Luke Renton
2540440

Assignment 03
State Machines and Phase-Space

APPM2023
2023-08-23

Question 1

In order to describe newtons second law as a state machine we are required to understand what it is. Newtons second law in words states that: "The acceleration of an object is directly proportional to the net force applied to it and inversely proportional to its mass." In mathematical terms it describes the following equation:

$$\vec{F} = ma$$

Which can be described by the following differential equation:

$$\vec{F} = m\left(\frac{d\vec{p}}{dt}\right) = m\left(\frac{d\vec{v}}{dt}\right) = m\left(\frac{d^2\vec{x}}{dt^2}\right)$$

These equation show us that the change of the state of a system is described by a singular differential equation.

It can also be noted that we have a first order differential equation in velocity and also a second order differential equation in position in order in order to solve the system we need to describe the object by the information needed to find the next state of the system. Notably the information needed is a position $\vec{x}$ as well as a velocity $\vec{v}$ which we can represent as a pair $(\vec{x}, \vec{v})$. This pair acts as an initial condition which provides a solution for the differential equations in both position and velocity.

Now, when we consider newtons second law in that when a force is applied to the object for the duration of a time δt a infinitesimal change in $d\vec{v}$ happens. This can be described by first principles as:

$$\frac{d\vec{v}}{dt} = \lim_{\delta t \to \infty} \frac{\vec{v}(t + \delta t) - \vec{v}(t)}{\delta t}$$

If we take a closer look at the discrete change in the velocity (not infinitesimally small) over the time taken δt we have that:

$$\delta\vec{v} = \frac{\vec{v}(t + \delta t) - \vec{v}(t)}{\delta t}$$

Recalling that $\frac{1}{m}\vec{F} = \left(\frac{d\vec{v}}{dt}\right)$ we can rearrange the terms to find that:

$$\vec{v}(t + \delta t) = \vec{v} + \delta t\delta\vec{v}(t) = \vec{v}(t) + \frac{1}{m}\delta t\vec{F}$$

Now we have derived something incredibly important: velocity at the following states is described as a function of the the velocity at time t plus the change in the velocity over the time δt. Similarly it can be shown that for position we have:

$$\vec{x}(t+\delta t) = \vec{x}(t) + \delta t \vec{v}(t)$$

Once again showing us that the position is described as a function of the current position plus some change in the velocity over a certain time δt.

Since both these functions are linear in δt the resulting action of each function is completely unique, meaning, in order to figure out the position and velocity of the system we only require the information at time t.

This leads us to the creation of a state machine whose transition rules between states at successive times t and $t + \delta t$ can be described as:

$$(\vec{x}(t), \vec{v}(t)) \rightarrow (\vec{x}(t+\delta t), \vec{v}(t+\delta t))$$

Thus the state of the system right after this rule has been applied gives us a unique state. Thus we have seen that Newtons Second Law gives rise to a state machine which is deterministic. Arguing for reversibility if we consider a decrease in time $t - \delta t$ in a similar way as shown above we can find that:

$$\vec{v}(t-\delta t) = \vec{v}(t) - \frac{1}{m}\delta t \vec{F}$$
$$\vec{x}(t-\delta t) = \vec{x}(t) - \delta t \vec{v}(t)$$

which uniquely defines a previous state with a transition rule:

$$(\vec{x}(t-\delta t), \vec{v}(t-\delta t)) \rightarrow (\vec{x}(t), \vec{v}(t))$$

Here we see reversing the current time t creates an association of each current state to it's previous one. Thus we find that the state machine for Newton's Second Law is also reversible.

Question 2

Using the paramatized equation of a line with positions $(x(t), y(t))$ we have:

$$\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y}$$

Owing to constant motion we know what:

$$\ddot{x} = \ddot{y} = 0$$

Meaning that each coordinate can be described by:

$$\begin{aligned} x(t) &= a_1 t + a_2 \\ y(t) &= b_1 t + b_2 \end{aligned}$$

Solving for t in our first equation we have that:

$$\begin{aligned} x &= a_1 t + a_2 \\ t &= \frac{x - a_2}{a_1} \end{aligned}$$

Similarly:

$$t = \frac{y - b_2}{b_1}$$

In our case we will be using the equation: $t = \frac{x-a_2}{a_1}$ since we are describing y as a function of t in term of x. Thus we can now re-describe the paramatized equation to be:

$$\vec{r}(t) = (a_1 t + a_2)\hat{x} + (b_1 t + b_2)\hat{y}$$

Subbing in for t and simplifying we have that:

$$\begin{aligned} \vec{r}(t) &= \left(a_1 \frac{x - a_2}{a_1} + a_2\right)\hat{x} + \left(b_1 \frac{x - a_2}{a_1} + b_2\right)\hat{y} \\ &= x\hat{x} + \left(\frac{b_1 x - b_1 a_2}{a_1} + \frac{a_1 b_2}{a_1}\right)\hat{y} \\ &= x\hat{x} + \left(\frac{b_1 x - b_1 a_2 + a_1 b_2}{a_1}\right)\hat{y} \\ &= x\hat{x} + \left(\frac{b_1}{a_1}x + \frac{a_1 b_2 - a_2 b_1}{a_1}\right)\hat{y} \end{aligned}$$

Now we let $m = \frac{b_1}{a_1}$ and $c = \frac{a_1 b_2 - a_2 b_1}{a_1}$ we have that:

$$\vec{r}(t) = x\hat{x} + (mx + c)\,\hat{y}$$

Therefore:

$$\vec{r}(t) = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ mx + c \end{pmatrix}$$

Now we can clearly for any given x on the path the y component is described by:

$$y = mx + c$$

Question 3

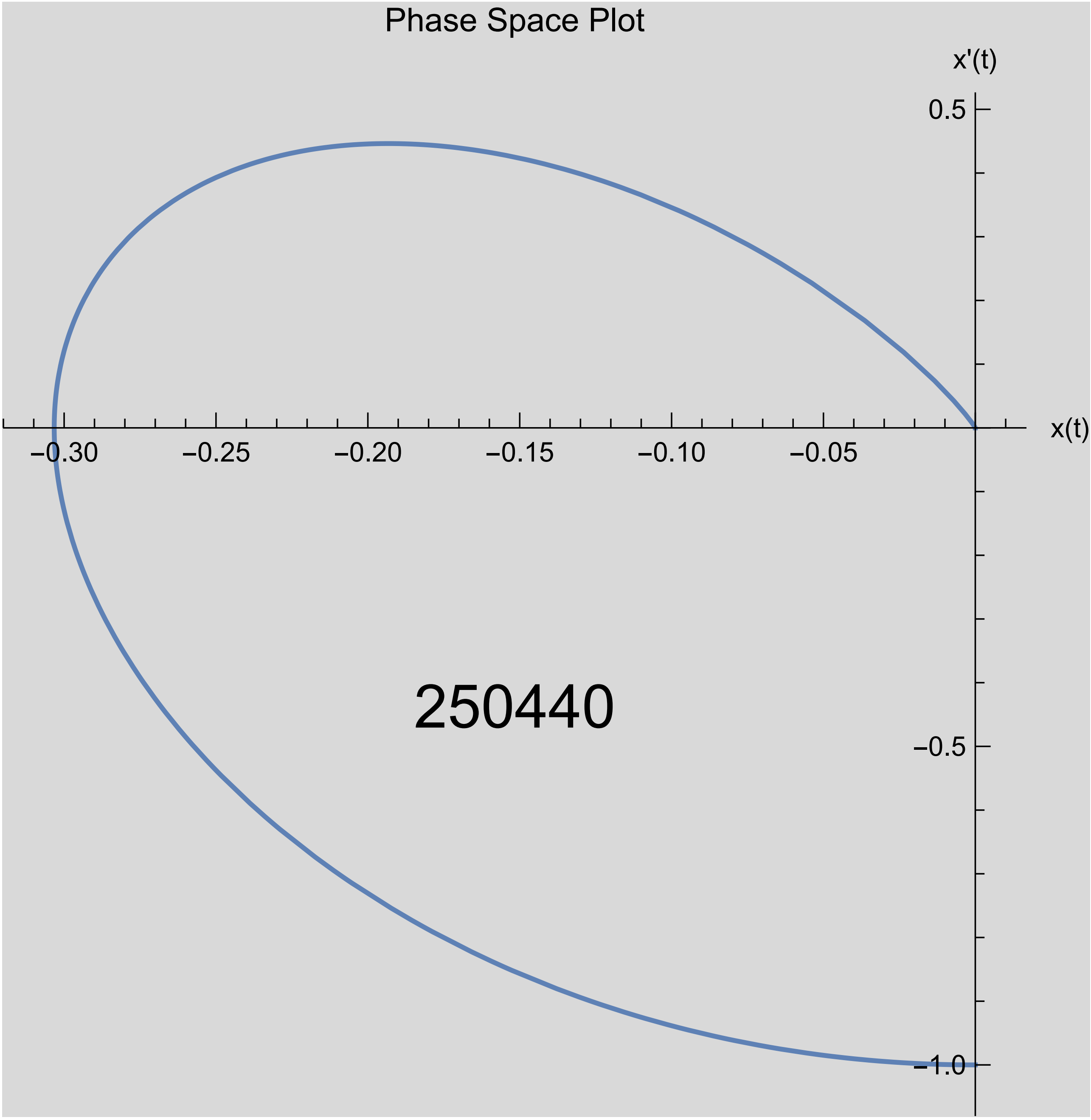

Figure 1: Plot of function: $x(t) = -e^{-at^2}t$ where $a > 1$

As $t \to \infty$ we see that the exponential term $-e^{-at^2}$ dominates the equation. Since $a > 1$, as $t \to \infty$, $-e^{-at^2}$ will tend to zero. This is because any positive value of a will cause the exponential term to decay rapidly as t grows. Consequently, the entire term $-e^{-at^2}t$ will also tend to zero. In other words, the system's behavior will approach zero as t becomes very large. While its corresponding differential equation will approach -1 as $t \to \infty$.

As a result the function will first decrease in the $x(t)$ for the first few time units before increasing (in an arcing manner) back toward zero. Further, the function will first increase in the $x'(t)$ direction before decreasing (in an arcing manner) downward towards -1.

Question 4

1. Inverting the embedding functions we have:
Solving for ρ:

since ρ is the radial distance from the origin it relates to the cartesian coordinate equation for a sphere:

$$r^2 = x^2 + y^2 + z^2$$

where here $r = \rho$ thus

$$\rho^2 = x^2 + y^2 + z^2$$

Finally:

$$\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

Solving for θ:

We see from the original embedding function:

$$z(\rho, \theta, \phi) = \rho\cos(\theta)$$

Therefore keeping in mind that $\rho(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ we have that:

$$\cos(\theta) = \frac{z}{\rho}$$
$$\theta = \arccos\left(\frac{z}{\rho}\right)$$

Finally:

$$\theta(x, y, z) = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

Solving for ϕ:

Simplifying the original functions x and y we have:

$$x(\rho, \theta, \phi) = \rho\sin(\theta)\cos(\phi)$$
$$\cos(\phi) = \frac{x}{\rho\sin(\theta)} \tag{1}$$

and

$$y(\rho, \theta, \phi) = \rho\sin(\theta)\sin(\phi)$$
$$\sin(\phi) = \frac{x}{\rho\sin(\theta)} \tag{2}$$

Dividing (2) through by (1) we have that:

$$\frac{\sin(\phi)}{\cos(\phi)} = \frac{x}{\rho\sin(\theta)} \div \frac{x}{\rho\sin(\theta)}$$

$$\tan(\phi) = \frac{y\rho\sin(\theta)}{x\rho\sin(\theta)}$$
$$\phi = \arctan\left(\frac{y}{x}\right)$$

Finally:

$$\phi(x,y,z) = \arctan\left(\frac{y}{x}\right)$$

To conclude we have found that after inverting the embedding functions:

$$\rho(x,y,z) = \sqrt{x^2+y^2+z^2}$$
$$\theta(x,y,z) = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$$
$$\phi(x,y,z) = \arctan\left(\frac{y}{x}\right)$$

2. Non-invertibility can occur according to the following cases:
Case I: When $\theta = 0$ or $\theta = \pi$ or any multiple thereof (i.e. at the poles of the sphere), resulting in $\sin(\theta) = 0$. At these points, there is an ambiguity in determining the azimuthal angle ϕ, leading to non-invertibility.
Case II: When $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$ or any multiple thereof resulting in $\cos(\theta) = 0$. Once again at these points there is an ambiguity in determining the azimuthal angle ϕ, leading to non-invertibility.

In both cases we lose our azimuthal angle owing to it being canceled out by 0 causing non-invertibility.

Case III: this is the degenerate case where $\rho = 0$ and thus we completely lose our θ and ϕ angles making it impossible to invert.

3. In the x-y-z coordinate plane we describe a sphere according to the following formula:

$$r^2 = x^2 + y^2 + z^2$$

Where r is a real constant. As we can see, the bead is forced onto the sphere by the holonomic constraint which causes the coordinates of the bead to satisfy the above equation. This essentially glues the bead to the surface of the sphere. We also find that if we have any two points on the x-y-z plane we can find some third point to satisfy that $r^2 = x^2 + y^2 + z^2$. In the case of knowing a x and y coordinate:

$$\forall x, \forall y, \exists z \ni r^2 = x^2 + y^2 + z^2$$

Similarly in the general case for when we are given any two points in the x-y-z plane there will be some corresponding third point which is completely determined by the other two points.

In the case of the coordinates (ρ, θ, ϕ) having two coordinates does not fix the third. This is because ϕ represents the azimuthal angle, which is a rotation in the x-y plane. Fixing ρ and θ only constrains the bead to a certain circular section of the sphere, but the bead can still rotate around the vertical axis (z) meaning there isn't one unique ϕ value corresponding to the two other given values.

4. Taking the partial derivative of our spherical coordinate embedding in terms of ρ, θ, ϕ and normalizing them we have that:

$$\hat{\rho} = \begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}, \hat{\theta} = \begin{pmatrix} \cos(\theta)\cos(\phi) \\ \cos(\theta)\sin(\phi) \\ -\sin(\theta) \end{pmatrix}, \hat{\phi} = \begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \\ 0 \end{pmatrix}.$$

Now, finding the derivatives of each unit vector in terms of the unit vectors:

$$\frac{d\hat{\rho}}{dt} = \frac{d}{dt}\begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} = \dot{\theta}\begin{pmatrix} \cos(\theta)\cos(\phi) \\ \cos(\theta)\sin(\phi) \\ -\sin(\theta) \end{pmatrix} + \dot{\phi}\begin{pmatrix} -\sin(\theta)\sin(\phi) \\ \sin(\theta)\cos(\phi) \\ 0 \end{pmatrix} = \dot{\theta}\hat{\theta} + \dot{\phi}\sin(\theta)\hat{\phi},$$

$$\frac{d\hat{\theta}}{dt} = \frac{d}{dt}\begin{pmatrix} \cos(\theta)\cos(\phi) \\ \cos(\theta)\sin(\phi) \\ -\sin(\theta) \end{pmatrix} = \dot{\theta}\begin{pmatrix} -\sin(\theta)\cos(\phi) \\ -\sin(\theta)\sin(\phi) \\ -\cos(\theta) \end{pmatrix} + \dot{\phi}\begin{pmatrix} -\cos(\theta)\sin(\phi) \\ \cos(\theta)\cos(\phi) \\ 0 \end{pmatrix} = -\dot{\theta}\hat{\rho} + \dot{\phi}\cos(\theta)\hat{\phi},$$

For the next derivative we are required to precompute an equation which allows us to simplify the derivative further. The equation we wish to precompute is:

$$\sin(\theta)\hat{\rho} + \cos(\theta)\hat{\theta}$$

Which we find to become:

$$\begin{aligned} \sin(\theta)\hat{\rho} + \cos(\theta)\hat{\theta} &= \sin(\theta)\begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} + \cos(\theta)\begin{pmatrix} \cos(\theta)\cos(\phi) \\ \cos(\theta)\sin(\phi) \\ -\sin(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \sin^2(\theta)\cos(\phi) + \cos^2(\theta)\cos(\phi) \\ \sin^2(\theta)\sin(\phi) + \cos^2(\theta)\sin(\phi) \\ \sin(\theta)\cos(\theta) + -\sin(\theta)\cos(\theta) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\phi)\left(\sin^2(\theta) + \cos^2(\theta)\right) \\ \sin(\phi)\left(\sin^2(\theta) + \cos^2(\theta)\right) \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \cos(\phi) \\ \sin(\phi) \\ 0 \end{pmatrix} \end{aligned} \tag{3}$$

Now finding the third and final derivative we have that:

$$\frac{d\hat{\phi}}{dt} = \frac{d}{dt}\begin{pmatrix} -\sin(\phi) \\ \cos(\phi) \\ 0 \end{pmatrix} = \dot{\phi}\begin{pmatrix} -\cos(\phi) \\ -\sin(\phi) \\ 0 \end{pmatrix} = -\dot{\phi}\begin{pmatrix} \cos(\phi) \\ \sin(\phi) \\ 0 \end{pmatrix} = -\dot{\phi}\left(\sin(\theta)\hat{\rho} + \cos(\theta)\hat{\theta}\right)$$

Now, using all the final answers we have derived from above we wish to compute the position vector:

$$\vec{p} = p\hat{\rho}$$

Deriving it once:

$$\vec{p} = \dot{p}\hat{\rho} + p\dot{\theta}\hat{\theta} + p\dot{\phi}\sin(\theta)\hat{\phi}.$$

Deriving it a second time we have that:

$$\begin{aligned}\ddot{\vec{p}} &= \dot{p}\hat{\rho} + \dot{p}\left(\dot{\theta}\hat{\theta} + \dot{\phi}\sin(\theta)\hat{\phi}\right) + \dot{p}\dot{\theta}\hat{\theta} + p\dot{\theta}\left(-\dot{\theta}\hat{\rho} + \dot{\theta}\hat{\rho} + \dot{\phi}\cos(\theta)\hat{\phi}\right) \\ &\quad + \dot{p}\dot{\phi}\sin(\theta)\hat{\phi} + p\ddot{\phi}\sin(\theta)\hat{\phi} + p\dot{\phi}\dot{\theta}\cos(\theta)\hat{\phi} - p\dot{\phi}\sin(\theta)\dot{\phi}\left(\sin(\theta)\hat{\rho} + \cos(\theta)\hat{\theta}\right)\end{aligned}$$

Grouping like terms we find that:

$$\begin{aligned}\ddot{\vec{p}} &= \left(\ddot{\rho} - \rho\dot{\theta}^2 - \rho\sin^2(\theta)\dot{\phi}^2\right)\hat{\rho} + \left(\rho\ddot{\theta} + 2\dot{\theta}\dot{\rho} - \rho\sin(\theta)\cos(\theta)\dot{\phi}^2\right)\hat{\theta} \\ &\quad + \left(\rho\sin(\theta)\ddot{\phi} + 2\rho\cos(\theta)\dot{\theta}\dot{\phi} + 2\sin(\theta)\dot{\rho}\dot{\phi}\right)\hat{\phi}\end{aligned}$$

Which is our final answer.

5. Getting our general force equation from the above derived equation we have that:

$$\begin{aligned}\vec{F}_{net} &= m\ddot{\vec{p}} \\ &= m\left(\ddot{p}_\rho\hat{\rho} + \ddot{p}_\theta\hat{\theta} + \ddot{p}_\phi\hat{\phi}\right),\end{aligned}$$

Getting our component forces we have that:

$$\begin{aligned}F_\theta &= m\left(\rho\ddot{\theta} + 2\dot{\theta}\dot{\rho} - \rho\sin(\theta)\cos(\theta)\dot{\phi}^2\right) \\ F_\phi &= m\left(\rho\sin(\theta)\ddot{\phi} + 2\rho\cos(\theta)\dot{\theta}\dot{\phi} + 2\sin(\theta)\dot{\rho}\dot{\phi}\right)\end{aligned}$$

However we have a restriction on ρ: it must be a constant (since it is a fixed radial distance) and thus we find that:

$$\dot{\rho} = \ddot{\rho} = 0$$

Which transforms our above component force equations to become:

$$\begin{aligned}F_\theta &= m\left(\rho\ddot{\theta} - \rho\sin(\theta)\cos(\theta)\dot{\phi}^2\right) \\ F_\phi &= m\left(\rho\sin(\theta)\ddot{\phi} + 2\rho\cos(\theta)\dot{\theta}\dot{\phi}\right)\end{aligned}$$

Now we also know that if a bead is on the surface of a sphere it means that the only force acting on it is a downward force on the $\hat{z}$ direction. This force being the weight of the bead which is:

$$\vec{W} = -mg\hat{z} = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix}$$

Which we can now use to construct a new system of equations, that is, the component forces using the weight. We get these new component forces by taking the inner product between the weight ($\vec{W}$) and the unit vectors $\hat{\theta}$ and $\hat{\phi}$:

$$\begin{aligned} F_\theta &= \hat{\theta} \cdot \vec{W} = mg\sin(\theta) \\ F_\phi &= \hat{\phi} \cdot \vec{W} = 0 \end{aligned}$$

Now we can create a system of equations subtracting the two corresponding component force equations like so:

$$\begin{aligned} F_\theta - F_\theta &= m\left(\rho\sin(\theta)\ddot{\phi} + 2\rho\cos(\theta)\dot{\theta}\dot{\phi}\right) - mg\sin(\theta) \\ 0 &= m\left(\rho\sin(\theta)\ddot{\phi} + 2\rho\cos(\theta)\dot{\theta}\dot{\phi} - g\sin(\theta)\right) \end{aligned}$$

Like wise:

$$\begin{aligned} F_\phi - F_\phi &= m\left(\rho\sin(\theta)\ddot{\phi} + 2\rho\cos(\theta)\dot{\theta}\dot{\phi}\right) - 0 \\ 0 &= m\left(\rho\sin(\theta)\ddot{\phi} + 2\rho\cos(\theta)\dot{\theta}\dot{\phi}\right) \end{aligned}$$

Which we can now create a system for in mathematica in order to solve for $\theta(t)$ and $\phi(t)$

NB: part 6 of question 4 follows on the next two pages

6a.

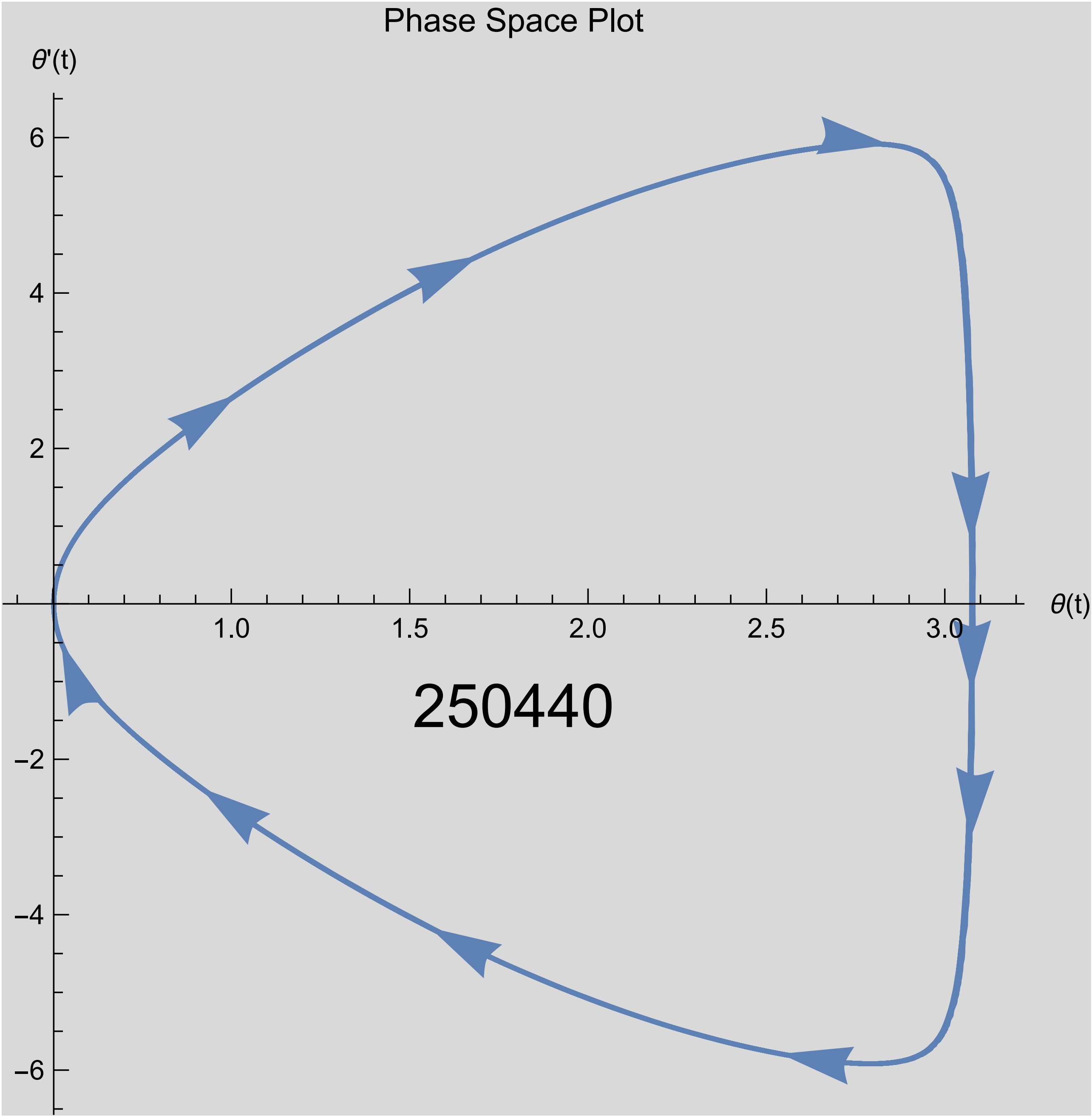

Figure 2: Phase portrait for $\theta(t)$

b.

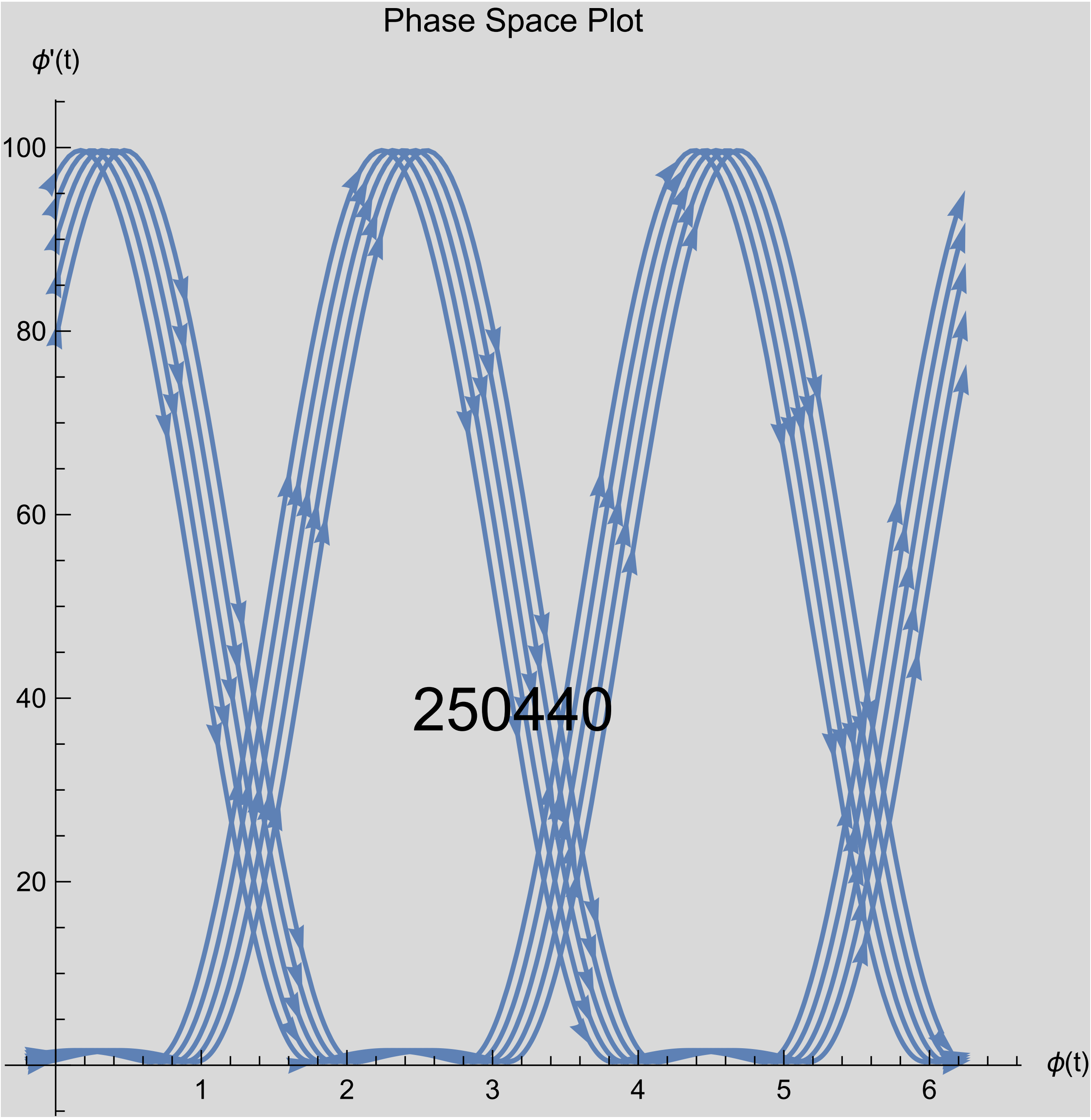

Figure 3: Phase portrait for $\phi(t)$