1.2 Bounded Sets, Suprema and Infima

Revision (Interval notation)

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$
- $(-\infty, b) = \{x \in \mathbb{R} : a < x < b\}$
- $(-\infty, b] = \{x \in \mathbb{R} : a < x \leq b\}$
- $(a, \infty) = \{x \in \mathbb{R} : a < x < b\}$
- $[a, \infty) = \{x \in \mathbb{R} : a \leq x < b\}$

Definition 1.1 (not unique, can be part of the set or not)

Let S be a nonempty subset of $\mathbb{R}$. Then

1. If there is a number $A \in \mathbb{R}$ such that $x \leq A$ for all $x \in S$, then A is said to be an *upper bound* of S, and S is said to be *bounded above*.
2. If there is a number $a \in \mathbb{R}$ such that $x \geq a$ for all $x \in S$, then a is said to be a *lower bound* of S, and S is said to be *bounded below*.
3. If S is bounded above and bounded below, then S is called *bounded*.

Example 1.1

Let $S = (-\infty, 1)$. Does S have upper and lower bounds?

Solution

➢ Upper bound (if exist, give and show that the definition is met)

Since $x < 1$ for all $x \in S$, 1 is an upper bound of S, and S is therefore bounded above.

But also 2 and 7.3 are upper bounds of S, for example.

➢ Lower bound (if not exist, proof by contradiction)

Assume that S has a lower bound m.

Then $m \leq 0$ since $0 \in S$, and $m - 1 < m \leq 0 < 1$ shows that $m - 1 \in S$.

So $m \leq m - 1$ since m is a lower bound of S.

But this is false, so that m cannot be a lower bound of S.

Therefore S has no lower bound, and S is therefore not bounded below. ■

Definition 1.2 (unique if it exists, must be part of the set)

Let S be a nonempty subset of $\mathbb{R}$.

1. If S has an upper bound M which is an element of S, then M is called the *greatest element* or *maximum* of S, and we write $M = \max S$.
2. If S has a lower bound m which is an element of S, then m is called the *least element* or *minimum* of S, and we write $m = \min S$.

From the definition, we immediately obtain the following proposition.

Proposition 1.1 (proof just use definitions)

Let S be a nonempty subset of $\mathbb{R}$. Then

1. $M = \max S \Leftrightarrow M \in S$ and $x \leq M$ for all $x \in S$,
2. $m = \min S \Leftrightarrow m \in S$ and $x \geq m$ for all $x \in S$.

We have already implicitly used in our notation that maximum and minimum are unique if they exist. The next result states this formally.

Proposition 1.2

Let S be a nonempty subset of $\mathbb{R}$. If the maximum or minimum of S exist, then they are unique.

Proof (assume there is two and prove they are equal)

- Maximum

Assume that S has maxima M_1 and M_2.

We must show $M_1 = M_2$.

Since $M_1 \in S$ and M_2 is an upper bound of S, we have $M_1 \leq M_2$.

Since $M_2 \in S$ and M_1 is an upper bound of S, we have $M_2 \leq M_1$.

Hence $M_1 = M_2$.

- Minimum

A similar proof holds for the minimum.

Assume that S has minima m_1 and m_2. We must show $m_1 = m_2$. Since $m_1 \in S$ and m_2 is a lower bound of S, we have $m_1 \geq m_2$. Since $m_2 \in S$ and m_1 is a lower bound of S, we have $m_2 \geq m_1$. Hence $m_1 = m_2$. ■

Example 1.2

Let $a, b \in \mathbb{R}$, $a < b$, and $S = [a, b)$. Then $\min S = a$, but S has no maximum.

Solution

- Minimum (if exist, give and show that the definition is met)

Clearly, $a \in S$ and $a \leq x$ for all $x \in S$, so that $a = \min S$.

- Maximum (if not exist, proof by contradiction)

Assume S has a maximum M.

Since $M \in S$, $a \leq M < b$.

Put $c = \frac{M+b}{2}$.

Then

$$a \leq M = \frac{2M}{2} = \frac{(1+1)M}{2} = \frac{M+M}{2} < \frac{M+b}{2} < \frac{b+b}{2} = b.$$

Therefore

$$a \leq M < c < b,$$

which shows $c \in S$ and $M < c$, contradicting that $M = \max S$ is an upper bound of S. ■

Definition 1.3 (unique if it exists, can be part of the set or not)

Let S be a nonempty subset of $\mathbb{R}$. A real number M is said to be the *supremum* or *least upper bound* of S, if

(a) M is an upper bound of S, and
(b) if L is any upper bound of S, then $M \leq L$.

The supremum of S is denoted by $\sup S$.

Definition 1.4 (unique if it exists, can be part of the set or not)

Let S be a nonempty subset of $\mathbb{R}$. A real number m is said to be the *infimum* or *greatest lower bound* of S, if

(a) m is a lower bound of S, and
(b) if l is any lower bound of S, then $m \geq l$.

The infimum of S is denoted by $\inf S$.

Note

Let S be a nonempty subset of $\mathbb{R}$.

1. By definition, if S has a supremum, then $\sup S$ is the minimum of the (nonempty) set of the upper bounds of S. Hence $\sup S$ is unique by Proposition 1.2.
2. Similarly, $\inf S$ is unique if it exists.

Proposition 1.3

Let S be a nonempty subset of $\mathbb{R}$.

1. If $\max S$ exists, then $\sup S$ exists, and $\sup S = \max S$.
2. If $\min S$ exists, then $\inf S$ exists, and $\inf S = \min S$.

Proof

- Part (1)

$\max S$ is an upper bound of S by definition.

Since $\max S \in S$, $\max S \leq L$ for any upper bound L of S.

Hence $\max S = \sup S$.

- Part (2)

The proof of (2.) is similar.

$\min S$ is a lower bound of S by definition. Since $\min S \in S$, $\min S \geq L$ for any lower bound L of S. Hence $\min S = \inf S$. ■

Example 1.3

Let $a < b$ and put $S = [a, b)$. Find $\inf S$ and $\sup S$ if they exist.

Solution

- Infimum (if exist, give and show that the definition is met)

By Proposition 1.3 and Example 1.2, $\inf S = \min S = a$ exists.

- Supremum (if exist, give and show that the definition is met)

From $x < b$ for all $x \in S$ we have that b is an upper bound of S.

If there were an an upper bound L of S with $L < b$, it would follow as in the solution to Example 1.2 that there is $c \in S$ with $L < c$, which contradicts the fact that L is an upper bound of S.

Hence $\sup S = b$. ∎

We now have the last axiom of the real number system.

C. The Dedekind completeness axiom

(C) Every nonempty subset of $\mathbb{R}$ which is bounded above has a supremum.

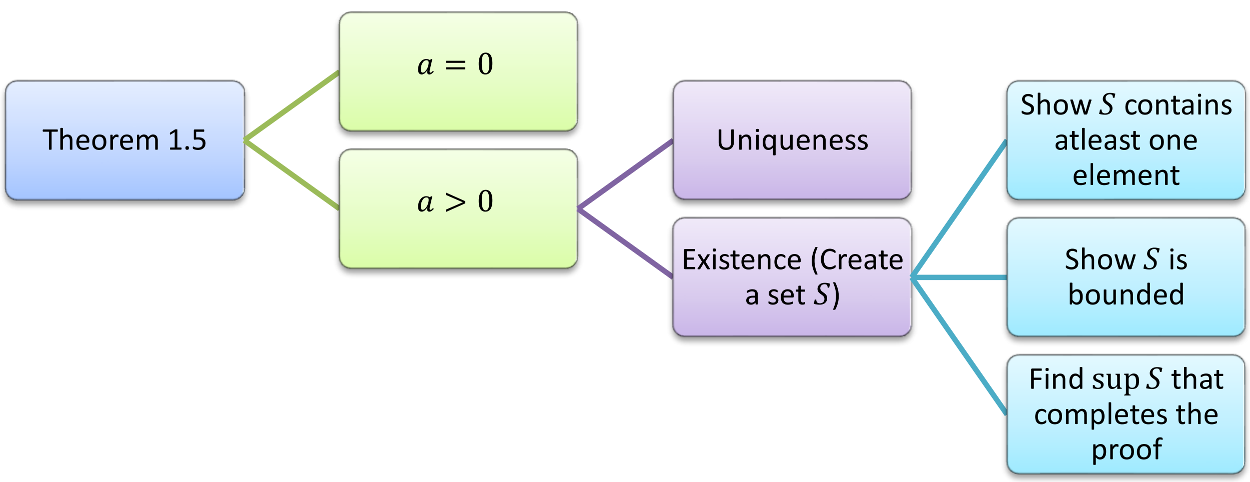
Additional Note

It can be shown that the real number system is uniquely determined by the axioms (A1)-(A4), (M1)-(M4), (D), (O1)-(O3) and (C).

Theorem 1.5 (Positive square root)

Let $a \geq 0$. Then there is a unique $x \geq 0$ such that $x^2 = a$. We write $x = \sqrt{a} = a^{1/2}$.

Skeleton of Proof (this helps to understand what we are going to do)



Proof

Case 1: $a = 0$

If $a = 0$, we have $0^2 = 0$ and $x^2 > 0$ if $x > 0$, so that $\sqrt{0} = 0$ is the unique number $x \geq 0$ such that $x^2 = 0$.

Case 2: $a > 0$

Now let $a > 0$. Note that $x \geq 0$ and $x^2 = a > 0$ gives $x > 0$.

(Uniqueness – assume there are two, prove they are equal)

For the uniqueness proof let $x_1 > 0$ and $x_2 > 0$ such that $x_1^2 = a$ and $x_2^2 = a$.

Then

$$0 = a - a = x_1^2 - x_2^2 = (x_1 + x_2)(x_1 - x_2).$$

From $x_1 + x_2 > 0$ it follows that $x_1 - x_2 = 0$, i.e., the uniqueness $x_1 = x_2$.

(Existence – create a bounded nonempty set that meet the requirements)

For the existence of the square root define

$$S_a = \{x \in \mathbb{R} : 0 < x, x^2 < a\}.$$

First we are going to show that $S_a \neq \emptyset$ and that S_a is bounded.

Part 1: $S_a \neq \emptyset$

- If $a < 1$, then $a^2 < a \cdot 1 = a$, so that $a \in S_a$. (pick $x = a$)
- If $a \geq 1$, then $\left(\frac{1}{2}\right)^2 < 1^2 = 1 \leq a$, so that $\frac{1}{2} \in S_a$. (pick $x = \frac{1}{2}$, can pick any that works)

Part 2: S_a is bounded

For $x > a + 1$ we have

$$x^2 > (a+1)^2 = a^2 + 2a + 1 > 2a > a,$$

so that any $x > a + 1$ does not belong to S_a.

Thus $x \leq a + 1$ for all $x \in S_a$.

We have shown that S_a is bounded above with upper bound $a + 1$.

(Show $\sup S_a = a$)

By the Dedekind completeness axiom there exists $M_a = \sup S_a$.

Note that $M_a > 0$ since S_a has positive elements.

To complete the proof we will show that $M_a^2 = a$.

By proof by contradiction, assume that $M_a^2 \neq a$.

Case I: $M_a^2 < a$.

Put

$$\varepsilon = \min\left\{\frac{a - M_a^2}{4M_a}, M_a\right\}.$$

Then $\varepsilon > 0$ and

$$\begin{aligned}(M_a + \varepsilon)^2 - a &= M_a^2 + 2M_a\varepsilon + \varepsilon^2 - a \\ &= M_a^2 - a + (2M_a + \varepsilon)\varepsilon \\ &\leq M_a^2 - a + 3M_a\varepsilon \\ &\leq M_a^2 - a + 3M_a \cdot \frac{a - M_a^2}{4M_a} \\ &= \frac{1}{4}(M_a^2 - a) \\ &< 0\end{aligned}$$

Thus

$$(M_a + \varepsilon)^2 < a$$

giving $M_a + \varepsilon \in S_a$, contradicting the fact that M_a is an upper bound of S_a.

Case II: $M_a^2 > a$.

Put

$$\varepsilon = \frac{M_a^2 - a}{2M_a}.$$

Then $0 < \varepsilon < \frac{1}{2}M_a$ and

$$\begin{aligned}(M_a - \varepsilon)^2 - a &= M_a^2 - 2M_a\varepsilon + \varepsilon^2 - a \\ &> M_a^2 - a - 2M_a\varepsilon \\ &= M_a^2 - a - 2M_a\frac{M_a^2 - a}{2M_a} \\ &= 0.\end{aligned}$$

Hence for all $x \geq M_a - \varepsilon > \frac{1}{2}M_a > 0$,

$$x^2 \geq (M_a - \varepsilon)^2 > a$$

so that any $x \geq M_a - \varepsilon$ does not belong to S_a. Hence $M_a - \varepsilon$ is an upper bound of S_a, contradicting the fact that M_a is the least upper bound of S_a.

So $M_a^2 \neq a$ is impossible, and $M_a^2 = a$ follows. ■

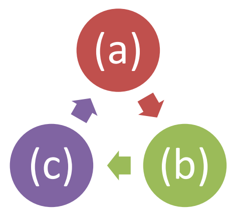
The completeness axiom can be thought of as ensuring that there are no ’gaps’ on the real line.

Theorem 1.6 (Characterizations of the supremum)

Let S be a nonempty subset of ℝ. Let M ∈ ℝ. The following are equivalent:

(a) $M = \sup S$
(b) M is an upper bound of S, and for each $\varepsilon > 0$, there is $s \in S$ such that $M - \varepsilon < s$
(c) M is an upper bound of S, and for each $x < M$, there exists $s \in S$ such that $x < s$.

Skeleton of Proof



Proof

(a) ⇒ (b): Since (a) holds, M is the least upper bound and thus an upper bound.

Let $\varepsilon > 0$.

Then $M - \varepsilon$ is not an upper bound of S since M is the least upper bound of S.

Hence $x \leq M - \varepsilon$ does not hold for all $x \in S$, and therefore there must be $s \in S$ such that $M - \varepsilon < s$.

(b) ⇒ (c): Let $x < M$.

Then $\varepsilon = M - x > 0$, and by assumption there is $s \in S$ with $M - \varepsilon < s$.

Then $x = M - \varepsilon < s$.

(c) ⇒ (a): By assumption, M is an upper bound of S.

Suppose that M is not the least upper bound.

Then there is an upper bound L of S with $L < M$.

By assumption (c), there is $s \in S$ such that $L < s$, contradicting that L is an upper bound of S.

So M must be indeed the least upper bound of S. ■

There is an apparent *asymmetry* in the Dedekind completion. However, there is a version for infima, which is obtained by reflection. To this end, if S is a (nonempty) subset of $\mathbb{R}$ set

$$-S = \{-x : x \in S\}.$$

Note that because of $-(-x) = x$ this can also can be written as

$$-S = \{x \in \mathbb{R} : -x \in S\}.$$

Since $x \leq y \Leftrightarrow -x \geq -y$ it is easy to see that the following properties hold:

Proposition 1.4

Let S be a nonempty subset of $\mathbb{R}$. Then

(a) $-S$ is bounded below if and only if S is bounded above, $\inf(-S) = -\sup S$, and if $\max S$ exists, then $\min(-S)$ exists and $\min(-S) = -\max S$.
(b) $-S$ is bounded above if and only if S is bounded below, $\sup(-S) = -\inf S$, and if $\min S$ exists, then $\max(-S)$ exists and $\max(-S) = -\min S$.
(c) $-S$ is bounded if and only if S is bounded.

Thus the Dedekind completeness axiom immediately gives:

Theorem 1.7

Every nonempty subset of $\mathbb{R}$ which is bounded below has an infimum.

Theorem 1.8 (Characterizations of the infimum)

Let S be a nonempty subset of $\mathbb{R}$. Let $m \in \mathbb{R}$. The following are equivalent:

(a) $m = \inf S$
(b) m is a lower bound of S, and for each $\varepsilon > 0$, there is $s \in S$ such that $s < m + \varepsilon$
(c) m is a lower bound of S, and for each $x > m$, there exists $s \in S$ such that $s < x$.

Theorem 1.9 (Dedekind cut)

Let A and B be nonempty subsets of $\mathbb{R}$ such that

i. $A \cap B = \emptyset$,
ii. $A \cup B = \mathbb{R}$,
iii. $\forall a \in A, \forall b \in B, a \leq b$.

Then there is $c \in \mathbb{R}$ such that $a \leq c \leq b$ for all $a \in A$ and $b \in B$.

Proof

A is nonempty and bounded above (any $b \in B$ is an upper bound of A), so $c = \sup A$ exists by the Dedekind completeness axiom, and $c \geq a$ for all $a \in A$ by definition of upper bound.

(iii) says that each $b \in B$ is an upper bound of A, and hence $c \leq b$ since c is the least upper bound of A. ∎

Note

1. The above theorem says that Dedekind completeness axiom implies the Dedekind cut property. Conversely, if the ordered fields axioms and the Dedekind cut property are satisfied, then the Dedekind completeness axiom holds (see Tutorial 1.2)
2. One can show that if $S \subset \mathbb{Z}$, $S \neq \emptyset$, and S is bounded below, then S has a minimum.

Revision

1. Number Systems
   - Natural numbers $\mathbb{N} = \{0,1,2,3, \dots\}$
   - Integers $\mathbb{Z} = \{x \in \mathbb{R} : x \in \mathbb{N} \; or -x \in \mathbb{N}\}$
   - Rational numbers $\mathbb{Q} = \left\{x \in \mathbb{R} : x = \frac{p}{q}, p, q \in \mathbb{Z}, q \neq 0\right\}$
   - Irrational numbers $\mathbb{Q}' = \{x \in \mathbb{R} : x \notin \mathbb{Q}\} = \mathbb{R} \backslash \mathbb{Q}$

2. Well-ordering principle (taught in TAM, proof not needed)

   Every nonempty subset of $\mathbb{N}$ *has a smallest element.*

Theorem 1.10 (The Archimedean principle)

For each $x \in \mathbb{R}$ there is $n \in \mathbb{N}$ such that $n > x$.

Proof (by contradiction)

Assume the Archimedean principle is false.

Then there is $x \in \mathbb{R}$ such that $n \leq x$ for all $n \in \mathbb{N}$.

That means, $\mathbb{N}$ is bounded above and therefore has a supremum M.

By Theorem 1.6 there is $n \in \mathbb{N}$ such that $M - 1 < n$.

Then $n + 1 > (M - 1) + 1 = M$

and $n + 1 \in \mathbb{N}$ contradict the fact that M is an upper bound of $\mathbb{N}$.

Hence the Archimedean principle must be true. ∎

Definition 1.5

A subset S of $\mathbb{R}$ is said to be *dense* in $\mathbb{R}$ if for all $x, y \in \mathbb{R}$ with $x < y$ there is $s \in S$ such that $x < s < y$.

Real numbers which are not rational numbers are called irrational numbers.

Note

1. $\sqrt{2}$ is irrational (see tutorials).
2. $\mathbb{Q}$ is an ordered field, i.e., $\mathbb{Q}$ satisfies all axioms of $\mathbb{R}$ except the Dedekind completeness axiom (see tutorials).
3. If a is rational and b is irrational, then $a + b$ is irrational (see tutorials).

Theorem 1.11

The set of rational numbers as well as the set of irrational numbers are dense in $\mathbb{R}$.

Proof

Let $x, y \in \mathbb{R}$, $x < y$.

By the Archimedean principle, there is a natural number n_0 such that

$$n_0 > \frac{2}{y - x} > 0.$$

Let $S = \{n \in \mathbb{Z} : n > n_0 x\}$.

By the Archimedean principle, $S \neq \emptyset$.

Also, S is bounded below.

So S has a minimum m.

Then $m - 1 \notin S$, so that $m - 1 \leq n_0 x$.

From $1 < 2 < 4$ it follows that $1 < \sqrt{2} < 2$. Then

x	$= \frac{n_0 x}{n_0}$	multiply with 1
	$< \frac{m}{n_0}$	$\min S = m$
	$< \frac{m-1+\sqrt{2}}{n_0}$	$1 < \sqrt{2} < 2$

$< \frac{m+1}{n_0}$	$\sqrt{2} - 1 < 1$
$\leq \frac{n_0 x + 2}{n_0}$	$m - 1 \leq n_0 x$
$= x + \frac{2}{n_0}$	divide with n_0
$< x + 2 \cdot \frac{y-x}{2}$	$n_0 > \frac{2}{y-x}$
$= y.$	simplify

Then

$$u = \frac{m}{n_0} \in \mathbb{Q}$$

$$v = \frac{m - 1 + \sqrt{2}}{n_0} \in \mathbb{R} \backslash \mathbb{Q}$$

and, by above, $x < u < v < y$. ■

Example 1.4

Let $S = \{x \in \mathbb{Q} : x < \sqrt{2}\}$ and $T = \{x \in \mathbb{Q} : x \leq \sqrt{2}\}$. Then $S = T$, and S and T have no supremum in $\mathbb{Q}$.

Note that $\sqrt{2} \notin T$ in $\mathbb{Q}$ and so $S = T$. ■

Example 1.5 (Bernoulli's inequality)

For all $x \in \mathbb{R}$ with $x \geq -1$ and all $n \in \mathbb{N}$,

$$(1 + x)^n \geq 1 + nx.$$

Proof

We proceed by induction on $n \in \mathbb{N}$.

- For $n = 0$,

$$LHS = (1 + x)^n = (1 + x)^0 = 1$$

$$RHS = 1 + nx = 1 + 0 \cdot x = 1$$

$$\therefore LHS = RHS.$$

- Let $n \in \mathbb{N}$ and assume the Bernoulli inequality holds for this n.
- Then,

$$\begin{aligned}(1+x)^{n+1} &= (1+x)(1+x)^n \\ &\geq (1+x)(1+xn) \\ &= 1+x+nx+nx^2 \\ &= 1+(n+1)x+nx^2 \\ &\geq 1+(n+1)x.\end{aligned}$$

By the principle of induction, the Bernoulli inequality holds for all $n \in \mathbb{N}$. ■

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