{q}} = \mathrm{J}^{-1}(\boldsymbol{q}) \begin{bmatrix} v \\ \omega \end{bmatrix}$

INVERSE JACOBIAN

- Must invert a square portion of the Jacobian
  - Consider the linear velocity
- Invert the matrix using the determinant
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- You can also invert the matrix using matlab (inv) or numpy (linalg.inv)

INVERSE JACOBIAN

- $\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -q_2 \sin(q_1) & \cos(q_1) \\ q_2 \cos(q_1) & \sin(q_1) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}$
- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
- $\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \frac{1}{-q_2} \begin{bmatrix} \sin(q_1) & -\cos(q_1) \\ -q_2 \cos(q_1) & -q_2 \sin(q_1) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$

INVERSE JACOBIAN

- $\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \frac{1}{-q_2} \begin{bmatrix} \sin(q_1) & -\cos(q_1) \\ -q_2\cos(q_1) & -q_2\sin(q_1) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$
- Joint 2 is extended to 3m, and the arm is oriented along the x axis. Find the joint velocities that would move the end effector by 2m/s in x and 3m/s in y

INVERSE JACOBIAN

- $\begin{bmatrix} \dot{q_1} \\ \dot{q_2} \end{bmatrix} = \frac{1}{-q_2} \begin{bmatrix} \sin(q_1) & -\cos(q_1) \\ -q_2\cos(q_1) & -q_2\sin(q_1) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$
- Joint 2 is extended to 3m, and the arm is oriented along the x axis. Find the joint velocities that would move the end effector by 2m/s in x and 3m/s in y
- $\begin{bmatrix} \dot{q_1} \\ \dot{q_2} \end{bmatrix} = \frac{1}{-3} \begin{bmatrix} \sin(0) & -\cos(0) \\ -3\cos(0) & -3\sin(0) \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
- $\begin{bmatrix} \dot{q_1} \\ \dot{q_2} \end{bmatrix} = \frac{1}{-3} \begin{bmatrix} 0 & -1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
- $\begin{bmatrix} \dot{q_1} \\ \dot{q_2} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

IN PRACTICE

- Note that because the Jacobian is a function of the joint states, and our control modifies the joint states, the Jacobian changes all the time
- Controlling the system is thus iterative, and the Jacobian must be recalculated on every iteration