Section 4.2: Continuous Functions

Definition 4.7 (Continuity of a function at a point)

Let f be a real function, $a \in \mathbb{R}$ and assume that the domain of f contains a neighborhood of a, that is, $f(x)$ is defined for all x in a neighborhood of a. We say that f is continuous at a if

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in (a-\delta, a+\delta), f(x) \in (f(a)-\varepsilon, f(a)+\varepsilon)$$

i.e.

$$\forall \varepsilon > 0, \exists \delta > 0, (|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon).$$

We realize that the definition of the existence of a limit of a function at a point and continuity at that point are very similar, but that there are subtle (and important) differences.

For limits, f does not need to be defined at a, and even if $f(a)$ exists, this value is not used at all when finding the limit of the function f at a.

We conclude

f is continuous at a

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, (|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon)$ by definition

$\Leftrightarrow \forall \varepsilon > 0, \exists \delta > 0, (0 < |x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon)$

$\therefore x = a \Rightarrow f(x) = f(a)$

$\Leftrightarrow \lim_{x \to a} f(x) = f(a).$

Hence, we have proven the following theorem.

Theorem 4.7

f is continuous at a if and only if the following three conditions are satisfied:

1. *$f(a)$ is defined, i.e., a is in the domain of f,*
2. *$\lim_{x \to a} f(x)$ exists, i.e., $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$, and*
3. *$f(a) = \lim_{x \to a} f(x)$.*

Example 4.6

1. Let

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 2 & \text{if } x = 2. \end{cases}$$

Then $\lim_{x \to 2} f(x) = 4$ exists, but this limit is different from $f(2) = 2$. Hence, f is not continuous at 2.

2. Let

$$f(x) = \frac{\sin x}{x}$$

for $x \neq 0$ while f is not defined at $x = 0$. Then $\lim_{x\to 0} f(x) = 1$ exists, but f is not defined at 0. Hence f is not continuous at 0.

3. Let

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Then $\lim_{x\to 0} f(x) = 1$ exists and $f(0) = 1$. Hence f is continuous at 0. ■

Theorem 4.8

If f and g are continuous at a and if $c \in \mathbb{R}$, then

1. *The sum $f + g$,*
2. *The difference $f - g$,*
3. *The product fg,*
4. *The quotient $\frac{f}{g}$ if $g(a) \neq 0$, and*
5. *The scalar multiple cf*

are functions that are also continuous at a.

Proof

The statements follow immediately from the limit laws, Theorem 4.3, and Theorem 4.7. For example, for (3.) we have

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a),$$

and then Theorem 4.3 gives

$$\lim_{x\to a} (fg)(x) = \lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = f(a) \cdot g(a) = (fg)(a).$$

Then Theorem 4.7 says that fg is continuous at a. ■

Recall that the composite $g \circ f$ of two functions f and g is defined by

$$(g \circ f)(x) = g\big(f(x)\big).$$

Theorem 4.9

If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.

Proof

Let $\varepsilon > 0$. Since g is continuous at $f(a)$, there is $\eta > 0$ such that

$$|y - f(a)| < \eta \Rightarrow \left|g(y) - g\big(f(a)\big)\right| < \varepsilon. \tag{1}$$

Since f is continuous at a, there is $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \eta. \tag{2}$$

Putting $y = f(x)$ in (1) it follows from (1) and (2) that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \eta \Rightarrow \left|g\big(f(x)\big) - g\big(f(a)\big)\right| < \varepsilon$$

that is,

$$|x - a| < \delta \Rightarrow |(g \circ f)(x) - (g \circ f)(a)| < \varepsilon.$$

Hence $g \circ f$ is continuous at a. ■

Definition 4.8

1. A function f is continuous from the right at a if $\lim_{x \to a^+} f(x) = f(a)$.
2. A function f is continuous from the left at a if $\lim_{x \to a^-} f(x) = f(a)$.

Example 4.7

Let

$$f(x) = \begin{cases} \dfrac{|x| + x}{2x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Determine the right and left continuity of f at $x = 0$.

Solution

$f(0) = 0$ whilst

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{|x| + x}{2x} = \lim_{x \to 0^-} \frac{-x + x}{2x} = \lim_{x \to 0^-} 0 = 0$$

and

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{|x| + x}{2x} = \lim_{x \to 0^+} \frac{x + x}{2x} = \lim_{x \to 0^+} 1 = 1.$$

Since

$$f(0) = 0 = \lim_{x \to 0^-} f(x),$$

f is continuous from the left at $x = 0$. Since

$$f(0) = 0 \neq 1 = \lim_{x \to 0^+} f(x),$$

f is not continuous from the right at $x = 0$. ∎

Note:

1. It is easy to show that f is continuous at a if and only if f is continuous from the right and continuous from the left at a.

2. If $a \in dom(f)$ and if there is $\varepsilon > 0$ such that
$$dom(f) \cap (a-\varepsilon, a+\varepsilon) = (a-\varepsilon, a],$$
then we say that f is continuous at a if
$$\lim_{x\to a^-} f(x) = f(a).$$

3. If $a \in dom(f)$ and if there is $\varepsilon > 0$ such that
$$dom(f) \cap (a-\varepsilon, a+\varepsilon) = [a, a+\varepsilon),$$
then we say that f is continuous at a if
$$\lim_{x\to a^+} f(x) = f(a).$$

4. The convention in (2.) and (3.) is consistent with what you will learn in General Topology about continuity. Just note that the condition $|f(x) - f(a)| < \varepsilon$ has to be checked for all $x \in dom(f)$ which satisfy $|x - a| < \delta$.

Lemma 4.1

If $f(x) \to b$ as $x \to a$ (a^+, a^-) and g is continuous at b, then $g\big(f(x)\big) \to g(b)$ as $x \to a$ (a^+, a^-), which can be written, e.g., as

$$\lim_{x\to a} g\big(f(x)\big) = g\left(\lim_{x\to a} f(x)\right).$$

Proof

The function

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in dom(f), x \neq a, \\ b & \text{if } x = a \end{cases}$$

is continuous (from the right, from the left) at a. Hence the result follows from Theorem 4.9.

∎

Definition 4.9

A function is continuous on a set $X \subset \mathbb{R}$ if f is continuous at each $x \in X$. Here continuity is understood in the sense of the above note with $X = dom(f)$. A function is said to be continuous if it is continuous on its domain.

Example 4.8

Show that $f(x) = \sqrt{x^2 - 4}$ is continuous.

Solution

The domain of f is

$$\{x \in \mathbb{R} : |x| \geq 2\} = (-\infty, -2] \cup [2, \infty).$$

By Theorem 4.8, the function $x \longmapsto x^2 - 4$ is continuous on $\mathbb{R}$, and by Theorem 4.3 (k), the square root is continuous at each positive number. So also, the composite function f is continuous on $(-\infty, -2) \cup (2, \infty)$. Also, the proof of Theorem 4.3 (k) can be easily adapted to show that the square root is continuous from the right at 0. Then it easily follows that f is continuous (from the right) at 2 and continuous (from the left) at -2. ∎

Theorem 4.10

The following functions are continuous on their domains.

1. *Polynomials* $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, $a_i \in \mathbb{R}$, $n \in \mathbb{N}$.
2. *Rational functions* $\frac{p(x)}{q(x)}$, p *and* $q \neq 0$ *polynomials.*
3. *Sums, differences, products, and quotients of continuous functions.*
4. *Root functions.*
5. *The trigonometric functions* $\sin x$, $\cos x$, $\tan x$, $\csc x$, $\sec x$, *and* $\cot x$.
6. *The exponential function* $\exp(x)$.
7. *The absolute value function* $|x|$.

Proof

(1), (2) and (3) easily follow from previous theorems on limits and continuity, as does (7). However, (7) can be easily proved directly:

For each $\varepsilon > 0$ let $\delta = \varepsilon$. Then, for $|x - a| < \delta$ we have

$$\big||x| - |a|\big| \leq |x - a| < \delta = \varepsilon.$$

The continuity of sin and cos follows from the sum of angles formulae and from the limits proved in Calculus I (the proofs used the Sandwich Theorem, which now has been proved). The continuity of the other trigonometric functions then follows from part (3).

Finally, the continuity of exp is a tutorial problem. ∎

Theorem 4.11

Let $a \in \mathbb{R}$ and let f be a real function which is defined in a neighborhood of a. Then f is continuous at a if and only if for each sequence (x_n) in $dom(f)$ with $\lim_{n\to\infty} x_n = a$ the sequence $f(x_n)$ satisfies $\lim_{n\to\infty} f(x_n) = f(a)$.

Proof

($\rightarrow$) Let (x_n) be a sequence in $dom(f)$ with $\lim_{n\to\infty} x_n = a$. We must show that

$$\lim_{n\to\infty} f(x_n) = f(a).$$

Hence, let $\varepsilon > 0$. Since f is continuous at a, there is $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Since $\lim_{n\to\infty} x_n = a$, there is $K \in \mathbb{R}$ such that for $n > K$, $|x_n - a| < \delta$. But then, by the previous implication, $|f(x_n) - f(a)| < \varepsilon$ for $n > K$.

($\leftarrow$) Assume that f is not continuous at a. Then

$$\exists \varepsilon > 0, \forall \delta > 0, \exists x \in dom(f), |x - a| < \delta \text{ and } |f(x) - f(a)| \geq \varepsilon.$$

In particular, for $\delta = \frac{1}{n}$, $n = 1,2, \ldots$ we find $x_n \in dom(f)$ such that $|x_n - a| < \frac{1}{n}$ and $|f(x) - f(a)| \geq \varepsilon$. But then $\lim_{n\to\infty} x_n = a$, whereas $\big(f(x_n)\big)$ does not converge to $f(a)$. ■

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Tutorial 4.2

1. Prove the following:
   a. The remainder of Theorem 4.8.
   b. The missing steps of Theorem 4.10.

2. Consider the function

$$f(x) = \begin{cases} \dfrac{\lfloor x \rfloor}{x} & \text{if } x \neq 0, \\ -1 & \text{if } x = 0. \end{cases}$$

   Investigate continuity from the left and right at $x = 0$, $x = \pi$ and $x = 1$.

3. Let $f(x) = x \sin\left(\frac{1}{x}\right)$ for $x \neq 0$ and

$$g(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

   Show that $f(x) \to 0$ as $x \to 0$ and that $g(x) \to 0$ as $x \to 0$, but that $g\big(f(x)\big)$ does not have a limit as $x \to 0$. Explain this behavior.

4. Find the values of a and b which make the function

$$f(x) = \begin{cases} x - 1 & \text{if } x \leq -2, \\ ax^2 + c & \text{if } -2 < x < 1, \\ x + 1 & \text{if } x \geq 1, \end{cases}$$

   continuous at $x = -2$ and $x = 1$.

5. Prove that if $\lim_{x \to 0^-} f(x)$ exists, then $\lim_{x \to 0^+} f(-x) = \lim_{x \to 0^-} f(x)$.

6. Prove that exp is continuous. You may use the following steps.
   a. The inequality $\exp(x) \geq 1 + x$ is true for all $x \in \mathbb{R}$.
   b. $\lim_{x \to 0^-} \exp(x) = 1$.
   c. $\lim_{x \to 0^+} \exp(x) = 1$. Hint: Use Problem 5 above.

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