Chapter 5: IDENTITY, INVERSE & WELL DEFINED MAPPINGS

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LEARNING OUTCOMES FOR THE LECTURE

By the end of this lecture, students will be able to:

- define when a mapping is 1-1 or onto
- determine whether a given function is 1-1 or onto or both
- given two functions, determine whether their composite exists or no
- determine whether two given functions are numerically equivalent or no
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Properties of mappings, compositions and bijections

Definition (5.2.1 (1))

$\alpha : A \to B$ and $\beta : A \to B$ *are equal* iff
$\alpha(a) = \beta(a), \qquad \forall a \in A.$

if the domains are not equal then the functions cannot be compared

Definition (5.2.1 (2))

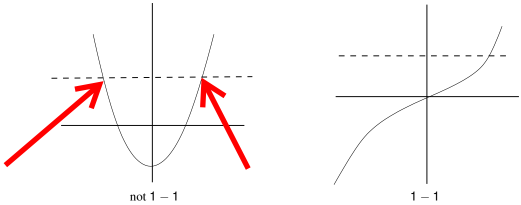
$\alpha : A \to B$, $A \subseteq B$ is called *an identity mapping* on A if
$\alpha(a) = a \quad \forall a \in A$ (denoted by $\alpha = 1_A$.)

the mapping fixes all elements in the domain

Definition (5.2.1 (3))

$\alpha : A \to B$ *is* *one to one,(1-1) or (injective)*, *if*
$\alpha(a_1) = \alpha(a_2) \Rightarrow \quad a_1 = a_2 \forall a_1, a_2 \in A$.

This is equivalent to the horizontal line test in the plane $\mathbb{R}^2$
to find out if a mapping is not 1-1 .
e.g.



α(s)=α(t) but s is not equal to t

Definition (5.2.1 (4))

$\alpha : A \to B$ is onto (surjective) if
$\forall b \in B \quad \exists a \in A \quad | \quad b = \alpha(a)$.
In this case Range α = $CoD(\alpha)$.

all elements in the codomain set have a corresponding element(s) in the domain set

Definition (5.2.1 (5))

If α is well defined, if α is *$1-1$ and onto, we say α is a bijection.*

Example (5.2.2 EQUALITY)

$\alpha : \mathbb{R} \to \mathbb{R}$ *such that* $\alpha(x) = x^2 + x + 1$. *and*
$\beta : \mathbb{R} \to \mathbb{R}$ *such that* $\beta(x) = (x-1)(x+2) + 3$.
Then $\alpha = \beta$ *since*

(i) $D(\alpha) = D(\beta) = \mathbb{R}$ *and*

same domain and same rule

(ii) $\alpha(x) = x^2 + x + 1$ *and*
$\beta(x) = (x-1)(x+2) + 3 = x^2 + x + 1$.
Thus $\alpha(x) = \beta(x) \quad \forall x \in D(\alpha) = D(\beta) = \mathbb{R}$.

Example (5.2.2 ONE TO ONE)

(i) $\alpha : \mathbb{N} \to \mathbb{N}; \qquad \alpha(n) = 2n + 1$ *is* $1 - 1$.

Let $m, n \in \mathbb{N} \quad \alpha(m) = \alpha(n)$

$\Rightarrow \quad 2m + 1 = 2n + 1$

$\Rightarrow \quad 2m = 2n$

$\Rightarrow \quad m = n.$

Therefore α *is* $1 - 1$.

(ii) $\alpha : \mathbb{R} \to \mathbb{R} \qquad \alpha(n) = n^2 + 1$ *and*
Let $m, n \in \mathbb{R} \quad \alpha(m) = \alpha(n)$
$\Rightarrow \quad m^2 + 1 = n^2 + 1$
$\Rightarrow \quad m^2 = n^2$
$\not\Rightarrow \quad m = n$
Since $(-m)^2 = (m)^2$ *but* $-m \neq m$. *Therefore* α *is not* $1 - 1$.
But $\alpha : \mathbb{N} \to \mathbb{N}; \qquad \alpha(n) = n^2 + 1$ *is* $1 - 1$.
Since $\quad m^2 = n^2 \quad \Rightarrow \quad m = n$ *in* $\mathbb{N}$.

(iii) $\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}; \quad \alpha(a, b) = ab$ *not* $1 - 1$ *(injective)*
since $\alpha(2, 3) = 6$ *and* $\alpha(1, 6) = 6$ *but* $(2, 3) \neq (1, 6)$.

Example (5.2.2 ONTO)

(i) $\alpha : \mathbb{R} \to \mathbb{R}; \quad \alpha(x) = 2x - 5$ *is onto since*
If $y = 2x - 5$ *then* $x = \frac{y+5}{2}$ *in* $\mathbb{R}\ \forall x \in \mathbb{R}$ for any y=2x-5 can we find (an) x value(s)?
Now $\alpha(\frac{y+5}{2}) = 2(\frac{y+5}{2}) - 5 = y$
Thus $\forall y \in \mathbb{R}, \quad \exists \quad \frac{y+5}{2} \in \mathbb{R} \quad | \quad \alpha(\frac{y+5}{2}) = y.$
But $\alpha : \mathbb{Z} \to \mathbb{Z}; \quad \alpha(x) = 2x - 5$ *is not onto since*
$\frac{y+5}{2} \notin \mathbb{Z} \qquad \forall y \in \mathbb{Z}$ *[CoD of* α*].*

(ii) $\alpha : \mathbb{N} \to \mathbb{N}; \quad \alpha(n) = 2n + 1.$
α *maps all natural numbers to odd numbers. Therefore*
α *not onto* codomain is the whole set of natural numbers but range=odd natural numbers.

when we make subject does the resultant expression represents an element in the domain?

Example (5.2.2 BIJECTION)

(i) $\alpha : A \to A$ *is a bijection if* α *is both* $1-1$ *and onto.*

(ii) $\alpha : \mathbb{R} \to \mathbb{R}; \quad \alpha(x) = 2x - 5$ *is a bijection since* α *is* $1-1$ *and onto.*

(iii) $\alpha : \mathbb{R} \to \mathbb{R}; \quad \alpha(x) = \sin x$ *is not a bijection since* α *is not* $1-1$ *since* $\sin x = \sin y \quad \not\Rightarrow x = y$ *e.g* $x = \dfrac{\pi}{6}; y = 2\pi + \dfrac{\pi}{6}$ *thus* $\sin\frac{\pi}{6} = \sin 2\pi + \frac{\pi}{6}$ *but* $\dfrac{\pi}{6} \neq 2\pi + \dfrac{\pi}{6}$. α *is not onto, Range* $\alpha = [-1, 1] \neq CoD(\alpha)$

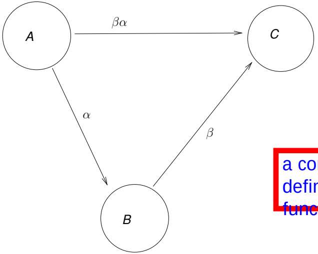
(iv) $\alpha : \mathbb{R} \to \mathbb{R}; \quad \alpha(x) = \tan x$ *is not a bijection since* α *is onto but* α *is not* $1-1$.

(v) $\alpha : \mathbb{R} \times \mathbb{R} \to \mathbb{R}; \quad \alpha(a, b) = ab$ *is not a bijection* α *is not injective* ($1-1$) *but* α *is onto* $\forall x \in \mathbb{R} \quad (1, x) \in \mathbb{R} \times \mathbb{R}$ *and* $\alpha(1, x) = 1.x = x$

do the horizontal line test on the graphs of the functions sin(x), tan(x) and cos(x) to confirm that the functions are not 1-1

Definition (5.2.1 (6))

If $\alpha : A \to B$ and $\beta : B \to C$ are well defined mappings, then $\beta\alpha : A \to C$ is a well defined mapping where
$\beta\alpha(a) = \beta(\alpha(a)) \quad : \alpha(a) \in B = D(\beta)$
$\beta\alpha$ is the composition of β and α and here $D(\beta) = CoD(\alpha)$.
$\beta\alpha$ start with α then β in that order.



a composite (if it exists) of well defined functions is a well defined function

Note that $\alpha\beta$ does not exist. $\beta : B \to C$ we can not apply α since $D(\alpha) = A, D(\alpha) \neq C, D(\alpha) \not\subseteq C$.

(i) If $\alpha : A \to A, \quad 1_A : A \to A$ then $\alpha 1_A = 1_A \alpha$. identity mapping
If $\alpha : A \to B$ and $1_B : B \to B$ then $\alpha 1_B$ does not exist.
$\alpha 1_A = \alpha$ and $1_B \alpha = \alpha$.

(ii) $\alpha : \mathbb{R} \to \mathbb{R} \quad ; \quad \alpha(x) = x + 1,$
$\beta : \mathbb{R} \to \mathbb{R} \quad ; \quad \beta(x) = x^2$
$\alpha\beta(x) = x^2 + 1$ and $\beta\alpha(x) = (x+1)^2$
$\alpha\beta \neq \beta\alpha$ composition of mappings is not commutative

Definition (5.2.1 (7))

If $\alpha : A \to B \quad ; \quad \beta : B \to C \quad ; \quad \delta : C \to D$
Composition of mappings is associative, $\delta(\beta\alpha) = (\delta\beta)\alpha$.

Definition (5.2.1 (8))

If $\alpha : A \to B$ *and* $\beta : B \to A$ *such that* $\alpha\beta = 1_B$ *and* $\beta\alpha = 1_A$. *Then* β *and* α *are inverse maps* $\beta = \alpha^{-1}$ *and* $\alpha = \beta^{-1}$. *If* β *exists, it is unique.*

Definition (5.2.1 (9))

If $\alpha : A \to B$ *is a bijection, then* A *and* B *are in one to one correspondence and* A *and* B *are said to be numerically equivalent. We have* $|A| = |B|$.