APPM2007
Lagrangian Mechanics

Tutorial 2

Question 1 (5 Points)

Consider the generic point $a \in \mathbb{R}^3$ that lies on the curve specified by the displacement vector

$$\vec{p} = \begin{pmatrix} z \\ x \\ y \end{pmatrix} = \begin{pmatrix} \rho\cos(\theta) \\ \rho\sin(\theta)\cos(\phi) \\ \rho\sin(\theta)\sin(\phi) \end{pmatrix}.$$

Perform the following construction

1. Construct the the set of unit vectors, tangent to the curve with respect to the co-ordinates $\{\rho, \theta, \phi\}$.
2. Show that these tangent vectors are mutually orthogonal.
3. Construct the metric $\mathbf{g}$.

Solution 1 (5 Points)

1. Construct the co-ordinate derivative of $\vec{p}$ as follows

$$\hat{e}_\rho = \partial_\rho \vec{p} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \end{pmatrix} = \hat{\rho},$$

$$\hat{e}_\theta = \partial_\theta \vec{p} = \begin{pmatrix} -\rho\sin(\theta) \\ \rho\cos(\theta)\cos(\phi) \\ \rho\cos(\theta)\sin(\phi) \end{pmatrix} = \rho \begin{pmatrix} -\sin(\theta) \\ \cos(\theta)\cos(\phi) \\ \cos(\theta)\sin(\phi) \end{pmatrix} = \rho\hat{\theta},$$

$$\hat{e}_\phi = \partial_\phi \vec{p} = \begin{pmatrix} 0 \\ -\rho\sin(\theta)\cos(\phi) \\ \rho\sin(\theta)\cos(\phi) \end{pmatrix} = \rho\sin(\theta) \begin{pmatrix} 0 \\ -\sin(\phi) \\ \cos(\phi) \end{pmatrix} = \rho\sin(\theta)\,\hat{\phi}.$$

Note that,

$$\hat{e}_\rho = \hat{\rho}, \quad \hat{e}_\theta = \rho\hat{\theta} \quad \text{and} \quad \hat{e}_\phi = \rho\sin(\theta)\,\hat{\phi}$$

where

$$\hat{\rho} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \end{pmatrix}, \quad \hat{\theta} = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta)\cos(\phi) \\ \cos(\theta)\sin(\phi) \end{pmatrix} \quad \text{and} \quad \hat{\phi} = \begin{pmatrix} 0 \\ -\sin(\phi) \\ \cos(\phi) \end{pmatrix}.$$

2. Note that

$$\hat{e}_\rho \cdot \hat{e}_\rho = \hat{\rho} \cdot \hat{\rho}, \quad \hat{e}_\theta \cdot \hat{e}_\theta = \rho^2 \hat{\theta} \cdot \hat{\theta} \quad \text{and} \quad \hat{e}_\phi \cdot \hat{e}_\phi = \rho^2 \sin^2(\theta)\, \hat{\phi} \cdot \hat{\phi}$$

and

$$\begin{aligned} \hat{e}_\rho \cdot \hat{e}_\theta &= \hat{e}_\theta \cdot \hat{e}_\rho = \rho \hat{\rho} \cdot \hat{\theta} \\ \hat{e}_\theta \cdot \hat{e}_\phi &= \hat{e}_\phi \cdot \hat{e}_\theta = \rho^2 \sin(\theta)\, \hat{\theta} \cdot \hat{\phi} \\ \hat{e}_\phi \cdot \hat{e}_\rho &= \hat{e}_\rho \cdot \hat{e}_\phi = \rho \sin(\theta)\, \hat{\phi} \cdot \hat{\rho}. \end{aligned}$$

It is easy to check that,

$$\hat{\rho} \cdot \hat{\rho} = \hat{\theta} \cdot \hat{\theta} = \hat{\phi} \cdot \hat{\phi} = 1 \quad \text{and} \quad \hat{\rho} \cdot \hat{\theta} = \hat{\theta} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{\rho} = 0.$$

3. Collect these inner products together in the form of the metric tensor

$$[\mathbf{g}] = \begin{pmatrix} \hat{e}_\rho \\ \hat{e}_\theta \\ \hat{e}_\phi \end{pmatrix} \cdot \begin{pmatrix} \hat{e}_\rho & \hat{e}_\theta & \hat{e}_\phi \end{pmatrix} = \begin{bmatrix} \hat{e}_\rho \cdot \hat{e}_\rho & \hat{e}_\rho \cdot \hat{e}_\theta & \hat{e}_\rho \cdot \hat{e}_\phi \\ \hat{e}_\theta \cdot \hat{e}_\rho & \hat{e}_\theta \cdot \hat{e}_\theta & \hat{e}_\theta \cdot \hat{e}_\phi \\ \hat{e}_\phi \cdot \hat{e}_\rho & \hat{e}_\phi \cdot \hat{e}_\theta & \hat{e}_\phi \cdot \hat{e}_\phi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & \rho^2 \sin^2(\theta) \end{bmatrix}.$$

$\square$

Question 2 (10 Points)

Consider the two co-ordinatisation of S^2 in $\mathbb{R}^3$,

$$\vec{a}(\rho,\theta,\phi) = \begin{pmatrix} \rho\cos(\theta) \\ \rho\sin(\theta)\cos(\phi) \\ \rho\sin(\theta)\sin(\phi) \end{pmatrix}$$

where $\{\rho,\theta,\phi\}$ are the radial, declination and azimuth positions in $\mathbb{R}^3$; and

$$\vec{a}(r,s) = \begin{pmatrix} \frac{2r}{r^2+s^2+1} \\ \frac{2s}{r^2+s^2+1} \\ \frac{r^2+s^2-1}{r^2+s^2+1} \end{pmatrix}$$

where $r,s \in \mathbb{R}^2$ are in the plane $z=0$. Show that in each co-ordinate system, the length of the of the curve passing through the north and south poles of S^2 is π. (Hint: construct appropriatly parametrised paths $\vec{a}(t)$ with t in the appropriate interval.)

Solution 2 (10 Points)

For the (ρ,θ,ϕ) co-ordinate system, the curve connecting the north and south poles is that curve, parametrised by t where

$$\vec{a}(t) = \vec{a}(\rho(t),\theta(t),\phi(t)) = \begin{pmatrix} \rho(t)\cos(\theta(t)) \\ \rho(t)\sin(\theta(t))\cos(\phi(t)) \\ \rho(t)\sin(\theta(t))\sin(\phi(t)) \end{pmatrix}.$$

In this paramterisation, the appropriate path parametrisation is $\rho(t)=1$, $\theta(t)=t$ and $\phi(t)=\phi$ is a constant. Then the vector tangent to the path at each time t is

$$\partial_t\vec{a}(t) = \begin{pmatrix} -\sin(t) \\ \cos(t)\cos(\phi) \\ \cos(t)\sin(\phi) \end{pmatrix}$$

where

$$\|\partial_t\vec{a}(t)\| = 1.$$

The length of the path connecting the north and south poles of the sphere is given by the interal over the tangent vector

$$l = \int_0^\pi \mathrm{d}t\,\|\partial_t\vec{a}(t)\| = \int_0^\pi \mathrm{d}t = \pi.$$

For the (r, s) co-ordinate system, the curve connecting the north and south poles is that curve, parametrised by t where, without loss of generality, the symmetry of this co-ordinatisation allows

$$\vec{a}(t) = \vec{a}(r(t), s(t)) = \begin{pmatrix} \frac{2r(t)}{r^2(t)+s^2(t)+1} \\ \frac{2s(t)}{r^2(t)+s^2(t)+1} \\ \frac{r^2(t)+s^2(t)-1}{r^2(t)+s^2(t)+1} \end{pmatrix}.$$

In this paramterisation, the appropriate path parametrisation is $r(t) = t$ and $s(t) = 0$. Then the vector tangent to the path at each time t is

$$\partial_t \vec{a}(t) = \begin{pmatrix} -2\frac{t^2-1}{\left(t^2+1\right)^2} \\ 0 \\ 4\frac{t^2-1}{\left(t^2+1\right)^2} \end{pmatrix}$$

where

$$\|\partial_t \vec{a}(t)\| = \frac{2}{t^2+1}.$$

The length of the path connecting the north and south poles of the sphere is given by the interal over the tangent vector

$$l = \int_0^\infty \mathrm{d}t\, \|\partial_t \vec{a}(t)\| = 2\int_0^\infty \mathrm{d}t\, \frac{1}{t^2+1} = 2\frac{\pi}{2} = \pi.$$

$\square$

Question 3 (10 Points)

Consider the two co-ordinatisation of S^2 in $\mathbb{R}^3$

$$\vec{a}(\rho,\theta,\phi) = \begin{pmatrix} \rho\cos(\theta) \\ \rho\sin(\theta)\cos(\phi) \\ \rho\sin(\theta)\sin(\phi) \end{pmatrix}$$

where $\{\rho,\theta,\phi\}$ are the radial, declination and azimuth positions in $\mathbb{R}^3$; and

$$\vec{a}(r,s) = \begin{pmatrix} \frac{2r}{r^2+s^2+1} \\ \frac{2s}{r^2+s^2+1} \\ \frac{r^2+s^2-1}{r^2+s^2+1} \end{pmatrix}$$

where $r,s \in \mathbb{R}^2$ are in the plane $z=0$. Show that in each co-ordinate system, the surface area of the unit sphere is 4π.

Solution 3 (10 Points)

For the (ρ,θ,ϕ) co-ordinate system, the area element is at a fixed radial distance $\rho = 1$ is given by

$$\begin{aligned}
A &= \int_0^{2\pi}\int_0^{\pi} \mathrm{d}\phi\,\mathrm{d}\theta\,\|\vec{e}_\theta \times \vec{e}_\phi\| = \int_0^{2\pi}\int_0^{\pi} \mathrm{d}\phi\,\mathrm{d}\theta\,\rho^2\sin(\theta) \\
&= 2\pi\int_0^{\pi} \mathrm{d}\theta\,\sin(\theta) \\
&= -2\pi[\cos(\theta)]_0^{\pi} \\
&= -2\pi(\cos(\pi) - \cos(0)) \\
&= -2\pi(-1-1) \\
&= 4\pi.
\end{aligned}$$

For the (r,s) co-ordinate system, the area element is computed from the cross-product of the tangent vectors,

$$\vec{e}_r = \partial_r\vec{a}(r,s) = \begin{pmatrix} -2\frac{r^2-s^2-1}{\left(r^2+s^2+1\right)^2} \\ -4\frac{rs}{\left(r^2+s^2+1\right)^2} \\ 4\frac{r}{\left(r^2+s^2+1\right)^2} \end{pmatrix} \quad \text{and} \quad \vec{e}_s = \partial_s\vec{a}(r,s) = \begin{pmatrix} -4\frac{rs}{\left(r^2+s^2+1\right)^2} \\ -2\frac{s^2-r^2-1}{\left(r^2+s^2+1\right)^2} \\ 4\frac{s}{\left(r^2+s^2+1\right)^2} \end{pmatrix}.$$

Note that $\vec{e}_r \cdot \vec{e}_s = 0$. The area element on the sphere is

$$\vec{e}_r \times \vec{e}_s = \begin{pmatrix} -8\frac{r}{\left(r^2+s^2+1\right)^2} \\ -8\frac{s}{\left(r^2+s^2+1\right)^2} \\ -4\frac{r^2+s^2-1}{\left(r^2+s^2+1\right)^2} \end{pmatrix},$$

so,

$$\|\vec{e}_r \times \vec{e}_s\|^2 = \frac{16}{\left(r^2+s^2+1\right)^4} \quad \text{or} \quad \|\vec{e}_r \times \vec{e}_s\| = \frac{4}{\left(r^2+s^2+1\right)^2}.$$

Therefore the total area on the sphere is given by

$$A = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}r\,\mathrm{d}s\,\frac{4}{\left(r^2+s^2+1\right)^2}.$$

By noting the rotational symmetry, we may compute this integral using the trigonometric substition, $r = t\cos(\zeta)$ and $s = t\sin(\zeta)$ to give

$$A = 4\int_0^{2\pi}\int_0^{\infty} \mathrm{d}\zeta\,\mathrm{d}t\,\frac{t}{\left(t^2+1\right)^2} = 4\cdot 2\pi\int_0^{\infty}\mathrm{d}t\,\frac{t}{\left(t^2+1\right)^2},$$

which is again soluble by substitution $u = t^2+1$ to give

$$A = 4\cdot 2\pi\int_1^{\infty}\mathrm{d}u\,\frac{1}{2u^2} = 4\pi\left[\frac{1}{u}\right]_1^{\infty} = 4\pi$$

$\square$

Question 4 (10 Points)

The *Cobb-Douglas production function* is used to model the number of units proxuced by variyng amounts of labour and capital. Let x define the units of labour and y denote the units of capital and C is a constant and $0 < a < 1$,

$$f(x, y) = Cx^a y^{1-a}.$$

The Cobb-Douglas production function for a particular manufacturer is given by

$$f(x, y) = 100x^{\frac{3}{4}} y^{\frac{1}{4}}.$$

Suppose that labour is charged at $R150$ per unit and capital is charged at $R250$ per unit. Suppose that the total cost of labour and capital is limited to $R50000$. Find the maximum production level for this manufacturer. (Hint: relate the rate of productivity to the rate of constraint using directional derivatives.)

Solution 4 (10 Points)

The limit on the cost of labour and capital gives a constraint,

$$g(x, y) = 150x + 250y = 50000.$$

The maximum prouduction level occurs when the constraint and production level function level sets are tangent. From the given production and constraint functions, we find tangent vectors

$$\begin{aligned}\nabla f(x, y) &= \partial_x f(x, y)\hat{x} + \partial_y f(x, y)\hat{y}\\ &= 75x^{-\frac{1}{4}} y^{\frac{1}{4}} \hat{x} + 25x^{\frac{3}{4}} y^{-\frac{3}{4}} \hat{y}\end{aligned}$$

and

$$\begin{aligned}\nabla g(x, y) &= \partial_x g(x, y)\hat{x} + \partial_y g(x, y)\hat{y}\\ &= 150\hat{x} + 250\hat{y}.\end{aligned}$$

If the tangent vectors are colinear, then $\lambda \nabla g(x, y) = \nabla f(x, y)$ where λ is some constant. This gives rise to a system of equations

$$\begin{aligned}75x^{-\frac{1}{4}} y^{\frac{1}{4}} \hat{x} =& 150\lambda\\ 25x^{\frac{3}{4}} y^{-\frac{3}{4}} \hat{x} =& 250\lambda,\end{aligned}$$

subject to

$$150x + 250y = 50000.$$

Solving for λ yields,

$$\lambda = \frac{x^{-\frac{1}{4}} y^{\frac{1}{4}}}{2}.$$

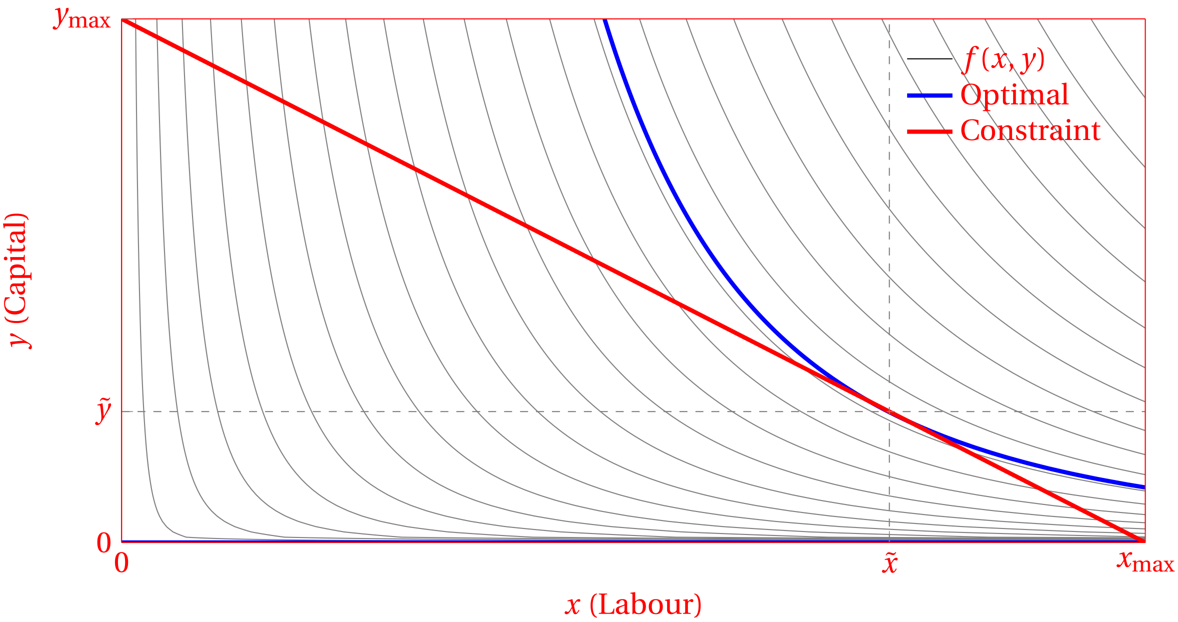
Substituting this back into the the component equations of $\lambda \nabla g(x, y) = \nabla f(x, y)$ yields optimail values

$$\tilde{x} = 250 \quad \text{and} \quad \tilde{y} = 50.$$

The maximum production level, subject to the constraint occurs at 250 units of labour and 50 units of capital,

$$f(\tilde{x}, \tilde{y}) = 16.719 \text{ units}.$$

The Lagrange multiplier λ is referred to as the *marginal proctivity of money* and describes the number of addition units of production for each unit of money spent on that production.



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