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UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

Course or topic No(s)	APPM 2007A, APPM2020A
Course or topic name(s) Paper Number & title	Numerical methods II
Examination/Test to be held during month(s) of *(delete as applicable)*	JUNE 2022 MAIN EXAMINATION
Year of Study *(Art & Science leave blank)*	SECOND
Degrees/Diplomas for which This course is prescribed (BSc (Eng) should indicate which branch)	BSc
Faculty/ies presenting Candidates	SCIENCE
Internal examiners(s) And telephone extension number(s)	Dr Walter Mudzimbabwe
External examiner(s)	Dr Rodrigue Yves M'pika Massoukou
Special materials required (graph/music/drawing paper) maps, diagrams, tables computer cards, etc.	NONE

Time allowance

Course No.(s)	APPM2007, APPM2020A	**Hours** 2 hrs

Instructions to candidates (Examiners may wish to use this space to indicate, *inter alia* the contribution made by this examination or test towards the year mark if appropriate)

ATTEMPT ALL QUESTIONS
CALCULATORS ARE PERMITTED
NO CELLPHONES ALLOWED
Total Marks Available = 57
100% = 57

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG
School of Computer Science and Applied Mathematics

APPM2007A, APPM2020A: Numerical methods II
June 2022, EXAMINATION

Instructions

1. This exam is 2 hours long.
2. There are 4 questions in this paper.
3. Answer **all** questions.
4. Start each question on a new page.
5. Where applicable, provide your final answer to 5 decimal places.
6. Formulae is given on page 4 and 5.

Total: [57 marks]

Question 1. [15 marks]

(a) Write down the Taylor series expansion of $f(x-h)$ and $f(x+3h)$. [2 marks]

(b) Use your answer to Question 1(a) to deduce a formula for $f'(x)$ that uses $f(x-h), f(x)$ and $f(x+3h)$, stating the truncation error and order of approximation. [4 marks]

(c) Use your answer to Question 1(a) to deduce a formula for $f''(x)$ that uses $f(x-h), f(x)$ and $f(x+3h)$, stating the truncation error and order of approximation. [5 marks]

(d) Show that the approximation you found in Question 1(a) is exact for $f(x) = x^2 + 2x + 1$. [4 marks]

Question 2. [12 marks]

(a) The formula

$$F_{n+1}(h) = \frac{F_n(h/2) - F_n(h)}{2^n - 1}$$

can be used to extrapolate an n^{th} order approximation F_n to an $(n+1)^{th}$ order approximation F_{n+1}. Use this formula to determine $F_2(h)$, an approximation for $f'(x)$ where $f(x) = 2^x \sin(x)$ at $x = 1.05$ and $h = 0.4$. [4 marks]

(b) Verify the order of your approximation in Question 2(a) by considering Taylor series expansions of f expressions in your approximation. [4 marks]

(c) Approximate the double integral

$$\int_{1}^{2}\int_{3}^{5} ye^{x}\,\mathrm{d}y\,\mathrm{d}x$$

using midpoint rule with $n_x = 4$ and $n_y = 5$. [4 marks]

Question 3. [19 marks]

(a) The following Python code implements a quadrature rule to estimate

$$\int_{a}^{b} f(x)\,\mathrm{d}x.$$

```
import numpy as np

def quadrature(f, a, b, n):
    h = (b - a) / n

    weights = np.zeros(n + 1)

    for i in np.arange(0, n + 1):
        if i == 0 or i == n:
            weights[i] = 0.5 * h
        else:
            weights[i] = h

    x = np.linspace(a, b, n + 1)

    return sum(weights * f(x))
```

(i) State the quadrature method that is being implemented by the Python function. [1 mark]

(ii) By considering every line of the code, give reasons to support your answer to (a)(i). [3 marks]

(b) Consider the following data.

x	0	2	4	6
y	1	-1	3	4

(i) Derive the Lagrange polynomial of suitable degree that interpolates the data in the table. [5 marks]

(ii) Derive the Newton divided differences polynomial of suitable degree that interpolates the data in the table. [6 marks]

(c) Without doing any calculations, explain how you would determine u such that

$$P(u) = \ln(1.9999),$$

where $P(x)$ is the polynomial you found in Question 3(b)(ii). [4 marks]

Question 4. [11 marks]

(a) Consider the initial value problem (IVP):

$$y' = t + \frac{3y}{t}, \quad 1 \leq t \leq 2, \quad y(1) = 0.$$

Use the midpoint rule to determine y_1, y_2, y_3 using $h = 0.1$. [2 marks]

(b) Consider the boundary value problem (BVP):

$$y'' + xy' - x^2y = 2x^2, \quad 0 \leq x \leq 1 \quad y(0) = 1, \quad y(1) = -1,$$

and $h = 0.25$.

(i) Write down the equations that y must satisfy in the form

$$a_i y_{i-1} + b_i y_i + c_i y_{i+1} = d_i,$$

by using a backward difference approximation for y' and central difference approximation for y''. [5 marks]

(ii) Write down the system in the form $\mathbf{Ay} = \mathbf{e}$. [2 marks]

(iii) Given that $\mathbf{AB} = \mathbf{I}_3$ where

$$\mathbf{B} = \begin{bmatrix} -0.0454 & -0.0303 & -0.0158 \\ -0.0276 & -0.0590 & -0.0308 \\ -0.0123 & -0.0263 & -0.0444 \end{bmatrix},$$

and $\mathbf{I}_3$ is a 3×3 identity matrix, solve the system in Question 4(b)(ii). [2 marks]

Formulae :

1. Central difference: $f'(x) = \dfrac{f(x+h) - f(x-h)}{2h}$, $f''(x) = \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$.

2. Richardson: $F_j^i = \dfrac{1}{4^j - 1}\left(4^j F_{j-1}^i - F_{j-1}^{i-1}\right)$, $\quad j = 1, 2, \ldots m,\ i = 1, 2, \ldots, n.$

3. Trapezoidal rule:

$$I \approx \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + \cdots + f_{n-1}) + f_n\right], \quad E_T = -\frac{(b-a)h^2}{12} f''(\epsilon),\ \epsilon \in [a, b]$$

4. Simpson rule:

$$I \approx \frac{h}{3}\left[f_0 + 4(f_1 + f_3 + \cdots f_{n-1}) + 2(f_2 + f_4 + \cdots + f_{n-2}) + f_n\right], \quad E_S = -\frac{(b-a)h^4}{180} f^{(4)}(\epsilon),\ \epsilon \in [a, l$$

5. Midpoint for double integration:

$$\int_a^b \int_c^d f(x, y) dy dx \approx h_x h_y \sum_{i=0}^{n_x - 1} \sum_{j=0}^{n_y - 1} f(a + \frac{h_x}{2} + ih_x, c + \frac{h_y}{2} + jh_y).$$

6. False position: $c = \frac{af(b) - bf(a)}{f(b) - f(a)}$.

7. Newton:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}, \quad i = 0, 1, 2, \cdot$$

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} x_i \\ y_i \end{bmatrix} - J^{-1}(x_i, y_i) \begin{bmatrix} f(x_i, y_i) \\ g(x_i, y_i) \end{bmatrix}, \quad i = 0, 1, 2, \cdot$$

8. Lagrange interpolation:

$$L_k(x) = \prod_{i=0, i \neq k}^{n} \frac{(x - x_i)}{(x_k - x_i)}.$$

9. Quadratic Interpolation: $a_0 = f(x_0)$, $a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$, $a_2 = \frac{\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(x_1) - f(x_0)}{x_1 - x_0}}{x_2 - x_0}$.

10 Polynomial least square equations:

$$\begin{aligned}
a_0 n + a_1 \sum_{i=1}^{n} x_i + \cdots + a_m \sum_{i=1}^{n} x_i^m &= \sum_{i=1}^{n} y_i \\
a_0 \sum_{i=1}^{n} x_i + a_1 \sum_{i=1}^{n} x_i^2 + \cdots + a_m \sum_{i=1}^{n} x_i^{m+1} &= \sum_{i=1}^{n} x_i y_i \\
\vdots \qquad \vdots \qquad \vdots \quad &\vdots \\
a_0 \sum_{i=1}^{n} x_i^m + a_1 \sum_{i=1}^{n} x_i^{m+1} + \cdots + a_m \sum_{i=1}^{n} x_i^{2m} &= \sum_{i=1}^{n} x_i^m y_i
\end{aligned}$$

11. Euler method: $y_{i+1} = y_i + hf(x_i, y_i)$.

12. Midpoint rule: $y_{i+1} = y_i + hf(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}})$, $\quad y_{i+\frac{1}{2}} = y_i + \frac{h}{2}f(x_i, y_i)$

13. Second order Runge-Kutta method:

$$y_{i+1} = y_i + (w_1k_1 + w_2k_2),\ k_1 = hf(x_i, y_i),\ k_2 = hf(x_i + h/2, y_i + k_1/2).$$

14. Fourth order Runge-Kutta method:

$$\begin{aligned} y_{i+1} &= y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4), \\ k_1 &= hf(x_i, y_i), \quad k_2 = hf(x_i + \frac{h}{2}, y_i + \frac{k_1}{2}) \\ k_3 &= hf(x_i + \frac{h}{2}, y_i + \frac{k_2}{2}), \quad k_4 = hf(x_i + h, y_i + k_3). \end{aligned}$$