Matrix Decompositions 2 of 2

Geometric Multiplicity

Geometric Multiplicity

Let λ_i be an eigenvalue of a square matrix $\mathbf{A}$.

- Then the geometric multiplicity of λ_i is the number of linearly independent eigenvectors associated with λ_i

Said differently, it is the dimensionality of the eigenspace spanned by the eigenvectors associated with λ_i.

- A specific eigenvalue's geometric multiplicity must be at least one because every eigenvalue has at least one associated eigenvector.
- An eigenvalue's geometric multiplicity cannot exceed its algebraic multiplicity, but it may be lower.

Geometric Multiplicity: Example

Consider

$$\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

Since this matrix is in upper triangle form it follows that its two repeated eigenvalue are $\lambda_1 = \lambda_2 = 2$

- The eigenvalue 2 has an algebraic multiplicity of 2

But the only unit eigenvector is

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- The eigenvalue has geometric multiplicity of 1

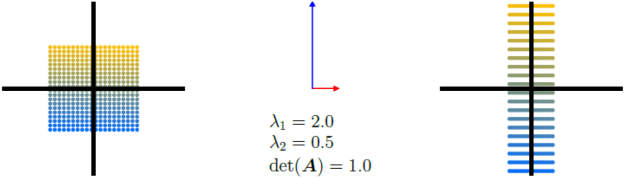
Graphical Intuition in Two Dimensions

In order to add a bit more of a tangible feel of the implications of determinants, eigenvectors, and eigenvalues we consider a couple of different linear mappings

- Consider

$$\mathbf{A}_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix} \tag{1}$$

  - It has eigenvalue $\lambda_1 = 2$ and $\lambda_2 = \frac{1}{2}$
  - The corresponding eigenvector are the canonical basis vectors of $\mathbb{R}^2$
  - $\det(\mathbf{A_1}) = 1$



Source: M.P. Deisenroth *et al*, Mathematics for Machine Learning (First Edition)

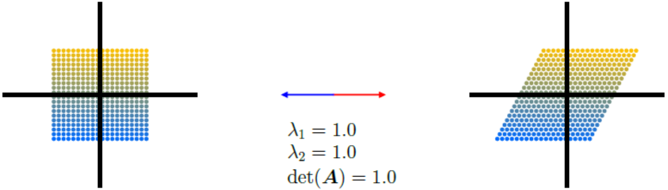
Graphical Intuition in Two Dimensions

- Consider

$$\mathbf{A}_2 = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \tag{2}$$

corresponds to a shearing mapping.

  - $\det(\mathbf{A_2}) = 1$ so it is area preserving
  - It has eigenvalues $\lambda_1 = \lambda_2 = 1$ and are repeated and the eigenvectors are collinear
  - This indicates that the mapping acts only along one direction (the horizontal axis).



Source: M.P. Deisenroth *et al*, Mathematics for Machine Learning (First Edition)

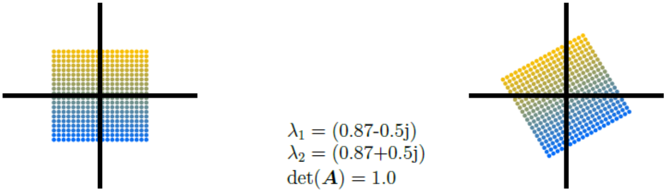
Graphical Intuition in Two Dimensions

- Consider

$$\mathbf{A}_3 = \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \tag{3}$$

which is a rotation of $\pi/6$ rad (30° degrees) counter-clockwise and has only complex eigenvalues

  - $\det(\mathbf{A_3}) = 1$ so it is area/volume preserving (as are all rotations)



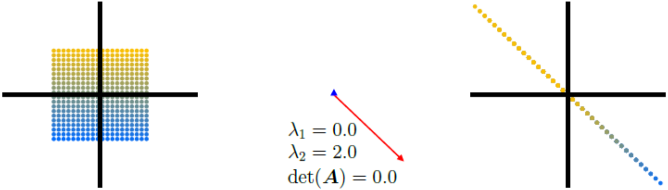
Source: M.P. Deisenroth *et al*, Mathematics for Machine Learning (First Edition)

Graphical Intuition in Two Dimensions

- Consider

$$\mathbf{A}_4 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{4}$$

  - The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 2$
  - The corresponding eigenvector are $\mathbf{v}_1 = [1, 1]$, $\mathbf{v}_2 = [-1, 1]$
  - $\det(\mathbf{A_4}) = 0$ so it is not area/volume preserving, and in fact collapses the area to 0.



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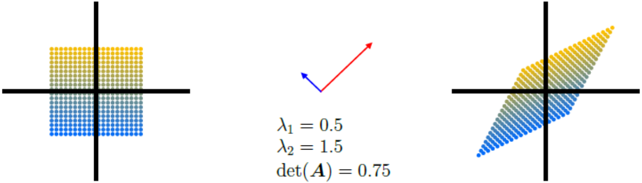
Graphical Intuition in Two Dimensions

- Consider

$$\mathbf{A}_4 = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \tag{5}$$

is a shear-and-stretch mapping

  - Specifically, since $\det(\mathbf{A}_4) = 0.75$ the area is scaled by 0.75.
  - It stretches space along the (red) eigenvector of λ_2 by a factor 1.5 and compresses it along the orthogonal (blue) eigenvector by a factor 0.5.



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Linearly Independent Eigenvectors

Theorem 4.12

The eigenvectors $\mathbf{v}_1, \ldots, \mathbf{x}_n$ of $\mathbf{A} \in \mathbb{R}^{n \times n}$ with n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ are are linearly independent.

This theorem means that that eigenvectors of a matrix with n distinct eigenvalues form a basis of $\mathbb{R}^n$.

Defective Matrices

Defective Matrix

A square matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is *defective* if it possesses fewer than n linearly independent eigenvectors.

Note that

- A non-defective matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ does not necessarily require n distinct eigenvalues, but it does require that the eigenvectors form a basis of $\mathbb{R}^n$.
- If a matrix has geometric multiplicity less than the corresponding algebraic multiplicity for an eigenvalue λ_i then the matrix is defective.

Important remark:

- A defective matrix cannot have n distinct eigenvalues, follows from 4.12, as distinct eigenvalues have linearly independent eigenvectors.

Positive Semidefinite Matrix

$S = S^T$ $\quad$ $\underline{\forall x \neq 0}$ $\; x^T S x \geq 0$

Theorem 4.14.

Given a matrix $\mathbf{A}^{m\times n}$, we can always obtain a symmetric, positive semidefinite matrix $\mathbf{S} \in \mathbb{R}^{n\times n}$ by defining

$$\mathbf{S} := \mathbf{A}^T\mathbf{A} \tag{6}$$

If $rk(\mathbf{A}) = n$, then $\mathbf{S} := \mathbf{A}^T\mathbf{A}$ is symmetric, positive definite.

$$x^T S x = x^T A^T A x = \overline{(Ax)^T Ax}$$

$$y = Ax \qquad y^T y > 0 \qquad y \neq 0$$

Positive Semidefinite Matrix

Motivation for theorem 4.14

- Symmetry: Recall that $\mathbf{S}$ is symmetric when $\mathbf{S} = \mathbf{S}^T$. Obverse that

$(BC)^T = C^T B^T$

$$
\begin{aligned}
S &= \mathbf{A}^T\mathbf{A} && (7)\\
&= \mathbf{A}^T(\mathbf{A}^T)^T && (8)\\
&= (((\mathbf{A}^T)^T)^T(\mathbf{A}^T)^T)^T && (9)\\
&= (\mathbf{A}^T\mathbf{A})^T && (10)\\
&= \mathbf{S}^T && (11)
\end{aligned}
$$

Positive Semidefinite Matrix

Motivation for theorem 4.14

- Positive semidefiniteness: let $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ then observe that

$$\begin{aligned}
\mathbf{x}^T\mathbf{S}\mathbf{x} &= \mathbf{x}^T\mathbf{A}^T\mathbf{A}\mathbf{x} && (12)\\
&= (\mathbf{x}^T\mathbf{A}^T)(\mathbf{A}\mathbf{x}) && (13)\\
&= (\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x}) && (14)\\
&\geq 0 && (15)
\end{aligned}$$

Since $(\mathbf{A}\mathbf{x})^T(\mathbf{A}\mathbf{x})$ is the dot product.

- Think about why we get > 0 if $rk(\mathbf{A}) = n$.

Spectral Theorem

Spectral Theorem

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be symmetric,

- there exists an orthonormal basis of the corresponding vector space V consisting of eigenvectors of $\mathbf{A}$, and each eigenvalue is real.

A direct implication of the spectral theorem is that the *eigendecomposition* of a symmetric matrix $\mathbf{A}$ exists (with real eigenvalues),

- and that we can find an ONB of eigenvectors so that $\mathbf{A} = \mathbf{PDP}^T$, where $\mathbf{D}$ is diagonal and the columns of $\mathbf{P}$ contain the eigenvectors.

Spectral Theorem: Worked example

Consider

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

The characteristic polynomial of $\mathbf{A}$, after factoring, is

$$p_{\mathbf{A}}(\lambda) = -(\lambda - 1)^2(\lambda - 7)$$

- So the eigenvalues are $\lambda_1 = 1$ with algebraic multiplicity of 2 and $\lambda_2 = 7$ with algebraic multiplicity of 1.

Spectral Theorem: Worked example

- Following our standard procedure for computing eigenvectors, we obtain the eigenspaces (see additional video for workings)

$$E_1 = span\left[\underbrace{\begin{bmatrix}-1\\1\\0\end{bmatrix}}_{\mathbf{x}_1}, \underbrace{\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{\mathbf{x}_2}\right], \quad E_7 = span\left[\underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{\mathbf{x}_3}\right]$$

$E_1 \perp E_7$

- Now $\mathbf{x}_3$ is orthogonal to $\mathbf{x}_1$ and $\mathbf{x}_2$ since

$$\mathbf{x}_3^T\mathbf{x}_1 = 0 \text{ and } \mathbf{x}_3^T\mathbf{x}_2 = 0 \tag{16}$$

but $\mathbf{x}_1$ and $\mathbf{x}_2$ are not

$$\mathbf{x}_1^T\mathbf{x}_2 = 1 \neq 0 \tag{17}$$

Spectral Theorem: Worked example

$Av = \lambda v$

We now need to form an orthonormal basis form $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ while the new basis vectors must still valid eigenvectors.

- Observe that since $\mathbf{x}_1$ and $\mathbf{x}_2$ are eigenvectors associated with the same eigenvalue λ, it follows that $\forall \alpha, \beta \in \mathbb{R}$ it holds that

$$\begin{aligned} \mathbf{A}\left(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2\right) &= \mathbf{A}\mathbf{x}_1\alpha + \mathbf{A}\mathbf{x}_2\beta && (18) \\ &= \lambda \mathbf{x}_1 \alpha + \lambda \mathbf{x}_2 \beta && (19) \\ &= \lambda(\alpha \mathbf{x}_1 + \beta \mathbf{x}_2) && (20) \end{aligned}$$

  so any linear combination of them is also a eigenvector.

  - This allows use to use The Gram-Schmidt algorithm safely to find eigenvectors associated with $\lambda_1 = 1$ that are orthogonal to each other (and to $\mathbf{x}_3$).

Spectral Theorem: Worked example

Specifically, we set

$$\mathbf{x}_1' = \mathbf{x}_1$$

and use the standard formula

$$\begin{aligned}
\mathbf{x}_2' &= \mathbf{x}_2 - \pi_{span[\mathbf{x}_1']}(\mathbf{x}_2) \\
&= \mathbf{x}_2 - \frac{\mathbf{x}_1'\mathbf{x}_1'^T}{\|\mathbf{x}_1'\|^2}\mathbf{x}_2 \\
&= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}
\end{aligned}$$

- Why is $\mathbf{x}_3$ still orthogonal to $\mathbf{x}_1'$ and $\mathbf{x}_2'$?

Theorem 4.16

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ then

$$\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i, \tag{21}$$

where $\lambda_i \in \mathbb{C}$ are (possibly repeated) eigenvalues of $\mathbf{A}$.

$\det(A) \in \mathbb{R}$

$\lambda_j = \alpha + ib$ $\quad i = \sqrt{-1}$

$\lambda_k = \alpha - ib$

Trace and Eigenvalues

Theorem 4.17

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ then

$$tr(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i, \tag{22}$$

where $\lambda_i \in \mathbb{C}$ are (possibly repeated) eigenvalues of $\mathbf{A}$.

$tr(A) \in \mathbb{R}$

Cholesky Decomposition

Cholesky Decomposition

A **symmetric**, **positive definite** matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ can be factorized into a product

$$\mathbf{A} = \mathbf{L}\mathbf{L}^T, \tag{23}$$

where $\mathbf{L}$ is a lower triangular matrix with positive diagonal elements:

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & \dots & 0 \\ \vdots & \ddots & \vdots \\ l_{n1} & \dots & l_{nn} \end{bmatrix} \begin{bmatrix} l_{11} & \dots & l_{n1} \\ \vdots & \ddots & \vdots \\ 0 & \dots & l_{nn} \end{bmatrix}$$

$\mathbf{L}$ is called the Cholesky factor of $\mathbf{A}$, and $\mathbf{L}$ is unique.

Cholesky Decomposition

We can derive the formula needed to build **L**. Let use consider a 3x3 case.

- Let $\mathbf{A} \in \mathbb{R}^{3\times 3}$ be symmetric, positive definite matrix.
- Note that

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \mathbf{L}\mathbf{L}^T = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

if we multiple out the right hand side we end up with

$$\mathbf{A} = \begin{bmatrix} l_{11}^2 & l_{21}l_{11} & l_{31}l_{11} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{31}l_{21} + l_{32}l_{22} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix} \tag{24}$$

$$\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{21}l_{11} & l_{31}l_{11} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{31}l_{21} + l_{32}l_{22} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix} \tag{25}$$

There are a couple of patterns present that can be used to calculate the entries of **L**

- In terms of the diagonals we can see that

$$l_{11} = \sqrt{a_{11}}, \quad l_{22} = \sqrt{a_{22} - l_{21}^2}, \quad l_{33} = \sqrt{a_{33} - (l_{31}^2 + l_{32}^2)} \tag{26}$$

- Similarly for the elements below the diagonal there is also a repeating pattern:

$$l_{21} = \frac{1}{l_{11}} a_{21}, \quad l_{31} = \frac{1}{l_{11}} a_{31}, \quad l_{32} = \frac{1}{l_{22}} (a_{32} - l_{31}l_{21}) \tag{27}$$

Thus, we constructed the *Cholesky decomposition* for any *symmetric, positive definite* 3×3 matrix.

Cholesky Decomposition:

One of the main upsides about the Cholesky decomposition is that

$$\begin{aligned}
\det(\mathbf{A}) &= \det(\mathbf{L}\mathbf{L}^T) \\
&= \det(\mathbf{L})\det(\mathbf{L}^T) \\
&= \det(\mathbf{L})\det(\mathbf{L}) \\
&= \det(\mathbf{L})^2
\end{aligned}$$

Since $\mathbf{L}$ is a triangular matrix, the determinant is simply the product of its diagonal entries so that

$$\det(\mathbf{A}) = \prod_i l_{ii}^2 \tag{28}$$

- The covariance matrix of a multivariate Gaussian variable is symmetric, positive definite.
- The Cholesky factorization of this covariance matrix allows us to generate samples from a Gaussian distribution.

Eigendecomposition and Diagonalization

Diagonal matrix

The matrix $\mathbf{D}$ is diagonal if it has value zero on all off-diagonal elements

For example

$$\mathbf{D} = \begin{bmatrix} c_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & c_n \end{bmatrix} \tag{29}$$

- Diagonal matrices are great because they allow fast computation of *determinants*, *powers*, and *inverses*.

Eigendecomposition and Diagonalization

Specifically, if $\mathbf{D}$ is diagonal

- $\det(\mathbf{D})$ is the product of its diagonal entries
- $\mathbf{D}^k$ is given by each diagonal element raised to the power k
- $\mathbf{D}^{-1}$ is the reciprocal of its diagonal elements if all of them are nonzero.

We will discuss how to transform matrices into diagonal form.

Eigendecomposition and Diagonalization

Diagonalizable

$\mathbf{A} \in \mathbb{R}^{n \times n}$ is *diagonalizable* if it is similar to a diagonal matrix.

- Specifically, if there exists an invertible $\mathbf{P} \in \mathbb{R}^{n \times n}$ such that

$$\mathbf{D} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} \tag{30}$$

Eigendecomposition and Diagonalization

We can actually get a constructive means of building $\mathbf{P}$ from $\mathbf{A} \in \mathbb{R}^{n \times n}$ by considering the following set up.

- Let $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$, and let $\mathbf{p}_1, \ldots, \mathbf{p}_n \in \mathbb{R}^n$.
- Define $\mathbf{P} := [\mathbf{p}_1, \ldots, \mathbf{p}_n]$ and let $\mathbf{D} \in \mathbb{R}^{n \times n}$ be diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$.
- Then we can show that

$$\mathbf{AP} = \mathbf{PD} \tag{31}$$

  if and only if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $\mathbf{A}$ and $\mathbf{p}_1, \ldots, \mathbf{p}_n$ are corresponding eigenvectors.

We can see that this statement holds because

$$\mathbf{AP} = \mathbf{A}[\mathbf{p}_1, \ldots, \mathbf{p}_n] = [\mathbf{A}\mathbf{p}_1, \ldots, \mathbf{A}\mathbf{p}_n] \tag{32}$$

$$\mathbf{PD} = [\mathbf{p}_1, \ldots, \mathbf{p}_n] \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix} = [\lambda_1 \mathbf{p}_1, \ldots, \lambda_n \mathbf{p}_n] \tag{33}$$

which implies that

$$\mathbf{A}\mathbf{p}_1 = \lambda_1 \mathbf{p}_1 \tag{34}$$

$$\vdots$$

$$\mathbf{A}\mathbf{p}_n = \lambda_n \mathbf{p}_n \tag{35}$$

Therefore, the columns of **P** must be eigenvectors of **A**.

Eigendecomposition and Diagonalization

The definition of diagonalization used in this course requires that $\mathbf{P}$ is invertible

- $\implies rk(\mathbf{P}) = n$
- $\implies$ we have n linearly independent eigenvectors $\mathbf{p}_1, \ldots, \mathbf{p}_n$ i.e., the $\mathbf{p}_i$ form a basis of $\mathbb{R}^n$.

Eigendecomposition and Diagonalization

Theorem 4.20: Eigendecomposition

$\mathbf{A} \in \mathbb{R}^{n \times n}$ can be factored into

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1} \tag{36}$$

where $\mathbf{P} \in \mathbb{R}^{n \times n}$ and $\mathbf{D}$ is a diagonal matrix whose diagonal entries are the eigenvalues of $\mathbf{A}$

- if and only if the eigenvectors of $\mathbf{A}$ form a basis of $\mathbf{R}^n$

In short only non-defective matrices can be diagonalized in this way.

Eigendecomposition and Diagonalization

Theorem 4.21

A symmetric matrix $\mathbf{S} \in \mathbb{R}^{n \times n}$ can always be diagonalized.

This follows from the spectral theorem:

- Specifically, the spectral theorem states that we can find an ONB of eigenvectors of $\mathbb{R}^n$.
- As such $\mathbf{P} = [\mathbf{p}_1, \ldots, \mathbf{p}_n]$ is an orthogonal matrix so that

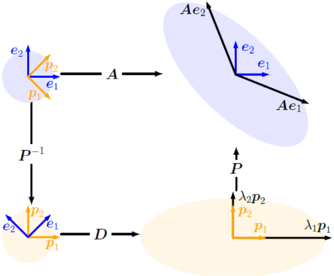
$$\mathbf{D} = \underline{\mathbf{P}^T\mathbf{AP}} \tag{37}$$

*

Geometric Intuition for the Eigendecomposition ($\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$)

We can interpret the eigendecomposition of a matrix as follows steps

- Let $\mathbf{A}$ be the transformation matrix of a linear mapping with respect to the standard basis.
- $\mathbf{P}^{-1}$ performs a basis change from the standard basis into the eigenbasis
  - This identifies the eigenvectors $\mathbf{p}_i$ (blue and orange arrows in the figure) onto the standard basis vectors $\mathbf{e}_i$

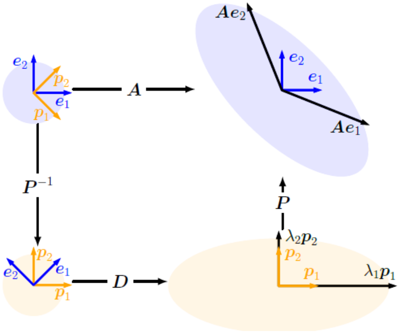


Source: M.P. Deisenroth *et al*, Mathematics for Machine Learning (First Edition)

Geometric Intuition for the Eigendecomposition ($\mathbf{A} = \mathbf{PDP}^{-1}$)

We can interpret the eigendecomposition of a matrix as follows steps

- Then, the diagonal matrix $\mathbf{D}$ scales the vectors along these axes by the eigenvalues λ_i
- Finally, $\mathbf{P}$ transforms these scaled vectors back into the standard/canonical coordinates yielding $\lambda_i \mathbf{p}_i$.



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Useful Properties of the Eigendecomposition ($\mathbf{A} = \mathbf{PDP}^{-1}$)

- We can find a matrix power for a matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ via the eigenvalue decomposition (if it exists) so that

$$\begin{aligned} \mathbf{A}^k &= (\mathbf{PDP}^{-1})^k && (38) \\ &= \mathbf{PDP}^{-1}\mathbf{PDP}^{-1} \dots \mathbf{PDP}^{-1} && (39) \\ &= \mathbf{PD}^k\mathbf{P}^{-1} && (40) \end{aligned}$$

  Computing $\mathbf{D}^k$ is efficient because we apply this operation individually to any diagonal element.

Useful Properties of the Eigendecomposition ($\mathbf{A} = \mathbf{PDP}^{-1}$)

$\det(P) = \frac{1}{\det(P^{-1})}$

- Assume that the eigendecomposition $\mathbf{A} = \mathbf{PDP}^{-1}$ exists. Then,

$$\det(\mathbf{A}) = \det(\mathbf{PDP}^{-1}) \tag{41}$$
$$= \det(\mathbf{P})\det(\mathbf{D})\det(\mathbf{P}^{-1}) \tag{42}$$
$$= \det(\mathbf{D}) \tag{43}$$
$$= \prod_i d_{ii} \tag{44}$$

allows for an efficient computation of the determinant of A

Non-Square decomposition

The eigenvalue decomposition requires square matrices.

- We will move on to the more general matrix decomposition technique, the singular value decomposition.

Eigendecomposition Examples

Wednesday, April 13, 2022 11:24 AM

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

① Find Eigenvalues

$$\det(A - \lambda I) = 0$$

$$= \det\left(\begin{bmatrix} 1-\lambda & 0 & 0 \\ 4 & 5-\lambda & 6 \\ 7 & 8 & 9-\lambda \end{bmatrix}\right) = 0$$

$$= (-1)^{1+1}(1-\lambda)\det\begin{bmatrix} 5-\lambda & 6 \\ 8 & 9-\lambda \end{bmatrix} + 0 + 0$$

$$= (1-\lambda)\left[(5-\lambda)(9-\lambda) - 48\right]$$

$$= (1-\lambda)(45 - 14\lambda + \lambda^2 - 48)$$

$$= (1-\lambda)(\lambda^2 - 14\lambda - 3)$$

Know the quadratic equation *

$$\therefore (1-\lambda)(\lambda^2 - 14\lambda - 3) = 0$$

$$\therefore (1-\lambda) = 0 \quad \Big| \quad (\lambda^2 - 14\lambda - 3) = 0$$

$$\therefore \lambda_1 = 1 \quad \Big| \quad (\lambda - 7 - 2\sqrt{13})(\lambda - 7 + 2\sqrt{13}) = 0$$

$$\therefore \lambda_2 = 7 + 2\sqrt{13}$$

$$\lambda_3 = 7 - 2\sqrt{13}$$

② Find Eigenvectors

$$(A - \lambda I)x = 0$$

For $\lambda_1 = 1$

$$(A - 1I)x = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 4 & 4 & 6 \\ 7 & 8 & 8 \end{bmatrix} x = 0$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -2\tfrac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -2\tfrac{1}{2} & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\therefore x_1 = \begin{bmatrix} 4 \\ -2\tfrac{1}{2} \\ -1 \end{bmatrix}$$

For $\lambda_2 = 7 + 2\sqrt{13}$

$$(A - (7 + 2\sqrt{13})I)x = 0$$

$$\begin{bmatrix} -6-2\sqrt{13} & 0 & 0 \\ 4 & -2-2\sqrt{13} & 6 \\ 7 & 8 & 2-2\sqrt{13} \end{bmatrix} x = 0$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{4} - \frac{\sqrt{13}}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore x_2 = \begin{bmatrix} 0 \\ \frac{1-\sqrt{13}}{4} \\ -1 \end{bmatrix}$$

For $\lambda_3 = 7 - 2\sqrt{13}$

$$(A - (7 - 2\sqrt{13})I)x = 0$$

$$\begin{bmatrix} -6+2\sqrt{13} & 0 & 0 \\ 4 & -2+2\sqrt{13} & 6 \\ 7 & 8 & 2+2\sqrt{13} \end{bmatrix} x = 0$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{4} + \frac{\sqrt{13}}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore x_3 = \begin{bmatrix} 0 \\ \frac{1+\sqrt{13}}{4} \\ -1 \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 3-\lambda & 2 & 2 \\ 2 & 3-\lambda & 2 \\ 2 & 2 & 3-\lambda \end{bmatrix}\right) = 0$$

$$(-1)^{1+1}(3-\lambda)\left[(3-\lambda)^2 - 4\right] + (-1)^{1+2}(2)\left[2(3-\lambda) - 4\right] + (-1)^{1+3}(2)\left[4 - 2(3-\lambda)\right] = 0$$

$$\Rightarrow -(1-\lambda)^2(\lambda - 7) = 0$$

$$\lambda_{1,2} = 1 \qquad AM = 2$$

$$\lambda_3 = 7 \qquad AM = 1$$

For $\lambda_3 = 7$

$$(A - 7I)x = 0$$

$$\left[\begin{array}{ccc|c} -4 & 2 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\therefore x_7 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

For $\lambda_{1,2} = 1$

$$(A - 1I)x = 0$$

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

$$\therefore x_1 = \left\{ \alpha_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} ; \alpha_1, \alpha_2 \in \mathbb{R} \right\}$$

AM for $\lambda = 1$ is 2

GM for $\lambda = 1$ is 2