Multivariable Calculus

MATH2007

2.4 Double Integrals and Fubini's Theorem (Part 1)

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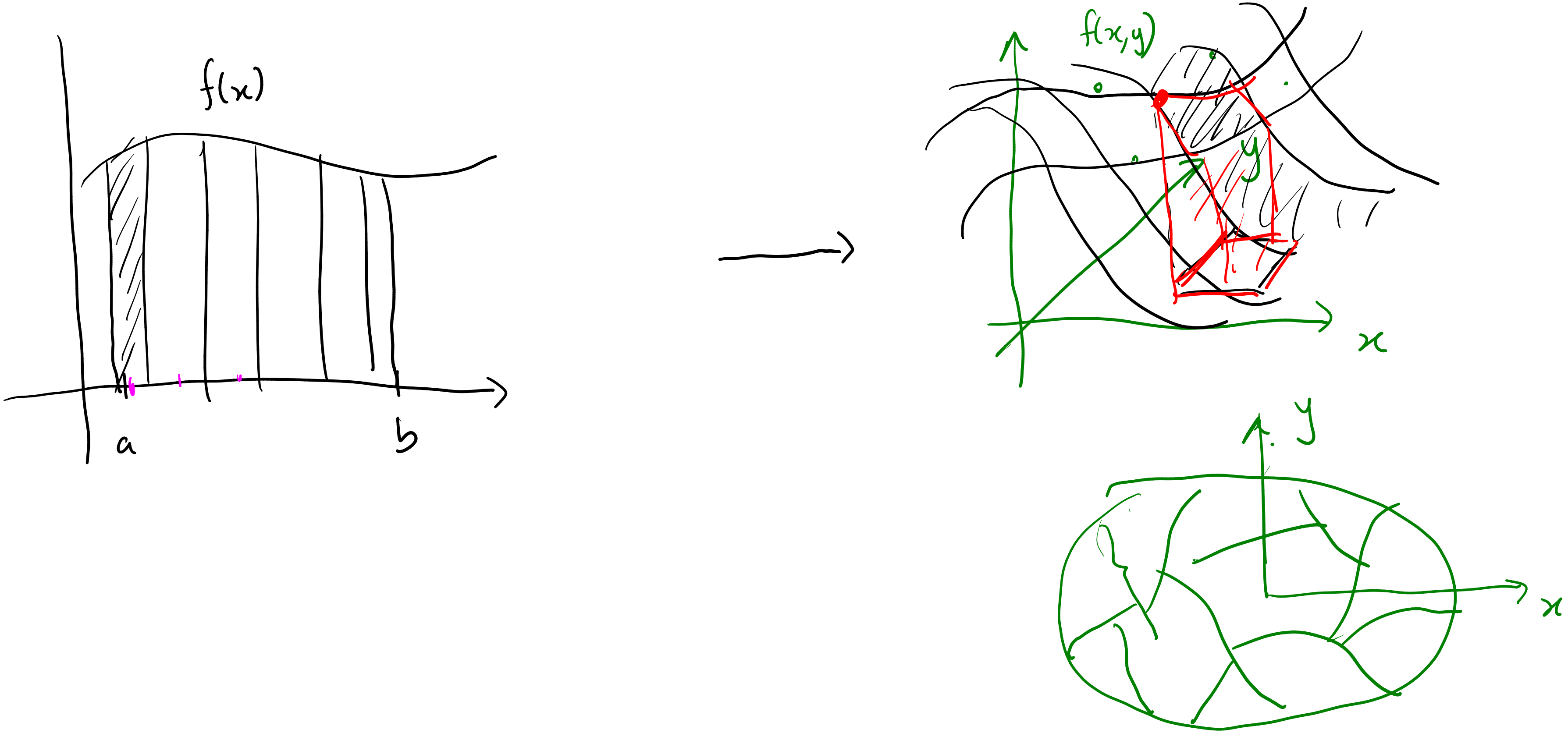
School of Mathematics


f(x)
a
b
f(x,y)
y
x
y
x

Definition (2.4.1)**.** Let D be a region in $\mathbb{R}^2$ with finite area and bounded by a piecewise smooth continuous closed curve. Let $f : D \to \mathbb{R}$ be continuous.

For each $N \in \mathbb{N}$, divide D into N subregions $D_i^N, i = 1, \ldots, N$, in such a manner that

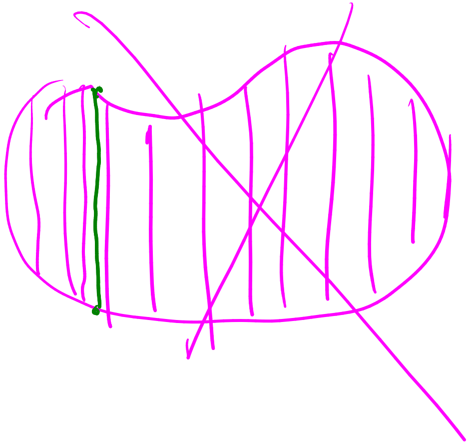
$$\lim_{N\to\infty} \max_{i=1,\ldots,N} \operatorname{diam}(D_i^N) = 0.$$

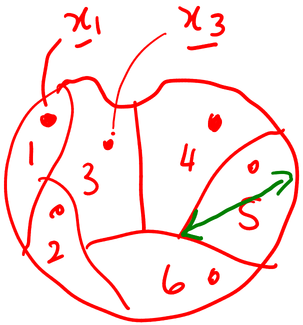
Let $\underline{x}_i^N \in D_i^N$. We define the integral of f over the region D by

$$\iint_D f\, da = \lim_{N\to\infty} \sum_{i=1}^{N} f(\underline{x}_i^N)\, \operatorname{Area}(D_i^N).$$

small area

$da \to dx\,dy$



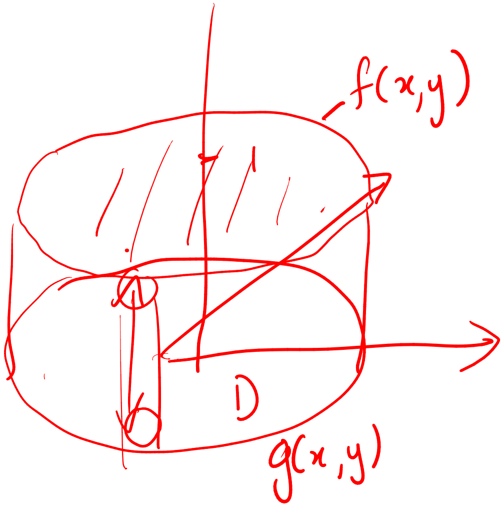


$\operatorname{diam}(D_5) =$ largest distance between points

Note.

1. $\displaystyle\iint_D 1 \; da = \text{Area}(D).$

2. If $f(\underline{x}) \geq g(\underline{x})$ for all $\underline{x} \in D$, then $\iint_D [f - g] \; da$ is the volume of the region over D bounded above by $f(\underline{x})$ and below by $g(\underline{x})$, $\underline{x} \in D$.



Note.

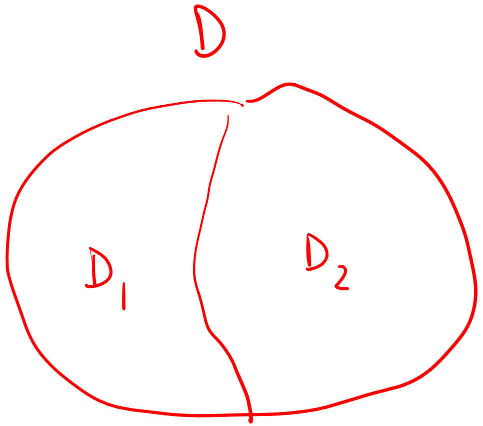
3. It can be easily seen that for α, $\beta \in \mathbb{R}$ and functions $f(x,y)$ and $g(x,y)$ we have:

$$\iint_D (\alpha f + \beta g)\; da = \alpha \iint_D f\; da + \beta \iint_D g\; da \qquad (\mathit{Linearity}).$$

4. If $D = D_1 \cup D_2$ where D_1 and D_2 are disjoint $(D_1 \cap D_2 = \emptyset)$, then

$$\iint_{D_1 \cup D_2} f\; da = \iint_{D_1} f\; da + \iint_{D_2} f\; da.$$

$$\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$$



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2.4 Double Integrals and Fubini's Theorem (Part 2)

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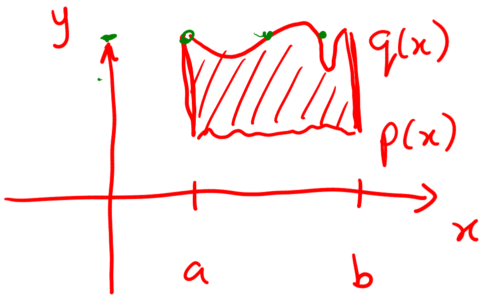
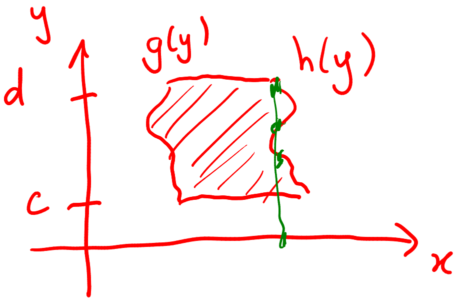
Types of regions:

Type 1: $D = \{(x,y) \mid p(x) \le y \le q(x), x \in [a,b]\}$

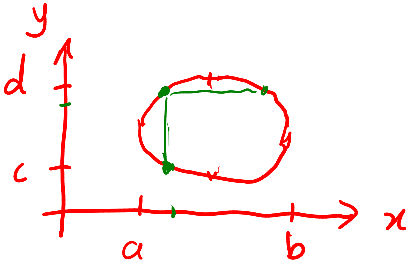


Type 2: $D = \{(x,y) \mid g(y) \le x \le h(y), y \in [c,d]\}$



Type 3: $D = \underbrace{\{(x,y) \mid p(x) \le y \le q(x), x \in [a,b]\}} = \overbrace{\{(x,y) \mid g(y) \le x \le h(y), y \in [c,d]\}}$

Type 1 and Type 2



Theorem (2.4.3 Fubini's Theorem)**.** Let $D \subset \mathbb{R}^2$ and $f : D \to \mathbb{R}$.

Type 1. If $D = \{(x, y) \,|\, p(x) \le y \le q(x), x \in [a, b]\}$ then

$$\iint_D f \; da = \int_a^b \underbrace{\left[\int_{p(x)}^{q(x)} f(x, y) \; dy \right]}_{\text{function in } x} dx.$$

Type 2. If $D = \{(x, y) \,|\, g(y) \le x \le h(y), y \in [c, d]\}$ then

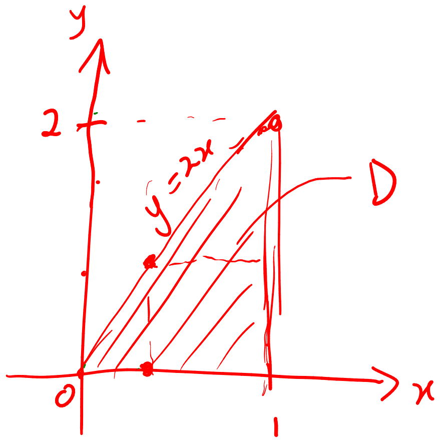
$$\iint_D f \; da = \int_c^d \underbrace{\left[\int_{g(y)}^{h(y)} f(x, y) \; dx \right]}_{\text{function in } y} dy.$$

Type 3. If $D = \{(x, y) \,|\, p(x) \le y \le q(x), x \in [a, b]\} = \{(x, y) \,|\, g(y) \le x \le h(y), y \in [c, d]\}$ then

$$\int_a^b \left[\int_{p(x)}^{q(x)} f(x, y) \; dy \right] dx = \iint_D f \; da = \int_c^d \left[\int_{g(y)}^{h(y)} f(x, y) \; dx \right] dy.$$

Note. Outer integration has constant limits. Inner integration may have variable limits.

Example. Let D be given by $x \in [0,1]$ and $0 \le y \le 2x$. Sketch the region D and evaluate $\iint_D (x^2+y)\,da$.



Type 3

$$\iint_D (x^2+y)\,da = \int_0^1 \Big(\underbrace{\int_0^{2x} x^2+y \; dy}_{\text{treat } x \text{ as a constant}} \Big)\, dx$$

$$= \int_0^1 \left[x^2 y + \frac{1}{2} y^2 \right]_{y=0}^{y=2x} dx$$

$$= \int_0^1 2x^3 + 2x^2 \; dx$$

$$= \left[\frac{1}{2} x^4 + \frac{2}{3} x^3 \right]_0^1$$

$$= \frac{1}{2} + \frac{2}{3} = \frac{7}{6}.$$

OR $\iint_D (x^2+y)\,da = \int_0^2 \left(\int_{\frac{1}{2}y}^{1} x^2+y \, dx \right) dy$

treat y as a constant

$$= \int_0^2 \left[\frac{x^3}{3} + yx \right]_{\frac{1}{2}y}^{1} dy$$

$x = 1$

$x = \frac{1}{2}y$

$$= \int_0^2 \frac{1}{3} + y - \frac{y^3}{24} - \frac{1}{2}y^2 \, dy$$

$$= \left[\frac{1}{3}y + \frac{1}{2}y^2 - \frac{y^4}{4 \cdot 24} - \frac{1}{6}y^3 \right]_0^2$$

$y = 2$

$y = 0$

$$= \frac{2}{3} + 2 - \frac{1}{6} - \frac{4}{3}$$

$$= \frac{7}{6}.$$

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2.4 Double Integrals and Fubini's Theorem
(Part 3)

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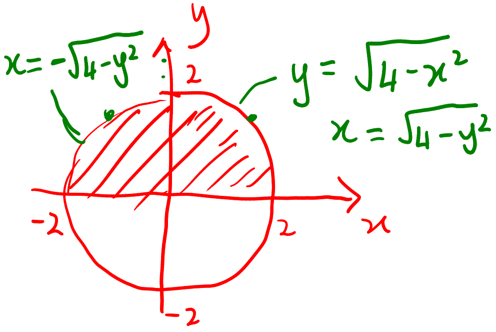
Example. Let D be part of the disk $x^2 + y^2 \leq 4$ with $y \geq 0$. Evaluate $\displaystyle\iint_D y\, da$.



Type 3

$$\iint_D y\, da = \int_{-2}^{2}\left(\int_0^{\sqrt{4-x^2}} y\, dy\right)dx$$

$$= \int_{-2}^{2} \left[\frac{1}{2}y^2\right]_0^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \frac{1}{2}(4-x^2)\, dx$$

$$= \left[2x - \frac{1}{6}x^3\right]_{-2}^{2}$$

$$= 2\left(2\cdot 2 - \frac{1}{6}\cdot 2^3\right) = 8 - \frac{8}{3} = \frac{16}{3}.$$

OR $\iint_D y\, dA = \int_0^2 \left(\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} y\, dx \right) dy$

$$= \int_0^2 \left[yx \right]_{x=-\sqrt{4-y^2}}^{x=\sqrt{4-y^2}} dy = \int_0^2 y\sqrt{4-y^2} - y\left(-\sqrt{4-y^2}\right) dy$$

$$= \int_0^2 2y\sqrt{4-y^2}\, dy$$

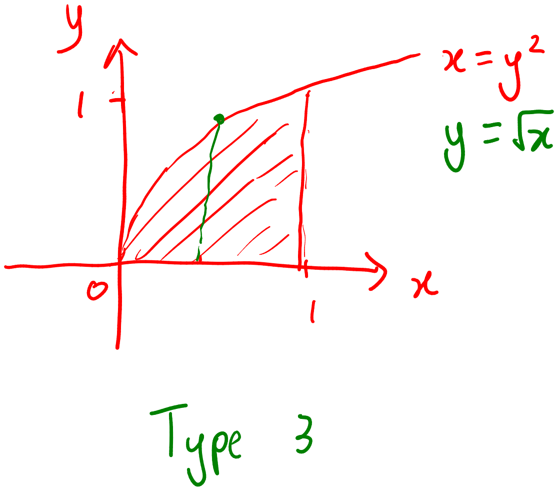
$u = 4 - y^2$ $\quad du = -2y\,dy$

$y=0 \Rightarrow u=4$ $\quad y=2 \Rightarrow u=0$

$$= -\int_4^0 \sqrt{u}\, du$$

$$= \int_0^4 \sqrt{u}\, du = \frac{2}{3} u^{3/2} \Big|_0^4 = \frac{16}{3}.$$

Example. By changing order of integration, evaluate $\int_0^1 \int_{y^2}^1 \sin(x^{\frac{3}{2}})\, dx\, dy$.



$$\int_0^1 \left(\int_{y^2}^1 \sin(x^{3/2})\, dx \right) dy = \int_0^1 \left(\int_0^{\sqrt{x}} \sin(x^{3/2})\, dy \right) dx$$

$$= \int_0^1 \Big[y \sin(x^{3/2}) \Big]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \sqrt{x} \sin(\sqrt{x}^3)\, dx$$

$u = \sqrt{x}^3 \quad du = \frac{3}{2}\sqrt{x}\, dx$

$x = 0 \Rightarrow u = 0$

$x = 1 \Rightarrow u = 1$

$$= \int_0^1 \frac{2}{3} \sin(u)\, du = \left[-\frac{2}{3}\cos(u) \right]_0^1$$

$$= \int_0^1 \frac{2}{3}\sin(u)\,du = \left[-\frac{2}{3}\cos(u)\right]_0^1$$

$$= -\frac{2}{3}\cos(1) + \frac{2}{3}.$$