14:00 hrs	18/06/13	EX: HALL

University of the Witwatersrand, Johannesburg

Course or topic numbers	MATH2001
Course or topic name(s) Paper Number & title	Basic Analysis
Examination to be held during month(s) of	June 2013
Year of Study	
Degrees/Diplomas for which this course is prescribed	
Faculty/ies presenting candidates	
Internal examiner(s) and telephone numbers	Prof. Manfred Möller – Ext 76220
Moderator	Prof. C. Labuschagne
Special materials required	
Time allowance	Course: MATH2001 Hours: 1
Instructions to candidates	60 marks in 60 minutes. No calculators are allowed.

Internal Examiners or Heads of Department are requested to sign the declaration overleaf

MATH2001–Basic Analysis Final Examination June 2013

Time: 60 minutes   Total marks: 60 marks

SECTION A Multiple choice

Answer the multiple choice questions on the computer card provided. There is ONLY ONE correct answer to each question. Please ensure that your student number is entered on the card, by pencilling in the requisite digit for each block.

Question 1 .... [2 marks]

Let $f, g : \mathbb{R} \to \mathbb{R}$, $a \in \mathbb{R}$. Which of the following statements is true?

A. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists.

B. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then $\lim_{x\to a} f(x)g(x)$ exists.

C. If $\lim_{x\to a} f(x)g(x)$ exists, then $\lim_{x\to a} f(x)$ exists.

D. If $\lim_{x\to a} f(x)g(x)$ exists, then $\lim_{x\to a} g(x)$ does not exist.

E. If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} \frac{f(x)}{g(x)}$ does not exist.

Question 2 .... [2 marks]

Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function, $a \in \mathbb{R}$. Which of the following statements is false?

A. $\lim_{x\to a^-} f(x) \le f(a)$.

B. $\lim_{x\to a^+} f(x) \ge f(a)$.

C. $f(x) - f(-x)$ is an increasing function.

D. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.

E. The function $g(x) = f(x) - f(a)$ is increasing.

Question 3 .... [3 marks]

Assume that I is an interval and that $f : I \to \mathbb{R}$ is continuous. Which of the following statements is true?

A. $g(I)$ is bounded, where $g(x) = \dfrac{1 - f^2(x)}{1 + f^2(x)}$.

B. $f(I)$ is bounded.

C. $f(I)$ is closed.

D. $h(I)$ is bounded, where $I = \mathbb{R}$ and $h(x) = f(x) - f(x-1)$.

E. $h(I)$ is closed, where $I = \mathbb{R}$ and $h(x) = f(x) - f(x-1)$.

Question 4 .... [3 marks]

A counterexample to the statement

"If $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and if $f'(\mathbb{R})$ is bounded, then $f(\mathbb{R})$ is closed"

is given by

A. $f(x) = x$.

B. $f(x) = \dfrac{1}{x}$.

C. $f(x) = \dfrac{x}{1+x^2}$.

D. $f(x) = \dfrac{x}{1-x^2}$.

E. $f(x) = \dfrac{1}{1+x^2}$.

Question 5 .... [4 marks]

Consider the following

Theorem. "Let f be continuous on the closed interval $[a, b]$ with $f(a) \neq f(b)$. Then for any number k between $f(a)$ and $f(b)$ there exists a number c in the open interval (a, b) such that $f(c) = k$."

Which part of its attempted proof below makes this proof incorrect?

A. Let $g(x) = f(x) - k$, $(x \in [a, b])$. Then g is continuous, and $g(a)$ and $g(b)$ have opposite signs: $g(a)g(b) < 0$.

B. Let $[a_0, b_0] = [a, b]$ and use bisection to define intervals $[a_n, b_n]$ as follows: If $[a_n, b_n]$ with $g(a_n)g(b_n) < 0$ has been found, let d be the midpoint of the interval $[a_n, b_n]$. If $g(d) = 0$, the result follows with $c = d$.

C. If $g(d)$ has the same sign as $g(b_n)$, then $g(a_n)$ and $g(d)$ have opposite signs, and putting $a_{n+1} = a_n$, $b_{n+1} = d$, we have $g(a_{n+1})g(b_{n+1}) < 0$. Otherwise, if $g(d)$ has the opposite sign to $g(b_n)$, we put $a_{n+1} = d$, $b_{n+1} = b_n$ and get again $g(a_{n+1})g(b_{n+1}) < 0$.

D. If this procedure does not stop, we obtain an increasing sequence (a_n) and a decreasing sequence (b_n), both of which converge. We observe that

$$b_n = a_n + \frac{1}{2}(b_{n-1} - a_{n-1}) = 2^{-n}(b - a).$$

Then

$$c := \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} 2^{-n}(b-a) = \lim_{n\to\infty} a_n.$$

E. Since $a \leq c \leq b$ and g is continuous at c, it follows that

$$g(c) = \lim_{n\to\infty} g(a_n) - \lim_{n\to\infty} g(b_n) = \lim_{n\to\infty} (g(a_n) - g(b_n)) \leq 0.$$

Therefore $g(c) = 0$, which gives $f(c) = k$.

Question 6 .... [3 marks]

Consider the sequence (a_n) of real numbers. Which of the following statements is false?

A. If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then (a_n) converges.

B. If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

C. If $a_n \neq 0$ for all n, then $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$.

D. If $\sum_{n=1}^{\infty} a_n$ converges conditionally, then $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = 1$.

E. If $\liminf_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

SECTION B

Answer this section in the answer book provided.

Question 1 .... [11 marks]

(a) Let $f : \mathbb{R} \to \mathbb{R}$. Write down the definition of $\lim_{x\to\infty} f(x) = -2$. (3)

(b) Prove from the definition that $\lim_{x\to\infty} \left(\frac{1-3x^2}{x^2+3} + 1 \right) = -2$. (8)

Question 2 .... [6 marks]

Let $a \in \mathbb{R}$ and suppose that f is continuous at a and g continuous at $f(a)$. Prove that the function $g \circ f$ is continuous at a.

Question 3 .... [9 marks]

Let $a \in \mathbb{R}$ and let f be a real function which is defined in a neighbourhood of a.

Show that f is continuous at a if and only if for each sequence (x_n) in $\mathrm{dom}(f)$ with $\lim_{n\to\infty} x_n = a$ the sequence $f(x_n)$ satisfies $\lim_{n\to\infty} f(x_n) = f(a)$.

Question 4 .... [7 marks]

(a) State the Intermediate Value Theorem. (2)

(b) Let $a < b$ and let f be a continuous function on $[a, b]$ such that $f([a, b]) \subset [a, b]$. Show that there is $x \in [a, b]$ such that $f(x) = x$. (3)

(c) Give an example of a noncontinuous function $f : [a, b] \to [a, b]$ such that $f(x) \neq x$ for all $x \in [a, b]$. (2)

Question 5 .... [5 marks]

Let (a_n) be a sequence of nonzero real numbers such that $\limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$.

Prove that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Question 6 .... [5 marks]

Let $a_n = \frac{5 + 4(-1)^n}{3^n}$.

(a) Find $\limsup_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$. (3)

(b) Does $\sum_{n=1}^{\infty} a_n$ converge? (2)
Justify your answer.