Multivariable Calculus

MATH2007

2.5 Change of variables
(Part 1)

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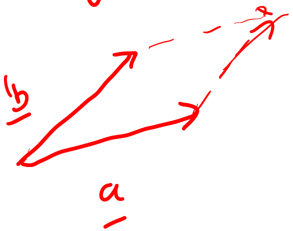
Theorem (2.5.1). Let $\underline{T} : \mathbb{R}^2 \to \mathbb{R}^2$ and let D be a region in $\mathbb{R}^2$. Suppose that D^* is a region in $\mathbb{R}^2$ such that $\underline{T}$ is one-to-one on D^* and $\underline{T}(D^*) = D$. Then

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(\underline{T}(u,v)) \underbrace{\left|\frac{\partial \underline{T}(u,v)}{\partial(u,v)}\right| du\,dv}.$$

$(x,y) = \underline{T}(u,v)$

$\begin{pmatrix}x\\ y\end{pmatrix} = \underline{T}\begin{pmatrix}u\\ v\end{pmatrix}$

$\boxed{D^*} \xrightarrow{\underline{T}} D$



area of parallelogram $= |\underline{a} \times \underline{b}| = |\det(\underline{a}\ \ \underline{b})| = |\det(\underline{T}'\underline{a}\ \ \underline{T}'\underline{b})|$

$\underline{a}, \underline{b}$ column vectors

$= |\det(\underline{T}'(\underline{a}\ \ \underline{b}))| = |(\det \underline{T}')\det(\underline{a}\ \underline{b})|$.

$$\begin{pmatrix}\Delta x\\ \Delta y\end{pmatrix} \approx \underline{T}\left(\begin{pmatrix}u\\ v\end{pmatrix} + \begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix}\right) - \underline{T}\begin{pmatrix}u\\ v\end{pmatrix} = \cancel{\underline{T}\begin{pmatrix}u\\ v\end{pmatrix}} + \underline{T}'\begin{pmatrix}u\\ v\end{pmatrix}\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix} - \cancel{\underline{T}\begin{pmatrix}u\\ v\end{pmatrix}}$$

(sev. var Taylor)

$$= \underline{T}'\begin{pmatrix}\Delta u\\ \Delta v\end{pmatrix}$$

Proof: omitted.

Note. In Theorem 2.5.1, D is the region of integration with respect to the co-ordinates (x, y) and D^* is the region in terms of the co-ordinates (u, v). In addition, if we write $\begin{pmatrix} x \\ y \end{pmatrix} = \underline{T} \begin{pmatrix} u \\ v \end{pmatrix}$ then Theorem 2.5.1 becomes

$$\iint_D f(x, y)\, dx\, dy = \iint_{D^*} f(\underline{T}(u, v)) \underbrace{\left| \frac{\partial(x, y)}{\partial(u, v)} \right|}\, du\, dv.$$

Note. If

$$\underline{T}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix} \quad \text{then} \quad \begin{pmatrix} u \\ v \end{pmatrix} = \underline{T}^{-1} \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix}$$

and

$$\frac{\partial \underline{T}^{-1}(x,y)}{\partial(x,y)} = \frac{1}{\dfrac{\partial \underline{T}(u,v)}{\partial(u,v)}}.$$

Useful tip:

If we cannot find x, y in terms of the new variables, we may calculate the Jacobian in terms of x, y then invert. It may then be possible to write in terms of the new variables or may cancel with the given integrand.

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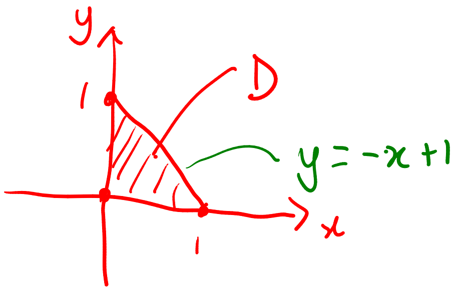
Example. Evaluate $\iint_D e^{\frac{y-x}{y+x}}\,dx\,dy$ where D is the triangle with vertices $\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}$ using the transformation $u = y - x$, $v = y + x$.



Type 3

$$\iint_D e^{\frac{y-x}{y+x}}\,dx\,dy = \int_0^1 \int_0^{-x+1} e^{\frac{y-x}{y+x}}\,dy\,dx$$

$$= \int_0^1 \int_0^{-y+1} e^{\frac{y-x}{y+x}}\,dy\,dx = ?$$

$u = y - x$ ① $\qquad v = y + x$ ②

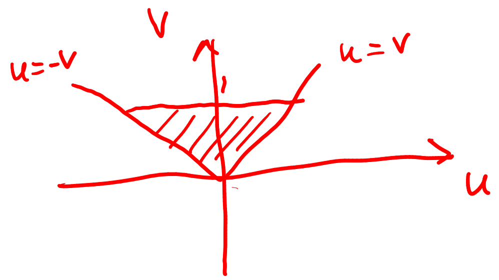
$y = \frac{1}{2}(u+v)$ ① + ②

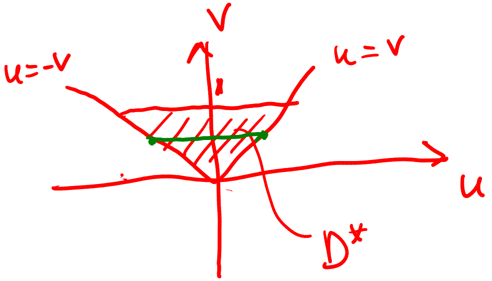
$x = \frac{1}{2}(v-u)$ ② − ①

$x \geq 0 \qquad y \geq 0 \qquad y \leq -x + 1$

$\frac{1}{2}(u+v) \geq 0 \qquad \frac{1}{2}(v-u) \geq 0 \qquad \frac{1}{2}(u+v) \leq \frac{1}{2}(u-v) + 1$

$u \geq -v \qquad v \geq u \qquad v \leq 1$





$$y = \tfrac{1}{2}(u+v)$$
$$x = \tfrac{1}{2}(v-u)$$
$$u = y-x$$
$$v = y+x$$

$$T(u,v) = \begin{pmatrix} \frac{1}{2}(v-u) \\ \frac{1}{2}(u+v) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\frac{\partial T}{\partial(u,v)} = \det\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = -\frac{1}{2}$$

$$\iint_D e^{\frac{y-x}{y+x}}\,dx\,dy = \iint_{D^*} e^{\frac{u}{v}} \left|\frac{\partial T}{\partial(u,v)}\right| du\,dv$$

$$= \iint_{D^*} e^{u/v}\,\frac{1}{2}\,du\,dv = \int_0^1 \left(\int_{-v}^{v} \frac{1}{2} e^{u/v}\,du\right) dv$$

$$= \int_0^1 \left[\frac{v}{2} e^{u/v}\right]_{u=-v}^{u=v} dv$$

$$\frac{\partial}{\partial u} \frac{v}{2} e^{u/v} = \frac{1}{v}\cdot\frac{v}{2} e^{\frac{u}{v}}$$

$$= \int_0^1 \frac{v}{2} e^{1} - \frac{v}{2} e^{-1}\,dv = \left[\frac{v^2}{4}\left(e - \frac{1}{e}\right)\right]_0^1 = \frac{1}{4}\left(e - \frac{1}{e}\right).$$

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2.5 Change of variables
(Part 3)

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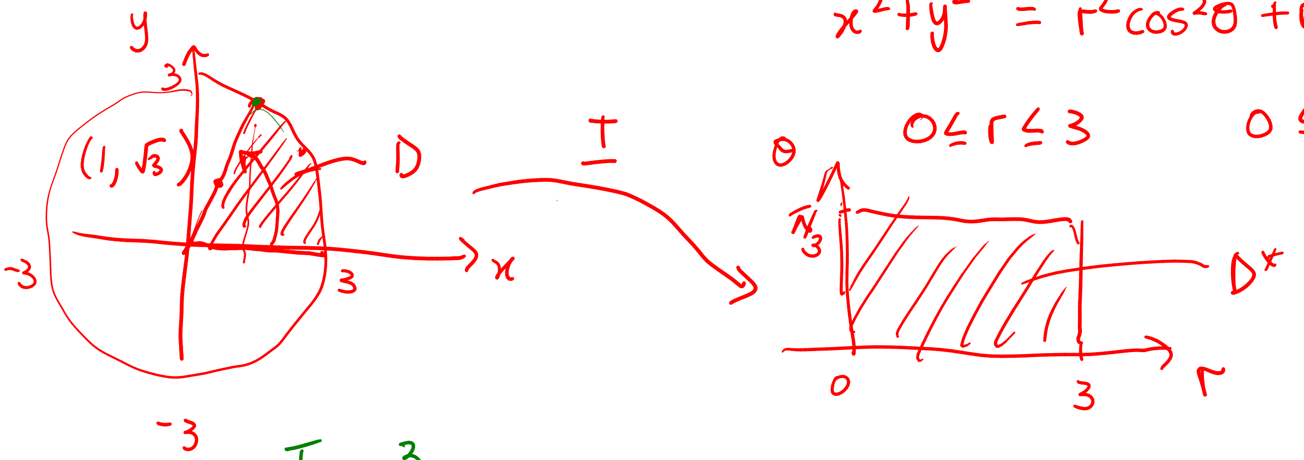
Example. Evaluate $\displaystyle\iint_D \frac{dx\,dy}{(1+x^2+y^2)}$ taken over D, the sector of the circle $x^2+y^2 \le 9$ from the positive x-axis to the ray in the direction of $\begin{pmatrix}1\\ \sqrt{3}\end{pmatrix}$ by changing to polar co-ordinates i.e. using the transformation $x = r\cos\theta$, $y = r\sin\theta$.

$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2 \le 9$$

$$0 \le r \le 3 \qquad 0 \le \theta \le \arctan(\sqrt{3}) = \pi/3$$



Type 3

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{pmatrix} = r$$

$$\iint_D \frac{dx\,dy}{(1+x^2+y^2)} = \iint_{D^*} \frac{1}{1+r^2\cos^2\theta+r^2\sin^2\theta}\cdot\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta = \iint_{D^*}\frac{1}{1+r^2}\cdot r\,dr\,d\theta$$

$0 \le r \le 3 \qquad 0 \le \theta \le \arctan(\sqrt{3}) = \pi/3$

$$\iint_D \frac{dx\,dy}{(1+x^2+y^2)} = \iint_{D^*} \frac{1}{1+r^2} \cdot r\,dr\,d\theta$$

$$= \int_0^{\pi/3} \int_0^3 \frac{r}{1+r^2}\,dr\,d\theta$$

$$= \int_0^{\pi/3} \left[\frac{1}{2} \ln(1+r^2) \right]_0^3 d\theta$$

$$= \int_0^{\pi/3} \frac{1}{2} \ln(10)\,d\theta = \frac{\theta}{2} \ln(10) \Big|_0^{\pi/3}$$

$$= \frac{\pi}{6} \ln(10).$$