Multivariable Calculus

MATH2007

1.5 Tangents and Normals

WITS UNIVERSITY

School of Mathematics

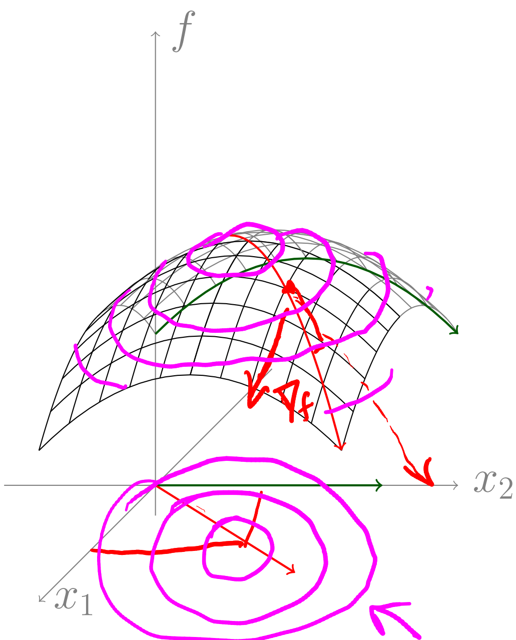
MC: Tangents and Normals

Revision: directional derivatives:

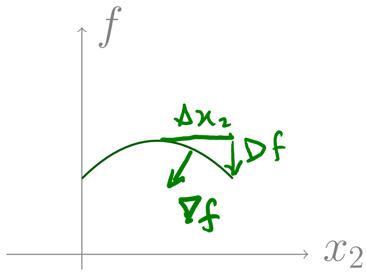
$$(x_1-1)^2 + (x_2-1)^2 = 4-c$$

$$f(x_1, x_2) = 4 - (x_1 - 1)^2 - (x_2 - 1)^2.$$

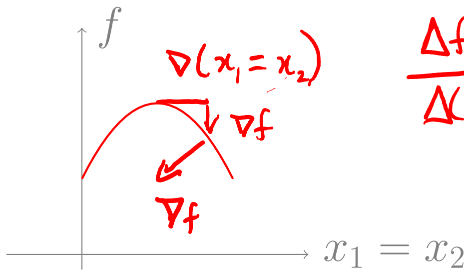
$$4-(x_1-1)^2 - (x_2-1)^2 = c$$



hypersurfaces



$$\frac{\Delta f}{\Delta x_2}$$



$$\frac{\Delta f}{\Delta(x_1=x_2)}$$

See the next page to see why the hypersurfaces are circles.

MC: Tangents and Normals

Definition (1.5.1)**.** Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ and $c \in \mathbb{R}$. A **hypersurface** in $\mathbb{R}^n$ is a set of the form

$$S = \{\underline{x} \in \mathbb{R}^n \mid \varphi(\underline{x}) = c\}.$$

A point $\underline{x} \in S$ is said to be a **regular point** of S if $\nabla\varphi(\underline{x})$ exists and is non zero, otherwise the point $\underline{x}$ is said to be **singular**.

- line: $y = mx + c \Rightarrow c = y - mx \quad \varphi(x,y) = y - mx$
- parabola: $y = ax^2 + b \Rightarrow -b = ax^2 - y \quad \varphi(x,y) = ax^2 - y \quad c = -b$
- circle: $(x-a)^2 + (y-b)^2 = r^2 \Rightarrow \varphi(x,y) = (x-a)^2 + (y-b)^2$
  $c = r^2$
- sphere: $(x-a)^2 + (y-b)^2 + (z-h)^2 = r^2$

  $\Rightarrow \varphi(x,y,z) = (x-a)^2 + (y-b)^2 + (z-h)^2 \quad c = r^2$

More "exotic" hypersurfaces

$n=1$ $\quad \varphi : \mathbb{R} \to \mathbb{R}$ $\quad \varphi(x) = x^2$ $\quad \varphi(x) = c \Rightarrow x^2 = c$

points $\quad S = \{1\}$ $\quad (c=1)$

$n=2$ $\quad \varphi : \mathbb{R}^2 \to \mathbb{R}$ $\quad \underline{\varphi(x,y)} = \begin{cases} 1 & (x,y) = (1,-1) \\ 0 & (x,y) \neq (1,-1) \end{cases}$

$$S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid \varphi(x,y) = 1 \right\} = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}. \quad \text{(point)}$$

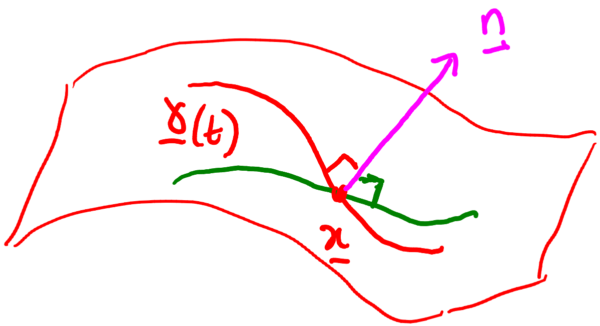
Note that $\varphi(x,y)$ (underlined) is not continuous!

$\underline{n} \neq \underline{0}$

Definition (1.5.2)**.** A vector $\underline{n}$ is said to be a **normal** to a hypersurface S at $\underline{x}$ if for each $\underline{\gamma} : \mathbb{R} \to S$ with $\underline{\gamma}(t_0) = \underline{x}$ for some $t_0 \in \mathbb{R}$ we have $\underline{n} \cdot \underline{\gamma}'(t_0) = 0$.

$S = \{ \underline{y} \in \mathbb{R}^n \mid \varphi(\underline{y}) = c \}$ $\qquad \varphi : \mathbb{R}^n \to \mathbb{R} \qquad c \in \mathbb{R}$

$\underline{\gamma} : \mathbb{R} \to S \subseteq \mathbb{R}^n$



Theorem (1.5.3). Let $\underline{x}_0$ be a regular point of S, where S is the hypersurface given by $S = \{\underline{x} \mid \varphi(\underline{x}) = c\}$. Then $\nabla\varphi(\underline{x}_0)$ is a normal to S at $\underline{x}_0$.

Proof. We know that $\nabla\varphi$ exists at $\underline{x}_0$ and $\nabla\varphi(\underline{x}_0) \neq \underline{0}$.

Let $\underline{\gamma} : \mathbb{R} \to S$ be arbitrary. such that $\underline{\gamma}(t_0) = \underline{x}_0$ for some $t_0 \in \mathbb{R}$.

Consider $\varphi(\underline{\gamma}(t)) = c$ (a constant), for all $t \in \mathbb{R}$

$$\frac{d}{dt}\varphi(\underline{\gamma}(t)) = 0$$

$\therefore$ $\nabla\varphi(\underline{\gamma}(t)) \cdot \underline{\gamma}'(t) = 0$ (chain rule).

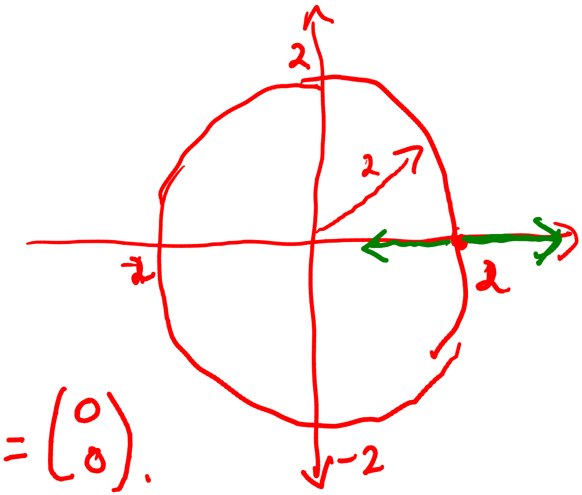
At $t = t_0$ $\nabla\varphi(\underline{\gamma}(t_0)) \cdot \underline{\gamma}'(t_0) = \nabla\varphi(\underline{x}_0) \cdot \underline{\gamma}'(t_0) = 0$.

This holds for all $\underline{\gamma} : \mathbb{R} \to S$ with $\underline{\gamma}(t_0) = \underline{x}_0$ (for some $t_0 \in \mathbb{R}$), and hence the proof is complete. $\square$

MC: Tangents and Normals

Example (a). Let $f(x, y) = x^2 + y^2$. Find a normal to the hypersurface $f(x, y) = 4$ at the point $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

$$S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, x^2 + y^2 = 4 \right\}$$



$$\nabla f(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \qquad \nabla f(2,0) = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

∇f exists at $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, and $\nabla f(2,0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$\therefore$ $\nabla f(2,0) = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$ is a normal to the hypersurface at $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

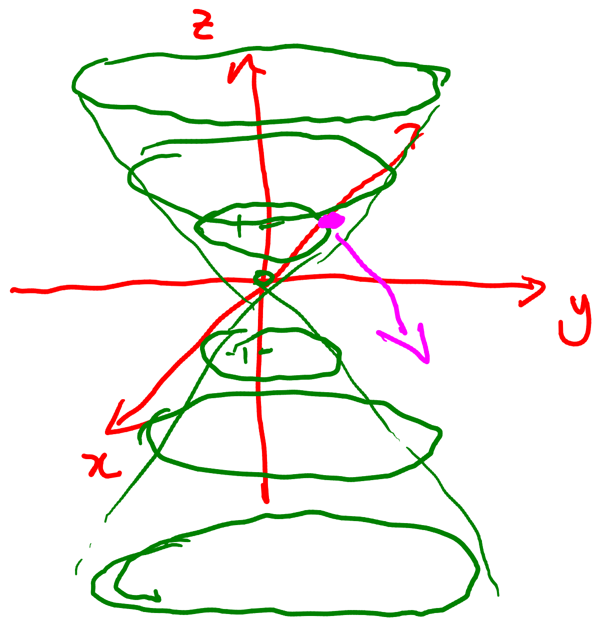
others include $\begin{pmatrix} 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \end{pmatrix} \dots$

$f(x,y,z)$

Example (b). Let $\cancel{f(x,}\, y, z) = x^2 + y^2 - z^2$. Find a normal to the hypersurface $f(x, y, z) = 0$ at the point $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x^2+y^2-z^2=0 \right\}$$

$$x^2+y^2 = z^2$$



$$\nabla f = \begin{pmatrix} 2x \\ 2y \\ -2z \end{pmatrix} \quad \text{exists at} \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\nabla f(0,1,1) = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is regular. A normal to S at $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ is $\nabla f(0,1,1) = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$.

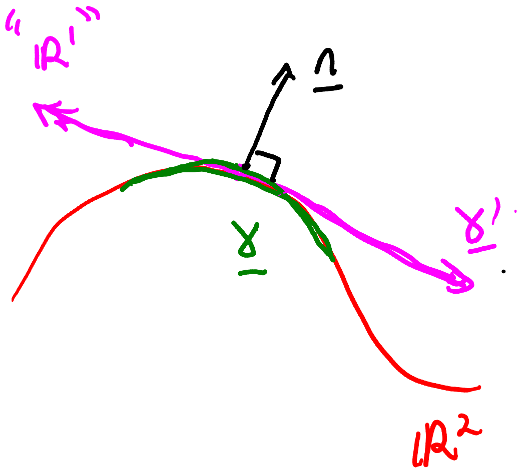
MC: Tangents and Normals

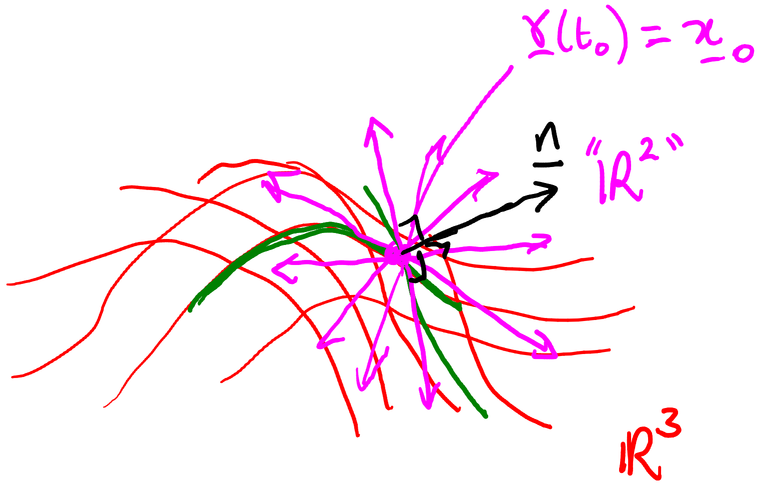
Definition (1.5.4)**.** We define the **set of tangent vectors** to the hypersurface S at $\underline{x}_0$ to be the set

$\underline{x}_0 \in S$

$$T_{\underline{x}_0} = \{\underline{\gamma}'(t_0) \mid \underline{\gamma} : \mathbb{R} \to S, \underline{\gamma}(t_0) = \underline{x}_0\}.$$

vector space with dimension $n-1$ when $S \subseteq \mathbb{R}^n$
(when $\underline{x}_0$ is a regular point)





$\nabla\varphi(\underline{x}_0) \neq \underline{0}$

Theorem (1.5.5)**.** If $\underline{x}_0$ is a regular point of $S = \{\underline{x} \,|\, \varphi(\underline{x}) = c\}$, then

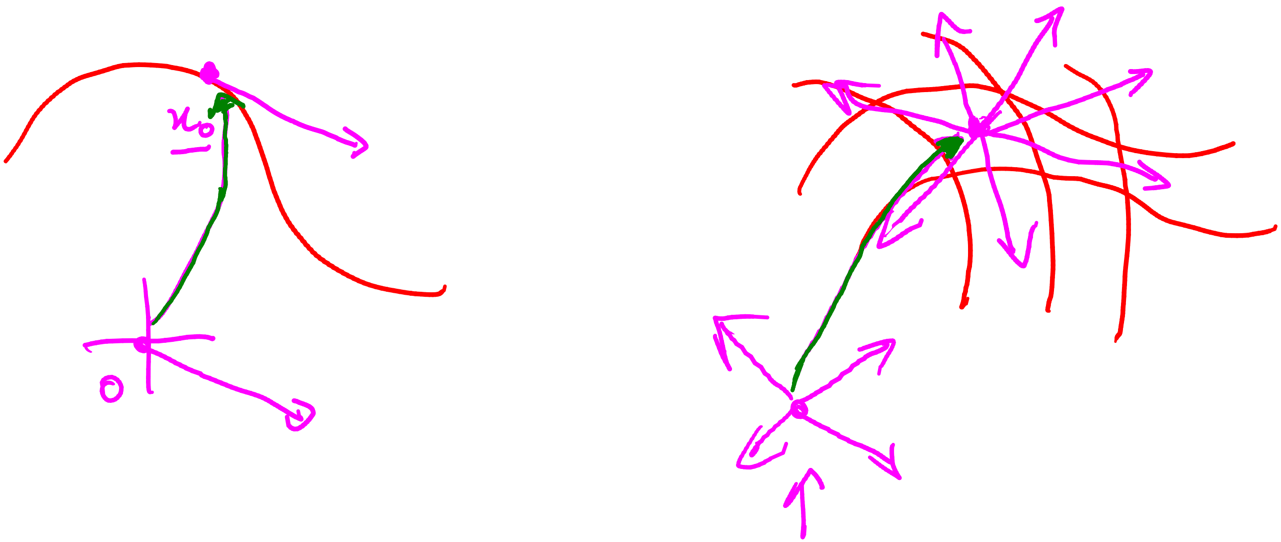
$$T_{\underline{x}_0} = \{\underline{v} \,|\, \underline{v} \cdot \nabla\varphi(\underline{x}_0) = 0\}.$$

orthogonal to the gradient of φ at $\underline{x}_0$.
vector
(equiv. orthog. to the normal at $\underline{x}_0$)

Proof: Omitted

Definition (1.5.6)**.** The **tangent (hyper)plane** to S at the regular point $\underline{x}_0$ of S is

$$\underline{x}_0 + T_{\underline{x}_0} = \{\underline{v} + \underline{x}_0 \,|\, \underline{v} \in T_{\underline{x}_0}\}.$$



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Example. Let $f(x,y) = y - x^2$ and $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix}^{\in \mathbb{R}^2} \mid f(x,y) = 0 \right\}$. Find $T_{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ and the tangent hyperplane to S at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

line

$y = x^2$

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is in S since $f(1,1) = 1 - 1^2 = 0$.

$\nabla f(x,y) = \begin{pmatrix} -2x \\ 1 \end{pmatrix}$

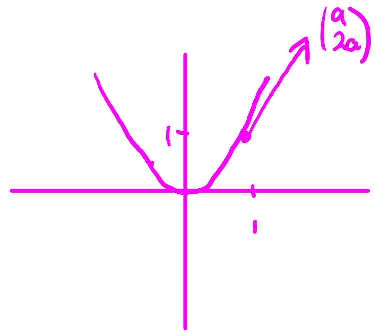
$\nabla f(1,1) = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

$$T_{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid \begin{pmatrix} a \\ b \end{pmatrix} \cdot \nabla f(1,1) = 0,\ a,b \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \end{pmatrix} = 0,\ a,b \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid -2a + b = 0,\ a,b \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ 2a \end{pmatrix} \mid a \in \mathbb{R} \right\}.$$



hyperplane to S at $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$: $\begin{pmatrix} 1 \\ 1 \end{pmatrix} + T_{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} = \left\{ \begin{pmatrix} a+1 \\ 2a+1 \end{pmatrix} \mid a \in \mathbb{R} \right\}$

Example. Let $f(x,y) = x^2 + y^2$ and $S = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \mid f(x,y) = 2 \right\}$. Find $T_{\begin{pmatrix} 1 \\ -1 \end{pmatrix}}$ and the tangent hyperplane to S at $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

line

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is in S since $f(1,-1) = 1^2 + (-1)^2 = 2$.

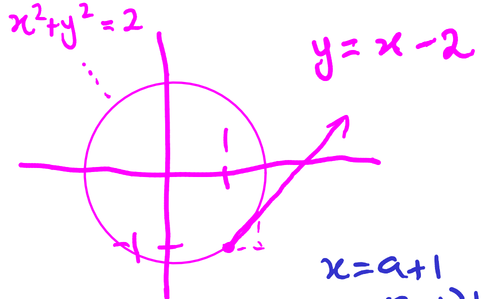
$\nabla f(x,y) = \begin{pmatrix} 2x \\ 2y \end{pmatrix} \qquad \nabla f(1,-1) = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$$T_{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid \begin{pmatrix} a \\ b \end{pmatrix} \cdot \nabla f(1,-1) = 0 \right\} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid \begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \end{pmatrix} = 0,\ a,b \in \mathbb{R} \right\}$$

$$= \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \mid 2a - 2b = 0,\ a,b \in \mathbb{R} \right\}$$

$a = b$

$$= \left\{ \begin{pmatrix} a \\ a \end{pmatrix} \mid a \in \mathbb{R} \right\}.$$



tangent hyperplane to S at $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is

$x = a+1$
$= (a-1)+2$
$= y+2$

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} + T_{\begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \left\{ \begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mid a \in \mathbb{R} \right\} = \left\{ \begin{pmatrix} \overbrace{a+1}^{x} \\ \underbrace{a-1}_{y} \end{pmatrix} \mid a \in \mathbb{R} \right\}$$

$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \ f(x,y,z)=0 \right\}$

Example. Let $f(x, y, z) = \underbrace{x^2 + y^2 - z^2}_{}$ ($\mathbb{R}^3$) and S given by $f(x, y, z) = 0$. Find $T_{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$ and the tangent hyperplane (plane) to S at $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

$x^2+y^2=z^2$

$f(0,1,1) = 0^2+1^2-1^2=0$

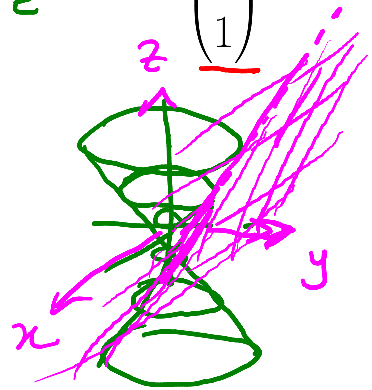
$\nabla f(x,y,z) = \begin{pmatrix} 2x \\ 2y \\ -2z \end{pmatrix} \qquad \nabla f(0,1,1) = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$

$T_{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \nabla f(0,1,1) = 0, \ a,b,c \in \mathbb{R} \right\}$

$= \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} = 2b-2c=0, \ a,b,c \in \mathbb{R} \right\}$

$b=c$

$= \left\{ \begin{pmatrix} a \\ b \\ b \end{pmatrix} \middle| \ a,b \in \mathbb{R} \right\}.$



$x=0 \Rightarrow y=z$

tangent plane to S at $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$: $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + T_{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} = \left\{ \begin{pmatrix} a \\ b+1 \\ b+1 \end{pmatrix} \middle| \ a,b \in \mathbb{R} \right\}$