Chapter 1: FINITE,COUNTABLE and INFINITE SETS

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LEARNING OUTCOMES FOR THE LECTURE

By the end of this lecture, students will be able to:

- identify the common notions used in set theory.
- use the notation of set theory.
- describe the elements of a set using set builder notation and by listing the elements, whichever is appropriate.
- define a subset and a proper subset
- prove whether or not one set is a subset (or proper subset) of another set
- form a new set from a given collection of sets by finding their union, intersection, difference or symmetric difference.
- use a venn diagram to represent sets

MOST CONCEPTS IN THIS CHAPTER LEARNT IN TAM MATH2025

There are in mathematics some terms that are called “Common Notions” and are not defined as to do so would cause us to violate the rules of “Circular reasoning". That is defining a term A using words that are subsequently defined in terms of A.

Example (1.1.1)

If I define a right angle as an angle of 90^0 *and subsequently define a* 90^0 *as a right angle!*

We use undefined terms:

- Sets $\cdots A$ (sets are collections of objects)
- belonging to the set $\cdots \in$
- equality $\cdots =$
- Congruent $\cdots \equiv$
- not belonging to $\cdots \notin$
- not equal to $\cdots \neq$

We use our common notion of these terms and negotiate a common meaning based on diagrams and previous experience.

NOTATION

- We use capital letters to represent a set.
- we use lower case letters to represent elements
- We use $|A|$ to represent the number of elements in a set A. This number is called order, or size or cardinality of the set A.

Example (1.1.2) *3 examples*

$\boxed{A \text{ is a set;}}$ $\boxed{a \in A \text{ or } a \text{ is an element of } A;}$ $\boxed{a = b \text{ or } A = B.}$

Definition (1.1.3)

Let A be a given set , $|A|$ is the order of A (number of elements in A).

- (i) $|A| = 0$ then $A = \emptyset$. (A is the empty set.)
- (ii) $|A| = 1$ then $A = \{a\}$ and A has a single member.
- (iii) $|A| = n, \quad n \in \mathbb{Z}^+$ then A is a finite set.
- (iv) $|A| = \infty$, then A is an infinite set.
  - (a) $|A| = |\mathbb{Z}| = \infty$, then A is countable. *(i.e We can put members in 1-1 corresponde with $\mathbb{N}$.)* How? This is one way
  - (b) $|A| \neq |\mathbb{Z}|$ and $|A| = \infty$, then A is not countable.

1 -> 0
2 -> 1
3 -> -1
4 -> 2
5 -> -2
and so on

In cases (ii), (iii) and (iv), $A \neq \emptyset$ or A is non emptyset.

A equals
the set of
all x in ℤ
such that
1 less than
equal to x
less than
equal to 4

LISTING AND SET BUILDER NOTATION

♡ $A = \{1, 2, 3, 4\} = \{4, 2, 1, 3\}$ is a list of 4 elements. No specified order. the line | means 'such that'

♡ $A = \{x \in \mathbb{Z} | 1 \leq x \leq 4\}$ is a Set Builder Notation.

In general: $A = \{x \in U | p(x)\}$ $p(x)$ is a statement of a property of x.

SUBSETS

SPECIAL SETS: We assume that a set with no elements exists.

(1) $\emptyset$ is the empty set and has no members. $|\emptyset| = 0$. no elements in here $\emptyset = \{ \}$

Example

♣ $A = \{x \in \mathbb{Z} | x^2 + 1 = 0\} = \emptyset$, $|A| = 0$.

♣ $B = \{x \in \mathbb{C} | x^2 + 1 = 0\} = \{i, -i\}$, $|B| = 2$.

(Note 1.3.2)

$\emptyset \subseteq A \; \forall$ sets A. The empty set is a subset of all sets. Infact it is a proper subset of all nonempty sets!

Challenge Question: What is the cardinality of A if $A = \{\{\emptyset\}\}$? (hint: for '{...}' read 'the set containing')

A

$\{\emptyset\}$

(2) $A = \{a\}$ then A is a singleton set.

Example (1.3.3)

$A = \{x \in \mathbb{R}^+ | x^2 - 1 = 0\} = \{1\}$, *$A$ is a singleton set.*

(3) The Universal set U, the set of all elements in a larger set where all sets under discussion are subsets.

|A|=?

A contains a set that contains the empty set

A does not contain the empty set...

Example (1.3.4)

If we consider A set of even integers and B set of odd integers. Then $\mathbb{Z}$ is the universal set.

PROPERTIES OF SETS

'subset of'

Let A and B be sets. $A \subseteq B$ if each element of A is also an element of B. Note that $A \subseteq B$ means $A \subset B$ or $A = B$. We say A is a subset of B. If $A \neq B$ then A is a proper subset of B. *Thus B is* ***NOT*** *a proper subset of itself* ***BUT*** *is a subset of itself!*

(i) If $A \subset B$ but $A \neq B$, then A is a proper subset of B. That is $A \subset B$. We may also write $A \underset{\neq}{\subset} B$. So A is a subset of itself but not a proper subset of itself. The emptyset $\emptyset$ is a proper subset of all non empty sets, but not a proper subset of itself. The empty set is referred to the trivial subset of a non empty set.

(ii) If A and B are sets and $A \subseteq B$ and $B \subseteq A$ then $A = B$. Certainly, if $A = B$ then $A \subset B$ and $B \subset A$. This principle is useful because it produces a method of showing that two sets are the same. That is, to show $A \subseteq B$ we must establish that; $\forall x \in A \Rightarrow x \in B$.

We can now prove A=B or A⊊B using this method.

For A=B we must show both directions...

To show $A = B$ we can prove $A \subseteq B$ and $B \subseteq A$ then $A = B$. or $\forall x \in A \Rightarrow x \in B$ and $\forall x \in B \Rightarrow x \in A$ then $A = B$.

We must prove implication in both directions

Example

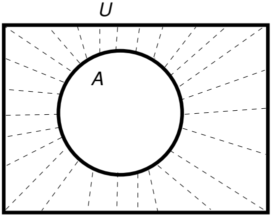
Show $\{x \in \mathbb{R} | x^2 - 1 = 0\} = \{\pm 1\}$. *Let* $A = \{x \in \mathbb{R} | x^2 - 1 = 0\}$ *and let* $B = \{\pm 1\}$.

one way $\boxed{\Rightarrow}$ $x \in A \Rightarrow \quad x^2 - 1 = 0 \Rightarrow \quad x^2 = 1 \Rightarrow \quad x = \pm 1 \Rightarrow$ $x \in B$.

opposite way $\boxed{\Leftarrow}$ $x \in B \Rightarrow \quad x = \pm 1 \Rightarrow \quad x^2 = 1 \Rightarrow \quad x^2 = 1 \Rightarrow$ $x^2 - 1 = 0 \Rightarrow \quad x \in A$.

CONSTRUCTING NEW SETS FROM OLD SETS

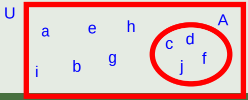
(1) If A is a subset of U then $A' = \{a \in U | a \notin A\}$ and is called the complement of A in U.



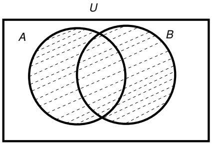
Example

Let $U = \{a, b, c, d, e, f, g, h, i, j\}$ and $A = \{c, d, f, j\}$. Find the complement A' of A in U.

$$A' = \{a, b, e, g, h, i\}$$

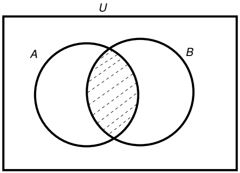


(2) If A and B are subsets of U, then the union of A and B, $A \cup B = \{a \in U | a \in A \text{ or } a \in B\}$.

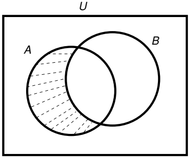


$A, B \subseteq U$ then $A \cup B = \{a \in U | a \in A \text{ or } a \in B\}$.

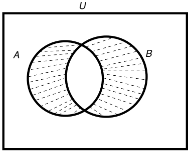
(3) If A and B are subsets of U, then the intersection of A and B, $A \cap B = \{a \in U | a \in A \text{ and } a \in B\}$.



(4) $A - B = A \backslash B = \{a \in U | a \in A \text{ and } a \notin B\}$ is called the difference between A and B.



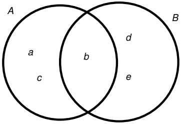
(5) $A \Delta B = (A - B) \cup (B - A)$ is called the symmetric difference between A and B. Thus $A \Delta B = \{a \in U | a \in (A - B) \cup (B - A)\} = \{a \in A \cup B | a \notin A \cap B\}$.



Example (1.5.2)

Given the set $A = \{a, b, c\}$ and $B = \{b, d, e\}$.

(i) Construct a venn diagram for A and B.



(ii)

- Find $A \cap B$, $A \cup B$, $A - B$ and $A \Delta B$.
- $A \cap B = \{b\}$
- $A \cup B = \{a, b, c, d, e\}$
- $A - B = \{a, c\}$
- $A \Delta B = \{a, c, d, e\}$

(6) It is often necessary to form unions and intersections of large classes of sets. Let $\{A_i\}$ be an entirely arbitrary class of sets indexed by a set I of subscripts. Then

$$\bigcup_{i \in I} A_i = \{a \in U | a \in A_i \text{ for at least one } \quad i \in I\}$$

and

$$\bigcap_{i \in I} A_i = \{a \in U | a \in A_i \text{ for all } \quad i \in I\}.$$

(7) For A and B non-empty sets, $A \times B = \{(a, b) | a \in A, \quad b \in B\}$ is called the direct (Cartesian) product of sets A and B. This definition of the product of two sets extends easily to the product of n sets, for n any positive integer. If $A_1, A_2, A_3, \cdots, A_n$ are non-empty sets, then their product $A_1 \times A_2 \times A_3 \times \cdots \times A_n$ is the set of all ordered n-tuples $(a_1, a_2, a_3, \cdots, a_n)$ where $a_i \in A_i$ for each subscript i.

We write

$$A_1 \times A_2 \times A_3 \times \cdots \times A_n = \prod_{i=1}^{n} = \{(a_1, a_2, a_3, \cdots, a_n) | a_i \in A_i\}.$$

We say 'A cross B'

AxB is the set of all ordered pairs with the first element in A and the second element in B

"AXB contains all ordered pairs with a in A and b in B"

Example (1.5.3)

Let $A = \{1, 2, 3\}$ *and* $B = \{a, b\}$. *Find* $A \times B$ *and* $|A \times B|$.
$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$.

(8) Suppose A is any set. Then the power set of A, denoted by $P(A)$, is the set of all subsets of A. That is: $P(A) = \{X | X \subseteq A\}$. Thus A and $\emptyset$ are in the power set of set A. If A is a finite set with n elements then $|P(A)| = 2^n$.

Example (1.5.4) "P(A) contains all possible subsets of A"

Let $A = \{a, b, c\}$.
$P(A) = \{\emptyset, \{a\},\{b\},\{c\},\{a, b\},\{a, c\},\{b, c\},\{a, b, c\}\}$.

(9) If A is a non-empty set, a family or collection Σ of subsets of A is a partition of A, with the elements in Σ called cells, if

- (i) no cell $A_i \in \Sigma$ is empty. That is $A_i \neq \emptyset$ for all $A_i \in \Sigma$.
- (ii) the cells are pair-wise disjoint. That is : $A_i \cap A_j = \emptyset$ for all A_i and A_j in the partition Σ.
- (iii) every element of A belongs to some cell. That is, $a \in A_i$ for some $A_i \in \Sigma$. By (ii) above a will belong to exactly one cell in the partition. We can write A as the union of the cells in the partition as follows: $A = \bigcup_{A_i \in \Sigma} A_i$.

Note: All 3 conditions must be satisfied for a collection of subsets to be a partition...
i) no empty cells, ii) no element can belong in 2 cells (the disjoint property), and iii) each element must be in a cell...

Example (1.5.5)

Let $A = \{a, b, c\}$.

- $\Sigma = \{\{a\}, \{b, c\}\}$ *is* Σ *a partition of* A? Yes, all 3 conditions satisfied
- $\Sigma = \{\{a, b\}, \{c\}\}$ *is* Σ *a partition of* A? Yes
- $\Sigma = \{\{a\}, \{c\}\}$ *is* Σ *a partition of* A? No, b is not in a subset. iii) not satisfied
- $\Sigma = \{\{a, b\}, \{a, c\}\}$ *is* Σ *a partition of* A? *No, a is in 2 subsets, ii) not satisfied*

SETS OF PERMUTATIONS:

Let A be a set, $A \neq \emptyset$. Then S_A is the set of permutations (arrangements) of elements in A.

Example

NOTE: S_A is a set of sets

Let $A = \{1, 2, 3\}$. $S_A =$
$\{\{1, 2, 3\}, \{3, 1, 2\}, \{2, 3, 1\}, \{1, 3, 2\}, \{3, 2, 1\}, \{2, 1, 3\}\}$.
$|A| = 3$ *and* $|S_A| = 6$.

S_A contains all possible rearrangements of 1, 2 and 3.