COMS 3003A
HW 6

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No new readings. This HW relies on understanding of Turing machines that you should have acquired from previous lectures and readings. Go back if you find these question hard—use the study break time for this.

We will now start describing TMs at a higher level, not wishing to be bogged down into the details of how to implement the machines we describe—this is what we expect you to do for this HW. Make sure, however, that, if pressed, you can translate your descriptions into detailed descriptions of TMs.

Mastering questions in this HW is essential to getting decent grades on subsequent tests and on the final exam.

1. Let M be a Turing machine.
   (a) Design a Turing machine M' that accepts exactly those string that M rejects and rejects exactly those strings that M accepts.
   (b) Under which conditions is M' going to be a decider?
2. Let Σ be an alphabet and let $L \subseteq \Sigma^*$. The *compliment* of L is the language
$$\bar{L} \;\; := \;\; \Sigma^* \setminus L = \{w \in \Sigma^* : w \notin L\}.$$
   (a) Prove that, if L is decidable, then $\bar{L}$ is decidable.
   (b) What can you say about $\bar{L}$ if L is undecidable?
3. Let Σ be an alphabet and let $L_1, L_2 \subseteq \Sigma^*$. Prove that, if L_1 and L_2 are decidable, then so are $L_1 \cap L_2$ and $L_1 \cup L_2$.
4. We want to give as input to a Turing machine with the binary input alphabet a binary encoding $\langle M \rangle$ of a Turing machine M and a binary word w. We, therefore, want to represent both as a single string $\langle M, w \rangle$ so that the machine that is given input $\langle M, w \rangle$ knows where $\langle M \rangle$ ends and w begins. How could we do this?

5. Assume that M_A is a Turing machine with the input alphabet $\{0,1\}$ that solves the following decision problem: given a binary encoding $\langle M\rangle$ of a Turing machine M and a binary word w, it decides if M accepts w, i.e.,

$$M_A(\langle M, w\rangle) = \begin{cases} 1 & \text{if } M(w) = 1; \\ 0 & \text{otherwise (what does this mean?).} \end{cases}$$

   Using M_A as a helper function, design a Turing machine that solves the following problem: given a binary encoding $\langle M\rangle$ of a Turing machine M, decide if M accepts its own encoding $\langle M\rangle$.

6. Assume that M_ϵ is a Turing machine with the input alphabet $\{0,1\}$ that solves the following decision problem: given a binary encoding $\langle M\rangle$ of a Turing machine M, it decides if M halts on the empty string ϵ, i.e.,

$$M_\epsilon(\langle M\rangle) = \begin{cases} 1 & M(\epsilon) \neq \infty; \\ 0 & \text{otherwise.} \end{cases}$$

   Using M_ϵ as a helper function, design a Turing machine M_H that solves the following problem: given a binary encoding $\langle M\rangle$ of a Turing machine M and a word w in its input alphabet, decide if M halts on w.

7. Assume that M_H is a Turing machine with the input alphabet $\{0,1\}$ that solves the following decision problem: given a binary encoding $\langle M\rangle$ of a Turing machine M and a word w in its input alphabet (which we may assume to be binary), it decides if M halts on w, i.e.,

$$M_H(\langle M, w\rangle) = \begin{cases} 1 & \text{if } M(w) \neq \infty; \\ 0 & \text{otherwise.} \end{cases}$$

   Using M_H as a helper function, design a Turing machine M_A that solves the following problem: given a binary encoding $\langle M\rangle$ of a Turing machine M and a word w in its input alphabet, decide if M accepts w.

8. Assume that $M_\sqcup$ is a Turing machine with the input alphabet $\{0,1\}$ that solves the following decision problem: given a binary encoding $\langle M\rangle$ of a Turing machine M, it decides if M writes a blank when running on it least one of its inputs, i.e.,

$$M_\sqcup(\langle M\rangle) = \begin{cases} 1 & \text{if there exists } w \text{ such that } M \text{ writes } \sqcup \text{ when running on } w; \\ 0 & \text{otherwise.} \end{cases}$$

   Using $M_\sqcup$ as a helper function, design a Turing machine M_H that solves the following problem: given a binary encoding $\langle M\rangle$ of a Turing machine M and a word w in its input alphabet, decide if M accepts w.

9. Assume that M_2 is a Turing machine with the input alphabet $\{0,1\}$ that solves the following decision problem: given a binary encoding $\langle M\rangle$ of a Turing machine M, it decides if M ever stays in the same state for two consecutive configurations, i.e.,

$$M_\sqcup(\langle M\rangle) = \begin{cases} 1 & M \text{ stays in the same state in } C_n \text{ and } C_{n+1}, \text{ for some } n \geqslant 0 \text{ and input } w; \\ 0 & \text{otherwise.} \end{cases}$$

   Using M_2 as a helper function, design a Turing machine M_H that solves the following problem: given a binary encoding $\langle M\rangle$ of a Turing machine M and a word w in its input alphabet, decide if M halts on w.