APPM2023
Mechanics II
2023

Class Test-01

Date: 06 April 2023 **Student Number:** __________

Duration: 60 minutes **Total:** 45 Points

Instructions

- Read all the questions carefully.
- This test comprises 3 questions.
- Answer all questions.
- Show all working in answer books provided.
- Start each question on a new page.
- There are 45 points available, and 45 points is 100%.

Question 1 — Basic Algebra **(25 Points)**

Suppose $\hat{a}$ and $\hat{b}$ are unit vectors in 3-dimensional space. Define the following vectors

$$\vec{a} = a\hat{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z} \quad \text{and} \quad \vec{b} = b\hat{b} = b_x\hat{x} + b_y\hat{y} + b_z\hat{z}$$

Answer the following questions.

[1.1] Prove that

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}.$$

Show all calculational steps. **(2 Points)**

[1.2] What conditions must $\hat{a}$ and $\hat{b}$ satisfy such that $\hat{a}+\hat{b}$ is a unit vector? **(4 Points)**

[1.3] Suppose $\alpha, \beta \in \mathbb{R}$. Use the definiton of the dot product to prove that

$$(\alpha\vec{a})\cdot(\beta\vec{b}) = (\beta\vec{a})\cdot(\alpha\vec{b}).$$

Show all calculational steps. **(3 Points)**

[1.4] Find the angle between any two adjacent diagonals of a cube. Draw a correctly labelled diagram to accompany your calculation. Show all calculational steps. **(6 Points)**

[1.5] Show by direct calculation that,

a1 b1
a2 b2
a3 b3

$$\vec{a}\times\vec{b} = \epsilon_{ijk}a_ib_j\hat{e}_k$$

and give a complete construction of the symbol ϵ_{ijk}, showing all steps. **(10 Points)**

Question 2 — Maximizing the Dot Product **(12 Points)**

Consider the vectors

$$\vec{v} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

where $\vec{u}$ is a unit vector, and define the Lagrange function

$$\mathscr{L}(x,y,z,\lambda) = f(x,y,z) - \lambda g(x,y,z).$$

for the generic function f, the constraint function g and the Lagrange multiplier λ. Here we shall consider a process to find the extrema (maxima or minima) of the function f by process of extremization subject to a constraint. Answer the following questions.

[2.1] Specify a constriant function $g(x,y,z)$ that the elements of $\vec{u}$ satisfy. Write this constraint function in the form

$$g(x,y,z) = 0.$$

(2 Points)

[2.2] Suppose that we want to extremize the value of the dot product between the vectors $\vec{u}$ and $\vec{v}$. Give an expression for the function $f(x,y,z)$ that we must extremize. **(2 Points)**

[2.3] Starting with the Lagrange function $\mathscr{L}$, show that the condition to extremize f is given by

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z).$$

u = a×(i+j+k)
v = a×(-i+j+k)

(2 Points)

[2.4] Use the expression $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ to show that the f is maximized when

$$x = \frac{2}{2\lambda} \quad y = \frac{3}{2\lambda} \quad \text{and} \quad z = \frac{1}{2\lambda}$$

such that

$$2\lambda = \frac{2}{x} = \frac{3}{y} = \frac{1}{z}.$$

(4 Points)

[2.5] Use the Lagrange multiplier λ to determine the the value of $\vec{u}$ that maximizes the $\vec{u}\cdot\vec{v}$ and the value of $\vec{u}$ that minimizes $\vec{u}\cdot\vec{v}$. What is the relationship between these values of $\vec{u}$? **(6 Points)**

Question 3 — Bead on a Surface **(8 Points)**

Consider the vector

$$\vec{r} = \vec{r}_1 + \vec{r}_2$$

where

$$\vec{r}_1 = \begin{array}{cc} \begin{pmatrix} \cos(\alpha) \\ \sin(\alpha) \\ 0 \end{pmatrix} & \begin{matrix} x \\ y \\ z \end{matrix} \end{array} \quad \text{and} \quad \vec{r}_2 = b \begin{array}{cc} \begin{pmatrix} \sin(\beta)\cos(\alpha) \\ \sin(\beta)\sin(\alpha) \\ -\cos(\beta) \end{pmatrix} & \begin{matrix} x \\ y \\ z \end{matrix} \end{array}$$

in $\mathbb{R}^3$ in the standard rectilinear xyz-coordinate system. Let b be a fixed, positive number, $\beta \in [0, 2\pi)$ and $\alpha \in [0, 2\pi)$, then $\vec{r}$ defines a surface in $\mathbb{R}^3$. Suppose that a bead slides on this surface following a path γ given by

$$\alpha = \alpha(t) \quad \text{and} \quad \beta = \beta(t)$$

Answer the following questions.

[3.1] What path does $\vec{r}_1$ trace as α varies over $[0, 2\pi)$? **(1 Points)**

[3.2] What path does $\vec{r}_2$ trace as β varies over $[0, 2\pi)$ and $\alpha = c$ is constant? **(1 Points)**

[3.3] Identify the surface traced by $\vec{r}$? **(2 Points)**

[3.4] Suppose $\alpha(t) = t$ and $\beta(t) = \frac{\pi}{2}$, the describe the path γ followed by the particle. **(2 Points)**

[3.5] Suppose $\alpha(t)=t$ and $\beta(t)=bt$, *the describe the path* γ **followed by the particle. (2 Points)**