ANALYSIS OF ALGORITHMS

LECTURE 4 : COMPLEXITY PROOFS

PROVING SET MEMBERSHIP

- So let's say you did some analysis of what the complexity is using hierarchical decomposition, and you ended up with $f(n) = 3n^2 + 2n + 6$
- Now, you want to decide which complexity class that function is in
- A good guess could be that $f(n) \in \Theta(n^2)$
- But.. Can you prove it?

PROVING SET MEMBERSHIP – USING THE DEFINITION

- Is $f(n) \in \Theta(n^2)$
- Well, that would mean that $f(n) \in O(n^2)$ and $f(n) \in \Omega(n^2)$
- $f(n) \in O(n^2) \Leftrightarrow \exists c, n_0 \ni f(n) \leq cn^2 \forall n \geq n_0$
- So we want to find a c and an n_0 that makes $3n^2 + 2n + 6 \leq cn^2$
- Well, we know that n is positive, so we know that $n < n^2$
- So $3n^2 + 2n + 6 < 3n^2 + 2n^2 + 6n^2$
- So $3n^2 + 2n + 6 < 11n^2$
- So we can pick $c = 11$ and $n_0 = 1$

PROVING SET MEMBERSHIP – FROM THE DEFINITION

- Is $f(n) \in \Theta(n^2)$
- Well, that would mean that $f(n) \in O(n^2)$ and $f(n) \in \Omega(n^2)$

PROVING SET MEMBERSHIP – FROM THE DEFINITION

- f$(n) \in \Omega(n^2) \Leftrightarrow \exists c, n_0 \ni f(n) \geq cn^2 \; \forall n \geq n_0$
- So we want to find a c and an n_0 that makes $3n^2 + 2n + 6 \geq cn^2$
- Well, we know that n^2 is positive, so we know that $3n^2 > n^2$
- So $3n^2 + 2n + 6 \geq n^2$
- So we can pick $c = 1$ and $n_0 = 1$
- So $f(n) \in \Omega(n^2)$, and therefore $f(n) \in \Theta(n^2)$

PROVING SET MEMBERSHIP – USING LIMITS

- There is an alternative method which is often more sensible and easier to do
- This is the way you should do it, mostly because it's easier
- The key to this method is keeping in mind that what we care about is the growth of the time taken relative to the problem size
- $O(f(n))$
  - Upper bound – all function in this class grow no faster than $f(n)$
- $\Omega(f(n))$
  - Lower bound – all function in this class grow no slower than $f(n)$
- $\Theta(f(n))$
  - Exact bound – all functions in this class don't grow faster or slower than $f(n)$

PROVING SET MEMBERSHIP – USING LIMITS

- Consider the ratio $\frac{f(n)}{g(n)}$
- Now, if $f(n)$ grows faster than $g(n)$, what would we expect $\lim_{n \to \infty} \left(\frac{f(n)}{g(n)}\right)$ to be?
- Well, if the numerator grows faster than the denominator, you'd expect the limit to be infinite
- Similarly if the denominator grows faster than the numerator, you'd expect the limit to be 0
- So let's look at the original $f(n) = 3n^2 + 2n + 6$
- How do we prove that $f(n) \in \Theta(n^2)$

PROVING SET MEMBERSHIP – USING LIMITS

- Is $f(n) \in \Theta(n^2)$
- Well, that would mean that $f(n) \in O(n^2)$ and $f(n) \in \Omega(n^2)$

PROVING SET MEMBERSHIP – USING LIMITS

- Is $3\mathrm{n}^2 + 2\mathrm{n} + 6 \in O(n^2)$
- $O(n^2)$
  - Upper bound – all function in this class grow no faster than n^2
- Consider the ratio $\frac{3n^2+2n+6}{n^2}$
- Let's look at $\lim_{n\to\infty}\left(\frac{3n^2+2n+6}{n^2}\right)$
- = $\lim_{n\to\infty}\left(\frac{3n^2}{n^2}+\frac{2n}{n^2}+\frac{6}{n^2}\right)$
- = 3

PROVING SET MEMBERSHIP – USING LIMITS

- Is $3\mathrm{n}^2 + 2\mathrm{n} + 6 \in O(n^2)$
- $O(n^2)$
  - Upper bound – all function in this class grow no faster than n^2
- $\lim_{n\to\infty}\left(\frac{3n^2+2n+6}{n^2}\right) = 3$
- Now if $3n^2 + 2n + 6$ grew faster than n^2, you'd expect the limit to be infinity
- So $3n^2 + 2n + 6$ grows no faster than n^2
- Which means that $3\mathrm{n}^2 + 2\mathrm{n} + 6 \in O(n^2)$

PROVING SET MEMBERSHIP – USING LIMITS

- Is $3\mathrm{n}^2 + 2\mathrm{n} + 6 \in \Omega(n^2)$
- $\Omega(n^2)$
  - Lower bound – all function in this class grow no slower than n^2
- Now, $\lim_{n\to\infty}\left(\frac{3n^2+2n+6}{n^2}\right) = 3$
- If $3n^2 + 2n + 6$ grew slower than n^2, you'd expect the limit to be 0
- So $3n^2 + 2n + 6$ grows no slower than n^2
- Which means that $3n^2 + 2n + 6 \in \Omega(n^2)$
- And then $3n^2 + 2n + 6 \in \Theta(n^2)$

WHAT DO THESE SETS LOOK LIKE?

- Consider $O(n^2)$ and $O(n)$
- Is every function in $O(n)$ in $O(n^2)$?
- I mean it makes sense, right? But can we prove it?
- Let's try to prove $O(n) \subseteq O(n^2)$

PROOF

- Let's try to prove $O(n) \subseteq O(n^2)$
- Let $f(n) \in O(n)$
- $\Leftrightarrow \exists c, n_0 \ni f(n) \leq cn \ \forall n \geq n_0$
- But we know that n is positive, so $n \leq n^2$
- $\Leftrightarrow \exists c, n_0 \ni f(n) \leq cn \leq cn^2 \ \forall n \geq n_0$
- $f(n) \leq cn^2 \ \forall n \geq n_0 \Leftrightarrow f(n) \in O(n^2)$
- And $f(n)$ was any arbitrary element of $O(n)$
- $So \ O(n) \subseteq \ O(n^2)$

PROOF

- We proved $O(n) \subseteq O(n^2)$
- What would you do if you had to prove $O(n) \subset O(n^2)$?