ANALYSIS OF ALGORITHMS

LECTURE 6 : GRAPH COLOURING

WHAT?

- Map Colouring Problem
  - Assign colours to countries on a map such that no adjacent countries have the same colour

WHY?

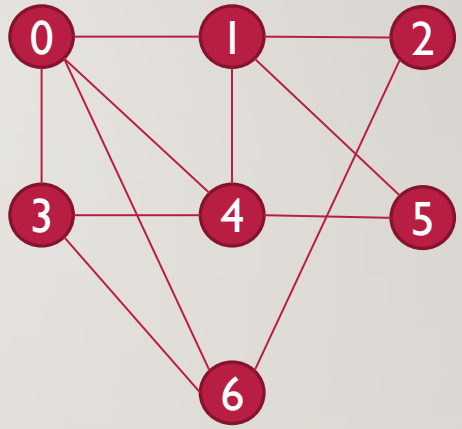
- Map Colouring Problem
  - Assign colours to countries on a map such that no adjacent countries have the same colour
- Why?
  - Visibility
  - Ease of production – Prefer solutions with a smaller number of colours

ABSTRACTION

- Let's examine the sentence again
- Assign **colours** to **countries on a map** such that no **adjacent countries** have the same **colour**
- We want to remove the problem domain from its specification to allow reuse
- Assign **colours** to **vertices** such that no **adjacent vertices** have the same colour

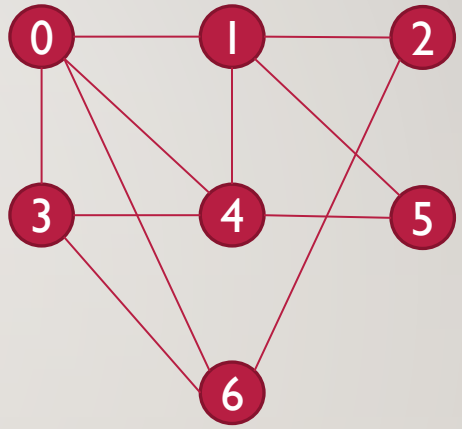
GRAPH COLOURING

- Assign colours to vertices such that no adjacent vertices have the same colour, using the minimum number of colours
- How do we do this?
  - Err… Nobody knows. Well not how to do it efficiently anyway.
  - NP-Complete



GRAPH COLOURING

- Assign colours to vertices such that no adjacent vertices have the same colour. Prefer solutions that use fewer colours.
- How do we do this?
  - Pretty much the first thing we think of gets a valid colouring, and doesn't do too badly in terms of the number of colours.
  - While there is some uncoloured vertex
    1. Pick any vertex
    2. Set its colour to the first colour that no neighbour is using



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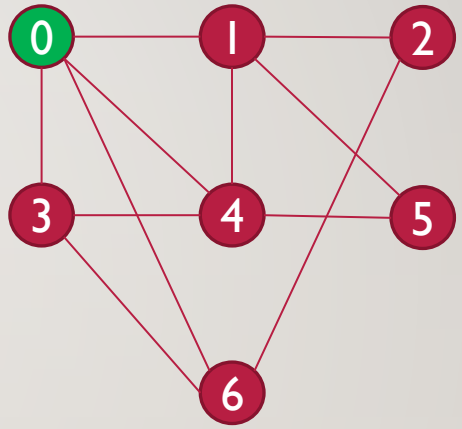

Green, Blue, Yellow, Orange, White

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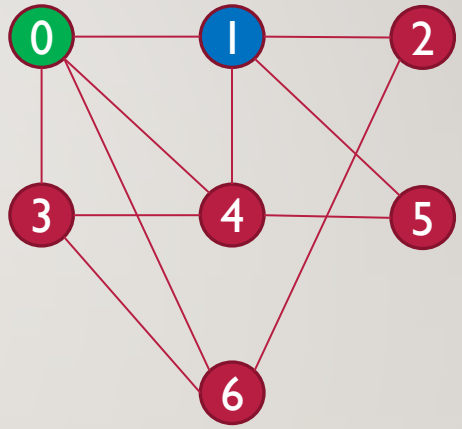

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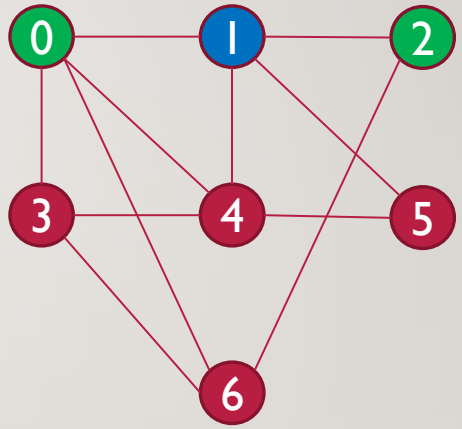

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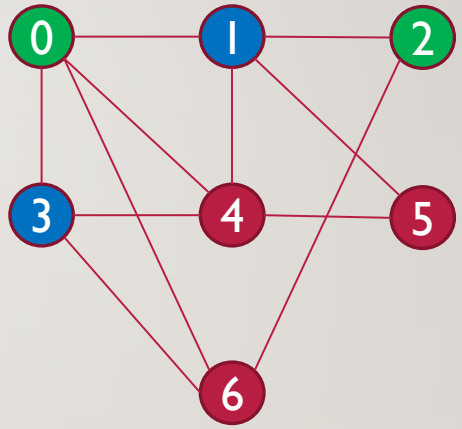

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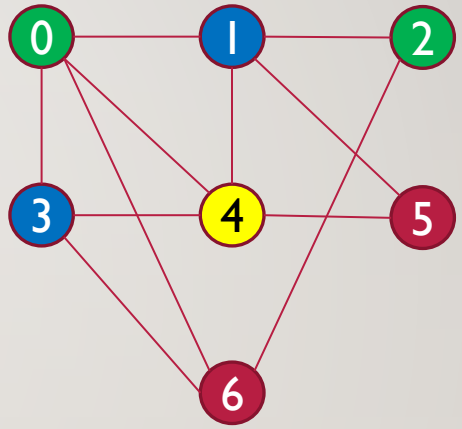

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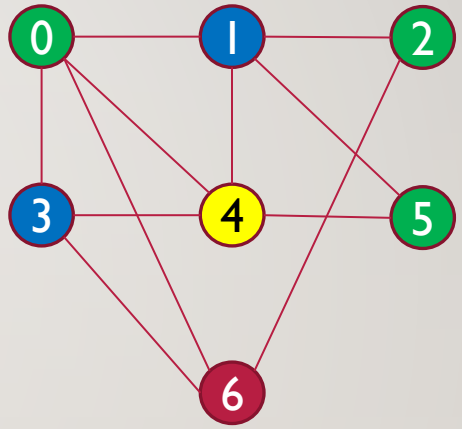

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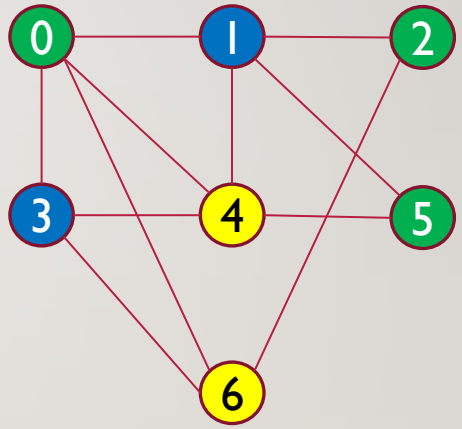

3 colours

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GRAPH COLOURING

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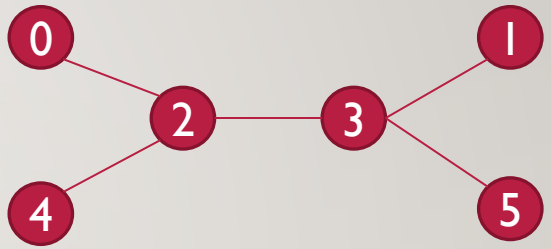

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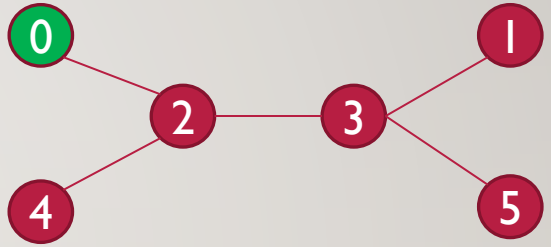

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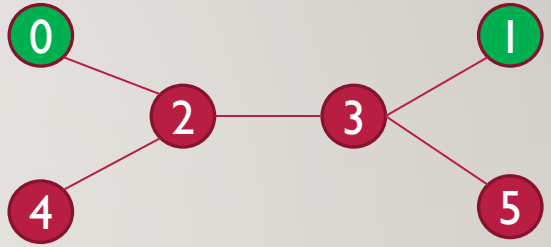

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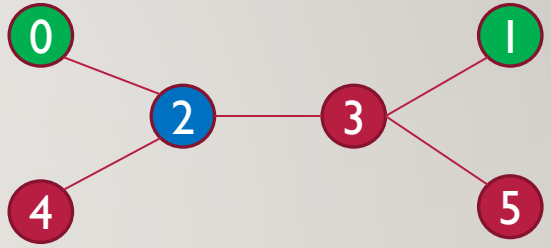

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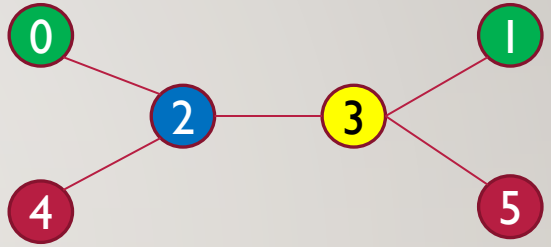

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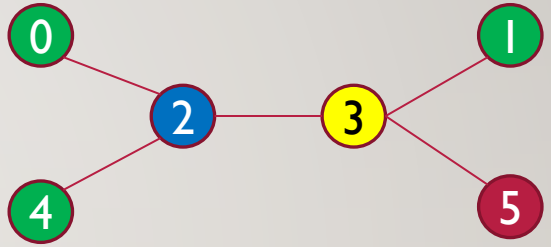

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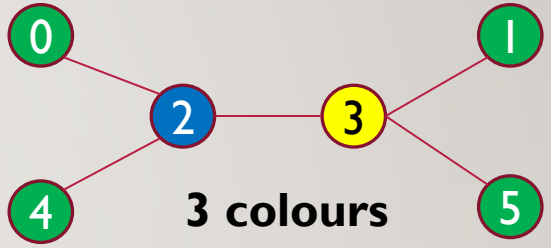

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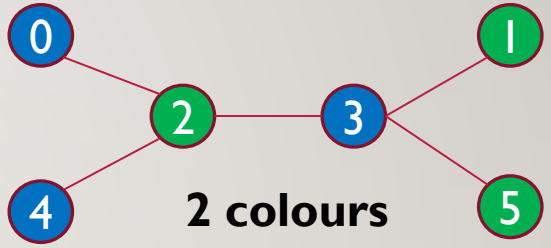

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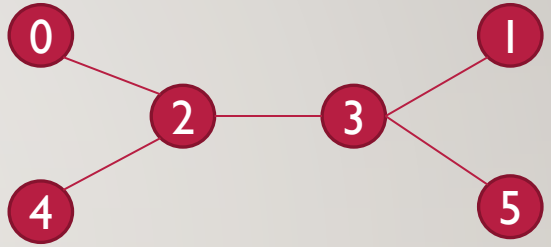
GRAPH COLOURING (IMPROVED)

- The vertices are harder to colour based on how many edges they have
- Sort by degree



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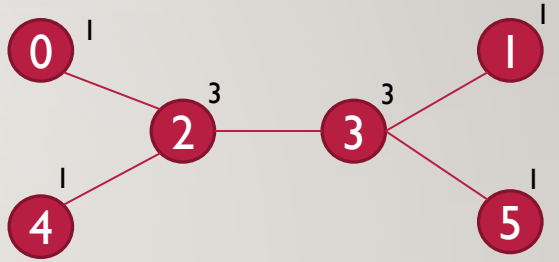

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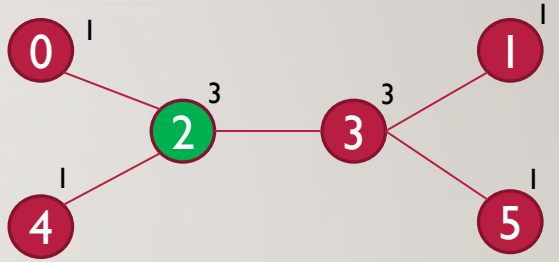

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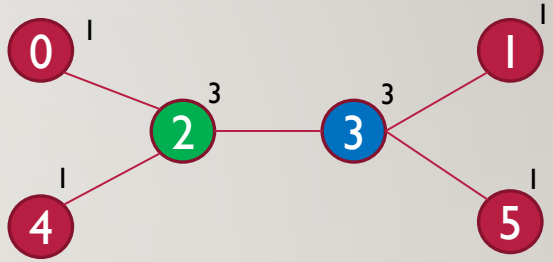

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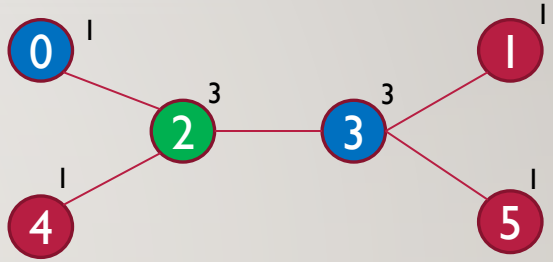

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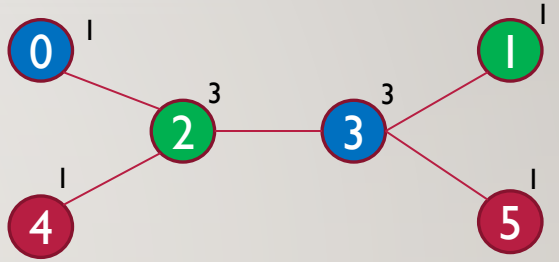

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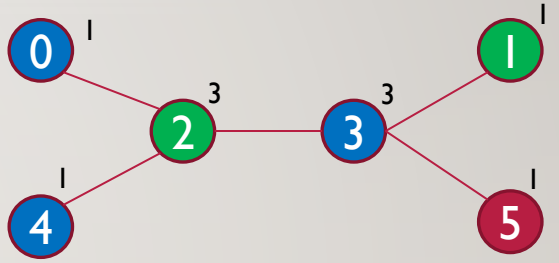

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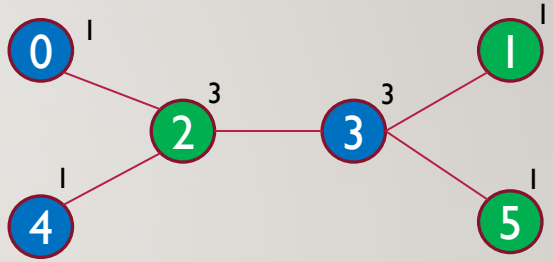

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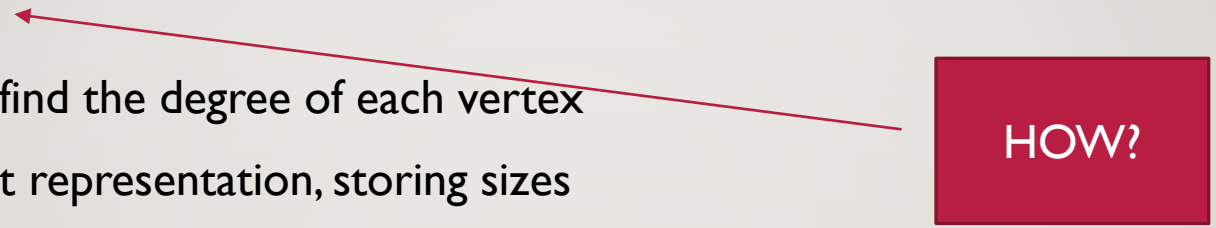
GRAPH COLOURING

- That is a fairly abstract algorithm…
- How do we make it more concrete?
- First, use numbers to represent colours
- Sort by degree
- While there is some uncoloured vertex
  1. Pick the uncoloured vertex with the largest degree
  2. Set its colour to the first colour that no neighbour is using

HOW?

- Sort by degree
  - Will have to find the degree of each vertex
  - Adjacency list representation, storing sizes
  - Then sort using some algorithm… Mergesort? Quicksort?
  - In Java, we can use the built-in **sort** function - $\Theta(nlogn)$

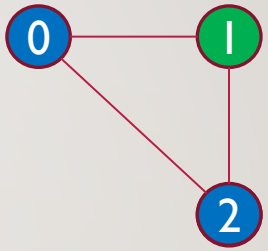


HOW? ATTEMPT 1

- Set its colour to the first colour that no neighbour is using
  - Could keep a variable for the smallest colour s, initially set to 0, indicating that no colours are used
  - Iterate through the neighbours, and increment s if a neighbour is using it
  - $\Theta(n)$

HOW? ATTEMPT 1

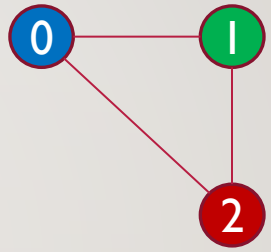
- Set its colour to the first colour that no neighbour is using
  - Could keep a variable for the smallest colour *s*, initially set to 0, indicating that no colours are used
  - Iterate through the neighbours, and increment *s* if a neighbour is using it
  - FAILS! In this case, the *s* will initially be 0. Iterating through the list, we see vertex 0 uses colour 1, so we leave s as *0*. Then we see vertex 1 uses colour 0, so we set *s* to 1.
  - Which means we will set vertex 2's colour to 1 (Blue) which is already used.



0. Green
1. Blue
2. Yellow

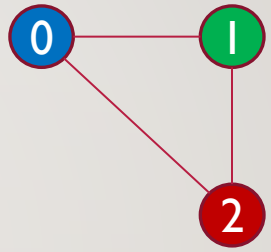
HOW? ATTEMPT 2

- Set its colour to the first colour that no neighbour is using
  - Could go back and check the colour every time s changes. So when s changes to 1, we start iterating from the top of the list again
  - Could be $\Theta(mn)$, potentially $\Theta(n^2)$ in the worst case



HOW? ATTEMPT 3

- Set its colour to the first colour that no neighbour is using
  - Could keep track of all colours in a used array, initially set to false.
  - Iterate through all neighbours, setting their colours to true in the list
  - Then search for the first false (unused)



0 (Green)	1 (Blue)	2 (Yellow)
F	F	F

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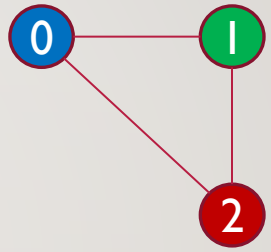

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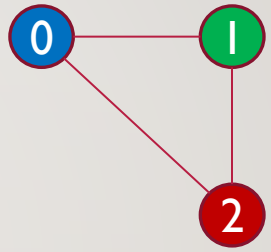

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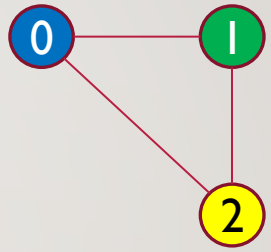
  - $O(n)$



0 (Green)	1 (Blue)	2 (Yellow)
T	T	F

GRAPH COLOURING (IMPROVED)

- Sort by degree $\Theta(nlogn)$
- While there is some uncoloured vertex – runs n times
  1. Pick the uncoloured vertex with the largest degree - $\Theta(1)$
  2. Set its colour to the first colour that no neighbour is using - $O(n)$
- Total - $O(nlogn + n^2) = O(n^2)$
- Don't worry if you're not sure if you understood it, because you have to code it this week, so you will understand it after <evil laugh>

HMM… INTERESTING…

- 4 colour theorem
- Next up:
  - Proof of *r*-colourability