Chapter 2: Sequence

2.1. Definitions, Examples and Theorems

Definition 2.1.

- A **(real) sequence** is an ordered list of infinitely many real numbers

$$(a_n)_{n=1}^{\infty} := a_1, a_2, a_3, a_4, \ldots$$

- a_n is the n**-th term** of the sequence and n is the **index**.

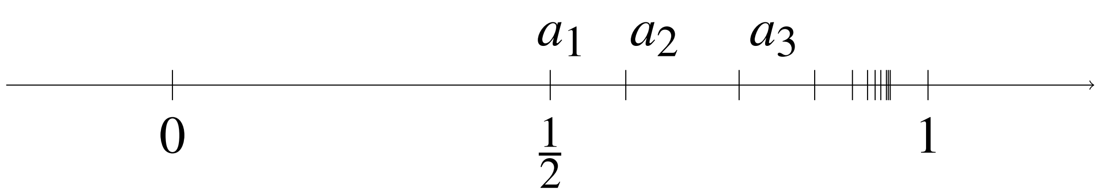
- One can write $(a_n)_{n=n_0}^{\infty}$, where $n_0 \in \mathbb{Z}$, or simply (a_n).

- For example, $(2n-3)_{n=1}^{\infty}$, $(2n-3)_{n=0}^{\infty}$, $(2n-3)_{n=-5}^{\infty}$

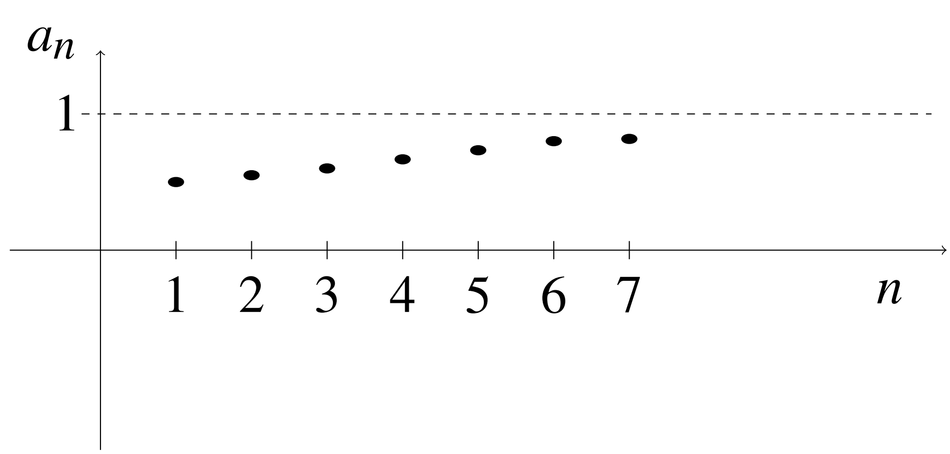
- A sequence as a function of integers:

$$a_n = f(n)$$

- Plotting the sequence $\left(\dfrac{n}{n+1}\right)_{n=1}^{\infty}$ as points on x-axis:



- Plotting $\left(\dfrac{n}{n+1}\right)_{n=1}^{\infty}$ as the graph of a function:



Convergence

- **Definition 2.2.** Let $(a_n)_{n=1}^{\infty}$ be a sequence.

  (1) Statement 'a_n tends to L as n tends to infinity',

  written as '$a_n \to L$ as $n \to \infty$', is defined by:

  $\forall\, \epsilon > 0 \;\; \exists\, K \in \mathbb{R} \;\; \forall\, n \in \mathbb{N}, \;\; n \geq K, \; |a_n - L| < \epsilon$

  (2) If $a_n \to L$ as $n \to \infty$, we say that

  $(a_n)_{n=1}^{\infty}$ **converges** to L, and we also write

  $\lim\limits_{n\to\infty} a_n = L$. That is, L is the **limit** of $(a_n)_{n=1}^{\infty}$.

  (3) The sequence $(a_n)_{n=1}^{\infty}$ is said to be **convergent**

  if it converges to some real number.

  Otherwise, $(a_n)_{n=1}^{\infty}$ is said to be **divergent**.

- $|a_n - L| < \epsilon$ means $L - \epsilon < a_n < L + \epsilon$

- K depends on ϵ and may be written K_ϵ

Example 2.1.

1. Prove that the sequence $(a_n) = \left(\dfrac{n}{n+1}\right)$ converges and find its limit .

   Solution.

   By guessing the limit (see the graph), take $L = 1$.

   Let $\epsilon > 0$. We must find K_ϵ such that for all n,

   $$n \geq K_\epsilon \implies |\, a_n - L \,| = \left| \left(\frac{n}{n+1}\right) - 1 \right| < \epsilon$$

   For this we first simplify:

   $$\left| \left(\frac{n}{n+1}\right) - 1 \right| = \left| \frac{n-(n+1)}{n+1} \right| = \left| \frac{-1}{n+1} \right| = \frac{1}{n+1}$$

   Hence, $|\, a_n - 1 \,| < \epsilon$ provided that $\dfrac{1}{n+1} < \epsilon$,

   which can be written as $n + 1 > \dfrac{1}{\epsilon}$ or $n > \dfrac{1}{\epsilon} - 1$.

   If we take $K_\epsilon = \dfrac{1}{\epsilon} - 1$, then $|\, a_n - 1 \,| < \epsilon$ for all $n > K_\epsilon$

   Since $\epsilon > 0$ was arbitrary, then $\dfrac{n}{n+1} \to 1$ as $n \to \infty$.

2. Show that the sequence $(a_n) = \left(\frac{1}{n}\right)$ converges to 0.

Solution

Given $\epsilon > 0$. We need to find K_ϵ such that for all n,

$$n \geq K_\epsilon \implies \left|\frac{1}{n} - 0\right| < \epsilon$$

Simplifying $\left|\frac{1}{n} - 0\right| < \epsilon \implies \frac{1}{n} < \epsilon$, or $n > \frac{1}{\epsilon}$.

Take $K_\epsilon = \frac{1}{\epsilon}$ then $\left|\frac{1}{n} - 0\right| < \epsilon$ for every $n > K_\epsilon$.

This shows that $\left(\frac{1}{n}\right)$ converges to 0.

Tutorial 2.1.1(1)

(a) Prove, using the definition of convergence, that the sequence $(a_n) = \left(\frac{n}{n+1}\right)$ does not converge to 2.

(b) Prove, using the definition of convergence, that the sequence $(a_n) = ((-1)^n)$ does not converge to any L.