Interpolation

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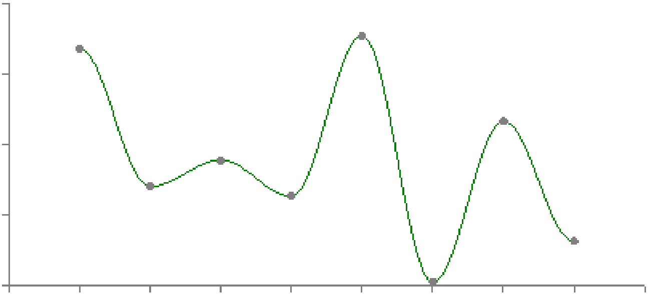
Motivation:

t_i, time	1	1.3	1.6	1.9	2.2
$V(t_i)$, Volume	0.7652	0.6201	0.4554	0.2818	0.1104

Question: What is $V(1.7)$?

Interpolation



Linear interpolation

Given only 2 points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ the interpolating function is a straight line passing through them.
Let $P_1(x) = a_0 + a_1 x = f(x)$.
The polynomial has to pass through these two points so

$$P_1(x_0) = a_0 + a_1 x_0 = f(x_0)$$
$$P_1(x_1) = a_0 + a_1 x_1 = f(x_1)$$

Therefore

$$a_0 = \frac{f(x_0)x_1 - f(x_1)x_0}{x_1 - x_0}, \quad a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Therefore:

$$P_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

which is a linear interpolating formula.

Example

Question: Estimate $\ln(2)$ using linear interpolation given $x_0 = 1$ and $x_1 = 6$.
Solution: Data: $(x_0, f(x_0)) = (1, \ln(1))$ and $(x_1, f(x_1)) = (6, \ln(6))$.
So

$$P_1(x) = f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0)$$

$$\ln(2) = P_1(2) = \ln 1 + \frac{\ln 6 - \ln 1}{6 - 1}(2 - 1)$$

$$= 0.3583519.$$

From calculator $\ln(2) = 0.6931472$.

Quadratic interpolation

Given 3 points $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$, a parabola

$$P_2(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)$$

can be used which passes through them.
How do we find a_0, a_1, a_2?
The polynomial has to pass through the 3 points.
Substituting $x = x_0$ in $P_2(x)$ results in $P_2(x_0) = a_0 = f(x_0)$.
Substituting $x = x_1$ in $P_2(x)$ results in
$P_2(x_1) = a_0 + a_1(x_1 - x_0) = f(x_0) + a_1(x_1 - x_0) = f(x_1)$.
Therefore

$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

What is a_2?

Substituting $x = x_2$ in $P_2(x)$ results in

$$
\begin{aligned}
P_2(x_2) = f(x_2) &= a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\
&= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) \\
&\vdots \\
&\vdots \\
&\vdots
\end{aligned}
$$

Finally,

$$
a_2 = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2 - x_0}.
$$

Example

Question: Estimate $\ln(2)$ using quadratic interpolation given $x_0 = 1$, $x_1 = 4$ and $x_2 = 6$.
Solution: Data: $(x_0, f(x_0)) = (1, \ln(1))$, $(x_1, f(x_1)) = (4, \ln(4))$ and $(x_2, f(x_2)) = (6, \ln(6))$.
So $P_2(x)$

$$\begin{aligned}
&= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\
&= f(x_0) + \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \\
&\quad + \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2 - x_0}(x - x_0)(x - x_1) \\
&= \ln 1 + \frac{\ln 4 - \ln 1}{4 - 1}(x - 1) \\
&\quad + \frac{\frac{\ln 6-\ln 4}{6-4} - \frac{\ln 4-\ln 1}{4-1}}{6 - 1}(x - 1)(x - 4)
\end{aligned}$$

Example Cont'd

Therefore $P_2(x) = 0 + 0.4621(x-1) - 0.0519(x-1)(x-4)$
So ln(2) is approximate by
$P_2(2) = 0 + 0.4621(2-1) - 0.0519(2-1)(2-4) = 0.5658.$
linear interpolation gives 0.3583519.
Exact value is $\ln(2) = 0.6931472$.

Lagrange Interpolating Polynomials

Suppose we want to determine a first degree polynomial that passes through two points (x_0, y_0) and (x_1, y_1). Let such a polynomial have the form:

$$\begin{aligned} P(x) &= \frac{(x - x_1)}{(x_0 - x_1)} y_0 + \frac{(x - x_0)}{(x_1 - x_0)}\ y_1 \\ &= L_0(x)y_0 + L_1(x)y_1. \end{aligned}$$

Note that

1. $P(x_0) = y_0$ and $P(x_1) = y_1$.
2. When $x = x_0,\ L_0(x_0) = 1$ and $L_1(x_0) = 0$.
3. When $x = x_1,\ L_0(x_1) = 0$ and $L_1(x_1) = 1$.

Lagrange Interpolating Polynomials

In general, to construct a polynomial of degree at most $P_n(x)$ that passes through the $n+1$ points $(x_0, f(x_0)),\ (x_1, f(x_1)), \ldots, (x_n, f(x_n))$, we need to construct for $k = 0, 1, \ldots, n$,

1. $L_{n,k}(x_i) = 0$ when $i \neq k$.
2. $L_{n,k}(x_k) = 1$.

Therefore, for Condition 1. to hold,

$$L_{n,k}(x) = \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{\cdots}$$

Finally for 2. to hold,

$$\begin{aligned} L_{n,k}(x) &= \frac{(x - x_0)\cdots(x - x_{k-1})(x - x_{k+1})\cdots(x - x_n)}{(x_k - x_0)\cdots(x_k - x_{k-1})(x_k - x_{k+1})\cdots(x_k - x_n)} \\ &= \prod_{i=0, i\neq k}^{n} \frac{(x - x_i)}{(x_k - x_i)}. \end{aligned} \tag{1}$$

Example

The lagrange interpolating polynomial is thus

$$P(x) = L_{n,0}(x)f(x_0) + L_{n,1}(x)f(x_1) + \ldots + L_{n,n}(x)f(x_n)$$

We write L_k instead of $L_{n,k}$. Question: Use a Lagrange interpolating polynomial of the first and second order to evaluate $f(2)$ on the basis of the data

$$(1, 0), (4, 1.386294), (6, 1.791760).$$

Solution:

$$L_k(x) = \prod_{i=0, i\neq k}^{n} \frac{x - x_i}{x_k - x_i}$$

Example

For $n = 1$:

$P_1(x)$ we have $L_0(x) = \frac{(x-x_1)}{(x_0-x_1)}$ and $L_1(x) = \frac{(x-x_0)}{(x_1-x_0)}$

$$P_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1).$$

$$\text{So} \qquad P_1(2) = \frac{2-4}{1-4}(0) + \frac{2-1}{4-1}(1.386294) = 0.4620981.$$

For $n = 2$:

$P_2(x)$ we have $L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$,

$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}$, $\qquad L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$

$$P_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2).$$

$$\text{So } P_2(2) = \frac{(2-4)(2-6)}{(1-4)(1-6)}(0) + \frac{(2-1)(2-6)}{(4-1)(4-6)}(1.386294) + \frac{(2-1)(2-4)}{(6-1)(6-4)}(1.791760) = 0.565844.$$

Example 2

Use the following data to approximate $f(1.5)$ using the Lagrange interpolating polynomial for $n = 4$.

x_i	1	1.3	1.6	1.9	2.2
$f(x_i)$	0.765198	0.620086	0.455402	0.281819	0.110362

Here $L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)}$ etc.
Clearly we need code to do such tasks.
Finally,

$$P_4(x) = (((0.0018251x + 0.0552928)x - 0.343047)\\ x + 0.0733913)x + 0.977735,$$

which gives,

$$P_4(1.5) = 0.508939.$$

Newton's Divided Differences

0. The zeroth divided difference of f w.r.t. x_i is $f[x_i] = f(x_i) = f_i$.
1. The first divided difference of f w.r.t. x_i and x_{i+1} is:
$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i} = \frac{f_{i+1} - f_i}{x_{i+1} - x_i}$$
2. The Second divided difference of f w.r.t. x_i, x_{i+1} and x_{i+2} is:
$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$
3. The k^{th} divided difference of f w.r.t. $x_i, x_{i+1}, \cdots, x_{i+k}$ is:
$$f[x_i, x_{i+1}, \cdots, x_{i+k-1}, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \cdots, x_{i+k}] - f[x_i, x_{i+1}, \cdots, x_{i+k-1}]}{x_{i+k} - x_i}$$

The polynomial

We fit a $P_n(x)$ to the $n+1$ data points $(x_i, f(x_i)),\ i = 0, 1, \cdots, n$ in the form:

$$P_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \cdots \\ + a_n(x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

Since the polynomial must pass through the points (x_i, f_i) we have:

- $x = x_0,\ P_n(x_0) = f(x_0)\ so,\ a_0 = f(x_0) = f[x_0]$.
- $x = x_1,\ P_n(x_1) = f(x_1)\ so,\ f[x_0] + a_1(x_1 - x_0) = f[x_1] \quad \Rightarrow a_1 = \frac{f[x_1]-f[x_0]}{x_1 - x_0} = f[x_0, x_1]$.
- $x = x_2,\ P_n(x_2) = f(x_2). \quad \ldots\ldots \quad a_2 = f[x_0, x_1, x_2]$.

In general:

$$a_k = f[x_0, x_1, \cdots, x_k]$$

"Tree"

Suppose:

$$P_3(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2).$$

We know that $a_0 = f[x_0]$, $a_1 = f[x_0, x_1]$, $a_2 = f[x_0, x_1, x_2]$ and $a_3 = f[x_0, x_1, x_2, x_3]$

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
x_0	$f[x_0]$			
		$f[x_0, x_1]$		
x_1	$f[x_1]$		$f[x_0, x_1, x_2]$	
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$
x_2	$f[x_2]$		$f[x_1, x_2, x_3]$	
		$f[x_2, x_3]$		
x_3	$f[x_3]$			

Example

Find a polynomial satisfied by $(-4, 1245), (-1, 33), (0, 5), (2, 9)$

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
-4	1245			
		-404		
-1	33		94	
		-28		-14
0	5		10	
		2		
2	9			

$$
\begin{aligned}
\therefore P_3(x) &= 1245 - 404(x+4) + 94(x+4)(x+1) \\
&\quad - 14(x+4)(x+1)x \\
&= -14x^3 + 24x^2 + 52x + 5.
\end{aligned}
$$