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University of the Witwatersrand, Johannesburg

Course or topic No(s)	MATH2001
Course or topic name(s) Paper number & title	Basic Analysis
Examination/Test* to be held during month(s) of (*delete as applicable)	June
Year of study (Art & Sciences leave blank)	Second Year
Degrees/Diplomas for which this course is prescribed (BSc (Eng) should indicate which branch)	Bsc, Bcom, BA
Faculty/ies presenting candidates	Science
Internal examinar(s) and telephone number(s)	Mr A Blecher Ext 76202 & Dr S Bau Ext 76215
External examiner(s)	Dr I Naidoo - UNISA
Calculator policy	

Time allowance

Course Nos	MATH2001	Time	90

Instruction to candidates (Examiners may wish to use this space to indicate, inter alia, the contribution made by this examination or test towards the year mark, if appropriate)

Answers to All questions must be legible.
Total : 90 marks
Duration : 1h30

Internal Examiners or Heads of Department are requested to sign the Declaration overleaf

Math2001–Basic Analysis Examination June 2010

Time: 90 minutes   Total marks: 90 marks

SECTION A Multiple choice

Answer the multiple choice questions on the computer card provided. There is ONLY ONE correct answer to each question. Please ensure that your student number is entered on the card, by pencilling in the requisite digit for each block.

Question 1 .... [2 points]

Let

$$f(x) = \begin{cases} \sin x & \text{if } x \leq 0 \\ x & \text{if } 0 < x \leq \pi \\ 1 & \text{if } \pi < x. \end{cases}$$

Which of the following is **false**?

A. $f(x)$ is continuous on the left at $x = 0$.

B. $f(x)$ is continuous on the left at $x = \pi$.

C. $f(x)$ is not continuous on the left at $x = 0$.

D. $f(x)$ is continuous on the right at $x = 0$.

E. $f(x)$ is not continuous on the right at $x = \pi$.

Question 2 .... [2 points]

Let $f(x)$ be a function defined on (a, b) that is not bounded above. Which of the following is necessarily true?

A. $f(x)$ is either an increasing function or a decreasing function.

B. If $f(x)$ is an increasing function then $\lim_{x \to a^+} f(x) = -\infty$ or $\lim_{x \to a^+} f(x) = \inf\{f(x) : x \in (a, b)\}$.

C. If $f(x)$ is an increasing function then $\lim_{x \to b^-} f(x) = \sup\{f(x) : x \in (a, b)\}$.

D. $f(x)$ is not decreasing because this would be in contradiction to the given information that it is not bounded above.

E. $f(x)$ is necessarily an increasing function.

Question 3 .... [2 points]

Let the sequence (a_n) be defined by

$$a_1 = 1,\ a_n = \sqrt{1 + a_{n-1}}\ (n \geq 2).$$

Then as $n \to \infty$, $a_n \to$

A. $\sqrt{2}$.

B. $\dfrac{\sqrt{5}}{2}$.

C. $\dfrac{1+\sqrt{5}}{2}$.

D. ∞.

E. None of the above.

Question 4 .... [2 points]

Which statement is **false**?

A. If a series is absolutely convergent then it is convergent.

B. The series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^s}$ is divergent for $s \leq 1$.

C. $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^s}$ is convergent for $s > 1$.

D. The condition $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$ guarantees that $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ is convergent.

E. $a_n \not\to 0$ implies that $\sum\limits_{n=1}^{\infty} a_n$ is divergent.

Question 5 .... [2 points]

Suppose that the series $\sum\limits_{n=1}^{\infty} a_n$ is convergent. Then

A. the series $\sum\limits_{n=1}^{\infty} |a_n|$ is convergent.

B. the series $\sum\limits_{n=1}^{\infty} a_n^2$ is convergent.

C. the series $\sum\limits_{n=1}^{\infty} \sqrt{a_n}$ is convergent.

D. the series $\sum\limits_{n=1}^{\infty} \sqrt{a_n}$ is divergent.

E. if $a_n > 0$ for all $n \in \mathbb{N}$ then $\sum\limits_{n=1}^{\infty} \log a_n$ is divergent.

Question 6 .... [2 points]

If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges. Then

A. $\sum_{n=1}^{\infty}(a_n + b_n)$ converges.

B. $\sum_{n=1}^{\infty}(a_n - b_n)$ converges.

C. $\sum_{n=1}^{\infty}(a_n + b_n)$ diverges.

D. $\sum_{n=1}^{\infty}(b_n - a_n)$ converges.

E. None of the above

SECTION B

Answer this section in the answer book provided.

Question 7 .... [6 points]

You may assume that $f(x)$ satisfies all conditions required by the following definitions. Define

(a) $\lim_{x \to c^+} f(x) = -\infty.$ (2)

(b) $\lim_{x \to c^-} f(x) = L.$ (2)

(c) $\lim_{x \to -\infty} f(x) = L.$ (2)

Question 8 .... [20 points]

Use the appropriate definition to show that

(a) $\lim_{x \to 2^+} \dfrac{1}{2-x} = -\infty.$ (6)

(b) For

$$f(x) = \begin{cases} x^2 + 2x + 1 & \text{if } x \geq 0 \\ -2 & \text{if } x < 0, \end{cases}$$

$f(x) \to 4$ as $x \to 1^-$. (8)

(c) $\lim_{x \to -\infty} \dfrac{x \cos x}{x^2 - 1} = 0.$

(Hint: $x \to -\infty$ means x is eventually negative.) (6)

Question 9 .... [18 points]

(a) Prove the Intermediate Value Thereom: If a real valued function $f(x)$ is continuous on $[a, b]$ and if $f(a) < 0 < f(b)$ then there exists $c \in (a, b)$ such that $f(c) = 0$. (12)

(b) Use the Intermediate Value Theorem to show that if $f : [0, 2] \to [0, 2]$ is continuous on $[0, 2]$ then there exists at least one number $c \in [0, 2]$ such that $f(c) = c$.

(4)

(c) Let $f(x) = \sqrt{2x}$. Show that there exists $c \in (0, 2)$ such that $\sqrt{2c} = c$. (You may assume that $\sqrt{2x}$ is continuous on $[0, 2]$.) (2)

Question 10 .... [12 points]

(a) State the definition of conditional convergence of a series and provide an example of a conditionally convergent series. (4)

(b) Suppose that if (a_n) is a nonincreasing sequence of positive real numbers which converges to zero. Show that series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} a_n$ is convergent. (8)

Question 11 .... [16 points]

Determine whether each of the following series is convergent or divergent, and if convergent also determine whether it is absolutely convergent or conditionally convergent.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ (4)

(b) $\displaystyle\sum_{n=1}^{\infty} \sin n$ (4)

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ (4)

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ (4)

Question 12 .... [6 points]

Find the intervals of convergence of the following series:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^n}$ (3)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n x^n}{n}$ (3)