APPM2023
Mechanics II
2023

Assignment-04

The Multi-Link Pendulum

Issued: 06 October 2023 2023 **Total:** 70

Due: 17:00, 23 October 2023 2023

Instructions

- Read all the instructions and questions carefully.
- Typeset the solution document using 'Assignment.cls' LaTeX document template. Submissions that have not used this template shall receive a zero grade.
- Use plain written English where necessary.
- Students are free to use whatever resources at their disposal to answer this assignment, including Computer Algebra and Graphing software. However, all necessary calculation steps and details should be include to obtain full credit.
- Students may use the `Mathematica` and LaTeX supplementary resources posted on the course Moodle page to complete this assignment. In particular, students should consult the example script files on the course Moodle site to help answer this assignment.
- Students are encouraged to work in groups. However, this is to be individual work and each student must submit their own report.
- Plagiarised submissions shall receive a zero grade.
- No late submissions shall be considered.
- Do not submit any `Mathematica` code for this assignment.

Introduction

In this assignment we shall consider the multi-link pendulum. Refer to Figure 1 for the questions that follow. The files

1. `n-link-pendulum.mp4` and
2. `n-link-pendulum-WRONG.mp4`

depict the simulated motion of identical 25-link pendulums, one with the correctly constructed Lagrangian, and the other with the wrong Lagrangian. Refer the these simulations in the final question. Students should refer to the `Mathematica` documentation for information on the following functions to help with this assignment: `D`, `NDSolveValue`, `Manipulate` and `Line`. Students should also refer to the `Mathematica` code from previous assignments to help with this assignment.

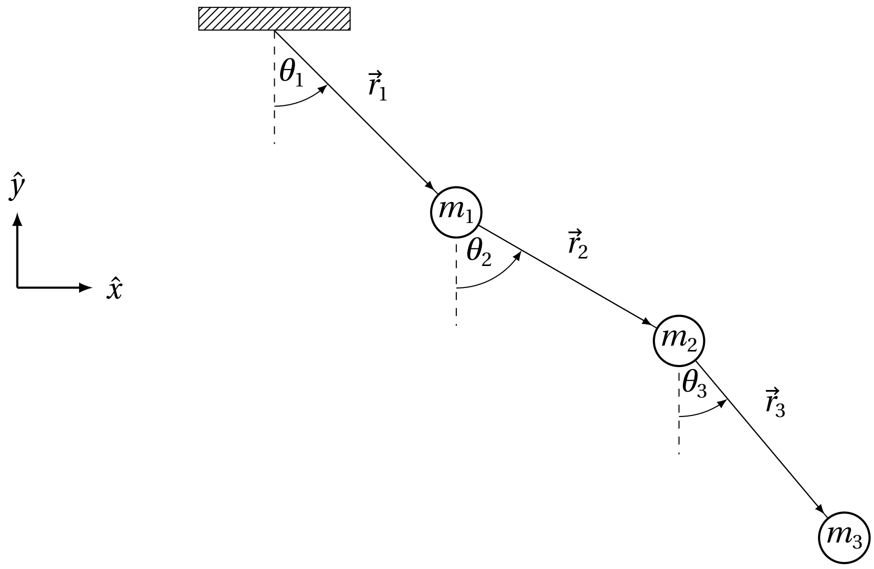

Figure 1: The multi-link pendulum, comprising a collection of masses m_i and massless, rigid rods of length r_i. Here we show an example of a 3-link pendulum, fixed at an anchor point. The relative position of m_{i+1} with respect to m_i is given by the vector $\vec{r}_{i+1}$. The weight of each mass acts in the $-\hat{y}$ direction.

Question 1 — The Simple Pendulum (10 Points)

Start by considering the simple pendulum comprising a single bob of mass m attached to the end of a rigid, massless rod of length r. Let $\vec{\rho}$ denote the position of the bob.

[**1.1**] Show that the potential energy of the simple pendulum is given by

$$V = -mgr\cos(\theta).$$

(1 Points)

[**1.2**] Show that the kinetic energy of the simple pendulum is given by

$$T = \frac{1}{2}m\left(r\dot{\theta}\right)^2.$$

(2 Points)

[**1.3**] Show that a Lagrangian for this system is given by

$$\mathscr{L} = \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos(\theta).$$

(1 Points)

[**1.4**] Show by direct calculation that the corresponding equation of motion for the simple pendulum is given by

$$0 = \ddot{\theta} + \frac{g}{r}\sin(\theta).$$

(6 Points)

Question 2 — The Double Pendulum (35 Points)

Now consider the double pendulum comprising mass m_1, suspended from an anchor point by a rigid massless rod of length r_1, and m_2, suspended from m_1 by a rigid massless rod of length r_2. It will be convenient to use the position vectors $\vec{\rho}_i$ to denote the position of m_i, relative to the anchor point.

[**2.1**] Show that the potential energy of the double pendulum is given by

$$V = -g\left\{(m_1+m_2)\,r_1\cos(\theta_1) + (m_2)\,r_2\cos(\theta_2)\right\}$$

(4 Points)

[**2.2**] Show that the kinetic energy of the double pendulum is given by

$$T = \frac{1}{2}\left\{(m_1+m_2)\,r_1^2\dot{\theta}_1^2 + (m_2)\,r_2^2\dot{\theta}_2^2 + (2m_2\cos(\theta_2-\theta_1))\,r_1r_2\dot{\theta}_1\dot{\theta}_2\right\}$$

(5 Points)

[**2.3**] Show that the potential energy of the double pendulum can be rewritten as

$$V = -g\, Y^T M$$

where Y is a column matrix with elements $Y^i = r_i \cos(\theta_i)$and M is a column matrix,

$$M = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

and determine μ_1 and μ_2. **(4 Points)**

[**2.4**] Show that the kinetic energy of the double pendulum can be rewritten as

$$T = \frac{1}{2}\dot{X}^\top \tilde{M} \dot{X}$$

where X is a column matrix with entries $X^i = r_i \theta_i$ and $\tilde{M}$ is a symmetric square matrix

$$\tilde{M} = \begin{pmatrix} \mu_1 & \mu_2 \cos(\theta_2 - \theta_1) \\ \mu_2 \cos(\theta_2 - \theta_1) & \mu_2 \end{pmatrix}$$

and determine μ_1 and μ_2. **(5 Points)**

[**2.5**] Construct an appropriate Lagrangian for the double pendulum. **(1 Points)**

[**2.6**] Show by direct calculation that the corresponding equations of motion for the double pendulum are given by

$$\begin{aligned} 0 &= \ddot{\theta}_1 + \frac{g}{r_1}\sin(\theta_1) + \frac{\mu_2}{\mu_1}\frac{r_2}{r_1}\left(\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)\right) \\ 0 &= \ddot{\theta}_2 + \frac{g}{r_2}\sin(\theta_2) + \frac{r_1}{r_2}\left(\ddot{\theta}_1 \cos(\theta_1 - \theta_2) + \dot{\theta}_1^2 \sin(\theta_2 - \theta_1)\right) \end{aligned}$$

(12 Points)

[**2.7**] Suppose that $r_1 = r_2 = r$ and $m_1 = m_2$. Describe how the equations of motion for the double pendulum compare with that of simple pendulum. Describe the similarities and differences. **(4 Points)**

Question 3 — The n-Link Pendulum (25 Points)

Now consider the generalization of the simple and double pendulum to n masses and rods. Suppose that a bob of mass m_1 is suspended from an anchor point by a massless, rigid rod of length r_1. Further, suppose that the i-th bob has a mass m_i and is attached to a rod of length r_i. Then the bob of mass m_{i+1} is suspended beneath the the bob of mass m_i from a rigid massless rod of length r_{i+1}. Denote the position of mass m_i by the vector $\vec{\rho}_i$ relative to the anchor point.

[**3.1**] Generalize the Lagrangian for the double pendulum system to include n masses and rods. Describe the general form of the vectors X, Y, M and the mass matrix $\tilde{M}$. (Hint: Try to spot a pattern, rather than direct computation.) **(10 Points)**

[**3.2**] The Lagrangian

$$\mathscr{L} = \frac{1}{2}\dot{X}^\top \tilde{M}\dot{X} + gY^\top M$$

does not make specific reference to the metric tensor $\mathbf{g}$ in the usual way that we expect when computing kinetic energy. How has the metric information been encoded in this epression? Be specific in your discussion. **(5 Points)**

[**3.3**] Consider `n-link-pendulum.mp4` and `n-link-pendulum-WRONG.mp4`. These are videos of the simulated motion of a heavy chain with 25 evenly spaced links. We model this chain as a 25-link pendulum, where $r_i = 1$ and $m_i = 1$ for $i \in \{1, \ldots, 25\}$. `n-link-pendulum.mp4` is generated using the correct form of the Lagrangian, while `n-link-pendulum-WRONG.mp4` is generated from a Lagrangian where the angular dependence in $\tilde{M}$ is removed. Describe the qualitative difference in the motion of each pendulum and then explain the effect of including the angular dependence in $\tilde{M}$. How do these simulations agree with your expectations of the motion of a swinging heavy chain. You should refer to Figure 1 and the multi-link pendulum Lagrangian in your discussion. **(5 Points)**

[**3.4**] Set $n = 3$, $r_i = 1$, $m_i = 1$ and $g = 9.81$. Use `Mathematica` to perform the following construction

1. Construct the equation of motion for the system.
2. Solve these equations numerically using the initial conditions $\theta_i(0) = \pi/3$ and $\dot{\theta}_i(0) = 0$ on the interval $t = [0;3]$.
3. Animate the pendulum using the same description of the multilink configuration presented in Figure 1. Use the `Line` function to draw the collection of links in this pendulum and the `Manipulate` to animate the simulation. (Hint: your animated plot should resemble the animation in `n-link-pendulum.mp4`.

Include only the equations of motion and the single plot of the pendulum in at $t = 2$ in your answer. **(5 Points)**