Singular value decomposition (SVD) and Cholesky decomposition

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Singular value decomposition (SVD)

Let $A \in \mathbb{R}^{m\times n}$ then there exist orthogonal matrices

$$U = [u_1, u_2, \cdots, \cdots, u_m] \in \mathbb{R}^{m\times m}$$
$$V = [v_1, v_2, \cdots, v_n] \in \mathbb{R}^{n\times n}$$

such that

$$U^T AV = \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_p) \tag{1}$$

where $p = \min\{m, n\}$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p \geq 0$.
We can write (1) as

$$A = U \operatorname{diag}(\sigma_1, \sigma_2, \cdots, \sigma_p)V^T$$

which is called the singular decomposition (SVD) of A.
The σ_i's are called <u>singular values</u> of A and vectors u_i and v_i are the i^{th} <u>left and right singular vectors</u> respectively.

Singular value decomposition (SVD)

We can also verify that

$$\mathsf{A}\mathsf{v}_i = \sigma_i \mathsf{u}_i$$

$$\mathsf{A}^T \mathsf{u}_i = \sigma_i \mathsf{v}_i$$

To do this we need to verify that

$$\mathsf{A} = \sum_{j=1}^{r} \sigma_j \mathsf{u}_j \mathsf{v}_j^T$$

which implies

$$\mathsf{A}^T = \sum_{j=1}^{r} \sigma_j \mathsf{v}_j \mathsf{u}_j^T$$

Singular value decomposition (SVD)

Therefore

$$
\begin{aligned}
\mathsf{A}\mathsf{v}_i &= \left(\sum_{j=1}^{r} \sigma_j \mathsf{u}_j \mathsf{v}_j^T\right) \mathsf{v}_i \\
&= \sum_{j=1}^{r} \sigma_j \mathsf{u}_j \mathsf{v}_j^T \mathsf{v}_i \\
&= \sigma_i \mathsf{u}_i \mathsf{I}_n \\
&= \sigma_i \mathsf{u}_i
\end{aligned}
$$

SVD Example

Verify that the SVD of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$$

comprises

$$U = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

and singular values 2 and 0.

SVD Example

Solution:

1.) Verify that $U^T AV = \text{diag}(2, 0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

2.) Verify that U and V are orthogonal.

Positive definite systems

$A \in \mathbb{R}^{m\times n}$ is positive definite if

$$x^T A x > 0, \quad \text{nonzero } x \in \mathbb{R}^n.$$

Cholesky decomposition: If $A \in \mathbb{R}^{m\times n}$ is symmetric and positive definite then there exists lower triangular matrix $G \in \mathbb{R}^{n\times n}$ with positive entries such that

$$A = GG^T.$$

Example: The matrix

$$A = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$$

is positive definite and has Cholesky decomposition

$$G = \begin{bmatrix} \sqrt{2} & 0 \\ -\sqrt{2} & \sqrt{3} \end{bmatrix}.$$

Exercise: Verify that A in the example is positive definite.

Construction of Cholesky decomposition

We can contruct the matrix G by comparing elements in the equation $A = GG^T$.
First note that $i \geq k$ we have

$$\begin{aligned} a_{ik} &= \sum_{p=1}^{k} g_{ip} g_{kp} \\ &= \sum_{p=1}^{k-1} g_{ip} g_{kp} + g_{ik} g_{kk}, \end{aligned}$$

this implies, $$g_{ik} = \left(a_{ik} - \sum_{p=1}^{k-1} g_{ip} g_{kp} \right) / g_{kk}, \qquad i > k.$$

and for $i = k$, $$g_{kk} = \left(a_{kk} - \sum_{p=1}^{k-1} g_{kp}^2 \right)^{1/2}.$$

Cholesky decomposition Algorithm

Given $A \in \mathbb{R}^{m\times n}$ is symmetric and positive definite then the following algorithm computes a lower triangular matrix $G \in \mathbb{R}^{n\times n}$ such that $A = GG^T$:

For $k = 1, 2 \ldots, n$

$$g_{kk} = \left(a_{kk} - \sum_{p=1}^{k-1} g_{kp}^2 \right)^{1/2}$$

For $i = k+1, k+2 \ldots, n$

$$g_{ik} = \left(a_{ik} - \sum_{p=1}^{k-1} g_{ip} g_{kp} \right) / g_{kk}$$