APPM3039A
Applied Mathematics

Class Test 3

Instructions:

- Start each question on a new page
- Answer all questions.
- Show all workings.
- Do not round off until the final answer.
- Give final answers rounded off to two decimal places.
- You may use a non-programmable calculator.

Date: 28 August 2024

Duration: 1 hour **Total:** 40 Marks

Question 1 - Eulerian and Lagrangian derivatives [15 marks]

Consider a two-dimensional flow where the path of each fluid element is described by the equation

$$(x_1(t), x_2(t)) = \left(\frac{X_1}{\sqrt{1+t}}, \frac{X_2}{(1+t)}\right),$$

and (X_1, X_2) are the initial coordinates, $(x_1(t), x_2(t))$ are the coordinates of the element at time t.

1. Calculate the Jacobian for the flow. (1 mark)

2. Derive the velocity of the fluid in terms of x_1, x_2 and t. (4 marks)

3. Calculate the Lagrangian time derivative of the velocity of the fluid (acceleration) in terms of x_1, x_2 and t. (4 marks)

4. Calculate the Eulerian time derivative of the velocity of the fluid in terms of x_1, x_2 and t. (2 marks)

5. Verify that

$$\frac{\partial v_1}{\partial t} + v_k \frac{\partial v_1}{\partial x_k} = \frac{D v_1}{D t}.$$

(4 marks)

Question 2 - Field equations **[10 marks]**

1. Derive the continuity equation

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \underline{v} = 0,$$

where ρ and $\underline{v}$ are the density and velocity field of the material. Assume that mass is conserved. (5 marks)

2. Given that the momentum principle on a volume element V gives

$$\frac{D}{Dt}\int_V \rho v_i dV = \int_V F_i dV + \int_{\partial V} T_i(\underline{n}) dS,$$

where ρ is the density of the volume element, v_i is the i-th component of the velocity, F_i is the i-th component of the body force per unit volume, $T_i(\underline{n})$ is the i-th component of the stress vector on the surface element dS. Show that

$$\rho\frac{Dv_i}{Dt} = F_i + \tau_{ki,k},$$

where τ_{ik} are components of the stress tensor. (5 marks)

Question 3 - Introduction to elasticity

[15 marks]

1. The generalized Hooke's law is

$$\tau_{ik} = C_{ikrs} e_{rs}.$$

For an isotropic material

$$C_{ikrs} = \lambda \delta_{ik} \delta_{rs} + \mu (\delta_{ir} \delta_{ks} + \delta_{is} \delta_{kr}),$$

where λ and μ are Lamé constants. Show that for an isotropic material

$$\tau_{ik} = \lambda \theta \delta_{ik} + 2\mu e_{ik}, \tag{1}$$

where

$$\theta = e_{rr}.$$

(3 marks)

2. Using (1), show that

$$\Theta = (3\lambda + 2\mu)\theta,$$

where $\Theta = \tau_{kk}$. (3 marks)

3. The Young modulus E and the Poisson ratio σ of a material can be written in terms of the Lamé constants λ and μ by

$$E = \frac{(3\lambda + 2\mu)\mu}{\lambda + \mu}, \qquad \sigma = \frac{\lambda}{2(\lambda + \mu)}.$$

Show that

$$\lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)}, \qquad \mu = \frac{E}{2(1+\sigma)}.$$

(5 marks)

4. An alloy of aluminium has a Young modulus of 69×10^9 Pa and a Poisson ratio of 0.33.

   (a) Calculate the Lamé constants for the aluminium alloy. (2 marks)

   (b) Estimate the pressure required to reduce a volume of the aluminium alloy by 2 %. (2 marks)

$$\mu = \frac{\lambda}{2\sigma} - \lambda$$

Formula sheet

Indicial identities:

- $\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$

Divergence theorem:

- $\int_{\partial V} C_{ik} n_k dS = \int_V C_{ik,k} dV$

Jacobian Evolution:

- $\frac{DJ}{Dt} = J \text{div}\, \underline{v}$

Cauchy's formula:

- $T_i(\underline{n}) = \tau_{ki} n_k$

END OF PAPER