Multivariable Calculus

MATH2007

1.1 Derivatives and Differentials (Part 1)

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Definition. Let $f : \mathbb{R}^n \to \mathbb{R}$. We define the **partial derivative** of f at $\underline{x} = (x_1, ..., x_n)$ with respect to x_j by

$$\frac{\partial f(\underline{x})}{\partial x_j} = \lim_{h \to 0} \frac{f(x_1, ..., x_{j-1}, x_j + h, x_{j+1}, ..., x_n) - f(x_1, ..., x_{j-1}, x_j, x_{j+1}, ..., x_n)}{h}.$$

$(x_1, ..., x_n)$

Note. (a) $\dfrac{\partial f(\underline{x})}{\partial x_j}$ is also denoted by $D_j f$ and by f_{x_j}.

(b) In terms of the standard basis:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_j + h \\ \vdots \\ x_n \end{pmatrix} = \underline{x} + h\underline{e_j} \qquad \underline{e_j} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\text{-th row}$$

$$\frac{\partial f(\underline{x})}{\partial x_j} = \lim_{h \to 0} \frac{f(\underline{x} + h\underline{e_j}) - f(\underline{x})}{h}.$$

Example. (a) Let $f(x_1, x_2, x_3) = x_3 \ln(x_1^2 + x_2^2)$. Then

$$\frac{\partial f(\underline{x})}{\partial x_1} = x_3 \frac{\partial}{\partial x_1} \ln(x_1^2+x_2^2) = x_3 \frac{1}{x_1^2+x_2^2} \frac{\partial}{\partial x_1}(x_1^2+x_2^2) = \frac{x_3}{x_1^2+x_2^2} \cdot 2x_1$$

(x_2, x_3 constants)

$$\frac{\partial f(\underline{x})}{\partial x_2} = x_3 \cdot \frac{1}{x_1^2+x_2^2} 2x_2$$

(x_1, x_3 constants)

$$\frac{\partial f(\underline{x})}{\partial x_3} = 1 \cdot \underbrace{\ln(x_1^2+x_2^2)}_{\text{constant}}$$

(x_1, x_2 constant)

(b) Let $f(u,v) = ue^{2v}$. Find $\left.\underbrace{\dfrac{\partial f}{\partial u}\right|_{(2,3)}}_{\frac{\partial f}{\partial u}(2,3)}$ and $\left.\dfrac{\partial f}{\partial v}\right|_{(2,3)}$.

$$\frac{\partial f}{\partial v}(2,3)$$

$$\frac{\partial f}{\partial u} = 1 \cdot e^{2v} = e^{2v}$$

$$\frac{\partial f}{\partial v} = u\left(2e^{2v}\right) = 2ue^{2v}$$

$$\left.\frac{\partial f}{\partial u}\right|_{(u,v)=(2,3)} = e^{6}$$

$$\left.\frac{\partial f}{\partial v}\right|_{(u,v)=(2,3)} = 4e^{6}.$$

Note. The following rules for derivatives hold for partial derivatives.

(a) $$\frac{\partial(\alpha f+\beta g)}{\partial x_j}=\alpha\frac{\partial f}{\partial x_j}+\beta\frac{\partial g}{\partial x_j},\qquad \alpha,\beta\in\mathbb{R},\qquad f,g:\mathbb{R}^n\to\mathbb{R}.$$

constant

(linearity)

(b) $$\frac{\partial(fg)}{\partial x_j}=f\frac{\partial g}{\partial x_j}+g\frac{\partial f}{\partial x_j},\qquad f,g:\mathbb{R}^n\to\mathbb{R}.$$

(product rule)

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1.1 Derivatives and Differentials
(Part 2)

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Definition. Let $\underline{F} : \underbrace{\mathbb{R}^n}_{\text{n-variables}} \to \mathbb{R}^m$, we define $\underline{F}'(\underline{x})$ the matrix **derivative** of $\underline{F}$ at $\underline{x}$ by

$$\underline{F}' = \underbrace{\begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}}_{n \text{ columns}} \left.\vphantom{\begin{bmatrix} \frac{\partial F_1}{\partial x_1} \\ \vdots \\ \frac{\partial F_m}{\partial x_1} \end{bmatrix}}\right\} m \text{ rows}.$$

$$\underline{F}(\underline{x}) = \begin{pmatrix} F_1(\underline{x}) \\ F_2(\underline{x}) \\ \vdots \\ F_m(\underline{x}) \end{pmatrix}$$

m×n matrix

Example. (a) Let $\underline{F}(x_1, x_2, x_3) = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} (x_1^2 + x_2^2)\cos x_3 \\ (x_1^2 + x_2^2)\sin x_3 \end{pmatrix}$.

$\underline{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$\underline{F}'(x_1, x_2, x_3)$ is a 2×3 matrix.

$$\underline{F}'(x_1, x_2, x_3) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} \end{pmatrix}$$

$$= \begin{pmatrix} 2x_1 \cos x_3 & 2x_2 \cos x_3 & -(x_1^2+x_2^2)\sin x_3 \\ 2x_1 \sin x_3 & 2x_2 \sin x_3 & (x_1^2+x_2^2)\cos x_3 \end{pmatrix}.$$

(b) Let $\phi(x_1, x_2) = x_1 e^{-x_2}$. Find $\phi'(2, 3)$.

$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^1$

$\phi'(x_1, x_2)$ is a 1×2 matrix.

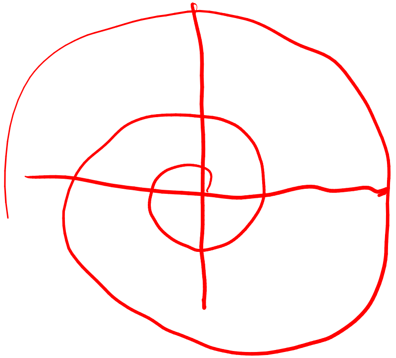
$$\phi'(x_1, x_2) = \begin{pmatrix} \frac{\partial \phi}{\partial x_1} & \frac{\partial \phi}{\partial x_2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-x_2} & -x_1 e^{-x_2} \end{pmatrix}$$

$$\phi'(2,3) = \phi'(x_1 = 2, x_2 = 3) = \begin{pmatrix} e^{-3} & -2e^{-3} \end{pmatrix}.$$

(c) Let

$$\underline{r}(t) = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} e^t \cos t \\ e^t \sin t \end{pmatrix}$$



$$\underline{r} : \mathbb{R}^1 \to \mathbb{R}^2$$

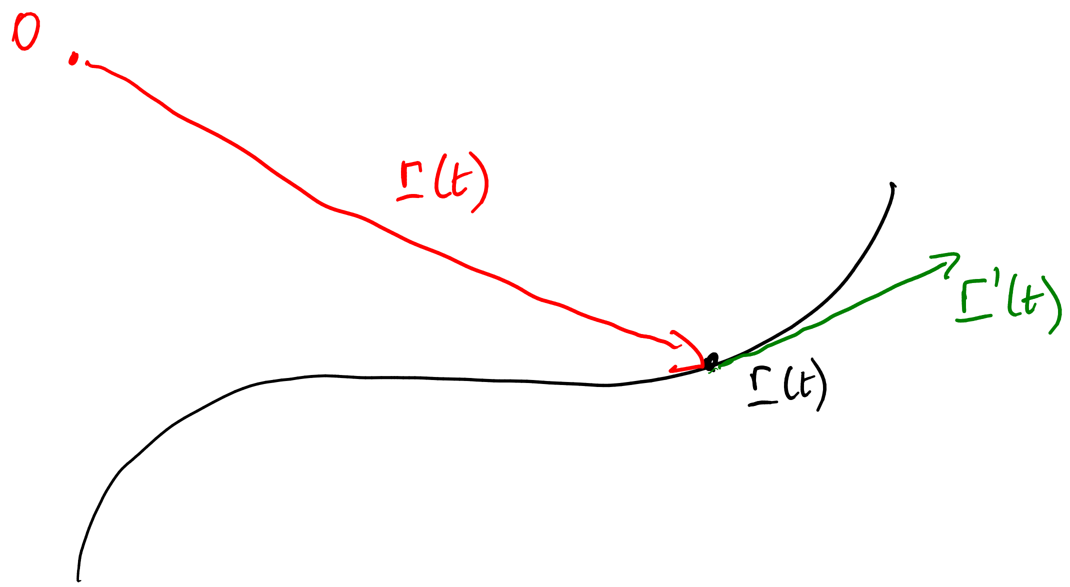
$$\underline{r}'(t) = \begin{pmatrix} \frac{\partial r_1}{\partial t} \\ \frac{\partial r_2}{\partial t} \end{pmatrix}$$

$$= \begin{pmatrix} e^t \cos t - e^t \sin t \\ e^t \sin t + e^t \cos t \end{pmatrix}.$$

Note.

(i) If $\underline{r}(t)$ gives the position of a particle at time t, then $\underline{r}'(t)$ is the velocity at time t and $\underline{r}''(t) = (\underline{r}'(t))'$ is the acceleration at time t.

(ii) $\underline{r}'(t)$ is a vector tangent to the curve traced out by $\underline{r}(t)$, $t \in \mathbb{R}$, at the point $\underline{r}(t)$.



Example. (a) Consider the curve $y = x^2$ in $\mathbb{R}^2$, this can be parametrized (written in parametric form) by

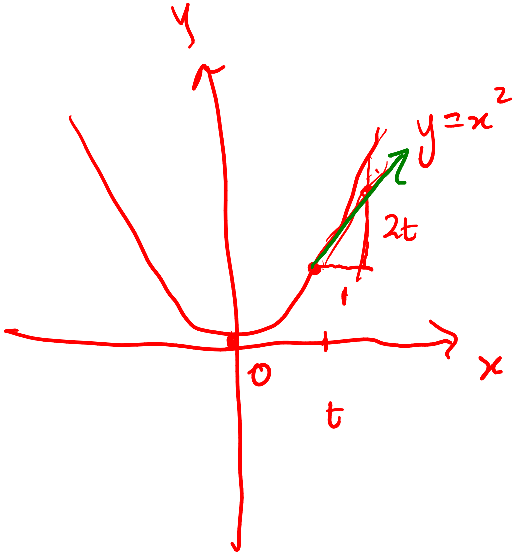
$$\underline{r}(t) = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

$x = t$

$y = t^2$

Illustrate $\underline{r}'(t)$.

$$\underline{r}'(t) = \begin{pmatrix} 1 \\ 2t \end{pmatrix} \qquad \left.\begin{matrix} x \text{ change} \\ y \text{ change} \end{matrix}\right\} \text{ for slope}$$



$y = x^2$ $\qquad \dfrac{dy}{dx} = 2x$

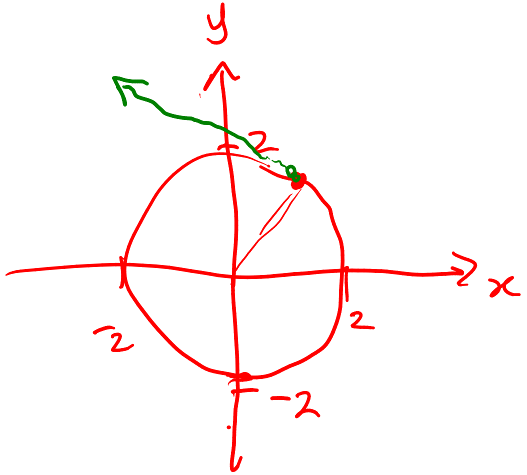
(b) Consider the circle centre $(0,0)$ radius 2. This can be parametrized by

$$\underline{r}(t) = \begin{pmatrix} 2\cos t \\ 2\sin t \end{pmatrix} \begin{matrix} x \\ y \end{matrix} \quad t \in [0, 2\pi] \qquad \text{i.e.} \qquad x = r\cos\theta, \qquad y = r\sin\theta.$$

Find the tangent line to this circle at $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} = \begin{pmatrix} 2\cos t_0 \\ 2\sin t_0 \end{pmatrix} = \underline{r}(t_0) \Rightarrow \quad t_0 = \frac{\pi}{3}$

tangent line: $\quad \underline{r}(t_0) + t\,\underline{r}'(t_0)$

$$\underline{r}'(t) = \begin{pmatrix} -2\sin t \\ 2\cos t \end{pmatrix} \qquad \underline{r}'\left(\frac{\pi}{3}\right) = \begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$$



equation for the tangent line:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix} + t\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} 1 - t\sqrt{3} \\ \sqrt{3} + t \end{pmatrix}$$

$y = \cdots\cdots$

Multivariable Calculus

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1.1 Derivatives and Differentials
(Part 3)

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Definition. Let $\underline{F} : \mathbb{R}^n \to \mathbb{R}^m$ and $\underline{a} \in \mathbb{R}^n$. We define the **differential** of $\underline{F}$ at $\underline{a}$ to be the **linear map** given by

$$d\underline{F}(\underline{a};\underline{h}) = \underline{F}'(\underline{a})\underline{h}$$

(*where $\underline{a}$ is the point of evaluation and $\underline{h}$ is a small change*).

Note. For small $\underline{h} \in \mathbb{R}^n$ we have

$$\underline{F}(\underline{a}+\underline{h}) \cong \underline{F}(\underline{a}) + d\underline{F}(\underline{a};\underline{h}) = \underline{F}(\underline{a}) + \underline{F}'(\underline{a})\underline{h}.$$

$f(a+h) \simeq f(a) + \underbrace{h f'(a)}_{df(a;h)}$

(i.e a small change in the domain $\underline{h}$ gives a small change in the range $\underline{F}'(\underline{a})\underline{h}$).

linear: (in $\underline{h}$) for all constants $\alpha, \beta \in \mathbb{R}$, and $\underline{h}_1, \underline{h}_2 \in \mathbb{R}^n$

$$d\underline{F}(\underline{a}, \alpha \underline{h}_1 + \beta \underline{h}_2) = \alpha\, d\underline{F}(\underline{a}, \underline{h}_1) + \beta\, d\underline{F}(\underline{a}, \underline{h}_2)$$

Note. If $\underline{u}$ and $\underline{v}$ are column vectors in $\mathbb{R}^n$, i.e

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} \text{ and } \underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

then

$$\underline{u} \cdot \underline{v} = \sum_{i=1}^{n} u_i v_i = \underline{u}^T \underline{v}.$$

If $\underline{u} \cdot \underline{v} \in \mathbb{C}^n$ then $\underline{u} \cdot \underline{v} = \underline{u}^T \overline{\underline{v}}$ but for $\underline{v} \in \mathbb{R}^n$ we have $\underline{v} = \overline{\underline{v}}$.

Theorem. Let $\underline{F}, \underline{G} : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}$, i.e.

$$\underline{F}(x_1, \ldots, x_n) = \begin{pmatrix} F_1 \\ \vdots \\ F_m \end{pmatrix} \quad \text{and} \quad \underline{G}(x_1, \ldots, x_n) = \begin{pmatrix} G_1 \\ \vdots \\ G_m \end{pmatrix},$$

then: $\underline{F}(\underline{x}) \cdot \underline{G}(\underline{x})$

(a) $(\underline{F} \cdot \underline{G})' = \underline{F}^T \underline{G}' + \underline{G}^T \underline{F}'$; (product rule $\to$ [vector–vector])

$1\times n$ $\quad 1\times m \quad m\times n \quad 1\times m \quad m\times n$

(b) $(\alpha \underline{F} + \beta \underline{G})' = \alpha \underline{F}' + \beta \underline{G}'$ for all $\alpha, \beta \in \mathbb{R}$; (linearity)

constants

$m\times n \quad m\times n \quad m\times 1 \quad 1\times n$

(c) $(g\underline{F})' = g\underline{F}' + \underline{F}g'$. (product rule $\to$ [scalar–vector])

$g(\underline{x})\underline{F}(\underline{x})$ $\qquad$ $g\underline{F} : \mathbb{R}^n \to \mathbb{R}^m$

order is important

Proof of (a). ∎

$\underline{F} \cdot \underline{G} : \mathbb{R}^n \to \mathbb{R}$ $\qquad$ $\underline{F}(\underline{x}) \cdot \underline{G}(\underline{x}) = \sum_{j=1}^{m} F_j(\underline{x}) G_j(\underline{x})$

$$\frac{\partial}{\partial x_k} \underline{F} \cdot \underline{G} = \sum_{j=1}^{m} \frac{\partial}{\partial x_k} F_j \cdot G_j = \sum_{j=1}^{m} \left(G_j \frac{\partial F_j}{\partial x_k} + F_j \frac{\partial G_j}{\partial x_k} \right) = \sum_{j=1}^{m} G_j \frac{\partial F_j}{\partial x_k} + \sum_{j=1}^{m} F_j \frac{\partial G_j}{\partial x_k}$$

$$(\underline{F}\cdot\underline{G})' = \left(\frac{\partial}{\partial x_1}\underline{F}\cdot\underline{G} \;\; \cdots \;\; \frac{\partial}{\partial x_n}\underline{F}\cdot\underline{G}\right)$$

$$= \left(\sum_{j=1}^{m} G_j\frac{\partial F_j}{\partial x_1} + \sum_{j=1}^{m} F_j\frac{\partial G_j}{\partial x_1} \;\; \cdots \;\; \sum_{j=1}^{m} G_j\frac{\partial F_j}{\partial x_n} + \sum_{j=1}^{m} F_j\frac{\partial G_j}{\partial x_n}\right)$$

$$= \overbrace{\left(\sum_{j=1}^{m} G_j\frac{\partial F_j}{\partial x_1} \;\; \cdots \;\; \sum_{j=1}^{m} G_j\frac{\partial F_j}{\partial x_n}\right)}^{\underline{G}^T\cdot\underline{F}'} + \underbrace{\left(\sum_{j=1}^{m} F_j\frac{\partial G_j}{\partial x_1} \;\; \cdots \;\; \sum_{j=1}^{m} F_j\frac{\partial G_j}{\partial x_n}\right)}_{\underline{F}^T\cdot\underline{G}'}$$

$$\underline{G}^T\cdot\underline{F}' = (G_1 \;\cdots\; G_m)\begin{pmatrix}\frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n}\\ \vdots & & \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n}\end{pmatrix} = \left(\sum_{j=1}^{m} G_j\frac{\partial F_j}{\partial x_1} \;\; \cdots \;\; \sum_{j=1}^{m} G_j\frac{\partial F_j}{\partial x_n}\right)$$

Thus $(\underline{F}\cdot\underline{G})' = \underline{G}^T\cdot\underline{F}' + \underline{F}^T\cdot\underline{G}'$. $\square$

Example. (a) Let $\underline{T}(u,v) = e^{-u+v}\begin{pmatrix} uv \\ u^2+v^2 \end{pmatrix}$. Find $\underline{T}'$ using part (c) with $g = e^{-u+v}$ and

$$\underline{F} = \begin{pmatrix} uv \\ u^2+v^2 \end{pmatrix}.$$

$$\underline{T}(u,v) = \begin{pmatrix} e^{-u+v}\, uv \\ e^{-u+v}(u^2+v^2) \end{pmatrix}$$

$$\underline{T}'(\underline{u},v) = g\,\underline{F}' + \underline{F}\,g'$$

$$= e^{-u+v}\begin{pmatrix} v & u \\ 2u & 2v \end{pmatrix} + \begin{pmatrix} uv \\ u^2+v^2 \end{pmatrix}\begin{pmatrix} -e^{-u+v} & e^{-u+v} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-u+v}v & e^{-u+v}u \\ 2e^{-u+v}u & 2e^{-u+v}v \end{pmatrix} + \begin{pmatrix} -e^{-u+v}uv & e^{-u+v}uv \\ -e^{-u+v}(u^2+v^2) & e^{-u+v}(u^2+v^2) \end{pmatrix}$$

$$= e^{-u+v}\begin{pmatrix} v-uv & u+uv \\ 2u-u^2-v^2 & 2v+u^2+v^2 \end{pmatrix}.$$

(b) If $\|\underline{F}\|$ is constant, show that $(\underline{F}')^T\underline{F} = \underline{0}$.

$\|\underline{F}\| = c$ (c a constant)

$\therefore \sqrt{\underline{F}\cdot\underline{F}} = c$ $\therefore$ $\underline{F}\cdot\underline{F} = c^2$ (for all $\underline{x}$)

$\therefore$ $(\underline{F}\cdot\underline{F})' = \underline{0}$ Thm (a) $\underline{F} = \underline{G}$

$\therefore$ $\underline{F}^T \overset{G'}{\underline{F}'} + \overset{G^T}{\underline{F}^T} \underline{F}' = \underline{0}$ (by Thm (a))

$\therefore$ $2\underline{F}^T\underline{F}' = \underline{0}$

$\therefore$ $\underline{F}^T\underline{F}' = \underline{0}$ $\therefore$ $(\underline{F}^T\underline{F}')^T = 0$

$(AB)^T = B^T A^T$

$(A^T)^T = A$

$\therefore$ $(\underline{F}')^T\underline{F} = \underline{0}$

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1.1 Derivatives and Differentials
(Part 4)

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Theorem. Let $\underline{r}, \underline{s} : \mathbb{R} \to \mathbb{R}^3$ i.e.

$$\left[\underline{r}(t) = \begin{pmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{pmatrix} \text{ and } \underline{s}(t) = \begin{pmatrix} s_1(t) \\ s_2(t) \\ s_3(t) \end{pmatrix} \right]$$

then

(a) $(\underline{r} \cdot \underline{s})' = \underline{r} \cdot \underline{s}' + \underline{r}' \cdot \underline{s}$

(b) $(\underline{r} \times \underline{s})' = \underline{r}' \times \underline{s} + \underline{r} \times \underline{s}'$.

order matters

Proof. $\square$

(a) $(\underline{r} \cdot \underline{s})' = \underline{r}^T \underline{s}' + \underline{s}^T \underline{r}'$ (by Thm 1.1.4(a))

$= \underline{r} \cdot \underline{s}' + \underline{s} \cdot \underline{r}'$ (property of $\cdot$)

$= \underline{r} \cdot \underline{s}' + \underline{r}' \cdot \underline{s}$ (commutativity of $\cdot$)

b) $\underline{r} \times \underline{s} = \begin{pmatrix} r_2 s_3 - s_2 r_3 \\ r_3 s_1 - s_3 r_1 \\ r_1 s_2 - s_1 r_2 \end{pmatrix}$

$$(\underline{r} \times \underline{s})' = \begin{pmatrix} r_2' s_3 + r_2 s_3' - s_2' r_3 - s_2 r_3' \\ r_3' s_1 + r_3 s_1' - s_3' r_1 - s_3 r_1' \\ r_1' s_2 + r_1 s_2' - s_1' r_2 - s_1 r_2' \end{pmatrix}$$ (product rule)

$$= \underline{r}' \times \underline{s} + \underline{r} \times \underline{s}'$$

$\underline{r}' = \begin{pmatrix} r_1' \\ r_2' \\ r_3' \end{pmatrix}$ $\underline{s}' = \begin{pmatrix} s_1' \\ s_2' \\ s_3' \end{pmatrix}$ $r_1' = \frac{\partial r_1}{\partial t} = \frac{dr_1}{dt}$.

Example. (a) If $\underline{r}$ has constant length, show that the vector $\underline{r}(t)$ is orthogonal to the path traced out by $\underline{r}(t)$, $t \in \mathbb{R}$.

$\|\underline{r}\| = c$ (c a constant)

$\underline{r}(t) \cdot \underline{r}(t) = c^2$ for all $t \in \mathbb{R}$

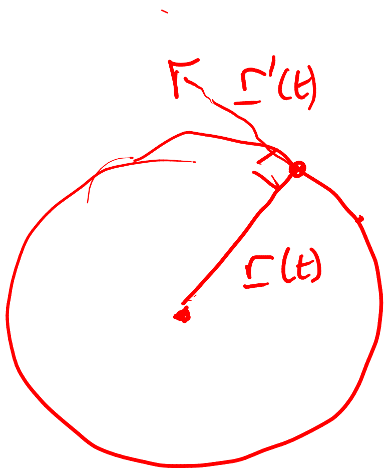
$(\underline{r}(t) \cdot \underline{r}(t))' = 0$

$2\,\underline{r}(t) \cdot \underline{r}'(t) = 0$ (by Thm (a)).

$\therefore\ \underline{r}(t) \cdot \underline{r}'(t) = 0$

$\therefore\ \underline{r}(t)$ is orthogonal to the tangent at $\underline{r}(t)$;

i.e. to the path traced out by $\underline{r}(t)$.



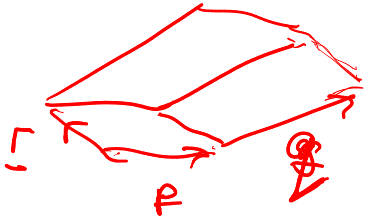
(b) Consider the parallelepiped with sides spanned by $\underline{p}(t)$, $\underline{q}(t)$, $\underline{r}(t)$. The volume of this solid is $|\underline{p}(t) \cdot (\underline{q}(t) \times \underline{r}(t))|$. Let

scalar

$$\underline{p} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \underline{q} = \begin{pmatrix} 0 \\ t \\ 1 \end{pmatrix}, \qquad \underline{r} = \begin{pmatrix} 0 \\ 2 \\ f(t) \end{pmatrix}.$$

Find the rate of change of this volume as a function of time and hence find a condition for a local max or min in volume at t_0.

$$g(t) = \underline{p}(t) \cdot (\underline{q}(t) \times \underline{r}(t)) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & t & 1 \\ 0 & 2 & f(t) \end{vmatrix}$$



$$= t f(t) - 2 \qquad g'(t) = f(t) + t f'(t).$$

using dot and cross product rules:

$$g'(t) = (\underline{p}(t) \cdot (\underline{q}(t) \times \underline{r}(t)))' = \underbrace{\cancel{\underline{p}' \cdot (\underline{q}(t) \times \underline{r}(t))}}_{0 \text{ since } \underline{p}' = \underline{0}} + \underline{p} \cdot (\underline{q}(t) \times \underline{r}(t))'$$

0 ine

$$g'(t) = \big(\underline{p}(t)\cdot(\underline{q}(t)\times\underline{r}(t))\big)' = \cancel{\underline{p}'\cdot(\underline{q}(t)\times\underline{r}(t))} + \underline{p}\cdot(\underline{q}(t)\times\underline{r}(t))'$$

$$= \underline{p}\cdot\big(\underline{q}'(t)\times\underline{r}(t) + \underline{q}(t)\times\underline{r}'(t)\big)$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\left[\begin{pmatrix}0\\1\\0\end{pmatrix}\times\begin{pmatrix}0\\2\\f(t)\end{pmatrix} + \begin{pmatrix}0\\t\\1\end{pmatrix}\times\begin{pmatrix}0\\0\\f'(t)\end{pmatrix}\right]$$

$$= f(t) + t f'(t).$$

Max / Min if $f(t_0) + t_0 f'(t_0) = 0$