14:00 hrs	08 / 11 /2010	HALL 29	Exams Office Use Only

University of the Witwatersrand, Johannesburg

Course or topic No(s)	MATH2016
Course or topic name(s) Paper number & title	Advanced Analysis
Examination/Test* to be held during month(s) of (*delete as applicable)	Final October 2010
Year of study (Art & Sciences leave blank)	Second Year
Degrees/Diplomas for which this course is prescribed (BSc (Eng) should indicate which branch)	BSc, BCom, BA
Faculty/ies presenting candidates	Science, Commerce, Humanities
Internal examinar(s) and telephone number(s)	Dr A Davison Ext 76240
External examiner(s)	Professor Ebrahim Momoniat
Calculator policy	

Time allowance

Course Nos	MATH2016	Time	90 minutes

Instruction to candidates (Examiners may wish to use this space to indicate, inter alia, the contribution made by this examination or test towards the year mark, if appropriate)	**Answers to All questions must be legible.** **Total : 90 marks** **Duration : 1:30**

Internal Examiners or Heads of Department are requested to sign the Declaration overleaf

Question 1 - 19 marks

(a) Recall that the First Mean Value Theorem (FMVT) says that if f is continuous and differentiable on a closed interval $[\alpha, \beta]$ then there exists a value $c \in (\alpha, \beta)$ such that

$$f'(c)(\beta - \alpha) = f(\beta) - f(\alpha).$$

Use the FMVT to prove the following: if f is differentiable on $[a, b]$ and has continuous derivative f', then

$$\int_a^b f'(t)\,dt = f(b) - f(a).$$

You may assume that f' is Riemann integrable on $[a, b]$. (12 marks)

(b) Arrange the following quantities in increasing order:

$$(b-a)\inf\{f(x)|x \in [a,b]\}$$
$$\sup\{f(x)|x \in [a,b]\}(b-a)$$
$$U(f,P)$$
$$L(f,P)$$
$$\int_a^b f(t)\,dt$$

(Your answer should look something like "First quantity $\leq$ Second quantity $\leq$ Third quantity etc".) (2 marks)

(c) Verify part (b) using $f(x) = x^2$ and $P = \{-1 < \frac{1}{2} < 0 < \frac{1}{2} < 1\}$. (5 marks)

Question 2 - 16 marks

Let S be the part of the plane $x + y + z = 1$ in the first octant.

(a) Parameterize S using parameters u and v. (3 marks)

(b) Evaluate

$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \quad \text{and} \quad \left\| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right\|.$$

(4 marks)

(c) Calculate the surface area of S. (4 marks)

(d) Calculate $\iint_S \mathbf{F} \cdot d\mathbf{a}$ where $\mathbf{F}(x,y,z) = \begin{pmatrix} 2y+z \\ x+z \\ y-x \end{pmatrix}$.

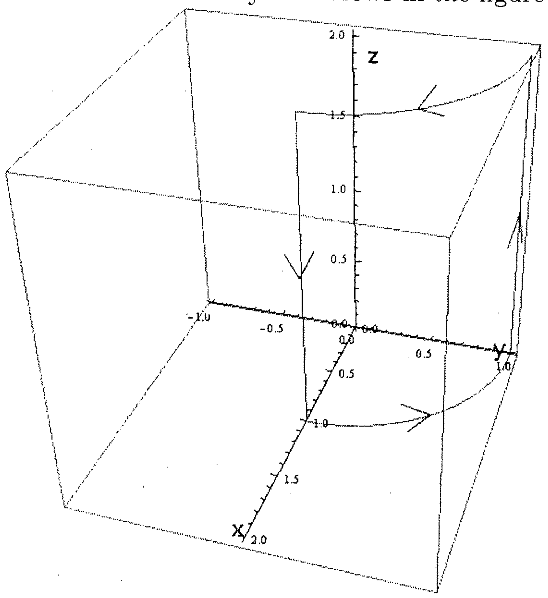
(5 marks)

Question 3 - 20 marks

Recall that Stokes' Theorem says that if S is a surface with boundary ∂S, oriented anticlockwise with respect to the normal of S, and $\mathbf{F}$ is a function $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ then

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{a} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

(a) Use Stokes' Theorem to evaluate the integral $\int_\Gamma \mathbf{F} \cdot d\mathbf{r}$ where Γ is the curve that bounds the part of the cylinder $x^2 + y^2 = 1$ in the first octant between $z = 0$ and $z = 2$, taken in the direction indicated by the arrows in the figure below.



NOTE: Do NOT evaluate the integral directly! (10 marks)

(b) Let S be the surface parameterized by

$$\mathbf{r}(u, v) = \begin{pmatrix} v \cos t \\ v \sin t \\ v^2 \cos 2t \end{pmatrix},$$

$t \in [0, 2\pi]$, $u \in [0, 2\pi]$, $v \in [0, 1]$, and let

$$\mathbf{F} = \begin{pmatrix} 0 \\ 0 \\ 3y - 3x \end{pmatrix} = \nabla \times \begin{pmatrix} 3xy \\ 3xy \\ 0 \end{pmatrix}.$$

Use Stokes' Theorem to evaluate $\iint_S \mathbf{F} \cdot d\mathbf{a}$.
NOTE: Again, do not try to do the integral directly! (10 marks)

Question 4 - 15 marks

(a) What is a gradient vector field? What is a potential function? How are these two things related? (3 marks)

(b) Given that $\mathbf{F}$ is conservative, i.e. that $\int_C \mathbf{F}\cdot d\mathbf{r}$ depends only on the endpoints of C, prove that $\int_\Gamma \mathbf{F}\cdot d\mathbf{r} = 0$ for any closed path Γ. (5 marks)

(c) Given that $\mathbf{F} = \begin{pmatrix} y^2 \sin z \\ 2xy\sin z \\ z + xy^2 \cos z \end{pmatrix} = \nabla\phi$, find the potential function ϕ and hence evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is a curve beginning at the origin and ending at $(2, -1, \pi)$. (7 marks)

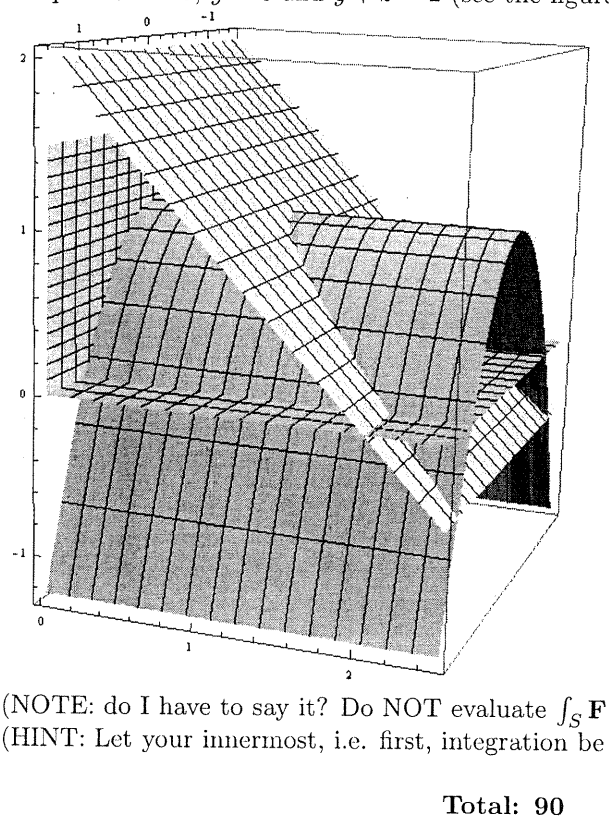
Question 5 - 20 marks

(a) Change the order of integration to evaluate

$$\int_0^1 \int_0^{y^2} \int_0^5 \sin\left(x^{3/2}\right) \mathrm{d}z \mathrm{d}x \mathrm{d}y$$

(8 marks)

(b) Use the Gauss divergence theorem to find $\int_S \mathbf{F}\cdot \mathbf{da}$, where $\mathbf{F} = \begin{pmatrix} xy \\ y^2 + e^{xz^2} \\ \sin xy \end{pmatrix}$ and S is the surface of the solid B being the region bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $z = 0$, $y = 0$ and $y + z = 2$ (see the figure below).



(NOTE: do I have to say it? Do NOT evaluate $\int_S \mathbf{F}\cdot \mathbf{da}$ directly.)
(HINT: Let your innermost, i.e. first, integration be with respect to y.) (12 marks)

Total: 90