Question 1 - The Simple Pendulum

First we construct our system as seen in the figure below:

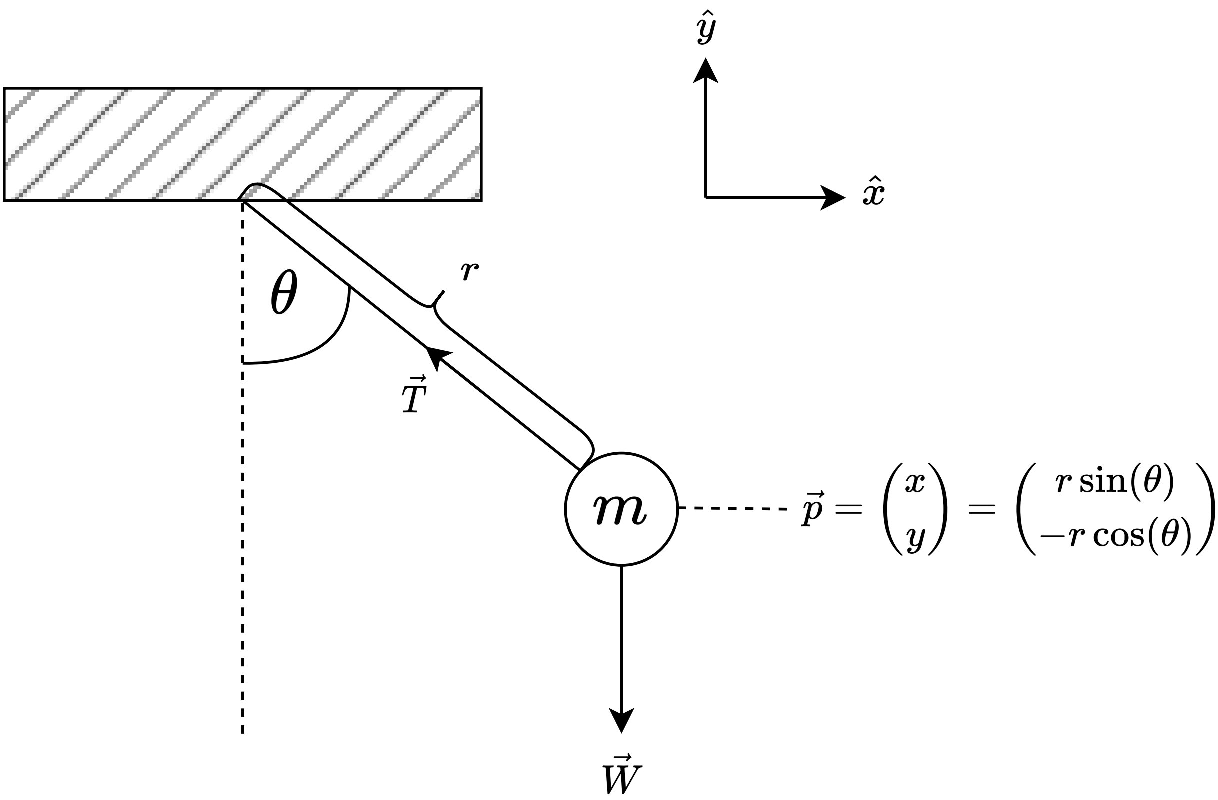

Figure 1: A simple pendulum

1.1 Using the above figure we find that the position of the bob is defined in polar coordinates as:

$$\vec{p} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\sin(\theta) \\ -r\cos(\theta) \end{pmatrix}$$

We know the equation of the potential energy is defined as

$$V = mgh$$

In our case h is defined as

$$h = -r\cos(\theta)$$

Thus we find that

$$V = \underrightarrow{-mgr\cos(\theta)}$$

1.2 We know that the Kinetic energy is defined as

$$T = \frac{1}{2}mv^2$$

In our system velocity is defined as:

$$\begin{aligned}
v^2 &= \dot{\vec{p}} \cdot \dot{\vec{p}} \\
&= \begin{pmatrix} r\cos(\theta)\dot{\theta} \\ r\sin(\theta)\dot{\theta} \end{pmatrix} \cdot \begin{pmatrix} r\cos(\theta)\dot{\theta} \\ r\sin(\theta)\dot{\theta} \end{pmatrix} \\
&= r^2\cos^2(\theta)\theta^2 + r^2\sin^2(\theta)\dot{\theta}^2 \\
&= r^2\dot{\theta}^2(\cos^2(\theta) + \sin^2(\theta)) \\
&= r^2\dot{\theta}^2
\end{aligned}$$

Thus we find that:

$$T = \underrightarrow{\frac{1}{2}m(r\dot{\theta})^2}$$

1.3

$$\begin{aligned}
\mathscr{L} &= T - V \\
&= \frac{1}{2}m(r\dot{\theta})^2 - (-mgr\cos(\theta)) \\
&= \frac{1}{2}mr^2\dot{\theta}^2 + mgr\cos(\theta)
\end{aligned}$$

1.4 Substituting our lagrangian in the euler-lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathscr{L}}{\partial \theta} = 0$$

Computing each compenent individually

$$\frac{\partial \mathscr{L}}{\partial \dot{\theta}} = mr^2\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta}$$

$$\frac{\partial \mathscr{L}}{\partial \theta} = -mgr\sin(\theta)$$

Thus our euler-lagrange equation becomes

$$\begin{aligned}
0 &= mr^2\ddot{\theta} - (-mgr\sin(\theta)) \\
mr^2\ddot{\theta} &= -(mgr\sin(\theta)) \\
\ddot{\theta} &= \frac{-mgr\sin(\theta)}{mr^2} \\
\ddot{\theta} &= \frac{-g}{r}\sin(\theta) \\
0 &= \ddot{\theta} + \frac{g}{r}\sin(\theta)
\end{aligned}$$

Question 2 is on the next page

Question 2 - The Double Pendulum

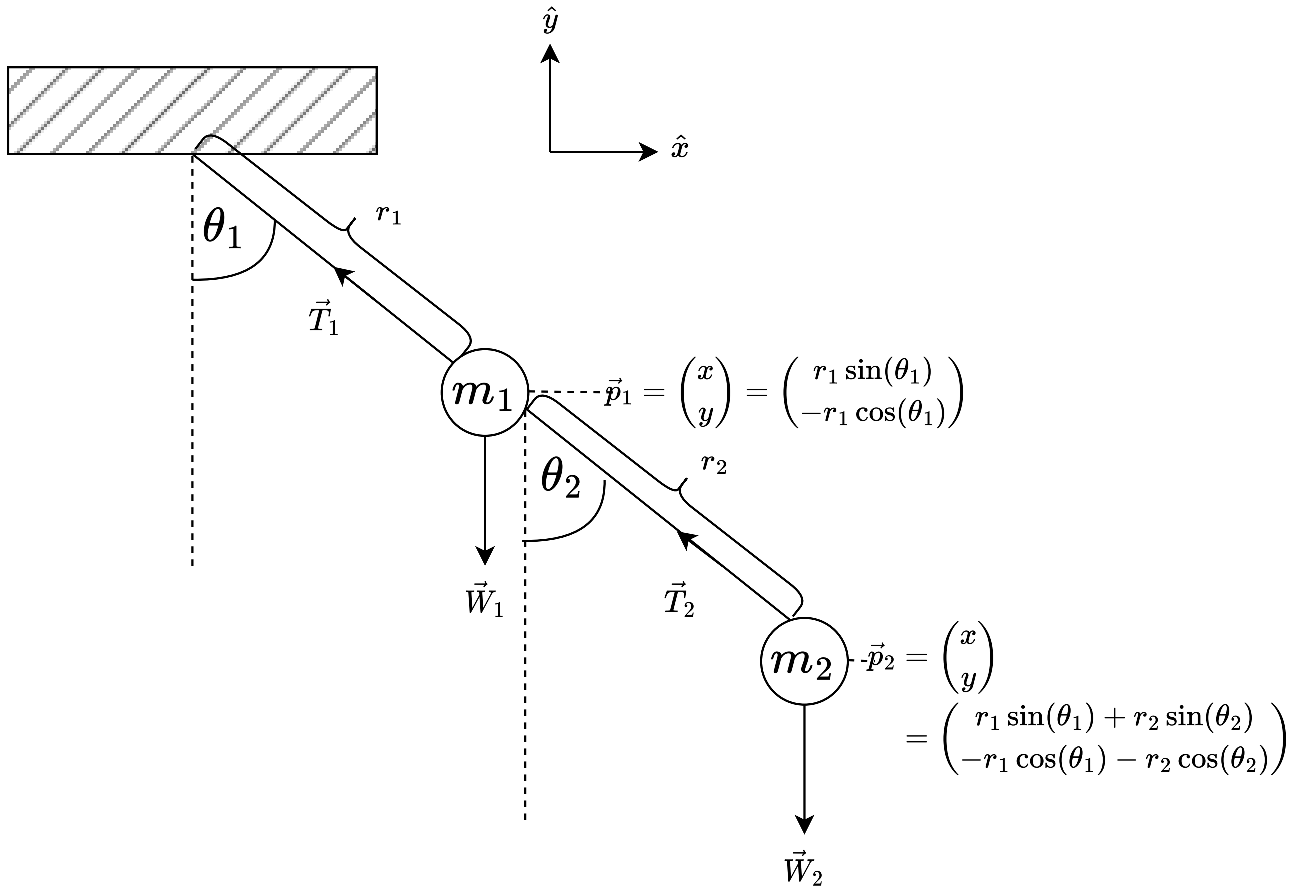

Figure 2: A simple pendulum

2.1 Recall that potential energy is defined as

$$V = mgh$$

Looking at each bob's potential energy, for our system we find that

$$\begin{aligned} V_1 &= -m_1 g r_1 \cos(\theta_1) \\ V_2 &= -m_2 g \left(r_1 \cos(\theta_1) + r_2 \cos(\theta_2)\right) \end{aligned}$$

Thus for our system we find that

$$\begin{aligned} V_{\text{system}} &= V_1 + V_2 \\ &= -m_1 g r_1 \cos(\theta_1) - m_2 g \left(r_1 \cos(\theta_1) + r_2 \cos(\theta_2)\right) \\ &= -m_1 g r_1 \cos(\theta_1) - m_2 g r_1 \cos(\theta_1) - m_2 g r_2 \cos(\theta_2) \\ &= -g \left((m_1 + m_2) r_1 \cos(\theta_1) + m_2 r_2 \cos(\theta_2)\right) \end{aligned}$$

2.2 Recall that the kinetic energy is defined as

$$T = \frac{1}{2}mv^2$$

Looking at each bob's velocity, for our system we find that for v_1

$$\begin{aligned}
v_1^2 &= \dot{\vec{p}}_1 \cdot \dot{\vec{p}}_1 \\
&= \begin{pmatrix} r\cos(\theta_1)\dot{\theta}_1 \\ r\sin(\theta_1)\dot{\theta}_1 \end{pmatrix} \cdot \begin{pmatrix} r\cos(\theta_1)\dot{\theta}_1 \\ r\sin(\theta_1)\dot{\theta}_1 \end{pmatrix} \\
&= r_1^2\cos^2(\theta_1)\dot{\theta_1}^2 + r_1^2\sin^2(\theta_1)\dot{\theta}_1^2 \\
&= r_1^2\dot{\theta_1}^2(\cos^2(\theta_1) + \sin^2(\theta_1)) \\
&= r_1^2\dot{\theta}_1^2
\end{aligned}$$

Thus

$$T_1 = \frac{1}{2}m_1 r_1^2\dot{\theta}_1^2$$

And for v_2 we have

$$\begin{aligned}
v_2^2 &= \dot{\vec{p}}_2 \cdot \dot{\vec{p}}_2 \\
&= \begin{pmatrix} r_1\cos(\theta_1)\dot{\theta}_1 + r_2\cos(\theta_2)\dot{\theta}_2 \\ r_1\sin(\theta_1)\dot{\theta}_1 + r_2\sin(\theta_2)\dot{\theta}_2 \end{pmatrix} \cdot \begin{pmatrix} r_1\cos(\theta_1)\dot{\theta}_1 + r_2\cos(\theta_2)\dot{\theta}_2 \\ r_1\sin(\theta_1)\dot{\theta}_1 + r_2\sin(\theta_2)\dot{\theta}_2 \end{pmatrix} \\
&= r_1\cos^2(\theta_1)\dot{\theta_1}^2 + 2r_1r_2\cos(\theta_1)\cos(\theta_2)\dot{\theta}_1\dot{\theta}_2 + r_2^2\sin^2(\theta_2)\dot{\theta}_2^2 \\
&\quad + r_1\sin^2(\theta_1)\dot{\theta_1}^2 + 2r_1r_2\sin(\theta_1)\sin(\theta_2)\dot{\theta}_1\dot{\theta}_2 + r_2^2\sin^2(\theta_2)\dot{\theta}_2^2
\end{aligned}$$

Grouping like terms and taking out a common factor we have that

$$\begin{aligned}
&= r_1^2\dot{\theta}_1^2\left(\cos^2(\theta_1) + \sin^2(\theta_1)\right) + 2r_1r_2\dot{\theta}_1\dot{\theta}_2\left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2)\right) \\
&\quad + r_2^2\dot{\theta}_2^2\left(\cos^2(\theta_2) + \sin^2(\theta_2)\right)
\end{aligned}$$

Using the trigonometric identities that

$$\cos^2(A) + \sin^2(A) = 1$$

and

$$\cos(A)\cos(B) + \sin(A)\sin(B) = \cos(A - B)$$

we find that

$$v_2 = r_1^2\dot{\theta}_1^2 + 2r_1r_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + r_2^2\dot{\theta}_2^2$$

Thus

$$T_2 = \frac{1}{2}m_2\left(r_1^2\dot{\theta}_1^2 + 2r_1r_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) + r_2^2\dot{\theta}_2^2\right)$$

Finally we find that

$$\begin{aligned}
T_{system} &= T_1 + T_2 \\
&= \frac{1}{2}m_1 r_1^2 \dot{\theta}_1^2 + \frac{1}{2}m_2\left(r_1^2\dot{\theta}_1^2 + 2r_1 r_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) + r_2^2 \dot{\theta}_2^2\right) \\
&= \frac{1}{2}\left(m_1 r_1^2 \dot{\theta}_1^2 + m_2 r_2^2 \dot{\theta}_2^2 + 2r_1 r_2 m_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right)
\end{aligned}$$

Also since cosine is symmetric

$$= \frac{1}{2}\left((m_1 + m_2)r_1^2\dot{\theta}_1^2 + m_2 r_2^2 \dot{\theta}_2^2 + (2m_2\cos(\theta_2 - \theta_1))\, r_1 r_2 \dot{\theta}_1 \dot{\theta}_2\right)$$

As required

2.3 Using the above equation

$$V = -g\left(r_1(m_1 + m_2)\cos(\theta_1) + m_2 r_2 \cos(\theta_2)\right)$$

Clearly we can see that each term contains an r_i and a $\cos(\theta_i)$ thus we may make the following construction

$$Y = \begin{pmatrix} r_1\cos(\theta_1) \\ r_2\cos(\theta_2) \end{pmatrix} \text{ where } Y^i = r_i\cos(\theta_i) \text{ for } i = 1, 2$$

But this does not fully rewritten our equation, we are still missing a part such that

$$\begin{aligned}
(r_1(m_1 + m_2)\cos(\theta_1) + m_2 r_2\cos(\theta_2)) &= YM \\
&= \begin{pmatrix} r_1\cos(\theta_1) \\ r_2\cos(\theta_2) \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \\
&= r_1\cos(\theta_1)\mu_1 + r_2\cos(\theta_2)\mu_2
\end{aligned}$$

Matching coefficients we find that

$$\begin{aligned}
\mu_1 &= m_1 + m_2 \\
\mu_2 &= m2
\end{aligned}$$

Therefore completing our rewrite of the original system we have that

$$V = -g\begin{pmatrix} r_1\cos(\theta_1) & r_2\cos(\theta_2) \end{pmatrix}\begin{pmatrix} m_1 + m_2 \\ m_2 \end{pmatrix}$$

2.4 Using the above equation

$$T = \frac{1}{2}\left((m_1 + m_2)r_1^2\dot{\theta}_1^2 + m_2 r_2^2\dot{\theta}_2^2 + 2r_1 r_2 m_2 \dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right)$$

Clearly we can see that each term contains an r_i and a $\dot{\theta}_i$ thus we may make the following construction

$$X^i = r_i\theta_i \text{ for } i = 1, 2 \text{ such that } X = \begin{pmatrix} r_1\theta_1 \\ r_2\theta_2 \end{pmatrix}$$

Therefore deriving X with respect to time we have that

$$\dot{X} = \begin{pmatrix} r_1\dot{\theta}_1 \\ r_2\dot{\theta}_2 \end{pmatrix}$$

But this does not fully rewritten our equation, we are still missing a part such that

$$\begin{aligned}
2T &= \dot{X}^T\tilde{M}\dot{X} \\
&= \begin{pmatrix} r_1\dot{\theta}_1 & r_2\dot{\theta}_2 \end{pmatrix} \begin{pmatrix} \mu_1 & \mu_2\cos(\theta_2-\theta_1) \\ \mu_2\cos(\theta_2-\theta_1) & \mu_2 \end{pmatrix} \begin{pmatrix} r_1\dot{\theta}_1 \\ r_2\dot{\theta}_2 \end{pmatrix} \\
&= \begin{pmatrix} r_1\dot{\theta}_1\mu_1 + r_2\dot{\theta}_2\mu_2\cos(\theta_2-\theta_1) & r_1\dot{\theta}_1\mu_2\cos(\theta_2-\theta_1) + r_2\dot{\theta}_2\mu_2 \end{pmatrix} \begin{pmatrix} r_1\dot{\theta}_1 \\ r_2\dot{\theta}_2 \end{pmatrix} \\
&= r_1^2\dot{\theta}_1^2\mu_1 + r_1r_2\dot{\theta}_1\dot{\theta}_2\mu_2\cos(\theta_2-\theta_1) + r_1r_2\dot{\theta}_1\dot{\theta}_2\mu_2\cos(\theta_2-\theta_1) + r_2^2\dot{\theta}_2^2\mu_2
\end{aligned}$$

Dividing through by $\frac{1}{2}$ we finally have that

$$T = \frac{1}{2}\left(r_1^2\dot{\theta}_1^2\mu_1 + r_2^2\dot{\theta}_2^2\mu_2 + 2r_1r_2\dot{\theta}_1\dot{\theta}_2\mu_2\cos(\theta_2-\theta_1)\right)$$

Now matching coefficients to our original equation we find that

$$\begin{aligned}
\mu_1 &= m_1 + m_2 \\
\mu_2 &= m_2
\end{aligned}$$

Thus we find our system to be

$$\begin{aligned}
T &= \frac{1}{2}\dot{X}^T\tilde{M}\dot{X} \\
&= \begin{pmatrix} r_1\dot{\theta}_1 & r_2\dot{\theta}_2 \end{pmatrix} \begin{pmatrix} m_1+m_2 & m_2\cos(\theta_2-\theta_1) \\ m_2\cos(\theta_2-\theta_1) & m_1+m_2 \end{pmatrix} \begin{pmatrix} r_1\dot{\theta}_1 \\ r_2\dot{\theta}_2 \end{pmatrix}
\end{aligned}$$

2.5 Constructing our Lagrangian we have

$$\begin{aligned}
\mathscr{L} &= T - V \\
&= \frac{1}{2}\left((m_1+m_2)r_1^2\dot{\theta}_1^2 + m_2r_2^2\dot{\theta}_2^2 + 2r_1r_2m_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right) \\
&\quad + g\left(r_1(m_1+m_2)\cos(\theta_1) + m_2r_2\cos(\theta_2)\right)
\end{aligned}$$

2.6 Substituting our lagrangian in the euler-lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial\dot{q}^i}\right) - \frac{\partial\mathscr{L}}{\partial q^i} = 0$$

where $i = 1$ corresponds to θ_1 and $i = 2$ corresponds to θ_2. For $i = 1$:

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}_1}\right) - \frac{\partial \mathscr{L}}{\partial \theta_1} = 0$$

Computing the different components we have

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial \dot{\theta}_1} &= \frac{1}{2}\left(2(m_1 + m_2)r_1^2\dot{\theta}_1 + 2r_1r_2m_2\cos(\theta_2 - \theta_1)\dot{\theta}_2\right) \\
&= (m_1 + m_2)r_1^2\dot{\theta}_1 + r_1r_2m_2\cos(\theta_2 - \theta_1)\dot{\theta}_2 \\
\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}_1}\right) &= (m_1 + m_2)r_1^2\ddot{\theta}_1 - r_1r_2m_2\sin(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1)\dot{\theta}_2 + r_1r_2m_2\cos(\theta_2 - \theta_1)\ddot{\theta}_2 \\
\frac{\partial \mathscr{L}}{\partial \theta_1} &= -r_1r_2m_2\sin(\theta_2 - \theta_1)\dot{\theta}_1\dot{\theta}_2 - g\left((m_1 + m_2)r_1\sin(\theta_1)\right)
\end{aligned}$$

Therefore our euler lagrange becomes

$$\begin{aligned}
0 = (m_1 + m_2)r_1^2\ddot{\theta}_1 &- r_1r_2m_2\sin(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1)\dot{\theta}_2 + r_1r_2m_2\cos(\theta_2 - \theta_1)\ddot{\theta}_2 \\
&+ r_1r_2m_2\sin(\theta_2 - \theta_1)\dot{\theta}_1\dot{\theta}_2 + g\left((m_1 + m_2)r_1\sin(\theta_1)\right)
\end{aligned}$$

Diving through by $(m_1 + m_2)r_1^2$

$$\begin{aligned}
0 &= \ddot{\theta}_1 - \frac{m_2r_2}{(m_1 + m_2)r_1}\sin(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1)\dot{\theta}_2 - \frac{m_2r_2}{(m_1 + m_2)r_1}\cos(\theta_2 - \theta_1)\ddot{\theta}_2 \\
&\quad + \frac{m_2r_2}{(m_1 + m_2)r_1}\sin(\theta_2 - \theta_1)\dot{\theta}_1\dot{\theta}_2 + \frac{g}{r_1}\sin(\theta_1) \\
&= \ddot{\theta}_1 + \frac{g}{r_1}\sin(\theta_1) + \frac{m_2r_2}{(m_1 + m_2)r_1}\left(-\sin(\theta_2 - \theta_1)\dot{\theta}_2^2 - \cancel{\sin(\theta_2 - \theta_1)\dot{\theta}_1\dot{\theta}_2}\right. \\
&\quad \left. + \cos(\theta_2 - \theta_1)\ddot{\theta}_2 + \cancel{\sin(\theta_2 - \theta_1)\dot{\theta}_1\dot{\theta}_2}\right) \\
&= \ddot{\theta}_1 + \frac{g}{r_1}\sin(\theta_1) + \frac{m_2r_2}{(m_1 + m_2)r_1}\left(\ddot{\theta}_2\cos(\theta_2 - \theta_1) - \dot{\theta}_2^2\sin(\theta_2 - \theta_1)\right) \\
&= \ddot{\theta}_1 + \frac{g}{r_1}\sin(\theta_1) + \frac{\mu_2}{\mu_1}\frac{r_2}{r_1}\left(\ddot{\theta}_2\cos(\theta_2 - \theta_1) - \dot{\theta}_2^2\sin(\theta_2 - \theta_1)\right)
\end{aligned}$$

For $i = 2$:

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}_2}\right) - \frac{\partial \mathscr{L}}{\partial \theta_2} = 0$$

Computing the different components we have

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial \dot{\theta}_1} &= m_2r_2^2\dot{\theta}_2 + r_1r_2m_2\cos(\theta_2 - \theta_1)\dot{\theta}_1 \\
\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{\theta}_2}\right) &= m_2r_2^2\ddot{\theta}_2 + r_1r_2m_2\cos(\theta_2 - \theta_1)\ddot{\theta}_1 - r_1r_2m_2\sin(\theta_2 - \theta_1)\dot{\theta}_1(\dot{\theta}_2 - \dot{\theta}_1) \\
\frac{\partial \mathscr{L}}{\partial \theta_2} &= -r_1r_2m_2\sin(\theta_2 - \theta_1)\dot{\theta}_1\dot{\theta}_2 - gr_2m_2\sin(\theta_2)
\end{aligned}$$

Therefore our euler-lagrange becomes

$$0 = m_2r_2^2\ddot{\theta}_2 + r_1r_2m_2\cos(\theta_2 - \theta_1)\ddot{\theta}_1 - r_1r_2m_2\sin(\theta_2 - \theta_1)\dot{\theta}_1(\dot{\theta}_2 - \dot{\theta}_1)$$

$$+ r_1r_2m_2\sin(\theta_2-\theta_1)\dot\theta_1\dot\theta_2 + gr_2m_2\sin(\theta_2)$$

Diving through by $m_2r_2^2$

$$\begin{aligned}
0 &= \ddot\theta_2 + \frac{r_1}{r_2}\cos(\theta_2-\theta_1)\ddot\theta_1 - \frac{r_1}{r_2}\sin(\theta_2-\theta_1)(\dot\theta_2-\dot\theta_1)\dot\theta_1 + \frac{r_1}{r_2}\sin(\theta_2-\theta_1)\dot\theta_1\dot\theta_2 + \frac{g}{r_2}\sin(\theta_2)\\
&= \ddot\theta_2 + \frac{g}{r_2}\sin(\theta_2) + \frac{r_1}{r_2}\left(\cos(\theta_2-\theta_1)\ddot\theta_1 \cancel{- \sin(\theta_2-\theta_1)\dot\theta_1\dot\theta_2} + \sin(\theta_2-\theta_1)\dot\theta_1^2 + \cancel{\sin(\theta_2-\theta_1)\dot\theta_1\dot\theta_2}\right)\\
&= \ddot\theta_2 + \frac{g}{r_2}\sin(\theta_2) + \frac{r_1}{r_2}\left(\ddot\theta_1\cos(\theta_2-\theta_1) + \dot\theta_1^2\sin(\theta_2-\theta_1)\right)
\end{aligned}$$

Thus we have shown by direct calculation that the corresponding equations of motion for the double pendulum are given by

$$\begin{aligned}
0 &= \ddot\theta_1 + \frac{g}{r_1}\sin(\theta_1) + \frac{\mu_2}{\mu_1}\frac{r_2}{r_1}\left(\ddot\theta_2\cos(\theta_1-\theta_2) + \dot\theta_2^2\sin(\theta_1-\theta_2)\right)\\
0 &= \ddot\theta_2 + \frac{g}{r_2}\sin(\theta_2) + \frac{r_1}{r_2}\left(\ddot\theta_1\cos(\theta_1-\theta_2) + \dot\theta_1^2\sin(\theta_2-\theta_1)\right)
\end{aligned}$$

As required

2.7 If $r_1 = r_2$ and $m_1 = m_2$ our equations of motion become

$$\begin{aligned}
0 &= \ddot\theta_1 + \frac{g}{r_1}\sin(\theta_1) + \frac{1}{2}\left(\ddot\theta_2\cos(\theta_1-\theta_2) + \dot\theta_2^2\sin(\theta_1-\theta_2)\right)\\
0 &= \ddot\theta_2 + \frac{g}{r_1}\sin(\theta_2) + \ddot\theta_1\cos(\theta_1-\theta_2) + \dot\theta_1^2\sin(\theta_2-\theta_1)
\end{aligned}$$

The equations of motion for a double pendulum and a single pendulum exhibit both similarities and differences
Similarities:
1. Both sets of equations describe the motion of a pendulum in a gravitational field.
2. In both cases, the accelerations and depend on the gravitational acceleration (g), which affects the pendulum's motion.
4.The single pendulum's equation is a special case of the double pendulum when $r_1 = r$ and $\theta_2 = 0$

Differences:
1. The double pendulum equations involve two generalized coordinates and their respective angular accelerations. In contrast, the single pendulum equation has only one generalized coordinate and one angular acceleration
2. In the double pendulum case, the equations are coupled because $\ddot\theta_1$ depends on θ_2 and $\ddot\theta_2$ and vice versa. This means the motion of the second bob is dependent on the first. However in the single pendulum there is no such coupling.
3. The double pendulum equations are more complex and may lead to chaotic behavior, while the single pendulum equation is easily analytically solvable and results in simple harmonic motion for small angles.

Question 3 - The n-Link Pendulum

3.1 We have seen in the case of the 2-Link Pendulum we were able to rewrite our kinetic energy and potential energy in matrix form according to the following:

$$T = \frac{1}{2}\dot{X}^T\tilde{M}\dot{X}$$
$$V = gY^TM$$

We attempt to generalise such a form for an n-Link Pendulum by finding patterns in each of the different components.

Looking at the components of V:

The column matrix M is seen to have each row to be represented as a summation of the masses. Let the first row correspond to the first bob and the last row the last bob. We find that the total mass acting downward on the first bob is the sum of the first bob's mass with all the masses below it. Similarly if we were to consider the second bob, the total mass acting downward on that bob is its current mass plus all the masses below it. Importantly this mass now excludes the first bob. This pattern continues up until the n-th bob where we find the total mass acting downward on the bob is only the mass of itself. As described in the below formulation:

$$M = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \vdots \\ \mu_{n-1} \\ \mu_n \end{pmatrix} = \begin{pmatrix} m_1 + m_2 + m_3 + \ldots + m_{n-1} + m_n \\ m_2 + m_3 + \ldots + m_{n-1} + m_n \\ m_3 + \ldots + m_{n-1} + m_n \\ \vdots \\ m_{n-1} + m_n \\ m_n \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n m_i \\ \sum_{i=2}^n m_i \\ \vdots \\ \sum_{i=n}^n m_i \end{pmatrix}$$

The column matrix Y is simply the y-component in each bob if we treated each bob as an isolated singular pendulum. Thus we find

$$Y = \begin{pmatrix} r_1\cos(\theta_1) \\ r_2\cos(\theta_2) \\ \vdots \\ r_n\cos(\theta_n) \end{pmatrix}$$

Thus we find for a n-Link Pendulum our potential energy for the system becomes:

$$V = -g\begin{pmatrix} r_1\cos(\theta_1) \\ r_2\cos(\theta_2) \\ \vdots \\ r_n\cos(\theta_n) \end{pmatrix}^T \begin{pmatrix} \sum_{i=1}^n m_i \\ \sum_{i=2}^n m_i \\ \vdots \\ \sum_{i=n}^n m_i \end{pmatrix}$$

Next, looking at the components of T: Similarly to the column matrix Y we find that the column matrix X is the x-components of each bob if we treated each bob as an isolated singular pendulum. Thus we find that the general form of it is:

$$X = \begin{pmatrix} r_1\theta_1 \\ r_2\theta_2 \\ \vdots \\ r_n\theta_n \end{pmatrix}$$

Deriving our matrix X with respect to time we find

$$\frac{d}{dt}X = \begin{pmatrix} r_1\dot{\theta}_1 \\ r_2\dot{\theta}_2 \\ \vdots \\ r_n\dot{\theta}_n \end{pmatrix}$$

Looking at the symmetrically square matrix $\tilde{M}$ we immediately notice that the diagonal of the matrix is μ_i where i is the current row we are in. Using the symmetric property of the cosine function ($\cos(A - B) = \cos(B - A)$) in our 2 link pendulum example, I will now show how else the $\tilde{M}$ may be represented:

$$\tilde{M} = \begin{pmatrix} \mu_1 & \mu_2\cos(\theta_1 - \theta_2) \\ \mu_2\cos(\theta_2 - \theta_1) & \mu_2 \end{pmatrix}$$

Now although a pattern is not immediately apparent we further use the property that

$$\cos(A - A) = \cos(0) = 1$$

To find

$$\tilde{M} = \begin{pmatrix} \mu_1\cos(\theta_1 - \theta_1) & \mu_2\cos(\theta_1 - \theta_2) \\ \mu_2\cos(\theta_2 - \theta_1) & \mu_2\cos(\theta_2 - \theta_2) \end{pmatrix}$$

Now we get a big indication of the pattern arising in the matrix. We see that if we describe our matrix entries in terms of row i and column j where the first entry in the matrix is at $(i, j) = (1, 1)$ that the μ_k is determined according to the biggest number between the current row and column i.e. μ_k where $k = \max\{i, j\}$. We also see that in the cosine term the theta values are determined by the current row we are for the first theta and the current column we are in for the second theta i.e. $\cos(\theta_i - \theta_j)$. Thus we see that the elements of $\tilde{M}$ for the n-Link Pendulum can be described as

$$\tilde{M}^i_j = \mu_{\max\{i,j\}}\cos(\theta_i - \theta_j)$$

Which in matrix form can we described as:

$$\tilde{M} = \begin{pmatrix} \mu_1\cos(\theta_1 - \theta_1) & \mu_2\cos(\theta_1 - \theta_2) & \dots & \mu_n\cos(\theta_1 - \theta_n) \\ \mu_2\cos(\theta_2 - \theta_1) & \mu_2\cos(\theta_2 - \theta_2) & \dots & \mu_n\cos(\theta_2 - \theta_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n\cos(\theta_n - \theta_1) & \mu_n\cos(\theta_n - \theta_2) & \dots & \mu_n(\theta_n - \theta_n) \end{pmatrix}$$

$$= \begin{pmatrix} \mu_1 & \mu_2\cos(\theta_1-\theta_2) & \dots & \mu_n\cos(\theta_1-\theta_n) \\ \mu_2\cos(\theta_2-\theta_1) & \mu_2 & \dots & \mu_n\cos(\theta_2-\theta_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n\cos(\theta_n-\theta_1) & \mu_n\cos(\theta_n-\theta_2) & \dots & \mu_n \end{pmatrix}$$

Thus for a n-Link Pendulum our kinetic energy for the system becomes:

$$T = \frac{1}{2}\begin{pmatrix} r_1\dot{\theta}_1 \\ r_2\dot{\theta}_2 \\ \vdots \\ r_n\dot{\theta}_n \end{pmatrix}^T \begin{pmatrix} \mu_1 & \mu_2\cos(\theta_1-\theta_2) & \dots & \mu_n\cos(\theta_1-\theta_n) \\ \mu_2\cos(\theta_2-\theta_1) & \mu_2 & \dots & \mu_n\cos(\theta_2-\theta_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n\cos(\theta_n-\theta_1) & \mu_n\cos(\theta_n-\theta_2) & \dots & \mu_n \end{pmatrix} \begin{pmatrix} r_1\dot{\theta}_1 \\ r_2\dot{\theta}_2 \\ \vdots \\ r_n\dot{\theta}_n \end{pmatrix}$$

Thus we have now shown and described the general form of the vectors X, Y M and the mass matrix $\tilde{M}$ and so our Lagrangian for the n-Link Pendulum is:

$$\begin{aligned} \mathscr{L} &= T - V \\ &= \frac{1}{2}\begin{pmatrix} r_1\dot{\theta}_1 \\ r_2\dot{\theta}_2 \\ \vdots \\ r_n\dot{\theta}_n \end{pmatrix}^T \begin{pmatrix} \mu_1 & \mu_2\cos(\theta_1-\theta_2) & \dots & \mu_n\cos(\theta_1-\theta_n) \\ \mu_2\cos(\theta_2-\theta_1) & \mu_2 & \dots & \mu_n\cos(\theta_2-\theta_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n\cos(\theta_n-\theta_1) & \mu_n\cos(\theta_n-\theta_2) & \dots & \mu_n \end{pmatrix} \begin{pmatrix} r_1\dot{\theta}_1 \\ r_2\dot{\theta}_2 \\ \vdots \\ r_n\dot{\theta}_n \end{pmatrix} \\ &\quad - g\begin{pmatrix} r_1\cos(\theta_1) \\ r_2\cos(\theta_2) \\ \vdots \\ r_n\cos(\theta_n) \end{pmatrix}^T \begin{pmatrix} \sum_{i=1}^n m_i \\ \sum_{i=2}^n m_i \\ \vdots \\ \sum_{i=n}^n m_i \end{pmatrix} \end{aligned}$$

3.2 In our Lagrangian the metric information is not explicitly stated as a metric tensor g, as you might see in general relativity, but it is implicitly encoded in the form of the mass matrix $\tilde{M}$, which depends on the generalized coordinates (θ_i) and the summation of masses coefficients (μ_i).

The elements of $\tilde{M}$ account for the distribution of mass, the distances between different points in the system and the angles between these points. The off-diagonal terms account for the interactions and coupling between different components of the system. It describes how changes in one coordinate affect the kinetic energy associated with the other coordinates.

In summary, the Lagrangian provided encodes metric information by representing the kinetic energy terms through the matrix $\tilde{M}$. This matrix reflects the geometric structure of the system, specifying how the kinetic energy depends on the chosen generalized coordinates, their rates, and the relationships between them.

3.3 When the correct Lagrangian is used (the Lagrangian which includes the angular dependence in the mass matrix $\tilde{M}$, as in the "n-link-pendulum.mp4") the simulation shows the interaction and coupling between different links in the chain. The angular dependence represented in $\tilde{M}$ accounts for the fact that the motion of one link affects the motion of the adjacent links. In a real heavy chain, as one link swings, it imparts forces and torques on the neighboring links which we see in the correct video. The motion of the pendulum is a wave-like motion.

In contrast, when we use the incorrect Lagrangian (the angular dependence in $\tilde{M}$ is removed, as in the "n-link-pendulum-WRONG.mp4"), the simulation treats each link as if it were an isolated pendulum, neglecting the interactions between them. This results in unrealistic, chaotic motion, where each link moves independently of the others, which is not representative of how a real heavy chain would behave. AS a result we see that the last link swings frantically around in a chaotic fashion.

3.4 Equations of motion for θ_1

$$0 = 3g\sin(\theta_1) + 3\ddot{\theta}_1 + 2\ddot{\theta}_2\cos(\theta_1 - \theta_2) + 2\dot{\theta}_2^2\sin(\theta_1 - \theta_2) + \ddot{\theta}_3\cos(\theta_1 - \theta_3) + \dot{\theta}_3^2\sin(\theta_1 - \theta_3)$$

Equations of motion for θ_2

$$0 = -2\dot{\theta}_1^2\sin(\theta_1 - \theta_2) + 2(g\sin(\theta_2) + \ddot{\theta}_1\cos(\theta_1 - \theta_2) + \ddot{\theta}_2 + \ddot{\theta}_3\cos(\theta_2 - \theta_3) + \dot{\theta}_3^2\sin(\theta_2 - \theta_3))$$

Equations of motion for θ_3

$$0 = g\sin(\theta_3) + \ddot{\theta}_1\cos(\theta_1 - \theta_3) + \ddot{\theta}_2\cos(\theta_2 - \theta_3) + \ddot{\theta}_3 - \dot{\theta}_1^2\sin(\theta_1 - \theta_3) - \dot{\theta}_2^2\sin(\theta_2 - \theta_3)$$

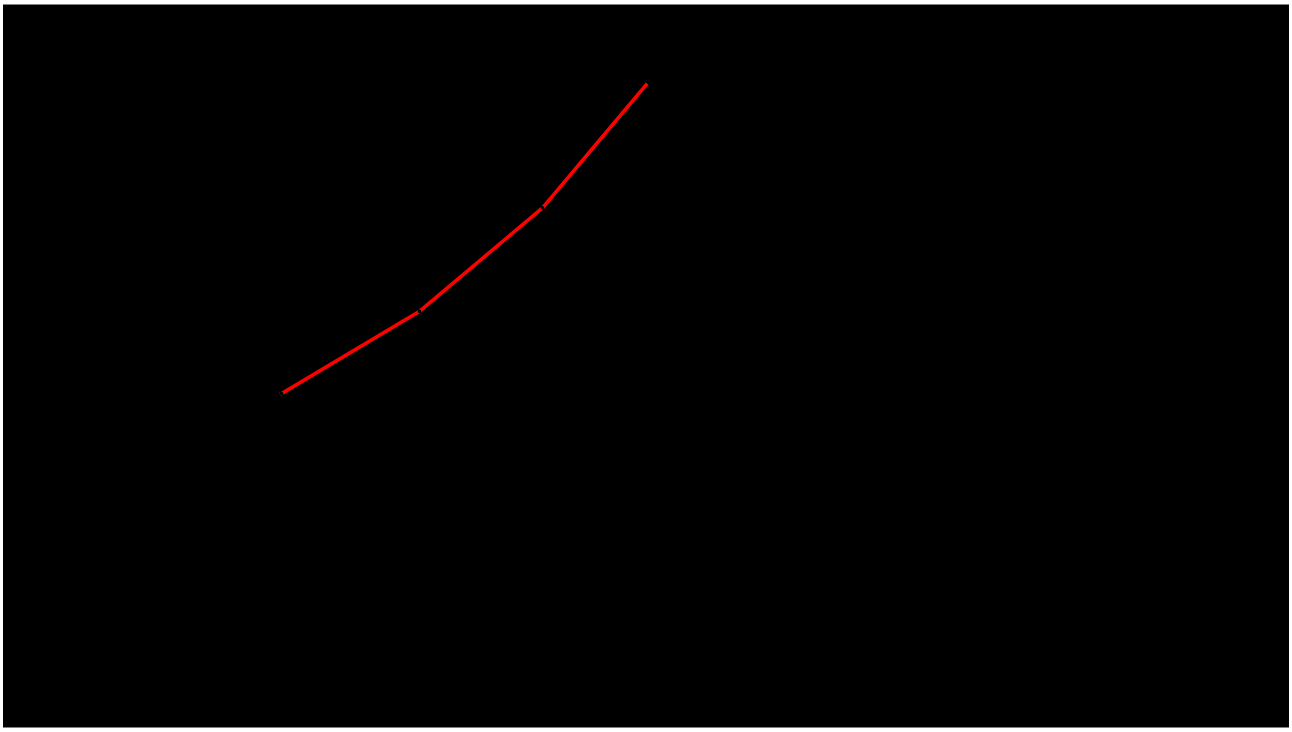

Figure 3: 3-Link Pendulum at t=2