MATH2001 (Basic Analysis)
Tutorial Test September 2020, MEMO

Question 1 : **[2 marks]**

Which one of the following statements is false:

A. The sequence (a_n) converges to $L \in \mathbb{R}$ if $\forall\, \epsilon > 0$, $\exists\, N_\epsilon \in \mathbb{R}$, such that $\forall\, n$,

$$n > N_\epsilon \implies |a_n - L| < \epsilon$$

B. If the sequence (a_n) converges to L then $\lim\limits_{n\to\infty} a_n = L$.

C. The sequence $\left(\dfrac{n+1}{n}\right)$ converges to 1.

D. The sequence $\left(\left|(-1)^{n+1}\right|\right)_{n=0}^{\infty}$ converges to 1.

E. None of the above.

Answer 1 : The statements A, B, C, and D are all correct. Thus the answer is E.

For A and B, see the study guide, Definition 2.2.

For C, $\lim\limits_{n\to\infty} \dfrac{n+1}{n} = \lim\limits_{n\to\infty} \dfrac{1+\frac{1}{n}}{1} = \dfrac{1+0}{1} = 1.$

For D, $\left(\left|(-1)^{n+1}\right|\right)_{n=0}^{\infty} = 1, 1, 1, \ldots, 1, 1, \ldots$

Question 2 : **[2 marks]**

Which one of the following statements is false:

A. The sequence $\left(\left(\dfrac{n+1}{n}\right)^n\right)$ is increasing.

B. $\lim\limits_{n\to\infty} n^2 - n + 10 = -\infty$.

C. The sequence $\left(\dfrac{n+1}{n}\right)$ is bounded below by 1.

D. Every Cauchy sequence converges.

E. None of the above.

Answer 2 : The statement B is false. Since, $\lim_{n\to\infty} n^2 - n + 10 = \lim_{n\to\infty} n(n - 1 + \frac{10}{n}) = \infty(\infty - 1 + 0) = \infty.\infty = \infty \neq -\infty$ (see the rules).

For A: $\frac{a_{n+1}}{a_n} = \frac{\left(\frac{(n+1)+1}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^n} = \left(\frac{n+2}{n+1}\right)^{n+1}\left(\frac{n}{n+1}\right)^n = \left(\frac{n+2}{n+1}\right)^{n+1}\left(\frac{n}{n+1}\right)^{n+1}\frac{n+1}{n} = \left(\frac{n(n+2)}{(n+1)^2}\right)^{n+1}\frac{n+1}{n} = \left(\frac{n^2+2n}{(n+1)^2}\right)^{n+1}\frac{n+1}{n} = \left(\frac{(n+1)^2-1}{(n+1)^2}\right)^{n+1}\frac{n+1}{n} = \left(1 - \frac{1}{(n+1)^2}\right)^{n+1}\frac{n+1}{n} \geq \left(1 - (n+1).\frac{1}{(n+1)^2}\right)\frac{n+1}{n} = \left(1 - \frac{1}{n+1}\right)\frac{n+1}{n} = \frac{n}{n+1}\frac{n+1}{n} = 1$, by using Bernoulli's inequality.

Hence, $a_{n+1} \geq a_n$, for all n.

For C: $a_n = \frac{n+1}{n} > 1 \implies a_n = \frac{n+1}{n} \geq 1$, since for all $n \geq 1$, $n + 1 > n$.

The statement D is Theorem 2.11 of the study guide.

Question 3 : **[2 marks]**

Let $a_n = \left(1 - \frac{x}{n}\right)^{-n}$ for all $x \in \mathbb{R}$. Then $\lim_{n\to\infty} a_n =$

A. e^x.

B. e^{-x}.

C. $e^{\frac{1}{x}}$.

D. $e^{-\frac{1}{x}}$.

E. None of the above.

Answer 3 : By Tutorial 2.2.1(8),

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(1 - \frac{x}{n}\right)^{-n} = \lim_{n\to\infty}\left(\left(1 + \frac{(-x)}{n}\right)^n\right)^{-1} = \left(e^{(-x)}\right)^{-1} = e^x. (A)$$

Question 4 : **[2 marks]**

Let $a_n = \frac{1-2n}{1+2n}$. Then $\lim_{n\to\infty} a_n =$

A. $-\infty$.

B. 1.

C. -1.

D. It does not exist.

E. None of the above.

Answer 4 : $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n(\frac{1}{n}-2)}{n(\frac{1}{n}+2)} = \lim_{n\to\infty} \frac{\frac{1}{n}-2}{\frac{1}{n}+2} = \frac{0-2}{0+2} = -1$. (C)

Question 5 : **[2 marks]**

Assume that the sequence

$$a_1 = 2, \quad a_{n+1} = \frac{72}{1+a_n}$$

converges. Then $\lim_{n\to\infty} a_n =$

A. -9.

B. 8.

C. 24.

D. 9.

E. None of the above.

Answer 5 : See Solution of Tutorial 2.2.1, 5(d). Since the sequence converges, $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} a_n = a$. Taking the limits as $n \to \infty$ in $a_{n+1} = \frac{72}{1+a_n}$, we have $a = \frac{72}{1+a}$. It follows $a^2 + a - 72 = (a+9)(a-8) = 0$. That is, $a = -9$ or $a = 8$. Because all the terms are nonnegative, $\lim_{n\to\infty} a_n = 8$. (B)

Total: 10 marks