COMS 3003A
HW 7

Dmitry Shkatov

Due 19 April, 2024

Reading: Leary & Kristiansen, Chapter 1.

1. **Let f be a function from a set X to a set Y. Write first-order sentences that are true just in case**

   (a) f **is an injection;**

$$\forall x \forall y \left(\neg(x = y) \to \neg(f(x) = f(y)) \right).$$

   (b) f **is a surjection;**

$$\forall y \exists x \, f(x) = y.$$

   (c) f **is a bijection.**

$$\forall x \forall y \left(\neg(x = y) \to \neg(f(x) = f(y)) \right) \wedge \forall y \exists x \, f(x) = y.$$

2. **The language of arithmetic contains the unary function symbol S ('successor'), the binary function symbols $+$ (addition) and $\cdot$ (multiplication), and a constant symbol 0. Define the following predicate letters in the language of arithmetic (i.e. write formulas that contain the same number of free variables as these letters and that are true precisely when the property or relation being defined holds):**

   (a) $x < y$;

$$x < y \quad := \quad \exists z \big(\neg(z = 0) \wedge (x + z = y)\big).$$

   (b) $x \leqslant y$;

$$x \leqslant y \quad := \quad x < y \vee x = y.$$

   (c) $x > y$;

$$x > y \quad := \quad \neg(x \leqslant y).$$

   (d) $x \geqslant y$;

$$x \geqslant y \quad := \quad \neg(x < y).$$

   (e) $\mathit{Divides}(x, y)$ (x divides y if y/x is a natural number; e.g., 3 divides 6, 9, and 15);

$$\mathit{Divides}(x, y) \quad := \quad \exists z(x \cdot z = y).$$

   (f) $\mathit{Prime}(x)$;

$$\begin{aligned} 1 \quad &:= \quad S(0); \\ \mathit{Prime}(x) \quad &:= \quad \neg(x = 0) \wedge \neg(x = 1) \wedge \forall y \big(\mathit{Divides}(y, x) \to y = 1 \vee y = x\big). \end{aligned}$$

   (g) $\mathit{Even}(x)$;

$$\begin{aligned} 2 \quad &:= \quad S(S(0)); \\ \mathit{Even}(x) \quad &:= \quad \exists y(x = y \cdot 2). \end{aligned}$$

   (h) $\mathit{Odd}(x)$.

$$\mathit{Odd}(x) \quad := \quad \neg \mathit{Even}(x).$$

3. **Translate into the language of arithmetic the following statements; for every formula you write, draw its formation tree.**

   (a) **There does not exist largest natural number.**

   $$\forall x \exists y \, (x < y).$$

   (b) **The number 0 is the smallest natural number.**

   $$\forall x \, (0 \leqslant x).$$

   (c) **Every number is either even or odd, and no number is both even and odd.**

   $$\forall x \, \big(Even(x) \vee Odd(x)\big) \wedge \neg \exists x \, \big(Even(x) \wedge Odd(x)\big).$$

   (d) **There are no natural numbers between a number and its immediate successor.**

   $$\forall x \, \neg \exists y \, (x < y \wedge y < S(x)).$$

   (e) **A natural number is either 0 or a successor of a natural number.**

   $$\forall x \, \big(x = 0 \vee \exists y \, (x = S(y))\big).$$

   (f) **There are infinitely many primes.**

   $$\exists x \, Prime(x) \wedge \forall x \, \big(Prime(x) \to \exists y \, (x < y \wedge Prime(y))\big).$$

   (g) **Every even number greater than 2 is a sum of two primes (this is Goldbach's conjecture).**

   $$\forall x \, \big((Even(x) \wedge x > 2) \to \exists y \exists z \, ((Prime(y) \wedge Prime(z)) \wedge y + z = x)\big).$$

   (h) **There are infinitely many pairs of prime numbers n and m such that $n - m = 2$ (this is a twin primes conjecture).**

   $$\begin{gathered} \exists x \exists y \, \big(((Prime(x) \wedge Prime(y)) \wedge y = x + 2) \wedge \\ \forall x \forall y \, \big(((Prime(x) \wedge Prime(y)) \wedge y = x + 2) \to \\ \exists z \exists v ((Prime(z) \wedge Prime(v)) \wedge (y < z \vee y = z) \wedge v = z + 2)\big). \end{gathered}$$

4. **The language of set theory contains a single binary predicate letter $\in$ ('member of'), no function symbols, and no constants. Translate into the language of set theory the following sentences:**

   (a) **If every element of X is an element of Y and vice versa, then X and Y are the same set.**

$$\forall x \forall y \left(\forall z \left((z \in x \leftrightarrow z \in y) \rightarrow x = y \right) \right).$$

   n

   (b) **There exists a set with no elements.**

$$\exists x \forall y \, \neg (y \in x).$$

   (c) **For every pair of sets, there exists a set containing precisely those elements that are in both sets.**

$$\forall x \forall y \exists z \forall u \left(u \in z \leftrightarrow u \in x \wedge u \in y \right),$$

   (d) **For every pair of sets, there exists a set containing precisely those elements that are in either set.**

$$\forall x \forall y \exists z \forall u \left(u \in z \leftrightarrow u \in x \vee u \in y \right),$$

   (e) **For every set X, there exists a set whose elements are precisely subsets of X.**

$$x \subseteq y := \forall z \, (z \in x \rightarrow z \in y);$$
$$\forall x \exists y \forall z \left(z \in y \leftrightarrow z \subseteq x \right).$$