APPM3039A
Applied Mathematics

Class Test 1

Instructions:

- Start each question on a new page
- Answer all questions.
- Show all workings.

Date: 9 April 2024

Duration: 1 hour

Total: 40 Marks

Question 1 **[12 Marks]**

1. Suppose that A and B are $n \times n$ matrices. Use the index notation to show that

   (a) $(AB)^T = B^T A^T$ (3 marks)

   (b) $tr(AB) = tr(BA)$ (3 marks)

2. Two $n \times n$ matrices A and B are said to be similar, if there is an $n \times n$ invertible matrix T such that

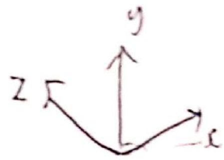
$$TAT^{-1} = B.$$

   Use the results from (1) to show that if two matrices are similar, their traces are equal. (2 marks)

3. Write the following matrix as the sum of a symmetric and skew symmetric matrix.

$$\begin{pmatrix} 7 & 9 & -6 \\ -8 & 0 & 9 \\ 9 & 3 & 4 \end{pmatrix}$$

(4 marks)



Question 2 [19 Marks]

1. Suppose that $\underline{x} = \begin{pmatrix} 3 & 1 & 1 \end{pmatrix}$, $\underline{y} = \begin{pmatrix} 4 & -2 & 5 \end{pmatrix}$, $\underline{z} = \begin{pmatrix} 2 & 8 & 7 \end{pmatrix}$.

   (a) Calculate $\left\|\underline{x} \times \underline{y}\right\|$ (4 marks)

   (b) Calculate the volume of the parallelopiped having adjacent sides $\underline{x}$, $\underline{y}$ and $\underline{z}$. (3 marks)

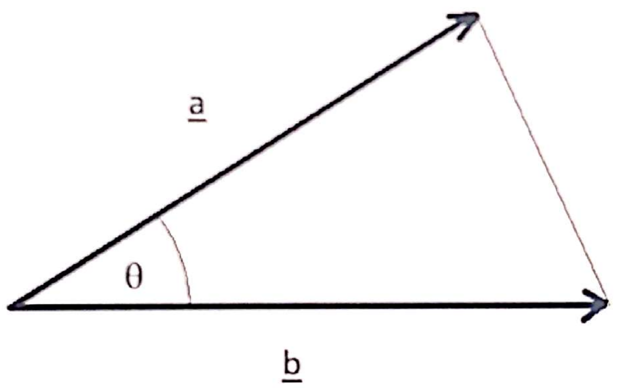
2. Suppose that $\underline{a} \in \mathbb{R}^3$ and $\underline{b} \in \mathbb{R}^3$. Use the index notation to show that

$$\|\underline{a} \times \underline{b}\|^2 = \|\underline{a}\|^2 \|\underline{b}\|^2 - (\underline{a} \cdot \underline{b})^2.$$

$$a_i a_i b_j b_j - (a_i b_i)(a_j b_j)$$

(8 marks)

3. Let $0 < \theta < \pi$ be the angle between two non-zero vectors $\underline{a}$ and $\underline{b}$ in the triangle below.



Suppose it can be shown that

$$\underline{a} \cdot \underline{b} = \|\underline{a}\| \|\underline{b}\| \cos(\theta).$$

Show, using the result in the previous question, that

$$\|\underline{a} \times \underline{b}\| = \|\underline{a}\| \|\underline{b}\| \sin(\theta).$$

(3 marks)

4. What is the geometrical interpretation of $\|\underline{a} \times \underline{b}\|$? (1 mark)

Question 3 [9 Marks]

1. Explain what the term *scalar field* means. (1 mark)

2. Give two physical examples of a scalar field. (2 marks)

3. Given that $\phi(x, y, z) = xyz$,

   (a) calculate gradϕ. (3 marks)

   (b) show that curl(gradϕ) $= 0$. (3 marks)

Formula Sheet

- $\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$
- $a_{ik}\delta_{kj} = a_{ij}$ (Substitution rule)

END OF PAPER

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Question 1
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9/12

Question 1

LHS $\qquad$ RHS

a) $(AB)^T$ $\qquad$ $B^T A^T$

let $AB = C$

$= \{(a_{ij} b_{jk})^T\}$ $\qquad$ $\{b_{jk}^T a_{ij}^T\}$

$\{a_{ij} b_{jk}\} = \{c_{ik}\}$ since $AB = C$ $\qquad$ $\{b_{kj} a_{ji}\}$

$= \{c_{ik}^T\}$

3/3

$= \{c_{ki}\}$

$= \{b_{kj} a_{ji}\}$

$\therefore$ LHS = RHS

b) $tr(AB)$

by ~~def~~ def. of trace & Matrix multiplication

$(a_{ij} b_{jk})_{nn}$

the multiplication is commutative

so $(b_{jk} a_{ij})_{nn}$

we switch dummy variables for multiplication

$(b_{kj} a_{ji})_{nn}$

$tr(BA)$

0/

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Question...
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2. if $TA\cancel{B}^{-1} = B$

then $tr(TA\cancel{B}^{-1}) = tr(B)$

$tr(TAT^{-1})$

by 1(b) $tr(AB) = tr(BA)$ for some A & $B \in \mathbb{R}^{n \times n}$

let $C = TA$ then $tr(TAT^{-1})$ is:

~~then~~ $= tr(CT^{-1}) =$

$= tr(T^{-1}C)$ (by above ~~theore~~ result)

$= tr(T^{-1}TA)$

$tr(IA)$

~~by definition of~~ :

by $tr(A)$ (since $AI = IA = A$)

2/2

$\therefore$ $tr(A) = tr(TAT^{-1}) = tr(B)$

so it follows that

$tr(A) = tr(B)$

3. ~~let~~ $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$$A = \begin{pmatrix} 7 & 9 & -6 \\ -8 & 0 & 9 \\ 9 & 3 & 4 \end{pmatrix} \quad A^T = \begin{pmatrix} 7 & -8 & 9 \\ 9 & 0 & 3 \\ -6 & 9 & 4 \end{pmatrix}$$

4/4

$$A = \begin{pmatrix} 7 & \frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & 0 & 6 \\ \frac{3}{2} & 6 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 8.5 & -7.5 \\ -8.5 & 0 & 3 \\ 7.5 & -3 & 0 \end{pmatrix}$$

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Question...
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19/19

Question 2

a) $||x \times y||$

$$\left|\left|\begin{bmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ 3 & 1 & 1 \\ 4 & -2 & 5 \end{bmatrix}\right|\right| = \begin{bmatrix} 7 \\ -11 \\ -10 \end{bmatrix}^T$$

$x \times y = \underline{e}_1(1\times 5 - 1\times -2) - \underline{e}_2(3\times 5 - 1\times 4) + \underline{e}_3(3\times -2 - 1\times 4)$

4/4

$$||x \times y|| = \sqrt{7^2 + (-11)^2 + (-10)^2}$$
$$= 3\sqrt{30}$$

b) Volume: $z \cdot (x \times y)$

$$\begin{pmatrix} 2 & 8 & 7 \end{pmatrix} \begin{bmatrix} 7 \\ -11 \\ -10 \end{bmatrix}$$

3/3

$|2\times 7 + 8\times -11 + 7\times -10|$

144 units^3

8/8

2 $||a \times b||^2$

$(a \times b)(a \times b)$

$\varepsilon_{ijk} e_k a_i b_j . \cancel{\varepsilon_{ijk}}\, \varepsilon_{klm} e_k a_l b_m$

$\varepsilon_{ijk}\varepsilon_{klm} = \varepsilon_{ijk}\varepsilon_{lmk} = \cancel{\delta_{im}\delta_{jl}} - \cancel{\delta_{in}}\ \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_i b_j a_l b_m$

$\delta_{il}\delta_{jm} a_i b_j a_l b_m - \delta_{im}\delta_{jl} a_i b_j a_l b_m$

$a_i \delta_{il} \cancel{a_i} b_j \delta_{jm} a_l b_m - a_i \delta_{im} b_j \delta_{jl} a_l b_m$

$a_l b_m a_l b_m - a_m b_l a_l b_m$

Substitution (next page)

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$a_l^2 b_m^2 - a_m b_m a_l b_l$

$\|a\|^2 \|b\|^2 - (a \cdot b)(a \cdot b)$

$\|\underline{a}\|^2 \|\underline{b}\|^2 - (a \cdot b)^2$

3) from 2.2

$\|a \times b\| = \sqrt{\|a\|^2 \|b\|^2 - (a \cdot b)^2}$ (sub in dot product)

$= \sqrt{\|a\|^2 \|b\|^2 - \|a\|^2 \|b\|^2 \cos^2(\theta)}$

$= \sqrt{\|a\|^2 \|b\|^2 (1 - \cos^2\theta)}$

use trig identity

$= \sqrt{\|a\|^2 \|b\|^2 \sin^2\theta}$

$= \|a\| \|b\| \sin\theta$

3/3

$\therefore \|a \times b\| = \|a\| \|b\| \sin\theta$

4) It is the magnitude of cross sectional area ~~betwee~~ of the parallelogram in 3d space created by the 2 vectors

1/1

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Question 3

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1) for a set U which is a subset of $\mathbb{R}^3$ a scalar field ϕ, assigns a scalar value to every point in that set U

$\phi(x_1\ x_2\ x_3) = \phi$

1/1

2) ~~$\phi(x,y,z) = x^2 + y^2 + z^2$~~

~~ϕc~~.

0/

3 a) grad ϕ ~~$\nabla\phi = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}$~~

grad $\phi = \phi_{,i}\, e_i$

$$\text{grad}\,\phi = \begin{pmatrix} \frac{\partial}{\partial x}\phi(x,y,z) \\ \frac{\partial}{\partial y}\phi(x,y,z) \\ \frac{\partial}{\partial z}\phi(x,y,z) \end{pmatrix} = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}$$

3/3

b) curl (grad ϕ)

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

3/3

$$e_1\left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right) - e_2\left(\frac{\partial}{\partial x}xy - \frac{\partial}{\partial z}yz\right) + e_3\left(\frac{\partial}{\partial y}yz - \frac{\partial}{\partial x}xz\right)$$

$$e_1(x - x) - e_2(y - y) + e_3(z - z)$$

$$e_1 0 - e_2 0 + e_3 0$$

$$0$$