Question 1 **[7 marks]**

Consider $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xe^y + z$ and the hypersurface

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \right\}.$$

(a) Find a normal vector to S at the point $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. **(2)**

Solution: A normal vector at every point on the surface is given by the gradient

$$\nabla f = \begin{pmatrix} e^y \\ xe^y \\ 1 \end{pmatrix}. \checkmark$$

At $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ we have the normal vector

$$\nabla f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \checkmark$$

(b) Find the directional derivative of f in the direction $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ at the point $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. **(2)**

Solution: The directional derivative is

$$D_{\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \nabla f\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \checkmark^{2}$$

(c) Find the tangent hyperplane to S at the point $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Give your answer as a set of points, **(3)**

i.e. **not** in equational form (simplify fully).

Solution:

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + T_{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot (\nabla f)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0 \right\} \\ &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 0 \right\} \checkmark \end{aligned}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,\middle|\, x + z = 0 \right\} \checkmark$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \left\{ \begin{pmatrix} x \\ y \\ -x \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \\ 1 - x \end{pmatrix} \,\middle|\, x, y \in \mathbb{R} \right\}. \checkmark$$

Question 2 **[8 marks]**

(a) Define what is meant by the vector path integral of $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$ along the oriented piecewise smooth curve $\Gamma \subseteq \mathbb{R}^n$. **(3)**

> **Solution:** Let $\Gamma = \{\, \mathbf{r}(t) \,:\, t \in [a, b] \,\}$ ✓ be an oriented piecewise smooth curve in $\mathbf{R}^n$. Let $\mathbf{v} : \Gamma \to \mathbf{R}^n$. We define the vector path integral of $\mathbf{v}$ along Γ, by
>
> $$\int_\Gamma \mathbf{v} \cdot d\mathbf{r} = \int_a^b \mathbf{v}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\; dt. \checkmark^{2}$$

(b) Prove or disprove: Let Γ be a parametric curve in $\mathbb{R}^n$. For every $\mathbf{v} : \Gamma \to \mathbb{R}^n$, there exists an $F : \Gamma \to \mathbb{R}$ such that **(5)**

$$\int_\Gamma \mathbf{v}\, d\mathbf{r} = \int_\Gamma F\, ds.$$

> **Solution:** The statement is true. Let $\Gamma = \{\, \mathbf{r}(t) \,:\, t \in [a, b] \,\}$. If we let $\mathbf{u}$ be the unit tangent to Γ at $\mathbf{r}(t)$, then we can define
>
> $$\int_\Gamma \mathbf{v} \cdot d\mathbf{r} = \int_\Gamma \mathbf{v} \cdot \mathbf{u}\; ds$$
>
> i.e. it is equal to scalar path integral of the form $\int_\Gamma F\, ds$ where $F = \mathbf{v} \cdot \mathbf{u}$.✓ If $\mathbf{u}$ is the unit tangent to Γ at $\mathbf{r}(t)$,
>
> $$\mathbf{u}(\mathbf{r}(t)) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \checkmark$$
>
> and $\dfrac{d\mathbf{r}}{dt} = \mathbf{r}'(t)$ i.e. $d\mathbf{r} = \mathbf{r}'(t)\, dt$ ✓½ and $ds = \|\mathbf{r}'(t)\|\, dt$, ✓½ so that
>
> $$\begin{aligned} \int_\Gamma \mathbf{v} \cdot d\mathbf{r} &= \int_\Gamma \mathbf{v} \cdot \mathbf{r}'(t)\, dt \checkmark^{1/2} \\ &= \int_\Gamma \mathbf{v} \cdot \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}\, \|\mathbf{r}'(t)\|\, dt \checkmark^{1/2} \\ &= \int_\Gamma (\mathbf{v} \cdot \mathbf{u})\, \|\mathbf{r}'(t)\|\, dt \checkmark^{1/2} \\ &= \int_\Gamma (\mathbf{v} \cdot \mathbf{u})\, ds \checkmark^{1/2} = \int_\Gamma F\, ds. \end{aligned}$$

Question 3 [14 marks]

Let $f : D \to \mathbb{R}$ be given by $f(x, y) = x^2 - y^2$. Consider the transformation $(x, y) = (u - v, u + v)$.

(a) Find D^* such that $\mathbf{T}(D^*) = D$, where $\mathbf{T}(u, v) = (u - v, u + v)$. **(4)**

> **Solution:** From $y \le 2 - x$, i.e. $(u + v) \le 2 - (u - v)$, we have $u \le 1$. ✓½ Since $2u = x + y \ge 0$, we find $0 \le u \le 1$. ✓½ Finally $0 \le u - v = y$ and $0 \le u + v = x$ provides $-u \le v \le u$. ✓ Consequently, $D^* = \{ (u, v) : u \in [0, 1],$ ✓ $v \in [-u, u]$ ✓ $\}$.

(b) Express $\iint_D f(x, y)\, dx\, dy$ as a double integral over D^*. **(10)**

Do not integrate. Leave your answer as a double integral.

> **Solution:** The Jacobian is
>
> $$\left| \frac{\partial(x, y)}{\partial(u, v)} \right| = \det \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \checkmark^{4} = 2. \checkmark$$
>
> Thus
>
> $$\iint_D f(x, y)\, dx\, dy = \iint_{D^*} f(\mathbf{T}(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dy \checkmark$$
> $$= \int_0^{1 \checkmark} \int_{-u}^{u \checkmark} 2uv\; dv\, du. \checkmark^{2}$$

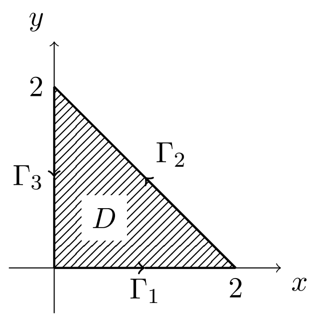

Figure 1: The region D for questions 3 and 4.

Question 4 **[31 marks]**

(a) Let $f : D \to \mathbb{R}$ be given by $f(x, y) = x^2 - y^2$. Evaluate $\iint_D f(x, y)\, dx\, dy$ using Fubini's theorem. **(8)**

Solution:

$$\begin{aligned}
\iint_D f(x, y)\, dx\, dy &= \int_0^2 \int_0^{2-y} (x^2 - y^2)\, dx\, dy \\
&= \int_0^2 \left[\frac{x^3}{3} - xy^2 \right]_0^{2-y} dy \\
&= \int_0^2 \frac{1}{3}(2-y)^3 - (2-y)y^2\, dy \\
&= \left[-\frac{1}{12}(2-y)^4 - \frac{2}{3}y^3 + \frac{1}{4}y^4 \right]_0^2 \\
&= -\frac{16}{3} + 4 + \frac{16}{12} = 0.
\end{aligned}$$

(b) Show that $\mathbf{F}(\Gamma_1) = \mathbf{F}(\Gamma_3) = \{(0, 0)\}$. **(6)**

Solution: On Γ_1 we have $y = 0$ so that

$$\mathbf{F}(x, y) = \mathbf{F}(x, 0) = (0, 0).$$

On Γ_3 we have $x = 0$ so that

$$\mathbf{F}(x, y) = \mathbf{F}(0, y) = (0, 0).$$

(c) Evaluate $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = (-x^2 y, -xy^2)$ and $\partial D = \Gamma_1 + \Gamma_2 + \Gamma_3$. **(11)**

Solution: Since

$$\Gamma_2 = \{\, \mathbf{r}_2(t) = (2 - t, t) \,:\, t \in [0, 2] \,\}$$

we find

$$\begin{aligned}
\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} &= \int_{\Gamma_1} \mathbf{F} \cdot d\mathbf{r}_1 + \int_{\Gamma_2} \mathbf{F} \cdot d\mathbf{r}_2 + \int_{\Gamma_3} \mathbf{F} \cdot d\mathbf{r}_3 \\
&= \int_{\Gamma_2} \mathbf{F} \cdot d\mathbf{r}_2 \\
&= \int_0^2 \mathbf{F}(2 - t, t) \cdot (-1, 1)\, dt \\
&= \int_0^2 (-(2-t)^2 t, -(2-t)t^2) \cdot (-1, 1)\, dt \\
&= \int_0^2 2t^3 - 6t^2 + 4t\, dt
\end{aligned}$$

$$= \left[\frac{t^4}{2} - 2t^3 + 2t^2\right]_0^2 = 0. \checkmark$$

(d) State Green's theorem. **(5)**

Solution: Let D be a region in $\mathbb{R}^2$ with boundary ∂D oriented clockwise (i.e. positive orientation), then for $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ we have

$$\iint_D \left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) dx_1\, dx_2 = \int_{\partial D} \mathbf{F} \cdot d\mathbf{r}.$$

(e) Given that $\nabla\times \mathbf{F} = (0, 0, f(x, y))$, where $f(x, y) = x^2 - y^2$, verify that Green's theorem holds by comparing your answer to (c) with your answer to (a). **(1)**

Solution: We found $\displaystyle\int_{\partial D} \mathbf{F} \cdot d\mathbf{r} = \iint_D \nabla\times \mathbf{F} \cdot \mathbf{e}_3\, dx\, dy = \iint_D f(x, y)\, dx\, dy = 0.$ ✓