Question 1 Linear Algebra [30 Marks]

1. Consider the set $\mathcal{S} = \mathbb{R}\backslash\{-8\}$ with the operator $\circ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that: [5]

$$a \circ b = 2ab + 4a + 2b \text{ with } a, b \in \mathbb{R}\backslash\{-8\}$$

State the properties of a group and prove that $(\mathcal{S}, \circ)$ is not a group.

2. Consider the set $\mathcal{S}$ of 3×3 matrices: [5]

$$\mathcal{S} = \{ \begin{bmatrix} x & 1 & 1 \\ 0 & y & 1 \\ 0 & 0 & z \end{bmatrix} \in \mathbb{R}^{3\times3} | x, y, z \in \mathbb{R}\}$$

Say with reasons whether each of the five properties of an Abelian group hold for the set and operator: $(\mathcal{G}, +)$ where $+$ denotes element-wise addition (i.e. you do not need to formally prove the properties, merely give justification for why the property holds or does not).

3. A company has 4 products. Each product uses a combination of the same 3 resources. Product x_1 uses $(1, 0, 0)$ of each resource respectively. Likewise the resources used for x_2, x_3 and x_4 are $(0, 1, 0)$, $(-1, 6, 2)$ and $(1, -3, 1)$ respectively. In total the company has $(2, 8, 10)$ of each resource. Find the general solution for the number of products produced $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ which ensures the company uses all of the available resources. [6]

4. Consider two subspaces of $\mathbb{R}^4$.

$$U_1 = span[\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}] = span\left[\begin{bmatrix}1\\0\\-3\\0\end{bmatrix}, \begin{bmatrix}1\\2\\-3\\2\end{bmatrix}, \begin{bmatrix}0\\2\\0\\2\end{bmatrix}\right];$$

$$U_2 = span[\mathbf{v_4}, \mathbf{v_5}, \mathbf{v_6}, \mathbf{v_7}] = span\left[\begin{bmatrix}-1\\-2\\0\\0\end{bmatrix}, \begin{bmatrix}6\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\2\\1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right]$$

(a) State the two properties of a basis for a vector space. [1]

(b) Determine a basis of U_1. [3]

(c) Determine a basis of U_2. [3]

(d) Determine a basis for the union $U_1 \cup U_2$. [2]

(e) Determine a basis for the intersection $U_1 \cap U_2$. [5]

Question 2 Analytic Geometry [30 Marks]

1. Compute the distance between

$$x = \begin{bmatrix}2\\1\\2\end{bmatrix}, \; y = \begin{bmatrix}-2\\-2\\5\end{bmatrix}$$

using:

(a)

$$< \mathbf{x}, \mathbf{y} >= x^T y$$

[2]

(b)

$$< \mathbf{x}, \mathbf{y} >= x^T \begin{bmatrix}2 & 0 & 0\\0 & 2 & -1\\0 & -1 & 3\end{bmatrix} y$$

[3]

2. Compute the angle between

$$x = \begin{bmatrix}2\\1\end{bmatrix}, \; y = \begin{bmatrix}-2\\-2\end{bmatrix}$$

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using:

(a)

$$<\mathbf{x}, \mathbf{y}>= x^T y$$

[2]

(b)

$$<\mathbf{x}, \mathbf{y}>= x^T \begin{bmatrix}2 & 3\\3 & 1\end{bmatrix} y$$

[3]

3. Using the Gram-Schmidt method, turn the basis $B = (b_1, b_2, b_3)$ of a three-dimensional

(b)

$$<\mathbf{x}, \mathbf{y}>= x^T \begin{bmatrix}2 & 3\\3 & 1\end{bmatrix} y$$

[3]

3. Using the Gram-Schmidt method, turn the basis $B = (b_1, b_2, b_3)$ of a three-dimensional
subspace $U \subseteq \mathbb{R}^3$ into an Ortho-normal Basis $C = (c_1, c_2, c_3)$ of U, where [10]

$$b_1 = \begin{bmatrix}2\\0\\-1\end{bmatrix},\ b_2 = \begin{bmatrix}3\\2\\3\end{bmatrix},\ b_3 = \begin{bmatrix}0\\1\\4\end{bmatrix}$$

using the inner product:

$$<\mathbf{x}, \mathbf{y}>= x^T \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} y$$

4 Let V be a vector space and π an endomorphism of V. Prove that π is a projection if and only if $id_V - \pi$ is a projection, where id_V is the identity endomorphism on V. [6]

5. You are given a set of data $X = \{(x_i, y_i) : i = 1...n\}$. On average when $x_i = 0, y_i \neq 0$ (in other words the data does not pass through the origin).

   (a) Does this data exist on a vector or affine space (use the most specific definition possible according to the textbook definitions)? Justify your answer. [2]

   (b) Assume there is a linear relationship between x_i and y_i. Based on your above answer would you include a bias parameter in a linear regression model? Justify your answer. [2]