Chapter 4: Limits and Continuous Functions

The single most important concept in all of analysis is that of a limit. Every single notion of analysis is encapsulated in one sense or another to that of a limit. In the previous chapter, you learnt about limits of sequences. Here this notion is extended to limits of functions, which leads to the notion of continuity. You have learnt an intuitive version in Calculus I. Here you will learn a precise definition and you will learn how to prove the results you have learnt in Calculus I.

In this course all functions will have domains and ranges which are subsets of $\mathbb{R}$ unless otherwise stated. Such functions are called real functions.

* * * * * * *

Section 4.1: Limits of Real Valued Functions (Part A)

Definition 4.1

Let $a \in \mathbb{R}$. An interval of the form (c, d) with $c < a < d$ is called a *neighborhood* of a, and the set $(c, d)\backslash\{a\}$ is called a deleted *neighborhood* of a.

In the definition of the limit of sequences you have seen formal definitions for 'n tends to ∞' and 'a_n tends to L', and the latter one has an obvious generalization to 'x tends to a' and '$f(x)$ tends to L', which you will encounter in the next definition.

Definition 4.2 (Limit of a function)

Let f be a real function, $a, L \in \mathbb{R}$ and assume that the domain of f contains a deleted neighborhood of a, that is, $f(x)$ is defined for all x in a deleted neighborhood of f. Then '$f(x) \to L$ as $x \to a$' is defined to mean:

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in (a - \delta, a + \delta)\backslash\{a\}, f(x) \in (L - \varepsilon, L + \varepsilon)$$

i.e.

$$\forall \varepsilon > 0, \exists \delta > 0, (0 < |x - a| < \delta \to |f(x) - L| < \varepsilon).$$

If $f(x) \to L$ as $x \to a$, then we write $\lim_{x \to a} f(x) = L$.

See Figure 4.1 on the next page for a visual representation of Definition 4.2.

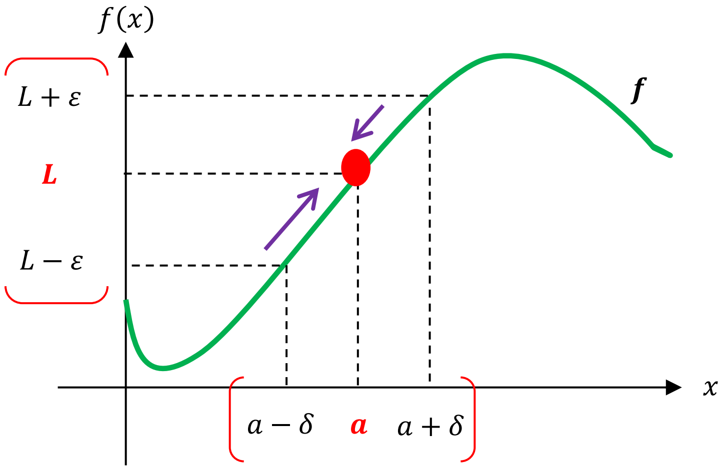

Figure 4.1

Note:

1. L in the definition above is also called the limit of the function at a. For the definition of the limit of a function at a, the number $f(a)$ is never used, and indeed, the function need not be defined at a.
2. Observe that the number δ will depend on ε, and also on a, if we vary a.

Example 4.1

Prove, from first principles, that $x^2 \to 4$ as $x \to 2$.

Let $\varepsilon > 0$. We must show that there is $\delta > 0$ such that $0 < |x-2| < \delta$ implies $|x^2-4| < \varepsilon$. Since we want to use $|x-2| < \delta$ to get $|x^2-4| < \varepsilon$, we try to factor out$|x-2|$ from $|x^2-4|$. Thus

$$\begin{aligned} |x^2-4| &= |(x-2)(x+2)| \\ &= |x-2| \cdot |x+2| \\ &= |x-2| \cdot |x-2+4| \\ &\leq |x-2|(|x-2|+4) \\ &< |x-2|(\delta+4) \end{aligned}$$

It is often convenient to make an initial restriction on δ, like $\delta \leq 1$. Then we can continue the above estimate to obtain from $|x-a| < \delta$ that

$$\begin{aligned} |x^2-4| &< |x-2|(\delta+4) \\ &\leq |x-2|(1+4) \\ &= 5|x-2|. \end{aligned}$$

If we now put

$$\delta = \min\left\{1, \frac{\varepsilon}{5}\right\},$$

then we can conclude for $0 < |x-2| < \delta$ that

$$\begin{aligned}|x^2-4| &\le 5|x-2|\\ &< 5\delta\\ &= 5\cdot\frac{\varepsilon}{5}\\ &= \varepsilon.\end{aligned}$$
■

To justify the notation $\lim_{x\to a} f(x) = L$ we have to show uniqueness.

Theorem 4.1

If $f(x) \to L$ as $x \to a$, then L is unique.

Proof

Let $f(x) \to L$ and $f(x) \to M$ as $x \to a$. We must show that $L = M$.

So let $\varepsilon > 0$. Then there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |x-a| < \delta_1 \to |f(x) - L| < \frac{\varepsilon}{2}$$

and

$$0 < |x-a| < \delta_2 \to |f(x) - M| < \frac{\varepsilon}{2}.$$

For $\delta = \min\{\delta_1, \delta_2\}$ and $0 < |x-a| < \delta$ it follows that

$$\begin{aligned}|L-M| &= |L - f(x) + f(x) - M|\\ &= |-(f(x)-L) + (f(x)-M)|\\ &\le |f(x)-L| + |f(x)-M|\\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\\ &= \varepsilon.\end{aligned}$$

Hence $L = M$ by Lemma 2.1. ■

Note:

1. Instead of using $\frac{\varepsilon}{2}$, just ε can also be used. If ε is used, then our final answer will be $|L-M| < 2\varepsilon$ which still gives the same conclusion. Any constant multiple of ε is allowed since we can always let $2\varepsilon = \varepsilon^*$ and so $|L-M| < \varepsilon^*$. In analysis people will usually choose constant multiples in the beginning, like $\frac{\varepsilon}{2}$ here, so that their final answer matches the definition. This is not a requirement and you can do either of the two.
2. The reason for choosing $\delta = \min\{\delta_1, \delta_2\}$ is so that our interval for $(a-\delta, a+\delta)$ overlaps on both the intervals $(a-\delta_1, a+\delta_1)$ and $(a-\delta_2, a+\delta_2)$.

Definition 4.3 (One sided limits)

1. Let f be a real function, $a, L \in \mathbb{R}$ and assume that the domain of f contains an interval (a, d) with $d > a$, that is, $f(x)$ is defined for all x in (a, d). Then '$f(x) \to L$ as $x \to a^+$' is defined to mean
$$\forall \varepsilon > 0, \exists \delta > 0, (a < x < a + \delta \to |f(x) - L| < \varepsilon).$$
If $f(x) \to L$ as $x \to a^+$, then we write $\lim_{x \to a^+} f(x) = L$.

2. Let f be a real function, $a, L \in \mathbb{R}$ and assume that the domain of f contains an interval (c, a) with $c < a$, that is, $f(x)$ is defined for all x in (c, a). Then '$f(x) \to L$ as $x \to a^-$' is defined to mean
$$\forall \varepsilon > 0, \exists \delta > 0, (a - \delta < x < a \to |f(x) - L| < \varepsilon).$$
If $f(x) \to L$ as $x \to a^-$, then we write $\lim_{x \to a^-} f(x) = L$.

Theorem 4.2

If $f(x)$ is defined in a deleted neighborhood of a, then

$$\lim_{x \to a} f(x) = L \leftrightarrow \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

Proof

$(\to)$ Assume that $\lim_{x \to a} f(x) = L$ and let $\varepsilon > 0$. Then

$$\exists \delta > 0, (x \in (a - \delta, a + \delta), x \neq a \to |f(x) - L| < \varepsilon)$$

gives

$$\exists \delta > 0, (x \in (a - \delta, a) \to |f(x) - L| < \varepsilon)$$

and

$$\exists \delta > 0, (x \in (a, a + \delta) \to |f(x) - L| < \varepsilon).$$

Hence $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

$(\leftarrow)$ Assume that $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$ and let $\varepsilon > 0$. Then there are $\delta_- > 0$ and $\delta_+ > 0$ such that

$$x \in (a - \delta_-, a) \to |f(x) - L| < \varepsilon$$

and

$$x \in (a, a + \delta_+) \to |f(x) - L| < \varepsilon.$$

Let $\delta = \min\{\delta_-, \delta_+\}$. Then

$$x \in (a-\delta, a+\delta), x \neq a \to x \in (a-\delta_-, a) \text{ or } x \in (a, a+\delta_+)$$
$$\to |f(x) - L| < \varepsilon$$

This proves that $\lim_{x \to a} f(x) = L$. ■

Example 4.2

Let $f(x) = \frac{x}{|x|}$ for $x \in \mathbb{R}\backslash\{0\}$. Then

- $f(x) = 1$ if $x > 0$ since $|x| = x$.
- $f(x) = -1$ if $x < 0$ since $|x| = -x$.
- $\lim_{x \to 0^+} f(x) = 1$. (Prove this using Definition 4.3)
- $\lim_{x \to 0^-} f(x) = -1$. (Prove this using Definition 4.3)

Now it follows from Theorem 4.2 that the function f does not have a limit as x tends to a.

* * * * * * *

Tutorial 4.1 Part A

1. Prove from the definitions that
   a. $2 + x + x^2 \to 8$ as $x \to 2$.
   b. $\frac{1}{x-2} \to -\frac{2}{3}$ as $x \to \frac{1}{2}$.
   c. $\lim_{x \to 1^-} \sqrt{1-x} = 0$.

2. By negating the definition of limit of a function show that the statement $f(x) \nrightarrow L$ as $x \to a$ is equivalent to the following:

   $\exists \varepsilon > 0, \forall \delta > 0, \exists x$ with $0 < |x - a| < \delta$ such that $|f(x) - L| \geq \varepsilon$.

3. For $x \in \mathbb{R}\backslash\{0\}$ ‘et $f(x) = \sin\left(\frac{1}{x}\right)$. Prove that f does not tend to any limit as $x \to 0$.

4. Let $f(x) = x - \lfloor x \rfloor$. For each integer n, find $\lim_{x \to n^-} f(x)$ and $\lim_{x \to n^+} f(x)$ if they exist.

5. Write out all the steps for Example 4.2.

* * * * * * *