Math2001 – Basic Analysis Test April 17 2015
MEMO

Axioms and some basic properties of real numbers

In this test, you may refer to the axioms of real numbers and those parts of Theorems 1.2 and 1.3 which are listed on the next two pages.

A. Axioms of addition

(A1) **Associative Law:** $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{R}$.

(A2) **Commutative Law:** $a + b = b + a$ for all $a, b \in \mathbb{R}$.

(A3) **Zero:** There is a real number 0 such that $a + 0 = a$ for all $a \in \mathbb{R}$.

(A4) **Additive inverse:** For each $a \in \mathbb{R}$ there is $-a \in \mathbb{R}$ such that $a + (-a) = 0$.

M. Axioms of multiplication

(M1) **Associative Law:** $a(bc) = (ab)c$ for all $a, b, c \in \mathbb{R}$.

(M2) **Commutative Law:** $ab = ba$ for all $a, b \in \mathbb{R}$.

(M3) **One:** There is a real number 1 such that $1 \neq 0$ and $a \cdot 1 = a$ for all $a \in \mathbb{R}$.

(M4) **Multiplicative inverse:** For each $a \in \mathbb{R}$ with $a \neq 0$ there is $a^{-1} \in \mathbb{R}$ such that $aa^{-1} = 1$.

D. The distributive law axiom

(D) **Distributive Law:** $a(b + c) = ab + ac$ for all $a, b, c \in \mathbb{R}$.

O. The order axioms

(O1) **Trichotomy:** For each $a \in \mathbb{R}$, exactly one of the following statements is true: $a > 0$ or $a = 0$ or $-a > 0$.

(O2) If $a > 0$ and $b > 0$, then $a + b > 0$.

(O3) If $a > 0$ and $b > 0$, then $ab > 0$.

C. The Dedekind completeness axiom

(C) Every nonempty subset of $\mathbb{R}$ which is bounded above has a supremum.

Theorem 1.2 (Basic field properties: Distributive laws)
(a) $\forall a, b, c \in \mathbb{R},\ (a+b)c = ac + bc$.
(b) $\forall a \in \mathbb{R},\ a \cdot 0 = 0$.
(c) $\forall a, b \in \mathbb{R},\ ab = 0 \Leftrightarrow a = 0$ or $b = 0$.
(d) $\forall a, b \in \mathbb{R},\ (-a)b = -(ab)$.
(e) $\forall a \in \mathbb{R},\ (-1)a = -a$.
(f) $\forall a, b \in \mathbb{R},\ (-a)(-b) = ab$.

Theorem 1.3 (Basic field properties: multiplication)
(a) The number 1 is unique.
(b) For all $a \in \mathbb{R}$ with $a \neq 0$, the number a^{-1} is unique.
(c) For all $a, b \in \mathbb{R}$ with $a \neq 0$, the equation $ax = b$ has a unique solution. This solution is $x = a^{-1}b$.

Question 1 . . . . [12 marks]

The correct answers to Question 1 are

(a): D

(b): C

(c): C

(d): A

Question 2 .... [6 marks]

Let $a \in \mathbb{R} \setminus \{0\}$. Prove that $(a^{-1})^{-1} = a$.
For each step of the proof, write down which of the axioms and properties listed on the first two pages you are using.

Solution.

By (M4), $aa^{-1} = 1$. ✓

Then (M2) gives $a^{-1}a = aa^{-1} = 1$. ✓

It follows from Theorem 1.2 (c) that $a^{-1} \neq 0$. ✓

Again by (M4), $a^{-1}(a^{-1})^{-1} = 1$. ✓

Hence $a^{-1}a = 1$ and $a^{-1}(a^{-1})^{-1}) = 1$. Thus a and $(a^{-1})^{-1}$ are solutions of $a^{-1}x = 1$. ✓

By Theorem 1.3 (c), this solution is unique, and $(a^{-1})^{-1} = a$ follows ✓.

Question 3 .... [14 marks]

Let S be a nonempty set of real numbers.

(a) Write down what it means that S is bounded below and write down the definition of $\inf S$. (2)

Answer: S is said to be **bounded below** if there is a number $a \in \mathbb{R}$ such that $x \geq a$ for all $x \in S$.

A real number m is said to be the **infimum** or **greatest lower bound** of S, denoted by $\inf S$, if
(a) m is a lower bound of S, and
(b) if l is any lower bound of S, then $m \geq l$.

(b) Prove that the following are equivalent: (12)
(i) $L = \inf S$,
(ii) L is a lower bound of S and for all $x > L$ there is $s \in S$ such that $s < x$.

Proof. (i) $\Rightarrow$ (ii): Since (i) holds, L is the greatest lower bound and thus a lower bound. Let $x > L$. Then x is not a lower bound of S since L is the greatest lower bound of S. Hence $x \leq a$ does not hold for all $a \in S$, and therefore there must be $s \in S$ such that $s < x$.

(ii) $\Rightarrow$ (i): By assumption, L is a lower bound of S. Suppose that L is not the greatest lower bound. Then there is a lower bound M of S with $L < M$. By assumption (ii), there is $s \in S$ such that $s < M$, contradicting the assumption that M is a lower bound of S. So L must be indeed the greatest lower bound of S.

Question 4 .... [20 marks]

Let (a_n) be a sequence of real numbers and let $L \in \mathbb{R}$.

(a) Define "$a_n \to L$ as $n \to \infty$". (2)

Answer: The statement '$a_n \to L$ as $n \to \infty$' is defined by:
$\forall\, \varepsilon > 0\; \exists\, K \in \mathbb{R}\; \forall n \in \mathbb{N}, n \geq K, |a_n - L| < \varepsilon$.

(b) Define "$a_n \to \infty$ as $n \to \infty$". (2)

Answer: '$a_n \to \infty$ as $n \to \infty$' if for every $A \in \mathbb{R}$ there is $K \in \mathbb{R}$ such that $a_n > A$ for all $n \geq K$.

(c) Show by using definitions only that $\lim\limits_{n\to\infty} \dfrac{n^2+3}{3n+1} = \infty$. (8)

Solution. We first estimate

$$\frac{n^2+3}{3n+1} > \frac{n^2}{3n+n} = \frac{n^2}{4n} = \frac{n}{4} \qquad (*)$$

Hence let $A > 0$ and put $K = 4A$. For $n \geq K$ we then conclude from (*) that

$$\frac{n^2+3}{3n+1} > \frac{n}{4} \geq \frac{K}{4} = \frac{4A}{4} = A.$$

Hence $\lim\limits_{n\to\infty} \dfrac{n^2+3}{3n+1} = \infty$.
[Note to marker: instead of $n \geq K$ also $n > K$ is correct.]

(d) Show by using definitions only that $\left(\dfrac{3n-1}{n+3}\right)$ converges. (8)

Solution. First we calculate

$$\begin{aligned}\left|\frac{3n-1}{n+3} - 3\right| = \left|\frac{3n-1-3n-9}{n+3}\right| &= \left|\frac{-10}{n+3}\right| \\ &= \frac{10}{n+3}.\end{aligned}$$

Now let $\varepsilon > 0$. Then

$$\begin{aligned}\left|\frac{3n-1}{n+3} - 3\right| < \varepsilon \Leftrightarrow \frac{10}{n+3} &< \varepsilon \\ \Leftrightarrow \frac{10}{\varepsilon} &< n+3 \\ \Leftrightarrow \frac{10}{\varepsilon} - 3 &< n. \qquad (**)\end{aligned}$$

Hence let $K > \dfrac{10}{\varepsilon} - 3$. It follows for $n \geq K$ that

$$n > \frac{10}{\varepsilon} - 3$$

and therefore

$$\left|\frac{3n-1}{n+3}-3\right|<\varepsilon.$$

by (**). By definition, this means that $\left(\dfrac{3n-1}{n+3}\right)$ converges to 3.

Question 5 .... [8 marks]

State and prove the Sandwich Theorem.

Solution.

Sandwich Theorem: If $a_n \le b_n \le c_n$ and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$, then $\lim\limits_{n\to\infty} b_n = L$.

PROOF. Let $\varepsilon > 0$ and choose k_1 and k_2 such that $|a_n - L| < \varepsilon$ if $n \ge k_1$ and $|c_n - L| < \varepsilon$ if $n \ge k_2$. In particular, for $n \ge K = \max\{k_1, k_2\}$,

$$L-\varepsilon < a_n < L+\varepsilon, \quad L-\varepsilon < c_n < L+\varepsilon$$

gives

$$L-\varepsilon < a_n \le b_n \le c_n < L+\varepsilon.$$

Hence $|b_n - L| < \varepsilon$ if $n \ge K$, which shows that $\lim\limits_{n\to\infty} b_n = L$.