Tutorial 2.2.1.

1. Prove Theorem 2.9.

2. Use suitable rules or first principles to find
   (a) $\lim_{n\to\infty} (n^2+2n-10)$ (b) $\lim_{n\to\infty} \left(n-\frac{1}{n}\right)$ (c) $\lim_{n\to\infty} \frac{n^3-3n^2}{n+1}$

3. Prove that if $\lim_{n\to\infty} |\, a_n \,| = \infty$, then (a_n) diverges.

4. Prove that if $p \in \mathbb{N}$, $p > 0$, then $n^p \to \infty$ as $n \to \infty$.

5. Define a sequence as follows:

$$a_0 = 0,\ a_1 = \frac{1}{2},\ a_{n+1} = \frac{1}{3}\left(1 + a_n + a_{n-1}^3\right) \quad \text{for } n \ge 2\,.$$

   (a) Use induction to show that $0 \le a_n \le \frac{2}{3}$ for all $n \in \mathbb{N}$.
   (b) Use induction to show that $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$.
   (c) Explain why we may conclude that (a_n) converges.
   (d) Using the fact that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n-1} = \lim_{n\to\infty} a_{n+1}$, find $\lim_{n\to\infty} a_n$.

6. Let $\lim_{n\to\infty} a_n = \infty$, $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} c_n = 0$. Show, by giving examples, that no general conclusion can be made about the behaviour of the following sequences:
   (a) $a_n - b_n$, (b) $a_n c_n$, (c) $\frac{a_n}{b_n}$, (d) $\frac{a_n}{c_n}$.

7. Let (a_n) and (b_n) be sequences such that $a_n \le b_n$ for all $n \in \mathbb{N}$. Show that

$$\lim_{n\to\infty} \inf a_n \le \lim_{n\to\infty} \inf b_n \text{ and } \lim_{n\to\infty} \sup a_n \le \lim_{n\to\infty} \sup b_n\,.$$

8. (a) Show that $\exp(x) = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n$ exists for all $x \in \mathbb{R}$ and that $\exp(1) = e$.
   Hint. Adapt the proof of Example 2.2.3.
   (b) Use Bernoulli's inequality to prove that

$$\lim_{n\to\infty} \left(\frac{1+\frac{x+y}{n}}{1+\frac{x+y}{n}+\frac{xy}{n^2}}\right)^n = 1$$

   for all $x, y \in \mathbb{R}$.
   (c) Use (b) to show that $\exp(x+y) = \exp(x)\exp(y)$ for all $x, y \in \mathbb{R}$.
   (d) Show that $\exp(x) \ge 1 + x$ for all $x > 0$.
   (e) Show that $\exp(x) > 0$ for all $x \in \mathbb{R}$.
   (f) Show that exp is strictly increasing.
   (g) Show that $\exp(n) = e^n$ for all $n \in \mathbb{Z}$.