Basic Analysis 2015 — Solutions of Tutorials

Section 1.1

Tutorial 1.1.1 (Theorem 1.3) [Basic field properties: multiplication]
(a) The number 1 is unique.
(b) For all $a \in \mathbb{R}$ with $a \neq 0$, the number a^{-1} is unique.
(c) For all $a, b \in \mathbb{R}$ with $a \neq 0$, the equation $ax = b$ has a unique solution. This solution is $x = a^{-1}b$.
(d) $\forall a \in \mathbb{R} \setminus \{0\},\ (a^{-1})^{-1} = a$.
(e) $\forall a, b \in \mathbb{R} \setminus \{0\},\ (ab)^{-1} = a^{-1}b^{-1}$.
(f) $\forall a \in \mathbb{R} \setminus \{0\},\ (-a)^{-1} = -a^{-1}$.
(g) $1^{-1} = 1$.

Proof. (a) Let $1, 1' \in \mathbb{R}$ such that $a \cdot 1 = a$ and $a \cdot 1' = a$ for all $a \in \mathbb{R}$. We must show that $1 = 1'$:

$$\begin{aligned} 1 &= 1 \cdot 1' \\ &= 1' \cdot 1 && \text{by (M2)} \\ &= 1' \end{aligned}$$

(b) Let $a \in \mathbb{R} \setminus \{0\}$ and $a, a' \in \mathbb{R}$ such that $a \cdot a' = 1$ and $a \cdot a'' = 1$. We must show that $a' = a''$:

$$\begin{aligned} a' &= a' \cdot 1 && \text{by (M3)} \\ &= a'(aa'') \\ &= (a'a)a'' && \text{by (M1)} \\ &= (aa')a'' && \text{by (M2)} \\ &= 1 \cdot a'' \\ &= a'' \cdot 1 && \text{by (M2)} \\ &= a'' && \text{by (M3)} \end{aligned}$$

(c) First we show that $x = a^{-1}b$ is a solution. So let $x = a^{-1}b$.
Then

$$\begin{aligned} ax &= a(a^{-1}b) \\ &= (aa^{-1})b && \text{by (M1)} \\ &= 1 \cdot b && \text{by (M4)} \\ &= b \cdot 1 && \text{by (M2)} \\ &= b && \text{by (M3)} \end{aligned}$$

To show that the solution is unique let $x \in \mathbb{R}$ such that $ax = b$.
Then

$$\begin{aligned} x = x \cdot 1 &= x(aa^{-1}) && \text{by (M3), (M4)} \\ &= (xa)a^{-1} && \text{by (M1)} \\ &= (ax)a^{-1} && \text{by (M2)} \\ &= ba^{-1} && \because ax = b \\ &= a^{-1}b && \text{by (M2)} \end{aligned}$$

This shows that the solution is unique.
(d) Note that

$$a^{-1}(a^{-1})^{-1} = 1 \quad \text{by (M4).}$$

On the other hand

$$a^{-1}a = aa^{-1} = 1 \quad \text{by (M2), (M4).}$$

By part (b), it follows that
$$(a^{-1})^{-1} = a.$$

(e)

$$\begin{aligned}
(ab)(a^{-1}b^{-1}) &= ((ba)(a^{-1})b^{-1} && \text{by (M1), (M2)}\\
&= (b(aa^{-1}))b^{-1} && \text{by (M1)}\\
&= (b \cdot 1)b^{-1} && \text{by (M4)}\\
&= bb^{-1} && \text{by (M3)}\\
&= 1 && \text{by (M4)}
\end{aligned}$$

By part (b), $(ab)^{-1} = a^{-1}b^{-1}$.
(f)

$$\begin{aligned}
(-a)^{-1} &= (-a)^{-1} \cdot 1 && \text{by (M3)}\\
&= (-a)^{-1}(aa^{-1}) && \text{by (M4)}\\
&= [(-a)^{-1}(-(-a))]a^{-1} && \text{by (M1), Theorem 1.1 (d)}\\
&= [-(-a)^{-1}(-a)]a^{-1} && \text{by (M2), Theorem 1.2 (d)}\\
&= (-1)a^{-1} && \text{by (M2), (M4)}\\
&= -a^{-1} && \text{by Theorem 1.2 (e)}
\end{aligned}$$

By part (b), $(ab)^{-1} = a^{-1}b^{-1}$.
(g) $1 \cdot 1 = 1$ by (M3), so that $1^{-1} = 1$ by (M4) and (b). □

Tutorial 1.1.2
1. Prove Theorem 1.4, (c)–(g):
Let $a, b, c, d \in \mathbb{R}$. Then
(c) $a < b \Rightarrow a + c < b + c$.
(d) $a < b$ and $c < d \Rightarrow a + c < b + d$.
(e) $a < b$ and $c > 0 \Rightarrow ca < cb$.
(f) $0 \le a < b$ and $0 \le c < d \Rightarrow ac < bd$.
(g) $a < b$ and $c < 0 \Rightarrow ca > cb$.

Proof. (c)

$$\begin{aligned}
a < b &\Rightarrow b - a > 0 && \text{by definition of } <\\
&\Rightarrow (b + c) - (a + c) > 0 && \text{by axioms of addition}\\
&\Rightarrow a + c < b + c && \text{by definition of } <
\end{aligned}$$

(d)

$$\begin{aligned}
a < b \text{ and } c < d &\Rightarrow b - a > 0 \text{ and } d - c > 0 && \text{by definition of } <\\
&\Rightarrow (b - a) + (d - c) > 0 && \text{by (O2)}\\
&\Rightarrow (b + d) - (a + c) > 0 && \text{by axioms and properties of addition}\\
&\Rightarrow a + c < b + d && \text{by definition of } <
\end{aligned}$$

(e)

$$\begin{aligned}
a < b \text{ and } c > 0 &\Rightarrow b - a > 0 \text{ and } c > 0 && \text{by definition of } <\\
&\Rightarrow c(b - a) > 0 && \text{by (O3)}\\
&\Rightarrow cb - ca > 0 && \text{by axioms and properties of addition and (D)}\\
&\Rightarrow ca < cb && \text{by definition of } <
\end{aligned}$$

(f)

$$\begin{aligned} 0 \le a < b \text{ and } 0 \le c < d &\Rightarrow b-a>0,\ b>0,\ d-c>0,\ d>0 && \text{by definition of } < \\ &\Rightarrow ad < bd && \text{by (e)} \end{aligned}$$

if $a = 0$: $ac = 0 = ad < bd \quad \therefore ac < bd$
if $a > 0$: $ac < ad < bd \quad \therefore ac < bd$

(g)

$$\begin{aligned} a < b \text{ and } c < 0 &\Rightarrow b-a > 0 \text{ and } -c > 0 && \text{by definition of } < \text{ and by (a)} \\ &\Rightarrow (-c)(b-a) > 0 && \text{by (O3)} \\ &\Rightarrow ca - cb > 0 && \text{by (D) and Theorem 1.2, (d), (f)} \\ &\Rightarrow ca > cb && \text{by definition of } > \end{aligned}$$

□

2. **The absolute value function.** Define the following function on $\mathbb{R}$:

$$|x| = \begin{cases} x & \text{if } x \ge 0, \\ -x & \text{if } x < 0. \end{cases}$$

Prove the following statements for $x, y \in \mathbb{R}$:
(a) $|x| \ge 0$,
(b) $|xy| = |x|\,|y|$,
(c) $|y| < x \Leftrightarrow -x < y < x$,
(d) $|x+y| \le |x| + |y|$.

Proof. (a) By (O1), we have to consider 3 cases:
Case I: $x > 0 \Rightarrow |x| = x > 0 \Rightarrow |x| \ge 0$
Case II: $x = 0 \Rightarrow |x| = x = 0 \Rightarrow |x| \ge 0$
Case I: $-x > 0 \Rightarrow |x| = -x > 0 \Rightarrow |x| \ge 0$

(b) Case I: $x = 0$ or $y = 0$: then $xy = 0$ by Theorem 1.2 (b). Hence

$$|xy| = |0| = 0 = |x|\,|y|.$$

Case II: $x > 0$, $y > 0$: Then $xy > 0$ by (O3). Hence

$$|xy| = xy = |x|\,|y|.$$

Case III: $x < 0$, $y > 0$: Then $xy < 0$ by Theorem 1.4 (g). Hence, by Theorem 1.2 (d),

$$|xy| = -xy = (-x)y = |x|\,|y|.$$

Case IV: $x > 0$, $y < 0$: Interchange x and y and apply Case III.
Case V: $x < 0$, $y < 0$: Then $-x > 0$ and $-y > 0$ by Theorem 1.4 (a). Applying Theorem 1.2 (f) and Case II give

$$|xy| = |(-x)(-y)| = (-x)(-y) = |x|\,|y|.$$

(c) Let $|y| < x$. Then $0 \le |y| < x$, so that $-x < 0$. If $y \ge 0$, then

$$-x < 0 \le y = |y| < x,$$

so that $-x < y < x$. If $y < 0$, then $|-y| = |(-1)y| = |y|$ by (b), and the above gives $-x < -y < x$, so that Theorem 1.4 (g) gives $-x < -(-y) < -(-x)$, that is, $-x < y < x$.
Conversely, let $-x < y < x$. If $y \ge 0$, then $|y| = y < x$.
If $y < 0$, then $-x < y$ gives $-y < x$, so that $|y| = -y < x$.

(d) If $x + y \ge 0$, then

$$|x+y| = x+y \le |x| + |y| \quad (\text{using } z \le |z|).$$

If $x + y < 0$, then

$$|x+y| = -(x+y) \le |x| + |y| \quad (\text{using } -z \le |z|).$$

□

3. Let $x, y, z \in \mathbb{R}$. Which of the following statements are **true** and which are **false**?
(a) $x \le y \Rightarrow xz \le yz$, **false**
(b) $0 < x \le y \Rightarrow \frac{1}{y} \le \frac{1}{x}$, **true**
(c) $x < y < 0 \Rightarrow \frac{1}{y} < \frac{1}{x}$, **true**
(d) $x^2 < 1 \Rightarrow x < 1$, **true**
(e) $x^2 < 1 \Rightarrow -1 < x < 1$, **true**
(f) $x^2 > 1 \Rightarrow x > 1$ **false**.

4. In each of the following questions fill in the $\square$ with $<$ or $>$.
(a) $a \ge 3 \Rightarrow \frac{a-2}{7} \square \frac{a}{7}$, $<$
(b) $a \ge 1 \Rightarrow \frac{3}{a+1} \square \frac{3}{a}$, $<$
(c) $a > 1 \Rightarrow \frac{9}{a} \square \frac{10}{a-1}$, $<$
(d) $a > 1 \Rightarrow \frac{1}{a^2} \square \frac{1}{a}$, $<$
(e) $a \ge 2 \Rightarrow \frac{1}{a^2-1} \square \frac{1}{a}$, $<$
(f) $a > 3 \Rightarrow \frac{-3}{a} \square \frac{-2}{a-1}$. $<$

5. Let $x \ge 0$ and $y \ge 0$. Show that $x < y \Leftrightarrow x^2 < y^2$.

Proof. Observe that
$$x^2 < y^2 \Leftrightarrow y^2 - x^2 > 0 \Leftrightarrow (y-x)(y+x) > 0$$
and
$$x < y \Leftrightarrow y - x > 0.$$
In either case, $y \ne x$, so that $x \ge 0$ and $y \ge 0$ gives that $x > 0$ or $y > 0$. It follows that $x + y > 0$. Hence
$$y - x > 0 \Rightarrow (y-x)(y+x) > 0$$
and
$$(y-x)(y+x) > 0 \Rightarrow y - x = (y-x)(y+x)(y+x)^{-1} > 0.$$
$\square$