MATH2001–Basic Analysis Final Examination June 2012

Time: 60 minutes   Total marks: 60 marks

SECTION A Multiple choice

Answer the multiple choice questions on the computer card provided. There is ONLY ONE correct answer to each question. Please ensure that your student number is entered on the card, by pencilling in the requisite digit for each block.

Question 1 .... [2 marks]

Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{R}) \subset \mathbb{Q}$. Then

A. f is not continuous.

B. f is continuous but not differentiable.

C. f is differentiable but not continuous.

D. f is differentiable and monotonic.

E. f is continuous if and only if f is constant.

Question 2 .... [3 marks]

Consider the following

Theorem. Let $a < b$ be real numbers and let f be an increasing function defined on the interval (a, b). Then
$\lim_{x \to b^-} f(x) = \sup\{f(x) : x \in (a, b)\}$.

and its attempted proof below. Circle the statement which makes this proof incorrect.

A. Let $L = \sup\{f(x) : x \in (a, b)\}$ and $\varepsilon > 0$.

B. Then there is $c \in (a, b)$ such that $f(c) < L - \varepsilon$.

C. Put $\delta = b - c > 0$.

D. Now let $x \in (c, b)$. Then $c < x$ gives $f(c) \le f(x)$ since f is increasing and $f(x) \le L$ for all $x \in (c, b) \subset (a, b)$ by definition of the supremum.

E. Then

$$L - \varepsilon < f(c) \le f(x) \le L < L + \varepsilon$$

for these x. By definition, this mean $f(x) \to L$ as $x \to b^-$.

Question 3 .... [4 marks]

Assume that I is an interval and that $f : I \to \mathbb{R}$ is continuous. Which of the following statements is incorrect?

A. $f(I)$ is an interval or a singleton.

B. If f is monotonic, then the function $f^{-1} : f(I) \to \mathbb{R}$ is continuous.

C. If I is bounded and closed, then $f(I)$ has a maximum.

D. The function h defined by $h(x) = \ln(1 + (f(x))^2)$ is continuous.

E. If I is bounded, then $f(I)$ is bounded.

Question 4 .... [4 marks]

Let $f : \mathbb{R} \to \mathbb{R}$ be infinitely differentiable, i. e., $f^{(n+1)} = \left(f^{(n)}\right)' : \mathbb{R} \to \mathbb{R}$ is defined for all $n \in \mathbb{N}$ where $f^{(0)} = f$. Assume that there is some $n \in \mathbb{N}$ with $n \geq 1$ such that $f^{(n+1)}(0) \neq 0$ and $f^{(j)}(0) = 0$ for all $j \in \mathbb{N}$ with $j \leq n$. For $j \in \mathbb{N}$ let $A(j)$ be the statement

$$\text{If } \lim_{x\to 0} \frac{f^{(j+1)}(x)}{f^{(j+2)}(x)} \text{ exists, then } \lim_{x\to 0} \frac{f^{(j)}(x)}{f^{(j+1)}(x)} \text{ exists, and } \lim_{x\to 0} \frac{f^{(j)}(x)}{f^{(j+1)}(x)} = \lim_{x\to 0} \frac{f^{(j+1)}(x)}{f^{(j+2)}(x)}.$$

One or both of the following sets may be used in a correct answer to the question below:

$$S = \{j \in \mathbb{N} : A(j) \text{ is true}\},$$
$$T = \left\{ j \in \mathbb{N} : j \leq n,\ \lim_{x\to 0} \frac{f^{(j)}(x)}{f^{(j+1)}(x)} \text{ exists} \right\}.$$

Below are five outlines of attempts to prove or disprove

$$\lim_{x\to 0} \frac{f(x)}{f'(x)} = 0. \qquad (*)$$

Which of these outlines is correct?

A. $0 \in S$ and $j \in S$ implies $j + 1 \in S$ by mathematical induction in view of l'Hôpital's rule. Hence $S = \mathbb{N}$, which proves $(*)$.

B. $0 \in T$ and $j \in T$ implies $j + 1 \in T$ by mathematical induction in view of l'Hôpital's rule. Hence $T = \mathbb{N}$, which proves $(*)$.

C. $T \neq \emptyset$. Hence $k = \min(T)$ exists by the well-ordering principle. $k > 0$ is impossible since then $k - 1 \in T$ by l'Hôpital's rule. Hence $\min(T) = 0$, which proves $(*)$.

D. $(*)$ is correct but none of the above attempts are correct since $S = \emptyset$ and $T = \emptyset$ are possible.

E. $(*)$ is false.

Question 5 .... [3 marks]

Consider the series $\sum_{n=1}^{\infty} a_n$. Which of the following statements is false?

A. If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

B. If $\limsup_{n\to\infty} \sqrt[n]{|a_n|} \geq 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.

C. If $a_n \neq 0$ for all n and if $\limsup_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

D. If $a_n = \frac{(-1)^n}{n}$, then $\sum_{n=1}^{\infty} a_n$ converges conditionally.

E. None of the above.

SECTION B

Answer this section in the answer book provided.

Question 1 .... [9 marks]

Write down the definitions of the following limits of functions where $a, L \in \mathbb{R}$ and f is a real-valued functions. Also write down the assumptions for the domain of f.

(a) $\lim_{x \to a} f(x) = L$. (3)

(b) $\lim_{x \to a^+} f(x) = -\infty$. (3)

(c) $\lim_{x \to \infty} f(x) = L$. (3)

Question 2 .... [14 marks]

Prove from the definitions that

(a) $\lim_{x \to -\infty} \dfrac{2x^2 - 1}{x^2 + 2} = 2$ (8)

(b) $\lim_{x \to -1^-} \dfrac{1}{x + 1} = -\infty$ (6)

Question 3 .... [6 marks]

Let $a \in \mathbb{R}$ and let f be continuous at a with $f(a) \neq 0$. Prove that the function $\dfrac{1}{f}$ is also continuous at a.

Question 4 .... [8 marks]

Let $f : (0, 1) \to \mathbb{R}$ be an increasing function such that $f((0, 1))$ is an interval. Show that f is continuos from the left. (f is also continuous from the right, but you do not need to prove this.)

Question 5 .... [7 marks]

(a) For which $x \in \mathbb{R}$ does the geometric series $\sum_{n=0}^{\infty} x^n$ converge? (1)

(You do not need to justify your answer.)

(b) Let $a \in \mathbb{R}$ with $|a| > 1$. For which $x \in \mathbb{R}$ does the series $\sum_{n=0}^{\infty} a x^n$ converge? (1)

(c) What are the radii of convergence of $\sum_{n=0}^{\infty} x^n$ and $\sum_{n=0}^{\infty} a x^n$? (1)

(d) Using your answer to part (c) or otherwise, find $\limsup_{n \to \infty} \sqrt[n]{|a|}$. (1)

(e) Find $\liminf_{n \to \infty} \sqrt[n]{|a|}$. (2)

(f) Show that $\lim_{n \to \infty} \sqrt[n]{|a|}$ exists and find its value. (1)

Question 6 .... [Bonus Question: 3 marks]

Elaborate on the outline of your answer to the MCQ question 4 from part A. Note that if your answer is D or E, then you would have to provide an example which illustrates your answer.