Question 1 Matrix Decomposition [30 Marks]

1. Compute the determinant of A where. [6]

$$A = \begin{bmatrix} 2 & 3 & 2 & 1 & 5 \\ 2 & -1 & 6 & 0 & 3 \\ -1 & 1 & 2 & 0 & 0 \\ -1 & 0 & 4 & 0 & 0 \\ 2 & 0 & 3 & 0 & 0 \end{bmatrix}$$

2. Let $B \in \mathbb{R}^{2\times 3}$ with a singular value decomposition of $B = U\tilde{B}V^T$. Let $x \in \mathbb{R}^3$ be a **coordinate** vector in terms of the canonical basis. Specifically:

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \tilde{B} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 6 & 0 \end{bmatrix} \quad V = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & \frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & \frac{-2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{-1}{3} \end{bmatrix} \quad x = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}\sqrt{18}$$

   (a) Determine the matrix B. [6]

   (b) Compute $\hat{x}$ where $\hat{x} = Bx$. [3]

   (c) Project x onto the basis defined by the right singular vectors (V). Call this new vector $\tilde{x}$. [4]

   (d) Compute $\hat{\tilde{x}} = \tilde{B}\tilde{x}$. Note $\hat{\tilde{x}}$ is a vector in terms of the basis defined by the left singular vectors. [3]

   (e) Project $\hat{\tilde{x}}$ onto the canonical basis **from** the basis defined by the left singular vectors (U). **Hint:** use your answer from question 2(d) [3]

3. Determine the SVD (remember to normalize Singular Vectors) of A where: [5]

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}[2]\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} + \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}[1]\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{bmatrix}$$

Question 2 Vector Calculus [30 Marks]

1. Compute the derivative $f'(x)$ for $f(x)$ shown below (note log refers to the natural log). [8]

$$f(x) = log(x)^3 cos(x-4)^2$$

2. Compute the third order Taylor polynomial T_3 of $f(x) = sin(x) + exp(x)$ [7] at $x_0 = \frac{\pi}{2}$ (you don't have to foil out all the powers of $x - x_0$).

   Remember that the Taylor polynomial of degree n of $f : \mathbb{R} \longrightarrow \mathbb{R}$ at x_0 is defined as

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

   where $f^{(k)}(x_0)$ is the kth derivative of f at x_0 (which we assume exists) and $\frac{f^{(k)}(x_0)}{k!}$ are the coefficients of the polynomial, according to **Definition 5.3** of the textbook.

3. Compute the derivative $f'(x)$ of the logistic (sigmoid) function $f(x)$. Write your final answer in terms of the original logistic function (manipulate your answer so that it contains $f(x)$ in it). [6]

$$f(x) = \frac{1}{1 + e^{-x}}$$

4. Consider the function $f(x) = \sqrt{(x^3 + \sin(x^3))} - (x^3 + \sin(x^3))^2 + \sin(2x^3)$. [9] Depict $f(x)$ as a data flow graph. Make sure to define all intermediate variables.