Basic Analysis 2015 — Solutions of Tutorials

Section 3.1

Tutorial 3.1.1.

1. Prove from the definitions that
(a) $2 + x + x^2 \to 8$ as $x \to 2$, (b) $\dfrac{1}{x-2} \to -\dfrac{2}{3}$ as $x \to \dfrac{1}{2}$, (c) $\lim\limits_{x\to 1^-} \sqrt{1-x} = 0$.

Proof. (a) Let $f(x) = 2 + x + x^2$. Then

$$\begin{aligned} |f(x) - 8| &= |2 + x + x^2 - 8| = |x^2 + x - 6| \\ &= |(x-2)(x+3)| = |x-2|\,|x-2+5| \\ &\le |x-2|(|x-2|+5). \end{aligned}$$

For $|x-2| \le 1$ it follows that

$$|f(x) - 8| \le |x-2|(1+5) = 6|x-2|. \tag{1}$$

Now let $\varepsilon > 0$ and put

$$\delta = \min\left\{1, \frac{\varepsilon}{6}\right\}.$$

For $0 < |x-a| < \delta$ it follows from (1) that

$$|f(x) - 2| \le 6|x-2| < 6\delta \le 6\frac{\varepsilon}{6} = \varepsilon.$$

Hence $2 + x + x^2 \to 8$ as $x \to 2$.

(b) Let $g(x) = \dfrac{1}{x-2}$. Then

$$\begin{aligned} \left|g(x) + \frac{2}{3}\right| &= \left|\frac{3 + 2(x-2)}{3(x-2)}\right| = \left|\frac{2x-1}{3(x-2)}\right| \\ &= \frac{2}{3}\,\frac{|x-\frac{1}{2}|}{|x-2|} \end{aligned}$$

and

$$|x-2| = \left|\left(x - \frac{1}{2}\right) - \frac{3}{2}\right| \ge \frac{3}{2} - \left|x - \frac{1}{2}\right|.$$

For $|x - \frac{1}{2}| \le \frac{1}{2}$ it follows that $|x-2| \ge \frac{3}{2} - \frac{1}{2} = 1$.
Now let $\varepsilon > 0$ and put

$$\delta = \min\left\{\frac{1}{2}, \varepsilon\right\}.$$

For $0 < |x-a| < \delta$ it follows that

$$\left|g(x) + \frac{2}{3}\right| \le \frac{2}{3}\left|x - \frac{1}{2}\right| < \frac{2}{3}\varepsilon < \varepsilon.$$

Hence $\dfrac{1}{x-2} \to -\dfrac{2}{3}$ as $x \to \dfrac{1}{2}$.

(c) Let $h(x) = \sqrt{1-x}$. Let $\varepsilon > 0$ and put $\delta = \min\{1, \varepsilon^2\}$. Then it follows for $x \in (1-\delta, 1)$ that

$$|h(x)| = \sqrt{1-x} < \sqrt{\delta} \le \varepsilon.$$

Hence $\lim\limits_{x\to 1^-} \sqrt{1-x} = 0$. $\square$

2. By negating the definition of limit of a function show that the statement $f(x) \not\to L$ as $x \to a$ is equivalent to the following:
$\exists\, \varepsilon > 0\ \forall\, \delta > 0\ \exists\, x$ with $0 < |x - a| < \delta$ such that $|f(x) - L| \geq \varepsilon$.

Proof. This follows immediately from the following, where X is a set and $A(x)$ is a statement for each $x \in X$:
$\neg(\forall\, x \in X\ A(x)) \Leftrightarrow \exists\, x \in X\ \neg A(x)$,
$\neg(\exists\, x \in X\ A(x)) \Leftrightarrow \forall\, x \in X\ \neg A(x)$. □

3. For $x \in \mathbb{R} \setminus \{0\}$ let $f(x) = \sin \frac{1}{x}$. Prove that f does not tend to any limit as $x \to 0$.

Proof. Let $\varepsilon = \frac{1}{2}$ and let $\delta > 0$. Then there is $n \in \mathbb{N}$ such that $n > \dfrac{1}{2\pi\delta}$.
Putting $x = \dfrac{1}{2\pi n}$ we have $x \in (0, \delta)$, and
putting $y = \dfrac{1}{\pi\left(2n + \frac{1}{2}\right)}$ we have $y \in (0, \delta)$.
Then

$$f(y) - f(x) = \sin\left(\pi\left(2n + \frac{1}{2}\right)\right) - \sin(2\pi n) = 1 - 0 = 1,$$

and for each $L \in \mathbb{R}$ we have

$$\begin{aligned} 2\varepsilon = 1 = |f(y) - f(x)| &= |(f(y) - L) - (f(x) - L)| \\ &\leq |f(y) - L| + |f(x) - L|, \end{aligned}$$

so that $|f(y) - L| \geq \varepsilon$ or $|f(x) - L| \geq \varepsilon$. By Question 2 of this tutorial, it follows that f does not tend to any limit as $x \to 0$. □

4. Let $f(x) = x - \lfloor x \rfloor$. For each integer n, find $\lim\limits_{x \to n^-} f(x)$ and $\lim\limits_{x \to n^+} f(x)$ if they exist.

Proof. For $x \in (n - 1, n)$, $x - \lfloor x \rfloor = x - (n - 1)$.
Let $\varepsilon > 0$ and put $\delta = \min\{1, \varepsilon\}$. Then $x \in (n - \delta, n)$ shows that $x \in (n - 1, n)$, and therefore

$$|x - \lfloor x \rfloor - 1| = |x - (n - 1) - 1| = n - x < \delta,$$

so that $\lim\limits_{x \to n^-} f(x) = 1$.
For $x \in (n, n + 1)$, $x - \lfloor x \rfloor = x - n$.
Let $\varepsilon > 0$ and put $\delta = \min\{1, \varepsilon\}$. Then $x \in (n, n + \delta)$ shows that $x \in (n, n + 1)$, and therefore

$$|x - \lfloor x \rfloor| = x - n < \delta,$$

so that $\lim\limits_{x \to n^+} f(x) = 0$. □