Chapter 5: Differentiation

Section 5.1: The Chain Rule and Inverse Functions

SELF-STUDY!

Take note that this section is for self-study as it only revises some of the important work covered in first year Calculus. Although its self-study, take note that the work here will also be assessed in the exam.

We have done this in all the years this course has been presented, both face-to-face and online. You are welcome to email me or post on Forums (Sakai) if you have any questions or queries.

The student manual has all the proofs in that you need to know. I will just highlight a few things here:

- Definition 5.1 gives us the formula to calculate the derivative of a function f:

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

  but you can also make use of the first principles formula you know from high school:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Theorem 5.1 tells us that differentiability implies continuity. Can we deduce the convers: continuity will imply differentiability? No, there are many functions which are continuous but not differentiable, for example, $f(x) = |x|$.
- Theorem 5.2 and Example 5.1 gives us some of the differentiation rules that we can use to find the derivative of a function. You can also make use of the rules that you learned in first year Calculus.
- Make sure you understand the wording, difference and applications of Theorem 5.3, Theorem 5.4 and Theorem 5.5. Theorems with names are always important, hence they get a name!

* * * * * * *

<u>Tutorial 5.1</u>

Part 1

Let $a < b$ be real numbers, $f, g: [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b).

1. Show that g is injective on $[a, b]$ if $g'(x) \neq 0$ for all $x \in (a, b)$.

2. Prove the **Second Mean Value Theorem**: If $g'(x) \neq 0$ for all $c \in (a, b)$, then there is $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Hint: Consider the function

$$h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)}\big(g(x) - g(a)\big).$$

3. Show that the statement of the Second Mean Value Theorem remains correct if one replaces the condition that $g'(x) \neq 0$ for all $x \in (a, b)$ with the weaker condition that $g(a) \neq g(b)$ and that there is no $x \in (a, b)$ with $g'(x) = f'(x) = 0$.

4. Prove the following one-sided version of l'Hôpital's Rule: If $f(a) = g(a) = 0$, $g'(x) \neq 0$ for x near a and

$$\lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

exists, then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.$$

Part 2

Let $-\infty \leq a < b \leq \infty$ and let $f, g: (a, b) \to \mathbb{R}$ be differentiable on (a, b) such that $g'(x) \neq 0$ for all $x \in (a, b)$. Prove the following one-sided versions of l'Hôpital's Rule.

5. If

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} g(x) = 0 \qquad \text{and} \qquad \lim_{x \to a^+} \frac{f'(x)}{g'(x)}$$

exists as a proper or improper limit, then

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)}.$$

6. If

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = \infty \qquad \text{and} \qquad \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$$

exists as a proper or improper limit, then

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

* * * * * * *

Section 5.2: The Chain Rule and Inverse Functions

Theorem 5.6 (Chain Rule)

Let I and J be intervals, $g: J \to \mathbb{R}$ and $f: I \to \mathbb{R}$ with $f(I) \subset J$, and let $a \in I$. Assume that f is differentiable at a and that g is differentiable at $f(a)$. Then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'\big(f(a)\big)f'(a).$$

Proof

Since $\lim_{x \to a} f(x) = f(a)$ (f is continuous at a) and $\lim_{x \to a} \frac{f(x)-f(a)}{x-a} = f'(a)$ (f differentiable at a), we have

$$\begin{aligned}(g \circ f)'(a) &= \lim_{x \to a} \frac{(g \circ f)(x) - (g \circ f)(a)}{x - a} \\ &= \lim_{x \to a} \frac{g\big(f(x)\big) - g\big(f(a)\big)}{x - a} \\ &= \lim_{x \to a} \frac{g\big(f(x)\big) - g\big(f(a)\big)}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \to a} \frac{g\big(f(x)\big) - g\big(f(a)\big)}{f(x) - f(a)} \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.\end{aligned}$$

Let $y = f(x)$ and $y \to f(a)$ as $x \to a$. Then

$$\begin{aligned}(g \circ f)'(a) &= \lim_{y \to f(a)} \frac{g(y) - g\big(f(a)\big)}{y - f(a)} \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= g'\big(f(a)\big)f'(a)\end{aligned}$$

since g is differentiable at $f(a)$. ■

Theorem 5.7 Let I be an interval, let $f: I \to \mathbb{R}$ be continuous and strictly increasing or decreasing, and $b \in f(I)$. Assume that f is differentiable at $a = f^{-1}(b)$ with $f'(a) \neq 0$. Then f^{-1} is differentiable at $b = f(a)$ and

$$(f^{-1})'(b) = \frac{1}{f'\big(f^{-1}(b)\big)}$$

or, equivalently,

$$(f^{-1})'\big(f(a)\big) = \frac{1}{f'(a)}.$$

Proof

f^{-1} is the inverse of $f \Rightarrow (f^{-1} \circ f)(x) = x$ and $(f^{-1} \circ f)(y) = y$.

Also, $a = f^{-1}(b) \Leftrightarrow f(a) = b$.

Let $y = f(x)$. Then $x \longrightarrow a$ and $y \longrightarrow f(a)$. Thus

$$\begin{aligned}
(f^{-1})'(b) &= \lim_{y \to b} \frac{f^{-1}(y) - f^{-1}(b)}{y - b} \\
&= \lim_{x \to a} \frac{f^{-1}\big(f(x)\big) - f^{-1}\big(f(a)\big)}{f(x) - f(a)} \\
&= \lim_{x \to a} \frac{x - a}{f(x) - f(a)} \\
&= \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a}\right]^{-1} \\
&= \left[\lim_{x \to a} \frac{f(x) - f(a)}{x - a}\right]^{-1} \\
&= [f'(a)]^{-1} \\
&= \frac{1}{f'(a)} \\
&= \frac{1}{f'\big(f^{-1}(b)\big)}.
\end{aligned}$$

■

This result now allows us to find the derivatives of arcsin and arctan.

Example 5.2

Show that $arcsin : [-1,1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is continuous on $[-1,1]$ and differentiable on $(-1,1)$, and find $\frac{d}{dx}\arcsin(x)$.

Solution

$sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$ is strictly increasing and continuous. By Theorem 4:15, $arcsin$ is continuous on $[-1,1]$. Since

$$\frac{d}{dx}\sin(x) = \cos(x) \neq 0 \text{ for } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

that is, if $\sin(x) \in (-1,1)$, it follows by Theorem 5.7 that $arcsin$ is differentiable on $(-1,1)$ and

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\cos(\arcsin(x))} = \frac{1}{\sqrt{1 - \sin^2(\arcsin(x))}} = \frac{1}{\sqrt{1 - x^2}}$$

since $\cos(t) > 0$ for $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. ■

Finally, in this section we shall show that $e^x = \exp(x)$ is differentiable and find its derivative. This will then allow us to find the derivative of $\ln x$ as an inverse function.

Theorem 5.8 (Derivative of e^x)

1. $e^x \geq 1 + x$ for $x \in \mathbb{R}$ and $e^x \leq \frac{1}{1-x}$ for $x < 1$.
2. e^x is differentiable and $\frac{d}{dx} e^x = e^x$.

Proof

1. Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$ with $n > |x|$. Then $\frac{x}{n} > -1$, and with the aid of Bernoulli's inequality we calculate

$$\left(1 + \frac{x}{n}\right)^n \geq 1 + n\left(\frac{x}{n}\right) = 1 + x$$

$$\Rightarrow e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n \geq 1 + x.$$

Thus we have shown that $e^x \geq 1 + x$ for all $x \in \mathbb{R}$. Replacing x with $-x$ gives $e^{-x} \geq 1 - x$. Taking inverses for $x < 1$, i.e., $1 - x > 0$, leads to

$$e^x = \frac{1}{e^{-x}} \leq \frac{1}{1-x}$$

for $x < 1$.

2. The above estimates give for $h < 1$ that

$$h \leq e^h - 1 \leq \frac{1}{1-h} - 1 = \frac{h}{1-h}$$

$$\Rightarrow \begin{cases} 1 \leq \dfrac{e^h - 1}{h} \leq \dfrac{1}{1-h} & \text{if } h > 0, \\ \dfrac{1}{1-h} \leq \dfrac{e^h - 1}{h} \leq 1 & \text{if } h < 0. \end{cases}$$

Application of the Sandwich Theorem leads to

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1.$$

Therefore

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h} = e^x \lim_{h\to 0}\frac{e^h - 1}{h} = e^x. \qquad \blacksquare$$

* * * * * * *

Tutorial 5.2

1. Find the derivatives of
   a. $\arctan x$
   b. $\arccos x$
   c. $\ln x$ and $\ln|x|$
   d. $\ln|f(x)|$ where $f: I \to \mathbb{R}\backslash\{0\}$ is differentiable on I.

2. Let $x > 0$ and let $r = \frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Define $x^r = (x^p)^{1/q}$. Show that $x^r = \exp(r \ln x)$

3. In view of Problem 2 above, we define $x^r = \exp(r \ln x)$ for all $x > 0$ and $r \in \mathbb{R}$.
   a. Show that $\lim_{x \to -\infty} \exp(x) = 0$.
   b. Show that $\lim_{x \to \infty} \frac{1}{x^r} = 0$ for all $r > 0$.

* * * * * * *