Basic Analysis 2015 — Solutions of Tutorials

Section 1.3

Tutorial 1.3.1 1. Show that if $S \subset \mathbb{Z}$, $S \neq \emptyset$, and S is bounded below, then S has a minimum.

Proof. Let $S \subset \mathbb{Z}$, $s \neq \emptyset$, be bounded. Let L be a lower bound of S. By the Archimedean principle there is $n \in \mathbb{N}$ such that $n > -L$. Let

$$T = \{k + n : k \in S\}.$$

Since $S \neq \emptyset$, there is $k \in S$. Then $k + n \in T$, that is, $T \neq \emptyset$. For all $k \in S$,

$$k + n > k - L \geq 0.$$

Here we have used that the lower bound L of S satisfies $L \leq k$. If $k \in \mathbb{N}$ then also $k + n \in \mathbb{N}$ by Theorem 1.14.3. If $-k \in \mathbb{N}$, then $n > -k$ gives $l \in \mathbb{N}$ such that $n = (-k) + l$ by Theorem 1.14.4. Hence $n + k = l \in \mathbb{N}$.
We have shown that $T \subset \mathbb{N}$, $T \neq \emptyset$. By the well-ordering principle, T has a minimum, say p. Thus $p \in T$ and $p \leq t$ for all $t \in T$. Since $S = \{l - n : l \in T\}$, we have that

$$m := p - n \in S,$$

and for all $k \in S$,

$$m = p - n \leq (k + n) - n = k.$$

Hence $m = \min S$. □

2. Show that $\sqrt{2}$ is irrational. **Hint.** Assume that $\sqrt{2} = \frac{p}{q}$ with positive integers p and q. You may assume that p and q do not have common factors, i. e., there are no positive integers k, p_1, q_1 with $k \neq 1$ such that $p = p_1 k$ and $q = q_1 k$.

Proof. By definition of the square root, $\sqrt{2} = \frac{p}{q}$ gives $2 = \frac{p^2}{q^2}$, that is, $2q^2 = p^2$. Hence p is divisible by 2, and so there is a positive integer p_1 such that $p = 2p_1$. Thus we have

$$2q^2 = (2p_1)^2 = 4p_1^2,$$

which gives $q^2 = 2p_1^2$. Hence we find an integer q_1 such that $q = 2q_1$. Therefore q and p would have a common factor 2, which is impossible. □

3. Show that the rational numbers satisfy the axioms (A1)–(A4), (M1)–(M4), (D), and (O1)–(O3).

Proof. The associative, commutative, distributive and order laws are satisfied by any subset of the real numbers and therefore also by $\mathbb{Q}$. Furthermore, 0 and 1 are natural numbers and therefore rational numbers. We therefore have to show
(i) if $a, b \in \mathbb{Q}$, then $a + b \in \mathbb{Q}$;
(ii) if $a \in \mathbb{Q}$, then $-a \in \mathbb{Q}$;
(iii) if $a, b \in \mathbb{Q}$, then $ab \in \mathbb{Q}$;
(iv) if $a \in \mathbb{Q}$ and $b \neq 0$, then $a^{-1} \in \mathbb{Q}$.
We will first show (i), (ii) and (iii) for integers. Therefore let $a, b \in \mathbb{Z}$.
(i) We may assume that $a \geq b$. If $a, b \in \mathbb{N}$, then $a + b \in \mathbb{N}$ by Theorem 1.14.4. If $-a, -b \in \mathbb{N}$, then $-a - b \in \mathbb{N}$, and therefore $a + b = -(-a - b) \in \mathbb{Z}$. If $a, -b \in \mathbb{N}$ and $a + b \geq 0$, then $a \geq -b$, and by Theorem 1.14.4 there is $k \in \mathbb{N}$ such that $a = -b + k$, so that $a + b = k \in \mathbb{N} \subset \mathbb{Z}$. Finally, if $a, -b \in \mathbb{N}$ and $a + b < 0$, then $-b > a$, and by Theorem 1.14.4 there is $k \in \mathbb{N}$ such that $-b = a + k$, so that $a + b = -k \in \mathbb{Z}$.
(ii) is obvious by definition of $\mathbb{Z}$.
(iii) For $n \in \mathbb{N}$ let $A(n)$ be the statement that $nb \in \mathbb{Z}$ for all $b \in \mathbb{Z}$. Since $0 \cdot b = 0$ for all $b \in \mathbb{Z}$, $A(0)$ is true. Assume $A(n)$ is true and let $b \in \mathbb{Z}$. Then

$$(n + 1)b = nb + b.$$

But $nb \in \mathbb{Z}$ by hypothesis, and then $nb + b \in \mathbb{Z}$ by part (i). Hence (iii) has been proved for $a \in \mathbb{N}$ by the principle of mathematical induction. Finally, if $-a \in \mathbb{N}$ and $b \in \mathbb{Z}$, then

$$ab = -[(-a)b] \in \mathbb{Z}$$

by what we have proved so far in (iii) and by (ii).

Now we are going to prove (i)–(iv) for $\mathbb{Q}$. Let $a, b \in \mathbb{Q}$ with $a = \frac{p}{q}$ and $b = \frac{r}{s}$, $p, q, r, s \in \mathbb{Z}$, $q \neq 0$, $s \neq 0$. (i) From properties (i) and (iii) for $\mathbb{Z}$ we have

$$a + b = \frac{p}{q} + \frac{r}{s} = \frac{ps + rq}{qs} \in \mathbb{Q}$$

because both the numerator and the denominator on the right hand side are integers.
(ii) From part (ii) for integers it immediately follows that

$$-a = -\frac{p}{q} = \frac{-p}{q} \in \mathbb{Z}.$$

(iii) From part (iii) for integers it immediately follows that

$$ab = \frac{p}{q}\frac{r}{s} = \frac{pr}{qs} \in \mathbb{Z}.$$

(iv) If $a \neq 0$, then $p \neq 0$, and $a^{-1} = \frac{q}{p} \in \mathbb{Q}$ follows. □

4. Let $a, b \in \mathbb{Q}$ with $b \neq 0$ and $r \in \mathbb{R} \setminus \mathbb{Q}$. Show that $a + br \in \mathbb{R} \setminus \mathbb{Q}$.

Proof. Assume $x = a + br \in \mathbb{Q}$. Then $br = x - a \in \mathbb{Q}$ and $r = \frac{x-a}{b} \in \mathbb{Q}$. This contradiction shows that $a - br \in \mathbb{R} \setminus \mathbb{Q}$. □

5. For $x \in \mathbb{R}$ and $n \in \mathbb{N}$, x^n is defined inductively by
(i) $x^0 = 1$,
(ii) $x^{n+1} = xx^n$ for $n \in \mathbb{N}$.
Show that (a) $x^n x^m = x^{n+m}$, (b) $(x^n)^m = x^{nm}$, (c) $x^n y^n = (xy)^n$.

Proof. We prove all statements by induction on n or m, where $n, m \in \mathbb{N}$.
(a) $n = 0$: $x^0 x^m = x^m = x^{0+m}$ by (i).
$n \Rightarrow n + 1$:

$$\begin{aligned} x^{n+1}x^m &= xx^n x^m && \text{(by (ii))} \\ &= xx^{n+m} && \text{(by induction hypothesis)} \\ &= x^{n+m+1} && \text{(by (ii))} \\ &= x^{(n+1)+m}. \end{aligned}$$

(b) $m = 0$: $(x^n)^0 = 1$ by (i).
$m \Rightarrow m + 1$:

$$\begin{aligned} (x^n)^{m+1} &= x^n(x^n)^m && \text{(by (ii))} \\ &= x^n x^{nm} && \text{(by induction hypothesis)} \\ &= x^{n+nm} && \text{(by (a))} \\ &= x^{n(m+1)}. \end{aligned}$$

(c) $n = 0$: $x^0 y^0 = 1 \cdot 1 = 1 = (xy)^0$ by (i).
$n \Rightarrow n + 1$:

$$\begin{aligned} x^{n+1}y^{n+1} &= xx^n yy^n && \text{(by (ii))} \\ &= (xy)(xy)^n && \text{(by induction hypothesis)} \\ &= (xy)^{n+1} && \text{(by (ii)).} \end{aligned}$$

Tutorial 2.1.1 1. (a) Prove, using the definition of convergence, that the sequence $\left(\frac{n}{n+1}\right)$ does not converge to 2.

Proof. We first estimate

$$\frac{n}{n+1} - 2 = \frac{n - 2(n+1)}{n} = \frac{-n-2}{n+1} = -1 - \frac{1}{n+1} < -1.$$

Let $\varepsilon = 1$. Then

$$\left|\frac{n}{n+1} - 2\right| > 1 = \varepsilon$$

for all $n \in \mathbb{N}$. Hence there is no K such that $\left|\frac{n}{n+1} - 2\right| < \varepsilon$ for all $n > K$. Therefore the sequence does not converge to 0. □

(b) Prove, using the definition of convergence, that the sequence $((-1)^n)$ does not converge to any L.

Proof. Asuume, by proof of contradiction, that the sequence converges to some $L \in \mathbb{R}$. Let $\varepsilon > 0$. Then there would exist K such that $|(-1)^n - L| < \varepsilon$ for $n > K$. In particular,

$$|(-1)^{2n} - L| < \varepsilon \quad \text{and} \quad |(-1)^{2n+1} - L| < \varepsilon.$$

But then

$$2 = |1 - (-1)| = |1 - L - (-1 - L)| \le |1 - L| + |-1 - L| = |(-1)^{2n} - L| + |(-1)^{2n+1} - L| < 2\varepsilon,$$

which is clearly false if $\varepsilon \le 1$. □

2. (a) Prove that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} |a_n| = |L|$. (Hint: use (and prove) the inequality $||x| - |y|| \le |x - y|$.)

Proof. For all $x, y \in \mathbb{R}$ we know by Tutorial 1.1.2, 2(d) that

$$|x| = |(x - y) + y| \le |x - y| + |y|.$$

Hence

$$|x| - |y| \le |x - y|.$$

Interchanging x and y we have

$$|y| - |x| \le |x - y|.$$

Hence, by definition of the absolute value,

$$||x| - |y|| \le |x - y|.$$

Now let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = L$, there is K such that $|a_n - L| < \varepsilon$ for $n > K$. But then

$$||a_n| - |L|| \le |a_n - L| < \varepsilon$$

for these n which proves $\lim_{n\to\infty} |a_n| = |L|$. □

(b) Give an example to show that the converse to part (a) is not true.

Solution. Let $a_n = (-1)^n$. From part 1 (b) we know that (a_n) does not converge. But $|a_n| = 1$, and therefore $(|a_n|)$ is a constant sequence, which converges by Theorem 2.3 (a).

3. **Contractive maps.** Suppose that for some $c \in \mathbb{R}$ with $0 < c < 1$, we have $|a_{n+1} - L| \le c|a_n - L|$ for all $n \in \mathbb{N}$.
(a) Use induction to prove that $|a_n - L| \le c^n|a_0 - L|$ for all $n \in \mathbb{N}$.

Proof. Induction base: For $n = 0$, we have $|a_0 - L| = c^0|a_0 - L|$.
Induction step: Assume the estimate is true for n. Then

$$\begin{aligned} |a_{n+1} - L| &\le c|a_n - L| \\ &\le cc^n|a_0 - L| \qquad \text{(by induction hypothesis)} \\ &= |c^{n+1}|a_0 - L|. \end{aligned}$$

Hence the estimate is true for $n + 1$. By the principle of mathematical induction, the estimate is true for all $n \in \mathbb{N}$. □

(b) Use the Sandwich Theorem and the fact that $\lim_{n\to\infty} c^n = 0$ to prove that $\lim_{n\to\infty} a_n = L$.

Proof. By (a),

$$0 \le |a_n - L| \le c^n |a_0 - L|$$

for all $n \in \mathbb{N}$. By the proof of Theorem 2.5, $\lim_{n\to\infty} c^n = 0$. Hence the limits on the left and the right in ther above estimate are both 0, and the Sandwich Theorem gives $\lim_{n\to\infty} |a_n - L| = 0$, which means that $\lim_{n\to\infty} a_n = L$. □

4. **Recursive algorithm for finding $\sqrt{a}$.** Let $a > 1$ and define

$$a_0 = a \quad \text{and} \quad a_n = \frac{1}{2}\left(a_{n-1} + \frac{a}{a_{n-1}}\right) \text{ for } n \ge 1.$$

(a) Prove that $0 < a_n - \sqrt{a} = \frac{1}{2a_{n-1}}(a_{n-1} - \sqrt{a})^2$ for $n \ge 1$.

Proof. Assuming $a_{n-1} \neq 0$ we get

$$\begin{aligned} a_n - \sqrt{a} &= \frac{1}{2}\left(a_{n-1} + \frac{a}{a_{n-1}}\right) - \sqrt{a} \\ &= \frac{1}{2a_{n-1}}\left(a_{n-1}^2 + a\right) - \sqrt{a} \\ &= \frac{1}{2a_{n-1}}\left(a_{n-1} - \sqrt{a}\right)^2. \end{aligned}$$

For $n = 0$, $a_0 = a \neq \sqrt{a}$.
By induction and the above identity it follows that $a_n - \sqrt{a} > 0$ for $n \ge 1$ and particularly $a_n > 0$. □

(b) Use (a) to prove that $0 \le a_n - \sqrt{a} \le \frac{1}{2}(a_{n-1} - \sqrt{a})$ for $n \ge 1$.

Proof. We have

$$0 \le \frac{1}{a_{n-1}}(a_{n-1} - \sqrt{a}) < 1,$$

and the identity in part (a) gives

$$a_n - \sqrt{a} = \frac{1}{2a_{n-1}}\left(a_{n-1} - \sqrt{a}\right)\left(a_{n-1} - \sqrt{a}\right) \le \frac{1}{2}\left(a_{n-1} - \sqrt{a}\right).$$

□

(c) Deduce that $\lim_{n\to\infty} a_n = \sqrt{a}$.

Proof. This immediately follows from (a), (b) and tutorial question 3. □

(d) Apply four steps of the recursive algorithm with $a = 3$ to approximate $\sqrt{3}$.
Solution. The values and their approximation are

$$a_0 = 3, \quad a_1 = 2, \quad a_2 = \frac{7}{4} = 1.75, \quad a_3 = \frac{97}{56} \approx 1.73214, \quad a_4 = \frac{18817}{10864} \approx 1.73205.$$