3.3 Power Series

- **Definition 3.4**

  Let a and c_n, $n \in \mathbb{N}$, be real numbers. A series

  $$\sum_{n=0}^{\infty} c_n (x-a)^n$$

  is called a **power series in** $(x-a)$ or a **power series centred at** a or a **power series about** a.

- **Notation.**

  The **radius of convergence** of the series $\sum_{n=0}^{\infty} c_n (x-a)^n$ is a number R or ∞ defined as follows:

  (i) $R = 0$ if $\limsup_{n\to\infty} \sqrt[n]{|c_n|} = \infty$,

  (ii) $R = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|} \cancel{\in \mathbb{R}}}$ if $0 < \limsup_{n\to\infty} \sqrt[n]{|c_n|} \in \mathbb{R}$,

  (iii) $R = \infty$ if $\limsup_{n\to\infty} \sqrt[n]{|c_n|} = 0$.

- **Theorem 3.11**

  There are three alternatives for the domain of a power series

$$\sum_{n=0}^{\infty} c_n(x-a)^n :$$

(i) If $R = 0$, then the series converges only for $x = a$.
(ii) If $R = \infty$, then the series converges absolutely for all $x \in \mathbb{R}$.
(iii) If $0 < R \in \mathbb{R}$, then the series converges absolutely if $|x-a| < R$ and diverges if $|x-a| > R$.
Note that anything may happen if $|x-a| = R$.
The domain of the series is called the **interval of convergence**.

Proof. see Notation (1) above for $R=0$

(i) If $\limsup_{n\to\infty} \sqrt[n]{|c_n|} = \infty$, then for $x \neq a$ $|x-a|>0$

$$\limsup_{n\to\infty} \sqrt[n]{|c_n(x-a)^n|} = \limsup_{n\to\infty} \sqrt[n]{|c_n|}\,|x-a| = \infty.$$

In view of the Root Test, this shows that the power series diverges if $x \neq a$. see Thm 3.10(ii) $L>1$

(ii), (iii): If $\limsup_{n\to\infty} \sqrt[n]{|c_n|} \in \mathbb{R}$, then

$$\limsup_{n\to\infty} \sqrt[n]{|c_n(x-a)^n|} = |x-a| \limsup_{n\to\infty} \sqrt[n]{|c_n|}.$$

Hence the power series converges for all $x \in \mathbb{R}$ if $\limsup_{n\to\infty} \sqrt[n]{|c_n|} = 0$, proving (ii). $L = 0 < 1$ abs conv $\Rightarrow$ conv

If finally $0 < \limsup_{n\to\infty} \sqrt[n]{|c_n|} \in \mathbb{R}$, then, by the Root Test, the series converges if $|x-a| \limsup_{n\to\infty} \sqrt[n]{|c_n|} < 1$, i.e., $|x-a| < \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}$ $\to R$

and the series diverges if $|x-a| > \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}$.

□

- **Tutorial 5.3.1.**

  1. (a) Let $c > 1$ and put $c_n = \sqrt[n]{c} - 1$.

  (i) Show that $c_n \geq 0$.

  (ii) Show that $\lim_{n\to\infty} \sup c_n \leq 0$. **Hint.** Use Bernoulli's inequality.

  (iii) Conclude that $\lim_{n\to\infty} \sqrt[n]{c} = 1$

  (b) Use (a) to show that $\lim_{n\to\infty} \sqrt[n]{c} = 1$ for all $c > 0$.

  2. Consider $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$ for all $n \in \mathbb{N}$. Show that

  $$\lim_{n\to\infty} \inf \left|\frac{a_{n+1}}{a_n}\right| \leq \lim_{n\to\infty} \inf \sqrt[n]{a_n}$$
  $$\leq \lim_{n\to\infty} \sup \sqrt[n]{a_n} \leq \lim_{n\to\infty} \sup \left|\frac{a_{n+1}}{a_n}\right|$$

  What can you say if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists or is ∞?

  3. Consider the power series $\sum_{n=1}^{\infty} a_n(x-a)^n$ with $a_n \neq 0$ for all $n \in \mathbb{N}$. Using tutorial problem 2 above or otherwise, prove that if $R = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ exists or is ∞, then R is the radius of convergence of the power series.

4. Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

5. Find the radius and interval of convergence for each of the following power series:

(a) $\sum_{n=1}^{\infty} \frac{(2x)^n}{n}$, (b) $\sum_{n=1}^{\infty} \frac{x^n}{n^n}$, (c) $\sum_{n=1}^{\infty} n^n x^n$,

(d) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{3^n \sqrt{n}}$, (e) $\sum_{n=1}^{\infty} \frac{(-2x)^n}{n^3}$, (f) $\sum_{n=1}^{\infty} (-1)^n x^n$.

(g) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$, (h) $\sum_{n=1}^{\infty} \frac{(x+3)^n}{n^3}$, (i) $\sum_{n=1}^{\infty} \frac{(nx)^n}{(2n)!}$

.