Lecture 2

Find the differential of $u = f(g(x))$

$$du = f'(x)\, dx$$
$$= f'(g(x))\, dx$$
$$= (f'(g(x)) \cdot g'(x)\, dx$$

With this idea, we can find the integral of a function using the reversal of chain rule and the idea of differentials. This method of integration is called integration by substitution.

Integration by Substitution

Given the integral $\int f(g(x)) \cdot g'(x)\, dx$

we make the substitution
$$u = g(x)$$
So that $du = g'(x)\, dx$

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du$$

thereby simplifying the integral $f(u)$ has an easily identifiable integral

Example
Evaluate the integral
$$\int 12(3\sin^2 x + 1)^{1/4} \sin x \cos dx$$

Solution

Let $u = 3\sin^2 x + 1$

so that $\frac{du}{dx} = 6\sin x\cos x$

$du = 6\sin x \cos x \, dx$

$dx = \frac{du}{6\sin x\cos x}$

$\int 12(3\sin^2 x+1)^{1/4} \sin x\cos x \, dx$

$= \int 12 u^{1/4} \cancel{\sin x\cos x} \cdot \frac{du}{6\cancel{\sin x\cos x}}$

$= \int \cancel{12}^{2} u^{1/4} \frac{du}{\cancel{6}_{1}} \Rightarrow \int 2u^{1/4}\, du$

$\Rightarrow 2\int u^{1/4} du = 2 \cdot \frac{u^{5/4}}{5/4} + c$ $\rightarrow$ $y' = \int x^n dx$, $y = \frac{x^{n+1}}{n+1} + c$

$= 2u^{5/4} \times \frac{4}{5} + c$

$= \frac{8}{5} u^{5/4} + c$

$= \frac{8}{5}(3\sin^2 x + 1)^{5/4} + c$

where c is a constant of integration.

If $f(u)$ can not be integrated easily, then another substitution or integration method may be required

Some common choices for $g(x)$

- Trigonometric functions
- Hyperbolic functions
- Power functions

Example 2 ÷ Find the integral of $\int x^2 \sqrt{x^3+1}\, dx$

Solution: Let $u = x^3 + 1$

$$\frac{du}{dx} = 3x^2$$

$$du = 3x^2\, dx$$

$$dx = \frac{du}{3x^2}$$

$$\int x^2 \sqrt{x^3+1}\, dx = \int \cancel{x^2} \cdot \sqrt{u} \cdot \frac{du}{3\cancel{x^2}}$$

$$= \int \sqrt{u}\, \frac{du}{3} \Rightarrow \frac{1}{3}\int \sqrt{u}\, du$$

$$= \frac{1}{3}\int u^{1/2}\, du$$

$$= \frac{1}{3} \cdot \frac{u^{3/2}}{3/2} + c$$

$$= \frac{1}{3} \times \frac{2}{3} u^{3/2} + c$$

$$= \frac{2}{9} u^{3/2} + c$$

$$= \frac{2}{9}(x^3+1)^{3/2} + c$$

Exercise ÷ Find $\int x\sqrt{1-x}\, dx$

Integration by parts

Recall product rule when differentiating a product of two functions.

$$[u(x) \cdot v(x)]' = u(x)\, v'(x) + u'(x)\, v(x)$$

Rearranging

$$u(x)\, v'(x) = [u(x)\, v(x)]' - u'(x)\, v(x)$$

upon integrating both sides:

$$\int u(x)\, v'(x)\, dx = u(x)\, v(x) - \int u'(x)\, v(x)\, dx$$

$$\int uv'\, dx = uv - \int v u'\, dx$$

This is called integration by parts formula.

Example: Integrate the function $f(x) = x^2 e^x$

Solution

$$\int x^2 e^x\, dx$$

Let $u = x^2$ $\qquad$ $v' = e^x$

$u' = 2x$ $\qquad$ $\int v' = \int e^x$

$v = e^x$

$$\int u(x)\, v'(x)\, dx = u(x)\, v(x) - \int v(x)\, u'(x)\, dx$$

$$\int x^2 e^x\, dx = x^2 e^x - \int e^x \cdot 2x\, dx$$

$$= x^2 e^x - 2\int x e^x\, dx$$

Again, use integration by parts on the last expression.

$2\int x e^x \, dx$

Let $u = x$

$u'(x) = 1$

$v' = e^x$

$\int v' = \int e^x$

$v = e^x$

$v(x) = e^x$

$$2\int x e^x \, dx = 2\left(x e^x - \int e^x \, dx\right)$$

$$= 2xe^x - 2e^x$$

Substituting into the main integral:

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + c$$

$$= e^x\left(x^2 - 2x + 2\right) + c$$

Exercise: Integrate the following functions

1. $f(x) = x \ln(x)$

2. $f(x) = x^3 e^{x^2}$

3. $f(x) = x \cos x$