Multivariable Calculus

MATH2007

2.2 Scalar Path Integrals (Part 1)

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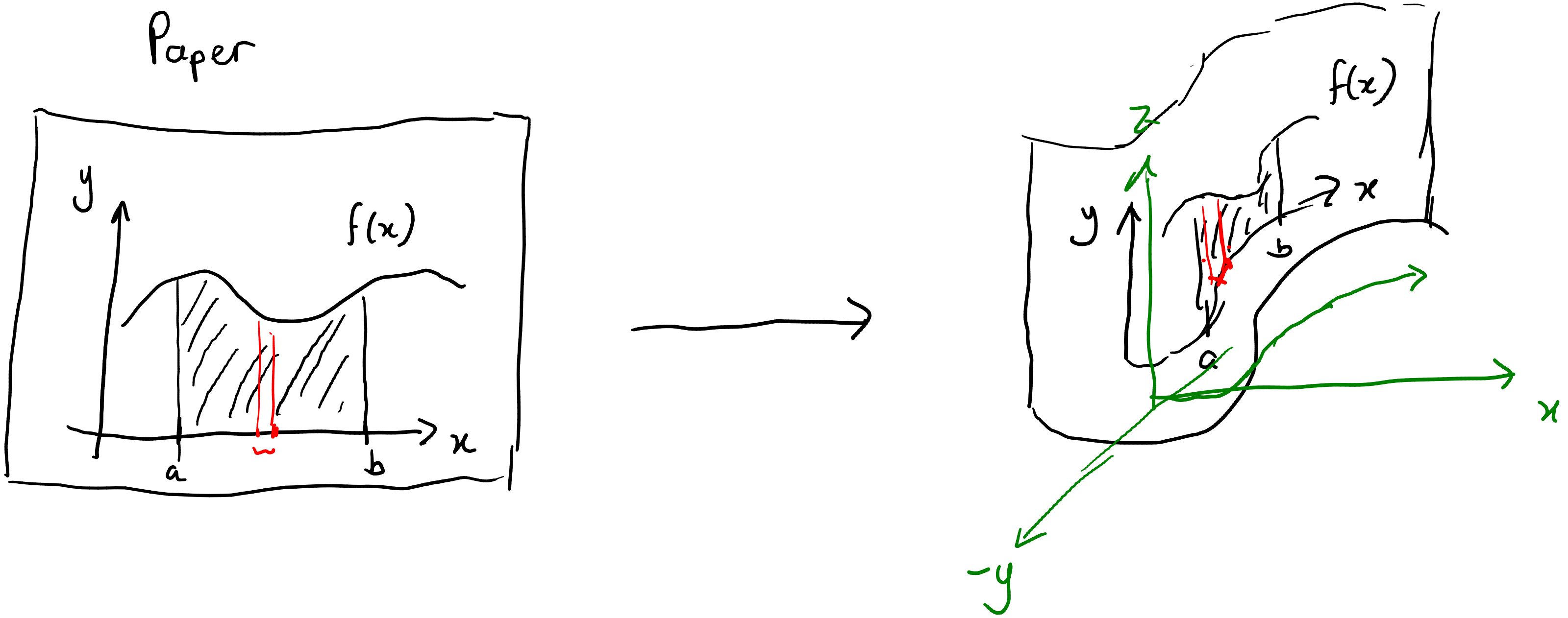
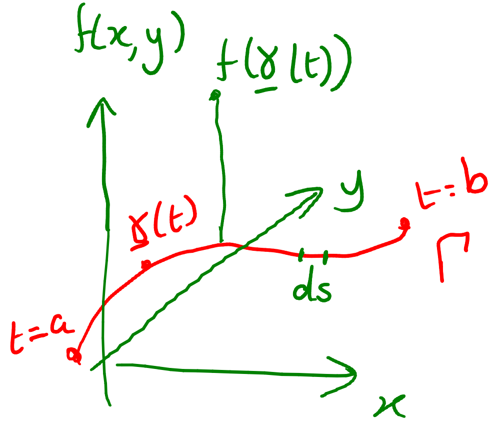

Paper
y
f(x)
x
a
b
f(x)
z
y
x
a
b
x
-y

Definition (2.2.1)**.** Let the curve Γ be parametrised by $\underline{\gamma}(t)$, $t \in [a, b]$. Let f be a real-valued function defined on Γ. We define the **scalar path integral** of f over Γ by:

$$\int_\Gamma f \; ds := \int_a^b f(\underline{\gamma}(t)) \; ||\underline{\gamma}'(t)|| \; dt.$$



Note. We may thus formally consider

$$ds = ||\underline{\gamma}'(t)|| \; dt.$$

Note. The length of the curve Γ is $\displaystyle\int_\Gamma 1 \; ds$.

Theorem (2.2.3)**.** The value of the scalar integral of $\cancel{F}$ f along Γ is independent of the orientation of Γ.

$$\int_{\Gamma} f \; ds = \int_{\Gamma^-} f \; ds.$$

Proposition. The value of the scalar integral of F along Γ is independent of the parametrization and orientation of Γ.

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2.2 Scalar Path Integrals
(Part 2)

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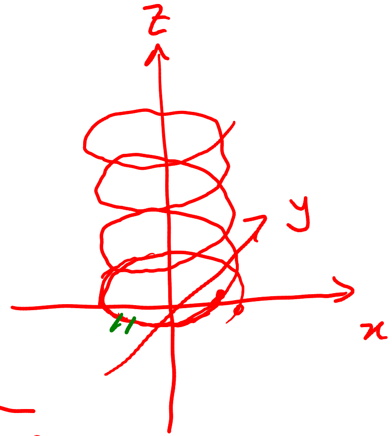
Example. Let $\underline{r}(t) = (a\cos t, a\sin t, bt), \quad t \in [0,3]$ and $F(x,y,z) = z^3$. Then $\Gamma = \{\underline{r}(t) : t \in [0,3]\}$ is a helix. Let F denote the mass ~~mass~~ per unit length of Γ. Find the mass of Γ.

$$\text{mass of } \Gamma \approx \sum_{\text{segments}} \text{length(segment)} \, (\text{mass per unit length})$$



$$\text{mass of } \Gamma = \int_\Gamma F \, ds = \int_0^3 F(\underline{r}(t)) \underbrace{\left\| \frac{d\underline{r}}{dt} \right\| dt}_{ds}$$

$$\frac{d\underline{r}}{dt} = (-a\sin t, a\cos t, b) \qquad \left\| \frac{d\underline{r}}{dt} \right\| = \sqrt{a^2\sin^2 t + a^2\cos^2 t + b^2}$$

$$= \sqrt{a^2+b^2}$$

$$\text{mass of } \Gamma = \int_0^3 F(\overset{x}{a\cos t}, \overset{y}{b\sin t}, \overset{z}{bt}) \cdot \sqrt{a^2+b^2} \, dt$$

$$= \sqrt{a^2+b^2} \int_0^3 (bt)^3 \, dt = \sqrt{a^2+b^2} \int_0^3 b^3 t^3 \, dt$$

$$\text{Mass of } r = \int_0^3 F(\overset{x}{a\cos t}, \overset{y}{b\sin t}, \overset{z}{bt}) \cdot \sqrt{a^2+b^2}\, dt$$

$$= \sqrt{a^2+b^2} \int_0^3 (bt)^3\, dt = \sqrt{a^2+b^2} \int_0^3 b^3 t^3\, dt$$

$$= b^3 \sqrt{a^2+b^2} \left[\frac{t^4}{4} \right]_0^3$$

$$= \frac{81 b^3}{4} \sqrt{a^2+b^2}.$$

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2.2 Scalar Path Integrals
(Part 3)

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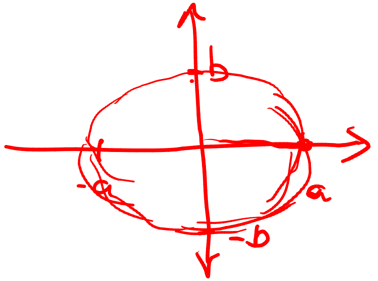
Example. Compute the scalar integral of $F(x,y)=\sqrt{\left(\frac{bx}{a}\right)^2+\left(\frac{ay}{b}\right)^2}$ over the ellipse $\underbrace{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}_{\Gamma}$.

$$\cos^2 t + \sin^2 t = 1 \qquad \underbrace{\frac{x^2}{a^2}}_{\cos^2 t} + \underbrace{\frac{y^2}{b^2}}_{\sin^2 t} = 1$$

$$\boxed{\begin{aligned} x &= a\cos t \\ y &= b\sin t \end{aligned}}$$



Let $\underline{\Gamma}(t) = \begin{pmatrix} a\cos t \\ b\sin t \end{pmatrix}$ $\quad t \in [0, 2\pi]$

$\underline{\Gamma}'(t) = \begin{pmatrix} -a\sin t \\ b\cos t \end{pmatrix}$ $\qquad \|\underline{\Gamma}'(t)\| = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$

$$\int_\Gamma F\,ds = \int_0^{2\pi} F(\underline{\Gamma}(t))\,\|\underline{\Gamma}'(t)\|\,dt = \int_0^{2\pi} F(\overset{x}{a\cos t}, \overset{y}{b\sin t})\sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt$$

$$= \int_0^{2\pi} \sqrt{b^2\cos^2 t + a^2\sin^2 t}\,\sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt$$

$$\int_\Gamma F\,ds = \int_0^{2\pi} F(\underline{r}(t))\,\|\underline{r}'(t)\|\,dt = \int_0^{2\pi} F(\overset{x}{a\cos t}, \overset{y}{b\sin t})\sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt$$

$$= \int_0^{2\pi} \sqrt{b^2\cos^2 t + a^2\sin^2 t}\,\sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt$$

$$= \int_0^{2\pi} (a^2\sin^2 t + b^2\cos^2 t)\,dt$$

$$= \int_0^{2\pi} \left(a^2\,\frac{1-\cos 2t}{2} + b^2\,\frac{1+\cos 2t}{2}\right)dt$$

$$= \left[\frac{a^2}{2}t - \frac{a^2}{4}\cancel{\sin 2t} + \frac{b^2}{2}t + \frac{b^2}{4}\cancel{\sin 2t}\right]_0^{2\pi}$$

$$= \frac{1}{2}(a^2+b^2)2\pi = (a^2+b^2)\pi.$$