Math2001–Basic Analysis Test April 2012: Reference Solutions

SECTION A Multiple Choice

All five questions are multiple choice questions and must be answered on the computer card provided. Please ensure that your student number is entered on the card, by pencilling in the requisite digit for each block. There is ONLY ONE correct answer to each question.

Questions	1	2	3	4	5
Answers	C	B	C	A	D

SECTION B

Answer this section in the answer book provided.

Question 1 . . . . [8 points]

Prove the *Archimedean principle*: *For each* $x \in \mathbb{R}$ *there exists* $n \in \mathbb{N}$ *such that* $x < n$.

Proof (This is in the lecture) Suppose, to the contrary, that there is $x \in \mathbb{R}$ such that for every $n \in \mathbb{N}$, $n \leq x$ ($\surd$). That is, the nonempty subset $\mathbb{N}$ of $\mathbb{R}$ is bounded above ($\surd$). By the completeness axiom, $\mathbb{N}$ has a supremum M ($\surd$). Since M is the smallest upper bound for $\mathbb{N}$, there exists $n \in \mathbb{N}$ such that $M - 1 < n$ ($\surd$ $\surd$). Then $n + 1 > (M - 1) + 1 = M$ ($\surd$). Since $n + 1 \in \mathbb{N}$, this contradicts the assumption that M is an upper bound for $\mathbb{N}$ ($\surd$). This proves the Archimedean principle ($\surd$).

Question 2 . . . . [8 points]

Prove that

$$\lim_{n \to \infty} n^{\frac{1}{n}} = 1.$$

Proof: Let $a_n = n^{\frac{1}{n}}$. Then for $n > 1$, $a_n > 1$ and write $a_n = 1 + a$ ($\surd$). Then

$$a_n^n = (1+a)^n = 1 + na + \frac{n(n-1)}{2}a^2 + \cdots + a^n > \frac{n(n-1)}{2}a^2. \; (\surd \; \surd)$$

For $n > 2$, $n - 1 > \dfrac{n}{2}$. Hence

$$a_n^n > \frac{n^2}{4}a^2 = \frac{n^2}{4}(a_n - 1)^2. \; (\surd)$$

That is

$$n > \frac{n^2}{4}\left(n^{\frac{1}{n}} - 1\right)^2. \; (\surd)$$

. Hence
$$0 < n^{\frac{1}{n}} - 1 < \frac{2}{\sqrt{n}}. \ (\surd)$$

Now $\lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$ ($\surd$). Hence by the "Sandwich Theorem",

$$\lim_{n\to\infty} n^{\frac{1}{n}} = 1. \ (\surd)$$

Question 3 .... [8 points]

Suppose that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = +\infty$ and $\lim_{n\to\infty} c_n = c > 0$. Using only the relevant definition, prove that

$$\lim_{n\to\infty} a_n b_n = +\infty, \ \lim_{n\to\infty} (a_n + c_n) = +\infty.$$

Proof: (a) Let $A \in \mathbb{R}$ (arbitrarily large positive real number). Since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = +\infty$, there exist $K_1, K_2 \in \mathbb{R}$ such that

$$\begin{array}{ll} (1) & n \geq K_1 \Rightarrow a_n > \sqrt{A} \ (\surd) \\ (2) & n \geq K_2 \Rightarrow b_n > \sqrt{A}. \ (\surd) \end{array}$$

Let $K = \max\{K_1, K_2\}$. If $n \geq K$ then both (1) and (2) are true ($\surd$). Hence

$$a_n b_n > \sqrt{A} \cdot \sqrt{A} = A.$$

Therefore, $\lim_{n\to\infty} a_n b_n = +\infty$ ($\surd$).

(b) Let $A \in \mathbb{R}$ (arbitrarily large positive real number). Since $\lim_{n\to\infty} c_n = c > 0$, the sequence (c_n) is bounded: there exist $a, b \in \mathbb{R}$ such that $a \leq c_n \leq b$ for all $n \in \mathbb{N}$ ($\surd$). Since $\lim_{n\to\infty} a_n = +\infty$, there exists $K \in \mathbb{R}$ such that if $n \geq K$ then $a_n > A - a$ ($\surd$). Hence $a_n + c_n > A - a + a = A$ ($\surd$).

Hence $\lim_{n\to\infty} a_n + c_n = +\infty$ ($\surd$).

Question 4 .... [6 points]

Let $a > 0$ and $a_n = a + (-1)^n n^2$. Prove that $\lim_{n\to\infty} |a_n| = +\infty$.

Proof: Let $A \in \mathbb{R}$ (arbitrarily large positive real number) ($\surd$). Let $K = \sqrt{A} + \sqrt{a}$ ($\surd$). Then for each $n \geq K$, ($\surd$)

$$|a_n| = \left|a + (-1)^n n^2\right| \geq \left|(-1)^n n^2\right| - a \geq K^2 - a > (K - \sqrt{a})^2 = A. \ (\surd\ \surd)$$

Hence $\lim_{n\to\infty} |a_n| = +\infty$ ($\surd$).

Question 5 .... [10 points]

Determine limits and justify your answer.

(1) $\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$;

(2) $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n+\left(\frac{n}{n+1}\right)^{n+1}\right]$.

Answer: (1)

$$\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}\left(\frac{1}{1+\dfrac{1}{n}}\right)^n=\frac{1}{\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n}=\frac{1}{e}.$$

(**5 marks**)

(2)

$$\begin{aligned}\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n+\left(\frac{n}{n+1}\right)^{n+1}\right] &=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n+\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{n+1}\\ &=e+\lim_{n\to\infty}\frac{1}{1+\dfrac{1}{n}}\cdot\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\\ &=e+\frac{1}{e}.\end{aligned}$$

(**5 marks**)

Note to markers: This memo is only for reference to the markers. Any valid proof based on concepts and results contained in Chapters 1 and 2 merits full marks. Except the last question, a tick ($\surd$) indicates suggestion for **1 mark**. Award of half marks are up to the discretion of colleagues who are marking.