# Basic Analysis 2015 — Solutions of Tutorials

## Section 1.2

**Tutorial 1.2.1**

1. In each of the following cases, state if the given set is bounded above or not. If a set is bounded above, give two different upper bounds for the set, give the supremum of the set and state if the set has a maximum or not.

(a) $(-3, 2)$ is bounded above, 4 and 5 are upper bounds, the supremum is 2, and $(-3, 2)$ has no maximum.

(b) $(1, \infty)$ is not bounded above.

(c) $[10, 11]$ is bounded above, 11 and 15 are upper bounds, the supremum is 11, and the maximum exists.

(d) $\{5, 4\}$ is bounded above, 11 and 5 are upper bounds, the supremum is 5, and the maximum exists.

(e) $\{10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, -1, -2, -5\}$ is bounded above, 11 and 10 are upper bounds, the supremum is 10, and the maximum exists.

(f) $(-\infty, 2]$ is bounded above, 3 and 10 are upper bounds, the supremum is 2, and the maximum exists.

(g) $\{x \in \mathbb{R} : x^2 < 3\}$ is bounded above, 3 and 10 are upper bounds, the supremum is $\sqrt{3}$, and the set has no maximum.

(h) $\{x \in \mathbb{R} : x^2 \leq 3\}$ is bounded above, 3 and 10 are upper bounds, the supremum is $\sqrt{3}$, and the maximum exists.

2. For each of the sets in Q. 1, state if the given set is bounded below or not. If a set is bounded below, give two different lower bounds for the set, give the infimum of the set and state if the set has a minumum or not.

(a) $(-3, 2)$ is bounded below, $-4$ and $-5$ are lower bounds, the infimum is $-3$, and $(-3, 2)$ has no minimum.

(b) $(1, \infty)$ is bounded below, $-4$ and $-5$ are lower bounds, the infimum is 1, and the set has no minimum.

(c) $[10, 11]$ is bounded below, 10 and 5 are lower bounds, the infimum is 10, and the minimum exists.

(d) $\{5, 4\}$ is bounded below, 1 and 4 are lower bounds, the infimum is 4, and the minimum exists.

(e) $\{10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, -1, -2, -5\}$ is bounded below, $-11$ and $-10$ are lower bounds, the infimum is $-5$, and the minimum exists.

(f) $(-\infty, 2]$ is not bounded below.

(g) $\{x \in \mathbb{R} : x^2 < 3\}$ is bounded below, $-3$ and $-10$ are lower bounds, the infimum is $-\sqrt{3}$, and the set has no minimum.

(h) $\{x \in \mathbb{R} : x^2 \leq 3\}$ $\{x \in \mathbb{R} : x^2 < 3\}$ is bounded below, $-3$ and $-10$ are lower bounds, the infimum is $-\sqrt{3}$, and the minimum exists.

3. Let $S$ be a nonempty subset of $\mathbb{R}$.
(a) If $a$ is the greatest element of $S$, then what is $\sup S$? $\sup S = a$.
(b) If $\sup S = a$, then what are the upper bounds of $S$? The upper bounds are all real numbers $x$ satisfying $x \geq a$.
(c) If $\sup S = a$, does $S$ have a maximum? We do not know without further information on the set.
(d) If $\sup S = a$ and $a \in S$, does $S$ have a maximum? Yes.

4. Prove Proposition 1.10.

*Proof.* (a) We have

$$\begin{aligned}
-S \text{ is bounded below } &\Leftrightarrow \exists\, l \in \mathbb{R}\ \forall\, x \in -S \quad l \leq x \\
&\Leftrightarrow \exists\, l \in \mathbb{R}\ \forall\, -y \in -S \quad l \leq -y \\
&\Leftrightarrow \exists\, l \in \mathbb{R}\ \forall\, y \in S \quad l \leq -y \\
&\Leftrightarrow \exists\, l \in \mathbb{R}\ \forall\, y \in S \quad y \leq -l \\
&\Leftrightarrow S \text{ is bounded above.}
\end{aligned}$$

Now let $S$ be bounded above and let $l$ be any lower bound of $-S$. Then

$$\begin{aligned}
&\forall\, x \in -S \quad l \leq x \\
\Rightarrow &\forall\, -y \in -S \quad l \leq -y \\
\Rightarrow &\forall\, -y \in -S \quad y \leq -l \\
\Rightarrow &\forall\, y \in S \quad y \leq -l.
\end{aligned}$$

Hence $-l$ is an upper bound of $S$, so that $M = \sup S \leq -l$. Therefore $l \leq -M$, and we have shown that $-M$ is the greatest lower bound of $S$. Hence

$$\inf(-S) = -M = -\sup S.$$

If $\max S$ exists, then $M = \sup S = \max S \in S$, and $\inf S = -M \in -S$ follows, which means that $\min(-S)$ exists and $\min(-S) = -\max S$.
(b) Replacing $S$ with $-S$ in part (a) we have

$$-S \text{ is bounded above } \Leftrightarrow -(-S) \text{ is bounded below } \Leftrightarrow S \text{ is bounded below .}$$

The identities stated in (b) now easily follow from (a).
(c) is clear from the definition of bounded sets, (a) and (b). $\square$

5. Prove Theorem 1.11.

*Proof.* This has been shown in the proof of part (a) of Proposition 1.10. $\square$

6. Prove Theorem 1.12.

*Proof.* We could adapt the proof of Theorem 1.9 to this case. Alternatively, we can apply Theorem 1.9 to the nonempty set $-S$, and use Proposition 1.10. Then

$$m = \inf S \Leftrightarrow m = -\sup(-S) = -M.$$

In the proof of Proposition 1.10 we have seen that $m$ is a lower bound of $S$ if and only $-m$ is an upper bound of $-S$, and observing

$$M - \varepsilon < -s \Leftrightarrow s < m + \varepsilon,$$

that $x > m$ means $-x < M$, and that $-x < -s$ for $s \in S$ means $s < x$, it follows that the equivalence in Theorem 1.12 follows from the equivalence in Theorem 1.9. $\square$

7. Let $S$ and $T$ be non-empty subsets of $\mathbb{R}$ which are bounded above. Use Theorem 1.9 to prove that $\sup(S+T) = \sup S + \sup T$.

*Proof.* Step 1: Let $K = \sup S$ and $M = \sup T$. For all $x \in S$ and $y \in T$ we have $x \leq K$ and $y \leq M$, so that $x + y \leq K + M$. But this shows that $z \leq K + M$ for all $z \in S + T$. Hence $K + M$ is an upper bound of $S + T$.
Step 2: Let $\varepsilon > 0$. Then $\frac{\varepsilon}{2} > 0$ as well. Since $\sup S = K$, by Theorem 1.9 there is $s \in S$ such that $K - \frac{\varepsilon}{2} < s$. Also, since $\sup T = M$, there is $t \in T$ such that $M - \frac{\varepsilon}{2} < t$. Then $u = s + t \in S + T$, and

$$K + M - \varepsilon = \left(K - \frac{\varepsilon}{2}\right) + \left(M - \frac{\varepsilon}{2}\right) < s + t = u.$$

Step 3: Hence $K + M$ satiesfies Theorem1.9 (b), and it follows from Theorem 1.9 that

$$\sup(S + T) = K + M = \sup S + \sup T. \qquad \square$$

8. **Some simple questions.** Decide which of the following statements are **true** and which are **false**.

| | |
|---|---|
| (a) $\frac{1}{2} \in \{0, 1\}$ **false** | (b) $3 \in (0, 3)$ **false** |
| (c) $17 \in [0, 17]$ **true** | (d) $17 \in (-3, 18)$ **true** |
| (e) $17 \in [16, 18]$ **true** | (f) $2 \in \{1, 3, 5, 7\}$ **false** |
| (g) $2.5 \in \{x \in \mathbb{R} : x^2 \geq 4\}$ **false** | (h) $-1 \in \{x \in \mathbb{R} : 2x + 7 < 5\}$ **false** |

9*. Assume that the Dedekind cut property, Theorem 1.13, as well as the ordered field axioms are satisfied. Show that the Dedekind completeness holds.

*Proof.* Let $S$ be nonempty and bounded above.
If $S$ has a maximum, say $M$, the the supremum exists by Proposition 1.7.
Now consider the case that $S$ has no maximum. Let $B$ be the set of all upper bounds of $S$. Then $B \neq \emptyset$ since $S$ is bounded above. Define $A = \mathbb{R} \setminus B$. Then clearly $(A \cap B) = \emptyset$ and $A \cup B = \mathbb{R}$. Also, since $S$ is nonempty, there is $s \in S$. It follows for all $x \in B$ that $s - 1 < s \leq x$, which means that $s - 1 \notin B$, that is, $s - 1 \in A$, and therefore $A$ is nonempty. Hence $A$ and $B$ satisfy the assumptions (i) and (ii) of Theorem 1.13.
To prove assumption (iii) assume by proof of contradiction that there are $a \in A$ and $b \in B$ such that $a > b$. Since $B$ is the set of upper bounds of $S$, we have $s \leq b$ for all $s \in S$. But then also $s \leq a$ for all $s \in S$. But this means that $a$ is an upper bound of $S$, and therefore $a \in B$ by definition of $B$. But this contradicts $A \cap B = \emptyset$. Hence also assumption (iii) of Theorem 1.13 is satisfied.
By Theorem 1.13, there is $c \in \mathbb{R}$ such that $a \leq c \leq b$ for all $a \in A$ and $b \in B$.
We want to show that $c$ is the least upper bound of $S$, which proves the existence of the supremum of $S$.
Since $S$ has no maximum, no upper bound of $S$ belongs to $S$ by definition of the maximum. Hence $s \notin B$ for all $s \in S$, which means $s \in A$ for all $s \in S$. But then (iii) says that $s \leq c$ for all $s \in S$, that is $c$ is an upper bound of $S$, and $c \leq b$ for all $b \in B$ says that $c$ is the least upper bound of $S$. □