Section 4.1: Limits of Real Valued Functions – Part C

Rather than calculating limits from the definition, in general one will use limit laws. In this section we state and prove some of these laws.

Theorem 4.3 (Limit Laws)

Let $a, c \in \mathbb{R}$ and suppose that the real functions f and g are defined in a deleted neighborhood of a and that $\lim_{x\to a} f(x) = L \in \mathbb{R}$ and $\lim_{x\to a} g(x) = M \in \mathbb{R}$ both exist. Then

a. $\lim_{x\to a} c = c$.
b. $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$.
c. $\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M$.
d. $\lim_{x\to a}[cf(x)] = c\lim_{x\to a} f(x) = cL$.
e. $\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right] = LM$.
f. If $M \neq 0$, $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$.
g. If $L \neq 0$ and $M = 0$, $\lim_{x\to a}\frac{f(x)}{g(x)}$ does not exist.
h. If $n \in \mathbb{N}$, $\lim_{x\to a}[f(x)^n] = \left[\lim_{x\to a} f(x)\right]^n = L^n$.
i. $\lim_{x\to a} x = a$.
j. If $n \in \mathbb{N}$, $\lim_{x\to a} x^n = a^n$.
k. If $n \in \mathbb{N}$, $\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{a}$. If n is even, we assume that $a \geq 0$.
l. If $n \in \mathbb{N}$, $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$.
m. If $\lim_{x\to a}|f(x)| = 0$, then $\lim_{x\to a} f(x) = 0$.

Proof

The proofs are similar to those in Theorem 2.2 and we will only prove (b), (e) and (f).

(b) $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$.

Let $\varepsilon > 0$. Then there are numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that

i. $|f(x) - L| < \frac{\varepsilon}{2}$ if $0 < |x - a| < \delta_1$,
ii. $|g(x) - M| < \frac{\varepsilon}{2}$ if $0 < |x - a| < \delta_2$.

Put $\delta = \min\{\delta_1, \delta_2\}$. For $0 < |x - a| < \delta$ we have $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$ and therefore

$$\begin{aligned}\left|\left(f(x)+g(x)\right)-(L+M)\right| &= |(f(x)-L)+(g(x)-M)| \\ &\le |f(x)-L|+|g(x)-M| \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \\ &= \varepsilon.\end{aligned}$$

Hence $f(x)+g(x)$ converges to $L+M$ as $x \to a$.

(e) $\lim_{x\to a}[f(x)g(x)] = \left[\lim_{x\to a} f(x)\right]\left[\lim_{x\to a} g(x)\right] = LM.$

We consider two cases: one special case to which then the general case is reduced.

Case 1: $L = M = 0$.

Let $\varepsilon > 0$. Then there are numbers δ_1 and δ_2 such that

i. $|f(x)-L| = |f(x)| < 1$ if $0 < |x-a| < \delta_1$,
ii. $|g(x)-M| = |g(x)| < \varepsilon$ if $0 < |x-a| < \delta_2$.

Put $\delta = \min\{\delta_1, \delta_2\}$. For $0 < |x-a| < \delta$ we have

$$|f(x)g(x)-0| = |f(x)-0|\cdot|g(x)-M| < 1\cdot\varepsilon = \varepsilon.$$

Case 2: *L and M are arbitrary.*

Then

$$f(x)g(x) = (f(x)-L)(g(x)-M) + L(g(x)-M) + f(x)M.$$

(To see why this is true, simplify the right-hand side of the equation.)

Since $f(x) \to L$ and $g(x) \to M$, we have

$$(f(x)-L) \to 0 \text{ and } (g(x)-M) \to 0 \tag{4.1}$$

as $x \to a$ and it follows that

$\lim_{x\to a} f(x)g(x)$

$= \lim_{x\to a}[(f(x)-L)(g(x)-M) + L(g(x)-M) + f(x)M]$

$= \lim_{x\to a}(f(x)-L)(g(x)-M) + \lim_{x\to a} L(g(x)-M) + \lim_{x\to a} f(x)M$ by part (b)

$= \lim_{x\to a}(f(x)-L)(g(x)-M) + L\left[\lim_{x\to a}(g(x)-M)\right] + M\left[\lim_{x\to a} f(x)\right]$ by part (d)

$= 0 + L\left[\lim_{x\to a}(g(x)-M)\right] + M\left[\lim_{x\to a} f(x)\right]$ by (4.1) and Case 1

$= 0 + L\cdot 0 + M\left[\lim_{x\to a} f(x)\right]$ by (4.1)

$= 0 + L \cdot 0 + ML$ since $f(x) \to L$

$= LM.$

(f) If $M \neq 0$, $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$.

First consider $f(x) = 1$. Since $M \neq 0$ and $g(x) \to M$ as $x \to a$, there is $\delta_0 > 0$ such that

$$|g(x) - M| < \frac{|M|}{2}$$

for $0 < |x - a| < \delta_0$. Then, for $0 < |x - a| < \delta_0$

$$\begin{aligned} |g(x)| &= |M + (g(x) - M)| \\ &\geq |M| - |g(x) - M| \\ &\geq |M| - \frac{|M|}{2} \\ &= \frac{|M|}{2} \end{aligned}$$

which implies

$$\begin{aligned} \left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| &= \left|\frac{1}{g(x)} - \frac{1}{M}\right| \\ &= \left|\frac{M - g(x)}{g(x)M}\right| \\ &= \frac{|g(x) - M|}{|g(x)| \cdot |M|} \\ &\leq \frac{|g(x) - M|}{\left(\frac{|M|}{2}\right) \cdot |M|} \\ &= \frac{2|g(x) - M|}{|M|^2}. \end{aligned}$$

Now let $\varepsilon > 0$ and $\delta_1 > 0$ such that $0 < |x - a| < \delta_1$ implies

$$|g(x) - M| \leq \frac{|M|^2}{2}\varepsilon.$$

(To get this use $\left|\frac{f(x)}{g(x)} - \frac{L}{M}\right| \leq \frac{2|g(x)-M|}{|M|^2} \leq \varepsilon$.)

Put $\delta = \min\{\delta_0, \delta_1\}$. It follows for $0 < |x - a| < \delta$ that

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| \leq \frac{2|g(x) - M|}{|M|^2} < \frac{2 \cdot \frac{|M|^2}{2}\varepsilon}{|M|^2} = \varepsilon.$$

(This gives us $\frac{1}{g(x)} \to \frac{1}{M}$ as $x \to a$.)

The general case now follows with (d):

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \left[f(x) \cdot \frac{1}{g(x)}\right] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} \frac{1}{g(x)} = L \cdot \frac{1}{M} = \frac{L}{M}. \qquad \blacksquare$$

Note: For (f) and (g), if both $L = 0$ and $M = 0$, then we have an indeterminate form – use L'Hopital.

Recall that a polynomial function is of the form

$$f(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_2 x^2 + b_1 x + b_0$$

with $b_i \in \mathbb{R}$ for $i = 1,2,\ldots,n$ and n any nonnegative integer. A rational function is of the form

$$f(x) = \frac{p(x)}{q(x)}$$

with $p(x)$ and $q(x)$ polynomials. We then have the following as a consequence of Theorem 2.2.

Corollary 4.1

If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a).$$

Corollary 4.2

All the limit rules in Theorem 4.3 remain true if $x \to a$ is replaced by any of the following: $x \to a^+$, $x \to a^-$, $x \to \infty$ and $x \to -\infty$.

Proof

For $x \to a^+$ and $x \to a^-$ one just has to replace $0 < |x - a| < \delta$ with $0 < x - a < \delta$ and $-\delta < x - a < 0$, respectively, in the proof of each of the statements.

For $x \to \infty$ and $x \to -\infty$, the proofs are very similar to those for sequences. $\blacksquare$

Similar rules hold if the functions have infinite limits. We state some of the results for $x \to a$, observing that there are obvious extensions as in Corollary 4.2.

Theorem 4.4

Assume that $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} g(x) = \infty$ and $\lim_{x\to a} h(x) = c \in \mathbb{R}$. Then

(a) $f(x) + g(x) \to \infty$ as $x \to a$.
(b) $f(x) + h(x) \to \infty$ as $x \to a$.
(c) $f(x)g(x) \to \infty$ as $x \to a$.
(d) $f(x)h(x) \to \begin{cases} \infty & if\ c > 0 \\ -\infty & if\ c < 0 \end{cases}$ as $x \to a$.

Proof

We prove (c) and leave the other parts as exercises.

Let $A > 0$. Then there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

i. $f(x) > 1$ if $0 < |x - a| < \delta_1$,
ii. $g(x) > A$ if $0 < |x - a| < \delta_2$.

With $\delta = \min\{\delta_1, \delta_2\}$ it follows for $0 < |x - a| < \delta$ that

$$f(x)g(x) > 1 \cdot A = A. \qquad \blacksquare$$

Theorem 4.5 (Sandwich Theorem)

Let $a \in \mathbb{R} \cup \{\infty, -\infty\}$ and assume that f, g and h are real functions defined in a deleted neighborhood of a. If $f(x) \le g(x) \le h(x)$ for x in a deleted neighborhood of a and

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x),$$

then

$$\lim_{x \to a} g(x) = L.$$

Proof

Note that $L \in \mathbb{R} \cup \{\infty, -\infty\}$. We will prove this theorem in the case $a \in \mathbb{R}$ and $L \in \mathbb{R}$. The other cases are left as an exercise.

Let $\varepsilon > 0$. Then there are δ_1 and δ_2 such that

i. $|f(x) - L| < \varepsilon$ if $0 < |x - a| < \delta_1$,
ii. $|h(x) - L| < \varepsilon$ if $0 < |x - a| < \delta_2$.

Put $\delta = \min\{\delta_1, \delta_2\}$. Then, for $0 < |x - a| < \delta$,

$$L - \varepsilon < f(x) < L + \varepsilon$$

and

$$L - \varepsilon < h(x) < L + \varepsilon$$

gives

$$L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon.$$

Hence $|g(x) - L| < \varepsilon$ if $0 < |x - a| < \delta$. ■

Theorem 4.6

Let f be defined on an interval (a, b), where $a = -\infty$ and $b = \infty$ are allowed. With the convenient notation $a^+ = -\infty$ if $a = -\infty$ and $b^- = \infty$ if $b = \infty$, we obtain

(a) *If f is increasing, then*

$$\lim_{x \to b^-} f(x) = \sup\{f(x) : x \in (a, b)\}$$

and

$$\lim_{x \to a^+} f(x) = \inf\{f(x) : x \in (a, b)\}.$$

(b) *If f is decreasing, then*

$$\lim_{x \to b^-} f(x) = \inf\{f(x) : x \in (a, b)\}$$

and

$$\lim_{x \to a^+} f(x) = \sup\{f(x) : x \in (a, b)\}.$$

Proof

Since all four cases have similar proofs, we only prove (a) in the case $b \in \mathbb{R}$.

Let $L = \sup\{f(x) : x \in (a, b)\}$

Case 1: $L \in \mathbb{R}$

Let $\varepsilon > 0$. By Theorem 1.6 there is $c \in (a, b)$ such that $L - \varepsilon < f(c)$. Put $\delta = b - c > 0$. Now let $b - \delta < x < b$, i.e., $x \in (c, b)$. Then $c < x$ gives $f(c) \leq f(x)$ since f is increasing and $f(x) \leq L$ for all $x \in (c, b) \subset (a, b)$ by definition of the supremum, so that

$$L - \varepsilon < f(c) \leq f(x) \leq L < L + \varepsilon$$

for these x. This means $f(x) \to L$ as $x \to b^-$ by definition.

Case 2: $L = \infty$

In this case, $\{f(x) : x \in (a, b)\}$ is not bounded above. Therefore, for each $A \in \mathbb{R}$ there is $c \in (a, b)$ such that $f(c) > A$. Since f is increasing, it follows for all $x \in (c, b)$ that $A < f(c) \leq f(x)$. Therefore $f(x) \to \infty$ as $x \to b^-$. ■

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Tutorial 4.1 – Part C

1. Prove the following:
    a. The remainder of the Limit Laws in Theorem 4.3.
    b. Corollary 4.1.
    c. The remainder of Theorem 4.4.

d. The other cases for Theorem 4.5.
e. The other cases for Theorem 4.6.

2. Let n be a positive integer. Prove that
   a. $\lim\limits_{x\to\infty} x^n = \infty$
   b. $\lim\limits_{x\to-\infty} x^n = \begin{cases} \infty & \text{if } n \text{ is even,} \\ -\infty & \text{if } n \text{ is odd.} \end{cases}$
   c. $\lim\limits_{x\to0^+} x^{-n} = \infty$
   d. $\lim\limits_{x\to0^-} x^{-n} = \begin{cases} \infty & \text{if } n \text{ is even,} \\ -\infty & \text{if } n \text{ is odd.} \end{cases}$

3.
   a. Let f, g be defined in a deleted neighborhood of a and assume that $f(x) < g(x)$ for all x in a deleted neighborhood of a. Show that if $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$ exist, then $L \le M$.
   b. Give examples for $L < M$ and for $L = M$ in (a).
   c. Formulate and prove the result corresponding to (a) for one-sided limits.

4. Using rules for limits, determine the behavior of $f(x)$ as x tends to the given limit:
   a. $f(x) = \frac{4x}{3-x}$ as $x \to 3^-$.
   b. $f(x) = \frac{(x-4)(x-1)}{x-2}$ as $x \to 2^+$.
   c. $f(x) = \frac{2x+1}{x^2-x}$ as $x \to 0^+$.

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