Quadrature

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Presentation Outline

Numerical Quadrature

Problem: Approximate the definite integral:

$$I = \int_a^b f(x)dx. \tag{1}$$

An n-point quadrature formula has the form:

$$I = \int_a^b f(x)dx = \sum_{i=1}^{n} w_i f(x_i) + R_n, \tag{2}$$

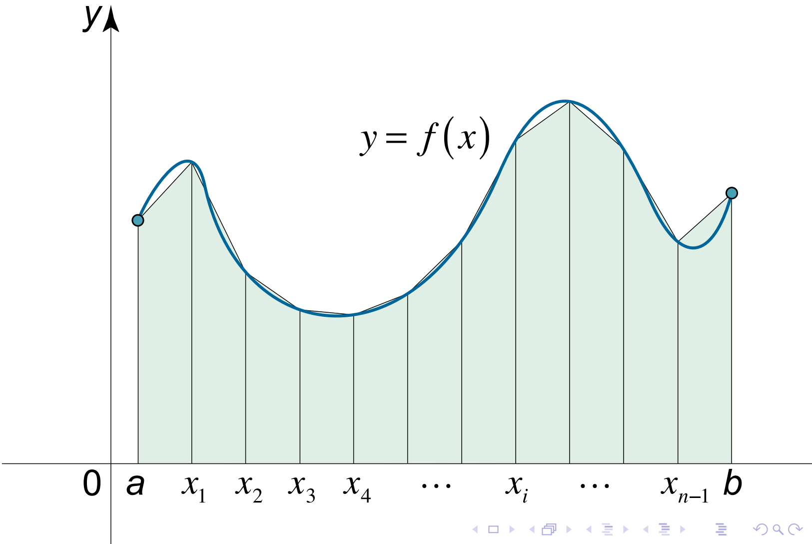
w_i - weights and remainder R_n. Therefore

$$I \approx \sum_{i=1}^{n} w_i f(x_i) \tag{3}$$

You can approximate f by a polynomial of degree n, P_n

$$I = \int_a^b f(x)dx \approx \int_a^b P_n(x)dx. \tag{4}$$

Trapezoidal Rule



Trapezoidal Rule

Partition $[a, b]$ into n subintervals of equal width
So there will be $n+1$ points: $x_0,\ x_1,\ \cdots,\ x_n$, where $x_0 = a$ and $x_n = b$.
Let

$$x_{i+1} - x_i = h = \frac{b-a}{n}, \quad i = 0, 1, 2, \cdots, n-1.$$

On each subinterval $[x_i, x_{i+1}]$, approximate $f(x)$ with a first degree polynomial,

$$\begin{aligned} P_1(x) &= f_i + \frac{f_{i+1} - f_i}{x_{i+1} - x_i}(x - x_i) \\ &= f_i + \frac{f_{i+1} - f_i}{h}(x - x_i). \end{aligned}$$

Trapezoidal Rule

Then we have:

$$\begin{aligned}\int_{x_i}^{x_{i+1}} f(x)dx &\approx \int_{x_i}^{x_{i+1}} P_1(x)dx \\ &= \int_{x_i}^{x_{i+1}} \left(f_i + \frac{f_{i+1}-f_i}{h}(x-x_i)\right) dx \\ &= \frac{h}{2}(f_i + f_{i+1})\end{aligned}$$

Summing over all subintervals and simplifying gives:

$$I = \int_a^b f(x)dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)dx \approx \sum_{i}^{n} \frac{f(x_{i-1}) + f(x_i)}{2} h, \tag{5}$$

or:

$$I \approx \frac{h}{2}\left[f_0 + 2(f_1 + f_2 + \cdots + f_{n-1}) + f_n\right], \tag{6}$$

which is known as the **Composite Trapezoidal**

Error of Trapezoidal rule

The error of Trapezoidal rule is:

$$E_T = \int_a^b f(x)dx - I, \tag{7}$$

It can be shown that

$$E_T = -\frac{(b-a)h^2}{12} f''(\epsilon), \qquad \epsilon \in [a, b], \tag{8}$$

We can also see that the error is of order $\mathcal{O}(h^2)$.

Example

Using the trapezoidal rule, evaluate:

$$\int_0^1 \frac{1}{1+x^2}dx.$$

use $n = 6$, i.e. we need 7 nodes.

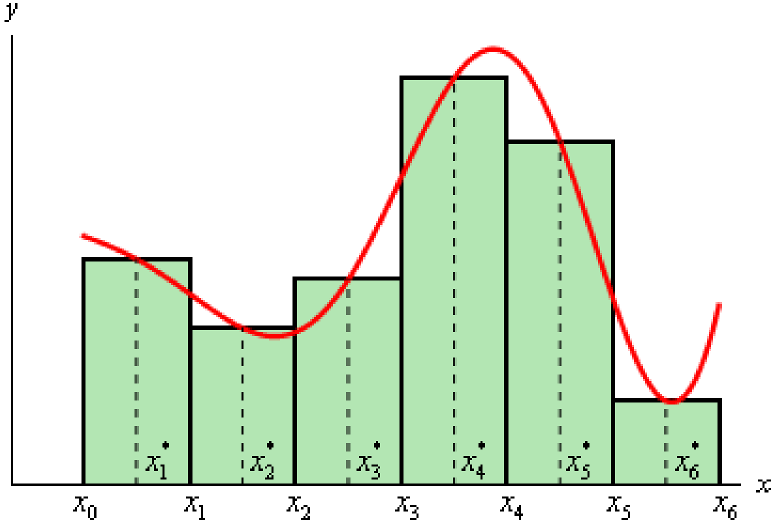
Solution:

Since $n = 6$ then $h = (1-0)/6 = 1/6$, therefore:

$$I \approx \frac{1}{12}\left[f_0 + 2(f_1 + f_2 + f_3 + f_4 + f_5) + f_6\right] = 0.784241$$

where $f_i = f(x_i)$ and $x_i = x_0 + ih = i/6, \quad i = 0, 1, 2, \cdots, 6.$
Exact value is $\pi/4 = 0.785398$, so approximation is correct to 2 decimals, not bad!

The Midpoint Method



The Midpoint Method

Partition $[a, b]$ into n subintervals of equal width.

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \ldots + \int_{x_{n-1}}^{x_n} f(x)dx,$$
$$\approx hf\left(\frac{x_0 + x_1}{2}\right) + hf\left(\frac{x_1 + x_2}{2}\right) + \ldots$$
$$+ hf\left(\frac{x_{n-1} + x_n}{2}\right),$$

This can be rewritten as:

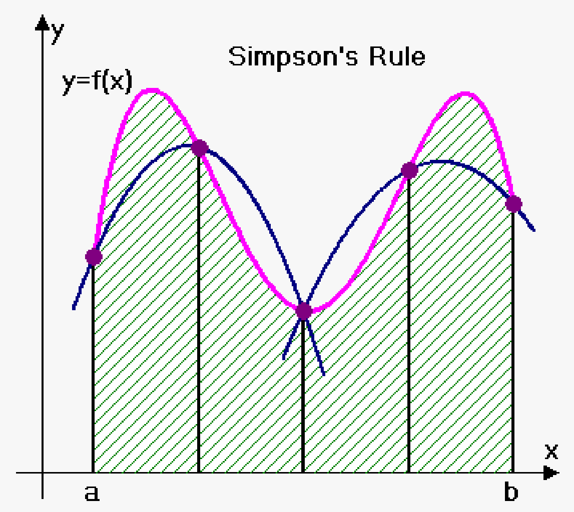
$$\int_a^b f(x)dx \approx h\sum_{i=0}^{n-1} f(m_i), \tag{9}$$

where $m_i = (a + h/2) + ih$.

Simpson's Rule

The trapezoidal rule approximates the area under a curve by summing over the areas of trapezoids formed by connecting successive f_i's with straight lines.
Simpson's rule a parabola to connect adjacent points. Simpson requires n to be **even**.

Simpson's Rule



Simpson's Rule

Therefore our approximation is:

$$I = \frac{h}{3}\left[f_{i-1} + 4f_i + f_{i+1}\right]. \tag{10}$$

Summing the definite integrals over each subinterval $[x_{i-1}, x_{i+1}]$ for $i = 1, 3, 5, \cdots, n-1$ provides the approximation:

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[(f_0 + 4f_1 + f_2) + (f_2 + 4f_3 + f_4) + \cdots \right.$$
$$\left. +(f_{n-2} + 4f_{n-1} + f_n)\right] \tag{11}$$

By simplifying this sum we obtain the approximation scheme:

$$\begin{aligned}\int_a^b f(x)dx &\approx \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{n-2} + 4f_{n-1} + f_n\right] \\ &\approx \frac{h}{3}\left[f_0 + 4(f_1 + f_3 + \cdots f_{n-1}) + 2(f_2 + f_4 + \cdots + f_{n-2}) \right. \\ &\quad \left. +f_n\right] \end{aligned} \tag{12}$$

Error of Simpson's Rule

The error for Simpson's rule is:

$$E_S = -\frac{(b-a)h^4}{180} f^4(\epsilon), \qquad \epsilon \in [a, b], \tag{13}$$

giving an error of $\mathcal{O}(h^4)$. Hence if the integrand is of degree $n \leq 3$, then the error is zero and we obtain the exact value. The same can be said for the trapezoidal rule the integrand is linear.

Example

Using the Simpson's rule, evaluate:

$$\int_0^1 \frac{1}{1+x^2}\,dx,$$

use $n = 6$, i.e. we need 7 nodes.

Solution:

Since $n = 6$ then $h = (1-0)/6 = 1/6$, therefore:

$$I \approx \frac{1}{18}[f_0 + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4) + f_6] = 0.785398$$

where $f_i = f(x_i)$ and $x_i = x_0 + ih, i = 0, 1, 2, \cdots, 6.$

Exact value is $\pi/4 = 0.785398$, so approximation is correct to 6 decimals, that's great!

Double integrals

Problem:

$$\int_a^b \int_c^d f(x,y)dydx.$$

Can we approximate this integral numerically on $[a,b] \times [c,d]$? We now have h_x and h_y given by

$$h_x = \frac{b-a}{n_x}, \quad h_y = \frac{d-c}{n_y}$$

Can be done by using any of the quadrature rules we have seen so far.

Double integrals using Midpoint rule

Let

$$g(x) = \int_c^d f(x,y)dy$$

Therefore

$$\int_a^b \int_c^d f(x,y)dydx = \int_a^b g(x)dx$$

Thus Midpoint rule applied to $g(x)$ results in:

$$g(x) = \int_c^d f(x,y)dy \approx h_y \sum_{j=0}^{n_y-1} f(x,\bar{y}_j), \qquad \bar{y}_j = c + \frac{1}{2}h_y + jh_y.$$

So, the double integral approximated by the midpoint method:

$$\int_a^b g(x)dx \approx h_x \sum_{i=0}^{n_x-1} g(\bar{x}_i), \quad \bar{x}_i = a + \frac{1}{2}h_x + ih_x.$$

Example of a double integral using Midpoint rule

Using $n_x = n_y = 5$, compute the integral:

$$\int_2^3 \int_0^2 (2x + y)dydx.$$

Solution: $h_x = \frac{b-a}{n_x} = \frac{3-2}{5}, \quad h_y = \frac{d-c}{n_y} = \frac{2-0}{5}$

$$\int_2^3 \int_0^2 (2x + y)dydx = \int_2^3 g(x)dx$$

Example of a double integral using Midpoint rule

Thus Midpoint rule applied to $g(x)$ results in:

$$
\begin{aligned}
g(x) = \int_0^2 (2x+y)dy \approx h_y \sum_{j=0}^{n_y-1} (2x+\bar{y}_j), \qquad \bar{y}_j = 0 + \frac{1}{2}\frac{2}{5} + j\frac{2}{5} \\
= \frac{2}{5}\sum_{j=0}^{4}(2x+\frac{1}{5}+j\frac{2}{5}) \\
= \frac{2}{5}\sum_{j=0}^{4}(2x+\frac{1}{5}) + \frac{2}{5}\sum_{j=0}^{4} j\frac{2}{5} \\
= 2(2x+\frac{1}{5}) + \frac{8}{5} \\
= 4x+2
\end{aligned}
\tag{14}
$$

Example of a double integral using Midpoint rule

So, the double integral approximated by the midpoint method:

$$\int_2^3 g(x)dx \approx h_x \sum_{i=0}^{n_x-1} g(\bar{x}_i),\ \bar{x}_i = 2 + \frac{1}{2}\frac{1}{5} + i\frac{1}{5} = \frac{21}{10} + i\frac{1}{5}.$$

$$= \frac{1}{5}\sum_{i=0}^{4}(4\bar{x}_i + 2) = \frac{4}{5}\sum_{i=0}^{4}\bar{x}_i + 2$$

$$= \frac{4}{5}\sum_{i=0}^{4}(\frac{21}{10} + i\frac{1}{5}) + 2$$

$$= \frac{4}{5}\sum_{i=0}^{4}\frac{21}{10} + \frac{4}{5}\frac{1}{5}\sum_{i=0}^{4} i + 2$$

$$= 12$$

Tripple integrals

Consider the triple integral:

$$\int_a^b \int_c^d \int_e^f g(x,y,z)dzdydx,$$

We split the integral into one-dimensional integrals:

$$\begin{aligned} p(x,y) &= \int_e^f g(x,y,z)dz \\ q(x) &= \int_c^d p(x,y)dy \\ \int_a^b \int_c^d \int_e^f g(x,y,z)dzdydx &= \int_a^b q(x)dx \end{aligned}$$

Tripple integrals using Midpoint rule

Next we apply the midpoint rule to each of these one-dimension integrals:

$$p(x,y) = \int_e^f g(x,y,z)dz \approx h_z \sum_{k=0}^{n_z-1} g(x,y,\bar{z}_k),$$

$$q(x) = \int_c^d p(x,y)dy \approx h_y \sum_{j=0}^{n_y-1} p(x,\bar{y}_j),$$

$$\int_a^b \int_c^d \int_e^f g(x,y,z)dzdydx = \int_a^b q(x)dx \approx h_x \sum_{i=0}^{n_x-1} q(\bar{x}_i),$$

where:

$$\bar{z}_k = e + \frac{1}{2}h_z + kh_z, \quad \bar{y}_j = c + \frac{1}{2}h_y + jh_y \quad \bar{x}_i = a + \frac{1}{2}h_x + ih_x.$$

Example of a tripple integral using Midpoint rule

Evaluate the following integral:

$$\int_2^3 \int_1^2 \int_0^1 8xyz \; dzdydx,$$

where $n_x = n_y = n_z = 5$.
Solution: $h_x = h_y = h_z = 1/5$

Tripple integrals using Midpoint rule

Next we apply the midpoint rule to each of these one-dimension integrals:

$$p(x,y) = \int_0^1 (8xyz)dz \approx \frac{1}{5}\sum_{k=0}^{4} 8xy\bar{z}_k,$$

$$q(x) = \int_1^2 p(x,y)dy \approx \frac{1}{5}\sum_{j=0}^{4} p(x,\bar{y}_j),$$

$$\int_2^3\int_1^2\int_0^1 g(x,y,z)dzdydx = \int_2^3 q(x)dx \approx \frac{1}{5}\sum_{i=0}^{4} q(\bar{x}_i),$$

where:

$$\bar{z}_k = 0 + \frac{1}{2}\frac{1}{5} + k\frac{1}{5}, \quad \bar{y}_j = 1 + \frac{1}{2}\frac{1}{5} + j\frac{1}{5} \quad \bar{x}_i = 2 + \frac{1}{2}\frac{1}{5} + i\frac{1}{5}.$$

Tripple integrals using Midpoint rule

Next we apply the midpoint rule to each of these one-dimension integrals:

$$p(x,y) = \frac{1}{5}\sum_{k=0}^{4} 8xy\bar{z}_k = \frac{1}{5}8xy\sum_{k=0}^{4}(\frac{1}{10}+\frac{k}{5}) = 4xy,$$

$$q(x) = \frac{1}{5}\sum_{j=0}^{4} p(x,\bar{y}_j) = \frac{1}{5}\sum_{j=0}^{4} 4x\bar{y}_j = \frac{4x}{5}\sum_{j=0}^{4}(\frac{11}{10}+\frac{j}{5}) = 6x,$$

$$\int_2^3\int_1^2\int_0^1 (8xyz)dzdydx = \int_2^3 q(x)dx \approx \frac{1}{5}\sum_{i=0}^{4} q(\bar{x}_i),$$

$$= \frac{6}{5}\sum_{i=0}^{4}\bar{x}_i = \frac{6}{5}\sum_{i=0}^{4}(\frac{21}{10}+\frac{i}{5}),$$

$$= 15$$