# Multivariable Calculus

MATH2007

## 1.4 Directional Derivatives (Part 1)

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**Definition** (1.4.1)**.**

Let $f : \mathbb{R}^n \to \mathbb{R}$ and let $\underline{u}$ be a **unit vector** in $\mathbb{R}^n$. We define the **directional derivative** of $f$ at $\underline{x}$ in the direction $\underline{u}$ by

$$D_{\underline{u}}f(\underline{x}) = \lim_{t\to 0} \frac{f(\underline{x} + t\underline{u}) - f(\underline{x})}{t}$$

for each $\underline{u}$ and $\underline{x}$ for which limit exists.

unit vector : $\|\underline{u}\| = 1$

**Note.**

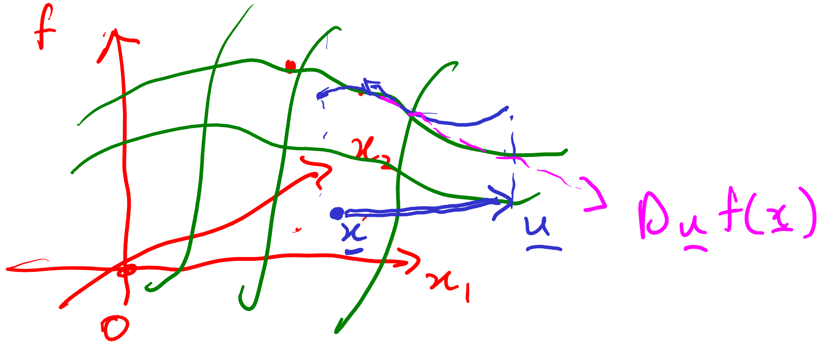
(a) It follows from the definition that $D_{\underline{u}} f(\underline{x})$ is the rate of increase of $f$ along the path $\underline{x} + t\underline{u}, \quad t \in \mathbb{R}$, at $t = 0$.

(b) The partial derivative with respect to the $k^{th}$ variable at $\underline{x}$ is the directional derivative in the direction of the $k^{th}$ basis vector, $e_k$.

b) $$\frac{\partial f}{\partial x_j} = D_{\underline{e}_j} f(\underline{x}) = \lim_{t \to 0} \frac{f(\underline{x} + t\underline{e}_j) - f(\underline{x})}{t}$$

$$\underline{e}_j = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow \text{j-th entry}$$

a)



**Example.** Let $f(x_1, x_2) = x_2 e^{x_1}$. Find the directional derivative of $f$ at $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ in the direction $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

$$D_{\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}} f\begin{pmatrix} 1 \\ 2 \end{pmatrix} = D_{\underline{u}} f\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

where $\underline{u} = \frac{1}{\left\|\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right\|}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$= \lim_{t \to 0} \frac{f\left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} + t\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}\right) - f\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{t}$$

$$= \lim_{t \to 0} \frac{f\begin{pmatrix} 1 - t/\sqrt{2} \\ 2 + t/\sqrt{2} \end{pmatrix} - f\begin{pmatrix} 1 \\ 2 \end{pmatrix}}{t}$$

$$= \lim_{t \to 0} \frac{(2 + t/\sqrt{2})e^{1 - t/\sqrt{2}} - 2e^1}{t}$$

$$\overset{\text{l'Hôpital}}{=} \lim_{t \to 0} \frac{\frac{1}{\sqrt{2}}e^{1 - t/\sqrt{2}} - \frac{1}{\sqrt{2}}(2 + t/\sqrt{2})e^{1 - t/\sqrt{2}}}{1} = \frac{1}{\sqrt{2}}e - \frac{2}{\sqrt{2}}e = -\frac{e}{\sqrt{2}}.$$

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**Example.** Find the directional derivative of $f(x,y) = (x+y)\cos(y^2)$ at $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ in the direction $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$.

$$D_{\underline{u}} f\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \lim_{t\to 0} \frac{f\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} + t\frac{1}{\sqrt{40}}\begin{pmatrix} 2 \\ 6 \end{pmatrix}\right) - f\begin{pmatrix} -1 \\ 1 \end{pmatrix}}{t}$$

where $\underline{u} = \frac{1}{\left\|\begin{pmatrix} 2 \\ 6 \end{pmatrix}\right\|}\begin{pmatrix} 2 \\ 6 \end{pmatrix} = \frac{1}{\sqrt{40}}\begin{pmatrix} 2 \\ 6 \end{pmatrix}$

$$= \lim_{t\to 0} \frac{f\begin{pmatrix} -1 + 2t/\sqrt{40} \\ 1 + 6t/\sqrt{40} \end{pmatrix} - f\begin{pmatrix} -1 \\ 1 \end{pmatrix}}{t}$$

$$= \lim_{t\to 0} \frac{8\cancel{t}/\sqrt{40}\cos\left(\left(1+6t/\sqrt{40}\right)^2\right) - 0}{\cancel{t}}$$

$$= \frac{8}{\sqrt{40}}\cos(1).$$

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1.4 Directional Derivatives
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**Theorem** (1.4.2)**.**
Let $f : \mathbb{R}^n \to \mathbb{R}$ and let $\underline{u}$ be a **unit vector** in $\mathbb{R}^n$. Then

$$D_{\underline{u}} f(\underline{x}) = \underline{u} \cdot \nabla f(\underline{x}).$$

dot product.

*Proof.* Omitted. □

$\frac{d}{dt} f(\underline{r}(t))$ where $\underline{r}(t) = \underline{x} + t\underline{u}$ (chain rule).

**Example.** Find the directional derivative of $f(x_1, x_2, x_3) = (1+x_2)e^{x_1 \sin(x_3)}$ at $\begin{pmatrix} 1 \\ 2 \\ \frac{\pi}{2} \end{pmatrix}$ in the direction $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

$$\underline{u} = \frac{1}{\left\|\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\|} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

$$\nabla f(x_1, x_2, x_3) = \begin{pmatrix} (1+x_2)\sin(x_3) e^{x_1 \sin(x_3)} \\ e^{x_1 \sin(x_3)} \\ (1+x_2) x_1 \cos(x_3) e^{x_1 \sin(x_3)} \end{pmatrix}$$

$$D_{\underline{u}} f\begin{pmatrix} 1 \\ 2 \\ \pi/2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \nabla f\begin{pmatrix} 1 \\ 2 \\ \pi/2 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3e \\ e \\ 0 \end{pmatrix}$$

$$= \frac{2}{\sqrt{3}} e.$$

**Theorem** (1.4.3)**.**
Let $f : \mathbb{R}^n \to \mathbb{R}$.

(i) The direction of maximum rate of increase of $f$ at $\underline{x}$ is $\nabla f(\underline{x})$ and the rate of increase of $f$ in this direction is $\|\nabla f(\underline{x})\|$.

(ii) The direction of minimum rate of increase of $f$ at $\underline{x}$ is $-\nabla f(\underline{x})$ and the rate of increase of $f$ in this direction is $-\|\nabla f(\underline{x})\|$.

*Proof.* □

$$D_{\underline{u}} f(\underline{x}) = \underline{u} \cdot \nabla f(\underline{x}) = \|\underline{u}\| \cdot \|\nabla f(\underline{x})\| \cos\theta \qquad \cos\theta \in [-1,1]$$

where $\theta$ is the angle between $\underline{u}$ and $\nabla f(\underline{x})$.

(i) Since $D_{\underline{u}} f(\underline{x})$ achieves its maximum at $\cos\theta = 1$ (when $\theta = 2k\pi$ $k \in \mathbb{Z}$) when $\underline{u}$ is in the same direction as $\nabla f(\underline{x})$; for which $\underline{u} = \frac{1}{\|\nabla f(\underline{x})\|} \nabla f(\underline{x})$.

$$D_{\underline{u}} f(\underline{x}) = \frac{1}{\|\nabla f(\underline{x})\|} \underbrace{(\nabla f(\underline{x})) \cdot (\nabla f(\underline{x}))}_{\| \cdots \|^2} = \|\nabla f(\underline{x})\|.$$

(ii) The direction of max. decrease is achieven when $\cos\theta = -1$ (i.e $\theta = \pi + 2k\pi$, $k \in \mathbb{Z}$); for which $\underline{u} = \frac{1}{\|-\nabla f(\underline{x})\|}(-\nabla f(\underline{x}))$

and

$$D_{\underline{u}} f(\underline{x}) = \frac{1}{\|-\nabla f(\underline{x})\|}(-\nabla f(\underline{x})) \cdot (\nabla f(\underline{x}))$$

$$= \frac{-1}{\|\nabla f(\underline{x})\|} \|\nabla f(\underline{x})\|^2$$

$$= -\|\nabla f(\underline{x})\|.$$

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1.4 Directional Derivatives
(Part 4)

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**Theorem** (1.4.3)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$.

(i) The direction of maximum rate of increase of $f$ at $\underline{x}$ is $\nabla f(\underline{x})$ and the rate of increase of $f$ in this direction is $\|\nabla f(\underline{x})\|$.

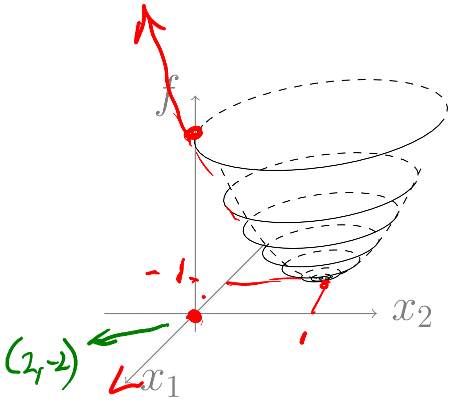
(ii) The direction of minimum rate of increase of $f$ at $\underline{x}$ is $-\nabla f(\underline{x})$ and the rate of increase of $f$ in this direction is $-\|\nabla f(\underline{x})\|$.

**Example.** Find the directions of maximum and minimum rates of increase of the following function at the given point. In each case also give the maximum and minimum rates of increase.

$f(x_1, x_2) = (x_1 + 1)^2 + (x_2 - 1)^2$ at $\begin{pmatrix}0\\0\end{pmatrix}$

$f(x_1, x_2) = r^2 \geq 0$

$(x_1+1)^2 + (x_2-1)^2 = r^2$




direction of max rate of increase at $\begin{pmatrix}0\\0\end{pmatrix}$

$$\nabla f(0,0) = \left.\begin{pmatrix}2(x_1+1)\\2(x_2-1)\end{pmatrix}\right|_{\substack{x_1=0\\x_2=0}} = \begin{pmatrix}2\\-2\end{pmatrix}$$

rate at $\begin{pmatrix}0\\0\end{pmatrix}$ of increase: $\left\|\begin{pmatrix}2\\-2\end{pmatrix}\right\| = \sqrt{8} = 2\sqrt{2}$.

direcction of min rate of increase at $\begin{pmatrix}0\\0\end{pmatrix}$ is $\begin{pmatrix}-2\\2\end{pmatrix}$

rate at $\begin{pmatrix}0\\0\end{pmatrix}$ of (min) increase: $-2\sqrt{2}$.

**Example.** Find the directions of maximum and minimum rates of increase of the following function at the given point. In each case also give the maximum and minimum rates of increase.

$$f(x, y, z) = (x^2 + y^2 - e^z) \text{ at } \begin{pmatrix} 1 \\ 1 \\ \ln 2 \end{pmatrix}.$$

$x^2+y^2-e^z \qquad z=\ln 2$

$x^2+y^2=2$

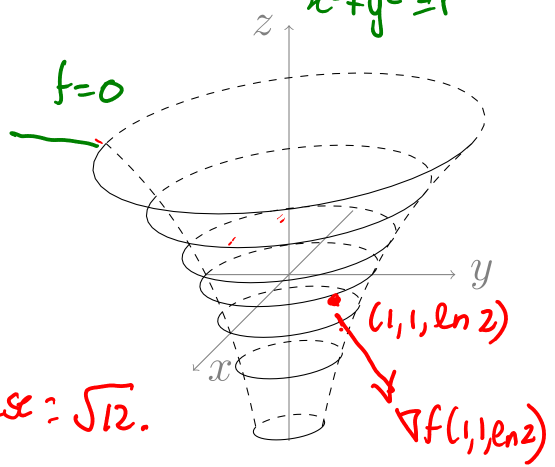
(*) $z=\ln r^2$

$\Rightarrow f=x^2+y^2-r^2$

$\Rightarrow x^2+y^2=r^2$

$f(1,1,\ln 2)=0$ (*)

Surface: $f=0$




Direction of max. Inc. at $\begin{pmatrix} 1 \\ 1 \\ \ln 2 \end{pmatrix}$ is

$$\nabla f\begin{pmatrix} 1 \\ 1 \\ \ln 2 \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ -e^z \end{pmatrix}\Bigg|_{\substack{x=1\\y=1\\z=\ln 2}} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}.$$

max rate of increase: $\sqrt{12}$.

Direction of min. rate of inc. at $\begin{pmatrix} 1 \\ 1 \\ \ln 2 \end{pmatrix}$ is $\begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix}$, rate of increase: $-\sqrt{12}$.