Chapter 7:
Groups of Symmetry

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LEARNING OUTCOMES FOR THE LECTURE

By the end of this lecture, students will be able to:

- ♣ define a permutation
- ♣ define the group of permutations of a non-empty set
- ♣ find all elements in S_n, the group of permutations of n elements
- ♣ write the Cayley table of S_3
- ♣ use table notation to represent a permutation

Permutations

Let X be a non empty set with n elements. We may choose to list these elements as follows: $X = \{a_1, a_2, \cdots, a_n\}$ where $|X| = n$.
A permutation is a rearrangement of these elements. In statistics we are often interested in the number of rearrangements of the elements in this set. In maths we are more interested in the rearrangements themselves and what happens as we follow one re-arrangement by another.
Since X may represent any set of n elements, for convenience we may represent X as follows;
$X = \{1, 2, 3, \cdots, n\}$ where r represents the r^{th} elements of the set. We formally define a permutation as follows:

SYMMETRY GROUPS

Definition of a permutation

If X is a non-empty set a permutation α on X is a bijection $\alpha : X \to X$.(That is α is a mapping, well defined, one to one and onto X.)

Definition (7.2.1)

If $X \neq \emptyset$ and $|X| = n$, we say S_X or just S_n is the group of permutations of X. $S_n = \{\alpha : X \to X \quad | \quad \alpha \quad \text{bijection}\}$.

We consider the operation of composition on S_X to make $S_X = S_n$ a group.

Example (7.2.2 (1))

Let $X = \{1\}$*. Then* $S_X = S_1 = \{e\}$ *where*
$e : X \to X$ *is the identity mapping taking 1 to 1.*
$|X| = 1$ *and* $|S_X| = |S_1| = 1$.

Example (7.2.2 (2))

Let $X = \{1, 2\}$*. Then* $S_X = S_2 = \{e, \alpha\}$ *where*

$$\begin{aligned} e : 1 &\to 1 \\ 2 &\to 2 \end{aligned} \qquad \textit{and} \qquad \begin{aligned} \alpha : 1 &\to 2 \\ 2 &\to 1 \end{aligned}$$

$|X| = 2$ *and* $|S_X| = 2$.

Example (7.2.2 (3))

$X = \{1, 2, 3\}$. *Then* $S_X = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ *where*

$$\alpha_1 = e : \begin{array}{l} 1 \to 1 \\ 2 \to 2 \\ 3 \to 3 \end{array} \quad ; \quad \alpha_2 = \rho : \begin{array}{l} 1 \to 2 \\ 2 \to 3 \\ 3 \to 1 \end{array} \quad ; \quad \alpha_3 = \rho^2 : \begin{array}{l} 1 \to 3 \\ 2 \to 1 \\ 3 \to 2 \end{array}$$

$$\alpha_4 = R : \begin{array}{l} 1 \to 1 \\ 2 \to 3 \\ 3 \to 2 \end{array} \quad ; \quad \alpha_5 = \rho R : \begin{array}{l} 1 \to 2 \\ 2 \to 1 \\ 3 \to 3 \end{array} \quad ; \quad \alpha_6 = \rho^2 R : \begin{array}{l} 1 \to 3 \\ 2 \to 2 \\ 3 \to 1 \end{array}$$

ρ is called a rotation.
R is called a reflection.

Check that $\rho^3 = e$, $R^2 = e$ and $\rho R = R\rho^2$.

(i) $\rho^3(1) = \rho^2(\rho(1)) = \rho^2(2) = \rho(\rho(2)) = \rho(3) = 1$.
Similarly, $\rho^3(2) = 2$; $\rho^3(3) = 3$.
$\therefore \quad \rho^3 = e$.

(ii) $R^2(1) = R(R(1)) = R(1) = 1$. Similarly,
$R^2(2) = R(R(2)) = R(3) = 2$ and $R^2(3) = R(2) = 3$.
$\therefore \quad R^2 = e$.

(iii) $\rho R(1) = \rho(1) = 2$ and $R\rho^2(1) = R(\rho(2)) = R(3) = 2$.
$\rho R(2) = \rho(3) = 1$ and $R\rho^2(2) = R(1) = 1$.
$\rho R(3) = \rho(2) = 3$ and $R\rho^2(3) = R(2) = 3$.
$\therefore \quad \rho R = R\rho^2$.

The equation $\rho^3 = R^2 = e$ and $\rho R = R\rho^2$ is called the presentation of S_3.

How can we write down the Cayley Table for S_3 under composition using the presentation of S_3.

0	e	ρ	ρ^2	R	ρR	$\rho^2 R$
e	e	ρ	ρ^2	R	ρR	$\rho^2 R$
ρ	ρ	ρ^2	e	ρR	$\rho^2 R$	R
ρ^2	ρ^2	e	ρ	$\rho^2 R$	R	ρR
R	R	$\rho^2 R$	ρR	e	ρ^2	ρ
ρR	ρR	R	$\rho^2 R$	ρ	e	ρ^2
$\rho^2 R$	$\rho^2 R$	ρR	R	ρ^2	ρ	e

Note $e^{-1} = e$; $\rho^{-1} = \rho^2$; $R^{-1} = R$; $(\rho R)^{-1} = \rho R$; $(\rho^2)^{-1} = \rho$; $(\rho^2 R)^{-1} = \rho^2 R$;

What about $R\,p$?

$R\,p = R\,p\,R^2 = R\,(p\,R)\,R = R\,(R\,p^2)\,R = R^2\,p^2\,R = p^2\,R$

($R^2 = e$) (associative) (from presentation)

Hence we see that $|S_3| = 6 = 3!$ and each element is a unit while e is unity. We consider

$K \subseteq S_3$ with $K = \{e, \rho, \rho^2\}$

$H \subseteq S_3$ with $H = \{e, R\}$

$J \subseteq S_3$ with $J = \{e, \rho R\}$

$I \subseteq S_3$ with $K = \{e, \rho^2 R\}$

o	e	ρR
e	e	ρR
ρR	ρR	e

Cayley table for J

K, H, J and I are each closed under the binary operation of composition and each has all elements units. Also each contains the unity e. Therefore K, H, J and I are subgroups of S_3.

TABLE NOTATION

(NOTE 7.2.3 (1))

$X = \{1, 2, \cdots, n\}$ and $\alpha : X \to X$ is a bijection. So if $r \in X$ then $\alpha(r) \in X$. We write

$$\alpha : \quad \begin{pmatrix} 1 & 2 & \cdots & n \\ \alpha(1) & \alpha(2) & \cdots & \alpha(n) \end{pmatrix}$$

Example $\rho \in S_3$ and $R \in S_3$ can be written as:

$$\rho = \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}; R = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\rho^2 R = \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}; \rho R = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

We can use this notation to calculate $\rho R \rho$

first map is ρ

$$\rho R \rho = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = R$$

Using ρ and R we can calculate $\rho R \rho$

$$\rho = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

i.e. as above

$1 \to 2 \to 3 \to 1$

$2 \to 3 \to 2 \to 3$

$3 \xrightarrow{\rho} 1 \xrightarrow{R} 1 \xrightarrow{\rho} 2$

(first map)