Tutorial Solutions of Chapter 2 Lecture 2

Tutorial 2.1.1(2)

(a) Prove that if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} |\, a_n \,| = |\, L \,|$.

Hint: Use (and prove) the inequality

$$|\, |\, x \,| - |\, y \,| \,| \leq |\, x - y \,|\,.$$

Use Tutorial 1.1.2,2(d) to prove this inequality.

Proof : For all $x, y \in \mathbb{R}$ we know by Tutorial 1.1.2,2(d) that

$$|\, x \,| = |\, (x - y) + y \,| \leq |\, x - y \,| + |\, y \,|\,.$$

Hence

$$|\, x \,| - |\, y \,| \leq |\, x - y \,|\,.$$

Interchanging x and y we have

$$|\, y \,| - |\, x \,| \leq |\, x - y \,|\,.$$

Hence, by definition of the absolute value,

$$\big|\, |\, x \,| - |\, y \,| \,\big| \leq |\, x - y \,|\,.$$

Now let $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = L$, there is K suc that $\ a_n - L \,| < \epsilon$ for $n > K$. But then

$$\big|\, |\, a_n \,| - |\, L \,| \,\big| \leq |\, a_n - L \,| < \epsilon$$

for these n which proves $\lim_{n\to\infty} |\, a_n \,| = |\, L \,|$ □

(b) Give an example to show that the converse to part (a) is not true. **Hint:** Use $a_n = (-1)^n$.

Proof : Let $a_n = (-1)^n$. From question Tutorial 2.1.1(1)(b), the sequence (a_n) does not converge. But $|a_n| = 1$, and therefore $|(a_n)|$ is a constant sequence, which converges by Theorem 2.3 (a).
□