Zeros of nonlinear equations

Walter Mudzimbabwe

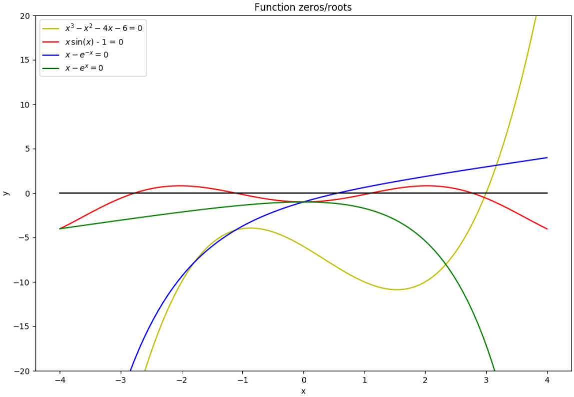
Zeros

We sometimes have to find points where a non-linear function is 0, i.e.

$$f(x) = 0.$$

Such points are called zeros or roots of $f(x) = 0$.
All numerical methods for finding roots are usually iterative.

Examples


Function zeros/roots
$x^3 - x^2 - 4x - 6 = 0$
$x\sin(x)$ - 1 = 0
$x - e^{-x} = 0$
$x - e^{x} = 0$
y
x
20
15
10
5
0
−5
−10
−15
−20
−4
−3
−2
−1
0
1
2
3
4

Methods

- Bracketing Methods: These successively reduce the interval $[a, b]$ that contains the root.
- Fixed Point Methods: They have form $x_{n+1} = g(x_n)$, ie., iterative.

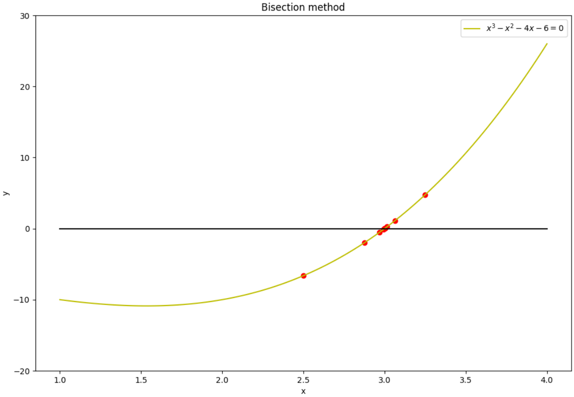
Bisection method

Suppose $f(a)f(b) < 0$ on $[a, b]$ then we know that a root of the equation lies in the interval $[a, b]$.
The mid point of $[a, b]$ is

$$c = \frac{(a+b)}{2}$$

- If $f(a)f(c) < 0$ then $f(a)$ and $f(c)$ have opposite signs and so the root must lie in [a, c].
- If $f(a)f(c) > 0$ then $f(a)$ and $f(c)$ have same signs and, so the root must lie in the interval $[c, b]$.

Pictorial representation



Bisection steps

1. $c = \frac{1}{2}(a+b)$.
2. If $f(a)f(c) < 0$ then $[a, b] = [a, c]$.
3. If $f(a)f(c) > 0$ then $[a, b] = [c, b]$.
4. Stop if root has been found otherwise go to 1.

Bisection method example

Problem: Perform two iterations of the bisection method on the function $f(x) = x^3 - x^2 - 4x - 6$ using $[1, 4]$.

Solution:

$f(1) = -10$ and $f(4) = 26$ also $f(1)$ and $f(4)$ have opposite signs so there is a root/zero in $[1, 4]$.

Iteration 1:

$[a, b] = [1, 4]$, $c = \frac{1}{2}(a + b) = 2.5$

$f(a)f(c) = f(1)f(2.5) > 0$ so $[a, b] = [c, b] = [2.5, 4]$.

Iteration 2:

$[a, b] = [2.5, 4]$, $c = \frac{1}{2}(a + b) = 3.25$

$f(a)f(c) = f(2.5)f(3.25) < 0$ so $[a, b] = [a, c] = [2.5, 3.25]$.

$\vdots$

Iteration 11:

$.....c = 3.0002$

so $[a, b] = [2.9995, 3.0010]$.

False position method or Regula Falsi

The bisection method is simple however it is slow. False position method gives a better way of finding c. The equation of the line through $(a, f(a))$ and $(b, f(b))$ is

$$y = f(a) + \frac{x-a}{b-a}(f(b) - f(a)).$$

False position method or Regula Falsi

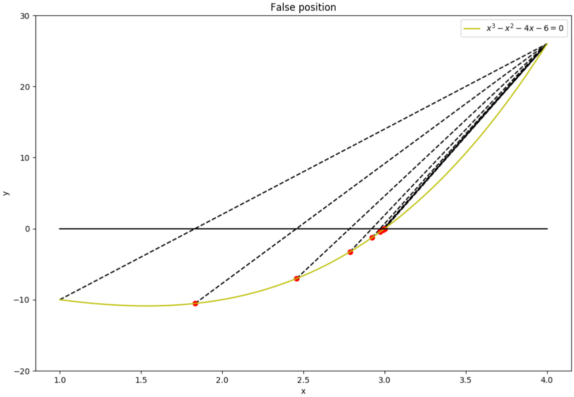
We require the point c where $y = 0$, i.e.

$$0 = f(a) + \frac{c-a}{b-a}(f(b) - f(a)),$$

from which we solve for c to get:

$$c = \frac{af(b) - bf(a)}{f(b) - f(a)}.$$

Pictorial representation



False position steps

1. $c = \frac{af(b) - bf(a)}{f(b) - f(a)}$.
2. If $f(a)f(c) < 0$ then $[a, b] = [a, c]$.
3. If $f(a)f(c) > 0$ then $[a, b] = [c, b]$.
4. Stop if root has been found otherwise go to 1.

We can use a stopping criteria such as $|c_n - c_{n-1}| < \epsilon$, where c_n is c at iteration n.

False position example

Problem: Perform two iterations of the False position method on the function $f(x) = x^3 - x^2 - 4x - 6$, using $[1, 4]$.

Solution:

$f(1) = -10$ and $f(4) = 26$ also $f(1)$ and $f(4)$ have opposite signs so there is a root/zero in $[1, 4]$.

Iteration 1:

$[a, b] = [1, 4]$, $c = \frac{af(b)-bf(a)}{f(b)-f(a)} = 1.8333, f(1.8333) = -10.53$

$f(a)f(c) = f(1)f(1.8333) > 0$ so $[a, b] = [c, b] = [1.8333, 4]$.

Iteration 2:

$[a, b] = [1.8333, 4]$, $c = \frac{af(b)-bf(a)}{f(b)-f(a)} = 2.4580,$

$f(2.4580) = -7.02$, $f(a)f(c) = f(1.8333)f(2.4580) > 0$ so

$[a, b] = [c, b] = [2.4580, 4]$.

$\vdots$

Iteration 8:

..... $c = 2.9996, ...$ $[a, b] = [2.9988, 4]$.

Fixed point methods

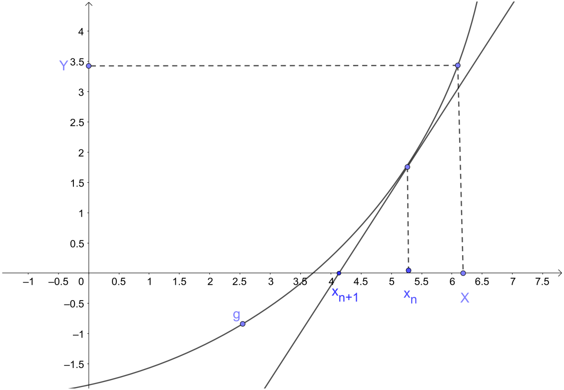
This simply involves a rewriting of the function $f(x) = 0$ into the form $x = g(x)$.
The fixed point method is then

$$x_{n+1} = g(x_n).$$

This rearrangement can often be done in several ways.

Example of fixed point method: Newton method

Problem: Find x such that $g(x) = 0$.



The derivative of $g(x)$ at x_n is $g'(x_n) = \dfrac{y - g(x_n)}{x - x_n}$.

Let $x = x_{n+1}$ when $y = 0$, i.e., x-intercept.

Newton formula

Therefore:

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}.$$

Newton method steps:

- Get initial guess x_0.
- Iterate $x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$ for $n = 0, 1, \ldots$

Example using Newton method

Example: Solve $e^{-x} = \ln(x)$ with $x_0 = 1$.

Solution:

$g(x) = \ln(x) - e^{-x}, g'(x) = \frac{1}{x} + e^{-x}$

Iteration 1:

$$n = 0,\ x_1 = x_0 - \frac{g(x_0)}{g'(x_0)} = x_0 - \frac{\ln(x_0) - e^{-x_0}}{\frac{1}{x_0} + e^{-x_0}} = 1.26894$$

Iteration 2:

$$n = 1,\ x_2 = x_1 - \frac{g(x_1)}{g'(x_1)} = x_1 - \frac{\ln(x_1) - e^{-x_1}}{\frac{1}{x_1} + e^{-x_1}} = 1.30911$$

Iteration 3:

$$n = 2,\ x_3 = x_2 - \frac{g(x_2)}{g'(x_2)} = x_2 - \frac{\ln(x_2) - e^{-x_2}}{\frac{1}{x_2} + e^{-x_2}} = 1.30980$$

Newton's Method for Systems of Nonlinear

Consider a system of two equations:

$$f(x, y) = 0, \quad g(x, y) = 0$$

Taylor's expansion of the two functions near (x, y):

$$f(x+h, y+k) = f(x, y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \dots \tag{1}$$

$$g(x+h, y+k) = g(x, y) + h\frac{\partial g}{\partial x} + k\frac{\partial g}{\partial y} + \dots \tag{2}$$

Newton's Method for Systems of Nonlinear

If we keep only the first order terms, we are looking for a couple (h, k) such that:

$$f(x+h, y+k) = 0 \approx f(x, y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \tag{3}$$

$$g(x+h, y+k) = 0 \approx g(x, y) + h\frac{\partial g}{\partial x} + k\frac{\partial g}{\partial y} \tag{4}$$

hence it is equivalent to the linear system:

$$\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = -\begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} \tag{5}$$

or

$$\begin{bmatrix} h \\ k \end{bmatrix} = -\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}^{-1} \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix} \tag{6}$$

Finally

$$\begin{bmatrix} x+h \\ y+k \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} h \\ k \end{bmatrix}$$

$$= \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}^{-1} \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} \quad (7)$$

in general,

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} x_n \\ y_n \end{bmatrix} - \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}^{-1}_{x=x_n, y=y_n} \begin{bmatrix} f(x_n,y_n) \\ g(x_n,y_n) \end{bmatrix} \quad (8)$$

i.e.,

$$\mathbf{x}_{n+1} = \mathbf{x}_n - J(\mathbf{x}_n)^{-1} f(\mathbf{x}_n)$$

Example Newton's method for Systems

Use Newton's method to find the root of

$$f(x,y) = x^3 - 3xy^2 - 1 = 0$$
$$g(x,y) = 3x^2y - y^3 = 0$$

with $(x_0, y_0) = (-0.6, 0.6)$

Solution:

$$J(x,y) = \begin{pmatrix} 3x^2 - 3y^2 & -6xy \\ 6xy & 3x^2 - 3y^2 \end{pmatrix}, \quad J(x_0,y_0) = \begin{pmatrix} 0 & 2.16 \\ -2.16 & 0 \end{pmatrix}$$

so

$$J^{-1}(x_0,y_0) = \begin{pmatrix} 0 & -0.463 \\ 0.463 & 0 \end{pmatrix}$$

Therefore $\mathbf{x}_1 = \mathbf{x}_0 - J(\mathbf{x}_0)^{-1} f(\mathbf{x}_0)$

$$= \begin{pmatrix} -0.6 \\ 0.6 \end{pmatrix} - \begin{pmatrix} 0 & -0.463 \\ 0.463 & 0 \end{pmatrix} \begin{pmatrix} -0.568 \\ 0.432 \end{pmatrix} = \begin{pmatrix} -0.4 \\ 0.863 \end{pmatrix}$$

Example Newton's method for Systems

$$J(x_1, y_1) = \begin{pmatrix} -1.754 & 2.071 \\ -2.071 & -1.754 \end{pmatrix}$$

so

$$J^{-1}(x_1, y_1) = \begin{pmatrix} -0.238 & -0.281 \\ 0.281 & -0.238 \end{pmatrix}$$

Therefore $\mathbf{x}_2 = \mathbf{x}_1 - J(\mathbf{x}_1)^{-1} f(\mathbf{x}_1)$

$$= \begin{pmatrix} -0.4 \\ 0.863 \end{pmatrix} - \begin{pmatrix} -0.238 & -0.281 \\ 0.281 & -0.238 \end{pmatrix} \begin{pmatrix} -0.170 \\ -0.228 \end{pmatrix} = \begin{pmatrix} -0.505 \\ 0.856 \end{pmatrix}$$