COMS 3003A
HW 5

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Watching:
Videos for week 5 (see Moodle).

(1) In this question, we're looking at bijections, injections, and surjections. If $f\colon X \to Y$ and $g\colon Y \to Z$, then the *composition* of f and g is the function $g \circ f\colon X \to Z$ defined by

$$(g \circ f)(x) \quad = \quad g(f(x)) \qquad \text{whenever } x \in X.$$

(a) Prove that a composition of bijections is an bijection.

(b) Prove that a composition of injections is an injection.

(c) What can we say about surjections?

(2) Prove that the set of prime numbers is countable.

(3) Prove that the set of points on the real plane that have integer coordinates is countable.

(4) Prove that the set of Turing machines is countable.

(5) We have shown in lecture that, for every set X, the set of Boolean-valued functions with domain X is strictly larger than X. Prove, using this fact and (4), that there exist undecidable decision problems.

(6) Prove that the set of all subsets of a set X is strictly larger than X itself.

(7) As we have seen, every Turing machine can be represented as a binary string. Thus, every Turing machine can be input to a Turing machine with a binary input alphabet. Which questions do you think we might like to ask about Turing machines that we might want to be answered by Turing machines?

(8) What kind of questions we might like to ask Turing machines about themselves? I.e., think of a meaningful question we might pose to a Turing machine M about M.