3.1 Def and Ex

$\lim_{n\to\infty} s_n = s \in \mathbb{R}$

* $\sum_{n=1}^{\infty} a_n = s$ if $(s_n) \to s$ where $s_n = \sum_{k=1}^{n} a_k$
* GEOM SERIES $|r|<1 \Rightarrow \sum_{n=1}^{\infty} a_n \to$ But $|r| \ge 1 \Rightarrow \sum_{n=1}^{\infty} a_n \not\to$
* $\sum_{n=1}^{\infty} a_n \to \Rightarrow a_n \to 0$
* TD $\lim_{n\to\infty} a_n \neq 0 \Rightarrow \sum_{n=1}^{\infty} a_n \not\to$

* Rules

Chapter 3: Series

3.2 Convergence Tests

- **Lemma 3.1**

  Let $a_n \ge 0$ for all $n \in \mathbb{N}^*$ and let $s_k = \sum_{n=1}^{k} a_n$. Then

  if all the terms of nonnegative

  $\sum_{n=1}^{\infty} a_n$ converges if and only if (s_n) is bounded.

  $\sum_{n=1}^{\infty} a_n = s$ $|s_n| \le M$

  $s_{n+1} = \sum_{k=1}^{n+1} a_k = \sum_{k=1}^{n} a_k + a_{n+1} = s_n + a_{n+1} \ge s_n$ since $a_n \ge 0$ $s_{n+1} \ge s_n$

  Proof. Since $s_{n+1} = s_n + a_{n+1} \ge s_n$, it follows that (s_n) is an increasing sequence. By Theorem 2.9, this sequence and hence the series converges if and only if the sequence (s_n) is bounded. □

  Thm 2.9 $(a_n)\nearrow$ $|a_n| \le M \Rightarrow (a_n) \longrightarrow$

  Thm 2.6 $(a_n) \to \Rightarrow |a_n| \le M$

- **Theorem 3.6 (Comparison Test)**

  Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with nonnegative terms and assume that $a_n \le b_n$ for all $n \in \mathbb{N}^*$.

(b_n) (a_n)

(i) If $\sum_{n=1}^{\infty} b_n$ converges, then also $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} a_n$ diverges, then also $\sum_{n=1}^{\infty} b_n$ diverges.

Proof. Let $s_k = \sum_{n=1}^{k} a_n$ and $t_k = \sum_{n=1}^{k} b_n$. Then both (s_k) and (t_k) are increasing sequences with $s_k \leq t_k$.

Proof of Lemma 3.1

Hence, if $\sum_{n=1}^{\infty} b_n$ converges, then (t_k) is bounded, say

Lemma 3.1

$t_k \leq M$ for all $k \in \mathbb{N}$, and $s_k \leq t_k \leq M$ for all $k \in \mathbb{N}$. Hence (s_k) is a bounded sequence and thus $\sum_{n=1}^{\infty} a_n$ converges by Lemma 3.1.

(ii) is the contrapositive of (i). $\square$

$\sum_{n=1}^{\infty} a_n$ D $\Longrightarrow$ $\sum_{n=1}^{\infty} b_n$ D

- **Example 3.5**

  Test the series $\sum_{n=1}^{\infty} \frac{\sin^2 n + 10}{n + 2^n}$ for convergence.

  Solution. Putting

$$a_n = \frac{\sin^2 n + 10}{n + 2^n}$$

$\sin^2 n + 10 \le 1 + 10 = 11$

$a_n = \frac{\sin^2 n + 10}{n + 2^n} \le \frac{11}{n + 2^n} < \frac{11}{2^n} = 11\left(\frac{1}{2}\right)^n = b_n$

C S $\sum_{n=1}^{\infty} b_n = \frac{11}{1-\frac{1}{2}} - 11$ S

it follows that $0 < a_n < 11\left(\frac{1}{2}\right)^n =: b_n$. By Theorem 3.1, the series with general term b_n converges. Hence $\sum_{n=1}^{\infty} \frac{\sin^2 n + 10}{n + 2^n}$ converges by the Comparison Test.

thm 3 6(1)

- **Definition 3.2**

  1. A series $\sum_{n=1}^{\infty} a_n$ is called **absolutely convergent** if the series of its absolute values $\sum_{n=1}^{\infty} |a_n|$ converges.

  2. A series $\sum_{n=1}^{\infty} a_n$ is called **conditionally convergent** if it is convergent but not absolutely convergent.

- **Theorem 3.7** Every absolutely convergent series is convergent.

  Proof. Let $\sum_{n=1}^{\infty} a_n$ be absolutely convergent. Let $\epsilon > 0$. Then, by Theorem 3.5, there is $K \in \mathbb{N}$ such that for all $m \geq k \geq K$, $\sum_{n=k}^{m} |a_n| < \epsilon$. Because of

  $\left|\sum_{n=k}^{m} |a_n|\right| = \sum_{n=k}^{m} |a_n|$

  $$\left|\sum_{n=k}^{m} a_n\right| \leq \sum_{n=k}^{m} |a_n| \qquad |a_1 + a_2| \leq |a_1| + |a_2|$$

  it follows that $\left|\sum_{n=k}^{m} a_n\right| < \epsilon$ for these k, m and therefore $\sum_{n=1}^{\infty} a_n$ converges by Theorem 3.5. $\square$

- **Definition 3.3** An **alternating series** is a series of the form

  $$\sum_{n=1}^{\infty} (-1)^n b_n \quad \text{or} \quad \sum_{n=1}^{\infty} (-1)^{n-1} b_n \quad \text{with } b_n \geq 0.$$

- **Theorem 3.8 (Alternating series test)**

  If the alternating series $\sum_{n=1}^{\infty}(-1)^n b_n$ or $\sum_{n=1}^{\infty}(-1)^{n-1} b_n$ satisfies

  (i) $b_n \geq b_{n+1}$ for all n,

  (ii) $\lim_{n\to\infty} b_n = 0$,

  then the series converges.

  Proof. For $k \in \mathbb{N}^*$ and $m \in \mathbb{N}$, we have

$(-1)^k(-1)^k = 1$

$(-1)^k(-1)^{k+1} = -1$

$$
\begin{aligned}
&(-1)^k \textstyle\sum_{n=k}^{k+2m}(-1)^n b_n \\
&= (b_k - b_{k+1}) + (b_{k+2} - b_{k+3}) + \cdots + (b_{k+2m-2} - b_{k+2m-1}) \\
&+ b_{k+2m} \quad \geq b_{k+2m} \\
&= b_k - (b_{k+1} - b_{k+2}) - \cdots - (b_{k+2m-1} - b_{k+2m}). \quad \leq b_k
\end{aligned}
$$

  Hence

$$0 \leq b_{k+2m} \leq (-1)^k \sum_{n=k}^{k+2m}(-1)^n b_n \leq b_k.$$

  Similarly,

$= (-1)^k \sum_{n=k}^{k+2m} (-1)^n b_n + (-1)^k (-1)^{k+2m+1} b_{k+2m+1}$

$$0 \leq (-1)^k \sum_{n=k}^{k+2m+1}(-1)^n b_n \leq b_k - b_{k+2m+1} \leq b_k.$$

Now let $\epsilon > 0$. Since $b_k \to 0$ as $k \to \infty$, there is $K \in \mathbb{N}$ such that $b_K < \epsilon$. Hence for all $l > k > K$:

$$\left|\sum_{n=k}^{l}(-1)^n b_n\right| \le b_k \le b_K < \epsilon.$$

by Thm 3.5

Hence the alternating series converges. □

- **Note.** $\lim_{n\to\infty} b_n = 0$ is necessary by Theorem 3.3 since $\lim_{n\to\infty}(-1)^n b_n = 0 \Leftrightarrow \lim_{n\to\infty} b_n = 0 \Leftrightarrow \lim_{n\to\infty}(-1)^{n-1} b_n = 0$

- **Example 3.6** The alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3n}{4n-1}$ does not converge since

  b_n

$$\lim_{n\to\infty}\frac{3n}{4n-1} = \frac{3}{4} \neq 0$$

and thus (ii) is not satisfied, which is necessary for convergence. See the note following the statement of the Alternating Series Test. (Div Test!)

- **Example 3.7** Find whether the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^3+1}$ is convergent.

  Solution. Clearly, we have an alternating series with

  $$\lim_{n\to\infty} b_n = \lim_{n\to\infty}\frac{n^2}{n^3+1} = 0.$$

  To show that $b_n \geq b_{n+1}$ we have at least the following 2 choices:

  1. Show the inequality directly by cross-multiplication and simplification $b_n / b_{n+1} \geq 1$?

  2. Show that $f(x) = \dfrac{x^2}{x^3+1}$ has a negative derivative for sufficiently large x. $f'(x) < 0$?

  Alternatively, write

  $$\frac{n^2}{n^3+1} = \frac{n^2+\frac{1}{n}}{n^3+1} - \frac{\frac{1}{n}}{n^3+1} = \frac{1}{n} - \frac{1}{n(n^3+1)}$$

  Then

  $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^3+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} + \sum_{n=1}^{\infty}\frac{(-1)^{n}}{n(n^3+1)}$$

is the sum of two alternating series which converge by the alternating series test; hence the series converges.

- **Theorem 3.9 (Ratio Test)**

  (i) If $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

  (ii) If $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = l > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

  Proof. Note that the ratio test assumes $a_n \neq 0$ for all $n \in \mathbb{N}$.

  (i) Let $\epsilon > 0$ such that $L + \epsilon < 1$.
  Then there is $K \in \mathbb{N}$ such that $\left|\frac{a_{n+1}}{a_n}\right| < L + \epsilon$ for all $n \geq K$. Hence, for $m > K$:

$$|a_m| = |a_K| \underbrace{\left|\frac{a_{K+1}}{a_K}\right| \cdots \left|\frac{a_m}{a_{m-1}}\right|}_{m-K} < |a_K|(L+\epsilon)^{m-K}. \qquad (*)$$

$0<L+\varepsilon<1$

Since $\sum_{m=K}^{\infty} |a_K|(L+\epsilon)^{m-K}$ is a convergent geometric series, it follows from (*) and the Comparison Test, Theorem 3.6, that $\sum_{m=K}^{\infty} a_m$ converges absolutely. Hence also $\sum_{n=1}^{\infty} a_n$ converges absolutely.

$l > l'$

(ii) Let $l' \in (1, l)$. Then there is $K \in \mathbb{N}$ such that $\left|\frac{a_{n+1}}{a_n}\right| > l'$ for all $n \geq K$. Hence for $m > K$:

$$|a_m| = |a_K|\left|\frac{a_{K+1}}{a_K}\right| \cdots \left|\frac{a_m}{a_{m-1}}\right| > |a_K|, \qquad |a_m| > \varepsilon$$

so that $a_n \not\to 0$ as $n \to \infty$. Hence $\sum_{n=1}^{\infty} a_n$ diverges by the Test of Divergence, Theorem 3.3. □

- **Theorem 3.10 (Root Test)**

(i) If $\lim_{n\to\infty} \sup \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) If $\lim_{n\to\infty} \sup \sqrt[n]{|a_n|} = L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Proof.

(i) Let $\epsilon > 0$ such that $L + \epsilon < 1$. Then there is $K \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} < L + \epsilon$ for all $n \geq K$. Hence $|a_n| < (L+\epsilon)^n$ for all $n \geq K$: Since $\sum_{n=K}^{\infty} (L+\epsilon)^n$ is a convergent geometric series, it follows from the comparison test, Theorem 3.6, that $\sum_{n=K}^{\infty} a_n$ converges absolutely. Hence also $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) Let $L' \in (1, L)$. Then for each $K \in \mathbb{N}$ there is $m \geq K$ such that $\sqrt[n]{|a_n|} > L'$. Hence $|a_m| > (L')^m > 1$ for this m, and we conclude $a_n \not\to 0$ as $n \to \infty$. Indeed, if $a_n \to 0$ as $n \to \infty$, for $\epsilon = 1$ there would be $K \in \mathbb{N}$ such that $|a_n| < 1$ for all $n > K$.
Hence $\sum_{n=1}^{\infty} a_n$ diverges by the Test of Divergence. $\square$

Tutorial

- ~~Theorem~~ **3.2.1.**

1. Test $\sum_{n=1}^{\infty}\left\{\left(\frac{1}{2}\right)^n + 5\left(\frac{3}{4}\right)^n\right\} n\sin\left(\frac{1}{n}\right)$ for convergence.

2. Prove that the sequence $(a_n)_{n=1}^{\infty}$ converges if and only if
(i) $(a_{2n})_{n=1}^{\infty}$ converges,
(ii) $(a_{2n-1})_{n=1}^{\infty}$ converges,
(i) $(a_n - a_{n-1}) \to 0$ as $n \to \infty$.

3. Use Tut 2 to prove that the Alternating Series.
Hint. Show that $(s_{2n})_{n=1}^{\infty}$ and $(s_{2n-1})_{n=1}^{\infty}$ are monotonic sequences.

4. Use the alternating series test, ratio test or root test to test for convergence:
(a) $\sum_{n=1}^{\infty}\left(\frac{(-1)^n n}{2n+1}\right)^{2n}$, (b) $\sum_{n=1}^{\infty}\frac{n!2^n}{(2n)!}$,
(c) $\sum_{n=1}^{\infty}(-1)^n\left(e-\left(1+\frac{1}{n}\right)^n\right)$, (d) $\sum_{n=1}^{\infty}\frac{n^n}{(n!)^2}$,
(e) $\sum_{n=1}^{\infty}\frac{2^n}{n}$.