Chapter 5:
IDENTITY, INVERSE & WELL DEFINED MAPPINGS

Mphako-Banda

UNIVERSITY OF THE WITWATERSRAND, JOHANNESBURG

SCHOOL OF MATHEMATICS

LEARNING OUTCOMES FOR THE LECTURE

By the end of this lecture, students will be able to:

- give proofs of properties of inverses of mappings
- state the Invertibility Theorem
- prove the Invertibility Theorem
-
-

Properties of the inverses of mapping

Theorem (5.3.1)

Let $\alpha : A \to B$ and $\beta : B \to C$ denote mappings

- **(i)** $1_A : A \to A$ *is invertible and* $(1_A)^{-1} = 1_A$
- **(ii)** *If α is invertible, then α^{-1} is unique and is invertible with* $(\alpha^{-1})^{-1} = \alpha$
- **(iii)** *If α and β are both invertible, then $\beta\alpha$ is invertible and* $(\beta\alpha)^{-1} = \alpha^{-1}\beta^{-1}$.

Proof:

(i) Let $1_A : A \to A$ be the mapping with $1_A(a) = a \quad \forall a \in A$. Then $1_A 1_A(x) = x \quad \forall x \in A$. so $1_A = (1_A)^{-1}$ as it satisfies the conditions of inverse in definition 5.2.1 part(8).

(ii) Let $\alpha : A \to B$ be invertible and assume both $\beta : B \to A$ and $\gamma : B \to A$ are inverses of α.

So $\gamma\alpha = 1_A$ and $\beta\alpha = 1_A$ by definition of inverse 5.2.1 part (8).
Also $\alpha\gamma = 1_B$ and $\alpha\beta = 1_B$
Thus $\gamma(\alpha\beta) = (\gamma\alpha)\beta = 1_A(\beta) = \beta$
but $\gamma(\alpha\beta) = \gamma(1_B) = \gamma$ so $\beta = \gamma$ and inverses are unique if they exist.

NOTE: $1_A\beta(x) = \beta(x) \quad \forall x \in B$ and $\gamma 1_B(y) = \gamma(y) \quad \forall y \in B$. That is two mappings are equal if they have the same value for all elements in the Domain.

$$\alpha^{-1}\alpha(a) = \alpha^{-1}(\alpha(a)) = 1_A(a) = a = \alpha\alpha^{-1}(a)$$

$$\therefore \quad (\alpha^{-1})^{-1} = \alpha$$

(iii) Assume $\alpha : A \to B$ and $\beta : B \to C$ are both invertible
$\beta\alpha : A \to C$ is a well defined mapping and
$\alpha^{-1}\beta^{-1} : C \to A$ is a well defined mapping (since
$\alpha^{-1} : B \to A$ and $\beta^{-1} : C \to B$.)
Now $(\beta\alpha)(\alpha^{-1}\beta^{-1}) = \beta(1_B)\beta^{-1} = \beta\beta^{-1} = 1_C$
and $(\alpha^{-1}\beta^{-1})(\beta\alpha) = \alpha^{-1}(\beta^{-1}\beta)\alpha = \alpha^{-1}(1_B)\alpha =$
$\alpha^{-1}\alpha = 1_A$
$\therefore (\beta\alpha)^{-1} = \alpha^{-1}\beta^{-1}$ as it is unique if it exists.

Theorem (5.3.2 INVERTIBILITY THEOREM)

A mapping $\alpha : A \to B$ is invertible if and only if it is a bijection. Thus if $\alpha : A \to B$ is a bijection then α^{-1} exists as a mapping from B to A and this inverse is unique. Further if a mapping is invertible then it is a bijection.

invertible=bijection=1-1 & surjective

f and only if is a double
mplication.
prove the following:
(i)if ɑ is invertible ⇒ α is 1-1
and surjective
(ii)⟸) ɑ is 1-1 and surjective
⇒ ɑ is a bijection

PROOF:

$\Rightarrow$: Assume $\alpha : A \to B$ is invertible so $\exists \beta : B \to A$ such that $\beta\alpha = 1_A$ and $\alpha\beta = 1_B$ and $\beta = \alpha^{-1}$. Show that α is a bijection that is α is $1-1$ and onto.

$\underline{\alpha, \quad 1-1 \quad}$: Let $x, y \in A$.

$$\begin{aligned} \alpha(x) = \alpha(y) &\Rightarrow \beta(\alpha(x)) = \beta(\alpha(y)) \text{ since } \beta \text{ is well defined} \\ &\Rightarrow (\beta\alpha)(x) = (\beta\alpha)(y) \text{ definition of composition} \\ &\Rightarrow 1_A(x) = 1_A(y) \text{ since } \beta\alpha = 1_A \\ &\Rightarrow x = y \text{ since } 1_A \text{ identity map} \end{aligned}$$

Therefore α is $1-1$ map.

α, onto :

Let $y \in B$ then $\beta(y) \in A$ since $D(\beta) = B$.
So $\alpha(\beta(y)) \in B$ since $D(\alpha) = A$.
But $\alpha(\beta(y)) = (\alpha\beta)(y) = 1_B(y) = y$ Since 1_B identity on B.
$\therefore \quad \alpha$ is onto B.
$\therefore \quad \alpha$ is a bijection of A to B.
$\Leftarrow$: Assume α is a bijection ($1-1$ and onto)
Define $\beta : B \to A$ by $\beta(y) = x$ iff $\alpha(x) = y$.

since α is 1-1 and onto

well defined

We check β is (W.D) map, $\alpha\beta = 1_B$ and $\beta\alpha = 1_A$.

or any y_1 and y_2 in B, there exist x_1 and x_2 in A

(i) $y_1 = y_2$ for $y_1, y_2 \in B \Rightarrow \quad \exists x_1, x_2 \in A$ with $\alpha(x_1) = y_1$, $\alpha(x_2) = y_2$, and $\alpha(x_1) = \alpha(x_2)$ since α is onto

$\Rightarrow \quad x_1 = x_2$ since α is $1-1$

$\Rightarrow \quad \beta(y_1) = \beta(y_2)$ by definition of β

$\Rightarrow \quad \beta$ is well defined.

we show that $\alpha\beta=1_B$

(ii) $\alpha\beta(y)=\alpha(x)$ where $x=\beta(y)$. Thus $\alpha(x)=y$ by definition of β

$\therefore \quad \alpha\beta(y)=\alpha(x)=y$ where $x=\beta(y)$

$\therefore \quad \alpha\beta=1_B$.

(iii) $\beta\alpha(x)=y$

$\Rightarrow \quad \beta(\alpha(x))=y \Rightarrow \quad \alpha(y)=\alpha(x)$ by definition of β

$\Rightarrow \quad x=y \quad (\alpha \quad is \quad 1-1)$

Hence $\beta(\alpha(x))=x \quad \forall x\in A$. Thus $\beta\alpha=1_A$.

Thus α is invertible iff α is a bijection.

we show that $\beta\alpha = 1_A$