Math 104, Midterm Examination 1

Instructor: Guoliang Wu

Time: July 9, 2:30-4:00pm

1. (a) (5 points) Use the limit theorems to find the following limit. Justify each step.

$$\lim_{n\to\infty} \frac{\sqrt{n^4+1}+2n}{2n^2-1}$$

Solution:

$$\begin{aligned}\lim_{n\to\infty} \frac{\sqrt{n^4+1}+2n}{2n^2-1} &= \lim_{n\to\infty} \frac{\sqrt{1+1/n^4}+2/n}{2-1/n^2}\\ &= \frac{1+0}{2-1} = \frac{1}{2}.\end{aligned}$$

(b) (5 points) Find the following limit and justify your answer.

$$\lim_{n\to\infty} \left(\sqrt{n^2+2n}-n\right)$$

Solution:

$$\begin{aligned}\lim_{n\to\infty} \left(\sqrt{n^2+2n}-n\right) &= \lim_{n\to\infty} \frac{\left(\sqrt{n^2+2n}-n\right)\left(\sqrt{n^2+2n}+n\right)}{\sqrt{n^2+2n}+n}\\ &= \lim_{n\to\infty} \frac{2n}{\sqrt{n^2+2n}+n}\\ &= \lim_{n\to\infty} \frac{2}{\sqrt{1+2/n}+1}\\ &= 1.\end{aligned}$$

2. (5 points $\times$ 5) Are the following statements true or false? Just circle your answer. No proof needed.

(a) Given any real-valued[1] sequence (s_n), the sequences $\sup\{s_m : m \geq n\}$ and $\inf\{s_m : m \geq n\}$ are monotone.

Solution: True. $\sup\{s_m : m \geq n\}$ is decreasing and $\inf\{s_m : m \geq n\}$ is increasing. (See textbook or notes.)

[1]Meaning, each s_n is a real number.

(b) For any real numbers a, b, c, we always have $|a - b + c| \leq |a| + |b| + |c|$.

Solution: True. This is because of the triangle inequality. $|a - b + c| \leq |a| + |-b| + |c| = |a| + |b| + |c|$.

(c) If x_1 and x_2 are irrational, then $x_1 x_2$ is also irrational.

Solution: False. For example, $x_1 = \sqrt{2}, x_2 = -\sqrt{2}$. Both are irrational, but the sum is 0 which is rational.

(d) If (s_n) is a Cauchy sequence, then it is bounded.

Solution: True. This is a theorem in the textbook. If you don't remember, you should remember that any Cauchy sequence is convergent and every convergent sequence is bounded. (In fact, in the textbook, the first step to prove a Cauchy sequence is convergent is to show that it is bounded.)

(e) The Completeness Axiom says that every nonempty subset of $\mathbb{R}$ that is bounded above has a maximum.

Solution: False. The Completeness Axiom only says that every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum. But there are some bounded sets that do not have a maximum. For instance, the open interval $(0, 1)$.

3. (a) (5 points) State the Rational Zeros Theorem.

Solution: See textbook.

(b) (10 points) Show that $a = \sqrt{1 + \sqrt[3]{3}}$ is not a rational number.

Proof.

$$a^2 = 1 + \sqrt[3]{3}$$
$$(a^2 - 1)^3 = 3$$
$$a^6 - 3a^4 + 3a^2 - 4 = 0.$$

According to the Rational Zeros Theorem, if the above equation has a rational solution $\frac{p}{q}$, then $p|(-4)$ and $q|1$. Therefore, the possible rational solutions are $\pm 1, \pm 2, \pm 4$. However,

$$(\pm 1)^6 - 3(\pm 1)^4 + 3(\pm 1)^2 - 4 = 1 - 3 + 3 - 4 = -3 \neq 0$$
$$(\pm 2)^6 - 3(\pm 2)^4 + 3(\pm 2)^2 - 4 = 64 - 48 + 12 - 4 \neq 0$$
$$(\pm 4)^6 - 3(\pm 4)^4 + 3(\pm 4)^2 - 4 = 256 + 48 - 4 \neq 0$$

Thus, none of them is a solution. Therefore, the equation has no rational solution and hence a is not rational. $\square$

4. (10 points) Use the definition of limits of sequences to show that if
$$\lim_{m\to\infty} s_{2m-1} = \lim_{m\to\infty} s_{2m} = s,$$
then the sequence $(s_n)_{n=1}^{\infty}$ is convergent and
$$\lim_{n\to\infty} s_n = s.$$

Solution: Given any $\epsilon > 0$, since $\lim_{m\to\infty} s_{2m-1} = \lim_{m\to\infty} s_{2m} = s$, there exists $N_1 > 0$ and $N_2 > 0$ such that
$$|s_{2m-1} - s| < \epsilon, \quad \forall m > N_1,$$
$$|s_{2m} - s| < \epsilon, \quad \forall m > N_2.$$

Let $N = \max\{(N_1 + 1)/2, N_2/2\}$, then $2N - 1 > N_1$ and $2N > N_2$. Thus for any $n > N$,
(i) if n is odd, suppose $n = 2m_1 - 1$ for some m_1. Then $n = 2m_1 - 1 > N > 2N_1 - 1 \Rightarrow m_1 > N_1$. Thus
$$|s_n - s| = |s_{2m_1} - s| < \epsilon.$$
(ii) if n is even, suppose $n = 2m_2$ for some m_2. Then $n = 2m_2 > N > 2N_2 \Rightarrow m_2 > N_2$. Thus
$$|s_n - s| = |s_{2m_2} - s| < \epsilon.$$
Therefore, by definition,
$$\lim_{n\to\infty} s_n = s.$$

5. The sequence $(s_n)_{n=1}^{\infty}$ is given as following
$$s_1 = 1, \quad s_{n+1} = 1 + \frac{1}{1 + s_n}.$$

(a) (5 points) Write down the terms s_2, s_3, and s_4 (in fractions):

$$s_2 = \underline{\frac{3}{2}}, s_3 = \underline{\frac{7}{5}}, s_4 = \underline{\frac{17}{12}}.$$

Based on these terms, is the sequence (s_n) monotone?
Solution: No, it is not monotone. $\frac{3}{2} > \frac{7}{5}$ but $\frac{7}{5} < \frac{17}{12}$.

(b) (4 points) The sequence (s_n) is in fact bounded. (*No proof of this statement is required.*) Please give a lower bound and an upper bound of the sequence.

Lower bound: $\underline{0}$, upper bound: $\underline{2}$

(c) (8 points) Consider the subsequence $(s_{2m-1})_{m=1}^{\infty} = (s_1, s_3, s_5, \cdots)$. Use mathematical induction to show that this subsequence is increasing. *Hint: Express s_{2m+1} in terms of s_{2m-1} first.*

Proof. We need to show $s_{2m+1} \geq s_{2m-1}$ for all $m = 1, 2, \cdots$

$$s_{2m+1} = 1 + \frac{1}{1 + s_{2m}} = 1 + \frac{1}{1 + 1 + \frac{1}{1+s_{2m-1}}} = 1 + \frac{1}{2 + \frac{1}{1+s_{2m-1}}}.$$

(i) If $m = 1$, $s_1 = 1$, and $s_3 = 7/5$. Thus, $s_3 > s_1$.
(ii) Assume $s_{2m+1} \geq s_{2m-1}$. Note that

$$s_{2m+3} = 1 + \frac{1}{2 + \frac{1}{1+s_{2m+1}}},$$

and

$$s_{2m+1} = 1 + \frac{1}{2 + \frac{1}{1+s_{2m-1}}}. \tag{1}$$

We have $\frac{1}{1+s_{2m+1}} \leq \frac{1}{1+s_{2m-1}}$, thus $\frac{1}{2+\frac{1}{1+s_{2m+1}}} \geq \frac{1}{2+\frac{1}{1+s_{2m-1}}}$.
Therefore, we have $s_{2m+3} \geq s_{2m+1}$.
By induction, the sequence s_{2m-1} is increasing. $\square$

(d) (5 points) Based on Parts (b) and (c), we can conclude that the subsequence (s_{2m-1}) is convergent. (*No proof of this statement is required.*) Please find its limit.

Proof. Since the subsequence s_{2m-1} is bounded (since s_n is) and monotonic, it is convergent. Assuming its limit is s, then

$$\lim s_{2m+1} = \lim s_{2m-1} = s.$$

So taking the limit in Equation (1), we get

$$s = 1 + \frac{1}{2 + \frac{1}{1+s}}.$$

Solving this equation, we have obtain two roots $s = \pm\sqrt{2}$. But since $s_{2m-1} \geq 0$ for all m, the limit s is also nonnegative. Therefore,

$$s = \sqrt{2}.$$

□

(e) (3 points) A similar argument as in (c) and (d) will show that the subsequence (s_{2m}) converges to the same limit as (s_{2m-1}) does. (*No proof of this statement is required.*) Based on this, can we conclude that the sequence (s_n) is convergent? If yes, please circle 'Yes' and write its limit in the blank. If not, simply circle 'No'.

Solution: The answer is **Yes**, because of the result we just proved in Problem 4.

6. (15 points) Prove that if a sequence (s_n) has no convergent subsequence, then $\lim_{n\to\infty} |s_n| = +\infty$.

Proof. Suppose not, then $|s_n|$ does not diverge to $+\infty$. (Recall that $\lim |s_n| = +\infty$ means that for any $M > 0$, there exists $N > 0$ such that for any $n > N$, we have $|s_n| > M_0$. We need to reverse this statement.) So there exists $M_0 > 0$, such that for any $N > 0$ there is an $n_0 > N$ with

$$|s_{n_0}| \leq M_0.$$

Thus, we can take $n_1 > 1$, such that $|s_{n_1}| \leq M_0$. (i.e., let $N = 1$ in the above statement.) Then take $n_2 > n_1$ such that $|s_{n_2}| \leq M_0$. (i.e., let $N = n_1$.) Repeating this argument, we obtain a subsequence $(s_{n_k})_{k=1}^{\infty}$ of (s_n), where

$$|s_{n_k}| \leq M_0, \quad \forall n = 1, 2, \cdots$$

This implies that the subsequence (s_{n_k}) is bounded.

According to Bolzano-Weierstrass Theorem, (s_{n_k}) has a convergent subsequence. This convergent subsequence is also a subsequence of the original sequence (s_n). We have a contradiction.

Therefore,

$$\lim_{n\to\infty} |s_n| = +\infty.$$

$\square$