Section 4.3: The Intermediate Value Theorem

The following important theorem on continuous functions tells us that the graph of a continuous function cannot jump from one side of a horizontal line $y = k$ to the other without intersecting the line at least once.

Theorem 4.12 (Intermediate Value Theorem (IVT))

Suppose that f is continuous on the closed interval $[a, b]$ with $f(a) \neq f(b)$. Then for any number k between $f(a)$ and $f(b)$ there exists a number c in the open interval (a, b) such that $f(c) = k$.

Proof

Let

$$g(x) = f(x) - k \qquad\qquad x \in [a, b].$$

Then g is continuous, and 0 lies between $g(a)$ and $g(b)$, that is, $g(a)$ and $g(b)$ have opposite signs: $g(a) \cdot g(b) < 0$.

Let $[a_0, b_0] = [a, b]$ and use bisection to define intervals $[a_n, b_n]$ as follows: If $[a_n, b_n]$ with $g(a_n) \cdot g(b_n) < 0$ has been found, let d be the midpoint of the interval $[a_n, b_n]$.

- If $g(d) = 0$, the result follows with $c = d$.
- If $g(d)$ has the same sign as $g(b_n)$, then $g(a_n)$ and $g(d)$ have opposite signs, and putting $a_{n+1} = a_n$, $b_{n+1} = d$, we have $g(a_{n+1}) \cdot g(b_{n+1}) < 0$.
- Otherwise, if $g(d)$ has the opposite sign to $g(b_n)$, we put $a_{n+1} = d$, $b_{n+1} = b_n$ and get again $g(a_{n+1}) \cdot g(b_{n+1}) < 0$.

If this procedure does not stop, we obtain an increasing sequence (a_n) and a decreasing sequence (b_n), both of which converge by Theorem 2.9. We observe that

$$b_n = a_n + \frac{1}{2}(b_{n-1} - a_{n-1}) = a_n + 2^{-n}(b - a).$$

Then

$$c = \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} 2^{-n}(b - a) = \lim_{n\to\infty} a_n.$$

Since $a \leq c \leq b$ and g is continuous at c, it follows in view of Theorem 4.11 that

$$[g(c)]^2 = \lim_{n\to\infty} g(a_n) \cdot \lim_{n\to\infty} g(b_n) = \lim_{n\to\infty} g(a_n) \cdot g(b_n) \leq 0.$$

Therefore $g(c) = 0$ (since a squared number less or equal to zero can only be zero), which gives $f(c) = k$. Since $f(a) \neq k$ and $f(b) \neq k$, it follows that $c \neq a$ and $c \neq b$, so that $a < c < b$. ■

Note: You have seen the definition of an interval in first year and you will recall that the definition required several cases, depending on whether the endpoints belong to the interval or not and whether the interval is bounded (above, below), namely

$$(a,b),\ [a,b),\ (a,b],\ [a,b],\ (-\infty,b),\ (-\infty,b],\ (a,\infty),\ [a,\infty),\ (-\infty,\infty)$$

where $a,b \in \mathbb{R}$ and $a < b$. However, intervals can be characterized by one common property. For this we need the following notion:

A sunset S of $\mathbb{R}$ is called a singleton if the set S has exactly one element.

Definition 4.10

1. A set $S \subset \mathbb{R}$ is called an interval if
   i. $S \neq \emptyset$,
   ii. S is not a singleton,
   iii. If $x, y \in S$, $x < y$, then each $z \in \mathbb{R}$ with $x < z < y$ satisfies $z \in S$.
2. An interval of the form $[a,b]$ with $a < b$ is called a closed bounded interval.

Note: A subset S of $\mathbb{R}$ is an interval if and only if it contains at least two elements and if all real numbers between any two elements in S also belong to S.

Definition 4.11

For a function $f: X \to Y$ and $A \subset X$, the set

$$f(A) = \{y \in Y :\ \exists x \in A \cap dom(f), f(x) = y\} = \{f(x) :\ x \in A \cap dom(f)\}$$

is called the image of A under f.

Corollary 4.3

Let I be an interval and let f be a continuous real function on I. Then $f(I)$ is either an interval or a singleton.

Proof

Since $I \neq \emptyset$, there is $x \in I$ and so $f(x) \in f(I)$. Hence $f(I) \neq \emptyset$.

Hence we have to show that if $f(I)$ is not a singleton, then it is an interval, that is, we must show that for any $x, y \in I$ with $f(x) < f(y)$ and any $k \in \mathbb{R}$ with $f(x) < k < f(y)$ there is $c \in I$ such that $f(c) = k$.

Indeed, from $f(x) \neq f(y)$, we have $x \neq y$. If $x < y$, then f is continuous on $[x,y]$, and by the intermediate value theorem, there is $c \in (x,y)$ with $f(c) = k$. A similar argument holds for $x > y$. ■

Example 4.9

Let $f(x) = x^2$. Then $f\big((-1,2)\big) = [0,4)$. Notice that $I = (-1,2)$ is an open interval, while $f(I)$ is not. (Draw the parabola for $x \in (-1,2)$, then you will see $y \in [0,4)$.)

Theorem 4.13

Let f be a real function which is continuous on $[a, b]$, where $a < b$. Then f is bounded on $[a, b]$, i.e., $f([a, b])$ is bounded.

Proof

Assume that $f([a, b])$ is unbounded. Let d be the midpoint of the interval $[a, b]$. Then at least one of the sets $f([a, d])$, $f([d, b])$ would be unbounded, because otherwise

$$f([a, b]) = f([a, d]) \cup f([d, b])$$

would be bounded. By induction, we find subintervals $[a_n, b_n]$ of $[a, b]$ such that (a_n) is increasing, (b_n) is decreasing, $f([a_n, b_n])$ is unbounded, and

$$b_n = a_n + \frac{1}{2}(b_{n-1} - a_{n-1}) = 2^{-n}(b - a),$$

see the proof of Theorem 4.12, and we infer that both sequences converge with

$$c := \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n \in [a, b].$$

Since f is continuous at c, there is $\delta > 0$ such that

$$f\big((c - \delta, c + \delta) \cap [a, b]\big) \subset (f(c) - 1, f(c) + 1).$$

Since $c = \lim\limits_{n\to\infty} b_n$, there is $K \in \mathbb{N}$ such that $|b_n - c| < \delta$ for $n > K$. Similarly, there is $M \in \mathbb{N}$ such that $|a_n - c| < \delta$ for $n > M$. Let $n > \max\{K, M\}$. Then

$$c - \delta < a_n \le c \le b_n < c + \delta,$$

and

$$f([a_n, b_n]) \subset f\big((c - \delta, c + \delta) \cap [a, b]\big) \subset (f(c) - 1, f(c) + 1)$$

would give the contradiction that $f([a_n, b_n])$ would have to be bounded as well as unbounded. ■

Theorem 4.14

A continuous function on a closed bounded interval achieves its supremum and infimum.

Proof

Let $[a, b]$ be a closed bounded interval and f be a continuous function on $[a, b]$. We must show that there are $x_1, x_2 \in [a, b]$ such that $f(x_1) = \inf f([a, b])$ and $f(x_2) = \sup f([a, b])$. We are going to show the latter; the proof of the first statement is similar.

By Theorem 4.13, $S = f([a, b])$ is bounded. Let $M = \sup S$. By proof of contradiction, assume that $f(x) \neq M$ for all $x \in [a, b]$. Define

$$g(x) = \frac{1}{M - f(x)}, \quad x \in [a, b].$$

By Theorem 4.10, g is continuous on $[a, b]$, and by Theorem 4.13, $g([a, b])$ is bounded. So there is $K \in \mathbb{R}$ such that $0 < g(x) \leq K$ for all $x \in [a, b]$. Hence for all $x \in [a, b]$:

$$\frac{1}{K} \leq \frac{1}{g(x)} = M - f(x) \Rightarrow f(x) \leq M - \frac{1}{K},$$

so that the number $M - \frac{1}{K} < M$ would be an upper bound of $f([a, b])$. This contradicts the fact that $M = \sup f([a, b])$. ■

Corollary 4.4

If f is continuous on $[a, b]$, $a < b$, then either $f([a, b])$ is a singleton or $f([a, b]) = [c, d]$ with $c < d$.

Proof

From Corollary 4.3 and Theorems 4.13 and 4.14 it follows that $f([a, b])$ is either a singleton or a bounded interval which contains both its infimum and supremum. But such an interval is of the form $[c, d]$ with $c < d$. ■

Theorem 4.15

Let I be an interval and $f: I \to \mathbb{R}$ be a strictly monotonic continuous function. Then $f(I)$ is an interval, and the inverse function $f^{-1}: f(I) \to \mathbb{R}$ is continuous.

Proof

By Corollary 4.3, $f(I)$ is an interval. Assume that f is strictly increasing. Then also f^{-1} is strictly increasing. Let $b \in f(I)$, i.e., $b = f(a)$ for some $a \in I$. If a is not the left endpoint of I, then b is not the left endpoint of $f(I)$, and for $y \in f(I)$ with $y < b = f(a)$ we have

$$f^{-1}(y) < f^{-1}\big(f(a)\big) = a.$$

Therefore, f^{-1} is bounded above and increasing on $f(I) \cap (-\infty, b)$, and thus

$$\alpha = \lim_{y \to b^-} f^{-1}(y)$$

exists and $\alpha \leq f^{-1}(b) = a$, see Theorem 4.6. Assume that $\alpha < a$. Since $a \in I$ is not the left endpoint of I, $I \cap (\alpha, a) \neq \emptyset$. Let $x_0 \in I \cap (\alpha, a)$. Then $f(x_0) < f(a) = b$ and hence

$$x_0 = f^{-1}\big(f(x_0)\big) \leq \alpha.$$

This gives the contradiction $x_0 > \alpha$ and $x_0 \leq \alpha$. Therefore it follows that $\alpha = a$, which gives

$$\lim_{y \to b^-} f^{-1}(y) = a = f^{-1}(b).$$

Hence f^{-1} is continuous from the left. Similarly, one can show that f^{-1} is continuous from the right. Therefore f^{-1} is continuous. The case f strictly decreasing is similar. ■

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Tutorial 4.3

1. Prove that a subset S of $\mathbb{R}$ is an interval if and only if S has the form

   $$(a,b),\ [a,b),\ (a,b],\ [a,b],\ (-\infty,b),\ (-\infty,b],\ (a,\infty),\ [a,\infty) \text{ or } (-\infty,\infty)$$

   where $a,b \in \mathbb{R}$ and $a < b$.

   Hint: Consider $\inf S$ and $\sup S$.

2. Let f be a real function and let $\emptyset \neq A \subset B \subset dom(f)$ such that for each $x \in A$ there is $\varepsilon > 0$ such that $(x-\varepsilon, x] \subset A$ or $[x, x+\varepsilon) \subset A$ or $(x-\varepsilon, x+\varepsilon) \subset A$, and the same property for B. Show that if f is continuous on B, then f is also continuous on A.

3. **A fixed point theorem**: Let $a < b$ and let f be a continuous function on $[a,b]$ such that $f([a,b]) \subset [a,b]$. Show that there is $x \in [a,b]$ such that $f(x) = x$.

4. Let I be an interval and f be a continuous function on I such that $f(I)$ is unbounded. What can you say about $f(I)$? Find examples which illustrate your answer.

5. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function which only assumes rational values. Show that f is constant.

6. Find a continuous function $f\colon [-1,1] \to \mathbb{R}$ which is one-to-one when restricted to rational numbers in $[-1,1]$ but which is not one-to-one on the whole interval $[-1,1]$.

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