Tutorial Solutions Ch1

Multivariable Calculus (University of the Witwatersrand, Johannesburg)

Chapter 1, Part 3: The Chain Rule

1.

$$
\begin{aligned}
A &= hg'(t+\delta) \\
B &= sf'(a+\epsilon) \\
C &= sf'(a+\epsilon) = hg'(t+\delta)f'(g(t)+t) \\
D &= g'(t)f'(g(t)).
\end{aligned}
$$

2. (a) $[f(\mathbf{G}(t))]' = f'(\mathbf{G}(t))\mathbf{G}'(t)$

   (b)

$$
[f(\mathbf{G}(t))]' = (2x_1, 2x_2, -1)\,|_{\mathbf{x}=\mathbf{G}(t)} \begin{pmatrix} G_1'(t) \\ G_2'(t) \\ G_3'(t) \end{pmatrix} = 2G_1(t)G_1'(t) + 2G_2(t)G_2'(t) - G_3'(t)
$$

   (c) As $\mathbf{G}(t) \in A\,\forall t$ we have $f(\mathbf{G}(t)) = k\,\forall t$ so $[f(\mathbf{G}(t))]' = 0$, giving $0 = \nabla f(\mathbf{G}(t)) \cdot \mathbf{G}'(t)$, Thus ∇f at $\mathbf{G}(t)$ is orthogonal to the curve given by $\mathbf{G}(t)$ at $\mathbf{G}(t)$.

3. (a) $\nabla f(x,y) = \begin{pmatrix} e^x \\ -2y \end{pmatrix}$

   (b) Since $f(\mathbf{r}(t)) = 0\,\forall\, t$ we have $[f(\mathbf{r}(t))]' = 0$ so $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0$, i.e.

$$
\begin{pmatrix} e^x \\ -2y \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix} = 0
$$

giving

$$\frac{\mathrm{d}x}{\mathrm{d}t}e^x - 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 0.$$

But, as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}},$$

we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{e^x}{2y}.$$

4. (a) $[f \circ \mathbf{G}]'(t) = f'(\mathbf{G}(t))\mathbf{G}'(t)$.

(b) Let $\mathbf{G}(t) = \begin{pmatrix} x(t) \\ y(t) \\ t \end{pmatrix}$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} f(x(t), y(t), t) = [f \circ \mathbf{G}]'(t),$$

so

$$\frac{\mathrm{d}f(x(t), y(t), t)}{\mathrm{d}t} = \nabla f(x, y, t) \cdot \begin{pmatrix} x(t) \\ y'(t) \\ 1 \end{pmatrix} = x'\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \frac{\partial f}{\partial t}.$$

(c) Since $f(x(t), y(t), t)$ is constant, $\frac{\mathrm{d}}{\mathrm{d}t} f(x(t), y(t), t) = 0$, giving

$$0 = x'\frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \frac{\partial f}{\partial t} = -e^{-t}\frac{\partial f}{\partial x} + 2t\frac{\partial f}{\partial y} + \frac{\partial f}{\partial t}.$$

But $x = e^{-t}$ and $y = t^2$ so

$$0 = -x\frac{\partial f}{\partial x} + 2\sqrt{y}\frac{\partial f}{\partial y} + \frac{\partial f}{\partial t}.$$

Another possible answer is

$$0 = -x\frac{\partial f}{\partial x} - 2\ln x\frac{\partial f}{\partial y} + \frac{\partial f}{\partial t}.$$

5. (a)

$$\nabla f = \begin{pmatrix} -2x\sin(x^2-3y) \\ \\ 3\sin(x^2-3y) \end{pmatrix} = \begin{pmatrix} -2x \\ \\ 3 \end{pmatrix}\sin(x^2-3y)$$

(b) $\nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) = 0$ so

$$\begin{pmatrix} x' \\ \\ y' \end{pmatrix} \cdot \begin{pmatrix} -2x \\ \\ 3 \end{pmatrix}\sin(x^2-3y) = 0,$$

i.e. $(3y' - 2xx')\sin(x^2-3y) = 0$ so either $x^2 = 3y$ or $\frac{y'}{x'} = \frac{2x}{3}$.

(c) Thus

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{3}.$$

6. (a)

$$\frac{\partial(F_i \circ \mathbf{G})}{\partial x_j} = \nabla F_i\big|_{\mathbf{G}(\mathbf{x})} \cdot \frac{\partial \mathbf{G}(\mathbf{x})}{\partial x_j} = \sum_{k=1}^{p} \frac{\partial F_i}{\partial y_k}\bigg|_{\mathbf{G}(\mathbf{x})} \frac{\partial G_k}{\partial x_j}\bigg|_{\mathbf{x}}$$

(b)

$$= \left(\frac{\partial F_i}{\partial y_1}\bigg|_{\mathbf{G}(\mathbf{x})}, \ldots, \frac{\partial F_i}{\partial y_p}\bigg|_{\mathbf{G}(\mathbf{x})} \right) \begin{pmatrix} \frac{\partial G_1}{\partial x_j}\Big|_{\mathbf{x}} \\ \\ \vdots \\ \\ \frac{\partial G_p}{\partial x_j}\Big|_{\mathbf{x}} \end{pmatrix}.$$

(c)

$$(\mathbf{F}\circ\mathbf{G})'(\mathbf{x}) = \begin{pmatrix} \frac{\partial(F_1\circ\mathbf{G})}{\partial x_1} & \cdots & \frac{\partial(F_1\circ\mathbf{G})}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial(F_m\circ\mathbf{G})}{\partial x_1} & \cdots & \frac{\partial(F_m\circ\mathbf{G})}{\partial x_n} \end{pmatrix}$$

(d)

$$\begin{aligned} &= \begin{pmatrix} \frac{\partial F_1}{\partial x_1}\Big|_{\mathbf{G}} & \cdots & \frac{\partial F_1}{\partial x_n}\Big|_{\mathbf{G}} \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial x_1}\Big|_{\mathbf{G}} & \cdots & \frac{\partial F_m}{\partial x_n}\Big|_{\mathbf{G}} \end{pmatrix} \begin{pmatrix} \frac{\partial G_1}{\partial x_1}\Big|_{\mathbf{x}} & \cdots & \frac{\partial G_1}{\partial x_n}\Big|_{\mathbf{x}} \\ \vdots & & \vdots \\ \frac{\partial G_p}{\partial x_1}\Big|_{\mathbf{x}} & \cdots & \frac{\partial G_p}{\partial x_n}\Big|_{\mathbf{x}} \end{pmatrix} \\ &= \mathbf{F}'(\mathbf{G}(\mathbf{x}))\mathbf{G}'(\mathbf{x}). \end{aligned}$$

7.

$$\mathbf{F}'(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}, \qquad \phi'(x,y) = (2x-6y, 2y-6x)$$

$$\begin{aligned} (\phi\circ\mathbf{F})'(r,\theta) &= \phi'(\mathbf{F}(r,\theta))\mathbf{F}'(r,\theta) \\ &= (2r\cos\theta - 6r\sin\theta, 2r\sin\theta - 6r\cos\theta)\begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \\ &= \Big(2r\cos^2\theta - 6r\cos\theta\sin\theta + 2r\sin^2\theta - 6r\cos\theta\sin\theta, \\ &\quad -2r^2\cos\theta\sin\theta + 6r^2\sin^2\theta + 2r^2\cos\theta\sin\theta - 6r^2\cos^2\theta\Big) \end{aligned}$$

etc.

8. (a) $[\mathbf{F}\circ\mathbf{G}]'(\mathbf{a})\mathbf{h} = \mathbf{F}'(\mathbf{G}(a))\mathbf{G}(\mathbf{a})\mathbf{h}$, so

$$\mathrm{d}(\mathbf{F}\circ\mathbf{G})[\mathbf{a};\mathbf{h}] = [\mathbf{F}\circ\mathbf{G}]'(\mathbf{a})\mathbf{h} = \mathbf{F}'(\mathbf{G}(\mathbf{a}))\mathbf{G}'(\mathbf{a})\mathbf{h}$$

$$\begin{aligned} &= \mathbf{F}'(\mathbf{G}(\mathbf{a}))\mathrm{d}\mathbf{G}[\mathbf{a};\mathbf{h}] \\ &= \mathrm{d}\mathbf{F}\Big[\mathbf{G}(\mathbf{a});\mathrm{d}\mathbf{G}[\mathbf{a};\mathbf{h}]\Big]. \end{aligned}$$

(b) Note that

$$\mathrm{d}\mathbf{F}[\mathbf{r}(t);\mathbf{r}'(t)] = \mathbf{F}'(\mathbf{r}(t))\mathbf{r}'(t) = [\mathbf{F}\circ\mathbf{r}]'(t).$$

Let $\mathbf{h}$ be tangent to the curve given by $\mathbf{r}(t)$, at the point $\mathbf{r}(t)$. Then $\mathbf{h} = k\mathbf{r}'(t)$ so $\mathrm{d}\mathbf{F}[\mathbf{r}(t);\mathbf{h}] = k[\mathbf{F}(\mathbf{r}(t))]'$. Thus $\mathrm{d}\mathbf{F}[\mathbf{r}(t);\mathbf{h}]$ is a scalar multiple of $[\mathbf{F}(\mathbf{r}(t))]'$, and so is tangent to the curve given by $\mathbf{F}\circ\mathbf{r}$ at the point $\mathbf{F}(\mathbf{r}(t))$. So the mapping $\mathbf{h}\mapsto \mathrm{d}\mathbf{F}[\mathbf{r}(t);\mathbf{h}]$ takes vectors tangent to the curve $\mathbf{r}(t)$ at $\mathbf{r}(t)$ to vectors tangent to the curve $\mathbf{F}\circ\mathbf{r}$ at $\mathbf{F}\circ\mathbf{r}(t)$.

9. Area(P) $= |\det[\mathbf{h}:\mathbf{k}]|$ (where $[\mathbf{h}:\mathbf{k}]$ is the matrix you get by gluing $\mathbf{h}$ and $\mathbf{k}$ together).

$$\begin{aligned} \text{Area(Q)} &= \Big|\det\Big[\mathrm{d}\mathbf{F}[\mathbf{a};\mathbf{h}] : \mathrm{d}\mathbf{F}[\mathbf{a};\mathbf{k}]\Big]\Big| \\ &= \Big|\det\Big[\mathbf{F}'(\mathbf{a})\mathbf{h} : \mathbf{F}'(\mathbf{a})\mathbf{k}\Big]\Big| \\ &= \Big|\det\left(\mathbf{F}'(\mathbf{a})[\mathbf{h}:\mathbf{k}]\right)\Big| \\ &= |\det \mathbf{F}'(\mathbf{a})|\,|\det[\mathbf{h}:\mathbf{k}]| \\ &= |\det \mathbf{F}'(\mathbf{a})|\,\text{Area(P)}. \end{aligned}$$

The analogous result for $\mathbb{R}^3$ follows similarly; just use three vectors, say $\mathbf{h}$, $\mathbf{k}$, and $\mathbf{u}$, etc.

10. (a) $[\mathbf{F}\circ\mathbf{G}]'(\mathbf{x}) = \mathbf{F}'(\mathbf{G}(\mathbf{x}))\mathbf{G}'(\mathbf{x})$.

(b) The identity matrix $I = (\mathbf{x})' = \mathbf{F}'(\mathbf{G}(\mathbf{x}))\mathbf{G}'(\mathbf{x})$, so $\mathbf{F}'(\mathbf{y})\mathbf{G}'(\mathbf{x}) = I$ giving

$$\det\Big(\mathbf{F}'(\mathbf{y})\Big)\det\Big(\mathbf{G}'(\mathbf{x})\Big) = 1$$

and thus

$$\frac{\partial \mathbf{F}}{\partial \mathbf{y}}\frac{\partial \mathbf{G}}{\partial \mathbf{x}} = 1.$$

(c)

$$\begin{aligned}
[\mathbf{F}\circ\mathbf{G}]'(\mathbf{x}) &= \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} & \frac{\partial F_1}{\partial y_3} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} & \frac{\partial F_2}{\partial y_3} \end{pmatrix} \begin{pmatrix} x_2 & x_1 \\ 2x_1 & -2x_2 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} x_2\frac{\partial F_1}{\partial y_1} + 2x_1\frac{\partial F_1}{\partial y_2} + \frac{\partial F_1}{\partial y_3} \\ x_1\frac{\partial F_2}{\partial y_1} - 2x_2\frac{\partial F_2}{\partial y_2} + \frac{\partial F_2}{\partial y_3} \end{pmatrix}
\end{aligned}$$

(d)

$$\begin{aligned}
[\mathbf{F}\circ\mathbf{G}]'(\mathbf{x}) &= \mathbf{F}'(\mathbf{G}(\mathbf{x}))\mathbf{G}'(\mathbf{x}) \\
&= \begin{pmatrix} G_2(\mathbf{x}) & G_1(\mathbf{x}) & 0 \\ 2 & 0 & 2G_3(\mathbf{x}) \end{pmatrix} \begin{pmatrix} \frac{\partial G_1}{\partial x_1} & \cdots & \frac{\partial G_1}{\partial x_4} \\ \vdots & & \vdots \\ \frac{\partial G_3}{\partial x_1} & \cdots & \frac{\partial G_3}{\partial x_4} \end{pmatrix},
\end{aligned}$$

but $\mathbf{F}\circ\mathbf{G}(\mathbf{x}) = \begin{pmatrix} G_1G_2 \\ 2G_1 + G_3^2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$, so

$$[\mathbf{F}\circ\mathbf{G}]'(\mathbf{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and thus, by looking at the $(1,1)$th entry of $[\mathbf{F} \circ \mathbf{G}]'(\mathbf{x})$ we see that

$$(G_2(\mathbf{x}),\, G_1(\mathbf{x}),\, 0) \begin{pmatrix} \frac{\partial G_1}{\partial x_1} \\ \\ \frac{\partial G_2}{\partial x_1} \\ \\ \frac{\partial G_3}{\partial x_1} \end{pmatrix} = 1$$

so $G_2 \frac{\partial G_1}{\partial x_1} + G_1 \frac{\partial G_2}{\partial x_1} = 1$.

The other identity has a typo in it. Sorry.