Basic Analysis 2015 — Solutions of Tutorials

Section 3.5

Tutorial 3.5.1 1. Prove that a subset S of $\mathbb{R}$ is an interval if and only if S has the form (a, b), $[a, b)$, $(a, b]$, $[a, b]$, $(-\infty, b)$, $(-\infty, b]$, (a, ∞), $[a, \infty)$ or $(-\infty, \infty)$ where $a, b \in \mathbb{R}$ and $a < b$.
Hint. Consider $\inf S$ and $\sup S$.

Proof. Let S be any of the sets of the form listed above. Clearly, $S \neq \emptyset$ and S is not a singleton. Indeed, in each case $\alpha := \inf S < \sup S =: \beta$, and $(\alpha, \beta) \subset S$. We can choose $x \in \mathbb{R}$ such that $\alpha < x < \beta$. Then we can choose $y \in \mathbb{R}$ such that $x < y < \beta$. Therefore, $\alpha, x < y < \beta$, which shows that x and y are two distinct elements in S. We have shown that $S \neq \emptyset$ and that S is not a singleton. For $x, y \in S$ and $z \in \mathbb{R}$ with $x < y < z$ we have $\alpha < z < \beta$ and therefore $z \in S$. This shows that S is an interval.
Conversely, let S be an interval and set

$$\alpha := \inf S, \quad \beta := \sup S.$$

Let $x \in \mathbb{R}$ such that $\alpha < x < \beta$. By definition of infimum and supremum, there are $a_1 \in S$ and $b_1 \in S$ such that $a_1 < x$ and $b_1 > x$. Hence $x \in S$, which shows that $(\alpha, \beta) \subset S$. If S has no minimum, then $x > \alpha$ for all $x \in S$, whereas if S has a minimum, then $\alpha \in S$ and $x \geq \alpha$ for all $x \in S$. If S has no maximum, then $x < \beta$ for all $x \in S$, whereas if S has a maximum, then $\beta \in S$ and $x \leq \beta$ for all $x \in S$. Hence we have four cases, namely, $S = (\alpha, \beta)$, $S = [\alpha, \beta)$, $S = (\alpha, \beta]$, and $S = [\alpha, \beta]$. In the cases $S = (\alpha, \beta)$ and $S = (\alpha, \beta]$, we have $\alpha = -\infty$ or $\alpha \in \mathbb{R}$, and in the cases $S = (\alpha, \beta)$ and $S = [\alpha, \beta)$, we have $\beta = \infty$ or $\beta \in \mathbb{R}$. This leads to the nine cases as stated. ☐

2. Let f be a real function and let $\emptyset \neq A \subset B \subset \operatorname{dom}(f)$ such that for each $x \in A$ there is $\varepsilon > 0$ such that $(x - \varepsilon, x] \subset A$ or $[x, x + \varepsilon) \subset A$ or $(x - \varepsilon, x + \varepsilon) \subset A$, and the same property for B. Show that if f is continuous on B, then f is also continuous on A.

Proof. We here use that the above assumptions allow us to consider continuity from the left, continuity from the right and continuity as defined in this course. For the proof of this tutorial problem we use the more general definition of continuity:
f is said to be continuous on $A \subset \operatorname{dom} f$ if for $x \in A$ and each $\eta > 0$ there is $\delta > 0$ such that $y \in A$ and $|y - x| < \delta$ imply that $|f(y) - f(x)| < \eta$. But if f is continuous on B and $x \in A \subset B$, then we can find for $\eta > 0$ a number $\delta > 0$ such that $y \in B$ and $|y - x| < \delta$ imply that $|f(y) - f(x)| < \eta$. This clearly also holds if we only consider $y \in A$. Hence continuity of f on B implies continuity of f on A. ☐

3. **A fixed point theorem**. Let $a < b$ and let f be a continuous function on $[a, b]$ such that $f([a, b]) \subset [a, b]$. Show that there is $x \in [a, b]$ such that $f(x) = x$.

Proof. Let $g(x) = f(x) - x$. Then

$$g(a) = f(a) - a \geq a - a = 0$$

and

$$g(b) = f(b) - b \leq b - b = 0.$$

Clearly g is continuous. By the intermediate value theorem, there is $x \in [a, b]$ such that $g(x) = 0$, which means $f(x) = x$. ☐

4. Let I be an interval and f be a continuous function on I such that $f(I)$ is unbounded. What can you say about $f(I)$? Find examples which illustrate your answer.

Solution. Since a singleton is bounded, it follows from Corollary 3.16 that $f(I)$ is an unbounded interval. In view of Question 1, $f(I)$ must be of the form $(-\infty, \infty)$, $(-\infty, b)$, $(-\infty, b]$, (a, ∞), $[a, \infty)$ with $a, b \in \mathbb{R}$. Considering $f(x) = x$ on any interval of this form, we see that all these intervals can occur as $f(I)$ for suitably chosen functions f and intervals I.
5. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function which only assumes rational values. Show that f is constant.

Proof. Assume that f is not constant. Then there are $x, y \in I$ such that $f(x) \neq f(y)$. By Theorem 1.17 we can choose an irrational number z between $f(x)$ and $f(y)$. Because $f(I)$ is an interval by Corollary 3.16, $z \in f(I)$. But this contradicts the fact that z is irrational, and the assumption must be wrong. ☐

6. Find a continuous function $f : [-1,1] \to \mathbb{R}$ which is one-to-one when restricted to rational numbers in $[-1,1]$ but which is not one-to-one on the whole interval $[-1,1]$.

Proof. Let $\alpha \in (0,1)$ be an irrational number and define

$$f(x) = \begin{cases} -\alpha x & \text{if } -1 \le x < 0, \\ x & \text{if } 0 \le x \le 1. \end{cases}$$

Then $f(\alpha) = \alpha = f(-1)$, which shows that f is not injective.
For $x \in [-1,1] \cap \mathbb{Q}$ we have then $f(x) = x \in \mathbb{Q}$ if $x \ge 0$ and $f(x) = -\alpha x \notin \mathbb{Q}$ if $x < 0$, see Tutorial 1.3.1,4. Now let $x, y \in [-1,1] \cap Q$ with $f(x) = f(y)$. If $x \ge 0$, then $f(y) = f(x) \in \mathbb{Q}$ shows that also $y \ge 0$ and therefore $y = f(y) = f(x) = x$. If $x < 0$, then $f(y) = f(x) \notin \mathbb{Q}$ shows that also $y < 0$ and therefore $y = -\frac{1}{\alpha} f(y) = -\frac{1}{\alpha} f(x) = x$. Hence f is injective. □