08/06/2018 14:00

Examinations and Graduation Office
Old Mutual Sports Hall

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

SCHOOL OF MATHEMATICS

# MULTIVARIABLE CALCULUS — MATH2021

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| STUDENT NO. | |
| ID/PASSPORT NO. | |
| SIGNATURE | |

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| Date | 8 June 2018 |
| Venue | |
| Row & Seat | |

**Internal examiner:** Prof. Y. Hardy (x76248)

**External examiner:** Prof. R. Brits

**Instructions to Candidates:**

- Complete the information above.
- Check that this paper has a cover page and **10 pages**.
- Please do not write in red ink.
- Work done in pencil or altered will not be remarked.
- Show all working, which must be legible.
- Approximate marks are indicated.
- No cell phones or other electronic devices are allowed – they may be confiscated and further action taken.
- If you need extra space to answer a question, write on the back of the page, and indicate clearly that you have done so.
- Time allowed for this exam is 2 hours.

| Markers only | |
|---|---|
| **Question** | **Mark** |
| 1 | / 25 |
| 2 | / 14 |
| 3 | / 17 |
| 4 | / 10 |
| 5 | / 9 |
| 6 | / 17 |
| 7 | / 10 |
| 8 | / 18 |
| **Total** | / 120 |

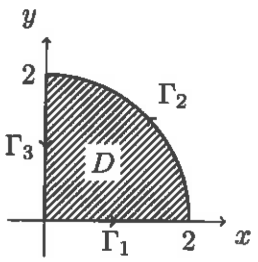


Figure 1: The region $D$, i.e. inside and on the circle of radius 2 with centre $(0,0)$ restricted to the first quadrant, for questions 1 and 2.

**Question 1** **[25 marks]**

(a) Let $f : D \to \mathbb{R}$ be given by $f(x,y) = 2x - \sqrt{4-y^2}$. Evaluate $\displaystyle\iint_D f(x,y)\,dx\,dy$ using Fubini's theorem. **(8)**

(b) Evaluate $\int_{\partial D} \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F}(x, y) = (0, x^2 - x\sqrt{4 - y^2})$ and $\partial D = \Gamma_1 + \Gamma_2 + \Gamma_3$. **(11)**

**Given:** $\int_{\Gamma_2} \mathbf{F} \cdot d\mathbf{r}_2 = \int_{\Gamma_3} \mathbf{F} \cdot d\mathbf{r}_3 = 0.$

(c) State Green's theorem. **(5)**

(d) Given that $\nabla \times \mathbf{F} = (0, 0, f(x, y))$, where $f(x, y) = 2x - \sqrt{4 - y^2}$, verify that Green's theorem holds by comparing your answer to (b) with your answer to (a). **(1)**

**Question 2** **[14 marks]**

Let $f : D \to \mathbb{R}$ be given by $f(x, y) = 2x - \sqrt{4 - y^2}$. Consider the polar form $(x, y) = (u \cos v, u \sin v)$.

(a) Find $D^*$ such that $\mathbf{T}(D^*) = D$, where $\mathbf{T}(u, v) = (u \cos v, u \sin v)$. **(4)**

(b) Express $\displaystyle\iint_D f(x, y)\, dx\, dy$ as a double integral over $D^*$. **(10)**

**Do not integrate. Leave your answer as a double integral.**

**Question 3** **[17 marks]**

(a) State and prove the Fundamental Theorem of Vector Calculus. **(12)**

(b) Determine whether $\mathbf{F}(x, y) = (2xy, x^2 + 1)$ a gradient vector field on $\mathbb{R}^2$. **(5)**

## Question 4 [10 marks]

(a) State the definition of the Scalar Surface Integral. **(5)**

(b) Prove or disprove: Let $S$ be a parametric surface in $\mathbb{R}^3$, defined on $D \subset \mathbb{R}^2$, with unit normal vector $\mathbf{n}(u,v)$ for $(u,v) \in D$. For every $f : S \to \mathbb{R}$, there exists an $\mathbf{F} : \mathbb{S} \to \mathbb{R}^3$ such that **(5)**

$$\iint_S f\, da = \iint_S \mathbf{F} \cdot d\mathbf{a}.$$

## Question 5

[9 marks]

Let $\mathbf{F}(x, y, z) = (0, 0, x)$ and let $S$ be the portion of the paraboloid $z = 1 - x^2 - y^2$ above and including the $x$-$y$ plane with normal pointing upwards (i.e. non-negative $z$-coordinate).

Find a parametrisation of the surface $S$ with normal pointing upwards.

**Question 6** **[17 marks]**

Let $\mathbf{F}(x, y, z) = (2x, z, y)^T$. Calculate $\iint_S \mathbf{F} \cdot d\mathbf{a}$ where $S = \left\{ \begin{pmatrix} uv \\ v \\ u \end{pmatrix} : u \in [0, 1],\ v \in [0, u] \right\}$.

**Question 7** **[10 marks]**

Let $\mathbf{F}(x, y, z) = (0, 0, x)$ and let $S$ be the portion of the paraboloid $z = 1 - x^2 - y^2$ above and including the $x$-$y$ plane with normal pointing upwards (i.e. non-negative $z$-coordinate).

(a) Show that the boundary of $S$ is the circle $x^2 + y^2 = 1$, $z = 0$. **(3)**

(b) Use Stokes' theorem and the parametrisation **(7)**

$$\partial S = \left\{ \begin{pmatrix} \cos t \\ \sin t \\ 0 \end{pmatrix} : 0 \leq t \leq 2\pi \right\}.$$

to calculate $\displaystyle\iint_S \mathbf{F} \cdot da$.

**Given:** $\nabla \times (0, \frac{1}{2}x^2, 0) = (0, 0, x)$ and $\displaystyle\int_0^{2\pi} \cos^3\theta \, d\theta = 0$.

**Question 8** **[18 marks]**

(a) Use Gauss' divergence theorem to calculate

$$\iint_S \mathbf{F} \cdot d\mathbf{a},$$

where $\mathbf{F}(x, y, z) = (0, 0, x)$ and $S$ is the surface bounding the volume above the $z = 0$ plane (positive $z$ coordinate) and below the paraboloid $z = 1 - x^2 - y^2$.

(b) Give the integration bounds for $x$ (in terms of $y$ and $z$), $y$ (in terms of $z$) and $z$ for the triple integral over the volume $B$ (bounded by $S$).

(c) Give a parametrisation of $S$, and state why the parametrisation is appropriate for Gauss' theorem.

(Extra space.)