COMS 3003A
HW 8

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Reading: Leary & Kristiansen, Chapter 1.

1. Consider the standard model of arithmetic ω, i.e. the set of natural numbers with the usual relations and operations. We assume that the language of arithmetic has been extended, using definitions, with binary predicate letters $<$ and $\leqslant$ (see HW 7 for details). Determine if the following sentences are true or false in ω:

   (a) $\forall x \exists y\, x < y$;

   (b) $\forall y \exists x\, x < y$;

   (c) $\exists x \forall y\, x \leqslant y$;

   (d) $\exists y \forall x\, x + y = x$;

   (e) $\exists x \forall y\, x + y = x$;

   (f) $\exists x \forall y\, \neg(S(y) = x)$;

   (g) $\exists y \forall x\, \neg(S(y) = x)$.

2. For each of the following formulas, find a model where the formula is true and a model where the formula is false:

   (a) $\forall x\, R(x, x)$;

   (b) $\forall x \forall y\, (R(x, y) \to R(y, x))$;

   (c) $\forall x \forall y\, \forall z\, (R(x, y) \wedge R(y, z) \to R(x, z))$;

   (d) $\forall x \forall y\, (R(x, y) \to \exists z\, (R(x, z) \wedge R(z, y)))$;

   (e) $\exists x\, P(x) \to \forall x\, P(x)$;

   (f) $\forall x \exists y\, R(x, y)$;

   (g) $\exists x \forall y R(y, x)$;

   (h) $\forall x \exists y R(x, y) \to \exists x \forall y R(y, x)$;

   (i) $\forall x \forall y \forall z\, (R(x, y) \wedge R(y, z) \to R(x, z)) \wedge \forall x \exists y R(x, y) \wedge \neg R(x, x)$.

3. Find out, for each of the following formulas, whether it is valid, i.e. true in every model, or not. Prove your claim.

   (a) $\forall x\, P(x) \land \forall x\, Q(x) \to \forall x\, (P(x) \land Q(x))$;

   (b) $\forall x\, (P(x) \lor Q(x)) \to \forall x\, P(x) \lor \forall x\, Q(x)$;

   (c) $\exists x\, P(x) \land \exists x\, Q(x) \to \exists x\, (P(x) \land Q(x))$;

   (d) $\exists x\, (P(x) \lor Q(x)) \to \exists x\, P(x) \lor \exists x\, Q(x)$;

   (e) $\exists x \forall y\, R(y, x) \to \forall x \exists y\, R(x, y)$;

   (f) $\forall x\, (P(x) \to Q(x)) \to (\forall x\, P(x) \to \forall x\, Q(x))$.

4. (a) Write a sentence φ without $=$ that has the following properties:

   - φ is true in every model with a single individual;
   - for every $n \geqslant 2$, there exists a model with n individuals where φ is false.

   (b) Write a formula with $=$ that is true precisely in models with two individuals.

   (c) Write a formula with $=$ that is true precisely in models with n individuals.

   (d) Does there exist a formula without $=$ that is true precisely in models with two individuals?

   (e) Write a formula without $=$ that is satisfiable only in models with infinite domains.

   (f) Write a formula without $=$ that is true in every model with a finite domain.

5. Let $\mathfrak{M}$ be a model, let α and β be assignments in $\mathfrak{M}$, and let φ be a sentence (i.e., a formula without free occurences of variables). Prove, by induction on φ, that $\mathfrak{M} \models \varphi[\alpha]$ if, and only if, $\mathfrak{M} \models \varphi[\beta]$.