Chapter 3: EQUIVALENCE RELATIONS

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LEARNING OUTCOMES FOR THE LECTURE

By the end of this lecture, students will be able to:

- ♣ Define an equivalence relation on a set A.
- ♣ Given a relation on a set A, check/show whether it is an equivalence relation.
- ♣
- ♣
- ♣

Definition (3.1.1 EQUIVALENCE RELATIONS)

Define a relation $\approx$ on a set A as an equivalence relation in A if $\forall a, b, c \in A$ we have

(i) $a \approx a \; \forall a \in A$. (*reflexive property*) "a is related to a"

(ii) $a \approx b \;\Rightarrow\; b \approx a$. (*symmetric property*) "if a is related to b then b is related to a"

(iii) $a \approx b \quad b \approx c \quad \Rightarrow a \approx c$. (*transitive property*) "a related to b abd b related to c implies a is related to c"

We say $a \approx b$ or "a equivalent to b."

Example: On $A = \mathbb{Z}$ define $a \approx b$ if $a - b$ is a multiple of 2.

"for any integers a and b, a is related to b if a-b=2t for some integer t"

In order to define an equivalence relation we need to

(i) define the set;

(ii) define the rule *(that satisfies the above three conditions (reflexive, symmetric and transitive properties))*

Definition (3.1.2 EQUIVALENCE CLASS)

Let A have $\approx$ as an equivalence relation. The equivalence class
$[a] = \bar{a} = \{x \in A | x \approx a\}$. "all x in A that are related to a"
$[a]$ is the equivalence class of A generated by a.

Example: On $A = \mathbb{Z}$ define $a \approx b$ iff $a - b = 2k$.
$[0] = \{0, \pm 2, \pm 4, \cdots\}$ "4 is related to 0 since 4-0=4=2t, t=2 integer"
$[1] = \{\pm 1, \pm 3, \cdots\}$ "3 is related to 1 since 3-1=2=2t, t=1"
NOTE: $[0] = [2] = [-2]$ and $[1] = [-1]$.

we say the class of 2 is equal to the class of 0 and is equal to the class of -2

<u>an euivalence class is a set</u>

Question: [1] ∩ [0]=?

[1] ∪ [0]=?

Do

Example (3.1.3 Examples)

- *Equality is an equivalence relation on any set A. Let* $a = b, \quad a, b \in \mathbb{Z}$. *Then* $[a] = \{a\}$. *Equivalence classes all singleton sets.*
- *Triangles in* $\mathbb{R}^2$, *congruence.*
- *Sets in* U, $A \approx B$ *iff* $|A| = |B|$.

can you show that these are equivalence relations?

♠ *We define* $\mathbb{Q} = \{\frac{p}{q} | q \neq 0, p, q \in \mathbb{Z}\}$. *On* $\mathbb{Q}$ *define* $\approx$ *as follows* $\frac{a}{b} \approx \frac{c}{d}$ *iff* $ad = bc$, *where* $b \neq 0, d \neq 0, a, b, c, d \in \mathbb{Z}$.
Show that $\approx$ *is an equivalence relation on* $\mathbb{Q}$.

(i) $\frac{a}{b} \approx \frac{a}{b}$ since $ab = ba, \forall a, b \in \mathbb{Z}$ ***(reflexive)***

(ii) $\frac{a}{b} \approx \frac{c}{d} \quad \Rightarrow \quad ad = bc \quad \Rightarrow \quad bc = ad \quad \Rightarrow \quad cb = da$

$\Rightarrow \quad \frac{c}{d} \approx \frac{a}{b}$. **(symmetric)**

(iii) $\frac{a}{b} \approx \frac{c}{d}; \quad \frac{c}{d} \approx \frac{e}{f} \quad \Rightarrow \quad ad = bc; \quad cf = de.$

$\Rightarrow \quad adf = bcf$ and $\quad bcf = bde$

$\Rightarrow \quad adf = bde; \Rightarrow \quad af = be$ since $d \neq 0$

$\Rightarrow \quad \frac{a}{b} \approx \frac{e}{f}$. ***(transitive)***

The equivalence classes are given by:

$$\begin{aligned}\left[\frac{a}{b}\right] &= \{\frac{x}{y} \in \mathbb{Q} | \frac{x}{y} \approx \frac{a}{b}\} \quad b \neq 0, y \neq 0. \\ &= \{\frac{x}{y} \in \mathbb{Q} | xb = ya\} \\ &= \{\frac{x}{y} \in \mathbb{Q} | \frac{x}{y} = \frac{a}{b} \text{ if } y \neq 0 \text{ and } b \neq 0\}\end{aligned}$$

e.g

$$\left[\frac{1}{2}\right] = \{\frac{-3}{-6}, \frac{-2}{-4}, \frac{-1}{-2}, \frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \cdots\}$$

Hence $\frac{24}{48} \approx \frac{1}{2}$ Equivalent not "really" equal.

"what is the equivalence class of 3?"