Definition 2.7.

1. With every sequence (a_n) which is bounded above, we can associate the sequence of numbers $\sup\{a_k : k \geq n\}$. Since
$\{a_k : k \geq n\} = \{a_n\} \cup \{a_k : k \geq n+1\} \supset \{a_k : k \geq n+1\}$,
this sequence is decreasing, and we denote its limit by
$\lim_{n\to\infty} \sup a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\}$.
If (a_n) is not bounded above, we put $\lim_{n\to\infty} \sup a_n = \infty$.

2. Similarly, if the sequence (a_n) is bounded below, the sequence of numbers $\inf\{a_k : k \geq n\}$ is increasing, and we denote its limit by $\lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\}$.
If (a_n) is not bounded below, we put $\lim_{n\to\infty} \inf a_n = -\infty$.

Note.

1. Since $\inf S \leq \sup S$ for every nonempty set S, it follows that $\lim_{n\to\infty} \inf a_n \leq \lim_{n\to\infty} \sup a_n$, where we write $-\infty < x$ and $x < \infty$ for each $x \in \mathbb{R}$.

2. (a_n) is bounded iff both $\lim_{n\to\infty} \inf a_n$ and $\lim_{n\to\infty} \sup a_n$ are real numbers (also called finite).

We have now the following characterization of the convergence of a sequence and its limit when it exists.

Theorem 2.11.

1. The sequence (a_n) converges iff $\lim_{n\to\infty} \inf\, a_n$ and $\lim_{n\to\infty} \sup\, a_n$ are finite and equal, and then

$$\lim_{n\to\infty} \inf\, a_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sup\, a_n$$

2. $\lim_{n\to\infty} a_n = \infty \Leftrightarrow \lim_{n\to\infty} \inf\, a_n = \infty$ and $\lim_{n\to\infty} \sup\, a_n = \infty$

3. $\lim_{n\to\infty} a_n = -\infty \Leftrightarrow \lim_{n\to\infty} \inf\, a_n = -\infty$ and $\lim_{n\to\infty} \sup\, a_n = -\infty$

Proof. For a sequence (a_n), denote $b_n = \inf\{a_k \,:\, k \geq n\}$ and $c_n = \sup\{a_k \,:\, k \geq n\}$.

1. Assume that (a_n) converges to L. Let $\epsilon > 0$.
Then there is $K \in \mathbb{N}$ such that $\forall k \geq K$, $L - \frac{\epsilon}{3} < a_k < L + \frac{\epsilon}{3}$
Hence, for $n \geq K$,

$$L - \frac{\epsilon}{3} \leq b_n \leq a_n \leq c_n \leq L + \frac{\epsilon}{3}\,.$$

Since (b_n) is increasing and (c_n) is decreasing, we have

$$L - \frac{\epsilon}{3} \leq \lim_{n\to\infty} \inf\, a_n \leq \lim_{n\to\infty} \sup\, a_n \leq L + \frac{\epsilon}{3}\,.$$

Hence

$$0 \leq \lim_{n\to\infty} \sup\, a_n - \lim_{n\to\infty} \sup\, a_n \leq \frac{2\epsilon}{3} < \epsilon\,.$$

From Lemma 2.1, we obtain that $\lim_{n\to\infty} \inf\, a_n = \lim_{n\to\infty} \sup\, a_n$.

Conversely, if $\lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} \sup a_n$, then it follows from $b_n \le a_n \le c_n$ and the Sandwich Theorem that (a_n) converges and that

$$\lim_{n\to\infty} \inf a_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \sup a_n .$$

$\square$

Another important concept is that of a Cauchy sequence:

Definition 2.8 (Cauchy sequence). A sequence (a_n) is called a Cauchy sequence if $\forall \epsilon > 0 \; \exists K \in \mathbb{R}$ such that $\forall n, m \in \mathbb{N}$ with $n, m \ge K$, $|\, a_n - a_m \,| < \epsilon$.

Theorem 2.12.
A sequence (a_n) converges iff it is a Cauchy sequence.

Proof. Let (a_n) be a convergent sequence with limit L .
Let $\epsilon > 0$ and let K such that $|\, a_n - L \,| < \dfrac{\epsilon}{2}$ for $n \ge K$.
Then it follows for $n, m \ge K$ that

$$|\, a_n - a_m| = |\,(a_n - L) - (a_m - L)\,| \le |\, a_n - L\,| + |\, a_m - L\,| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\, .$$

Conversely, assume that (a_n) is a Cauchy sequence. Let $\epsilon > 0$ and choose K such that $|a_n - a_m| < \dfrac{\epsilon}{3}$ for all $n, m \geq K$. Then

$$a_m - \frac{\epsilon}{3} < a_n < a_m + \frac{\epsilon}{3}\,.$$

In particular, choosing $m = K$,

$$\{a_n \,:\, n \geq K\} \subset \left(a_K - \frac{\epsilon}{3}, a_K + \frac{\epsilon}{3}\right)\,.$$

Thus (a_n) is bounded and

$$a_K - \frac{\epsilon}{3} \leq \lim_{n\to\infty} \inf\, a_n \leq \lim_{n\to\infty} \sup\, a_n \leq a_K + \frac{\epsilon}{3}$$

for $n \geq K$, which gives

$$0 \leq \lim_{n\to\infty} \sup\, a_n - \lim_{n\to\infty} \inf\, a_n \leq \left(a_K + \frac{\epsilon}{3}\right) - \left(a_K - \frac{\epsilon}{3}\right) = \frac{2\epsilon}{3} < \epsilon\,.$$

By Lemma 2.1, $\lim\limits_{n\to\infty} \inf\, a_n = \lim\limits_{n\to\infty} \sup\, a_n$, and an application of Theorem 2.11, part 1, completes the proof. $\square$