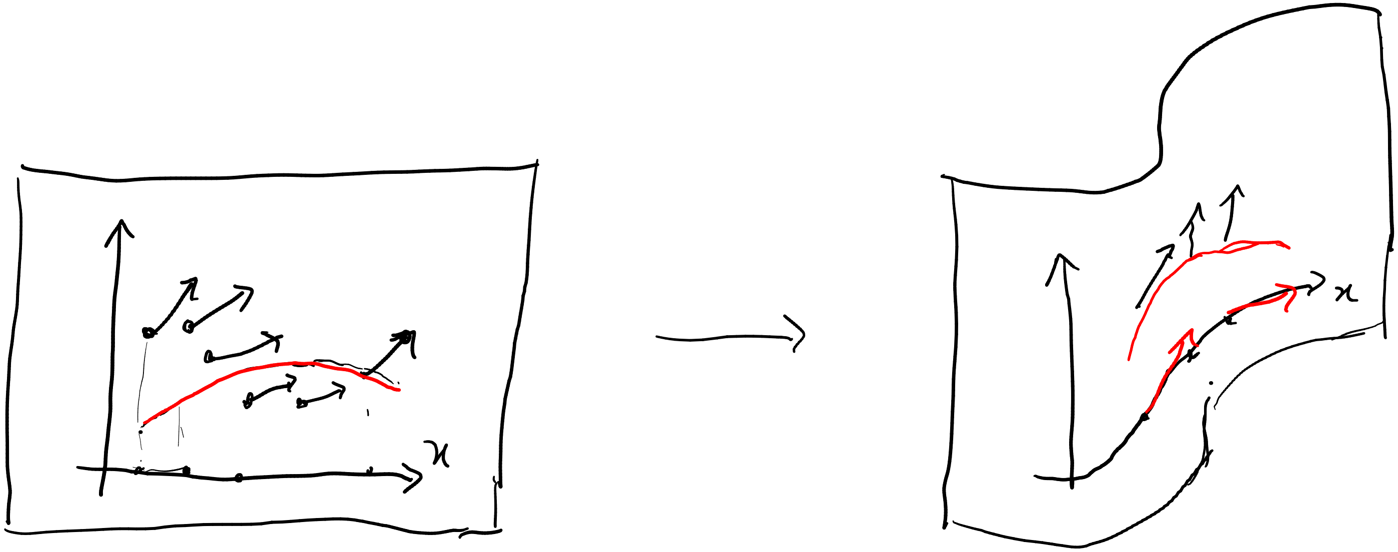
Multivariable Calculus

MATH2007

2.3 Vector Path Integrals (Part 1)

WITS UNIVERSITY

School of Mathematics


x
x

Definition (2.3.1)**.** Let the curve Γ be parametrised by $\underline{r}(t), t \in [a, b]$. Let $\underline{v} : \Gamma \to \mathbb{R}^n$. We define the **vector path integral** of $\underline{v}$ over Γ by:

dot product   dot product

$$\int_\Gamma \underline{v} \cdot d\underline{r} := \int_a^b \underline{v}(\underline{r}(t)) \cdot \underbrace{\underline{r}'(t)}_{\text{tangent}}\, dt.$$

$$\frac{d\underline{r}}{dt} \to d\underline{r}$$

Note. We may thus formally consider $d\underline{r} = \underline{r}'(t)\, dt$ and consequently, formally we have

$$d\underline{r} = \underline{u}(t)\, ds$$

$$\underline{r}'(t) = \underbrace{\|\underline{r}'(t)\|}_{ds} \cdot \underbrace{\frac{\underline{r}'(t)}{\|\underline{r}'(t)\|}}_{\underline{u}(t)}$$

where $\underline{u}(t)$ is the unit tangent vector to Γ at $\underline{r}(t)$ in the direction of the orientation of Γ, and

$$\int_\Gamma \underline{v} \cdot d\underline{r} = \int_\Gamma \underbrace{(\underline{v} \cdot \underline{u})}_{f}\, ds.$$

$\underline{u} = \underline{u}(t)$

Proof. □

$$\text{RHS} = \int_\Gamma \left[\underbrace{(\underline{v}(\underline{r}(t))}_{\underline{v}} \cdot \underbrace{\frac{\underline{r}'(t)}{\|\underline{r}'(t)\|}}_{\underline{u}} \right] ds = \int_a^b \left(\underline{v}(\underline{r}(t)) \cdot \frac{\underline{r}'(t)}{\|\underline{r}'(t)\|} \right) \|\underline{r}'(t)\|\, dt$$

$$\text{RHS} = \int_\Gamma \left[\underset{\underline{v}}{\underline{v}(\underline{r}(t))} \underset{\cdot}{\cdot} \underset{\underline{u}}{\frac{\underline{r}'(t)}{\|\underline{r}'(t)\|}} \right] ds = \int_a^b \left(\underline{v}(\underline{r}(t)) \cdot \frac{\underline{r}'(t)}{\cancel{\|\underline{r}'(t)\|}} \right) \cancel{\|\underline{r}'(t)\|}\, dt$$

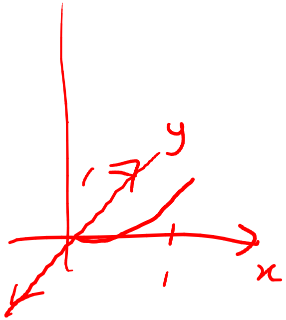
$$= \int_a^b \left[\underline{v}(\underline{r}(t)) \cdot \underline{r}'(t) \right] dt$$

$$= \int_\Gamma \underline{v} \cdot d\underline{r}.$$

Challenge: Can every scalar path integral be written as a vector path integral?

Example. Compute $\int_\Gamma \underline{v} \cdot d\underline{r}$ where $\underline{v}(x,y) = \begin{pmatrix} x^2 \\ xy \end{pmatrix}$ and $\Gamma = \{\underline{r}(t) \,|\, t \in [0,1]\}$ where $\underline{r}(t) = (t, t^2)$.

$\underline{v} : \mathbb{R}^2 \to \mathbb{R}^2$



$$\frac{d\underline{r}}{dt} = \underline{r}'(t) = (1, 2t)$$

$$\int_\Gamma \underline{v} \cdot d\underline{r} = \int_0^1 \underbrace{\underline{v}(t,t^2)}_{\underline{v}(\underline{r}(t))} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt$$

$$= \int_0^1 \begin{pmatrix} t^2 \\ t^3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt = \int_0^1 t^2 + 2t^4 \, dt$$

$$= \left[\frac{t^3}{3} + \frac{2}{5} t^5 \right]_0^1$$

$$= \frac{1}{3} + \frac{2}{5} = \frac{11}{15}.$$

Note. If $\underline{v}$ represents force, then $\int_\Gamma \underline{v} \cdot d\underline{r}$ is the work done on a particle by the force $\underline{v}$ in traversing Γ.

Note. Let Γ be parametrised by $\underline{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, $t \in [a,b]$, and let $\mathbf{v}(x,y) = \begin{pmatrix} P(x,y) \\ Q(x,y) \end{pmatrix}$, then one may write $d\underline{r} = \begin{pmatrix} dx \\ dy \end{pmatrix}$ and thus

$$\int_\Gamma \underline{v} \cdot d\underline{r} = \int_\Gamma P\ dx + Q\ dy.$$

$$\begin{pmatrix} P \\ Q \end{pmatrix} \cdot \begin{pmatrix} dx \\ dy \end{pmatrix}$$

Note. By the expression $\displaystyle\int_\Gamma P\,dx + Q\,dy + R\,dz$ we mean $\int_\Gamma \underline{v} \cdot d\underline{r}$ where $\underline{v} = (P, Q, R)$.

Example. Compute $\int_\Gamma x^2y\,dx + \frac{x^3}{3}\,dy$ where Γ is given by $y = x^2$ from $(0,0)$ to $(1,1)$.

$P\,dx \qquad Q\,dy \qquad \underline{v}(x,y) = \begin{pmatrix} x^2y \\ x^3/3 \end{pmatrix} \qquad \underline{r}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix},\ t \in [0,1]$

$\frac{d\underline{r}}{dt} = \begin{pmatrix} 1 \\ 2t \end{pmatrix} \qquad d\underline{r} = \begin{pmatrix} dt \\ 2t\,dt \end{pmatrix}$

$dx = dt \qquad dy = 2t\,dt$

$t^4\,dt + t^3/3 \cdot 2t\,dt$

$$\int_\Gamma x^2y\,dx + \frac{x^3}{3}\,dy$$

$$= \int_0^1 \underline{v}(\underline{r}(t)) \cdot \underline{r}'(t)\,dt$$

$$= \int_0^1 \underline{v}(t, t^2) \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt = \int_0^1 \begin{pmatrix} t^4 \\ t^3/3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2t \end{pmatrix} dt$$

$$= \int_0^1 t^4 + \frac{2}{3}t^4\,dt = \int_0^1 \frac{5}{3}t^4\,dt = \left[\frac{1}{3}t^5\right]_0^1$$

$$= \frac{1}{3}.$$

Multivariable Calculus

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2.3 Vector Path Integrals
(Part 3)

Wits University
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Theorem (2.3.3)**.** Let Γ be a piecewise smooth oriented curve, $\Gamma = \{\underline{r}(t) \,|\, t \in [0,1]\}$, then $\int_\Gamma \underline{v} \cdot d\underline{r}$ is invariant under orientation preserving reparametrization of Γ. As before, let Γ^- denoted Γ with reversed orientation. Then

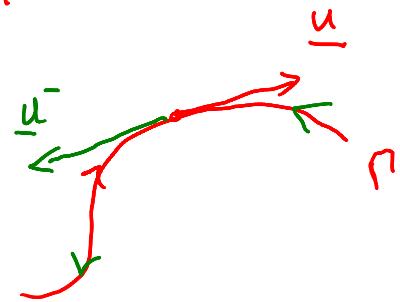
$$-\int_{\Gamma^-} \underline{v} \cdot d\underline{r} = \int_\Gamma \underline{v} \cdot d\underline{r}$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Proof. □

$$\int_\Gamma \underline{v} \cdot d\underline{r} = \int_\Gamma (\underline{v} \cdot \underline{u})\, ds \quad \text{where} \quad \underline{u} = \frac{\underline{r}'(t)}{\|\underline{r}'(t)\|}.$$

vector path int.   scalar path int.



$\underline{u}$ is independent of the parametrization.

for Γ^-: $\underline{u}^- = -\underline{u}$; thus

$$\int_{\Gamma^-} \underline{v} \cdot d\underline{r} = \int_{\Gamma^-} (\underline{v} \cdot \underline{u}^-)\, ds = \int_{\Gamma^-} \underline{v} \cdot (-\underline{u})\, ds = -\int_{\Gamma^-} \underline{v} \cdot \underline{u}\, ds.$$

$$\int_{\Gamma^-} \underline{v} \cdot d\underline{r} = \int_{\Gamma^-} (\underline{v} \cdot \underline{u}^-)\, ds = \int_{\Gamma^-} \underline{v} \cdot (-\underline{u})\, ds = -\int_{\Gamma^-} \underline{v} \cdot \underline{u}\, ds.$$

$$= -\int_{\Gamma} (\underline{v} \cdot \underline{u})\, ds$$

(Scalar path integrals are independent of orientation).

$$= -\int_{\Gamma} \underline{v} \cdot d\underline{r}$$

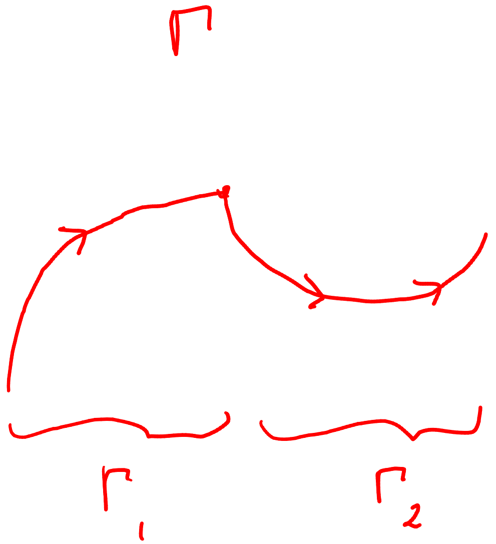
$$\therefore \quad -\int_{\Gamma^-} \underline{v} \cdot d\underline{r} = \int_{\Gamma} \underline{v} \cdot d\underline{r}.$$

Note. Scalar path integrals are independent of **orientation** and **parametrization** whereas vector path integrals are only invariant if the orientation is preserved (can have different parametrization but must have same orientation). If orientation is reversed the vector path integral changes sign.

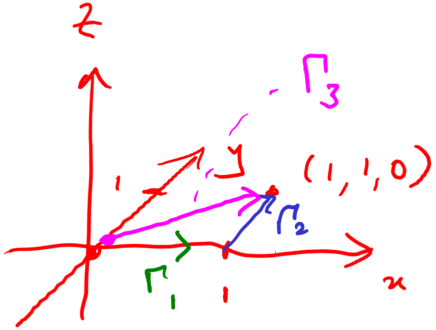
Note. If Γ is represented as the sum of two distinct curves Γ_1 and Γ_2, we may write $\Gamma = \Gamma_1 + \Gamma_2$ and

$$\int_{\Gamma} \underline{F} \cdot d\underline{r} = \int_{\Gamma_1} \underline{F} \cdot d\underline{r} + \int_{\Gamma_2} \underline{F} \cdot d\underline{r}.$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



Example. Let $\underline{F}(x,y,z) = (3x^2 - 6yz, 2y + 3x, 1 - 4z^3)$ compute $\int_\Gamma \underline{F} \cdot d\underline{r}$ where $\Gamma = \Gamma_1 + \Gamma_2$ where Γ_1 is the straight line from $(0,0,0)$ to $(1,0,0)$; Γ_2 is the straight line from $(1,0,0)$ to $(1,1,0)$. Give the value of $\int_\Gamma \underline{F} \cdot d\underline{r}$.



$$\Gamma = \Gamma_1 + \Gamma_2$$

$$\Gamma_1 : \quad \underline{r}_1(t) = (0,0,0) + t\left[(1,0,0) - (0,0,0)\right]$$
$$= (t,0,0) \qquad t \in [0,1]$$

$$\Gamma_2 : \quad \underline{r}_2(t) = (1,0,0) + t\left[(1,1,0) - (1,0,0)\right]$$
$$= (1,t,0) \qquad t \in [0,1]$$

$$\int_{\Gamma_1} \underline{F} \cdot d\underline{r}_1 = \int_0^1 \underline{F}(t,0,0) \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} dt = \int_0^1 \begin{pmatrix}3t^2\\3t\\1\end{pmatrix} \cdot \begin{pmatrix}1\\0\\0\end{pmatrix} dt = \int_0^1 3t^2\, dt = 1$$

$$\int_{\Gamma_2} \underline{F} \cdot d\underline{r}_2 = \int_0^1 \underline{F}(1,t,0) \cdot \begin{pmatrix}0\\1\\0\end{pmatrix} dt = \int_0^1 \begin{pmatrix}3\\2t+3\\1\end{pmatrix} \cdot \begin{bmatrix}0\\1\\0\end{bmatrix} dt = \int_0^1 2t+3\, dt = 4$$

$$\int_{\Gamma_1} \underline{F} \cdot d\underline{r}_1 = \int_0^1 \underline{F}(t,0,0) \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dt = \int_0^1 \begin{pmatrix} 3t^2 \\ 3t \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} dt = \int_0^1 3t^2 \, dt = 1$$

$$\int_{\Gamma_2} \underset{\sim}{F} \cdot d\underline{r}_2 = \int_0^1 \underline{F}(1,t,0) \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} dt = \int_0^1 \begin{pmatrix} 3 \\ 2t+3 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} dt = \int_0^1 2t+3 \, dt = 4$$

$$\int_{\Gamma} \underline{F} \cdot d\underline{r} = \int_{\Gamma_1} \underline{F} \cdot d\underline{r}_1 + \int_{\Gamma_2} \underset{\sim}{F} \cdot d\underline{r}_2$$

$$= 1 + 4$$

$$= 5.$$

$$\int_{\Gamma^-} \underline{F} \cdot d\underset{\sim}{r} = -5 \qquad \text{(exercise)}$$

$$\int_{\Gamma} \underline{F} \cdot d\underline{r} \overset{?}{=} \int_{\Gamma_3} \underline{F} \cdot d\underline{r}$$