Basic Analysis 2015 — Solutions of Tutorials

Section 3.3

Tutorial 3.3.1

1. Let n be a positive integer. Prove that

(a) $\lim\limits_{x\to\infty} x^n = \infty$,

(b) $\lim\limits_{x\to-\infty} x^n = \begin{cases} \infty & \text{if } n \text{ is even,} \\ -\infty & \text{if } n \text{ is odd,} \end{cases}$

(c) $\lim\limits_{x\to 0^+} x^{-n} = \infty$,

(d) $\lim\limits_{x\to 0^-} x^{-n} = \begin{cases} \infty & \text{if } n \text{ is even,} \\ -\infty & \text{if } n \text{ is odd.} \end{cases}$

Proof. (a) Let $A > 1$ and put $K = A$. For $x > K$ we have $x^n \geq x > K = A$, and the statement follows. Alternatively, use that the statement is trivial if $n = 1$ and then use mathematical induction and Theorem 3.6(c).

(b) We conclude form part (a) and Theorem 3.6(d) that

$$\lim_{x\to-\infty} x^n = \lim_{x\to\infty} (-x^n) = \lim_{x\to\infty} (-1)^n x^n = \begin{cases} \infty & \text{if } n \text{ is even,} \\ -\infty & \text{if } n \text{ is odd.} \end{cases}$$

(c) Since $x \to 0^+$ if and only if $\frac{1}{x} \to \infty$, it follows from (a) that

$$\lim_{x\to 0^+} x^{-n} = \lim_{x\to\infty} \left(\frac{1}{x}\right)^{-n} = \lim_{x\to\infty} x^n = \infty.$$

(d) Since $x \to 0^-$ if and only if $\frac{1}{x} \to -\infty$, it follows from (b) that

$$\lim_{x\to 0^-} x^{-n} = \lim_{x\to-\infty} \left(\frac{1}{x}\right)^{n} = \begin{cases} \infty & \text{if } n \text{ is even,} \\ -\infty & \text{if } n \text{ is odd.} \end{cases}$$

□

2. (a) Let f, g be defined in a deleted neighbourhood of a and assume that $f(x) < g(x)$ for all x in a deleted neighbourhood of a. Show that if $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$ exist, then $L \leq M$.

Proof. Let $\varepsilon > 0$. Then there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$(0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \varepsilon) \quad \text{and} \quad (0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \varepsilon).$$

Putting $\delta = \min\{\delta_1, \delta_2\}$ it follows for $0 < |x-a| < \delta$ that

$$L - \varepsilon < f(x) < g(x) < M + \varepsilon$$

and therefore

$$0 < g(x) - f(x) < M - L + 2\varepsilon.$$

Assume, by proof of contradiction, that $M < L$. Then we can choose $\varepsilon = \frac{1}{2}(L - M) > 0$, which would lead to the contradiction

$$0 < M - L + 2\varepsilon = M - L + L - M = 0.$$

This contradiction shows that $L \leq M$. □

(b) Give examples for $L < M$ and for $L = M$ in (a).

Solution. With $f(x) = 0$, $g(x) = e^x$ and $a = 0$, we have $f(x) < g(x)$ for all $x \in \mathbb{R}$ and $L = 0 < 1 = M$.
With $f(x) = 0$, $g(x) = x^2$ and $a = 0$, we have $f(x) < g(x)$ for all $x \in \mathbb{R} \setminus \{0\}$ and $L = 0 = M$.

(c) Formulate and prove the result corresponding to (a) for one-sided limits.

Solution. This is straightforward by replacing limits with one-sided limits and $0 < |x - a| < \delta$ with $a - \delta < x < a$ or $a < x < a + \delta$, respectively.

3. Using rules for limits, determine the behaviour of $f(x)$ as x tends to the given limit:

(a) $f(x) = \dfrac{4x}{3-x}$ as $x \to 3^-$,

(b) $f(x) = \dfrac{(x-4)(x-1)}{x-2}$ as $x \to 2^+$,

(c) $f(x) = \dfrac{2x+1}{x^2-x}$ as $x \to 0^+$.

Solution. (a) From $\lim\limits_{x\to 3^-} 4x = 12$ and $\lim\limits_{x\to 3^-} \frac{1}{3-x} = \infty$ it follows from Theorem 3.6(d) that $\lim\limits_{x\to 3^-} \dfrac{4x}{3-x} = \infty$.

(b) From $\lim\limits_{x\to 2^+} (x-4)(x-1) = -2$ and $\lim\limits_{x\to 2^+} \frac{1}{x-2} = \infty$ it follows by Theorem 3.6(d) that $\lim\limits_{x\to 2^+} \dfrac{(x-4)(x-1)}{x-2} = -\infty$.

(b) From $\lim\limits_{x\to 0^+} \dfrac{2x+1}{x-1} = -1$ and $\lim\limits_{x\to 0^+} \frac{1}{x} = \infty$ it follows from Theorem 3.6(d) that $\lim\limits_{x\to 0^+} \dfrac{2x+1}{x^2-x} = -\infty$.