COMS 3003A
Solutions to HW 8

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Reading: Leary & Kristiansen, Chapter 1.

1. **Consider the standard model of arithmetic ω, i.e. the set of natural numbers with the usual relations and operations. We assume that the language of arithmetic has been extended, using definitions, with binary predicate letters $<$ and $\leqslant$ (see HW 7 for details). Determine if the following sentences are true or false in ω:**

   (a) $\forall x \exists y \, x < y$;

   True: Choose an arbitrary $n \in \mathbb{N}$ as the value for x; surely, we can find $m \in \mathbb{N}$, as the value for y, such that $n < m$ is true (for example, pick $m = n + 1$).

   (b) $\forall y \exists x \, x < y$;

   False: Assign 0 to y; we cannot find the value for x that makes $x < y$ true.

   (c) $\exists x \forall y \, x \leqslant y$;

   True: Choose an arbitrary $n \in \mathbb{N}$ as the value for x; surely, we can find $m \in \mathbb{N}$, as the value for y, such that $n < m$ is true (for example, pick $m = n + 1$).

   (d) $\exists y \forall x \, x + y = x$;

   True: Assign 0 to y; then, whichever value we pick for x, it is true that $x + y = x$.

   (e) $\exists x \forall y \, x + y = x$;

   False: Whichever value we pick for x, we can always find a valued for y that makes $x + y = x$ false; indeed, suppose we picked n for the value of x; then, picking 1 for y will make $x + y = x$ false.

(f) $\exists x \forall y \neg(S(y) = x)$;

True: Assign 0 to x; then, whichever value we pick for y, it is false that $S(y) = x$, and hence true that $\neg(S(y) = x)$.

(g) $\exists y \forall x \neg(S(y) = x)$.

False: Suppose we assign $n \in \mathbb{N}$ to y. Then, surely, there exists a number m such that $S(n) = m$, namely $m = n + 1$. Hence, it's not true that, for every value of x, it is true that $\neg(S(y) = x)$.

2. For each of the following formulas, find a model where the formula is true and a model where the formula is false:

(a) $\forall x\, R(x, x)$;

True: $M = (D, I)$, where $D = \{a\}$ and $I(R) = \{\langle a, a\rangle\}$.

False: $M = (D, I)$, where $D = \{a\}$ and $I(R) = \varnothing$.

(b) $\forall x \forall y\, (R(x, y) \to R(y, x))$;

True: $M = (D, I)$, where $D = \{a, b\}$ and $I(R) = \{\langle a, b\rangle, \langle b, a\rangle\}$.

False: $M = (D, I)$, where $D = \{a, b\}$ and $I(R) = \{\langle a, b\rangle\}$.

(c) $\forall x \forall y \forall z\, (R(x, y) \wedge R(y, z) \to R(x, z))$;

True: $M = (D, I)$, where $D = \{a, b\}$ and $I(R) = \{\langle a, b\rangle, \langle b, b\rangle\}$.

False: $M = (D, I)$, where $D = \{a, b\}$ and $I(R) = \{\langle a, b\rangle, \langle b, a\rangle\}$.

(d) $\forall x \forall y\, (R(x, y) \to \exists z\, (R(x, z) \wedge R(z, y)))$;

True: $M = (D, I)$, where $D = \{a, b\}$ and $I(R) = \{\langle a, a\rangle, \langle a, b\rangle\}$.

False: $M = (D, I)$, where $D = \{a, b\}$ and $I(R) = \{\langle a, b\rangle\}$.

(e) $\exists x\, P(x) \to \forall x\, P(x)$;

True: $M = (D, I)$, where $D = \{a\}$ and $I(P) = \{a\}$.

False: $M = (D, I)$, where $D = \{a, b\}$ and $I(P) = \{a\}$.

(f) $\forall x \exists y\, R(x, y)$;

True: $M = (D, I)$, where $D = \{a\}$ and $I(R) = \{\langle a, a\rangle\}$.

False: $M = (D, I)$, where $D = \{a\}$ and $I(R) = \varnothing$.

(g) $\exists x \forall y R(y, x)$;

True: $M = (D, I)$, where $D = \{a\}$ and $I(R) = \{\langle a, a\rangle\}$.
False: $M = (D, I)$, where $D = \{a, b\}$ and $I(R) = \{\langle a, b\rangle\}$.

(h) $\forall x \exists y R(x, y) \to \exists x \forall y R(y, x)$;

True: $M = (D, I)$, where $D = \{a\}$ and $I(R) = \{\langle a, a\rangle\}$.
False: $M = (D, I)$, where $D = \{a, b\}$ and $I(R) = \{\langle a, b\rangle\}$.

(i) $\forall x \forall y \forall z\, (R(x, y) \wedge R(y, z) \to R(x, z)) \wedge \forall x \exists y R(x, y) \wedge \neg \forall x\, R(x, x)$.

True: $M = (D, I)$, where $D = \mathbb{N}$ and $I(R) = \{\langle n, m\rangle : n < m\}$.
False: $M = (D, I)$, where $D = \{a\}$ and $I(R) = \{\langle a, a\rangle\}$.

3. **Find out, for each of the following formulas, whether it is valid, i.e. true in every model, or not. Prove your claim.**

(a) $\forall x\, P(x) \wedge \forall x\, Q(x) \to \forall x\, (P(x) \wedge Q(x))$;

This formula is valid. Indeed, suppose otherwise. Then, there exists a model $M = (D, I)$ and an assignment α in M such that $M \not\models \forall x\, P(x) \wedge \forall x\, Q(x) \to \forall x\, (P(x) \wedge Q(x))[\alpha]$. Then,

(1) $M \models \forall x\, P(x)[\alpha]$.
(2) $M \models \forall x\, Q(x)[\alpha]$.
(3) $M \not\models \forall x\, (P(x) \wedge Q(x))[\alpha]$.

By (3), there exists $\alpha' \stackrel{x}{=} \alpha$ such that

(4) $M \not\models P(x) \wedge Q(x)[\alpha']$,

and so either

(5a) $M \not\models P(x)[\alpha']$

or

(5b) $M \not\models Q(x)[\alpha']$.

But in either case, we obtain a contradiction: since $\alpha' \stackrel{x}{=} \alpha$, from (1) we obtain

(6a) $M \models P(x)[\alpha']$,

while from form (2) we obtain

(6b) $M \models Q(x)[\alpha']$.

(b) $\forall x\, (P(x) \vee Q(x)) \to \forall x\, P(x) \vee \forall x\, Q(x)$;

This formula is not valid. Consider the model $M = (D, I)$ where $D = \{a, b\}$, $I(P) = \{a\}$, and $I(Q) = \{b\}$. Let α be an arbitrary assignment in M. Then, $M \models \forall x\, (P(x) \vee Q(x))[\alpha]$, but $M \not\models \forall x\, P(x)[\alpha]$ and $M \not\models \forall x\, Q(x)[\alpha]$, and so $M \not\models \forall x\, P(x) \vee \forall x\, Q(x)[\alpha]$.

(c) $\exists x\, P(x) \wedge \exists x\, Q(x) \to \exists x\,(P(x) \wedge Q(x))$;

This formula is not valid. To see this, consider the model from the previous question.

(d) $\exists x\,(P(x) \vee Q(x)) \to \exists x\, P(x) \vee \exists x\, Q(x)$;

This formula is valid. Indeed, suppose otherwise. Then, there exists a model $M = (D, I)$ and an assignment α in M such that $M \not\models \exists x\,(P(x) \vee Q(x)) \to \exists x\, P(x) \vee \exists x\, Q(x)[\alpha]$. Then,

(1) $M \models \exists x\,(P(x) \vee Q(x))[\alpha]$.

(2) $M \not\models \exists x\, P(x)[\alpha]$.

(3) $M \not\models \exists x\, Q(x)[\alpha]$.

By (3), there exists $\alpha' \stackrel{x}{=} \alpha$ such that

(4) $M \models P(x) \vee Q(x)[\alpha']$,

and so either

(5a) $M \models P(x)[\alpha']$

or

(5b) $M \models Q(x)[\alpha']$.

But in either case, we obtain a contradiction: since $\alpha' \stackrel{x}{=} \alpha$, from (1) we obtain

(6a) $M \not\models P(x)[\alpha']$,

while from form (2) we obtain

(6b) $M \not\models Q(x)[\alpha']$.

(e) $\exists x \forall y\, R(y, x) \to \forall x \exists y\, R(x, y)$;

This formula is valid. Indeed, suppose otherwise. Then, there exists a model $M = (D, I)$ and an assignment α in M such that $M \not\models \exists x \forall y\, R(y, x) \to \forall x \exists y\, R(x, y)[\alpha]$. Then,

(1) $M \models \exists x \forall y\, R(y, x)[\alpha]$.

(2) $M \not\models \forall x \exists y\, R(x, y)[\alpha]$.

By (1), there exists $\beta \stackrel{x}{=} \alpha$ such that

(3) $M \models \forall y\, R(y, x)[\beta]$.

By (2), there exists $\gamma \stackrel{x}{=} \alpha$ such that

(4) $M \not\models \exists y\, R(x, y)[\gamma]$.

By (4), $M \not\models \exists y\, R(x, y)[\gamma']$, for every assignment $\gamma' \stackrel{y}{=} \gamma$. In particular, if we consider the assinment δ defined by $\delta(y) = \beta(x)$ and $\delta(z) = \gamma(z)$, for every $z \in \mathit{Var} - \{y\}$, then, since $\delta \stackrel{y}{=} \gamma$,

(5) $M \not\models R(x, y)[\delta]$, i.e., $\langle \delta(x), \delta(y) \rangle \notin I(R)$.

Since $\delta(y) = \beta(x)$, it follows from (5) that

(6) $\langle \delta(x), \beta(x) \rangle \notin I(R)$.

Now, consider the assignment β' defined by $\beta'(y) = \delta(x)$ and $\beta'(z) = \beta(z)$, for every $z \in Var - \{y\}$. Then, $\beta' \stackrel{y}{=} \beta$, and so, by (3),

(7) $M \models R(y, x)[\beta']$, i.e., $\langle \beta'(y), \beta'(x) \rangle \in I(R)$.

Since $\beta'(y) = \delta(x)$ and $\beta'(x) = \beta(x) = \delta(y)$, by (7),

(8) $\langle \delta(x), \delta(y) \rangle \in I(R)$,

in contradiction with (5).

(f) $\forall x\,(P(x) \to Q(x)) \to (\forall x\, P(x) \to \forall x\, Q(x))$.

This formula is valid. Indeed, suppose otherwise. Then, there exists a model $M = (D, I)$ and an assignment α in M such that $M \not\models \forall x\,(P(x) \to Q(x)) \to (\forall x\, P(x) \to \forall x\, Q(x))[\alpha]$. Then,

(1) $M \models \forall x\,(P(x) \to Q(x))[\alpha]$.

(2) $M \models \forall x\, P(x)[\alpha]$.

(3) $M \not\models \forall x\, Q(x)[\alpha]$.

By (3), there exists $\alpha' \stackrel{x}{=} \alpha$ such that

(4) $M \not\models Q(x)[\alpha']$,

Since $\alpha' \stackrel{x}{=} \alpha$, it follows, by (1), that

(5) $M \models P(x) \to Q(x)[\alpha']$, and so

either

(5a) $M \not\models P(x)[\alpha']$.

or

(5b) $M \models Q(x)[\alpha']$.

Note that (5b) contradicts (4). In addition, it is not hard to see that (5a) contradicts (1): since $\alpha' \stackrel{x}{=} \alpha$, (1) implies that

(6) $M \models P(x)[\alpha']$,

in contradiction with (1). Thus, in either case, we get a contradiction.

4. (a) **Write a sentence φ without $=$ that has the following properties:**
   - **φ is true in every model with a single individual;**
   - **for every $n \geqslant 2$, there exists a model with n individuals where φ is false.**

$$\exists x\, P(x) \rightarrow \forall x\, P(x).$$

(b) **Write a formula with $=$ that is true precisely in models with two individuals.**

$$\exists x \exists y\, \neg(x = y) \wedge \forall z\, (z = x \vee z = y).$$

(c) **Write a formula with $=$ that is true precisely in models with n individuals.**

$$\exists x_1 \ldots \exists x_n \bigwedge_{i \neq j} \neg(x_i = x_j) \wedge \forall y \bigvee_{1 \leqslant i \leqslant n} y = x_i.$$

(d) **Does there exist a formula without $=$ that is true precisely in models with two individuals?**

Such a formula does not exist. Suppose $M = (D, I)$ is a model and φ is a formula without $=$ such that $M \models \varphi$. Pick any $a \in D$ and define a model $M' = (D', I')$ by adding to M a clone of a, i.e., define $D' = D \cup \{a'\}$, and make a' behave in M' exactly as a behaves in M with respect to the interpretation of all non-logical symbols. Then, $M' \models \varphi$.

(e) **Write a formula without $=$ that is satisfiable only in models with infinite domains.**

$$\forall x \forall y \forall z\, (R(x, y) \wedge R(y, z) \rightarrow R(x, z)) \wedge \forall x \exists y R(x, y) \wedge \neg \forall x\, R(x, x).$$

(f) **Write a formula without $=$ that is true in only in models with a finite domain.**

The formulas from the previous question is true only in models with infinite domains. Therefore, its negation

$$\forall x \forall y \forall z\, (R(x, y) \wedge R(y, z) \rightarrow R(x, z)) \wedge \forall x \exists y R(x, y) \wedge \neg \forall x\, R(x, x)$$

can only be true in models with a finite domain.

5. **Let $\mathfrak{M}$ be a model, let α and β be assignments in $\mathfrak{M}$, and let φ be a sentence (i.e., a formula without free occurences of variables). Prove, by induction on φ, that $\mathfrak{M} \models \varphi[\alpha]$ if, and only if, $\mathfrak{M} \models \varphi[\beta]$.**