Complexity

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- Complexity theory studies these questions

Complexity theory – Analysis

We can look at

- Worst case analysis
- Best case analysis
- Average case analysis
- Expected case analysis
- Amortised analysis

We won't cover the last two points in this course.

Complexity theory – Bounds

We can look at

- Upper Bounds
- Lower Bounds
- Tight bounds

We start by looking at how to describe these bounds.

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For example, if our function was $f(n) = 6n^3 + 2n^2 + 20n + 45$ then the function would be asymptotically n^3.

More formally

Let f and g be functions $f, g : \mathrm{N} \rightarrow \mathrm{R}^{+}$. Then we say that $f(n) \in$O$(g(n))$ if there exist positive integers c and n_0 such that for every integer $n \geq n_0$, $f(n) \leq cg(n)$.

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o is a *loose upper bound* ($<$ rather than $\leq$)

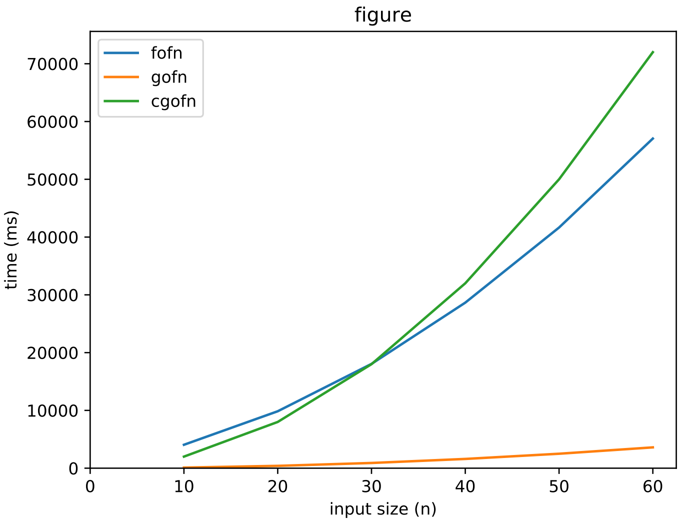
figure
fofn
gofn
cgofn
time (ms)
input size (n)
0
10000
20000
30000
40000
50000
60000
70000
0
10
20
30
40
50
60

Other bounds

Big Ω : tight lower bound (e.g. $\geq$)
Small ω : loose lower bound (e.g. $>$)
Θ : tight upper+lower (e.g. combine Ω, O)

Classes of problems

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Now we shift the argument to looking at *problems* to determine which class a given problem would fall into...

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Any *optimisation problem* has a related decision problem so in some sense they are equivalent.
Note: In the discussion below we are typically talking about *decision problems*.

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Example: $n = 1000$, then n^3 is 1 billion but 2^n is very large (larger than the number of atoms in the universe).
Polynomial time algorithms are fast enough for many purposes.
Exponential time algorithms are rarely useful.

Definition for P Class

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You have seen examples of these problems.

The class NP

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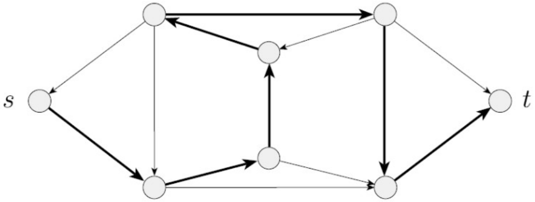
These problems typically have exponential time algorithms.

The Hamiltonian Path problem

A Hamiltonian path in directed graph G is a directed path that visits each node exactly once.

Decision problem: Are two nodes in G connected with a Hamilton path?



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NP comes from *nondeterministic polynomial time*
Here a nondeterministic algorithm is an algorithm that, even for the same input, can exhibit different behaviours on different runs
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A language is in NP if it is decided by some nondeterministic polynomial time Turing machine.

The P versus NP Question

Is $P = NP$*?* is one of the greatest unsolved problems in theoretical computer science.

Current thinking is that they are not the same.

That implies that P is a subset of NP.

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Important because

- Theoretical side – results about NP-Complete problems apply to the whole class
- Practical side – proving a new problem is NP-Complete gives us information about the new problem.

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For example, prove that vertex cover is NP-Complete by a reduction from 3SAT.

Details omitted!

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Underlying model is the Turing Machine – we look at the maximum number of tape cells used for an input of size n
This is analogous to looking at the amount of memory needed to solve a problem on a computer.
Similar theoretical results apply.
We won't discuss these here.

Conclusion

- Complexity classes allow us to group algorithms or problems independent of the computational model.
- Polynomial vs exponential.
- Polynomial is “easy”.
- P $=$ NP? Probably not...
- Completeness and reduction: solve one efficiently, solve them all!
- Space complexity – a different way of looking at things
- Relationship between time and space
- So, there are many unanswered questions!