29 Lagrangian Mechanics I

> Nature is thrifty in all its actions.
>
> (attributed to *Pierre Louis Moreau de Maupertuis*)

> When I was in high school, my physics teacher called me down one day after class and said, "You look bored, I want to tell you something interesting". Then he told me something I have always found fascinating. Every time the subject comes up I work on it.
>
> (*Richard Feynman*)

The subject referred to is *the principle of least action*[1] which can be summarised as

> ... the laws of Newton could be stated, not in the form $F = ma$, but in the form: the average kinetic energy less the average potential energy is as little as possible for the path of an object going from one point to another.
>
> (*Richard Feynman*)

[1] As a companion to this chapter you can listen to Feynmann's lecture on the subject.

This chapter is concerned with Lagrangian Mechanics which involves using the principle of least action, rather than Newton's second law, to study Classical Mechanics. To get a feel for how the principle works, consider the a particle, of mass m, moving in a gravitational field from some initial time t_1 to some later time t_2. We will make things simple by only considering a particle moving in the vertical direction. We will denote the vertical direction using the variable x. A schematic plot of $x(t)$, as predicted by Newton's laws, is shown in Figure 29.1. The kinetic energy of the particle is, $\frac{1}{2}m\dot{x}^2$, and the potential energy is $V(x) = mgx$. Now, if we take the difference between the kinetic and potential energy and integrate it over the trajectory,

in the space between motion & potentiality, action exists

$$S = \int_{t_1}^{t_2} \left(\tfrac{1}{2}m\dot{x}^2 - mgx\right) \mathrm{d}t \ , \qquad (29.1)$$

we will get some quantity, S, which we are going to call the **action**. What the principle of least action tells us, is that if we consider a slightly different trajectory, as shown in Figure 29.2, one that is not predicted by Newtons laws, if we calculate the action, (29.1), we will get a bigger quantity. Turning this around, the trajectory which minimises the action, is precisely the trajectory one gets by solving Newton's equations.

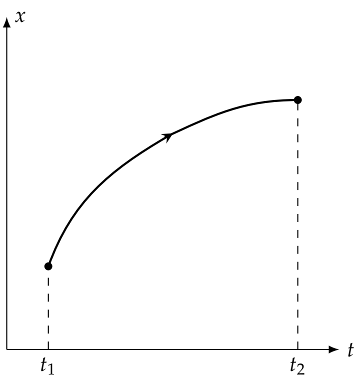

Figure 29.1: Trajetory of a particle in a gravitational field.

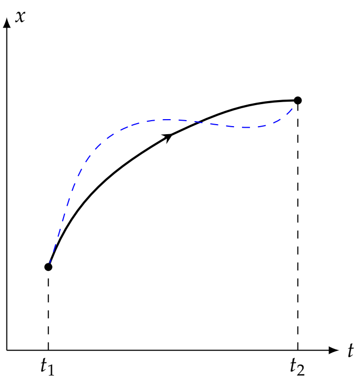

Figure 29.2: Trajetory of a particle in a gravitational field. The blue-dashed trajectory, not predicted by Newton's laws, has a larger action.

💬 Discussion 29.1

What are the dimensions of the action, S, in (29.1)? Can you think of an important physical constants that has these dimensions?

You may wonder why we are introducing a whole new formalism rather than sticking with Newton's laws. A huge advantage of Lagrangian mechanics is that it uses energy, which is a scalar. This means we do not need to worry as much about $\overrightarrow{\text{vectors}}$. This also makes working with different coordinate systems much easier.

Euler-Lagrange equations

1 particle in 1D

The starting point for Lagrangian mechanics, is the Lagrangian. We begin by looking at a single particle in one dimension. The Lagrangian of is then defined as the difference between the kinetic and potential energy

$$L(\dot{x}, x) = T(\dot{x}) - V(x) \, , \tag{29.2}$$

where $T = \frac{1}{2}m\dot{x}^2$. To implement the principle of least action, we need to find the trajectory that minimises the action

$$S = \int_{t_1}^{t_2} L(\dot{x}, x)\,\mathrm{d}t \, . \tag{29.3}$$

Now, you will have previously seen how to find the point at which a particular function, $V(x)$, has a minimum[2] – we just need solve the equation $V'(x) = 0$ in terms of x. Here we are confronted with a much more difficult problem – we need to find, not just a point, but a function $x(t)$ that minimises (29.3)[3].

To start, consider an arbitrary small shift or perturbation away from the solution, $x(t)$, which we write as[4]

$$x_{\mathrm{pt}}(t) = x(t) + \epsilon f(t) \, , \tag{29.4}$$

where $x_{\mathrm{pt}}(t)$ is the perturbed trajectory, ϵ is some small parameter and $f(t)$ is an arbitrary function. Figure 29.3 shows an example of a perturbed trajectory. Since we want to start and end our trajectories at the same points, we require that $f(t_1) = f(t_2) = 0$. Now, when ϵ is small, we can approximate the Lagrangian for the perturbed trajectory

$$L_{\mathrm{pt}} = L(x_{\mathrm{pt}}, \dot{x}_{\mathrm{pt}}) = L(x(t) + \epsilon f, \dot{x} + \epsilon \dot{f}) \, , \tag{29.5}$$

using Taylor's theorem:

$$L_{\mathrm{pt}} = \cancel{L(x, \dot{x})} + \frac{\partial L}{\partial x}(\epsilon f) + \frac{\partial L}{\partial \dot{x}}(\epsilon \dot{f}) + \mathcal{O}(\epsilon^2) \, . \tag{29.6}$$

This tells us that

$$\frac{\partial L}{\partial \epsilon} = \lim_{\epsilon \to 0} \frac{L_{\mathrm{pt}} - L}{\epsilon} = \frac{\partial L}{\partial x} f + \frac{\partial L}{\partial \dot{x}} \dot{f} \tag{29.7}$$

[2] or more accurately an extremum or critical point

[3] Mathematicians refer to the subject devoted to solving this class of problem as the "calculus of variations".

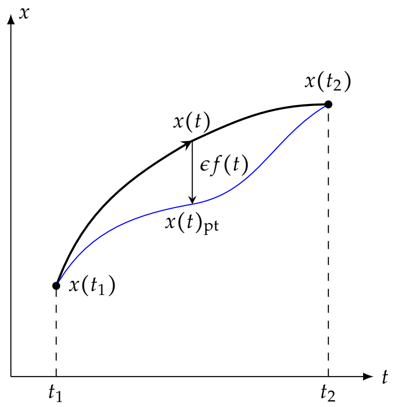

Figure 29.3: Trajetory of a particle $x(t)$ together with a possible perturbed trajectory $x_{\mathrm{pt}}(t)$.

[4] Most people will use the notation, $\epsilon f(t) = \delta x(t)$, but inspired by chapter 8 of [Moore 2013], I prefer this notation which allows us to talk about partial derivatives of the action as in (29.8) without having to delve into the more thorny subject of *functional derivatives*, which involve taking the derivative of one function with respect to another.

Remember that the quantity we want to minimise is the action, which means we want, $\frac{\partial S}{\partial \epsilon} = 0$, so looking at,

$$0 = \frac{\partial S}{\partial \epsilon} = \frac{\partial}{\partial \epsilon}\int_{t_1}^{t_2} L\,\mathrm{d}t = \int_{t_1}^{t_2} \frac{\partial L}{\partial \epsilon}\,\mathrm{d}t = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x}f + \frac{\partial L}{\partial \dot{x}}\frac{\mathrm{d}f}{\mathrm{d}t}\right)\mathrm{d}t\ . \tag{29.8}$$

Now, using integration by parts on the second term,

$$\begin{aligned}\int_{t_1}^{t_2}\frac{\partial L}{\partial \dot{x}}\frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}t &= \int_{t_1}^{t_2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}f\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)f\right)\mathrm{d}t \\ &= \left[\frac{\partial L}{\partial \dot{x}}f\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)f\,\mathrm{d}t \\ &= -\int_{t_1}^{t_2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)f\,\mathrm{d}t\ .\end{aligned} \tag{29.9}$$

Discussion 29.2

Justify the steps in (29.9).

Then, substituting (29.9) in (29.8) gives us

$$0 = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right)\right)f\,\mathrm{d}t\ , \tag{29.10}$$

Now, since (29.10) holds for any arbitrary function $f(t)$, we can conclude that

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0\ . \tag{29.11}$$

Discussion 29.3

Explain how we can conclude that since (29.10) holds for arbitrary f, (29.11) must be true.

Equation (29.11) is called the **Euler-Lagrange equation**. Then using (29.2) we find that

$$\begin{aligned}\frac{\partial L}{\partial x} &= -\frac{\partial V}{\partial x} = F_x \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial(\frac{1}{2}m\dot{x}^2)}{\partial \dot{x}}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(m\dot{x}) = \frac{\mathrm{d}p_x}{\mathrm{d}t}\end{aligned} \tag{29.12}$$

which, when substituted into (29.11), gives us NII in one dimension

$$\frac{\mathrm{d}p_x}{\mathrm{d}t} = F_x\ . \tag{29.13}$$

We see that, as claimed, the principle of least action is equivalent to Newton's second law [5].

[5] Strictly speaking, we have not shown that the action has to be minimum, but rather that we need it to be a critical point (also called an extremum).

1 particle in 3D

It is now relatively easy to generalise the principle of least action to a particle in three dimensions. Firstly, in (29.2), our kinetic energy is $T = \frac{1}{2}m\dot{\vec{r}}\cdot\dot{\vec{r}}$ and $V = V(\vec{r})$. One of the promised advantages of Lagrangian mechanics was the we could avoid dealing with vectors.

For now we'll simply replace the position of the particle, $\vec{r}$, with 3 coordinates, x^A, where, $x^1 = x$, $x^2 = y$, $x^3 = z$. We also write $p_1 = m\dot{x}$, $p_2 = m\dot{y}$ and $p_3 = m\dot{z}$. In this notation, Newton's second law corresponds to the 3 equations:

$$\dot{p}_A = -\frac{\partial V}{\partial x^A} \quad \text{where} \quad A = 1,2,3 \tag{29.14}$$

Discussion 29.4

Explain how (29.14) is a rewriting of $\vec{F} = \frac{d\vec{p}}{dt}$.

We can now run through our derivation of the Euler-Lagrange equation in one dimension with some minor modifications. Firstly, we consider a perturbation of all the coordinates

$$x^A_{\text{pt}}(t) = x^A(t) + \epsilon f^A(t) \,, \tag{29.15}$$

where $x^A_{\text{pt}}(t)$ is the perturbation of x^A, ϵ is some small parameter and the $f^A(t)$ are arbitrary functions. Then, as the Lagrangian is now a function of the x^A and $\dot{x}^A$, rather than just x and $\dot{x}$, (29.7) becomes

$$\frac{\partial L}{\partial \epsilon} = \lim_{\epsilon \to 0} \frac{L_{\text{pt}} - L}{\epsilon} = \sum_{A=1}^{3} \frac{\partial L}{\partial x^A} f^A + \sum_{A=1}^{3} \frac{\partial L}{\partial \dot{x}^A} \dot{f}^A \,. \tag{29.16}$$

Discussion 29.5

Fill in the details leading up to (29.16).

Then substituting (29.16) in (29.8) gives,

$$0 = \frac{\partial S}{\partial \epsilon} = \frac{\partial}{\partial \epsilon} \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} \left(\sum_{A=1}^{3} \frac{\partial L}{\partial x^A} f^A + \sum_{A=1}^{3} \frac{\partial L}{\partial \dot{x}^A} \dot{f}^A \right) dt \,, \tag{29.17}$$

and following analogous steps that lead from (29.8) to (29.11), we find an Euler-Lagrange equation for each coordinate.

$$\frac{\partial L}{\partial x^A} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^A} \right) = 0 \,, \quad \text{where } A = 1,2,3 \tag{29.18}$$

Discussion 29.6

Fill in the details that lead from (29.17) to (29.18).

Discussion 29.7

Show how (29.18), with $L = T - V = \sum_{A=1}^{3} \frac{1}{2} m \left(\dot{x}^A\right)^2 - V(x^1, x^2, x^3)$, leads to (29.14).

N particles in 3D (or is it 3N particles in 1D?)

Now we are finally able to generalise to N particles in three dimensions. We define $3N$ coordinates, x^A, with $A = 1, 2, \ldots, 3N$, called **generalised coordinates**. Each successive a group of three generalised coordinates correspond to the three coordinates of a particle as shown below:

$$
\begin{aligned}
&\vec{r}_{\textcircled{1}} \quad \begin{cases} r_{\textcircled{1}x} = x^1 \\ r_{\textcircled{1}y} = x^2 \\ r_{\textcircled{1}z} = x^3 \end{cases} \\
&\vec{r}_{\textcircled{2}} \quad \begin{cases} r_{\textcircled{2}x} = x^4 \\ r_{\textcircled{2}y} = x^5 \\ r_{\textcircled{2}z} = x^6 \end{cases} \\
&\vdots \\
&\vec{r}_{\textcircled{N}} \quad \begin{cases} r_{\textcircled{N}x} = x^{3N-2} \\ r_{\textcircled{N}y} = x^{3N-1} \\ r_{\textcircled{N}z} = x^{3N} \;, \end{cases}
\end{aligned}
\tag{29.19}
$$

so for example $x^7 = r_{\textcircled{3}x}$, e.t.c.. It will also be convenient to label the masses of our particles with these generalised coordinates in sets of three:

$$
\begin{aligned}
m_{\textcircled{1}} &= m_1 = m_2 = m_3 \\
m_{\textcircled{2}} &= m_4 = m_5 = m_6 \\
&\vdots \\
\vec{m}_{\textcircled{N}} &= m_{3N-2} = m_{3N-1} = m_{3N} \;.
\end{aligned}
\tag{29.20}
$$

Discussion 29.8

To which particle and which coordinate does x^{14} correspond? What is m_{14}?

Discussion 29.9

Explain how one can view the generalised coordinates as having replaced N particles in 3 spacial dimensions with a $3N$ particles in 1 spacial dimension. (It just so happens that these particles in one dimension come in triplets with identical masses).

We can then repeat (29.15),(29.16), (29.17) leading to (29.18), except that now A runs from 1 to $3N$ rather than from 1 to 3, so we have the Euler-Lagrange equations,

$$
\frac{\partial L}{\partial x^A} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^A}\right) = 0 \;, \quad \text{where } A = 1, 2, \ldots, 3N \tag{29.21}
$$

and our Lagrangian is

$$
L = L(x^A, \dot{x}^A) = T(\dot{x}^A) - V(x^A) = \sum_{A=1}^{3N} \tfrac{1}{2} m_A \left(\dot{x}^A\right)^2 - V(x^A) \;, \tag{29.22}
$$

where m_A is the mass of the corresponding particle[6].

[6] The notation $L(x^A, \dot{x}^A)$ is short hand for saying that L is a function of the x^A and $\dot{x}^A$, i.e. $L = L(x^1, x^2, \ldots, x^{3N}, \dot{x}^1, \dot{x}^2, \ldots, \dot{x}^{3N})$. Similarly, $T = T(\dot{x}^A)$ is short hand for saying that T is a function of the $\dot{x}^A$, and $V = V(x^A)$ is short hand for saying that V is a function of the x^A.

Discussion 29.10

What do you think we mean when we say that, "m_A is the mass of the corresponding particle", above. Remember we really have N particles not $3N$ particles.

Discussion 29.11

How do the Euler-Lagrange equations (29.21) and the Lagrangian (29.22) give us NII for our N particles

$$\vec{F}_n = \frac{\mathrm{d}\vec{p}_n}{\mathrm{d}t} \ ?$$

Note that $\vec{F}_n$ corresponds to the *total* force on the n-th particle.