Math 104, Practice Midterm 1 Solution

Please try to solve these problems by yourselves first. Please let me know if you find any typos/mistakes.

1. Use limit theorems to evaluate the following limits. Justify each step.

   (a) $\lim_{n\to\infty} \frac{5n^2+\sin n}{n^2+1}$
   Solution:
   $$\lim_{n\to\infty} \frac{5n^2+\sin n}{n^2+1} = \lim_{n\to\infty} \frac{5+\frac{\sin}{n^2}}{1+\frac{1}{n^2}} = 5,$$
   because, clearly, $\lim_{n\to\infty} \frac{1}{n^2} = 0$, and by squeeze theorem
   $$-\frac{1}{n^2} \le \frac{\sin n}{n^2} \le \frac{1}{n^2} \Rightarrow \lim_{n\to\infty} \frac{\sin n}{n^2} = 0.$$

   (b) $\lim_{n\to\infty} \left(\sqrt{9n^2+n} - 3n\right)^2$
   Solution:
   $$\begin{aligned}\sqrt{s_n} = \sqrt{9n^2+n} - 3n &= \frac{(\sqrt{9n^2+n}-3n)(\sqrt{9n^2+n}+3n)}{\sqrt{9n^2+n}+3n} \\ &= \frac{n}{\sqrt{9n^2+n}+3n} \\ &= \frac{1}{\sqrt{9+1/n}+3} \to \frac{1}{6}\end{aligned}$$
   Thus, $\lim s_n = \frac{1}{36}$.

   (c) $\lim_{n\to\infty} \frac{1+2^{(-1)^n}}{n}$
   Solution: $0 \le \frac{1+2^{(-1)^n}}{n} \le \frac{3}{n} \Rightarrow \lim \frac{1+2^{(-1)^n}}{n} = 0.$

2. Show that $\sqrt[3]{1+\sqrt{2}}$ is irrational.

   Proof. Denote by $a = \sqrt[3]{1+\sqrt{2}}$. Then
   $$\begin{aligned}a^3 &= 1+\sqrt{2}, \\ (a^3-1)^2 &= 2 \\ a^6 - 2a^3 - 1 &= 0.\end{aligned}$$

If $x = \frac{p}{q}$ (reduced) is a rational solution of the above equation, then, by the rational zeros theorem, $q|1$, $p|-1$ and the possible values of $\frac{p}{q}$ are ± 1. We check that ± 1 are not roots of the equation:

$$1^6 - 2(1)^3 - 1 \neq 0, (-1)^6 - 2(-1)^3 - 1 \neq 0.$$

So the equation has no rational roots and a is irrational. $\square$

3. Suppose $\lim_{n\to\infty} s_n = \infty$ and $\lim_{n\to\infty} t_n < 0$. Prove that $\lim_{n\to\infty}(s_n t_n) = -\infty$.

Proof. Denote by $t = \lim t_n < 0$. Then, by definition, for $\epsilon = -t/2 > 0$, there exists $N_1 > 0$ such that $|t_n - t| < \epsilon = -t/2$ for all $n > N_1$. Therefore,

$$t_n < t + \epsilon = t/2 < 0, \quad \forall n > N_1.$$

For any $M < 0$, let $M_1 = 2M/t > 0$. Since $\lim s_n = \infty$, there exists N_2 such that

$$s_n > M_1 = 2M/t, \quad \forall n > N_2.$$

Let $N = \max\{N_1, N_2\}$, then for any $n > N$, we have

$$s_n t_n < (t/2)(2M/t) = M.$$

By definition, $\lim s_n t_n = -\infty$. $\square$

4. Prove that the sequence $\sin\left(\frac{n\pi}{2}\right)$ diverges.

Proof. Suppose $\sin\left(\frac{n\pi}{2}\right)$ converge, then any subsequence should converge to the same limit. However,

$$s_{2m} = \sin\left(\frac{2m\pi}{2}\right) = 0,$$

$$s_{4m+1} = \sin\left(\frac{(4m+1)\pi}{2}\right) = 1.$$

We have two sequences converging to different limits. Thus, $\sin\left(\frac{2m\pi}{2}\right)$ diverges.

Another approach is to use definition. See Ross, Example 4 on P39. $\square$

5. True/False questions.

   (a) The field $\mathbb{Q}$ of rational numbers satisfy the completeness axiom.
   Solution: False. $\sup\{x \in \mathbb{Q} : x \leq \sqrt{2}\} \notin \mathbb{Q}$.

   (b) For a bounded sequence (s_n) in $\mathbb{R}$,

   $$\limsup s_n = \sup\{s_n : n \in \mathbb{N}\}.$$

   Solution: False. For instance, $(s_n) = (3, 1, 1, 1, 1, \cdots)$. Then

   $$\limsup s_n = 1.$$

   But $\sup\{s_n : n \in \mathbb{N}\} = 3$. Note that $\leq$ always holds.

   (c) If $r \in \mathbb{Q}$ and x is irrational, then $r + x$ is irrational.
   Solution: True. See homework 1 for a proof.

6. State what it means for a sequence (s_n) to diverge to infinity. (Notation: $\lim_{n\to\infty} s_n = \infty$.)

   Solution: A sequence (s_n) diverges to infinity if for any $M > 0$ there exists N such that

   $$s_n > M, \quad \forall n > N.$$

   How about a sequence does not diverge to infinity? does not converge?

7. Prove that if $\lim_{n\to\infty} \sqrt[n]{s_n} = L < 1$, then

   $$\lim_{n\to\infty} s_n = 0.$$

   Sorry, I forgot to type absolute value. It should be *if* $\lim_{n\to\infty} \sqrt[n]{|s_n|} = L < 1$*, then* ... Otherwise $\sqrt[n]{|s_n|}$ may not be real.

   Proof. Since $\lim_{n\to\infty} \sqrt[n]{|s_n|} = L < 1$, for $\epsilon_1 = \frac{1}{2}(1 - L) > 0$, there exists $N_1 > 0$ such that

   $$|\sqrt[n]{|s_n|} - L| < \epsilon_1, \forall n > N_1.$$

   Thus

   $$\sqrt[n]{|s_n|} < L + \epsilon_1 = \frac{1}{2}(1 + L), \forall n > N_1.$$

So

$$|s_n| < a^n, \forall n > N_1,$$

where $a = \frac{1}{2}(1+L) < 1$ since $L < 1$.

Recall that $\lim a^n = 0$ if $|a| < 1$. Thus, for any $\epsilon > 0$, there exists $N_2 > 0$ such that

$$a^n < \epsilon, \quad \forall n > N_2.$$

Let $N = \max\{N_1, N_2\}$, then for any $n > N$, we have

$$|s_n| < a^n < \epsilon.$$

By definition, $\lim s_n = 0$. □

8. If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}}, \forall n \geq 1,$$

prove that (s_n) converges.

Proof. We first prove that the sequence is bounded, namely, for all $n \geq 1$,

$$0 < s_n < 2.$$

Clearly $s_n > 0$ for all n. We use induction to show that $s_n \leq 2$.

When $n = 1$, $s_1 = \sqrt{2} < 2$.

Suppose $s_n < 2$. Then

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \leq \sqrt{2 + \sqrt{2}} < 2.$$

Thus s_n is bounded.

Second, we show that it is increasing by induction. I.e., for any $n \geq 1$,

$$s_{n+1} < s_n.$$

When $n = 1$, $s_2 = \sqrt{2 + \sqrt[4]{2}} > \sqrt{2} = s_1$.

Suppose $s_n < s_{n+1}$, then

$$\begin{aligned} &\sqrt{s_n} < \sqrt{s_{n+1}} \\ \Rightarrow &\sqrt{2 + \sqrt{s_n}} < \sqrt{2 + \sqrt{s_{n+1}}} \\ \Rightarrow &s_{n+1} < s_{n+2}. \end{aligned}$$

Thus, s_n is increasing.

By the monotone convergence theorem, the sequence (s_n) is convergent. $\square$

9. Show that if every subsequence of (s_n) has a further subsequence that converges to the same s, then (s_n) converges to s.

Proof. Suppose not, i.e., s is not the limit of (s_n). Then there exists $\epsilon_0 > 0$ such that for any $N > 0$, there is an $m > N$ satisfying

$$|s_m - s| \geq \epsilon_0.$$

We select a subsequence s_{n_k} of s_n in the following way. Take $n_1 = 1$, for any $k \geq 1$, n_{k+1} is chosen such that

$$n_{k+1} > n_k, \text{ and } |s_{n_{k+1}} - s| \geq \epsilon_0.$$

Then (s_{n_k}) form a subsequence and for any $k \geq 2$,

$$|s_{n_k} - s| \geq \epsilon_0.$$

By assumption, this subsequence has a further subsequence (denoted by t_k) that converges to s. But we have

$$|t_k - s| \geq \epsilon_0, \forall k$$

This is a contradiction to $\lim t_k = s$. Therefore,

$$\lim_{n \to \infty} s_n = s.$$

$\square$