Tutorial 3.3.1.

1. (a) Let $c > 1$ and put $c_n = \sqrt[n]{c} - 1$.
(i) Show that $c_n \geq 0$.
(ii) Show that $\limsup_{n\to\infty} c_n \leq 0$. **Hint.** Use Bernoulli's inequality.
(iii) Conclude that $\lim_{n\to\infty} \sqrt[n]{c} = 1$
(b) Use (a) to show that $\lim_{n\to\infty} \sqrt[n]{c} = 1$ for all $c > 0$.

Proof.

(a) (i) Assume $\sqrt[n]{c} \leq 1$. Then $c = \left(\sqrt[n]{c}\right)^n \leq 1$ which contradicts the assumption on c. Therefore $\sqrt[n]{c} > 1$ and thus $c_n = \sqrt[n]{c} - 1 > 0$. This implies $c_n \geq 0$.

(ii) In view of (i), Bernoulli's inequality is applicable to c_n and we obtain

$$c = \left(\sqrt[n]{c}\right)^n = \left(1 + \sqrt[n]{c} - 1\right)^n = (1 + c_n)^n \geq 1 + nc_n,$$

which can be rewritten as

$$c_n \leq \frac{c-1}{n}.$$

For $n \geq m$ it follows that

$$c_n \leq \frac{c-1}{n} \leq \frac{c-1}{m}.$$

Therefore

$$\begin{aligned}
\lim_{n\to\infty} \sup c_n &= \lim_{n\to\infty} \sup\{c_n \ : \ n \geq m\} \\
&\leq \lim_{n\to\infty} \sup \left\{ \frac{c-1}{n} \ : \ n \geq m \right\} \\
&= \lim_{m\to\infty} \frac{c-1}{m} = 0.
\end{aligned}$$

(iii) Since $c_n \geq 0$, $\lim_{n\to\infty} \inf c_n \geq 0$. Hence

$$0 \geq \lim_{n\to\infty} \inf c_n \leq \lim_{n\to\infty} \sup c_n \leq 0,$$

and all inequalities are equalities. Therefore $\lim_{n\to\infty} c_n = 0$ by Theorem 2.11.

(b) The case $c > 1$ has been proved in part (a), and $c = 1$ is trivial since $\sqrt[n]{1} = 1$. For $0 < c < 1$ we have $\frac{1}{c} > 1$, and so

$$\lim_{n\to\infty} \sqrt[n]{c} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{\frac{1}{c}}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{\frac{1}{c}}} = 1$$

by part (a)(iii). $\square$

2. Consider $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$ for all $n \in \mathbb{N}$. Show that

$$\lim_{n\to\infty} \inf \left|\frac{a_{n+1}}{a_n}\right| \leq \lim_{n\to\infty} \inf \sqrt[n]{a_n} \leq \lim_{n\to\infty} \sup \sqrt[n]{a_n} \leq \lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right|$$

What can you say if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists or is ∞?

Proof.

The middle inequality is trivial. Next we want to show the left inequality. Let

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

It is clear that all terms in the inequalities to be shown are nonnegative, so that the case $L = 0$ is obvious. Now let

$L \in (0, \infty]$ and choose $L' \in (0, L)$, $L'' \in (L', L)$, and $\epsilon > 0$ such that $\epsilon < 1 - \dfrac{L'}{L''}$. For natural numbers $m > k \geq 1$ we have

$$\frac{|a_m|}{|a_k|} = \left|\frac{a_m}{a_{m-1}}\right| \cdot \left|\frac{a_{m-1}}{a_{m-2}}\right| \cdots \left|\frac{a_{k+2}}{a_{k+1}}\right| \cdot \left|\frac{a_{k+1}}{a_k}\right|.$$

By definition of lim inf there is $K \in \mathbb{N}$ such that for natural numbers $n \geq K$ we have

$$\inf \left\{ \left|\frac{a_{n+1}}{a_n}\right| \, : \, k \geq n \right\} \geq L''$$

Then it follows for $m > k \geq K$ that

$$\frac{|a_m|}{|a_k|} \geq (L'')^{m-k},$$

which can be rewritten as

$$|a_m| \geq (L'')^m \frac{|a_k|}{(L'')^k}.$$

In view of part 1(b) there is a natural number K_1 such that

$$\sqrt[m]{\frac{|a_K|}{(L'')^K}} \geq 1 - \epsilon$$

for all $m \geq \max\{K, K_1\}$. Hence

$$\sqrt[m]{|a_m|} \geq L''(1-\epsilon) > L',$$

which implies

$$\lim_{n\to\infty} \inf \sqrt[m]{|a_m|} \geq L'.$$

Since $L' \in (0, L)$ was arbitrary, it follows that

$$\lim_{n\to\infty} \inf \sqrt[n]{|a_n|} \geq L.$$

Note that this includes the case $L = \infty$.
For any sequence (b_n) of positive numbers we have

$$\begin{aligned}
\lim_{n\to\infty} \inf b_n &= \lim_{n\to\infty} \inf \{b_m : m \geq n\} = \lim_{n\to\infty} \frac{1}{\sup\left\{\dfrac{1}{b_m} : m \geq n\right\}} \\
&= \frac{1}{\lim\limits_{n\to\infty} \sup\left\{\dfrac{1}{b_m} : m \geq n\right\}} = \frac{1}{\lim\limits_{n\to\infty} \sup \dfrac{1}{b_n}}.
\end{aligned}$$

Hence, applying the part we have already proved to the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ one gets

$$\begin{aligned}
\limsup_{n\to\infty} \sqrt[n]{|a_n|} &= \frac{1}{\liminf_{n\to\infty} \sqrt[n]{\frac{1}{|a_n|}}} \\
&\leq \frac{1}{\liminf_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|} \\
&= \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|
\end{aligned}$$

which proves the right hand inequality of the statement in this part 2.

If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists or is ∞, then all four term in that chain of inequalities are equal, and hence

$$\boxed{\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \sqrt[n]{|a_n|}\,.}$$

□

3. Consider the power series $\sum_{n=1}^{\infty} a_n(x-a)^n$ with $a_n \neq 0$ for all $n \in \mathbb{N}$. Using tutorial problem 2 above or otherwise, prove that if $\boxed{R = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|}$ exists or is ∞, then R is the radius of convergence of the power series.

Proof.

By the last part of the proof of part 2,

$$R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{|a_n|}} = \frac{1}{\liminf_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

$\square$

4. Prove that $\boxed{\lim_{n\to\infty} \sqrt[n]{n} = 1.}$

Proof.

This follows from

$$\lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = 1$$

and the last part of the proof of part 2.

5. Find the radius and interval of convergence for each of the following power series:

(a) $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$, (b) $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$, (c) $\sum_{n=1}^{\infty} n^n x^n$,

(d) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n \sqrt{n}}$, (e) $\sum_{n=1}^{\infty} \frac{(-2x)^n}{n^3}$, (f) $\sum_{n=1}^{\infty} (-1)^n x^n$.

(g) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} x^n$, (h) $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n^3}$, (i) $\sum_{n=1}^{\infty} \frac{(nx)^n}{(2n)!}$.

Solutions. When using the Ratio Test, we will use part 3 above, if applicable, to find the radius of convergence.

(a) By the Ratio Test, the radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{\frac{2^n}{n}}{\frac{2^{n+1}}{n+1}} \right| = \lim_{n\to\infty} \frac{n+1}{2n} = \frac{1}{2}$$

For $x = \frac{1}{2}$, the series is the harmonic series, which diverges, and for $x = -\frac{1}{2}$, the series is the alternating harmonic series, which converges. Hence the interval of convergence is $\left[-\frac{1}{2}, \frac{1}{2}\right)$.

(b) By the Ratio Test, the radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{\frac{1}{n^n}}{\frac{1}{(n+1)^{n+1}}} \right| = \lim_{n\to\infty} (n+1)\left(1+\frac{1}{n}\right)^n = \infty \,.\, e = \infty \,.$$

Hence the interval of convergence is $\mathbb{R}$.

(c) By the Root Test, the radius of convergence is

$$R = \frac{1}{\lim\limits_{n\to\infty} \left| \sqrt[n]{n^n} \right|} = \frac{1}{\lim\limits_{n\to\infty} n} = 0\,.$$

Hence the power series only converges for $x = 0$.

(d) By the Ratio Test, the radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{3^{n+1}\sqrt{n+1}}{3^n\sqrt{n}} \right| = \lim_{n\to\infty} 3\sqrt{1+\frac{1}{n}} = 3\,.$$

For $x = 4$, the series is a divergent p-series, where as for $x = -2$, the series is a convergent alternating series. Hence the interval of convergence is $[-2, 4)$.

(e) By the Ratio Test, the radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{\frac{(-2)^n}{n^3}}{\frac{(-2)^{n+1}}{(n+1)^3}} \right| = \lim_{n\to\infty} \frac{1}{2}\left(1 + \frac{1}{n}\right) = \frac{1}{2}\,.$$

For $x = -\frac{1}{2}$, the series is a convergent p-series, and for $x = \frac{1}{2}$, the series is of the absolute values is this p-series and therefore also converges. Hence the interval of convergence is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

(f) By the Root Test, the radius of convergence is

$$R = \frac{1}{\lim\limits_{n\to\infty} \left| \sqrt[n]{(-1)^n} \right|} = 1\,.$$

For $x = 1$ and $x = -1$, the series is a divergent geometric series. Hence the interval of convergence is $(-1, 1)$.

(g) By the Root Test, the radius of convergence is

$$R = \frac{1}{\left|\sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}}\right|} = \frac{1}{\lim\limits_{n\to\infty}\left(\frac{n}{n+1}\right)^{n}} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e\,.$$

For $x = e$ and $x = -e$, let a_n be the general term of the series. Then

$$|a_n| = \left(\frac{n}{n+1}\right)^{n^2} e^n = \left(1+\frac{1}{n}\right)^{-n^2} e^n = \left(\left(1+\frac{1}{n}\right)^{-n} e\right)^{n}\,.$$

From Example 2.4, we conclude that $\left(\left(1+\frac{1}{n}\right)^{-n} e\right)$ is a decreasing sequence which converges to 1. Hence

$$|a_n| = \left(\left(1+\frac{1}{n}\right)^{-n} e\right)^{n} \geq 1\,,$$

and the Test for Divergence shows that the series over a_n, that is, the given series at the end points, does not converge. Hence the interval of convergence is $(-e, e)$.

(h) By the Ratio Test, the radius of convergence is

$$R = \lim_{n\to\infty}\left|\frac{(n+1)^3}{n^3}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3} = 1\,.$$

For $x = -2$ and $x = -4$, the series of the absolute values is a convergent p-series. Hence the interval of convergence is $[-4, -2]$.

(i) By the Ratio Test, the radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{n^n (2(n+1))!}{(n+1)^{n+1}(2n)!} \right| = \lim_{n\to\infty} \frac{2(n+1)(2n+1)}{(n+1)\left(1+\frac{1}{n}\right)^n} = \infty\,.$$

Hence the interval of convergence is $\mathbb{R}$.