APPM2023
Mechanics II
2023

Assignment-01

Curvilinear Coordinate Systems

Issued: 16 March 2023 **Total:** 45

Due: 17:00, 24 March 2023

Instructions

- Read all the instructions and questions carefully.
- Typeset the solution document using 'Assignment.cls' LaTeX document template. Submissions that have not used this template shall receive a zero grade.
- Use plain written English where necessary.
- Students may use the `Mathematica` and LaTeX supplementary resources posted on the course Moodle page to complete this assignment.
- Students are encouraged to work in groups. However, this is to be individual work and each student must submit their own report.
- Plagiarised submissions shall receive a zero grade.
- No late submissions shall be considered.
- Do not submit any `Mathematica` code for this assignment.

Introduction

The objective of this assignment is to develop an understanding of the properies of parametric mappings. Students shall use the computer algebra system Mathematica to perform computations on vectors and functions, and to construct graphical representations of parametric points, curves, surfaces and other objects.

Consider the parametric $(r.s)$-plane R at $z = 0$ in Euclidean 3-space $\mathbb{R}^3$. We construct a function taking the pair of numbers r and s as inputs and constructing a point in $\mathbb{R}^3$,

$$R(r,s) = \begin{pmatrix} r \\ s \\ 0 \end{pmatrix}.$$

The pair of numbers $(r.s)$ is mapped to R as a point $(r, s, 0)$ in $\mathbb{R}^3$. Use the following code to define this function

```
R[r_, s_] := {r, s, 0};
```

Similarly, we construct a projective mapping from the R to the unit sphere S

$$S(r,s) = \begin{pmatrix} \frac{2r}{r^2+s^2+1} \\ \frac{2s}{r^2+s^2+1} \\ \frac{r^2+s^2-1}{r^2+s^2+1} \end{pmatrix}.$$

The pair of numbers $(r.s)$ is mapped to S as a point in $\mathbb{R}^3$. Use the following code to define this function

```
S[r_, s_] := {
             2 r / (r^2 + s^2 + 1),
             2 s / (r^2 + s^2 + 1),
             (r^2 + s^2 - 1) / (r^2 + s^2 + 1)
             };
```

Use the following code to construct a graphical representation of a portion of a 1-dimensional parametric funcions f

```
ParametricPlot3D[ f[t], {t, MINT, MAXT}]
```

where `MINT` and `MAXT` are the minimum and maximum t parametre values. Similarly, use the following code to construct a graphical representation of a portion of R

```
ParametricPlot3D[ R[r, s], {r, MINR, MAXR}, {s, MINS, MAXS}]
```

where `MINR`, `MAXR`, `MINS` and `MAXS` are the minimum r, maximum r, minimum s and maximum s parametre values, respectively. A graphical some portion of S may be constructed by replacing `R[r, s]` with `S[r, s]`.

A white point marker at (a, b, c) in $\mathbb{R}^3$ can be placed into a scene using the following code,

```
Graphics3D[ { White, Sphere[coords, 0.05] } ]
```

where

```
coords = {a, b, c};
```

Similarly, red arrow markers extending from (a, b, c) to (d, e, f) can be placed in the scene using

```
Graphics3D[{Red, Arrow[{start, end}]}]
```

where

```
start = {a, b, c};
end = {d, e, f};
```

are the initial and terminal point of the arrow.

The `Manipulate` function can be used to generate dynamic interactive graphics with control parametres. Students should experiment using

```
Manipulate[Plot[Sin[s x], {x, MINX, MAXX}], {s, MINS, MAXS}]
```

where `s` in an interactivly adjustable parametre.

Students may find the functions `D`, `Dot` and `Cross` useful. Students are encouraged to consult the `Mathematica` documentation for complete information on these functions. In addition, the `Show` function can be used to display multiple graphics objects together in a single plot, as follows

```
Show[
    Graphics3D[ { Red, Arrow[{start, end}] } ],
    Graphics3D[ { White, Sphere[coords, 0.05] } ],
    ParametricPlot3D[ f[t], {t, MINT, MAXT}]
    ]
```

The `Show` function can also be include inside a `Manipulate` function. Students can find a complete listing of options for all graphics related functions in the documentation and are encouraged to experiment with each function and option.

The following parametric functions will be used in the questions that follow:

Parametric Curve $p_1 = (r_1, s_1)$: Define $\omega_1 = \frac{\pi}{4}$, then

$$\rho(t) = \tan\left(\frac{\pi}{2} t\right) \qquad r_1(t, \theta) = \rho(t)\cos(\theta) \qquad s_1(t, \theta) = \rho(t)\sin(\theta)$$

$$r_1(t) = r_1(t, \omega_1) \qquad s_1(t) = s_1(t, \omega_1)$$

Parametric Curve $p_2 = (r_2, s_2)$: Define $\omega_2 = 2\pi$, then

$$r_2(t, \theta) = 2\cos(\theta\, t) \qquad s_2(t, \theta) = 2\sin(2\theta\, t)$$

$$r_2(t) = r_2(t, \omega_2) \qquad s_2(t) = s_2(t, \omega_2)$$

Parametric Curve $p_3 = (r_3, s_3)$: Define $\omega_3 = 2\pi$, then

$$r_3(t, \theta) = (1 + 2t)\cos(2\theta\, t) \qquad s_3(t, \theta) = (1 + 2t)\sin(2\theta\, t)$$

$$r_3(t) = r_3(t, \omega_3) \qquad s_3(t) = s_3(t, \omega_3)$$

In each case, $t \in [0, 1]$.

Parametric Curves and Surfaces

Question 1 (5 Points)

Construct a single graphic containing the the following objects in to demonstrate the parametric mappings

1. The portion of R bounded by $r, s \in [-3, 3]$.
2. The entirety of S. (Hint: use an appropriately modified set of co-ordinates to include the entire sphere.)

Describe the relationship between the co-ordinate grids on the R and S. Include a picture of your output.

Question 2 (5 Points)

Construct construct a single graphic containing R bounded by $r, s \in [-3, 3]$ and the entirety of S. Demonstrate the following parametric mappings

1. The point $a_1(t) = p_1(t)$ on R.
2. The point $b_1(t) = p_1(t)$ on S.
3. The 1-dimensional parametric path $p_1(t)$ on R from from $p_1(0)$ until $a_1(t)$.
4. The 1-dimensional parametric path projection of the path $p_1(t)$ on S from $p_1(0)$ until $a_1(t)$.
5. The circle on S marking the height of $b_1(t)$ in $\mathbb{R}^3$.
6. The directed line segment (arrow) joining the north pole of S to the $b_1(t)$.
7. The directed line segment (arrow) joining the $a_1(t)$ to $b_1(t)$.

Include an image of this construction at some arbitrary time $0 < t < 1$.

Question 3 (5 Points)

Construct construct a single graphic containing R bounded by $r, s \in [-3, 3]$ and the entirety of S. Demonstrate the following parametric mappings

1. The point $a_2(t) = p_2(t)$ on R.
2. The point $b_2(t) = p_2(t)$ on S.
3. The 1-dimensional parametric path $p_2(t)$ on R from from $p_2(0)$ until $a_2(t)$.
4. The 1-dimensional parametric path projection of the path $p_2(t)$ on S from $p_2(0)$ until $a_2(t)$.

5. The circle on S marking the height of $b_2(t)$ in $\mathbb{R}^3$.

6. The directed line segment (arrow) joining the north pole of S to the $b_2(t)$.

7. The directed line segment (arrow) joining the $a_2(t)$ to $b_2(t)$.

Include an image of this construction at some arbitrary time $0 < t < 1$.

Question 4 (5 Points)

Construct construct a single graphic containing R bounded by $r, s \in [-3, 3]$ and the entirety of S. Demonstrate the following parametric mappings

1. The point $a_3(t) = p_3(t)$ on R.

2. The point $b_3(t) = p_3(t)$ on S.

3. The 1-dimensional parametric path $p_3(t)$ on R from from $p_3(0)$ until $a_3(t)$.

4. The 1-dimensional parametric path projection of the path $p_3(t)$ on S from $p_3(0)$ until $a_3(t)$.

5. The circle on S marking the height of $b_3(t)$ in $\mathbb{R}^3$.

6. The directed line segment (arrow) joining the north pole of S to the $b_3(t)$.

7. The directed line segment (arrow) joining the $a_3(t)$ to $b_3(t)$.

Include an image of this construction at some arbitrary time $0 < t < 1$.

Tangents to Curves

Question 5 (5 Points)

Compute the following path lengths

1. $p_1(t)$ on S.

2. $p_2(t)$ on S.

3. $p_3(t)$ on S.

In each case let $t \in [0, 1]$. Use the `NIntegrate` to help in the computations.

Question 6 (5 Points)

Construct a local rectilinear co-ordinate system along the projection of the parametric curve $p_1(t)$ on S. Construct the following unit vectors

1. Unit tangent vector $\vec{T}_1(t)$.
2. Unit acceleration $\vec{N}_1(t)$.
3. Unit bi-normal $\vec{B}_1(t) = \vec{T}_1(t) \times \vec{N}_1(t)$.

Construct construct a single graphic containing R bounded by $r, s \in [-3, 3]$ and the entirety of S. Demonstrate the following parametric mappings

1. The directed line segment (arrow) depicting $\vec{T}_1(t)$.
2. The directed line segment (arrow) depicting $\vec{N}_1(t)$.
3. The directed line segment (arrow) depicting $\vec{B}_1(t)$.

Include an image of this construction at some arbitrary time $0 < t < 1$. Describe the relative orientation of $\vec{T}_1(t)$, $\vec{N}_1(t)$ and $\vec{B}_1(t)$. Describe the motion of $\vec{T}_1(t)$, $\vec{N}_1(t)$ and $\vec{B}_1(t)$ along the S.

Question 7 (5 Points)

Construct a local rectilinear co-ordinate system along the projection of the parametric curve $p_2(t)$ on S. Construct the following unit vectors

1. Unit tangent vector $\vec{T}_2(t)$.
2. Unit acceleration $\vec{N}_2(t)$.
3. Unit bi-normal $\vec{B}_2(t) = \vec{T}_2(t) \times \vec{N}_2(t)$.

Construct construct a single graphic containing R bounded by $r, s \in [-3, 3]$ and the entirety of S. Demonstrate the following parametric mappings

1. The directed line segment (arrow) depicting $\vec{T}_2(t)$.
2. The directed line segment (arrow) depicting $\vec{N}_2(t)$.
3. The directed line segment (arrow) depicting $\vec{B}_2(t)$.

Include an image of this construction at some arbitrary time $0 < t < 1$. Describe the relative orientation of $\vec{T}_2(t)$, $\vec{N}_2(t)$ and $\vec{B}_2(t)$. Describe the motion of $\vec{T}_2(t)$, $\vec{N}_2(t)$ and $\vec{B}_2(t)$ along the S.

Question 8 (5 Points)

Construct a local rectilinear co-ordinate system along the projection of the parametric curve $p_3(t)$ on S. Construct the following unit vectors

1. Unit tangent vector $\vec{T}_3(t)$.
2. Unit acceleration $\vec{N}_3(t)$.
3. Unit bi-normal $\vec{B}_3(t) = \vec{T}_3(t) \times \vec{N}_3(t)$.

Construct construct a single graphic containing R bounded by $r, s \in [-3, 3]$ and the entirety of S. Demonstrate the following parametric mappings

1. The directed line segment (arrow) depicting $\vec{T}_3(t)$.
2. The directed line segment (arrow) depicting $\vec{N}_3(t)$.
3. The directed line segment (arrow) depicting $\vec{B}_3(t)$.

Include an image of this construction at some arbitrary time $0 < t < 1$. Describe the relative orientation of $\vec{T}_3(t)$, $\vec{N}_3(t)$ and $\vec{B}_3(t)$. Describe the motion of $\vec{T}_3(t)$, $\vec{N}_3(t)$ and $\vec{B}_3(t)$ along the S.

Question 9 (5 Points)

Plot on a single set of axes the length of the tangent vectors $\vec{T}_1(t)$, $\vec{T}_2(t)$, $\vec{T}_3(t)$ on the interval $t \in [0, 1]$. Describe and explain the shape of each plot.