# MATH2001–Basic Analysis Final Examination June 2011

Time: 90 minutes      Total marks: 90 marks

## MEMO

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**SECTION A  Multiple choice**

| **Question** | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| **Answer** | C | A | B | E | B | E | A | E | B |
| **Marks** | 3 | 3 | 1 | 3 | 2 | 3 | 3 | 3 | 3 |

**SECTION B**

| **Question** | 1 and 2 | 3 and 4 | 5 and 6 |
|---|---|---|---|
| **Marker** | M Mohlala | S Bau | J-C Ndogmo |

**Question** 1 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . [9 marks]

Write down the definitions of the following limits of functions where $a, L \in \mathbb{R}$ and $f$ is a real-valued functions. Also write down the assumptions for the domain of $f$.

(a) $\lim\limits_{x\to a^-} f(x) = L$. (3)

**Solution.** Assume there is $c < a$ such that $(c, a) \subset \mathrm{dom}(f)$. ✓ Then $\lim\limits_{x\to a^-} f(x) = L$ if and only if

$$\forall \varepsilon > 0\ \exists \delta > 0\ (a - \delta < x < a \Rightarrow |f(x) - L| < \varepsilon) \quad ✓✓$$

(b) $\lim\limits_{x\to a^+} f(x) = -\infty$. (3)

**Solution.** Assume there is $c > a$ such that $(a, c) \subset \mathrm{dom}(f)$. ✓ Then $\lim\limits_{x\to a^+} f(x) = -\infty$ if and only if

$$\forall\, A(< 0)\ \exists \delta > 0\ (a < x < a + \delta \Rightarrow f(x) < A) \quad ✓✓$$

(c) $\lim\limits_{x\to \infty} f(x) = L$. (3)

**Solution.** Assume there is $a \in \mathbb{R}$ such that $(a, \infty) \subset \mathrm{dom}(f)$. ✓ Then $\lim\limits_{x\to \infty} f(x) = L$ if and only if

$$\forall \varepsilon > 0)\ \exists K > 0\ (x > K \Rightarrow |f(x) - L| < \varepsilon) \quad ✓✓$$

**Question 2** ............................................................ [14 marks]

Prove from the definitions that

(a) $\lim_{x\to -1^-} \dfrac{1}{x+1} = -\infty$ (6)

**Solution.** Let $A < 0$. $(\checkmark)$

We have to find $\delta > 0$ such that $-1-\delta < x < -1$ implies $\dfrac{1}{x+1} < A$. $\checkmark$

Now let $x < -1$. Since $x+1 < -1+1 < 0$, $\checkmark$

$$\begin{aligned}\frac{1}{x+1} < A &\Leftrightarrow x+1 > \frac{1}{A}\\ &\Leftrightarrow x > -1 + \frac{1}{A} \quad \checkmark\end{aligned}$$

Now put $\delta = -\dfrac{1}{A}$. Then $\delta > 0$ (since $A < 0$). $\checkmark$

And by the above calculations, $-1-\delta < x < 1$ implies $-1+\dfrac{1}{A} < x < -1$ $(\checkmark)$

and thus $\dfrac{1}{x+1} < A$. $\checkmark$

(b) $\lim_{x\to -\infty} \dfrac{x^2+1}{x^2-1} = 1$ (8)

**Solution.** Let $\varepsilon > 0$. $(\checkmark)$

First we calculate

$$\begin{aligned}\left|\frac{x^2+1}{x^2-1} - 1\right| &= \left|\frac{x^2+1-(x^2-1)}{x^2-1}\right| \quad \checkmark\\ &= \frac{2}{|x^2-1|} \quad \checkmark\end{aligned}$$

For $x < -1$, $x^2 > 1$, and therefore

$$\begin{aligned}\left|\frac{x^2+1}{x^2-1} - 1\right| < \varepsilon &\Leftrightarrow \frac{2}{x^2-1} < \varepsilon \quad \checkmark\\ &\Leftrightarrow x^2 - 1 > \frac{1}{2\varepsilon} \quad \checkmark\\ &\Leftrightarrow x^2 > \frac{1}{2\varepsilon} + 1. \quad \checkmark\end{aligned}$$

Choosing $A = -\sqrt{\dfrac{1}{2\varepsilon} + 1}$ $\checkmark$

or alternatively $A = -\dfrac{1}{2\varepsilon} - 1$ $\checkmark$

we get

$$
\begin{aligned}
x < A(< -1) &\Rightarrow x^2 > \frac{1}{2\varepsilon} + 1 \quad (\checkmark) \\
&\Rightarrow \left| \frac{x^2+1}{x^2-1} - 1 \right| < \varepsilon \quad \checkmark
\end{aligned}
$$

or alternatively

$$
\begin{aligned}
x < A(< -1) &\Rightarrow x^2 > A^2 > |A| = \frac{1}{2\varepsilon} + 1 \quad (\checkmark) \\
&\Rightarrow \left| \frac{x^2+1}{x^2-1} - 1 \right| < \varepsilon \quad \checkmark
\end{aligned}
$$

**Question** 3 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . [10 marks]

Let $a \in \mathbb{R}$ and let $f$ and $g$ be continuous at $a$. Prove that the product $fg$ is also continuous at $a$.

**Solution.** One can use first principles.

Alternatively, one can use that a function $h$ is continuous at $a$ if and only $\lim_{x\to a} h(x) = h(a)$.

From the rules of limits, applied to the continuous functions $f$ and $g$, one has

$$
\begin{aligned}
\lim_{x\to a}(fg)(x) &= \lim_{x\to a} f(x)g(x) \\
&= \lim_{x\to a} f(x) \lim_{x\to a} g(x) \\
&= f(a)g(a)[= (fg)(a)]
\end{aligned}
$$

**Question 4** . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . [17 marks]

Recall that $e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$.

(a) Show that $e^x \ge 1 + x$ for all $x \in \mathbb{R}$. (6)

**Hint.** You may use Bernoulli's inequality.

**Solution.** Let $x \in \mathbb{R}$ and $n \in \mathbb{N}$ with $n > |x|$. Then $\frac{x}{n} > -1$, and with the aid of Bernoulli's inequality we calculate

$$\begin{aligned} \left(1+\frac{x}{n}\right)^n &\ge 1 + n\frac{x}{n} = 1 + x \\ \Rightarrow e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &\ge 1 + x. \end{aligned}$$

Thus $e^x \ge 1 + x$ for all $x \in \mathbb{R}$.

(b) Using part (a) or otherwise show that $e^x \le \dfrac{1}{1-x}$ for $x > -1$. (3)

**Solution.** Replacing $x$ with $-x$ in the result of (a), $e^{-x} \ge 1 - x$. Taking inverses for $x < 1$, i. e., $1 - x > 0$, leads to $e^x = \dfrac{1}{e^{-x}} \le \dfrac{1}{1-x}$ for $x < 1$.

(c) Show that $\lim_{h\to 0} \dfrac{e^h - 1}{h}$ exists, and find this limit. (5)

**Solution.**
(a) and (b) give for $h < 1$ that

$$h \le e^h - 1 \le \frac{1}{1-h} - 1 = \frac{h}{1-h} \Rightarrow \begin{cases} 1 \le \dfrac{e^h-1}{h} \le \dfrac{1}{1-h} & \text{if } h > 0 \\ \dfrac{1}{1-h} \le \dfrac{e^h-1}{h} \le 1 & \text{if } h < 0 \end{cases}$$

Application of the Sandwich Theorem leads to

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1$$

since $\lim_{h\to 0} \dfrac{1}{1-h} = 1$.

(d) Show that $e^x$ is differentiable and find its derivative. (3)

**Solution.** By (c),

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0}\frac{e^x e^h - e^x}{h} = e^x \lim_{h\to 0}\frac{e^h-1}{h} = e^x$$

**Question** 5 .............................................................. [6 marks]

Let $f : [0, 2] \to \mathbb{R}$ be continuous with $f(0) = f(2)$. Show that there is $c \in [0, 1]$ such that $f(c) = f(c + 1)$.

**Solution.**

Consider the function

$$g(x) = f(x) - f(x + 1), \quad x \in [0, 1]. \quad \checkmark$$

Then $g$ is continuous on $[0, 1]$ $\quad (\checkmark)$

with

$$g(0) = f(0) - f(1) \quad (\checkmark)$$

and

$$g(1) = f(10 - f(2) = f(1) - f(0) = -g(0) \quad \checkmark$$

If $g(0) = 0$, then $g(1) = -g(0) = 0 = g(0)$, so that we can take $c = 0$. $\quad (\checkmark)$

If $g(0) \neq 0$, then $g(0)$ and $g(1) = -g(0)$ have opposite signs, $\quad (\checkmark)$

and by the Intermediate Value Theorem, there is $c \in (0, 1)$ such that $g(c) = 0$. $\quad \checkmark$

Hence

$$0 = g(c) = f(c) - f(c + 1), \quad (\checkmark)$$

so that

$$f(c) = f(c + 1). \quad (\checkmark)$$

**Question** 6 . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . [10 marks]

(a) Let $(a_n)$ be a sequence of real numbers with $a_n \neq 0$ for all $n$. Prove that (7)

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|.$$

**Solution.**

The result is clear if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \infty.$ (✓)

So let $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < \infty$ and let $\varepsilon > 0$. (✓)

Then there is $K \in \mathbb{N}$ such that

$$\left|\frac{a_{n+1}}{a_n}\right| \leq L + \varepsilon \text{ if } n \geq K. \quad \checkmark$$

Hence, for $n \geq K$,

$$\begin{aligned} \left|\frac{a_n}{a_K}\right| &= \left|\frac{a_n}{a_{n-1}}\right| \left|\frac{a_{n-1}}{a_{n-2}}\right| \cdots \left|\frac{a_{K+1}}{a_K}\right| \\ &\leq (L+\varepsilon)^{n-K}. \quad \checkmark \end{aligned}$$

Thus

$$\begin{aligned} |a_n| &\leq |a_K|(L+\varepsilon)^{n-K} \\ &= \frac{|a_K|}{(L+\varepsilon)^K}(L_\varepsilon)^n \quad \checkmark \end{aligned}$$

NB: For $c > 0$, $\ln \sqrt[n]{c} = \frac{1}{n} \ln c \to 0$ as $n \to \infty$,
so that $\sqrt[n]{c} \to 1$ as $n \to \infty$ .
Hence there is $K_1 \geq K$ such that

$$\frac{|a_K|}{(L+\varepsilon)^K} \leq (1+\varepsilon)^n \text{ for } n \geq K_1. \quad \checkmark$$

Therefore,

$$\sqrt[n]{|a_n|} \leq (1+\varepsilon)(L+\varepsilon) \text{ for } n \geq K_1 \quad (\checkmark)$$

This gives

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} \leq (1+\varepsilon) \quad \checkmark$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} \leq L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|. \quad (\checkmark)$$

(b) Give an example for a sequence $(a_n)$ where the inequality in (a) is strict. (3)

Write down the values of for $\limsup_{n\to\infty} \sqrt[n]{|a_n|}$ and $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ for your example. No proof is required.

**Solution.** Let $a_n = 1$ if $n$ is odd and $a_n = 2$ if $n$ is even. Then

$$\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1 \quad \text{and} \quad \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 2.$$