Chapter 7:
Groups of Symmetry

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LEARNING OUTCOMES FOR THE LECTURE

(sigma)

By the end of this lecture, students will be able to:

- define the set of elements moved by the permutation σ
- prove a property of the set of moved elements
- prove that a product of disjoint permutations is commutative
- find the inverse of a permutation written in cycle notation
- use cycle notation to determine a product of two permutations

Lemma (7.4.2)

Let $M_\sigma = \{x \in X | \sigma(x) \neq x\}$ be the set of elements that are moved by σ.
Then if k is moved by σ (i.e. $k \in M_\sigma$) then $\sigma(k)$ is also moved by σ.
That is $k \in M_\sigma \Rightarrow \sigma(k) \in M_\sigma$.

PROOF: $\sigma \in S_n$ so σ is a bijection and is invertible, so $\sigma^{-1} \in S_n$.
If σk is fixed by σ, then $\sigma(\sigma(k)) = \sigma(k)$.
$\sigma^{-1}[\sigma(\sigma(k))] = \sigma^{-1}\sigma(k)$ and hence $\sigma^{-1}\sigma(\sigma(k)) = k$.
$\sigma(k) = k$ and $k \notin M_\sigma$ contradiction.

σ = sigma τ = tau

Theorem (7.4.3)

If σ and τ in S_n are disjoint, then $\sigma\tau = \tau\sigma$.

PROOF: Let $k \in X$ where $|X| = n$. There are 4 cases to consider:

- (i) σ and τ both fix k.
- (ii) σ moves k and τ fixes k.
- (iii) σ fixes k and τ moves k.
- (iv) σ and τ both move k. *(cannot exist by assumption).*

CASE(i): $\sigma(k) = k$ and $\tau(k) = k$.
$\sigma\tau(k) = \sigma(k) = k$ and $\tau\sigma(k) = \tau k = k$.
$\therefore \sigma\tau = \tau\sigma$.
CASE(ii) Let $\sigma(k) = p$ and $k \in M_\sigma$. By Lemma above
$p = \sigma(k) \in M_\sigma$.
$\tau(k) = k \Rightarrow \sigma\tau(k) = \sigma(k) = p$
Also $\tau\sigma(k) = \tau(p)$.
But $p \in M_\sigma$ and is moved by σ so $p \notin M_\tau$ since σ and $\tau$are disjoint.
$\therefore \quad \tau(p) = p$.
$\therefore \quad \tau\sigma(k) = \tau(p) = p$ and $\tau\sigma = \sigma\tau$.
CASE(iii) as in case (ii)
CASE(iv) Cannot exist since τ and σ are disjoint.

Theorem (7.4.4)

If σ is an r-cycle, then σ^{-1} is also an r-cycle. Infact if $\sigma = (k_1 \quad k_2 \quad \cdots \quad k_r)$ then $\sigma^{-1} = (k_r \quad k_{r-1} \quad \cdots \quad k_1)$.

~~**PROOF:**~~

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}$$

$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma^{-1}(1) & \sigma^{-1}(2) & \cdots & \sigma^{-1}(n) \end{pmatrix}.$$

NOTE 7.4.5: Cycle Representation

(1) $\sigma = (1 \quad 3 \quad 2 \quad 4)$ or $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 2 & 1 & 5 \end{pmatrix}$.

$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 1 & 2 & 5 \end{pmatrix}$ or $\sigma^{-1} = (1 \quad 4 \quad 2 \quad 3) = (4 \quad 2 \quad 3 \quad 1)$.

(2) $\sigma = (1 \quad 2 \quad 4)$ in S_4 fixes 3 but in S_5 fixes 3 and 5.

(3) $\sigma = (1 \quad 3 \quad 7 \quad 2)(4 \quad 6)(8 \quad 5 \quad 10)(9)$.

$[1] = \{1, 3, 7, 2\}$; $[4] = \{4, 6\}$; $[8] = \{8, 5, 10\}$;

$[9] = \{9\}$

$X = [1] \cup [4] \cup [8] \cup [9]$.

Examples on multiplication of cycles of permutations

NB: When faced with the product α ~~on disjoint~~ of cycles follow the following procedure in S_n :

(i) Choose any number $1 \leq r_0 \leq n$. Conveniently choose r_0. Check in the most right cycle for r_0. If it appears, find where r_0 moves to, say r_1. If r_0 does not appear check the next cycle. If it never appears then α leaves r_0 fixed.

$(\ r_2\ r_3)(\quad)(\ r_1\ r_2\ )(\ r_0\ r_1\ )$

(ii) Having located r_1, check for an appearance of r_1 only to the left of its first position. Once located find where r_0 moves to. If r_1 does not appear to the left of its first appearance, then r_0 is mapped to r_1.
Start procedure again with extreme right cycle and r_1.

Eg) $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 4 & 6 & 1 \end{pmatrix}$ $\qquad \alpha = (1 \quad 3 \quad 2 \quad 5 \quad 6)$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 1 & 2 & 4 & 5 \end{pmatrix} \qquad \beta = (1 \quad 6 \quad 5 \quad 4 \quad 2 \quad 3)$$

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 6 & 5 & 4 \end{pmatrix} \qquad \gamma = (1 \quad 2)(4 \quad 6)$$

Find the products below:

$$\begin{aligned} \alpha\gamma &= (1 \quad 3 \quad 2 \quad 5 \quad 6)(1 \quad 2)(4 \quad 6) \\ &= (1 \quad 5 \quad 6 \quad 4)(2 \quad 3) \end{aligned}$$

(1 5 6 4)(2 3)

$$\begin{aligned}\gamma\alpha &= (1\ \ 2)(4\ \ 6)(1\ \ 3\ \ 2\ \ 5\ \ 6)\\ &= (1\ \ 3)(4\ \ 6\ \ 2\ \ 5)\end{aligned}$$

$$\begin{aligned}\alpha\beta &= \overset{\alpha}{(1\ \ 3\ \ 2\ \ 5\ \ 6)}\overset{\beta}{(1\ \ 6\ \ 5\ \ 4\ \ 2\ \ 3)}\\ &= (1)(2)(3)(4\ \ 5)(6)\\ &= (4\ \ 5)\end{aligned}$$

$$\begin{aligned}\beta\alpha &= (1\ \ 6\ \ 5\ \ 4\ \ 2\ \ 3)(1\ \ 3\ \ 2\ \ 5\ \ 6)\\ &= (1)(2\ \ 4)(3)(5)(6)\\ &= (2\ \ 4)\end{aligned}$$

Eg) $(1\ \ 3\ \ 4\ \ 5)(3\ \ 2\ \ 6)(2\ \ 4\ \ 1)(3\ \ 2) =$ $(1\,6\,4\,3\,5)(2)$
$(1\ \ 6\ \ 4\ \ 3\ \ 5)(2)$.