MATH2001–Basic Analysis Final Examination June 2012

Time: 60 minutes Total marks: 60 marks

MEMO

SECTION A Multiple choice

Answers: 1E, 2B, 3E, 4C, 5B.

SECTION B

Answer this section in the answer book provided.

Question 1 .... [9 marks]

Write down the definitions of the following limits of functions where $a, L \in \mathbb{R}$ and f is a real-valued functions. Also write down the assumptions for the domain of f.

(a) $\lim_{x\to a} f(x) = L.$ (3)

Solution. Assume there are b, c such that $b < a < c$ and $(b, a) \cup (a, c) \subset \operatorname{dom}(f)$. ✓ Then $\lim_{x\to a} f(x) = L$ if and only if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; (0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon) \quad ✓✓$$

(b) $\lim_{x\to a^+} f(x) = -\infty.$ (3)

Solution. Assume there is $c > a$ such that $(a, c) \subset \operatorname{dom}(f)$. ✓ Then $\lim_{x\to a^+} f(x) = -\infty$ if and only if

$$\forall A (< 0) \; \exists \delta > 0 \; (a < x < a + \delta \Rightarrow f(x) < A) \quad ✓✓$$

(c) $\lim_{x\to \infty} f(x) = L.$ (3)

Solution. Assume there is $a \in \mathbb{R}$ such that $(a, \infty) \subset \operatorname{dom}(f)$. ✓ Then $\lim_{x\to \infty} f(x) = L$ if and only if

$$\forall \varepsilon > 0 \; \exists K > 0 \; (x > K \Rightarrow |f(x) - L| < \varepsilon) \quad ✓✓$$

Question 2 .... [14 marks]

Prove from the definitions that

(a) $\lim_{x\to-\infty} \frac{2x^2-1}{x^2+2} = 2$ (8)

Let $\varepsilon > 0$. (✓)

First we calculate

$$\left|\frac{2x^2-1}{x^2+2} - 2\right| = \left|\frac{2x^2-1-2(x^2+2)}{x^2+2}\right| \quad ✓$$
$$= \frac{5}{x^2+2} \quad ✓$$

Then

$$\left|\frac{2x^2-1}{x^2+2} - 2\right| < \varepsilon \Leftrightarrow \frac{5}{x^2+2} < \varepsilon \quad ✓$$
$$\Leftrightarrow x^2+2 > \frac{5}{\varepsilon} \quad ✓$$
$$\Leftrightarrow x^2 > \frac{5}{\varepsilon} - 2. \quad ✓$$

Put $\varepsilon_1 = \min\{\varepsilon, 1\}$ and choose $A = -\sqrt{\frac{5}{\varepsilon_1} - 2}$ ✓ (Note that $\frac{5}{\varepsilon_1} - 2 \geq 3 > 0$)

Then

$$x < A(< 0) \Rightarrow x^2 > A^2 = \frac{5}{\varepsilon_1} - 2 \geq \frac{5}{\varepsilon} - 2 \quad ✓$$
$$\Rightarrow \left|\frac{2x^2-1}{x^2+2} - 2\right| < \varepsilon \quad (✓)$$

(b) $\lim_{x\to-1^-} \frac{1}{x+1} = -\infty$ (6)

Solution. Let $A < 0$ and $x < -1$. (✓)

We have to find $\delta > 0$ such that $-1-\delta < x < -1$ implies $\frac{1}{x+1} < A$. ✓

Now let $x < -1$. Since $x + 1 < -1 + 1 < 0$, ✓

$$\frac{1}{x+1} < A \Leftrightarrow x+1 > \frac{1}{A}$$
$$\Leftrightarrow x > -1 + \frac{1}{A} \quad ✓$$

Now put $\delta = -\frac{1}{A}$. Then $\delta > 0$ (since $A < 0$). ✓

And by the above calculations, $-1-\delta < x < -1$ implies $-1+\frac{1}{A} < x < -1$ ✓

and thus $\frac{1}{x+1} < A$. (✓)

Question 3 .... [8 marks]

Let $a \in \mathbb{R}$ and let f be continuous at a with $f(a) \neq 0$. Prove that the function $\dfrac{1}{f}$ is also continuous at a.

Proof. Since $f(a) \neq 0$, there is $\delta_0 > 0$ such that $|f(x) - f(a) < \dfrac{|f(a)|}{2}$ for $|x-a| < \delta_0$. ✓

Then, for $|x-a| < \delta_0$,

$$|f(x)| \geq |f(a)| - |f(x) - f(a)| \geq |f(a)| - \frac{|f(a)|}{2} = \frac{|f(a)|}{2} \quad \checkmark$$
$$\Rightarrow \left| \frac{1}{f(x)} - \frac{1}{f(a)} \right| = \frac{|f(a) - f(x)|}{|f(x)f(a)|} \leq \frac{2|f(x) - f(a)|}{|f(a)|^2}. \quad \checkmark$$

Now let $\varepsilon > 0$ and δ_1 such that $|x-a| < \delta_1$ implies $|f(x) - f(a)| \leq \dfrac{|f(a)|^2}{2}\varepsilon$. ✓

Put $\delta = \min\{\delta_0, \delta_1\}$. (✓) It follows for $|x-a| < \delta$ that

$$\left| \frac{1}{f(x)} - \frac{1}{f(a)} \right| \leq \frac{2|f(x) - f(a)|}{|f(a)|^2} < 2\frac{|f(a)|^2}{2}\varepsilon \frac{1}{|f(a)|^2} = \varepsilon. \quad \checkmark(\checkmark)$$

Question 4 .... [6 marks]

Let $f : (0,1) \to \mathbb{R}$ be an increasing function such that $f((0,1))$ is an interval. Show that f is continuos from the left. (f is also continuous from the right, but you do not need to prove this.)

Proof. Let $a \in (0,1)$. We are going to show that $\lim\limits_{x \to a^-} f(x) = f(a)$. (✓)

Since f is increasing, $f(x) \leq f(a)$ for all $x \in (0,a)$. (✓)

Hence $A = \{f(x) : x \in (0,a)\}$ is bounded above by $f(a)$. (✓)

Therefore $\sup(f(A))$ exists by the Dedekind completeness axiom, and $\sup(f(A)) \leq f(a)$. (✓)

If $\sup(f(A)) < f(a)$, then choose $c \in (\sup(f(A)), f(a))$. (✓) But

$$f\left(\frac{a}{2}\right) \leq \sup(f(A)) < c < f(a)$$

and the assumption that $f((0,1))$ is an interval give that $c \in f((0,1))$. ✓

Thus $c = f(x)$ for some x. (✓)

From $c > \sup(f(A))$ it follows that $x \notin (0,a)$ and therefore $x \in [a,1)$, which implies $c = f(x) \geq f(a)$ since f is increasing. ✓

This contradiction proves $\sup(f(A)) = f(a)$. (✓)

Now let $\varepsilon > 0$. Then there is $y \in (0,a)$ such that $f(a) - \varepsilon = \sup(f(A)) - \varepsilon < f(y)$. ✓

Let $\delta = a - y > 0$. (✓) Then, for $a - \delta < x < a$, i. e., $y < x < a$,

$$f(a) - \varepsilon < f(y) \leq f(x) \leq f(a) < f(a) + \varepsilon. \quad \checkmark$$

Question 5 .... [7 marks]

(a) For which $x \in \mathbb{R}$ does the geometric series $\sum_{n=0}^{\infty} x^n$ converge? (1)

(You do not need to justify your answer.)

Solution. $x \in (-1, 1]$.

(b) Let $a \in \mathbb{R}$ with $|a| > 1$. For which $x \in \mathbb{R}$ does the series $\sum_{n=0}^{\infty} ax^n$ converge? (1)

Solution. Since

$$\sum_{n=0}^{\infty} ax^n = a \sum_{n=0}^{\infty} x^n,$$

the series in (b) converges if and only the series in (a) converges, i. e., for $x \in (-1, 1]$.

(c) What are the radii of convergence of $\sum_{n=0}^{\infty} x^n$ and $\sum_{n=0}^{\infty} ax^n$? (1)

Solution. $R = 1$ in both cases.

(d) Using your answer to part (c) or otherwise, find $\limsup_{n\to\infty} \sqrt[n]{|a|}$. (1)

Solution. We know $1 = \frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a|}$ for the series in (b).

(e) Find $\liminf_{n\to\infty} \sqrt[n]{|a|}$. (2)

Solution. We know $\liminf_{n\to\infty} \sqrt[n]{|a|} \leq \limsup_{n\to\infty} \sqrt[n]{|a|}$. Also, $1 \leq |a|$ gives $1 \leq \sqrt[n]{|a|}$ and thus $1 \leq \liminf_{n\to\infty} \sqrt[n]{|a|}$. Altogether, with part (d), $\liminf_{n\to\infty} \sqrt[n]{|a|} = 1$.

(f) Show that $\lim_{n\to\infty} \sqrt[n]{|a|}$ exists and find its value. (1)

Solution. We know $\liminf_{n\to\infty} \sqrt[n]{|a|} = \limsup_{n\to\infty} \sqrt[n]{|a|} = 1$, whence $\lim_{n\to\infty} \sqrt[n]{|a|} = 1$.

Question 6 . . . . [Bonus Question: 3 marks]

Elaborate on the outline of your answer to the MCQ question 4 from part A. Note that if your answer is D or E, then you would have to provide an example which illustrates your answer.

Solution. To justify (C), we observe that

$$\lim_{x\to 0} \frac{f^{(n)}(x)}{f^{(n+1)}(x)} = \frac{\lim\limits_{x\to 0} f^{(n)}(x)}{\lim\limits_{x\to 0} f^{(n+1)}(x)} = \frac{0}{f^{(n+1)}(0)}.$$

Hence $n \in T$. By the well-ordering principle, T has a minumum k. If $k > 0$, then $k-1 \in T$ by l'Hôpital's rule. This limit is 0; strictly speaking, one should argue that the same reasoning as above holds for

$$T = \left\{ j \in \mathbb{N} : j \leq n,\ \lim_{x\to 0} \frac{f^{(j)}(x)}{f^{(j+1)}(x)} = 0 \right\}.$$