Multivariable Calculus

MATH2007

1.2 Vector Analysis
(Part 1)

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Definition (1.2.1. Gradient)**.**

nabla

Let $f : \mathbb{R}^n \to \underline{\mathbb{R}}$, we define the **gradient** of f, denoted either by grad$f$ or by ∇f, by

scalar valued

$$\text{grad} f = \nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}.$$

$$f' = \left(\frac{\partial f}{\partial x_1} \; \frac{\partial f}{\partial x_2} \; \cdots \; \frac{\partial f}{\partial x_n} \right) \qquad \nabla f = (f')^T \qquad f' = (\nabla f)^T$$

Example.

(a) Let $f(x_1, x_2) = x_2 e^{x_1}$ then $\nabla f(x_1, x_2) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \end{pmatrix} = \begin{pmatrix} x_2 e^{x_1} \\ e^{x_1} \end{pmatrix}.$

(b) Let $f(x, y, z) = (x^2 - y^2)e^z$. Find $\nabla f(\underbrace{2}_{x=2}, \underbrace{1}_{y=1}, \underbrace{-1}_{z=-1})$.

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} = \begin{pmatrix} 2x e^z \\ -2y e^z \\ (x^2 - y^2) e^z \end{pmatrix} \qquad \nabla f(2, 1, -1) = \begin{pmatrix} 4e^{-1} \\ -2e^{-1} \\ 3e^{-1} \end{pmatrix}.$$

Note. As we will prove later if $f : \mathbb{R}^n \to \mathbb{R}$ then:

(a) $\nabla f(\underline{x})$ is the direction we must move from $\underline{x}$ for f to increase fastest and $\|\nabla f(\underline{x})\|$ is the rate of increase at f as one moves in the direction of $\nabla f(\underline{x})$ from $\underline{x}$.

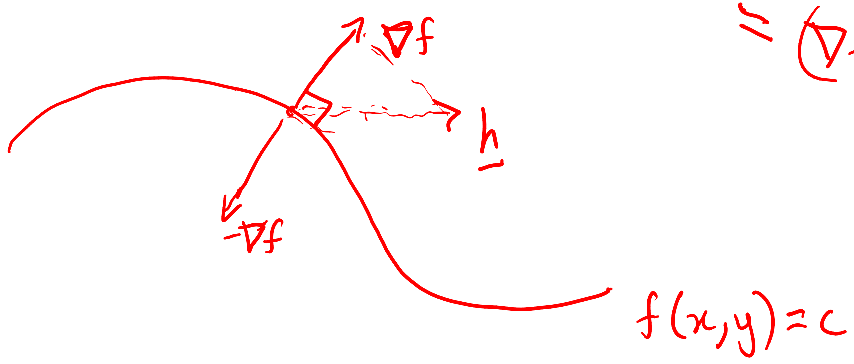
(b) If we consider the implicit curve given by $f(x, y) = c$ where c is a constant, let (x_0, y_0) be a point on this curve, then $\nabla f(x_0, y_0)$ is a normal to the curve at (x_0, y_0).

(c) If we consider the implicit surface given by $f(x, y, z) = c$. Let (x_0, y_0, z_0) be a point on this surface, then $\nabla f(x_0, y_0, z_0)$ is a normal to this surface at (x_0, y_0, z_0).

(d) Special case: For $f : \mathbb{R}^n \to \mathbb{R}$ and $h \in \mathbb{R}^n$ we have

$$df[\underline{a}; \underline{h}] = f'(\underline{a})h = \nabla f(\underline{a}) \cdot \underline{h}.$$

$$= (\nabla f(\underline{a}))^T \underline{h}$$



Note. We can think symbolically of $\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix}$ then

$$\nabla f = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} \neq \begin{pmatrix} \frac{\partial}{\partial x_1} f \\ \vdots \\ \frac{\partial}{\partial x_n} f \end{pmatrix}$$

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1.2 Vector Analysis
(Part 2)

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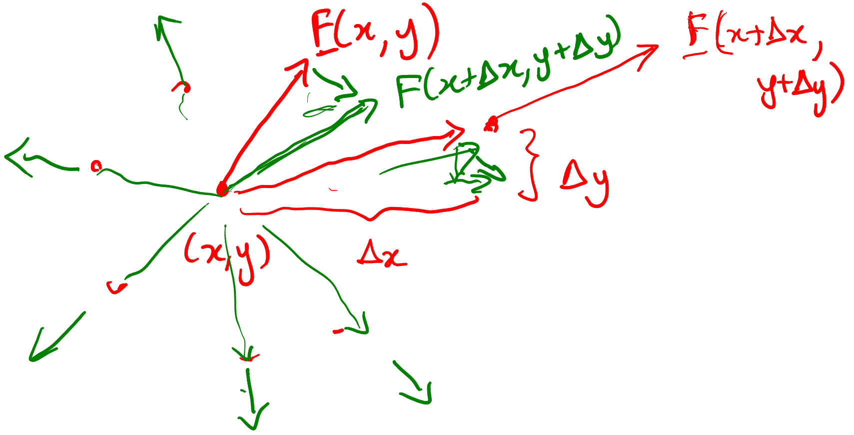
Definition (1.2.2. Divergence)**.**
Let $\underline{F} : \mathbb{R}^n \to \mathbb{R}^n$ we define the **divergence** of $\underline{F}$, denoted div$\underline{F}$ or $\nabla \cdot \underline{F}$, by

vector valued

$$\text{div } \underline{F} = \nabla \cdot \underline{F} = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}.$$

$\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3 \qquad \text{div } \underline{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2} + \frac{\partial F_3}{\partial x_3}$

$\underline{F} : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \text{div } \underline{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}$

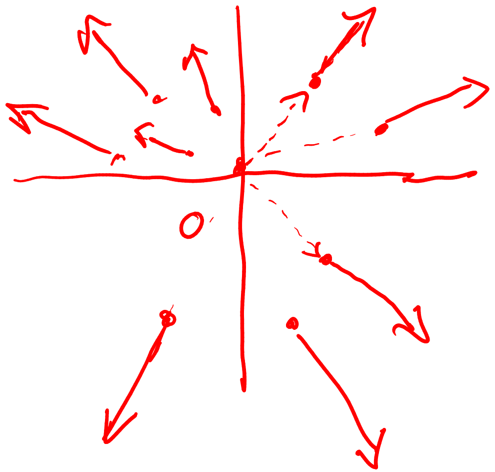


$\underline{F}$: velocity of fluid:

div $\underline{F} > 0$ "source" (more out than in)

div $\underline{F} < 0$ "sink" (more in than out)

Example. (a) Let $\underline{F}(x_1, x_2) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ i.e $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$.



$$\operatorname{div} \underline{F}(x_1, x_2) = \nabla \cdot \underline{F}(x_1, x_2)$$

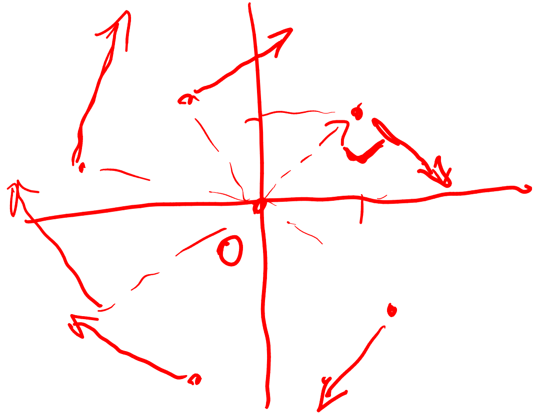
$$= \frac{\partial}{\partial x_1} F_1 + \frac{\partial}{\partial x_2} F_2$$

$$= \frac{\partial x_1}{\partial x_1} + \frac{\partial x_2}{\partial x_2}$$

$$= 2$$

(positive, "outflow" "source")

(b) Let $\underline{F}(x_1, x_2) = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}$ i.e $\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$.



$$\operatorname{div} \underset{\sim}{F} = \frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}$$

$$= \frac{\partial x_2}{\partial x_1} + \frac{\partial}{\partial x_2}(-x_1)$$

$$= 0 + 0$$

$$= 0$$

(no "inflow"/"outflow")

Note. (a) $\nabla \cdot \underline{F}(\underline{x})$ gives a measure of the amount at a fluid being created $(\nabla \cdot \underline{F} > 0)$ or destroyed $(\nabla \cdot \underline{F} < 0)$ per unit area at $\underline{x}$, as indicated in the two examples.

(b) Again, symbolically we can think of

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \qquad \text{and} \qquad \nabla \cdot \underline{F} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \vdots \\ \frac{\partial}{\partial x_n} \end{pmatrix} \cdot \begin{pmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{pmatrix} = \sum_{j=1}^{n} \frac{\partial F_j}{\partial x_j}.$$

Definition (1.2.3. Laplacian, harmonic)**.**
Let $f : \mathbb{R}^n \to \mathbb{R}$, we define the **Laplacian** of f, denoted $\nabla^2 f$, by

Scalar valued

$$\nabla^2 f = \nabla \cdot \nabla f = \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j^2}. \qquad (\text{ scalar})$$

We say that f is **harmonic** on a set Ω if $\nabla^2 f \equiv 0$ for all $\underline{x} \in \Omega$.

Example. (a) Let $f(x, y) = x + 2y + e^x \cos y$. Is f harmonic?

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial x}(1 + e^x \cos y) + \frac{\partial}{\partial y}(2 - e^x \sin y)$$

$$= e^x \cos y - e^x \cos y$$

$$= 0. \qquad \therefore\ f \text{ is harmonic.}$$

(b) Let $f(x, y, z) = x + xz - e^y$. Is f harmonic?

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial x}(1 + z) + \frac{\partial}{\partial y}(-e^y) + \frac{\partial}{\partial z}(x)$$

$$= 0 - e^y + 0$$

$$= -e^y \neq 0 \qquad \therefore\ f \text{ is not harmonic.}$$

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1.2 Vector Analysis
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Definition (1.2.4. Curl)**.**
Let $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$, we define the **curl** of $\underline{F}$, denoted $\operatorname{curl}\underline{F}$ or $\nabla \times \underline{F}$, by

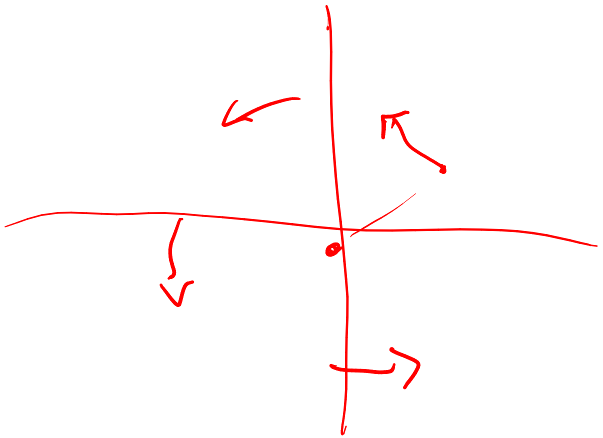
$$\operatorname{curl}\underline{F} = \nabla \times \underline{F} = \begin{bmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{bmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \underline{i}\left(\frac{\partial}{\partial x_2} F_3 - \frac{\partial}{\partial x_3} F_2\right)$$

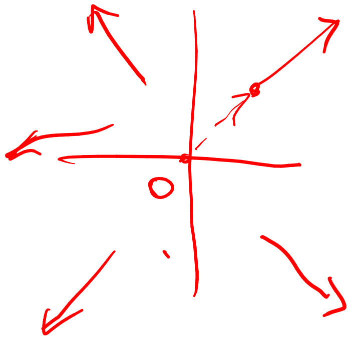
$$+ \underline{j}\left(\frac{\partial}{\partial x_3} F_1 - \frac{\partial}{\partial x_1} F_3\right)$$

$$+ \underline{k}\left(\frac{\partial}{\partial x_1} F_2 - \frac{\partial}{\partial x_2} F_1\right).$$

Note. $\nabla \times \underline{F}$ gives a measure of the local rotation of a fluid. (To make this concept rigorous we need Green's Theorem and Stoke's Theorem)



Example. (a) Let $\underline{F}(x_1, x_2, x_3) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ i.e $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$.



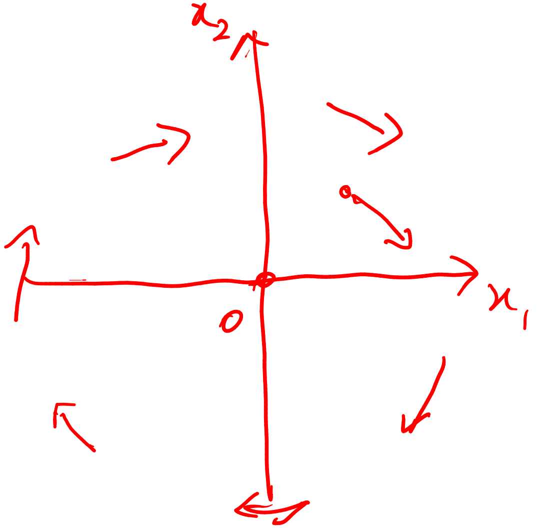
$$\operatorname{curl} \underset{\sim}{F} = \nabla \times \underline{F}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underset{\sim}{k} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_1 & x_2 & x_3 \end{vmatrix} \begin{matrix} \underline{i} & \underline{j} \\ \partial/\partial x_1 & \partial/\partial x_2 \\ x_1 & x_2 \end{matrix}$$

$$= \underline{i}\left(\frac{\partial}{\partial x_2} x_3 - \frac{\partial}{\partial x_3} x_2\right) + \underline{j}\left(\frac{\partial}{\partial x_3} x_1 - \frac{\partial}{\partial x_1} x_3\right) + \underline{k}\left(\frac{\partial}{\partial x_1} x_2 - \frac{\partial}{\partial x_2} x_1\right)$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(b) Let $\underline{F}(x_1, x_2, x_3) = \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix}$.



$$\operatorname{curl} \underline{F} = \nabla \times \underline{F}$$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 \\ x_2 & -x_1 & 0 \end{vmatrix}$$

$$= \underline{i}\left(\frac{\partial}{\partial x_2} 0 - \frac{\partial}{\partial x_3}(-x_1)\right) + \underline{j}\left(\frac{\partial}{\partial x_3} x_2 - \frac{\partial}{\partial x_1} 0\right) + \underline{k}\left(\frac{\partial}{\partial x_1}(-x_1) - \frac{\partial}{\partial x_2} x_2\right)$$

$$= \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$$

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1.2 Vector Analysis
(Part 4)

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Theorem (1.2.5)**.**

For $a, b \in \mathbb{R}$, $f, g : \mathbb{R}^n \to \mathbb{R}$ and $\underline{F}, \underline{G} : \mathbb{R}^n \to \mathbb{R}^n$ we have: $\nabla(af(\underline{x}) + bg(\underline{x})) = a\nabla f(\underline{x}) + b\nabla g(\underline{x})$

✓(a) $\nabla(af + bg) = a\nabla f + b\nabla g$ (∇ is linear)

✓(b) $\nabla(fg) = g\nabla f + f\nabla g$ (∇ product rule)

(c) $\nabla \cdot (a\underline{F} + b\underline{G}) = a\nabla \cdot \underline{F} + b\nabla \cdot \underline{G}$ (div is linear)

(d) $\nabla \cdot (g\underline{F}) = (g\nabla) \cdot \underline{F} + (\nabla g) \cdot \underline{F}$ (div product rule)

$g(\nabla \cdot \underline{F})$

and if $n = 3$ we in addition have:

(e) $\nabla \times (a\underline{F} + b\underline{G}) = a\nabla \times \underline{F} + b\nabla \times \underline{G}$ (curl is linear)

(f) $\nabla \times (g\underline{F}) = (\nabla g) \times \underline{F} + g\nabla \times \underline{F}$. (curl product rule)

(g) $\nabla \cdot (\underline{F} \times \underline{G}) = (\nabla \times \underline{F}) \cdot \underline{G} - (\nabla \times \underline{G}) \cdot \underline{F}$; (div → curl product rule)

Proof. Exercise! $\square$

Example. Let $g, f : \mathbb{R}^n \to \mathbb{R}$.

$\nabla f : \mathbb{R}^n \to \mathbb{R}^n$

(a) Show that $\nabla \cdot (g \nabla f) = \nabla g \cdot \nabla f + g \nabla^2 f$.

Let $\underline{F} = \nabla f$.

$$\nabla \cdot (g\,\underline{F}) = g\,(\nabla \cdot \underline{F}) + (\nabla g) \cdot \underline{F} \qquad \text{(Thm 1.2.5 (d))}$$

$$= g\,((\nabla \cdot \nabla) f) + (\nabla g) \cdot (\nabla f)$$

$$= (\nabla g) \cdot (\nabla f) + g\,(\nabla^2 f).$$

(b) If f is harmonic prove that $\nabla^2 gf = 2\nabla g \cdot \nabla f + f\nabla^2 g$.

$\nabla^2 f = 0.$

$$\nabla^2 gf = \nabla \cdot (\nabla gf)$$

$$= \nabla \cdot ((\nabla g) \cdot f + g(\nabla f)) \quad \text{(Thm 1.25 (b))}$$

$$= \nabla \cdot (f(\nabla g)) + \nabla \cdot (g \nabla f)) \quad \text{(Thm 1.25 (c))}$$

$$= f \cdot \nabla^2 g + (\nabla f) \cdot (\nabla g)$$

$$+ g \cdot \cancel{\nabla^2 f} + (\nabla g) \cdot (\nabla f) \quad \text{(Thm 1.25. (d))}$$

$$= 2(\nabla f) \cdot (\nabla g) + f \nabla^2 g.$$

Theorem (1.2.6). Equivalence of mixed partial derivatives

Let $f : \mathbb{R}^n \to \mathbb{R}$, then $\dfrac{\partial^2 f}{\partial x_i \partial x_j} = \dfrac{\partial^2 f}{\partial x_j \partial x_i}$ for all $i, j = 1, ..., n$.

Proof. See tutorial Q6 for a proof when $n = 2$. $\square$

$$\frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} f = \frac{\partial}{\partial x_j} \frac{\partial}{\partial x_i} f .$$

Example. Let $f(x, y) = xe^{2y}$.

$$\frac{\partial f}{\partial x} = e^{2y}$$

$$\frac{\partial f}{\partial y} = 2xe^{2y}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} e^{2y}$$

$$= 2e^{2y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} 2xe^{2y}$$

$$= 2e^{2y}$$

$$\therefore \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Theorem (1.2.7)**.**
Let $g : \mathbb{R}^3 \to \mathbb{R}$ and $\underline{F} : \mathbb{R}^3 \to \mathbb{R}^3$ then

1. $\nabla \times \nabla g \equiv \underline{0}$ curl(grad g) $\equiv 0$
2. $\nabla \cdot (\nabla \times \underline{F}) \equiv 0$ div(curl g) $\equiv 0$

identically equal

Proof. Exercise! $\square$

Example. Verify that $\nabla \cdot (\nabla \times \underline{F}) = 0$ for $\underline{F}(x, y, z) = \begin{pmatrix} x^2 \\ xy \\ ye^z \end{pmatrix}$.

$$\nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & ye^z \end{vmatrix} = \begin{pmatrix} e^z - 0 \\ 0 - 0 \\ y - 0 \end{pmatrix} = \begin{pmatrix} e^z \\ 0 \\ y \end{pmatrix}$$

$$\nabla \cdot (\nabla \times \underline{F}) = \frac{\partial}{\partial x} e^z + \frac{\partial}{\partial y} 0 + \frac{\partial}{\partial z} y = 0 + 0 + 0$$

$$= 0 \quad \checkmark$$