UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG
School of Computer Science and Applied Mathematics

APPM2023
Mechanics II
2023

Class Test-02

Date: 25 May 2023

Student Number: ______________________

Duration: 60 minutes

Total: 50 Points

Instructions

- Read all the questions carefully.
- This test comprises 3 questions.
- Answer all questions.
- Show all working in answer books provided.
- Start each question on a new page.
- There are 50 points available, and 50 points is 100%.

Question 1 — Basic Algebra (17 Points)

Consider the 3-dimensional Euclidean space $\mathbb{R}^3$, equipped with the standard rectilinear $x-y-z$-coordinates. Assume

$$\hat{x}=\begin{pmatrix}1\\0\\0\end{pmatrix}\quad \hat{y}=\begin{pmatrix}0\\1\\0\end{pmatrix}\quad \text{and}\quad \hat{z}=\begin{pmatrix}0\\0\\1\end{pmatrix}$$

and let

$$\vec{a}=\begin{pmatrix}a^x\\a^y\\a^z\end{pmatrix}\quad \text{and}\quad \vec{b}=\begin{pmatrix}b^x\\b^y\\b^z\end{pmatrix}.$$

Answer the questions that follow.

[**1.1**] Consider a parallelogram whose edges given by the vectors $\vec{a}$ and $\vec{b}$. Suppose that the smallest angle between $\vec{a}$ and $\vec{b}$ is θ. Prove that the area of the parallelogram is given by

$$a\,b\sin(\theta).$$

Include a properly labelled diagram as part of your proof. **(7 Points)**

[**1.2**] Define

$$M_x=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}\quad M_y=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}\quad \text{and}\quad M_z=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}$$

Prove that

$$\vec{a}\times\vec{b}=\left((\vec{a}\cdot\hat{x})M_x+(\vec{a}\cdot\hat{y})M_y+(\vec{a}\cdot\hat{z})M_z\right)\vec{b}.$$

(10 Points)

Question 2 — Arc-Length (20 Points)

Assume the the underlying space is 2-dimensional Euclidean space in standard rectilinear $x-y$-coordinates. Define

$$f(x)=-\sqrt{1-x^2}.$$

Answer the following questions.

[**2.1**] Show that the length ℓ of the curve defined by the generic function $h(x)$ is given by

$$\ell=\int_{x_i}^{x_f}\mathrm{d}x\,\sqrt{1+(h')^2(x)}$$

where the prime denotes differentiation with respecrt to x, and x_i and x_f mark the ends of the interval bounding the curve. **(7 Points)**

[2.2] Sketch the curve of the function $f(x)$. Include correctly labelled axes and all points of intersection between the curve and the axes. **(4 Points)**

[2.3] Compute the arc-length of the curve defined by $f(x)$ on the interal $x \in [-1, 1]$. Use

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan\left(\frac{x}{\sqrt{1-x^2}}\right) = \frac{1}{\sqrt{1-x^2}}$$

to help in your computations. **(9 Points)**

Question 3 — Surface Area **(13 Points)**

Consider the embedding of a surface in $\mathbb{R}^3$, where each point on the surface i given by the vector

$$\vec{q}(s,t) = \begin{pmatrix} s\cos(t) \\ s\sin(t) \\ -\frac{s^2}{2} \end{pmatrix}$$

Answer the folowing questions.

[3.1] Sketch the surface described by $\vec{q}$ by sketching a collection of coordinate curves. Include a properly labelled set of axes. **(4 Points)**

[3.2] Construct the metric tensor on the surface parametrised be $\vec{q}$. **(6 Points)**

[3.3] Compute area element on surface traced by $\vec{q}$. **(3 Points)**