MATH2001–Basic Analysis Examination 2016

Time: 60 minutes Total marks: 60 marks plus 6 bonus marks

Questions 1(a), 1(b) and 5(c) are multiple choice questions. There is exactly one correct answer for each of these MCQ questions. You will get full marks if you write down the letter which corresponds to the correct answer. However, you may add reasoning for your answer. This additional work will be ignored if you have written down the letter which corresponds to the correct answer. However, if this letter does not correspond to the correct answer, the reasoning you have written down may warrant partial credit for the MCQ question.

Question 1 .... [12 marks]

(a) Let (3)

$$f(x) = \begin{cases} \dfrac{1}{x^2+1} & \text{if } x > 0, \\ \cos x & \text{if } x < 0, \end{cases}$$

$$g(x) = \begin{cases} f(x) & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Which of the following is false?

A. f is continuous.

B. f is continuous at 0.

C. g is differentiable.

D. fg is differentiable.

E. $f + g$ is continuous.

(b) Let f, g, h be functions such that (3)
$\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to\infty} g(x) = -\infty$, $\lim_{x\to\infty} h(x) = -1$.

Which of the following may be false?

A. $\lim_{x\to\infty} (f(x) + g(x)) = \infty$;

B. $\lim_{x\to\infty} (f(x) - g(x)) = \infty$;

C. $\lim_{x\to\infty} (f(x)g(x)) = -\infty$;

D. $\lim_{x\to\infty} (f(x)h(x)) = -\infty$;

E. $\lim_{x\to\infty} (g(x)h(x)) = \infty$.

(c) **(Bonus question)** Give an example which illustrates your answer to part (b). (2)

(d) **(Bonus question)** Prove one of the correct statements in part (b). (4)

Question 2 .... [8 marks]

Let f be continuous at $a \in \mathbb{R}$ and let g be continuous at $f(a)$.
Prove that $g \circ f$ is continuous at a.

Question 3 .... [10 marks]

Statement I. Let I be an interval and let f be a continuous real function on I which is not constant. Then $f(I)$ is an interval.

(a) Name and state the theorem which you can use to prove Statement I. (4)

(b) Prove Statement I. (6)

Question 4 .... [6 marks]

Let $a < b$ be real numbers and let f, g be real functions defined on $[a, b)$. Assume that $f(x) \leq g(x)$ for all $x \in [a, b)$ and that $\lim\limits_{x \to b^-} f(x) = \infty$.

Prove that $\lim\limits_{x \to b^-} g(x) = \infty$.

Question 5 .... [12 marks]

Statement II. Let $a < b$ be real numbers and let $g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then g is injective (one-to-one) on $[a, b]$ if $g'(x) \neq 0$ for all $x \in (a, b)$.

(a) Name and state the theorem which you can use to prove Statement II. (4)

(b) Prove Statement II. (4)

(c) For $x \in [0, \pi]$, let $f(x) = x - \cos x$ and $g(x) = x + \cos x$. (4)
Which of the following is false?

A. f is strictly increasing on $\mathrm{dom}(f) = [0, \pi]$, and $\mathrm{dom}(f^{-1}) = [-1, \pi + 1]$.
B. g is strictly increasing on $\mathrm{dom}(g) = [0, \pi]$, and $\mathrm{dom}(g^{-1}) = [1, \pi - 1]$.
C. The domain of $(f^{-1})'$ is $[-1, \pi + 1]$.
D. The domain of $(g^{-1})'$ is $[1, \pi - 1]$.
E. g^2 is strictly increasing on $\mathrm{dom}(g) = [0, \pi]$.

Question 6 .... [10 marks]

(a) Give the definition of a series of real numbers. (2)

(b) Give a definition of convergence of a series. (2)

(c) Using the Cauchy criterion for sequences, state and prove (6)
a general criterion for convergence of a series which does not require knowledge of the limit of that series.

Question 7 .... [8 marks]

(a) Give the definition of the radius of convergence of a power series. (4)

(b) Find the radius of convergence of the power series (4)

$$\sum_{n=0}^{\infty} 3^n x^{2n}.$$