MATH2001 (Basic Analysis)
Memo of Quiz 3.1 September 2020

Question 1.1 : **[2 marks]**

Which one of the following statements is true?

A. The series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n$ converges.

B. The series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ diverges.

C. Every convergent series is absolutely convergent.

D. The series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

E. None of these.

Answer: D.

Question 1.2 : **[2 marks]**

Which one of the following statements is true?

A. The series $\sum_{n=1}^{\infty} n\sin\left(\frac{1}{n}\right)$ converges.

B. The series $\sum_{n=1}^{\infty}\frac{1}{n}$ converges.

C. The series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ is absolutely convergent.

D. If $\lim_{n\to\infty}\sum_{k=1}^{n} a_k \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

E. None of these.

Answer: E.

Question 1.3 : **[2 marks]**

Which one of the following statements is true?

A. If $a_n \to 0$ as $n \to \infty$, then the series $\sum_{n=1}^{\infty} a_n$ converges.

B. The series $\sum_{n=1}^{\infty} \frac{n^2 - 1}{n - 50n^2}$ is absolutely convergent.

C. The series $\sum_{n=1}^{\infty} \frac{n + 1}{n}$ diverges.

D. The series $\sum_{n=1}^{\infty} (-1)^{n+1}$ converges.

E. None of these.

Answer: C.

Question 1.4 : **[2 marks]**

Which one of the following statements is true?

A. The series $\sum_{n=1}^{\infty} n^2$ converges.

B. If the series $\sum_{n=1}^{\infty} a_n$ diverges, then $a_n \not\to 0$ as $n \to \infty$.

C. The series $\sum_{n=1}^{\infty} (-1)^n$ converges.

D. The series $\sum_{n=1}^{\infty} \frac{-n}{2n + 5}$ is absolutely convergent.

E. None of these.

Answer: E.

Question 2.1 : **[2 marks]**

$$1 + \frac{3}{100} + \frac{9}{100^2} + \frac{27}{100^3} + \cdots =$$

A. $\frac{100}{97}$.

B. $\frac{97}{100}$.

C. $\frac{3}{97}$.

D. $\frac{97}{3}$.

E. None of these.

Answer: A. $1 + \frac{3}{100} + \frac{9}{100^2} + \frac{27}{100^3} + \cdots = \sum_{n=0}^{\infty} \left(\frac{3}{100}\right)^n = \frac{1}{1 - \left(\frac{3}{100}\right)} = \frac{100}{97}$.

Question 2.2 : **[2 marks]**

$$1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \cdots =$$

A. $\frac{3}{2}$.

B. $\frac{2}{3}$.

C. $\frac{3}{5}$.

D. $\frac{5}{3}$.

E. None of these.

Answer: D. $1 + \frac{2}{5} + \frac{4}{25} + \frac{8}{125} + \cdots = \sum_{n=0}^{\infty} \left(\frac{2}{5}\right)^n = \frac{1}{1 - \left(\frac{2}{5}\right)} = \frac{5}{3}$.

Question 3.1 : **[2 marks]**

Which one of the statements A – D is false? If none, choose E.

A. The series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{(-1)^{n+1}n}{3n-2}\right)^{2n}$ converges.

B. The series $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)^n}{(n!)^2}$ converges.

C. The series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n}$ converges.

D. The series $\displaystyle\sum_{n=1}^{\infty}\frac{2^n+1}{3^n}$ converges.

E. None of these.

Answer: C. $\displaystyle\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n}=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{n}=e^{-1}\neq 0 \implies \sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n}$ diverges .

Question 3.2 : **[2 marks]**

Which one of the statements A – D is false? If none, choose E.

A. The series $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\left(e^2-\left(1+\frac{2}{n}\right)^{n}\right)$ converges.

B. The series $\displaystyle\sum_{n=1}^{\infty}\frac{(n+1)!3^n}{(2n)!}$ converges.

C. The series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)^{n}$ diverges.

D. The series $\displaystyle\sum_{n=1}^{\infty}\frac{3^n-1}{4^n}$ diverges.

E. None of these.

Answer: D. $\displaystyle\sum_{n=1}^{\infty}\frac{3^n-1}{4^n}=\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^{n}-\sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^{n}$, a combination of two convergent series.

Question 4.1 : **[4 marks]**

The radius R and the interval of convergence I of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(3x+1)^{n+1}}{2n+2}$ are:

A. $R = \dfrac{1}{3}$ and $I = \left(-\dfrac{2}{3}, 0\right)$.

B. $R = \dfrac{1}{3}$ and $I = \left(-\dfrac{2}{3}, 0\right]$.

C. $R = \dfrac{1}{3}$ and $I = \left[-\dfrac{2}{3}, 0\right]$.

D. $R = \dfrac{1}{3}$ and $I = \left[-\dfrac{2}{3}, 0\right)$.

E. None of these.

Answer: D. $\dfrac{(3x+1)^{n+1}}{2n+2} = \dfrac{3^{n+1}\left(x+\frac{1}{3}\right)^{n+1}}{2n+2} \implies a_n = \dfrac{3^{n+1}}{2n+2}$.

$R = \displaystyle\lim_{n\to\infty} \frac{3^{n+1}}{2n+2} \cdot \frac{2n+4}{3^{n+2}} = \lim_{n\to\infty} \frac{2n+4}{3(2n+2)} = \frac{1}{3}$ is the radius of convergence.

The series converges over $-\dfrac{1}{3} < x + \dfrac{1}{3} < \dfrac{1}{3}$, that is, $-\dfrac{2}{3} < x < 0$.

For $x = 0$, the series is the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+2}$, which diverges, and for $x = -\dfrac{2}{3}$, the series is the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+2}$, which converges. Hence the interval of convergence is $I = \left[-\dfrac{2}{3}, 0\right)$.

Question 4.2 : **[4 marks]**

The radius R and the interval of convergence I of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(5x+3)^{n+1}}{2n+1}$ are:

A. $R = \dfrac{1}{5}$ and $I = \left(-\dfrac{4}{5}, -\dfrac{2}{5}\right)$.

B. $R = \dfrac{1}{5}$ and $I = \left[-\dfrac{4}{5}, -\dfrac{2}{5}\right)$.

C. $R = \dfrac{1}{5}$ and $I = \left(-\dfrac{4}{5}, -\dfrac{2}{5}\right]$.

D. $R = \dfrac{1}{5}$ and $I = \left[-\dfrac{4}{5}, -\dfrac{2}{5}\right]$.

E. None of these.

Answer: B. $\dfrac{(5x+3)^{n+1}}{2n+1} = \dfrac{5^{n+1}\left(x+\frac{3}{5}\right)^{n+1}}{2n+1} \implies a_n = \dfrac{5^{n+1}}{2n+1}$.

$R = \displaystyle\lim_{n\to\infty} \frac{5^{n+1}}{2n+1} \cdot \frac{2n+3}{5^{n+2}} = \lim_{n\to\infty} \frac{2n+3}{5(2n+1)} = \frac{1}{5}$ is the radius of convergence.

The series converges over $-\dfrac{1}{5} < x + \dfrac{3}{5} < \dfrac{1}{5}$, that is, $-\dfrac{4}{5} < x < -\dfrac{2}{5}$.

For $x = -\dfrac{4}{5}$, the series is the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2n+1}$, which converges, and for $x = -\dfrac{2}{5}$, the series is the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+1}$, which diverges. Hence the interval of convergence is $I = \left[-\dfrac{4}{5}, -\dfrac{2}{5}\right)$.

Total: 10 marks