UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG
Computer Science and Applied Mathematics

APPM2007
Lagrangian Mechanics

Tutorial 2

Question 1 (5 Points)

Consider the generic point $a \in \mathbb{R}^3$ that lies on the curve specified by the displacement vector

$$\vec{p} = \begin{pmatrix} z \\ x \\ y \end{pmatrix} = \begin{pmatrix} \rho\cos(\theta) \\ \rho\sin(\theta)\cos(\phi) \\ \rho\sin(\theta)\sin(\phi) \end{pmatrix}.$$

Perform the following construction

1. Construct the the set of unit vectors, tangent to the curve with respect to the co-ordinates $\{\rho, \theta, \phi\}$.
2. Show that these tangent vectors are mutually orthogonal.
3. Construct the metric **g**.

Question 2 (10 Points)

Consider the two co-ordinatisation of S^2 in $\mathbb{R}^3$,

$$\vec{a}(\rho,\theta,\phi) = \begin{pmatrix} \rho\cos(\theta) \\ \rho\sin(\theta)\cos(\phi) \\ \rho\sin(\theta)\sin(\phi) \end{pmatrix}$$

where $\{\rho, \theta, \phi\}$ are the radial, declination and azimuth positions in $\mathbb{R}^3$; and

$$\vec{a}(r,s) = \begin{pmatrix} \frac{2r}{r^2+s^2+1} \\ \frac{2s}{r^2+s^2+1} \\ \frac{r^2+s^2-1}{r^2+s^2+1} \end{pmatrix}$$

where $r, s \in \mathbb{R}^2$ are in the plane $z = 0$. Show that in each co-ordinate system, the length of the of the curve passing through the north and south poles of S^2 is π. (Hint: construct appropriatly parametrised paths $\vec{a}(t)$ with t in the appropriate interval.)

Question 3 (10 Points)

Consider the two co-ordinatisation of S^2 in $\mathbb{R}^3$

$$\vec{a}(\rho,\theta,\phi) = \begin{pmatrix} \rho\cos(\theta) \\ \rho\sin(\theta)\cos(\phi) \\ \rho\sin(\theta)\sin(\phi) \end{pmatrix}$$

where $\{\rho,\theta,\phi\}$ are the radial, declination and azimuth positions in $\mathbb{R}^3$; and

$$\vec{a}(r,s) = \begin{pmatrix} \frac{2r}{r^2+s^2+1} \\ \frac{2s}{r^2+s^2+1} \\ \frac{r^2+s^2-1}{r^2+s^2+1} \end{pmatrix}$$

where $r, s \in \mathbb{R}^2$ are in the plane $z = 0$. Show that in each co-ordinate system, the surface area of the unit sphere is 4π.

Question 4 (10 Points)

The *Cobb-Douglas production function* is used to model the number of units proxuced by variyng amounts of labour and capital. Let x define the units of labour and y denote the units of capital and C is a constant and $0 < a < 1$,

$$f(x,y) = Cx^a y^{1-a}.$$

The Cobb-Douglas production function for a particular manufacturer is given by

$$f(x,y) = 100x^{\frac{3}{4}} y^{\frac{1}{4}}.$$

Suppose that labour is charged at $R150$ per unit and capital is charged at $R250$ per unit. Suppose that the total cost of labour and capital is limited to $R50000$. Find the maximum production level for this manufacturer. (Hint: relate the rate of productivity to the rate of constraint using directional derivatives.)