Chapter 8: Symmetries of Regular Polygons

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LEARNING OUTCOMES FOR THE LECTURE

By the end of this lecture, students will be able to:

- describe a geometric figure
- describe a rigid motion as a permutation of a figure
- find the set of all symmetries for a given geometric figure
- write down the Cayley table for a given figure
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In this chapter we discuss the symmetries of geometric figures. By a figure we mean a finite set of points called vertices, some pairs of which are joined by straight lines or edges.

Let F be a figure. A symmetry of F is a permutation of its vertices (edges) that can be realised by a rigid motion. The motion of symmetry preserves distance between the vertices (edges) it permutes. Each rigid motion is a permutation (bijection) of n objects. But not all permutations can be realised as rigid symmetries of a plane figure (or a body in space).

Let $S(F)$ denote the set containing symmetries of a figure F. Then

- $S(F)$ is closed under composition of functions. i.e. $\forall \sigma, \tau \in S(F), \quad \sigma \circ \tau \in S(F)$.
- $S(F)$ contains an identity symmetry e such that for each $\sigma \in S(F), \quad \sigma \circ e = e \circ \sigma = \sigma$.
- For each $\sigma \in S(F)$, there is an inverse symmetry $\sigma^{-1} \in S(F)$ such that $\sigma \circ \sigma^{-1} = \sigma^{-1} \circ \sigma = e$.
- composition of symmetries is assosiative, i.e. $\forall \sigma, \tau, \gamma \in S(F), \quad \sigma \circ (\tau \circ \gamma) = (\sigma \circ \tau) \circ \gamma$.

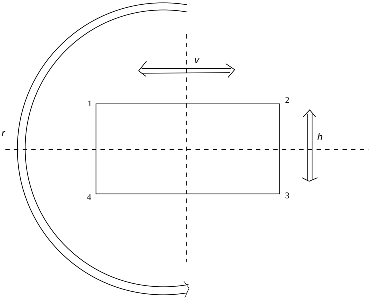
When considering all possible symmetries for a figure it is very useful to draw up a Cayley table. A Cayley table shows all combinations of how each symmetry is composed with every symmetry. We will use the next example to show how symmetries for a rectangle are identified and named and how to set up a Cayley table.

Example (8.0.1 Symmetries of a Rectangle)

- *What are all the symmetries of a non-square rectangle?*
- *Show the Cayley table for these symmetries.*

There are two axes of symmetry and one point of rotation: These are the lines through the midpoints of opposite sides and a point of rotation through π radians: this point at the centre of the rectangle.

If we label the corners of the rectangle 1,2,3 and 4 as in the diagram below, we can label the symmetries as follows:



$e = (1)(2)(3)(4)$ is the identity symmetry. The symmetry $v = (1 \quad 2)(3 \quad 4)$ and $h = (1 \quad 4)(2 \quad 3)$ result from reflecting the rectangle through vertical and horizontal axes of symmetry, respectively.

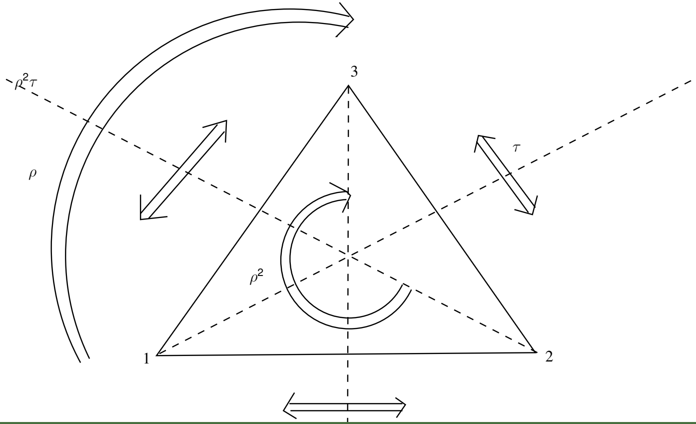
If we find $v \circ h = (1 \quad 2)(3 \quad 4)(1 \quad 4)(2 \quad 3) = (1 \quad 3)(2 \quad 4)$.
$r = (1 \quad 3)(2 \quad 4)$ is the result of rotating the rectangle through π radians about its centre point.
$r \circ h = (1 \quad 3)(2 \quad 4)(1 \quad 4)(2 \quad 3) = (1 \quad 2)(3 \quad 4) = v$.
If fact e, v, h and r are all the possible symmetries of the rectangle. The Cayley table for the rectangle is as follows:

$\circ$	e	r	v	h
e	e	r	v	h
r	r	e	h	v
v	v	h	e	r
h	h	v	r	e

A regular n-gon has rotational and reflectional symmetries.

Example (8.0.2 Symmetries of an Equilateral Triangle)

An equilateral triangle has six symmetries:
Three rotations and three reflections.



The elements of $S(\Delta)$ are:

ROTATIONS

$e = (1)(2)(3)$

$$\rho = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1 \quad 2 \quad 3)$$

$$\rho^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1 \quad 3 \quad 2)$$

REFLECTIONS

$$\tau = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (2 \quad 3)$$

$$\rho^2\tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1 \quad 3)$$

$$\rho\tau = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1 \quad 2)$$

$S(\Delta)$	e	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$(2\ 3)$	$(1\ 2)$	$(1\ 3)$
e	e	$(1\ 2\ 3)$	$(1\ 3\ 2)$	$(2\ 3)$	$(1\ 2)$	$(1\ 3)$
$(1\ 2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$	e	$(1\ 2)$	$(1\ 3)$	$(2\ 3)$
$(1\ 3\ 2)$	$(1\ 3\ 2)$	e	$(1\ 2\ 3)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2)$
$(2\ 3)$	$(2\ 3)$	$(1\ 3)$	$(1\ 2)$	e	$(1\ 3\ 2)$	$(1\ 2\ 3)$
$(1\ 2)$	$(1\ 2)$	$(2\ 3)$	$(1\ 3)$	$(1\ 2\ 3)$	e	$(1\ 3\ 2)$
$(1\ 3)$	$(1\ 3)$	$(1\ 2)$	$(2\ 3)$	$(1\ 3\ 2)$	$(1\ 2\ 3)$	e

$S(\Delta)$	e	ρ	ρ^2	τ	$\rho\tau$	$\rho^2\tau$
e	e	ρ	ρ^2	τ	$\rho\tau$	$\rho^2\tau$
ρ	ρ	ρ^2	e	$\rho\tau$	$\rho^2\tau$	τ
ρ^2	ρ^2	e	ρ	$\rho^2\tau$	τ	$\rho\tau$
τ	τ	$\rho^2\tau$	$\rho\tau$	e	ρ^2	ρ
$\rho\tau$	$\rho\tau$	τ	$\rho^2\tau$	ρ	e	ρ^2
$\rho^2\tau$	$\rho^2\tau$	$\rho\tau$	τ	ρ^2	ρ	e