Study Guides in Mathematics 2019

School of Mathematics

University of the Witwatersrand

BASIC ANALYSIS
STUDY GUIDE
FOR
MATH 2001

Contents

Introduction

Aims

Basic Analysis is a fundamental part of the mathematics curriculum and the results and methods of this course will be indispensable in any further course in the School which involves Analysis. You will have seen many of the definitions and theorems in Calculus I. So you may ask what is the difference between Calculus I and Introductory Analysis?

In Calculus you have learnt definitions and theorems which then were used to solve standard problems. Whenever it was not too hard, proofs were provided, but often concepts were explained rather vaguely with some sketches. In particular the fundamental concept of limit and the definition of real numbers have not been given formally. Indeed, it is the step from rational numbers to real numbers which is crucial in Real Analysis (and also in Calculus).

You have seen proof techniques and axioms in Transition to Abstract Mathematics and Algebra I. This combined with the axiom of the real numbers gives you the basis for all definitions, theorems and proofs in this course, spelt out rigorously using mathematical logic.

Students therefore have to know definitions and theorems and have to know the proofs which are provided in class. It is more important to understand definitions, theorems and proofs than learning them by heart. Any valid reformulation in your own words will be fully accepted.

You have seen some proofs in Calculus I, Transition to Abstract Mathematics (or even Algebra I) of results which you will re-encounter in Introductory Analysis. These proofs are provided in this study guide and may not be presented in class. They will be clearly indicated by their reference to first year. Although these proofs will not be examined directly, you nevertheless should recollect them.

In this course, you will only see naïve logic and naïve set theory, as much as needed to understand Introductory Analysis.

If you think that the concepts in Basic Analysis are not that obvious, you are right! Recall that Newton and Leibniz are credited as having created calculus at the end of the 17^{th} century, whereas the rigorous definitions which you will encounter in Basic Analysis were only introduced by Bolzano and Cauchy in the early 19^{th} century, and the axiomatic definition of the real number system only evolved in the second half of the 19^{th} century.

Throughout this study guide you will find additional notes. This is material which will not be examined and, in general, not discussed in class. However, some notation may be introduced there which may be used somewhere else in the course.

Outcomes

At the end of this course, students should be able to

- state the axioms of the real numbers
- state and explain definitions given in this course
- distinguish between axioms and definitions
- state and identify theorems given in this course
- prove theorems from this course
- find examples and counter-examples to illustrate concepts
- solve problems and perform computations using concepts and methods based on the theory
- solve theoretical problems as in tutorial examples
- apply theory

- form hypotheses
- reason mathematically
- describe and explain the hierarchical build up of the subject

Recommended textbooks

H. Amann, J. Escher, Analysis I, Springer (electronic).

K. A. Ross, Elementary Analysis, Springer.

M. Spivak, Calculus, Cambridge University Press.

In its presentation, the book by Ross is closest to this course. However, if students would like to have more reading on the transition from Calculus to Analysis, they may find Spivak's book useful. As this is only recommended reading, you may use any edition of these textbooks. The electronic version of Amann's textbook can be accessed via the Wits library catalogue. Volume I covers more than what will be covered in this course, and also gives an alternative definition of the real numbers. Volumes I to III are recommended as supplementary reading for this and further Analysis courses up to Honours level.

Chapter 1

The Real Number System

1.1 Definition and Properties of Real Numbers

We will give an axiomatic definition of the real numbers. We will define the set of real numbers, denoted by $\mathbb{R}$, as a set with two operations $+$ and $\cdot$, called addition and multiplication, as well as an ordering $<$, which satisfy the laws of addition (A), the laws of multiplication (M), the distributive law (D), the order laws (O) and the Dedekind Completeness Axiom (C). These laws and axioms will be given and discussed below.

Additional Note. An alternative way to define the real numbers is to use Peano's axiom to define the natural numbers, and then construct the integers and rational numbers in turn. Then the axioms (A), (M), (D) and (O) become properties of rational numbers, so that we call them laws.

Addition and multiplication are maps which assign to every two elements in $a, b \in \mathbb{R}$ an element in $\mathbb{R}$ which is denoted by $a + b$ and $a \cdot b$ (in general, written ab), respectively. We require that these operations satisfy the following axioms.

A. Axioms of addition

(A1) **Associative Law:** $a + (b + c) = (a + b) + c$ for all $a, b, c \in \mathbb{R}$.

(A2) **Commutative Law:** $a + b = b + a$ for all $a, b \in \mathbb{R}$.

(A3) **Zero:** There is a real number 0 such that $a + 0 = a$ for all $a \in \mathbb{R}$.

(A4) **Additive inverse:** For each $a \in \mathbb{R}$ there is $-a \in \mathbb{R}$ such that $a + (-a) = 0$.

Notation. For $a, b \in \mathbb{R}$ one writes $a - b := a + (-b)$.

Additional Notes. 1. Any set with the operation + satisfying (A1), (A3), (A4) is called a group. If also (A2) is satisfied, the group is called an Abelian (or commutative) group.
2. You will encounter detailed discussions of (Abelian) groups in algebra courses.
3. The additive inverse of a number is also called the negation of a number.

Theorem 1.1 (Basic group properties)
(a) *The number* 0 *is unique.*
(b) *For all* $a \in \mathbb{R}$*, the number* $-a$ *is unique.*
(c) *For all* $a, b \in \mathbb{R}$*, the equation* $a+x = b$ *has a unique solution. This solution is* $x = b-a$.
(d) $\forall a \in \mathbb{R},\ -(-a) = a$.
(e) $\forall a, b \in \mathbb{R},\ -(a + b) = -a - b$.
(f) $-0 = 0$.

Proof

(a) Let $0, 0' \in \mathbb{R}$ such that $a + 0 = a$ and $a + 0' = a$ for all $a \in \mathbb{R}$. We must show that $0 = 0'$:

$$\begin{aligned} 0 &= 0 + 0' \\ &= 0' + 0 \qquad \text{by (A2)} \\ &= 0' \end{aligned}$$

(b) Let $a \in \mathbb{R}$ and $a', a'' \in \mathbb{R}$ such that $a + a' = 0$ and $a + a'' = 0$. We must show that

$a' = a''$:

$$\begin{aligned} a' &= a' + 0 && \text{by (A3)} \\ &= a' + (a + a'') \\ &= (a' + a) + a'' && \text{by (A1)} \\ &= (a + a') + a'' && \text{by (A2)} \\ &= 0 + a'' \\ &= a'' + 0 && \text{by (A2)} \\ &= a'' && \text{by (A3)} \end{aligned}$$

(c) First we show that $x = b - a$ is a solution. So let $x = b - a$.

Then

$$\begin{aligned} a + x = a + (b - a) &= a + (b + (-a)) \\ &= a + ((-a) + b) && \text{by (A2)} \\ &= (a + (-a)) + b && \text{by (A1)} \\ &= 0 + b && \text{by (A4)} \\ &= b + 0 && \text{by (A2)} \\ &= b && \text{by (A3)} \end{aligned}$$

To show that the solution is unique let $x \in \mathbb{R}$ such that $a + x = b$.

Then

$$\begin{aligned} x &= x + 0 = x + (a + (-a)) && \text{by (A3), (A4)} \\ &= (x + a) + (-a) && \text{by (A1)} \\ &= (a + x) + (-a) && \text{by (A2)} \\ &= b - a && \because\ a + x = b \end{aligned}$$

This shows that the solution is unique.

(d) Note that

$$-a + (-(-a)) = 0 \quad \text{by (A4).}$$

On the other hand

$$-a + a = a + (-a) = 0 \quad \text{by (A2), (A4).}$$

By part (b), it follows that

$$-(-a) = a.$$

Alternatively:

$$\begin{aligned}
-(-a) &= -(-a) + 0 = -(-a) + (a + (-a)) && \text{by (A3), (A4)} \\
&= (a + (-a)) + (-(-a)) && \text{by (A2)} \\
&= a + ((-a) + (-(-a))) && \text{by (A1)} \\
&= a + 0 && \text{by (A4)} \\
&= a && \text{by (A3)}
\end{aligned}$$

(e)

$$\begin{aligned}
(a + b) + (-a - b) &= (a + b) + ((-a) + (-b)) \\
&= ((b + a) + (-a)) + (-b) && \text{by (A1), (A2)} \\
&= (b + (a + (-a))) + (-b) && \text{by (A1)} \\
&= (b + 0) + (-b) && \text{by (A4)} \\
&= b + (-b) && \text{by (A3)} \\
&= 0 && \text{by (A4)}
\end{aligned}$$

By part (b), $-(a + b) = -a - b$.

(f) $0 + 0 = 0$ by (A3), so that $-0 = 0$ by (A4) and (b).

(d), (e), (f): In class. □

M. Axioms of multiplication

(M1) Associative Law: $a(bc) = (ab)c$ for all $a, b, c \in \mathbb{R}$.

(M2) **Commutative Law:** $ab = ba$ for all $a, b \in \mathbb{R}$.

(M3) **One:** There is a real number 1 such that $1 \neq 0$ and $a \cdot 1 = a$ for all $a \in \mathbb{R}$.

(M4) **Multiplicative inverse:** For each $a \in \mathbb{R}$ with $a \neq 0$ there is $a^{-1} \in \mathbb{R}$ such that $aa^{-1} = 1$.

D. The distributive law axiom

(D) **Distributive Law:** $a(b + c) = ab + ac$ for all $a, b, c \in \mathbb{R}$.

Additional Notes. 1. Any set with the operations $+$, $\cdot$ satisfying the axioms (A1)–(A4), (M1)–(M4) and (D) is called a field.

2. You will encounter detailed discussions of fields in algebra courses.

3. The multiplicative inverse of a number is also briefly called the inverse of a number.

4. The set of nonzero real numbers, $\mathbb{R} \setminus \{0\}$, is an Abelian group with respect to multiplication.

Theorem 1.2 (Basic field properties: Distributive laws)

(a) $\forall a, b, c \in \mathbb{R},\ (a + b)c = ac + bc.$

(b) $\forall a \in \mathbb{R},\ a \cdot 0 = 0.$

(c) $\forall a, b \in \mathbb{R},\ ab = 0 \Leftrightarrow a = 0 \text{ or } b = 0.$

(d) $\forall a, b \in \mathbb{R},\ (-a)b = -(ab).$

(e) $\forall a \in \mathbb{R},\ (-1)a = -a.$

(f) $\forall a, b \in \mathbb{R},\ (-a)(-b) = ab.$

Proof (a) By (M2) and (D),

$$(a + b)c = c(a + b) = ca + cb = ac + bc.$$

(b) By (A3) and (D),

$$a \cdot 0 = a(0 + 0) = a \cdot 0 + a \cdot 0,$$

and by (A3), (A2), $a \cdot 0 = 0 + a \cdot 0$. Then Theorem 1.1,(c) gives $a \cdot 0 = 0$.

(c) If $b = 0$, then $ab = a \cdot 0 = 0$ by (b).

If $a = 0$, then $ab = ba = b \cdot 0 = 0$ by (M2) and (b).

Now assume that $ab = 0$. If $b = 0$, the property "$a = 0$ or $b = 0$" follows. So now assume $b \neq 0$. Then

$$a = a \cdot 1 = a(bb^{-1}) = (ab)b^{-1} = 0 \cdot b^{-1} = 0$$

by (M3), (M4), (M1), (M2) and (b).

(d) Using field laws, we get

$$ab + (-a)b = (a + (-a))b = 0 \cdot b = 0,$$

and from Theorem 1.1(b), $(-a)b = -ab$.

(e) is a special case of (d).

(f) From (d) and other laws and rules (which one's?) we find

$$(-a)(-b) = -[a(-b)] = -[(-b)a] = -[-ba] = ba = ab.$$

$\square$

Theorem 1.3 (Basic field properties: multiplication)

(a) *The number* 1 *is unique.*

(b) *For all $a \in \mathbb{R}$ with $a \neq 0$, the number a^{-1} is unique.*

(c) *For all $a, b \in \mathbb{R}$ with $a \neq 0$, the equation $ax = b$ has a unique solution. This solution is $x = a^{-1}b$.*

(d) $\forall a \in \mathbb{R} \setminus \{0\},\ (a^{-1})^{-1} = a.$

(e) $\forall a, b \in \mathbb{R} \setminus \{0\},\ (ab)^{-1} = a^{-1}b^{-1}.$

(f) $\forall a \in \mathbb{R} \setminus \{0\},\ (-a)^{-1} = -a^{-1}.$

(g) $1^{-1} = 1.$

Proof See tutorials. The proofs are similar to those of Theorem 1.1.

Next we give the axioms for the set of positive real numbers. It is convenient to use the notation $a > 0$ for positive numbers a.

O. The order axioms

(O1) **Trichotomy:** For each $a \in \mathbb{R}$, exactly one of the following statements is true: $a > 0$ or $a = 0$ or $-a > 0$.

(O2) If $a > 0$ and $b > 0$, then $a + b > 0$.

(O3) If $a > 0$ and $b > 0$, then $ab > 0$.

The definition of positivity of real numbers gives rise to an order relation for real numbers:

Definition. Let $a, b \in \mathbb{R}$. Then a is called larger than b, written $a > b$, if $a - b > 0$.

Notes. 1. Since $a - 0 = a$, the notation $a > 0$ is consistent.

2. It is convenient to introduce the following notations:

$$a \geq b \Leftrightarrow a > b \text{ or } a = b,$$

$$a < b \Leftrightarrow b > a,$$

$$a \leq b \Leftrightarrow a < b \text{ or } a = b.$$

3. We will define general powers later. Below we use the notation $a^2 = a \cdot a$.

Theorem 1.4 (Basic order properties) *Let $a, b, c, d \in \mathbb{R}$. Then*

(a) $a < 0 \Leftrightarrow -a > 0$.

(b) $a < b$ *and* $b < c \Rightarrow a < c$.

(c) $a < b \Rightarrow a + c < b + c$.

(d) $a < b$ *and* $c < d \Rightarrow a + c < b + d$.

(e) $a < b$ *and* $c > 0 \Rightarrow ca < cb$.

(f) $0 \leq a < b$ *and* $0 \leq c < d \Rightarrow ac < bd$.

(g) $a < b$ *and* $c < 0 \Rightarrow ca > cb$.

(h) $a \neq 0 \Rightarrow a^2 > 0$.

(i) $a > 0 \Rightarrow a^{-1} > 0$ *and* $a < 0 \Rightarrow a^{-1} < 0$.

(j) $0 < a < b \Rightarrow b^{-1} < a^{-1}$.

(k) $1 > 0$.

Proof

(a)

$$\begin{aligned} a < 0 \Leftrightarrow 0 > a & \qquad \text{by definition of } < \\ \Leftrightarrow 0 - a > 0 & \qquad \text{by definition of } 0 > a \\ \Leftrightarrow -a > 0 & \qquad \because\ 0 - a = -a + 0 = -a \end{aligned}$$

(b)

$$\begin{aligned} a < b \text{ and } b < c \Rightarrow b - a > 0 \text{ and } c - b > 0 & \qquad \text{by definition} \\ \Rightarrow (b-a) + (c-b) > 0 & \qquad \text{by (O2)} \\ \Rightarrow c - a > 0 & \qquad \text{by (A1)-(A4)} \\ \Rightarrow a < c & \qquad \text{by definition} \end{aligned}$$

(h) Since $a \neq 0$, either $a > 0$ or $a < 0$.

If $a > 0$, then $a^2 = aa > 0$ by (O3).

If $a < 0$, then $-a > 0$ by (a) and $a^2 = (-a)(-a)$ by Theorem 1.2, (f).

Hence $a^2 = (-a)(-a) > 0$ by (O3).

(i) Since $a \neq 0$, a^{-1} exists with $aa^{-1} = 1$ by (M4). Then $a^{-1} \neq 0$ by Theorem 1.2, (c). Hence $(a^{-1})^2 > 0$ by (h). Thus, if $a > 0$,

$a^{-1} = a(a^{-1})^2 > 0$ by (O3).

Similarly, use (g) if $a < 0$.

(j) By (i), $a^{-1} > 0$ and $b^{-1} > 0$. Hence $a^{-1}b^{-1} > 0$ by (O3). Then

$$\begin{aligned} b^{-1} &= a(a^{-1}b^{-1}) < b(a^{-1}b^{-1}) \qquad \text{by (e)} \\ &= (bb^{-1})a^{-1} = a^{-1}. \end{aligned}$$

(k) $1 = 1 \cdot 1 = 1^2 > 0$ by (h).

(b), (h), (i), (j), (k) In class. (c)–(g): See tutorials.

2. **The absolute value function.** Define the following function on $\mathbb{R}$:

Prove the following statements for $x, y \in \mathbb{R}$:

(a) $|x| \geq 0$,

(b) $|xy| = |x|\,|y|$,

(c) $|y| < x \Leftrightarrow -x < y < x$ whenever $x > 0$,

(d) $|x + y| \leq |x| + |y|$.

Proof (a) By (O1), we have to consider 3 cases:

Case I: $x > 0 \Rightarrow |x| = x > 0 \Rightarrow |x| \geq 0$

Case II: $x = 0 \Rightarrow |x| = x = 0 \Rightarrow |x| \geq 0$

Case I: $-x > 0 \Rightarrow |x| = -x > 0 \Rightarrow |x| \geq 0$

(b) Case I: $x = 0$ or $y = 0$: then $xy = 0$ by Theorem 1.2 (b). Hence

$$|xy| = |0| = 0 = |x|\,|y|.$$

Case II: $x > 0$, $y > 0$: Then $xy > 0$ by (O3). Hence

$$|xy| = xy = |x|\,|y|.$$

Case III: $x < 0$, $y > 0$: Then $xy < 0$ by Theorem 1.4 (g). Hence, by Theorem 1.2 (d),

$$|xy| = -xy = (-x)y = |x|\,|y|.$$

Case IV: $x > 0$, $y < 0$: Interchange x and y and apply Case III.

Case V: $x < 0$, $y < 0$: Then $-x > 0$ and $-y > 0$ by Theorem 1.4 (a). Applying Theorem 1.2 (f) and Case II give

$$|xy| = |(-x)(-y)| = (-x)(-y) = |x|\,|y|.$$

(c) Let $|y| < x$. Then $0 \leq |y| < x$, so that $-x < 0$. If $y \geq 0$, then

$$-x < 0 \leq y = |y| < x,$$

so that $-x < y < x$. If $y < 0$, then $|-y| = |(-1)y| = |y|$ by (b), and the above gives $-x < -y < x$, so that Theorem 1.4 (g) gives $-x < -(-y) < -(-x)$, that is, $-x < y < x$.

Conversely, let $-x < y < x$. If $y \geq 0$, then $|y| = y < x$.

If $y < 0$, then $-x < y$ gives $-y < x$, so that $|y| = -y < x$.

(d) If $x + y \geq 0$, then

$$|x + y| = x + y \leq |x| + |y| \quad (\text{using } z \leq |z|).$$

If $x + y < 0$, then

$$|x + y| = -(x + y) \leq |x| + |y| \quad (\text{using } -z \leq |z|).$$

3. Let $x, y, z \in \mathbb{R}$. Which of the following statements are **true** and which are **false**?

(a) $x \leq y \Rightarrow xz \leq yz$,

(b) $0 < x \leq y \Rightarrow \frac{1}{y} \leq \frac{1}{x}$,

(c) $x < y < 0 \Rightarrow \frac{1}{y} < \frac{1}{x}$,

(d) $x^2 < 1 \Rightarrow x < 1$,

(e) $x^2 < 1 \Rightarrow -1 < x < 1$,

(f) $x^2 > 1 \Rightarrow x > 1$.

4. In each of the following questions fill in the $\square$ with $<$ or $>$.

(a) $a \geq 3 \Rightarrow \frac{a-2}{7} \square \frac{a}{7}$,

(b) $a \geq 1 \Rightarrow \frac{3}{a+1} \square \frac{3}{a}$,

(c) $a > 1 \Rightarrow \frac{9}{a} \square \frac{10}{a-1}$,

(d) $a > 1 \Rightarrow \frac{1}{a^2} \square \frac{1}{a}$,

(e) $a \geq 2 \Rightarrow \frac{1}{a^2-1} \square \frac{1}{a}$,

(f) $a > 3 \Rightarrow \frac{-3}{a} \square \frac{-2}{a-1}$

5. Let $x \geq 0$ and $y \geq 0$. Show that $x < y \Leftrightarrow x^2 < y^2$.

Proof Observe that

$$x^2 < y^2 \Leftrightarrow y^2 - x^2 > 0 \Leftrightarrow (y - x)(y + x) > 0$$

and

$$x < y \Leftrightarrow y - x > 0.$$

In either case, $y \neq x$, so that $x \geq 0$ and $y \geq 0$ gives that $x > 0$ or $y > 0$. It follows that $x + y > 0$. Hence

$$y - x > 0 \Rightarrow (y - x)(y + x) > 0$$

and

$$(y - x)(y + x) > 0 \Rightarrow y - x = (y - x)(y + x)(y + x)^{-1} > 0.$$

1.2 Bounded Sets, Suprema and Infima

Definition 1.1 *Let S be a nonempty subset of $\mathbb{R}$. Then*

*1. If there is a number $A \in \mathbb{R}$ such that $x \leq A$ for all $x \in S$, then A is said to be an **upper bound of** S, and S is said to be **bounded above**.*

*2. If there is a number $a \in \mathbb{R}$ such that $x \geq a$ for all $x \in S$, then a is said to be a **lower bound of** S, and S is said to be **bounded below**.*

*3. If S is bounded above and bounded below, then S is called **bounded**.*

Example 1.1 *Let $S = (-\infty, 1)$. Does S have upper and lower bounds?*

Solution. Since $x < 1$ for all $x \in S$, 1 is an upper bound of S, and S is therefore bounded above. But also 2 and 7.3 are upper bounds of S, for example.

Assume that S has a lower bound m. Then $m \leq 0$ since $0 \in S$, and $m - 1 < m \leq 0 < 1$ shows that $m - 1 \in S$. So $m \leq m - 1$ since m is a lower bound of S. But this is false, so that m cannot be a lower bound of S. Therefore S has no lower bound, and S is therefore not bounded below.

Definition 1.2 *Let S be a nonempty subset of $\mathbb{R}$.*

*1. If S has an upper bound M which is an element of S, then M is called the **greatest element** or **maximum** of S, and we write $M = \max S$.*

2. If S has a lower bound m which is an element of S, then m is called the ***least element*** *or* ***minimum*** *of S, and we write $m = \min S$.*

From the definition, we immediately obtain

Proposition 1.1 *Let S be a nonempty subset of $\mathbb{R}$. Then*
1. $M = \max S \Leftrightarrow M \in S$ and $x \leq M$ for all $x \in S$,
2. $m = \min S \Leftrightarrow m \in S$ and $x \geq m$ for all $x \in S$.

We have already implicitly used in our notation that maximum and minimum are unique if they exist. The next result states this formally.

Proposition 1.2 *Let S be a nonempty subset of $\mathbb{R}$. If maximum or minimum of S exist, then they are unique.*

Proof Assume that S has maxima M_1 and M_2. We must show $M_1 = M_2$. Since $M_1 \in S$ and M_2 is an upper bound of S, we have $M_1 \leq M_2$. Since $M_2 \in S$ and M_1 is an upper bound of S, we have $M_2 \leq M_1$. Hence $M_1 = M_2$.

A similar proof holds for the minimum.

Example 1.2 *Let $a, b \in \mathbb{R}$, $a < b$, and $S = [a, b)$. Then $\min S = a$, but S has no maximum.*

Definition 1.3 *Let S be a nonempty subset of $\mathbb{R}$.*
A real number M is said to be the ***supremum*** *or* ***least upper bound*** *of S, if*
(a) M is an upper bound of S, and
(b) if L is any upper bound of S, then $M \leq L$.
The supremum of S is denoted by $\sup S$.

Definition 1.4 *Let S be a nonempty subset of $\mathbb{R}$.*
A real number m is said to be the ***infimum*** *or* ***greatest lower bound*** *of S, if*

(a) m is a lower bound of S, and

(b) if l is any lower bound of S, then $m \geq l$.

The infimum of S is denoted by $\inf S$.

Note. Let S be a nonempty subset of $\mathbb{R}$.

1. By definition, if S has a supremum, then $\sup S$ is the minimum of the (nonempty) set of the upper bounds of S. Hence $\sup S$ is unique by Proposition 1.2.

2. Similarly, $\inf S$ is unique if it exists.

Proposition 1.3 *Let S be a nonempty subset of $\mathbb{R}$.*

1. If $\max S$ exists, then $\sup S$ exists, and $\sup S = \max S$.

2. If $\min S$ exists, then $\inf S$ exists, and $\inf S = \min S$.

Proof 1. $\max S$ is an upper bound of S by definition. Since $\max S \in S$, $\max S \leq L$ for any upper bound L of S. Hence $\max S = \sup S$.

The proof of 2. is similar.

Example 1.3 *Let $a < b$ and put $S = [a, b)$. Find $\inf S$ and $\sup S$ if they exist.*

Solution. By Proposition 1.3 and Example 1.2, $\inf S = \min S = a$ exists.

From $x < b$ for all $x \in S$ we have that b is an upper bound of S. If there were an an upper bound L of S with $L < b$, it would follow as in the solution to Example 1.2 that there is $c \in S$ with $L < c$, which contradicts the fact that L is an upper bound of S.

Hence $\sup S = b$.

C. The Dedekind completeness axiom

(C) Every nonempty subset of $\mathbb{R}$ which is bounded above has a supremum.

Additional Note. It can be shown that the real number system is uniquely determined by the axioms (A1)-(A4), (M1)-(M4), (D), (O1)-(O3), (C).

Theorem 1.5 (Positive square root) *Let $a \geq 0$. Then there is a unique $x \geq 0$ such that $x^2 = a$. We write $x = \sqrt{a} = a^{\frac{1}{2}}$.*

Proof. If $a = 0$, we have $0^2 = 0$ and $x^2 > 0$ if $x > 0$, so that $\sqrt{0} = 0$ is the unique number $x \geq 0$ such that $x^2 = 0$.

Now let $a > 0$. Note that $x \geq 0$ and $x^2 = a > 0$ gives $x > 0$. For the uniqueness proof let $x_1 > 0$ and $x_2 > 0$ such that $x_1^2 = a$ and $x_2^2 = a$. Then

$$0 = a - a = x_1^2 - x_2^2 = (x_1 + x_2)(x_1 - x_2).$$

From $x_1 + x_2 > 0$ it follows that $x_1 - x_2 = 0$, i. e., the uniqueness $x_1 = x_2$.

For the existence of the square root define

$$S_a = \{x \in \mathbb{R} : 0 < x,\ x^2 < a\}.$$

First we are going to show that $S_a \neq \emptyset$ and that S_a is bounded.

If $a < 1$, then $a^2 < a \cdot 1 = a$, so that $a \in S_a$.

If $a \geq 1$, then $(\frac{1}{2})^2 < 1^2 = 1 \leq a$, so that $\frac{1}{2} \in S_a$.

For $x > a + 1$ we have

$$x^2 > (a+1)^2 = a^2 + 2a + 1 > 2a > a,$$

so that any $x > a+1$ does not belong to S_a. Thus $x \leq a+1$ for all $x \in S_a$. We have shown that S_a is bounded above with upper bound $a + 1$.

By the Dedekind completeness axiom there exists $M_a = \sup S_a$. Note that $M_a > 0$ since S_a has positive elements. To complete the proof we will show that $M_a^2 = a$.

By proof by contradiction, assume that $M_a^2 \neq a$.

Case I: $M_a^2 < a$. Put

$$\varepsilon = \min \left\{ \frac{a - M_a^2}{4M_a}, M_a \right\}.$$

Then $\varepsilon > 0$ and

$$\begin{aligned}(M_a+\varepsilon)^2 - a &= M_a^2 + 2M_a\varepsilon + \varepsilon^2 - a \\ &= M_a^2 - a + (2M_a+\varepsilon)\varepsilon \\ &\le M_a^2 - a + 3M_a\varepsilon \\ &\le M_a^2 - a + 3M_a\frac{a-M_a^2}{4M_a} \\ &= \frac{1}{4}(M_a^2 - a) < 0.\end{aligned}$$

Thus

$$(M_a+\varepsilon)^2 < a,$$

giving $M_a + \varepsilon \in S_a$, contradicting the fact that M_a is an upper bound of S_a.

Case II: $M_a^2 > a$. Put

$$\varepsilon = \frac{M_a^2 - a}{2M_a}.$$

Then $0 < \varepsilon < \frac{1}{2}M_a$ and

$$\begin{aligned}(M_a-\varepsilon)^2 - a &= M_a^2 - 2M_a\varepsilon + \varepsilon^2 - a \\ &> M_a^2 - a - 2M_a\varepsilon \\ &= M_a^2 - a - 2M_a\frac{M_a^2-a}{2M_a} \\ &= 0.\end{aligned}$$

Hence for all $x \ge M_a - \varepsilon > \frac{1}{2}M_a > 0$,

$$x^2 \ge (M_a-\varepsilon)^2 > a$$

so that any $x \ge M_a - \varepsilon$ does not belong to S_a. Hence $M_a - \varepsilon$ is an upper bound of S_a, contradicting the fact that M_a is the least upper bound of S_a.

So $M_a^2 \ne a$ is impossible, and $M_a^2 = a$ follows. $\square$

The completeness axiom can be thought of as ensuring that there are no 'gaps' on the real line.

Theorem 1.6 (Characterizations of the supremum) *Let S be a nonempty subset of $\mathbb{R}$. Let $M \in \mathbb{R}$. The following are equivalent:*

(a) $M = \sup S$*;*

(b) M *is an upper bound of* S*, and for each* $\varepsilon > 0$*, there is* $s \in S$ *such that* $M - \varepsilon < s$*;*

(c) M *is an upper bound of* S*, and for each* $x < M$*, there exists* $s \in S$ *such that* $x < s$*.*

Proof. (a) $\Rightarrow$ (b): Since (a) holds, M is the least upper bound and thus an upper bound.

Let $\varepsilon > 0$. Then $M - \varepsilon$ is not an upper bound of S since M is the least upper bound of S. Hence $x \leq M - \varepsilon$ does not hold for all $x \in S$, and therefore there must be $s \in S$ such that $M - \varepsilon < s$.

(b) $\Rightarrow$ (c): Let $x < M$. Then $\varepsilon = M - x > 0$, and by assumption there is $s \in S$ with $M - \varepsilon < s$. Then $x = M - \varepsilon < s$.

(c) $\Rightarrow$ (a): By assumption, M is an upper bound of S. Suppose that M is not the least upper bound. Then there is an upper bound L of S with $L < M$. By assumption (c), there is $s \in S$ such that $L < s$, contradicting that L is an upper bound of S. So M must be indeed the least upper bound of S. $\square$

There is an apparent asymmetry in the Dedekind completion. However, there is a version for infima, which is obtained by reflection. To this end, if S is a (nonempty) subset of $\mathbb{R}$ set

$$-S = \{-x : x \in S\}.$$

Note that because of $-(-x) = x$ this can also can be written as

$$-S = \{x \in \mathbb{R} : -x \in S\}.$$

Since $x \leq y \Leftrightarrow -x \geq -y$ it is easy to see that the following properties hold:

Proposition 1.4 *Let S be be nonempty subset of $\mathbb{R}$. Then*

(a) $-S$ *is bounded below if and only if* S *is bounded above,* $\inf(-S) = -\sup S$*, and if* $\max S$

exists, then $\min(-S)$ *exists and* $\min(-S) = -\max S$.

(b) $-S$ *is bounded above if and only if* S *is bounded below,* $\sup(-S) = -\inf S$*, and if* $\min S$ *exists, then* $\max(-S)$ *exists and* $\max(-S) = -\min S$.

(c) $-S$ *is bounded if and only if* S *is bounded.*

Thus the Dedekind completeness axiom immediately gives

Theorem 1.7 *Every nonempty subset of* $\mathbb{R}$ *which is bounded below has an infimum.*

Theorem 1.8 (Characterizations of the infimum) *Let* S *be a nonempty subset of* $\mathbb{R}$*. Let* $m \in \mathbb{R}$*. The following are equivalent:*

(a) $m = \inf S$*;*

(b) m *is a lower bound of* S*, and for each* $\varepsilon > 0$*, there is* $s \in S$ *such that* $s < m + \varepsilon$*;*

(c) m *is a lower bound of* S*, and for each* $x > m$*, there exists* $s \in S$ *such that* $s < x$.

Theorem 1.9 (Dedekind cut) *Let* A *and* B *be nonempty subsets of* $\mathbb{R}$ *such that*

(i) $A \cap B = \emptyset$,

(ii) $A \cup B = \mathbb{R}$,

(iii) $\forall\, a \in A\, \forall\, b \in B,\ a \leq b$.

Then there is $c \in \mathbb{R}$ *such that* $a \leq c \leq b$ *for all* $a \in A$ *and* $b \in B$.

Proof. A is nonempty and bounded above (any $b \in B$ is an upper bound of A), so $c = \sup A$ exists by the Dedekind completeness axiom, and $c \geq a$ for all $a \in A$ by definition of upper bound. (iii) says that each $b \in B$ is an upper bound of A, and hence $c \leq b$ since c is the least upper bound of A. □

Note. One can show that if $S \subset \mathbb{Z}$, $S \neq \emptyset$, and S is bounded below, then S has a minimum.

Theorem 1.10 (The Archimedean principle) *For each* $x \in \mathbb{R}$ *there is* $n \in \mathbb{N}$ *such that* $n > x$.

Proof Assume the Archimedian principle is false. Then there is $x \in \mathbb{R}$ such that $n \leq x$ for all $n \in \mathbb{N}$. That means, $\mathbb{N}$ is bounded above and therefore has a supremum M.

By Theorem 1.6 there is $n \in \mathbb{N}$ such that $M - 1 < n$. Then

$$n + 1 > (M - 1) + 1 = M$$

and $n + 1 \in \mathbb{N}$ contradict the fact that M is an upper bound of $\mathbb{N}$. Hence the Archimedian principle must be true. $\square$

Definition 1.5 *A subset S of $\mathbb{R}$ is said to be **dense** in $\mathbb{R}$ if for all $x, y \in \mathbb{R}$ with $x < y$ there is $s \in S$ such that $x < s < y$.*

Real numbers which are not rational numbers are called **irrational** numbers.

Note. 1. $\sqrt{2}$ is irrational (see tutorials).

2. $\mathbb{Q}$ is an ordered field, i. e., $\mathbb{Q}$ satisfies all axioms of $\mathbb{R}$ except the Dedekind completeness axiom (see tutorials).

3. If a is rational and b is irrational, then $a + b$ is irrational (see tutorials).

Theorem 1.11 *The set of rational numbers as well as the set of irrational numbers are dense in $\mathbb{R}$.*

Proof. In class.

Example 1.4 *Let $S = \{x \in \mathbb{Q} : x < \sqrt{2}\}$ and $T = \{x \in \mathbb{Q} : x \leq \sqrt{2}\}$. Then $S = T$, and S and T have no supremum in $\mathbb{Q}$.*

Example 1.5 (Bernoulli's inequality) *For all $x \in \mathbb{R}$ with $x \geq -1$ and all $n \in \mathbb{N}$, $(1 + x)^n \geq 1 + nx$.*

Chapter 2

Sequences

2.1 Definitions, Examples and Theorems

Definition 2.1 *A* ***(real) sequence*** *is an ordered list of infinitely many real numbers*

$$a_1, a_2, a_3, a_4, \ldots .$$

We usually denote a sequence by $(a_n) = a_1, a_2, a_3, \ldots .$

The number a_n is called the n-th term of the sequence. The subscript n is called the index.

Note. The index of the sequence does not have to start at 1, and therefore one also writes $(a_n)_{n=n_0}^{\infty}$, where n_0 can be any integer, e. g., $(2n-3)_{n=1}^{\infty}$, $(2n-3)_{n=0}^{\infty}$, $(2n-3)_{n=-5}^{\infty}$.

Note. (i) A sequence can be identified with a function on the (positive) integers:

$$a_n = f(n).$$

(ii) A sequence can be plotted as points of the real axis or as the graph of the function, see (i).

Below is a plot for the sequence $\left(\dfrac{n}{n+1}\right)_{n=1}^{\infty}$.

Definition 2.2 *Let (a_n) be a sequence and $L \in \mathbb{R}$.*

1. The statement 'a_n tends to L as n tends to infinity', which we write as '$a_n \to L$ as $n \to \infty$', is defined by:
$\forall \varepsilon > 0 \; \exists K \in \mathbb{R} \; \forall n \in \mathbb{N}, n \geq K, |a_n - L| < \varepsilon.$

2. If $a_n \to L$ as $n \to \infty$, we say that (a_n) converges to L, and we also write $\lim_{n\to\infty} a_n = L$.

*3. The sequence (a_n) is said to be **convergent** if it converges to some real number. Otherwise, the sequence (a_n) is said to be **divergent**.*

Note. 1. It is useful to note that $|a_n - L| < \varepsilon$ is equivalent to $L - \varepsilon < a_n < L + \varepsilon$.

2. The number K depends on ε. We may write K_ε to emphasize the dependence on ε.

3. If one wants to prove convergence of a sequence from first principles, then one has to 'guess' a limit L and then prove that it is indeed the limit.

4. The 'first few terms' do not matter for convergence and the limit. That is, the sequence $(a_n)_{n=n_0}^{\infty}$ converges if and only if the sequence $(a_n)_{n=m_0}^{\infty}$ converges, and their limits coincide.

5. By the Archimedean principle, there are only finitely many natural numbers such that $n < K$, and one may also assume $K \in \mathbb{N}$, without loss of generality.

Example 2.1 *Prove that the sequence $(a_n) = \left(\frac{n}{n+1}\right)$ converges and find its limit.*

Lemma 2.1 *If $L, M \in \mathbb{R}$ such that $|L - M| < \varepsilon$ for all $\varepsilon > 0$, then $L = M$.*

Proof. Assume $L \neq M$. Then either $L < M$ or $L > M$. But

$$L < M \Rightarrow |L - M| = M - L > 0,$$
$$M < L \Rightarrow |L - M| = L - M > 0,$$

so that $|L - M| > 0$. Then
$$0 < \frac{|L - M|}{2} < |L - M|,$$
which contradicts $|L - M| < \varepsilon$ for $\varepsilon = \frac{|L-M|}{2}$. Hence the assumption $L \neq M$ must be false, and $L = M$ follows. □

Theorem 2.1 *If the sequence (a_n) converges, then its limit is unique.*

Proof. Assume that (a_n) converges to L and M. Let $\varepsilon > 0$. Then there are numbers k_L and k_M such that

(i) $|a_n - L| < \frac{\varepsilon}{2}$ if $n \geq k_L$,

(ii) $|a_n - M| < \frac{\varepsilon}{2}$ if $n \geq k_M$.

Put $K = \max\{k_L, k_M\}$. For positive integers $n \geq K$ we have $n \geq k_L$ and $n \geq k_M$ and therefore

$$|L - M| = |(L - a_n) + (a_n - M)| \leq |a_n - L| + |a_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

By Lemma 2.1, $L = M$. $\square$

We are now going to prove the rules you have learnt in first year.

Theorem 2.2 (Limit Laws) *Let $c \in \mathbb{R}$ and suppose that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = M$ both exist. Then*

(a) $\lim\limits_{n\to\infty} c = c$.

(b) $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n = L + M$.

(c) $\lim\limits_{n\to\infty} (ca_n) = c \lim\limits_{n\to\infty} a_n = cL$.

(d) $\lim\limits_{n\to\infty} (a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right)\left(\lim\limits_{n\to\infty} b_n\right) = LM$.

(e) If $M \neq 0$, $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \frac{L}{M}$.

(f) If $L \neq 0$ *and* $M = 0$, *then* $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ *does not exist.*

(g) If $k \in \mathbb{Z}^+$, $\lim\limits_{n\to\infty} a_n^k = \left(\lim\limits_{n\to\infty} a_n\right)^k = L^k$.

(h) If $k \in \mathbb{Z}^+$, $\lim\limits_{n\to\infty} \sqrt[k]{a_n} = \sqrt[k]{\lim\limits_{n\to\infty} a_n} = \sqrt[k]{L}$. *If k is even, we assume $a_n \geq 0$ and $L \geq 0$.*

(i) If $\lim\limits_{n\to\infty} |a_n| = 0$, *then* $\lim\limits_{n\to\infty} a_n = 0$.

Proof. (a) For all $\varepsilon > 0$ and all indices n, with $a_n = c$, $|a_n - c| = |c - c| = 0 < \varepsilon$.

(b) Let $\varepsilon > 0$. Then there are numbers k_L and k_M such that

(i) $|a_n - L| < \frac{\varepsilon}{2}$ if $n \geq k_L$,

(ii) $|b_n - M| < \frac{\varepsilon}{2}$ if $n \geq k_M$.

Put $K = \max\{k_L, k_M\}$. For positive integers $n \geq K$ we have $n \geq k_L$ and $n \geq k_M$ and therefore

$$|(a_n + b_n) - (L + M)| = |(a_n - L) + (b_n - M)| \leq |a_n - L| + |b_n - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $(a_n + b_n)$ converges to $L + M$.

(c) Let $\varepsilon > 0$. Then there is a number K such that
$|a_n - L| < \frac{\varepsilon}{1+|c|}$ if $n \geq K$.
For positive integers $n \geq K$ we have

$$|ca_n - cL| = |c(a_n - L)| = |c|\,|a_n - L| \leq |c|\frac{\varepsilon}{1 + |c|} < \varepsilon.$$

(d) We consider 2 cases: one special case, to which then the general case is reduced.
Case I: $L = M = 0$. Let $\varepsilon > 0$. Then there are numbers k_L and k_M such that

(i) $|a_n - 0| < \varepsilon$ if $n \geq k_L$,

(ii) $|b_n - 0| < 1$ if $n \geq k_M$.

Put $K = \max\{k_L, k_M\}$. For positive integers $n \geq K$ we have $n \geq k_L$ and $n \geq k_M$ and therefore

$$|a_n b_n| = |a_n|\,|b_n| < \varepsilon \cdot 1 = \varepsilon.$$

Case II: L and M are arbitrary. Then

$$a_n b_n = (a_n - L)(b_n - M) + L(b_n - M) + a_n M.$$

By (a) and (b), $(a_n - L) \to 0$ and $(b_n - M) \to 0$ as $n \to \infty$, and by (b), (c), and Case I, it follows that $(a_n b_n)$ converges with

$$\lim_{n\to\infty} a_n b_n = 0 + L \cdot 0 + LM = LM.$$

(e) First consider $a_n = 1$. Then

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| = \frac{|M - b_n|}{|b_n M|}.$$

We must assure that b_n stays away from 0, and hence we estimate

$$|M| = |b_n + M - b_n| \leq |b_n| + |b_n - M|,$$

which gives

$$|b_n| \geq |M| - |b_n - M|.$$

Since $b_n \to M \neq 0$, there is k_1 such that $|b_n - M| < \frac{|M|}{2}$ for $n \geq k_1$, and hence

$$|b_n| \geq |M| - \frac{|M|}{2} = \frac{|M|}{2}.$$

If we therefore let $\varepsilon > 0$ and choose K such that

$$|b_n - M| < \min\left\{\frac{|M|}{2}, \frac{M^2}{2}\varepsilon\right\}$$

for $n \geq K$, it follows that

$$\left|\frac{1}{b_n} - \frac{1}{M}\right| = \frac{|M - b_n|}{|b_n|\,|M|} \leq |M - b_n|\frac{2}{|M|^2} < \frac{|M|^2}{2}\varepsilon\,\frac{2}{|M|^2} = \varepsilon.$$

The general case now follows with (d):

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} \frac{1}{b_n} = L \cdot \frac{1}{M}.$$

(f) Assume that $P = \lim\limits_{n\to\infty} \frac{a_n}{b_n}$ exists. Then, by (d),

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{a_n}{b_n}\, b_n\right) = \lim_{n\to\infty} \frac{a_n}{b_n} \cdot \lim_{n\to\infty} b_n = P \cdot M = P \cdot 0 = 0,$$

which contradicts $L \neq 0$.

(g) Follows from (d) by induction.

(h) We will only prove the case $k = 2$ as we have not yet established the existence of the k-th root.

If $L = 0$, let $\varepsilon > 0$ and choose K such that $a_n < \varepsilon^2$ for $n \geq K$. Then $\sqrt{a_n} < \varepsilon$ for these n, and $\lim\limits_{n\to\infty} \sqrt{a_n} = 0 = \sqrt{L}$.

If $L > 0$, then

$$\left|\sqrt{a_n} - \sqrt{L}\right| = \left|\frac{a_n - L}{\sqrt{a_n} + \sqrt{L}}\right| = \frac{|a_n - L|}{\sqrt{a_n} + \sqrt{L}} \leq \frac{1}{\sqrt{L}}|a_n - L|,$$

and choosing K such that $|a_n - L| < \sqrt{L}\varepsilon$ for $n \geq K$, it follows that $\left|\sqrt{a_n} - \sqrt{L}\right| < \varepsilon$ for $n \geq K$.

(i) is trivial. $\square$

Theorem 2.3 (Sandwich Theorem) *If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.*

Proof. Let $\varepsilon > 0$ and choose k_1 and k_2 such that $|a_n - L| < \varepsilon$ if $n \geq k_1$ and $|c_n - L| < \varepsilon$ if $n \geq k_2$. In particular, for $n \geq K = \max\{k_1, k_2\}$,

$$L - \varepsilon < a_n < L + \varepsilon, \quad L - \varepsilon < c_n < L + \varepsilon$$

gives

$$L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon.$$

Hence $|b_n - L| < \varepsilon$ if $n \geq K$. $\square$

Theorem 2.4 *Let $x \in \mathbb{R}$. Then x^n converges if and only if $-1 < x \leq 1$.*

Proof. For $x = 1$, $1^n = 1$, so that $\lim_{n\to\infty} 1^n = 1$ by Theorem 2.2 (a).

Let $0 < x < 1$ and let $\varepsilon > 0$. Then $0 < 1 < \frac{1}{x}$ and therefore

$$y := \frac{1}{x} - 1 > 0.$$

Then

$$\frac{1}{x^n} = \left(\frac{1}{x}\right)^n = (1 + y)^n \geq 1 + ny$$

by Bernoulli's inequality. Put $K = \dfrac{1-\varepsilon}{y\varepsilon}$. Then, for $n > K$,

$$0 < x^n \leq \frac{1}{1 + ny} < \frac{1}{1 + \frac{1-\varepsilon}{\varepsilon}} = \varepsilon.$$

Hence $x^n \to 0$ as $n \to \infty$. Now let $x > 1$. If x^n would converge to some L as $n \to \infty$, then

$$1 = \lim_{n\to\infty} 1^n = \lim \left(\frac{1}{x}\right)^n \cdot \lim_{n\to\infty} x^n = 0 \cdot L = 0,$$

which is impossible.

The remaining cases are left as an exercise. $\square$

Theorem 2.5 *If $r > 0$, then*

$$\lim_{n\to\infty} \frac{1}{n^r} = 0.$$

Proof. We are only going to prove the case that r is a positive integer.

Let $r = 1$ and choose $\varepsilon > 0$. Let $K = \frac{2}{\varepsilon}$. Then, for $n \geq K$,

$$0 < \frac{1}{n} \leq \frac{1}{K} = \frac{\varepsilon}{2} < \varepsilon,$$

whence $\left|\frac{1}{n}\right| < \varepsilon$.

Now assume the statement holds for an integer $r > 0$. Then

$$\lim_{n\to\infty} \frac{1}{n^{r+1}} = \lim_{n\to\infty} \frac{1}{n} \cdot \lim_{n\to\infty} \frac{1}{n^r} = 0 \cdot 0 = 0.$$

So the result follows for all positive integers by induction. $\square$

2.2 Bounded and Monotonic Sequences

Definition 2.3 *1. A sequence (a_n) is said to be **bounded above** if there is a number $M \in \mathbb{R}$ such that $a_n \leq M$ for all indices n.*
*2. A sequence (a_n) is said to be **bounded below** if there is a number $m \in \mathbb{R}$ such that $a_n \geq m$ for all indices n.*
*3. A sequence (a_n) is said to be **bounded** if it is a bounded above and bounded below.*

Note. A sequence $(a_n)_{n=n_0}^{\infty}$ is bounded (above, below) if and only if the set $\{a_n : n \in \mathbb{Z},\, n \geq n_0\}$ is bounded (above, below).

Definition 2.4 *A sequence* $\{a_n\}_{n=1}^{\infty}$ *is called*

increasing *if* $a_n \le a_{n+1}$ *for all indices* n*,*

strictly increasing *if* $a_n < a_{n+1}$ *for all indices* n*,*

decreasing *if* $a_n \ge a_{n+1}$ *for all indices* n*,*

strictly decreasing *if* $a_n > a_{n+1}$ *for all indices* n*,*

monotonic *if it is either increasing or decreasing,*

strictly monotonic *if it is either strictly increasing or strictly decreasing.*

Theorem 2.6 *Every convergent sequence is bounded.*

Proof. Here we use the fact, which is easy to prove (by induction), that a set of the form $\{a_n : n \in \mathbb{Z},\, n_0 \le n \le n_1\}$ is bounded (and indeed has minimum and maximum). We also use that the union of two bounded sets is bounded.
Since (a_n) converges, there are L and k such that $|a_n - L| < 1$ for all $n \ge k$. Hence $\{a_n : n < k\}$ and $\{a_n : n \ge k\} \subset (L-1, L+1)$ are bounded. So the sequence is bounded by the above note. $\square$

Although each unbounded sequence diverges, there are some divergent sequences which have a limiting behaviour which we want to exploit. Recall that we use the notation '$n \to \infty$' in the definition of the limit. This extends to '$a_n \to \infty$', and we have therefore the

Definition 2.5 *1. We say that* a_n *tends to infinity as* n *tends to infinity and write* $a_n \to \infty$ *as* $n \to \infty$ *if for every* $A \in \mathbb{R}$ *there is* $K \in \mathbb{R}$ *such that* $a_n > A$ *for all* $n \ge K$*.*

2. We say that a_n *tends to minus infinity as* n *tends to infinity and write* $a_n \to -\infty$ *as* $n \to \infty$ *if for every* $A \in \mathbb{R}$ *there is* $K \in \mathbb{R}$ *such that* $a_n < A$ *for all* $n \ge K$*.*

Notes. 1. For the definition of $a_n \to \infty(-\infty)$ as $n \to \infty$, we may restrict A to $A > A_0$ ($A < A_0$) for suitable numbers A_0.
2. For $a_n \to \infty$ as $n \to \infty$ we also write $\lim\limits_{n\to\infty} a_n = \infty$ and say that (a_n) diverges to ∞.

3. For $a_n \to -\infty$ as $n \to \infty$ we also write $\lim_{n\to\infty} a_n = -\infty$ and say that (a_n) diverges to $-\infty$.

Example 2.2 *1. Let $a_n = (-1)^n$. Then (a_n) is bounded but not convergent.*

2. Let $a_n = n$. Then (a_n) is unbounded (by the Archimedean principle). Hence (a_n) diverges.

Example 2.3 *Prove that $n^2 - n^3 + 10 \to -\infty$ as $n \to \infty$.*

Theorem 2.7 1. *Let (a_n) be a sequence with $a_n > 0$ for all indices n. Then*

$$\lim_{n\to\infty} a_n = \infty \Leftrightarrow \lim_{n\to\infty} \frac{1}{a_n} = 0.$$

2. *Let (a_n) be a sequence with $a_n < 0$ for all indices n. Then*

$$\lim_{n\to\infty} a_n = -\infty \Leftrightarrow \lim_{n\to\infty} \frac{1}{a_n} = 0.$$

Proof. 1. Assume that $\lim_{n\to\infty} a_n = \infty$. To show that $\lim_{n\to\infty} \frac{1}{a_n} = 0$ let $\varepsilon > 0$. Put $A = \frac{1}{\varepsilon}$. Then there is K such that $0 < A < a_n$ for all $n \geq K$. Then

$$0 < \frac{1}{a_n} < \frac{1}{A} = \varepsilon$$

for these n, which shows that $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

Conversely, assume that $\lim_{n\to\infty} \frac{1}{a_n} = 0$. Let $A \in \mathbb{R}$ and put $A_0 = \max\{A, 1\}$. Then $A_0 > 0$ and put $\varepsilon = \frac{1}{A_0} > 0$. Hence there is $K \in \mathbb{R}$ such that $\frac{1}{a_n} = \left|\frac{1}{a_n}\right| < \varepsilon$ for all $n \geq K$. This give $a_n > \frac{1}{\varepsilon} = A_0 \geq A$ for all $n \geq K$.

The proof of part 2 is similar. $\square$

Apart from this rule, there are more rules for infinite limits. Some of these rules are listed below. [Note to lecturers: do not write this down, just refer the students to the notes.]

Theorem 2.8 *Consider $k \in \mathbb{R}$ and sequences with the following properties as $n \to \infty$:*

$a_n \to \infty, \quad b_n \to \infty, \quad c_n \to c \in \mathbb{R}, \quad d_n \to -\infty.$

Then, as $n \to \infty$,

(a) $ka_n \to \begin{cases} \infty & \text{if } k > 0, \\ -\infty & \text{if } k < 0, \\ 0 & \text{if } k = 0. \end{cases}$

(b) $a_n + b_n \to \infty$,

(c) $a_n + c_n \to \infty$,

(d) $-d_n \to \infty$,

(e) $a_n c_n \to \begin{cases} \infty & \text{if } c > 0, \\ -\infty & \text{if } c < 0. \end{cases}$

(f) $a_n b_n \to \infty$.

Proof. See tutorials.

It is convenient to extend the notions of supremum and infimum to unbounded set.

Definition 2.6 *Let* $A \subset \mathbb{R}$ *be nonempty. If* A *is not bounded above, we write* $\sup A = \infty$, *and if* A *is not bounded below, we write* $\inf A = -\infty$.

We know that not every sequence converges, and it may be hard to decide if a sequence converges. However, the situation is different with monotonic sequences:

Theorem 2.9 1. Let (a_n) be an increasing sequence. If (a_n) is bounded, then (a_n) converges, and
$\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$. If (a_n) is not bounded, then (a_n) diverges to ∞.
2. Let (a_n) be a decreasing sequence. If (a_n) is bounded, then (a_n) converges, and
$\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$. If (a_n) is not bounded, then (a_n) diverges to $-\infty$.

Proof. 1. Assume (a_n) is bounded. Put $L = \sup\{a_n : n \in \mathbb{N}\}$ and let $\varepsilon > 0$.

We must show that there is $K \in \mathbb{N}$ such that $n \geq K$ gives $L - \varepsilon < a_n < L + \varepsilon$.

Indeed by the definition of the supremum we have $a_n \leq L < L+\varepsilon$ for all n and by Theorem 1.6 there is K such that $L - \varepsilon < a_K$. Then we have for all $n \geq K$ that

$$L - \varepsilon < a_K \leq a_n.$$

Hence $L - \varepsilon < a_n < L + \varepsilon$ for all $n \geq K$, which proves $a_n \to L$ as $n \to \infty$.

Now assume that (a_n) is not bounded. Since $a_0 \leq a_n$ for all n, (a_n) is bounded below. Hence (a_n) is not bounded above. Let $A \in \mathbb{R}$. Then there is an index K such that $a_K > A$, and thus $a_n \geq a_K > A$ for all $n \geq K$. $\square$

Example 2.4 *Let* $a_n = \left(1 + \frac{1}{n}\right)^n$ *and* $b_n = \left(1 + \frac{1}{n}\right)^{n+1}$. *Then* $a_n < b_n$ *for all* $n \in \mathbb{N}$, (a_n) *is an increasing sequence,* (b_n) *is a decreasing sequence, both sequences converge to the same number, and the limit is denoted by* e, *Euler's number.*

Definition 2.7 *1. With every sequence* (a_n) *which is bounded above, we can associate the sequence of numbers*

$$\sup\{a_k : k \geq n\}.$$

Since $\{a_k : k \geq n\} = \{a_n\} \cup \{a_k : k \geq n+1\} \supset \{a_k : k \geq n+1\}$, *this sequence is decreasing, and we denote its limit by*

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\}.$$

If (a_n) *is not bounded above, we put* $\limsup\limits_{n\to\infty} a_n = \infty$.

2. Similarly, if the sequence (a_n) *is bounded below, the sequence of numbers*

$$\inf\{a_k : k \geq n\}$$

is increasing, and we denote its limit by

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\}.$$

If (a_n) *is not bounded below, we put* $\liminf\limits_{n\to\infty} a_n = -\infty$.

Note. 1. Since $\inf S \leq \sup S$ for every nonempty set S, it follows that $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$, where we write $-\infty < x$ and $x < \infty$ for each $x \in \mathbb{R}$.

2. The sequence (a_n) is bounded if and only if both $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ are real numbers (also called finite).

We have now the following characterization of the convergence of a sequence and its limit when it exists.

Theorem 2.10 1. *The sequence (a_n) converges if and only if $\liminf_{n\to\infty} a_n$ and $\limsup_{n\to\infty} a_n$ are finite and equal, and then*

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

2. $\lim_{n\to\infty} a_n = \infty \Leftrightarrow \liminf_{n\to\infty} a_n = \infty$ *and* $\limsup_{n\to\infty} a_n = \infty$.

3. $\lim_{n\to\infty} a_n = -\infty \Leftrightarrow \liminf_{n\to\infty} a_n = -\infty$ *and* $\limsup_{n\to\infty} a_n = -\infty$.

Proof. For a sequence (a_n), denote $b_n = \inf\{a_k : k \geq n\}$ and $c_n = \sup\{a_k : k \geq n\}$. Note that $b_n \leq a_n \leq c_n$.

1. Assume that (a_n) converges to L. Let $\varepsilon > 0$. Then there is $K \in \mathbb{N}$ such that for all $k \geq K$, $L - \frac{\varepsilon}{3} < a_k < L + \frac{\varepsilon}{3}$. Hence, for $n \geq K$,

$$L - \frac{\varepsilon}{3} \leq b_n \leq a_n \leq c_n \leq L + \frac{\varepsilon}{3}.$$

Since (b_n) is increasing and (c_n) is decreasing, we have

$$L - \frac{\varepsilon}{3} \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq L + \frac{\varepsilon}{3}.$$

Hence

$$0 \leq \limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n \leq \frac{2\varepsilon}{3} < \varepsilon.$$

From Lemma 2.1 we obtain that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$.

Conversely, if $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$, then it follows from $b_n \leq a_n \leq c_n$ and the Sandwich Theorem that (a_n) converges and that

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n = \limsup_{n\to\infty} a_n.$$

2., 3. Exercise. $\square$

Another important concept is that of a Cauchy sequence:

Definition 2.8 (Cauchy sequence) *A sequence (a_n) is called a Cauchy sequence if for all $\varepsilon > 0$ there is $K \in \mathbb{R}$ such that for all $n, m \in \mathbb{N}$ with $n, m \geq K$, $|a_n - a_m| < \varepsilon$.*

Theorem 2.11 *A sequence (a_n) converges if and only if it is a Cauchy sequence.*

Proof. Let (a_n) be a convergent sequence with limit L. Let $\varepsilon > 0$ and let K such that $|a_n - L| < \frac{\varepsilon}{2}$ for $n \geq K$. Then it follows for $n, m \geq K$ that

$$|a_n - a_m| = |(a_n - L) - (a_m - L)| \leq |a_n - L| + |a_m - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Conversely, assume that (a_n) is a Cauchy sequence. Let $\varepsilon > 0$ and choose K such that $|a_n - a_m| < \frac{\varepsilon}{3}$ for all $m, n \geq K$. Then

$$a_m - \frac{\varepsilon}{3} < a_n < a_m + \frac{\varepsilon}{3}.$$

In particular, choosing $m = K$,

$$\{a_n : n \geq K\} \subset \left(a_K - \frac{\varepsilon}{3}, a_K + \frac{\varepsilon}{3}\right).$$

Thus (a_n) is bounded and

$$a_K - \frac{\varepsilon}{3} \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq a_K + \frac{\varepsilon}{3} \text{ for } n \geq K,$$

which gives

$$0 \leq \limsup_{n\to\infty} a_n - \liminf_{n\to\infty} a_n \leq \left(a_K + \frac{\varepsilon}{3}\right) - \left(a_K - \frac{\varepsilon}{3}\right) = \frac{2\varepsilon}{3} < \varepsilon.$$

By Lemma 2.1, $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$, and an application of Theorem 2.10, part 1, completes the proof. $\square$

Chapter 3

Series

3.1 Definitions and Examples

Given a sequence $\{a_n\}_{n=1}^{\infty}$ of real numbers, the symbol

$$\sum_{n=1}^{\infty} a_n$$

is called a **series** (of real numbers).

Definition 3.1 *Let* $\sum_{n=1}^{\infty} a_n$ *be a series.*

1. The number $s_n = \sum_{i=1}^{n} a_i$ *is called the* n***-th partial sum of the series****.*

2. The series $\sum_{n=1}^{\infty} a_n$ *is said to* ***converge*** *if the sequence* (s_n) *converges.*
In this case, the number $s = \lim_{n\to\infty} s_n$ *is called the* ***sum of the series*** *and we write*

$$\sum_{n=1}^{\infty} a_n = s$$

A series which does not converge is said to ***diverge*** *or be* ***divergent****.*

Note. 1. Observe that $\sum_{n=1}^{\infty} a_n$ denotes a series (convergent or divergent) as well as its sum if it converges.

2. A series does not have to start at $n = 1$. The change of notation is obvious for other starting indices.

Example 3.1 (See Calculus I) *[Note to lecturers: Students have seen this in Calculus I, so one does not need to write down much.] Consider the **geometric series***

$$\sum_{n=0}^{\infty} ar^n, \quad a \neq 0,\ r \in \mathbb{R}.$$

Recall the partial sums for $r \neq 1$:

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^n, \\ rs_n &= \phantom{a + {}} ar + ar^2 + \cdots + ar^n + ar^{n+1}. \end{aligned}$$

Subtracting these equations gives

$$s_n - rs_n = a - ar^{n+1},$$

and so

$$s_n = \frac{a(1 - r^{n+1})}{1 - r}.$$

By Theorem 2.4, (s_n) and thus the geometric series converges if $|r| < 1$, with

$$\sum_{n=0}^{\infty} ar^n = \lim_{n\to\infty} s_n = \frac{a}{1-r},$$

and diverges if $|r| > 1$ or $r = -1$. Finally, for $r = 1$, $s_n = a(n + 1)$, and thus $s_n \to \infty$ as $n \to \infty$.

Thus we have shown

Theorem 3.1 *The geometric series*

$$\sum_{n=0}^{\infty} ar^n, \quad a \neq 0,$$

is convergent if $|r| < 1$, with sum

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

If $|r| \geq 1$, the geometric series diverges.

Example 3.2 *Is the series* $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ *convergent or divergent?*

Solution. The n-th term of the series is obviously of the form ar^n:

$$a_n = 2^{2n}3^{1-n} = \frac{3 \cdot 4^n}{3^n} = 3\left(\frac{4}{3}\right)^n .$$

So $r = \frac{4}{3} > 1$, and the series diverges.

Example 3.3 *Write the number* $5.4\overline{417} = 5.4417417417\ldots$ *as a ratio of integers.*

Solution. Write

$$5.4\overline{417} = 5.4 + \frac{417}{10^4} + \frac{417}{10^7} + \frac{417}{10^{10}} + \ldots .$$

Starting with the second term we have a geometric series with $a = \frac{417}{10^{-4}}$ and $r = 10^{-3}$. Hence the series converges, and

$$\begin{aligned} 5.4\overline{417} &= 5.4 + \frac{\frac{417}{10^{-4}}}{1-10^{-3}} = 5.4 + \frac{417}{10^4-10} \\ &= \frac{54}{10} + \frac{417}{9990} = \frac{539877}{9990} = \frac{179959}{3330} . \end{aligned}$$

The following theorem has been proved in Calculus I.

Theorem 3.2 *If the series* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

Proof. Let

$$s_n = a_1 + a_2 + \cdots + a_n .$$

Then

$$a_n = s_n - s_{n-1} .$$

Since $\sum_{n=1}^{\infty} a_n$ converges,

$$\lim_{n\to\infty} s_n = s$$

exists. Since also $n-1 \to \infty$ as $n \to \infty$,

$$\lim_{n\to\infty} s_{n-1} = s.$$

Hence

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0.$$

□

The contrapositive statement to Theorem 3.2 is very useful:

Theorem 3.3 (Test for Divergence) *If* $\lim_{n\to\infty} a_n$ *does not exist or if* $\lim_{n\to\infty} a_n \neq 0$, *then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Note. From $\lim_{n\to\infty} a_n = 0$ nothing can be concluded about the convergence of the series $\sum_{n=1}^{\infty} a_n$.

From Theorem 2.2 we immediately infer

Theorem 3.4 (Sum Laws) *Let* $c \in \mathbb{R}$ *and suppose that* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *both converge. Then also* $\sum_{n=1}^{\infty}(ca_n)$, $\sum_{n=1}^{\infty}(a_n + b_n)$ *and* $\sum_{n=1}^{\infty}(a_n - b_n)$ *converge, and*

1. $\sum_{n=1}^{\infty}(ca_n) = c\sum_{n=1}^{\infty} a_n$,
2. $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$,
3. $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$,

Example 3.4 *Find the sum of the series* $\sum_{n=1}^{\infty}\left(\frac{5}{n(n+1)} + \frac{1}{5^n}\right)$.

Theorem 3.5 *The series* $\sum_{n=1}^{\infty} a_n$ *converges if and only if for each* $\varepsilon > 0$ *there is* $K \in \mathbb{N}$ *such that for all* $m \geq k \geq K$, $\left|\sum_{n=k}^{m} a_n\right| < \varepsilon$.

Proof. Let $s_k = \sum_{n=1}^{k} a_n$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} a_n \text{ converges } &\Leftrightarrow (s_n)_{n=1}^{\infty} \text{ converges} \qquad \text{(by definition)} \\
&\Leftrightarrow \forall\, \varepsilon > 0\ \exists K \in \mathbb{N}\ \forall m \geq k \geq K,\ |s_m - s_{k-1}| < \varepsilon \qquad \text{(by Theorem 2.11)} \\
&\Leftrightarrow \forall\, \varepsilon > 0\ \exists K \in \mathbb{N}\ \forall m \geq k \geq K,\ \left|\sum_{n=k}^{m} a_n\right| < \varepsilon.
\end{aligned}$$

□

3.2 Convergence Tests

Lemma 3.1 *Let $a_n \geq 0$ for all $n \in \mathbb{N}^*$ and let $s_k = \sum_{n=1}^{k} a_n$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if (s_n) is bounded.*

Proof. Since $s_{n+1} = s_n + a_{n+1} \geq s_n$, it follows that (s_n) is an increasing sequence. By Theorem 2.9, this sequence and hence the series converges if and only if the sequence (s_n) is bounded. □

Theorem 3.6 (Comparison Test) *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series with nonnegative terms and assume that $a_n \leq b_n$ for all $n \in \mathbb{N}^*$.*
(i) *If $\sum_{n=1}^{\infty} b_n$ converges, then also $\sum_{n=1}^{\infty} a_n$ converges.*
(ii) *If $\sum_{n=1}^{\infty} a_n$ diverges, then also $\sum_{n=1}^{\infty} b_n$ diverges.*

Proof. Let $s_k = \sum_{n=1}^{k} a_n$ and $t_k = \sum_{n=1}^{k} b_n$. Then both (s_k) and (t_k) are increasing sequences with $s_k \leq t_k$. Hence, if $\sum_{n=1}^{\infty} b_n$ converges, then (t_k) is bounded, say $t_k \leq M$ for all $k \in \mathbb{N}$, and $s_k \leq t_k \leq M$ for all $k \in \mathbb{N}$. Hence (s_k) is a bounded sequence and thus converges by Lemma 3.1.
(ii) is the contrapositive of (i). □

Example 3.5 *Test the series* $\sum_{n=1}^{\infty} \frac{\sin^2 n + 10}{n + 2^n}$ *for convergence.*

Definition 3.2 *1. A series* $\sum_{n=1}^{\infty} a_n$ *is called* ***absolutely convergent*** *if the series of its absolute values* $\sum_{n=1}^{\infty} |a_n|$ *converges.*

2. A series $\sum_{n=1}^{\infty} a_n$ *is called* ***conditionally convergent*** *if it is convergent but not absolutely convergent.*

Theorem 3.7 *Every absolutely convergent series is convergent.*

Proof. Let $\sum_{n=1}^{\infty} a_n$ be absolutely convergent. Let $\varepsilon > 0$. Then, by Theorem 3.5, there is $K \in \mathbb{N}$ such that for all $m \geq k \geq K$, $\sum_{n=k}^{m} |a_n| < \varepsilon$. Because of

$$\left| \sum_{n=k}^{m} a_n \right| \leq \sum_{n=k}^{m} |a_n|$$

it follows that $\left| \sum_{n=k}^{m} a_n \right| < \varepsilon$ for these k, m, and therefore $\sum_{n=1}^{\infty} a_n$ converges by Theorem 3.5.

Alternative proof. By assumption and Theorem 3.4, both $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} 2|a_n|$ converge. From

$$0 \leq a_n + |a_n| \leq 2|a_n|$$

and the comparison test, it follows that also $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges. Again by Theorem 3.4 it follows that

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} ((a_n + |a_n|) - |a_n|)$$

converges. $\square$

Recall from Calculus I that there are convergent series which are not absolutely convergent.

Definition 3.3 *An **alternating series** is a series of the form*

$$\sum_{n=1}^{\infty}(-1)^n b_n \qquad \text{or} \qquad \sum_{n=1}^{\infty}(-1)^{n-1}b_n \qquad \text{with } b_n \geq 0.$$

Theorem 3.8 (Alternating series test) *If the alternating series* $\sum_{n=1}^{\infty}(-1)^n b_n$ *or* $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ *satisfies*

(i) $b_n \geq b_{n+1}$ *for all* n,

(ii) $\lim_{n\to\infty} b_n = 0$,

then the series converges.

Proof. For $k \in \mathbb{N}^*$ and $m \in \mathbb{N}$ we have

$$\begin{aligned}(-1)^k \sum_{n=k}^{k+2m}(-1)^n b_n &= (b_k - b_{k+1}) + (b_{k+2} - b_{k+3}) + \cdots + (b_{k+2m-2} - b_{k+2m-1}) + b_{k+2m} \\ &= b_k - (b_{k+1} - b_{k+2}) - \cdots - (b_{k+2m-1} - b_{k+2m}).\end{aligned}$$

Hence

$$0 \leq b_{k+2m} \leq (-1)^k \sum_{n=k}^{k+2m}(-1)^n b_n \leq b_k.$$

Similarly,

$$0 \leq (-1)^k \sum_{n=k}^{k+2m+1}(-1)^n b_n \leq b_k - b_{k+2m+1} \leq b_k.$$

Now let $\varepsilon > 0$. Since $b_k \to 0$ as $k \to \infty$, there is $K \in \mathbb{N}$ such that $b_K < \varepsilon$. Hence for all $l \geq k \geq K$:

$$\left|\sum_{n=k}^{l}(-1)^n b_n\right| \leq b_k \leq b_K < \varepsilon.$$

Hence the alternating series converges. $\square$

Note. $\lim_{n\to\infty} b_n = 0$ is necessary by Theorem 3.3 since $\lim_{n\to\infty}(-1)^n b_n = 0 \Leftrightarrow \lim_{n\to\infty} b_n = 0 \Leftrightarrow \lim_{n\to\infty}(-1)^{n-1}b_n = 0$.

Example 3.6 *The alternating series* $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{3n}{4n-1}$ *does not converge since*

$$\lim_{n\to\infty}\frac{3n}{4n-1} = \frac{3}{4} \neq 0$$

and thus (ii) is not satisfied, which is necessary for convergence. See the note following the statement of the Alternating Series Test.

Example 3.7 *Find whether the series* $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n^2}{n^3+1}$ *is convergent.*

Theorem 3.9 (Ratio Test) (i) *If* $\limsup_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

(ii) *If* $\liminf_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = l > 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof. Note that the ratio test assumes $a_n \neq 0$ for all $n \in \mathbb{N}$.

(i) Let $\varepsilon > 0$ such that $L + \varepsilon < 1$.

Then there is $K \in \mathbb{N}$ such that $\left|\frac{a_{n+1}}{a_n}\right| < L + \varepsilon$ for all $n \geq K$. Hence for $m > K$:

$$|a_m| = |a_K|\,\left|\frac{a_{K+1}}{a_K}\right| \cdot \cdots \cdot \left|\frac{a_m}{a_{m-1}}\right| < |a_K|(L+\varepsilon)^{m-K}. \tag{*}$$

Since $\sum_{m=K}^{\infty} |a_K|(L+\varepsilon)^{m-K}$ is a convergent geometric series, it follows from (*) and the Comparison Test, Theorem 3.6, that $\sum_{m=K}^{\infty} a_m$ converges absolutely. Hence also $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) Let $l' \in (1, l)$.

Then there is $K \in \mathbb{N}$ such that $\left|\frac{a_{n+1}}{a_n}\right| > l'$ for all $n \geq K$. Hence for $m > K$:

$$|a_m| = |a_K|\,\left|\frac{a_{K+1}}{a_K}\right| \cdot \cdots \cdot \left|\frac{a_m}{a_{m-1}}\right| > |a_K|,$$

so that $a_n \not\to 0$ as $n \to \infty$. Hence $\sum_{n=1}^{\infty} a_n$ diverges by the Test for Divergence, Theorem 3.3. □

Theorem 3.10 (Root Test) (i) *If* $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

(ii) *If* $\limsup_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$*, then the series* $\sum_{n=1}^{\infty} a_n$ *diverges.*

Proof. (i) Let $\varepsilon > 0$ such that $L + \varepsilon < 1$.

Then there is $K \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} < L+\varepsilon$ for all $n \geq K$. Hence $|a_n| < (L+\varepsilon)^n$ for $n \geq K$: Since $\sum_{n=K}^{\infty} (L+\varepsilon)^n$ is a convergent geometric series, it follows from the comparison test, Theorem 3.6, that $\sum_{n=K}^{\infty} a_n$ converges absolutely. Hence also $\sum_{n=1}^{\infty} a_n$ converges absolutely.

(ii) Let $L' \in (1, L)$.

Then for each $K \in \mathbb{N}$ there is $m \geq K$ such that $\sqrt[m]{|a_m|} > L'$. Hence $|a_m| > (L')^m > 1$ for this m, and we conclude $a_n \not\to 0$ as $n \to \infty$.

Indeed, if $a_n \to 0$ as $n \to \infty$, for $\varepsilon = 1$ there would be $K \in \mathbb{N}$ such that $|a_n| < 1$ for all $n > K$.

Hence $\sum_{n=1}^{\infty} a_n$ diverges by the Test for Divergence, Theorem 3.3. $\square$

3.3 Power Series

Definition 3.4 *Let* a *and* c_n, $n \in \mathbb{N}$, *be real numbers. A series*

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

is called a ***power series in*** $(x-a)$ *or a* ***power series centred at*** a *or a* ***power series about*** a.

Note that $(x-a)^0 = 1$. For $x = a$, all terms from the second onwards are 0, so the series converges to c_0 for $x = a$.

Each power series defines a function whose domain is those $x \in \mathbb{R}$ for which the series converges.

Notation. With each power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ we associate a number R or ∞, called the **radius of convergence** of the series, which is defined as follows:

(i) $R = 0$ if $\limsup_{n\to\infty} \sqrt[n]{|c_n|} = \infty$,

(ii) $R = \dfrac{1}{\limsup_{n\to\infty} \sqrt[n]{|c_n|}}$ if $0 < \limsup_{n\to\infty} \sqrt[n]{|c_n|} \in \mathbb{R}$,

(iii) $R = \infty$ if $\limsup_{n\to\infty} \sqrt[n]{|c_n|} = 0$.

Theorem 3.11 *There are three alternatives for the domain of a power series*

$$\sum_{n=0}^{\infty} c_n (x-a)^n :$$

(i) *If $R = 0$, then the series converges only for $x = a$.*

(ii) *If $R = \infty$, then the series converges absolutely for all $x \in \mathbb{R}$.*

(iii) *If $0 < R \in \mathbb{R}$, then the series converges absolutely if $|x-a| < R$ and diverges if $|x-a| > R$.*

Note that (iii) says that the series converges for x in the interval $(a-R, a+R)$ and diverges outside $[a-R, a+R]$. For $x = a-R$ and $x = a+R$ anything can happen, see Calculus I. In any case, the domain of the series is an interval, called the **interval of convergence**.

Proof. In class.

Chapter 4

Limits and Continuous Functions

The single most important concept in all of analysis is that of a limit. Every single notion of analysis is encapsulated in one sense or another to that of a limit. In the previous chapter, you learnt about limits of sequences. Here this notion is extended to limits of functions, which leads to the notion of continuity. You have learnt an intuitive version in Calculus I. Here you will learn a precise definition and you will learn how to prove the results you have learnt in Calculus I.

In this course all functions will have domains and ranges which are subsets of $\mathbb{R}$ unless otherwise stated. Such functions are called real functions.

4.1 Limits of real valued functions

Definition 4.1 *Let $a \in \mathbb{R}$. An interval of the form (c, d) with $c < a < d$ is called a* ***neighbourhood of*** *a, and the set $(c, d) \setminus \{a\}$ is called a* ***deleted neighbourhood of*** *a.*

In the definition of the limit of sequences you have seen formal definitions for 'n tends to ∞' and 'a_n tends to L', and the latter one has an obvious generalization to 'x tends to a' and '$f(x)$ tends to L', which you will encounter in the next definition.

Definition 4.2 (Limit of a function) *Let f be a real function, $a, L \in \mathbb{R}$ and assume that the domain of f contains a deleted neighbourhood of a, that is, $f(x)$ is defined for all x in a deleted neighbourhood of f.*

Then '$f(x) \to L$ as $x \to a$' is defined to mean:

$\forall \varepsilon > 0 \, \exists \delta > 0 \, \forall x \in (a - \delta, a + \delta) \setminus \{a\}, \ f(x) \in (L - \varepsilon, L + \varepsilon)$,
i. e., $\forall \varepsilon > 0 \, \exists \delta > 0 \, (0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon)$.

If $f(x) \to L$ as $x \to a$, then we write $\lim\limits_{x \to a} f(x) = L$.

Note. 1. L in the definition above is also called the limit of the function at a. For the definition of the limit of a function at a, the number $f(a)$ is never used, and indeed, the function need not be defined at a.

2. Observe that the number δ will depend on ε, and also on a, if we vary a.

Example 4.1 *Prove, from first principles, that $x^2 \to 4$ as $x \to 2$.*

Solution. Let $\varepsilon > 0$. We must show that there is $\delta > 0$ such that $0 < |x - 2| < \delta$ implies $|x^2 - 4| < \varepsilon$. Since we want to use $|x - 2| < \delta$ to get $|x^2 - 4| < \varepsilon$, we try to factor out $|x - 2|$ from $|x^2 - 4|$. Thus

$$|x^2 - 4| = |x - 2| \, |x + 2| \leq |x - 2| \, (|x - 2| + 4).$$

It is often convenient to make an initial restriction on δ, like $\delta \leq 1$. Then we can continue the above estimate to obtain from $|x - a| < \delta$ that

$$|x^2 - 4| \leq |x - 2| \, (1 + 4) = 5|x - 2|.$$

If we now put

$$\delta = \min\left\{1, \frac{\varepsilon}{5}\right\},$$

then we can conclude (note that we already used $\delta \leq 1$) for $0 < |x - 2| < \delta$ that

$$|x^2 - 4| \leq 5|x - 2| < 5\frac{\varepsilon}{5} = \varepsilon.$$

To justify the notation $\lim\limits_{x \to a} f(x) = L$ we have to show

Theorem 4.1 *If $f(x) \to L$ as $x \to a$, then L is unique.*

Proof. Let $f(x) \to L$ and $f(x) \to M$ as $x \to a$. We must show that $L = M$.

So let $\varepsilon > 0$. Then there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2} \quad \text{and} \quad 0 < |x - a| < \delta_2 \Rightarrow |f(x) - M| < \frac{\varepsilon}{2}.$$

For $\delta = \min\{\delta_1, \delta_2\}$ and $0 < |x - a| < \delta$ (we indeed only need one such x here) it follows that

$$|L - M| = |(L - f(x)) - (f(x) - M)| \leq |f(x) - L| + |f(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence $L = M$ by Lemma 2.1. □

Definition 4.3 (One sided limits) *1. Let f be a real function, $a, L \in \mathbb{R}$ and assume that the domain of f contains an interval (a, d) with $d > a$, that is, $f(x)$ is defined for all x in (a, d).*

Then '$f(x) \to L$ as $x \to a^+$' is defined to mean:

$\forall \varepsilon > 0 \; \exists \delta > 0 \; (a < x < a + \delta \Rightarrow |f(x) - L| < \varepsilon)$.

If $f(x) \to L$ as $x \to a^+$, then we write $\lim_{x \to a^+} f(x) = L$.

2. Let f be a real function, $a, L \in \mathbb{R}$ and assume that the domain of f contains an interval (c, a) with $c < a$, that is, $f(x)$ is defined for all x in (c, a).

Then '$f(x) \to L$ as $x \to a^-$' is defined to mean:

$\forall \varepsilon > 0 \; \exists \delta > 0 \; (a - \delta < x < a \Rightarrow |f(x) - L| < \varepsilon)$.

If $f(x) \to L$ as $x \to a^-$, then we write $\lim_{x \to a^-} f(x) = L$.

Theorem 4.2 *If $f(x)$ is defined in a deleted neighbourhood of a, then*

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

Proof. $\Rightarrow$: Assume that $\lim_{x\to a} f(x) = L$ and let $\varepsilon > 0$. Then

$$\exists\, \delta > 0\ (x \in (a-\delta, a+\delta),\, x \neq a \Rightarrow |f(x) - L| < \varepsilon)$$

gives

$$\exists\, \delta > 0\ (x \in (a-\delta, a) \Rightarrow |f(x) - L| < \varepsilon) \quad \text{and} \quad \exists\, \delta > 0\ (x \in (a, a+\delta) \Rightarrow |f(x) - L| < \varepsilon).$$

Hence $\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L$.

$\Leftarrow$: Assume that $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$ and let $\varepsilon > 0$. Then there are $\delta_- > 0$ and $\delta_+ > 0$ such that

$$x \in (a - \delta_-, a) \Rightarrow |f(x) - L| < \varepsilon \quad \text{and} \quad x \in (a, a + \delta_+) \Rightarrow |f(x) - L| < \varepsilon.$$

Let $\delta = \min\{\delta_-, \delta_+\}$. Then

$$\begin{aligned} x \in (a-\delta, a+\delta),\, x \neq a &\Rightarrow x \in (a - \delta_-, a) \text{ or } x \in (a, a+\delta_+) \\ &\Rightarrow |f(x) - L| < \varepsilon. \end{aligned}$$

This proves that $\lim_{x\to a} f(x) = L$. $\square$

Example 4.2 *Let* $f(x) = \dfrac{x}{|x|}$ *for* $x \in \mathbb{R} \setminus \{0\}$*. Then* $f(x) = 1$ *if* $x > 0$ *and* $f(x) = -1$ *if* $x < 0$*. It is easy to prove from the definition that* $\lim_{x\to 0^+} f(x) = 1$ *and* $\lim_{x\to 0^-} f(x) = -1$*. Now it follows from Theorem 4.2 that the function* f *does not have a limit as* x *tends to* a*.*

In $x \to a$ the inequalities $0 < |x - a| < \delta$ occur. For $x \to \infty$, this has to be replaced by $x > K$. Hence we have the following

Definition 4.4 *1. Let* f *be a real function defined on a set containing an interval of the form* (c, ∞)*. Then '*$f(x) \to L$ *as* $x \to \infty$*' is defined to mean:*

$\forall\, \varepsilon > 0\ \exists\, K(> 0)\ (x > K \Rightarrow |f(x) - L| < \varepsilon)$.

If $f(x) \to L$ *as* $x \to \infty$*, then we write* $\lim_{x\to\infty} f(x) = L$*.*

2. Let f *be a real function defined on a set containing an interval of the form* $(-\infty, c)$*.*

Then '$f(x) \to L$ as $x \to -\infty$' is defined to mean:

$\forall\, \varepsilon > 0\; \exists\, K(< 0)\; (x < K \Rightarrow |f(x) - L| < \varepsilon)$.

If $f(x) \to L$ as $x \to -\infty$, then we write $\lim\limits_{x\to-\infty} f(x) = L$.

Example 4.3 *Let $f(x) = \dfrac{1}{x}$. Find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.*

For infinite limits we have similar definitions:

Definition 4.5 *Let f be a real function whose domain includes a deleted neighbourhood of the number a.*

1. '$f(x) \to \infty$ as $x \to a$' is defined to mean:

$\forall\, K(> 0)\; \exists\, \delta > 0\; (0 < |x - a| < \delta \Rightarrow f(x) > K)$.

If $f(x) \to \infty$ as $x \to a$, then we write $\lim\limits_{x\to a} f(x) = \infty$.

2. '$f(x) \to -\infty$ as $x \to a$' is defined to mean:

$\forall\, K(< 0)\; \exists\, \delta > 0\; (0 < |x - a| < \delta \Rightarrow f(x) < K)$.

If $f(x) \to -\infty$ as $x \to a$, then we write $\lim\limits_{x\to a} f(x) = -\infty$.

Example 4.4 *Prove that $\dfrac{2}{x^2} \to \infty$ as $x \to 0$.*

Definition 4.6 *Let f be a real function defined on a set containing an interval of the form (c, ∞).*

Then '$f(x) \to \infty$ as $x \to \infty$' is defined to mean:

$\forall\, A(> 0)\; \exists\, K(> 0)\; (x > K \Rightarrow f(x) > A)$.

If $f(x) \to \infty$ as $x \to \infty$, then we write $\lim\limits_{x\to\infty} f(x) = \infty$.

Similar definition hold with all other combinations of limits involving ∞ and $-\infty$, including one-sided limits.

Example 4.5 *Let* $f(x) = \dfrac{1}{x}$. *Find* $\lim\limits_{x \to 0^-} f(x)$, $\lim\limits_{x \to 0^+} f(x)$, $\lim\limits_{x \to 0} f(x)$.

Solution. *Let* $K > 0$ *and put* $\delta = \dfrac{1}{K}$. *Then, for* $0 < x < \delta$, $\dfrac{1}{x} > \dfrac{1}{\delta} = K$. *Thus* $\lim\limits_{x \to 0^+} f(x) = \infty$.

Now let $K < 0$ *and put* $\delta = -\dfrac{1}{K} > 0$. *Then, for* $-\delta < x < 0$, $\dfrac{1}{x} < \dfrac{1}{-\delta} = K$. *Thus* $\lim\limits_{x \to 0^-} f(x) = -\infty$.

Since $\lim\limits_{x \to 0^-} f(x) \neq \lim\limits_{x \to 0^+} f(x)$, $\lim\limits_{x \to 0} f(x)$ *does not exist.*

Rather than calculating limits from the definition, in general one will use limit laws. In this section we state and prove some of these laws.

Theorem 4.3 (Limit Laws) *Let* $a, c \in \mathbb{R}$ *and suppose that the real functions* f *and* g *are defined in a deleted neighbourhood of* a *and that* $\lim\limits_{x \to a} f(x) = L \in \mathbb{R}$ *and* $\lim\limits_{x \to a} g(x) = M \in \mathbb{R}$ *both exist. Then*

1. $\lim\limits_{x \to a} c = c$.
2. $\lim\limits_{x \to a} [f(x) + g(x)] = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x) = L + M$.
3. $\lim\limits_{x \to a} [f(x) - g(x)] = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x) = L - M$.
4. $\lim\limits_{x \to a} [cf(x)] = c \lim\limits_{x \to a} f(x) = cL$.
5. $\lim\limits_{x \to a} [f(x)g(x)] = \left[\lim\limits_{x \to a} f(x)\right] \left[\lim\limits_{x \to a} g(x)\right] = LM$.
6. *If* $M \neq 0$, $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{L}{M}$.
7. *If* $L \neq 0$ *and* $M = 0$, $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ *does not exist.*
8. *If* $n \in \mathbb{N}$, $\lim\limits_{x \to a} [f(x)^n] = \left[\lim\limits_{x \to a} f(x)\right]^n = L^n$.
9. $\lim\limits_{x \to a} x = a$.
10. *If* $n \in \mathbb{N}$, $\lim\limits_{x \to a} x^n = a^n$.

11. If $n \in \mathbb{N}$, $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$. *If* n *is even, we assume that* $a > 0$.

12. If $n \in \mathbb{N}$, $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$.

13. If $\lim_{x\to a} |f(x)| = 0$, *then* $\lim_{x\to a} f(x) = 0$.

Proof. The proofs are similar to those in Theorem 2.2 and we will only prove (b), (e), (f)

(b) Let $\varepsilon > 0$. Then there are numbers $\delta_1 > 0$ and $\delta_2 > 0$ such that

(i) $|f(x) - L| < \frac{\varepsilon}{2}$ if $0 < |x - a| < \delta_1$,

(ii) $|g(x) - M| < \frac{\varepsilon}{2}$ if $0 < |x - a| < \delta_2$.

Put $\delta = \min\{\delta_1, \delta_2\}$. For $0 < |x - a| < \delta$ we have $0 < |x - a| < \delta_1$ and $0 < |x - a| < \delta_2$ and therefore

$$|(f(x)+g(x))-(L+M)| = |(f(x)-L)+(g(x)-M)| \leq |f(x)-L|+|g(x)-M| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

Hence $f(x) + g(x)$ converges to $L + M$ as $x \to a$.

(e) We consider 2 cases: one special case, to which then the general case is reduced.

Case I: $L = M = 0$. Let $\varepsilon > 0$. Then there are numbers δ_1 and δ_2 such that

(i) $|f(x)| < 1$ if $0 < |x - a| < \delta_1$,

(ii) $|g(x)| < \varepsilon$ if $0 < |x - a| < \delta_2$.

Put $\delta = \min\{\delta_1, \delta_2\}$. For $0 < |x - a| < \delta$ we have

$$|f(x)g(x)| = |f(x)|\,|g(x)| < 1 \cdot \varepsilon = \varepsilon.$$

Case II: L and M are arbitrary. Then

$$f(x)g(x) = (f(x) - L)(g(x) - M) + L(g(x) - M) + f(x)M.$$

By (a), (c), $(f(x) - M) \to 0$ and $(g(x) - M) \to 0$ as $x \to a$, and by (b), (d), and Case I, it follows that

$$\lim_{x\to a} f(x)g(x) = 0 + L \cdot 0 + LM = LM.$$

(f) First consider $f = 1$. Since $M \neq 0$ and $g(x) \to M$ as $x \to a$, there is $\delta_0 > 0$ such that $|g(x) - M| < \dfrac{|M|}{2}$ for $0 < |x-a| < \delta_0$. Then, for $0 < |x-a| < \delta_0$,

$$\begin{aligned} & |g(x)| = |M + (g(x) - M)| \geq |M| - |g(x) - M| \geq |M| - \frac{|M|}{2} = \frac{|M|}{2} \\ \Rightarrow\ & \left| \frac{1}{g(x)} - \frac{1}{M} \right| = \frac{|M - g(x)|}{|g(x)M|} \leq \frac{2|g(x) - M|}{|M|^2}. \end{aligned}$$

Now let $\varepsilon > 0$ and $\delta_1 > 0$ such that $0 < |x-a| < \delta_1$ implies $|g(x) - M| \leq \dfrac{|M|^2}{2}\,\varepsilon$. Put $\delta = \min\{\delta_0, \delta_1\}$. It follows for $0 < |x-a| < \delta$ that

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| \leq \frac{2|g(x) - M|}{|M|^2} < 2\frac{|M|^2}{2}\,\varepsilon\,\frac{1}{|M|^2} = \varepsilon.$$

The general case now follows with (d):

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} f(x) \cdot \lim_{x \to a} \frac{1}{g(x)} = L \cdot \frac{1}{M}.$$

□

Recall that a polynomial function is of the form

$$f(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_2 x^2 + b_1 x + b_0$$

with $b_i \in \mathbb{R}$ for $i = 1, 2, \ldots, n$ and n any non-negative integer. A rational function is of the form $f(x) = \dfrac{p(x)}{q(x)}$ with $p(x)$ and $q(x)$ polynomials. We then have the following as a consequence of Theorem 2.2, (b),(d),(i),(j).

Corollary 4.1 *If f is a polynomial or a rational function and a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$.*

Corollary 4.2 *All the limit rules in Theorem 4.3 remain true if $x \to a$ is replaced by any of the following: $x \to a^+$, $x \to a^-$, $x \to \infty$, $x \to -\infty$.*

Proof. For $x \to a^+$ and $x \to a^-$ one just has to replace $0 < |x-a| < \delta$ with $0 < x - a < \delta$ and $-\delta < x - a < 0$, respectively, in the proof of each of the statements. For $x \to \infty$ and $x \to -\infty$, the proofs are very similar to those for sequences.

Similar rules hold if the functions have infinite limits. We state some of the results for $x \to a$, observing that there are obvious extensions as in Corollary 4.2. □

Theorem 4.4 *Assume that* $\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} g(x) = \infty$ *and* $\lim_{x\to a} h(x) = c \in \mathbb{R}$. *Then*

(a) $f(x) + g(x) \to \infty$ *as* $x \to a$,

(b) $f(x) + h(x) \to \infty$ *as* $x \to a$,

(c) $f(x)g(x) \to \infty$ *as* $x \to a$,

(d) $f(x)h(x) \to \begin{cases} \infty & \text{if } c > 0 \\ -\infty & \text{if } c < 0 \end{cases}$ *as* $x \to a$.

Proof. We prove (c) and leave the other parts as exercises.

Let $A > 0$. Then there are $\delta_1 > 0$ and $\delta_2 > 0$ such that

(i) $f(x) > 1$ if $0 < |x - a| < \delta_1$,
(ii) $g(x) > A$ if $0 < |x - a| < \delta_2$.

With $\delta = \min\{\delta_1, \delta_2\}$ it follows for $0 < |x - a| < \delta$ that

$$f(x)g(x) > 1 \cdot A = A.$$

□

Theorem 4.5 (Sandwich Theorem) *Let* $a \in \mathbb{R} \cup \{\infty, -\infty\}$ *and assume that* f, g *and* h *are real functions defined in a deleted neighbourhood of* a. *If* $f(x) \le g(x) \le h(x)$ *for* x *in a deleted neighbourhood of* a *and*
$\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$, *then* $\lim_{x\to a} g(x) = L$.

Proof. Note that $L \in \mathbb{R} \cup \{\infty, -\infty\}$. We will prove this theorem in the case $a \in \mathbb{R}$ and $L \in \mathbb{R}$. The other cases are left as an exercise.

Let $\varepsilon > 0$. Then there are δ_1 and δ_2 such that

(i) $|f(x) - L| < \varepsilon$ if $0 < |x - a| < \delta_1$,
(ii) $|h(x) - L| < \varepsilon$ if $0 < |x - a| < \delta_2$.

Put $\delta = \min\{\delta_1, \delta_2\}$. Then, for $0 < |x - a| < \delta$,

$$L - \varepsilon < f(x) < L + \varepsilon, \quad L - \varepsilon < h(x) < L + \varepsilon$$

gives

$$L - \varepsilon < f(x) \le g(x) \le h(x) < L + \varepsilon.$$

Hence $|g(x) - L| < \varepsilon$ if $0 < |x - a| < \delta$. $\square$

Theorem 4.6 *Let f be defined on an interval (a, b), where $a = -\infty$ and $b = \infty$ are allowed. With the convenient notation $a^+ = -\infty$ if $a = -\infty$ and $b^- = \infty$ if $b = \infty$, we obtain*

(a) *if f is increasing, then*

$$\lim_{x \to b^-} f(x) = \sup\{f(x) : x \in (a, b)\} \text{ and } \lim_{x \to a^+} f(x) = \inf\{f(x) : x \in (a, b)\};$$

(b) *if f is decreasing, then*

$$\lim_{x \to b^-} f(x) = \inf\{f(x) : x \in (a, b)\} \text{ and } \lim_{x \to a^+} f(x) = \sup\{f(x) : x \in (a, b)\}.$$

Proof. Since all four cases have similar proofs, we only prove (a) in the case $b \in \mathbb{R}$.

Let $L = \sup\{f(x) : x \in (a, b)\}$.

Case I: $L \in \mathbb{R}$.

Let $\varepsilon > 0$. By Theorem 1.6 there is $c \in (a, b)$ such that $L - \varepsilon < f(c)$. Put $\delta = b - c > 0$. Now let $b - \delta < x < b$, i. e., $x \in (c, b)$. Then $c < x$ gives $f(c) \le f(x)$ since f is increasing and $f(x) \le L$ for all $x \in (c, b) \subset (a, b)$ by definition of the supremum, so that

$$L - \varepsilon < f(c) \le f(x) \le L < L + \varepsilon$$

for these x. By definition, this mean $f(x) \to L$ as $x \to b^-$.

Case II: $L = \infty$. In this case, $\{f(x) : x \in (a, b)\}$ is not bounded above. Therefore, for each $A \in \mathbb{R}$ there is $c \in (a, b)$ such that $f(c) > A$. Since f is increasing, it follows for all $x \in (c, b)$ that $A < f(c) \le f(x)$. Therefore $f(x) \to \infty$ as $x \to b^-$. $\square$

4.2 Continuous functions

Definition 4.7 (Continuity of a function at a point) *Let f be a real function, $a \in \mathbb{R}$ and assume that the domain of f contains a neighbourhood of a, that is, $f(x)$ is defined* *for all x in a neighbourhood of a.*

We say that f is ***continuous at*** *a if*

$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in (a-\delta, a+\delta),\; f(x) \in (f(a)-\varepsilon, f(a)+\varepsilon)$,
i. e., $\forall \varepsilon > 0 \; \exists \delta > 0 \; (|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon)$.

We realize that the definition of the existence of a limit of a function at a point and continuity at that point are very similar, but that there are subtle (and important) differences:

For limits, f does not need to be defined at a, and even if $f(a)$ exists, this value is not used at all when finding the limit of the function f at a.

We conclude

$$\begin{aligned} f \text{ is continuous at } a &\Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 \; (|x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon) \quad \text{(by definition)} \\ &\Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 \; (0 < |x-a| < \delta \Rightarrow |f(x)-f(a)| < \varepsilon) \quad (\because\; x = a \Rightarrow f(x) = f(a)) \\ &\Leftrightarrow \lim_{x \to a} f(x) = f(a). \end{aligned}$$

Hence we have shown

Theorem 4.7 *f is continuous at a if and only if the following three conditions are satisfied:*

1. $f(a)$ is defined, i. e., a is in the domain of f,

2. $\lim_{x\to a} f(x)$ exists, i.e., $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$, and

3. $f(a) = \lim_{x\to a} f(x)$.

Example 4.6 *1. Let*

$$f(x) = \begin{cases} x^2 & if \quad x \neq 2 \\ 2 & if \quad x = 0. \end{cases}$$

Then $\lim_{x\to 2} f(x) = 4$ exists, but this limit is different from $f(2) = 2$. Hence f is not continuous at 2.

2. Let $f(x) = \dfrac{\sin x}{x}$ for $x \neq 0$ while f is not defined at $x = 0$. Then $\lim_{x\to 0} f(x) = 1$ exists, but f is not defined at 0. Hence f is not continuous at 0.

3. Let

$$f(x) = \begin{cases} \dfrac{\sin x}{x} & if \quad x \neq 0 \\ 1 & if \quad x = 0. \end{cases}$$

Then $\lim_{x\to 0} f(x) = 1$ *exists, and* $f(0) = 1$*. Hence* f *is continuous at* 0*.*

Theorem 4.8 *If* f *and* g *are continuous at* a *and if* $c \in \mathbb{R}$*, then*

1. *the sum* $f + g$*,*
2. *the difference* $f - g$*,*
3. *the product* fg*,*
4. *the quotient* $\dfrac{f}{g}$ *if* $g(a) \neq 0$ *and*
5. *the scalar multiple* cf

are functions that are also continuous at a*.*

Proof. The statements follow immediate from the limit laws, Theorem 4.3, and Theorem 4.7. For example, for 3. we have

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a),$$

and then Theorem 4.3 gives

$$\lim_{x\to a}(fg)(x) = \lim_{x\to a} f(x)g(x) = \lim_{x\to a} f(x) \lim_{x\to a} g(x) = f(a)g(a) = (fg)(a).$$

Then Theorem 4.7 says that fg is continuous at a.

Recall that the composite $g \circ f$ of two functions f and g is defined by $(g \circ f)(x) = g(f(x))$.□

Theorem 4.9 *If* f *is continuous at* a *and* g *is continuous at* $f(a)$*, then* $g \circ f$ *is continuous at* a*.*

Proof. Let $\varepsilon > 0$. Since g is continuous at $f(a)$, there is $\eta > 0$ such that

$$|y - f(a)| < \eta \Rightarrow |g(y) - g(f(a))| < \varepsilon. \tag{1}$$

Since f is continuous at a, there is $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \eta. \tag{2}$$

Putting $y = f(x)$ in (1) it follows from (1) and (2) that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \eta \Rightarrow |g(f(x)) - g(f(a))| < \varepsilon$$

that is,

$$|x - a| < \delta \Rightarrow |(g \circ f)(x) - (g \circ f)(a)| < \varepsilon.$$

Hence $g \circ f$ is continuous at a. $\square$

Definition 4.8 *1. A function f is* ***continuous from the right at*** *a if $\lim\limits_{x\to a^+} f(x) = f(a)$.*

2. A function f is ***continuous from the left at*** *a if $\lim\limits_{x\to a^-} f(x) = f(a)$.*

Example 4.7 *1. Let*

$$f(x) = \begin{cases} \dfrac{|x| + x}{2x} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Determine the right and left continuity of f at $x = 0$.

Solution. $f(0) = 0$ whilst

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{|x| + x}{2x} = \lim_{x\to 0^-} \frac{-x + x}{2x} = \lim_{x\to 0^-} 0 = 0$$

and

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{|x| + x}{2x} = \lim_{x\to 0^+} \frac{x + x}{2x} = \lim_{x\to 0^-} 1 = 1.$$

Since $f(0) = 0 = \lim\limits_{x\to 0^-} f(x)$, f is continuous from the left at $x = 0$. Since $f(0) = 0 \neq 1 = \lim\limits_{x\to 0^+} f(x)$, f is not continuous from the right at $x = 0$.

Note. 1. It is easy to show that f is continuous at a if and only if f is continuous from the right and continuous from the left at a.

2. If $a \in dom(f)$ and if there is $\varepsilon > 0$ such that $dom(f) \cap (a-\varepsilon, a+\varepsilon) = (a-\varepsilon, a]$, then we say that f is continuous at a if $\lim\limits_{x \to a^-} f(x) = f(a)$.

3. If $a \in dom(f)$ and if there is $\varepsilon > 0$ such that $dom(f) \cap (a-\varepsilon, a+\varepsilon) = [a, a+\varepsilon)$, then we say that f is continuous at a if $\lim\limits_{x \to a^+} f(x) = f(a)$.

4. The convention in 2 and 3 is consistent with what you will learn in General Topology about continuity. Just note that the condition $|f(x) - f(a)| < \varepsilon$ has to be checked for all $x \in dom(f)$ which satisfy $|x - a| < \delta$.

Lemma 4.1 *If $f(x) \to b$ as $x \to a$ (a^+, a^-) and g is continuous at b, then $g(f(x)) \to g(b)$ as $x \to a$ (a^+, a^-), which can be written, e. g., as*

$$\lim_{x \to a} g(f(x)) = g\left(\lim_{x \to a} f(x)\right).$$

Proof. The function

$$\widetilde{f}(x) = \begin{cases} f(x) & if \quad x \in dom(f), x \neq a \\ b & \qquad x = a, \end{cases}$$

is continuous (from the right, from the left) at a. Hence the result follows from Theorem 4.9.

Definition 4.9 *A function is continuous on a set $X \subset \mathbb{R}$ if f is continuous at each $x \in X$. Here continuity is understood in the sense of the above note with $X = dom(f)$. A function is said to be continuous if it is continuous on its domain.*

Example 4.8 *Show that $f(x) = \sqrt{x^2 - 4}$ is continuous.*

Solution. The domain of f is $\{x \in \mathbb{R} : |x| \geq 2\} = (-\infty, -2] \cup [2, \infty)$. By Theorem 4.8, the function $x \mapsto x^2 - 4$ is continuous on $\mathbb{R}$, and by Theorem 4.3 (k), the square root is continuous at each positive number. So also the composite function f is continuous on $(-\infty, -2) \cup (2, \infty)$. Also, the proof of Theorem 4.3 (k) can be easily adapted to show that the square root is continuous from the right at 0. Then it easily follows that f is continuous (from the right) at 2 and continuous (from the left) at -2.

Theorem 4.10 *The following functions are continuous on their domains.*

1. *Polynomials* $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$, $a_i \in \mathbb{R}$, $n \in \mathbb{N}$.
2. *Rational functions* $\frac{p(x)}{q(x)}$, p *and* $q \neq 0$ *polynomials.*
3. *Sums, differences, products and quotients of continuous functions.*
4. *Root functions.*
5. *The trigonometric functions* $\sin x$, $\cos x$, $\tan x$, $\csc x$, $\sec x$ *and* $\cot x$.
6. *The exponential function* $\exp(x)$.
7. *The absolute value function* $|x|$.

Proof. 1, 2 and 3 easily follow from previous theorems on limits and continuity, as does 7. However, 7 can be easily proved dircetly: For each $\varepsilon > 0$ let $\delta = \varepsilon$. Then, for $|x - a| < \delta$ we have

$$\big|\,|x| - |a|\,\big| \leq |x - a| < \delta = \varepsilon.$$

The continuity of sin and cos follows from the sum of angles formulae and from the limits proved in Calculus I (the proofs used the Sandwich Theorem, which now has been proved). The continuity of the other trigonometric functions then follows from part 3.

Finally, the continuity of exp is a tutorial problem. $\square$

Theorem 4.11 *Let* $a \in \mathbb{R}$ *and let* f *be a real function which is defined in a neighbourhood of* a. *Then* f *is continuous at* a *if and only if for each sequence* (x_n) *in* $dom(f)$ *with* $\lim_{n\to\infty} x_n = a$ *the sequence* $f(x_n)$ *satisfies* $\lim_{n\to\infty} f(x_n) = f(a)$.

Proof. $\Rightarrow$: Let (x_n) be a sequence in $dom(f)$ with $\lim_{n\to\infty} x_n = a$. We must show that $\lim_{n\to\infty} f(x_n) = f(a)$. Hence let $\varepsilon > 0$. Since f is continuous at a, there is $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon. \tag{1}$$

Since $\lim_{n\to\infty} x_n = a$, there is $K \in \mathbb{R}$ such that for $n > K$, $|x_n - a| < \delta$. But then, by (1), $|f(x_n) - f(a)| < \varepsilon$ for $n > K$.

$\Leftarrow$: (indirect proof) Assume that f is not continuous at a. Then

$$\exists\, \varepsilon > 0\ \forall\, \delta > 0\ \exists\, x \in dom(f),\ |x - a| < \delta \text{ and } |f(x) - f(a)| \geq \varepsilon.$$

In particular, for $\delta = \frac{1}{n}$, $n = 1, 2, \ldots$ we find $x_n \in dom(f)$ such that $|x_n - a| < \dfrac{1}{n}$ and $|f(x_n) - f(a)| \geq \varepsilon$. But then $\lim_{n\to\infty} x_n = a$, whereas $(f(x_n))$ does not converge to $f(a)$. $\square$

4.3 The Intermediate Value Theorem

The following important theorem on continuous functions tells us that the graph of a continuous function cannot jump from one side of a horizontal line $y = k$ to the other without intersecting the line at least once.

Theorem 4.12 (Intermediate Value Theorem (IVT)) *Suppose that f is continuous on the closed interval $[a, b]$ with $f(a) \neq f(b)$. Then for any number k between $f(a)$ and $f(b)$ there exists a number c in the open interval (a, b) such that $f(c) = k$.*

Proof. Let

$$g(x) = f(x) - k, \quad (x \in [a, b]).$$

Then g is continuous, and 0 lies between $g(a)$ and $g(b)$, that is, $g(a)$ and $g(b)$ have opposite signs: $g(a)g(b) < 0$.

Let $[a_0, b_0] = [a, b]$ and use bisection to define intervals $[a_n, b_n]$ as follows: If $[a_n, b_n]$ with $g(a_n)g(b_n) < 0$ has been found, let d be the midpoint of the interval $[a_n, b_n]$. If $g(d) = 0$, the result follows with $c = d$. If $g(d)$ has the same sign as $g(b_n)$, then $g(a_n)$ and $g(d)$

have opposite signs, and putting $a_{n+1} = a_n$, $b_{n+1} = d$, we have $g(a_{n+1})g(b_{n+1}) < 0$. Otherwise, if $g(d)$ has the opposite sign to $g(b_n)$, we put $a_{n+1} = d$, $b_{n+1} = b_n$ and get again $g(a_{n+1})g(b_{n+1}) < 0$.

If this procedure does not stop, we obtain an increasing sequence (a_n) and a decreasing sequence (b_n), both of which converge by Theorem 2.9. We observe that

$$b_n = a_n + \frac{1}{2}(b_{n-1} - a_{n-1}) = a_n + 2^{-n}(b - a).$$

Then

$$c := \lim_{n\to\infty} b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} 2^{-n}(b - a) = \lim_{n\to\infty} a_n.$$

Since $a \le c \le b$ and g is continuous at c, it follows in view of Theorem 4.11 that

$$g(c)^2 = \lim_{n\to\infty} g(a_n) \lim_{n\to\infty} g(b_n) = \lim_{n\to\infty} g(a_n)g(b_n) \le 0.$$

Therefore $g(c) = 0$, which gives $f(c) = k$.

Since $f(a) \neq k$ and $f(b) \neq k$, it follows that $c \neq a$ and $c \neq b$, so that $a < c < b$. □

Note. You have seen the definition of interval in first year and you will recall that that the definition required several cases, depending on whether the endpoints belong to the interval or not and whether the interval is bounded (above, below), namely (a, b), $[a, b)$, $(a, b]$, $[a, b]$, $(-\infty, b)$, $(-\infty, b]$, (a, ∞), $[a, \infty)$, $(-\infty, \infty)$ where $a, b \in \mathbb{R}$ and $a < b$. However, intervals can be characterized by one common property. For this we need the following notion: A subset S of $\mathbb{R}$ is called a **singleton** if the set S has exactly one element.

Definition 4.10 *1. A set $S \subset \mathbb{R}$ is called an interval if*

(i) $S \neq \emptyset$,

(ii) S is not a singleton,

(iii) if $x, y \in S$, $x < y$, then each $z \in \mathbb{R}$ with $x < z < y$ satisfies $z \in S$.

2. An interval of the form $[a, b]$ with $a < b$ is called a closed bounded interval.

Note. A subset S of $\mathbb{R}$ is an interval if and only if it contains at least two elements and if all real numbers between any two elements in S also belong to S.

Definition 4.11 *For a function $f : X \to Y$ and $A \subset X$, the set*

$$f(A) = \{y \in Y : \exists\, x \in A \cap dom(f),\ f(x) = y\} = \{f(x) : x \in A \cap dom(f)\}$$

*is called the **image of** A **under** f.*

Corollary 4.3 *Let I be an interval and let f be a continuous real function on I. Then $f(I)$ is either an interval or a singleton.*

Proof. In class. □

Example 4.9 *Let $f(x) = x^2$. Then $f((-1, 2)) = [0, 4)$. Notice that $I = (-1, 2)$ is an open interval, while $f(I)$ is not.*

Theorem 4.13 *Let f be a real function which is continuous on $[a, b]$, where $a < b$. Then f is bounded on $[a, b]$, i. e., $f([a, b])$ is bounded.*

Proof. In Class. □

Theorem 4.14 *A continuous function on a closed bounded interval achieves its supremum and infimum.*

Corollary 4.4 *If f is continuous on $[a, b]$, $a < b$, then either $f([a, b])$ is a singleton or $f([a, b]) = [c, d]$ with $c < d$.*

Proof. In Class. □

Theorem 4.15 *Let I be an interval and $f : I \to \mathbb{R}$ be a stricly monotonic continuous function. Then $f(I)$ is an interval, and the inverse function $f^{-1} : f(I) \to \mathbb{R}$ is continuous.*

Proof. In Class. □

Chapter 5

Differentiation

5.1 Definitions and Properties

In this section we recall definitions and results which have been stated and proved in first year calculus. These results must be known but will not be examined directly.

In this section let $A \subset \mathbb{R}$ such that for each $a \in A$ there is $\varepsilon > 0$ such that $(a-\varepsilon, a] \subset A$, $[a, a+\varepsilon) \subset A$ or $(a-\varepsilon, a+\varepsilon) \subset A$.

Definition 5.1 *Let $f : A \to \mathbb{R}$ and $a \in A$.*

1. *f is called differentiable at a if the limit*

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. The number $f'(x)$ is called the derivative of f.

2. *f is called differentiable on A if f is differentiable at each $a \in A$.*

Note. We also use the notations $\dfrac{df}{dx}$ or $\dfrac{d}{dx}f$ for f'.

Theorem 5.1 *Let $f : A \to \mathbb{R}$ and $a \in A$. If f is differentiable at a, then f is continuous at a.*

Proof. From

$$f(x) = f(a) + \frac{f(x) - f(a)}{x - a}(x - a)$$

it follows that

$$\lim_{x \to a} f(x) = f(a) + \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \lim_{x \to a}(x - a) = f(a) + [f'(a)](0) = f(a).$$

$\square$

Theorem 5.2 (Rules for the derivative) *Let $f, g : A \to \mathbb{R}$, $a \in A$, f and g differentiable at a, and $c \in \mathbb{R}$. Then*

1. *Linearity of the derivative: $f + g$ and cf are differentiable at a, and*

$$(f + g)'(a) = f'(a) + g'(a) \quad \textit{and} \quad (cf)'(a) = cf'(a).$$

2. *Product Rule: fg is differentiable at a, and*

$$(fg)'(a) = f'(a)g(a) + f(a)g'(a).$$

3. *Quotient Rule: If $\frac{f}{g}$ is defined at a, i. e., $g(a) \neq 0$, then $\frac{f}{g}$ is differentiable at a, and*

$$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.$$

Proof. 1. From first principles and rules of limits, we have

$$\begin{aligned}
(f + g)'(a) &= \lim_{x \to a} \frac{(f + g)(x) - (f + g)(a)}{x - a} \\
&= \lim_{x \to a} \frac{[f(x) + g(x)] - [f(a) + g(a)]}{x - a} \\
&= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} + \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \\
&= f'(a) + g'(a)
\end{aligned}$$

and

$$(cf)'(a) = \lim_{x \to a} \frac{(cf)(x) - (cf)(a)}{x - a} = \lim_{x \to a} \frac{cf(x) - cf(a)}{x - a} = c \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = cf'(a).$$

2. Observing Theorem 5.1 we have

$$\begin{aligned}
(fg)'(a) &= \lim_{x\to a} \frac{(fg)(x) - (fg)(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x-a} \\
&= \lim_{x\to a} \frac{[f(x) - f(a)]g(x)}{x-a} + \lim_{x\to a} \frac{f(a)[g(x) - g(a)]}{x-a} \\
&= \lim_{x\to a} \frac{f(x) - f(a)}{x-a} \lim_{x\to a} g(x) + f(a) \lim_{x\to a} \frac{g(x) - g(a)}{x-a} \\
&= f'(a)g(a) + f(a)g'(a).
\end{aligned}$$

3. We first consider the case $f = 1$:

$$\begin{aligned}
\left(\frac{1}{g}\right)'(a) &= \lim_{x\to a} \frac{\left(\frac{1}{g}\right)(x) - \left(\frac{1}{g}\right)(a)}{x-a} \\
&= \lim_{x\to a} \frac{\frac{1}{g(x)} - \frac{1}{g(a)}}{x-a} \\
&= \lim_{x\to a} \frac{g(a) - g(x)}{g(x)g(a)(x-a)} \\
&= \lim_{x\to a} \frac{g(a) - g(x)}{x-a} \frac{1}{g(a) \lim_{x\to a} g(x)} \\
&= -\frac{g'(a)}{[g(a)]^2}.
\end{aligned}$$

To prove the statement for general f, we use the product rule:

$$\begin{aligned}
\left(\frac{f}{g}\right)'(a) &= \left(f\frac{1}{g}\right)'(a) = f'(a)\left(\frac{1}{g}\right)(a) + f(a)\left(\frac{1}{g}\right)'(a) = \frac{f'(a)}{g(a)} + f(a)\left(-\frac{g'(a)}{[g(a)]^2}\right) \\
&= \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}.
\end{aligned}$$

□

Example 5.1 *1. If $c \in \mathbb{R}$, then $\frac{dc}{dx} = 0$.*

2. For $n \in \mathbb{N}^*$ we have $\dfrac{d}{dx}x^n = nx^{n-1}$.

The proof of the differentiability of e^x in first year calculus was incomplete; it will be given in the next section.

Also, the differentiability of the inverse trigonometric functions, $\ln x$ and x^r for $r \in \mathbb{R} \setminus \mathbb{N}$ will be postponed to the next section.

Theorem 5.3 (Fermat's Theorem) *Let I be an interval, $f : I \to \mathbb{R}$, and let c be in the interior of I. If f has a local maximum or local minimum at c and f is differentiable at c, then $f'(c) = 0$.*

Proof. Since c is an interior point of I, there is $\varepsilon_0 > 0$ such that $(c - \varepsilon_0, c + \varepsilon_0) \subset I$. Assume that f has a local maximum at c. Then there is $\varepsilon \in (0, \varepsilon_0)$ such that $f(x) \le f(c)$ for all $x \in (c - \varepsilon, c + \varepsilon)$.

Therefore

$$\frac{f(x) - f(c)}{x - c} \ge 0 \quad for \quad x \in (c - \varepsilon_0, c)$$

and

$$\frac{f(x) - f(c)}{x - c} \le 0 \quad for \quad x \in (c, c + \varepsilon_0).$$

Hence, since f is differentiable at c,

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \ge \lim_{x \to c^-} 0 = 0$$

and

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \le \lim_{x \to c^+} 0 = 0.$$

Therefore $0 \le f'(c) \le 0$, which proves $f'(c) = 0$.

The case of a local minimum at c can be dealt with in a similar way, or we may use the fact that then $-f$ has a local maximum at c and therefore $-f'(c) = 0$ by the first part of the proof. □

Theorem 5.4 (Rolle's Theorem) *Let $a < b$ be real numbers and $f : [a, b] \to \mathbb{R}$ be a function having the following properties:*

1. *f is continuous on the closed interval $[a, b]$,*

2. *f is differentiable on the open interval (a, b),*

3. $f(a) = f(b)$.

Then there is $c \in (a, b)$ such that $f'(c) = 0$.

Proof. If f is constant, then $f' = 0$, and the statement is true for any $c \in (a, b)$. If f is not constant, then there is $x_0 \in [a, b]$ such that $f(x_0) \neq f(a)$. We may assume $f(x_0) > f(a)$. Otherwise, $f(x_0) < f(a)$, and we can consider $-f$. Since f is continuous on $[a, b]$ by property 1, f has a maximum on $[a, b]$, see Corollary 4.4. That is, there is some $c \in [a, b]$ such that $f(c) \geq f(x)$ for all $x \in [a, b]$. In particular, $f(c) \geq f(x_0) > f(a)$. Since $f(a) = f(b)$, we have $c \neq a$ and $c \neq b$, and therefore $c \in (a, b)$. Hence $f'(c) = 0$ by Fermat's Theorem. $\square$

Theorem 5.5 (First Mean Value Theorem) *Let $a < b$ be real numbers and $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b).*

Then there is $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof. The function

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a)$$

is continuous on $[a, b]$, differentiable on (a, b), and $g(a) = f(a) = g(b)$. Hence g satisfies the assumptions of Rolle's theorem. Therefore, there is $c \in (a, b)$ such that $g'(c) = 0$. But

$$g'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

and substituting c completes the proof. $\square$

5.2 The Chain Rule and Inverse Functions

Theorem 5.6 (Chain Rule) *Let I and J be intervals, $g : J \to \mathbb{R}$ and $f : I \to \mathbb{R}$ with $f(I) \subset J$, and let $a \in I$. Assume that f is differentiable at a and that g is differentiable at $f(a)$. Then $g \circ f$ is differentiable at a and*

$$(g \circ f)'(a) = g'(f(a))f'(a).$$

Proof. In Class. $\square$

Theorem 5.7 *Let I be an interval, let $f : I \to \mathbb{R}$ be continuous and strictly increasing or decreasing, and $b \in f(I)$. Assume that f is differentiable at $a = f^{-1}(b)$ with $f'(a) \neq 0$. Then f^{-1} is differentiable at $b = f(a)$ and*

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \quad \textit{or, equivalently,} \quad (f^{-1})'(f(a)) = \frac{1}{f'(a)}.$$

Proof. In Class. $\square$

This result now allows us to find the derivatives of arcsin and arctan.

Example 5.2 *Show that* $\arcsin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ *is continuous on* $[-1, 1]$ *and differentiable on* $(-1, 1)$*, and find* $\dfrac{d}{dx}\arcsin$.

Solution. In Class. $\square$

Finally, in this section we shall show that $e^x = \exp(x)$ is differentiable and find its derivative. This will then allow us to find the derivative of $\ln x$ as an inverse function.

Theorem 5.8 (Derivative of e^x)

1. $e^x \geq 1 + x$ *for* $x \in \mathbb{R}$ *and* $e^x \leq \dfrac{1}{1-x}$ *for* $x < 1$.
2. e^x *is differentiable and* $\dfrac{d}{dx}e^x = e^x$.

Proof. In Class. $\square$