Section 4.1: Limits of Real Valued Functions (Part B)

In $x \to a$ the inequalities

$$0 < |x - a| < \delta$$

occur. For $x \to \infty$, this has to be replaced by $x > K$. Hence we have the following:

Definition 4.4

1. Let f be a real function defined on a set containing an interval of the form (c, ∞). Then '$f(x) \to L$ as $x \to \infty$' is defined to mean:
$$\forall \varepsilon > 0, \exists K > 0, (x > K \to |f(x) - L| < \varepsilon).$$
If $f(x) \to L$ as $x \to \infty$, then we write $\lim_{x\to\infty} f(x) = L$.

2. Let f be a real function defined on a set containing an interval of the form $(-\infty, c)$. Then '$f(x) \to L$ as $x \to -\infty$' is defined to mean:
$$\forall \varepsilon > 0, \exists K < 0, (x < K \to |f(x) - L| < \varepsilon).$$
If $f(x) \to L$ as $x \to -\infty$, then we write $\lim_{x\to-\infty} f(x) = L$.

Example 4.3

Let $f(x) = \frac{1}{x}$. Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$.

Writing out the sequences $(\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ and $-\frac{1}{1}, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, -\frac{1}{5}, \ldots)$, it appears that the limit might be zero. Therefore, let $\varepsilon > 0$ and put $K = \frac{1}{\varepsilon}$. Then $x > K$ gives

$$|f(x) - L| = \left|\frac{1}{x} - 0\right| = \frac{1}{x} < \frac{1}{K} = \varepsilon.$$

So $\lim_{x\to\infty} f(x) = 0$. Similarly, let $\varepsilon > 0$ and put $K = -\frac{1}{\varepsilon}$. Then $x < K$ gives

$$|f(x) - L| = \left|\frac{1}{x} - 0\right| = -\frac{1}{x} < -\frac{1}{K} = \varepsilon.$$

So $\lim_{x\to-\infty} f(x) = 0$.

For infinite limits we have similar definitions. ■

Definition 4.5

Let f be a real function whose domain includes a deleted neighborhood of the number a.

1. '$f(x) \to \infty$ as $x \to a$' is defined to mean:
$$\forall K > 0, \exists \delta > 0, (0 < |x - a| < \delta \to f(x) > K).$$
If $f(x) \to \infty$ as $x \to a$, then we write $\lim_{x\to a} f(x) = \infty$.

2. '$f(x) \to -\infty$ as $x \to a$' is defined to mean:

$$\forall K < 0, \exists \delta > 0, (0 < |x - a| < \delta \to f(x) < K).$$

If $f(x) \to -\infty$ as $x \to a$, then we write $\lim_{x \to a} f(x) = -\infty$.

Example 4.4

Prove that $\frac{2}{x^2} \to \infty$ as $x \to 0$.

We may choose K to be sufficiently large. So choose $K > 1$ and let $\delta = \frac{1}{K}$. Then, for $0 < |x| < \delta$, and because of $\delta < 1$ we have

$$f(x) = \frac{2}{x^2} = \frac{1}{|x|} \cdot \frac{2}{|x|} > \frac{1}{\delta} \cdot \frac{2}{\delta} = 2K^2 > K.$$

■

Definition 4.6

Let f be a real function defined on a set containing an interval of the form (c, ∞). Then '$f(x) \to \infty$ as $x \to \infty$' is defined to mean:

$$\forall A > 0, \exists K > 0, (x > K \to f(x) > A).$$

If $f(x) \to \infty$ as $x \to \infty$, then we write $\lim_{x \to \infty} f(x) = \infty$.

Similar definition hold with all other combinations of limits involving ∞ and $-\infty$, including one-sided limits. (Exercise: Write out all these limits.)

Example 4.5

Let $f(x) = \frac{1}{x}$. Find $\lim_{x \to 0^-} f(x)$, $\lim_{x \to 0^+} f(x)$ and $\lim_{x \to 0} f(x)$.

Let $K > 0$ and put $\delta = \frac{1}{K}$. Then, for $0 < x < \delta$,

$$f(x) = \frac{1}{x} > \frac{1}{\delta} = K.$$

Thus $\lim_{x \to 0^+} f(x) = \infty$. Now let $K < 0$ and put $\delta = -\frac{1}{K} > 0$. Then, for $-\delta < x < 0$,

$$f(x) = \frac{1}{x} < -\frac{1}{\delta} = K.$$

Thus $\lim_{x \to 0^-} f(x) = -\infty$. Since

$$\lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x),$$

$\lim_{x \to 0} f(x)$ does not exist. ■

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<u>Tutorial 4.1 – Part B</u>

1. Prove from the definitions that

   a. $\lim_{x \to -\infty} \frac{1-3x}{2x-1} = -\frac{3}{2}$.
   b. $\frac{1}{x-1} \to \infty$ as $x \to 1^+$.
   c. $\frac{1}{x-1} \to -\infty$ as $x \to 1^-$.

2. Prove that $f(x) = \frac{x}{10} + \frac{10 \sin x}{x} \to \infty$ as $x \to \infty$.

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