UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG
School of School of Computer Science and Applied Mathematics

APPM2023
Mechanics II
2023

Assignment-02

Arc Length, Area, Volume

Issued: 11 May 2023 **Total:** 46

Due: 17:00, 19 May 2023

Instructions

- Read all the instructions and questions carefully.
- Typeset the solution document using 'Assignment.cls' LaTeX document template. Submissions that have not used this template shall receive a zero grade.
- Use plain written English where necessary.
- Students may use the supplementary resources posted on the course Moodle page to complete this assignment.
- Students are encouraged to work in groups. However, this is to be individual work and each student must submit their own report.
- Plagiarised submissions shall receive a zero grade.
- No late submissions shall be considered.

Question 1 (6 Points)

A coin is rolled around the edge of another stationery coin without slipping. Each coin has radius r.

1. Explain why the outer coin will rotate twice on its journey around the inner coin. (3 Points)
2. Generalise your argument to predict the number of times a coin of radius $\frac{r}{2}$ will rotate when rolled around a coin of radius r. (3 Points)

Question 2 (22 Points)

Consider the surface of a torus where a point on the surface of the torus is given by the position vector

$$\vec{p} = \alpha r \vec{p}_1 + r \vec{p}_2,$$

where $0 < r$ is the radius of the smallest cycle on the torus, $1 < \alpha$ is a constant, and

$$\vec{p}_1 = \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \\ 0 \end{pmatrix} \quad \text{and} \quad \vec{p}_2 = \begin{pmatrix} \cos(\theta)\cos(\phi) \\ \cos(\theta)\sin(\phi) \\ \sin(\theta) \end{pmatrix}.$$

and $\theta \in [0, 2\pi)$ and $\phi \in [0, 2\pi)$. Answer the following questions

1. Show that the vectors tangent to the embedding coordinate curves are given by

$$\vec{e}_r = \begin{pmatrix} \cos(\phi)(\alpha + \cos(\theta)) \\ \sin(\phi)(\alpha + \cos(\theta)) \\ \sin(\theta) \end{pmatrix} \quad \vec{e}_\theta = r \begin{pmatrix} -\sin(\theta)\cos(\phi) \\ -\sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} \quad \text{and} \quad \vec{e}_\phi = r \begin{pmatrix} -\sin(\phi)(\alpha + \cos(\theta)) \\ \cos(\phi)(\alpha + \cos(\theta)) \\ 0 \end{pmatrix}$$

   Show all steps of the calculation. (6 Points)

2. Show that the area element for the outer surface of the torus is

$$\mathrm{d}A = r^2(\alpha + \cos(\theta))\,\mathrm{d}\theta\,\mathrm{d}\phi$$

   Show all steps of the calculation. (4 Points)

3. Show that the volume element for the region contained within the torus is

$$\mathrm{d}V = r^2(\alpha + \cos(\theta))(1 + \alpha\cos(\theta))\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}r.$$

   Show all steps of the calculation. (4 Points)

4. Show that the surface area of the torus is

$$A = 4\pi^2 \alpha r^2.$$

   Show all steps of the calculation. (2 Points)

5. Show that the volume of the region contained within the torus is

$$V = 2\pi^2 \alpha r^3.$$

Show all steps of the calculation. (3 Points)

6. Suppose that we "cut-open" and "flatten" the torus into a rectangle. What are the dimensions of the rectangle such that the torus and rectangle have identical surface area? Explain any conversion factors relating the parameters that control the size of the torus with those that control the size of the rectangle. (3 Points)

Question 3 (18 Points)

The General Theory of Relativity is a generalisation of Newton's Theory of Universal Gravitation in which the force of gravity is replaced by geometry. Consider a metric tensor in *Boyer-Lindquist* co-ordinates $\{t, r, \theta, \phi\}$ in the region about a rotating black-hole with mass M, rotation a and Newton's Gravitational constant G,

$$\mathbf{g} = \begin{pmatrix} g_{tt} & 0 & 0 & g_{t\phi} \\ 0 & g_{rr} & 0 & 0 \\ 0 & 0 & g_{\theta\theta} & 0 \\ g_{\phi t} & 0 & 0 & g_{\phi\phi} \end{pmatrix},$$

where g_{tt}, g_{rr}, $g_{\theta\theta}$, $g_{\phi\phi}$, $g_{\phi t}$ and $g_{t\phi}$ are all functions of G, M, a, r and θ.

Let λ denote the independent time parameter of motion. Answer the following questions

1. What information is stored in $\mathbf{g}$? (4 points)

2. Prove that $g_{\phi t} = g_{t\phi}$. Explain all steps. (3 points)

3. Why is $\mathbf{g}$ a useful quantity in Geometry? (2 points)

4. Suppose

$$g_{tt} = -\left(1 - \frac{Rr}{\Sigma}\right), \quad g_{rr} = \frac{\Sigma}{\Delta}, \quad g_{\theta\theta} = \Sigma, \quad \text{and} \quad g_{\phi\phi} = \sin^2(\theta)\left(r^2 + a^2 \sin^2(\theta)\frac{Rr}{\Sigma}\right)$$

and

$$g_{\phi t} = g_{t\phi} = -a \sin^2(\theta)\frac{Rr}{\Sigma}$$

where

$$R = 2MG, \quad \Sigma = r^2 + a^2 \cos^2(\theta) \quad \text{and} \quad \Delta = r^2 + a^2 - Rr.$$

Construct the arc-length for a free particle of mass m moving in this black-hole geometry. (Hint: remember, λ is the independent time parameter in this space.) (4 points)

5. Consider arc-length in the the special case of $a = 0$, $R \ll r$. Compare this arc-length with that of Euclidean space. What are the consequences of having an arc-length defined like this, explain? (Hint: use the squared line-element $\mathrm{d}s^2$ to simplify your discussion.) (5 points)