8:30 hrs	24/ 10 /2013	HALL 29	Exams Office Use Only

University of the Witwatersrand, Johannesburg

Course or topic No(s)	MATH2016
Course or topic name(s) Paper number & title	Advanced Analysis
Examination/Test* to be held during month(s) of (*delete as applicable)	November Exams
Year of study (Art & Sciences leave blank)	Second Year
Degrees/Diplomas for which this course is prescribed (BSc (Eng) should indicate which branch)	BSc; BCom, BA
Faculty/ies presenting candidates	Science, Commerce, Humanities
Internal examiner(s) and telephone number(s)	Prof S Bau Ext 76215 Dr J Alt Ext 76201
External examiner(s)	Prof C Labuschagne
Calculator policy	

Time allowance

Course No's	MATH2016	Hours	1hour

Instruction to candidates (Examiners may wish to use this space to indicate, inter alia, the contribution made by this examination or test towards the year mark, if appropriate)

Answer *ALL* questions.
Show all working

Internal Examiners or Heads of Department are requested to sign the Declaration overleaf

University of the Witwatersrand, School of Mathematics
MATH2016 Advanced Analysis Final Exam, October 2013

This exam is **one page** and contains **six questions** worth a total of 60 marks. You have 60 minutes to attempt all the questions. Please write your answers clearly in the answer book, explaining all assumptions and results you are using.

Question 1 (8 marks): Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Give the definitions of the lower and upper integrals of f over $[a, b]$, and write what it means for f to be Riemann integrable over $[a, b]$. (Explain all of the notation you use.)

Question 2 (12 marks): Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Prove that for any $\varepsilon > 0$, there is a subdivision P of the interval $[a, b]$ such that

$$U(f, P) - L(f, P) < \varepsilon.$$

Question 3 (10 marks): Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and suppose that for any $\delta > 0$, there exists a subdivision $P_\delta = \{a = x_0 < \ldots < x_n = b\}$ of $[a, b]$ such that $|f(x) - f(x_j)| < \delta$ for all $j = 1, \ldots, n$ and any $x \in [x_{j-1}, x_j]$. Show that f is Riemann integrable over $[a, b]$.

Question 4 (8 marks):

(a) Write down the definition of a metric space.

(b) If (X, d) is a metric space, show that $d(x, y) \geq 0$ for any $x, y \in X$.

Question 5. (10 marks):

(a) If (X, d) is a metric space, write down the definition of a Cauchy sequence (x_k) of (X, d).

(b) Let $d_1, d_2 : X \times X \to \mathbb{R}$ be two metrics on X and suppose that some (positive) constants $K_1, K_2 \in \mathbb{R}$ exist such that

$$d_1(x, y) \leq K_1 d_2(x, y) \leq K_2 d_1(x, y)$$

for all $x, y \in X$. Show that a sequence (x_k) is a Cauchy sequence of (X, d_1) if and only if it is a Cauchy sequence of (X, d_2).

(c) Conclude that (X, d_1) is complete if and only if (X, d_2) is complete, for d_1, d_2 metrics as in (b).

Question 6. (12 marks):

(a) Let (X, d) be a metric space, and $T : X \to X$ a bounded mapping. Write down the definition of $\|T\|$.

(b) What are the definitions of a contraction $T : X \to X$ and a generalized contraction $T : X \to X$?

(c) Let $f : [0, \infty) \to [0, \infty)$ be a differentiable function satisfying $|f'(x)| \leq \beta$ for some constant $\beta < 1$. Show that f is a contraction with respect to the usual metric on $\mathbb{R}$. (*Hint:* Use the Mean Value Theorem.)

(d) Use the Banach Fixed Point Theorem to show that the equation $x = \sqrt{c + x}$ has a unique real solution $a \in [0, \infty)$ for any fixed constant $c > \frac{1}{4}$. (You do not need to find the solution a.)