Multivariable Calculus

MATH2007

1.6 Maxima and Minima
(Part 1)

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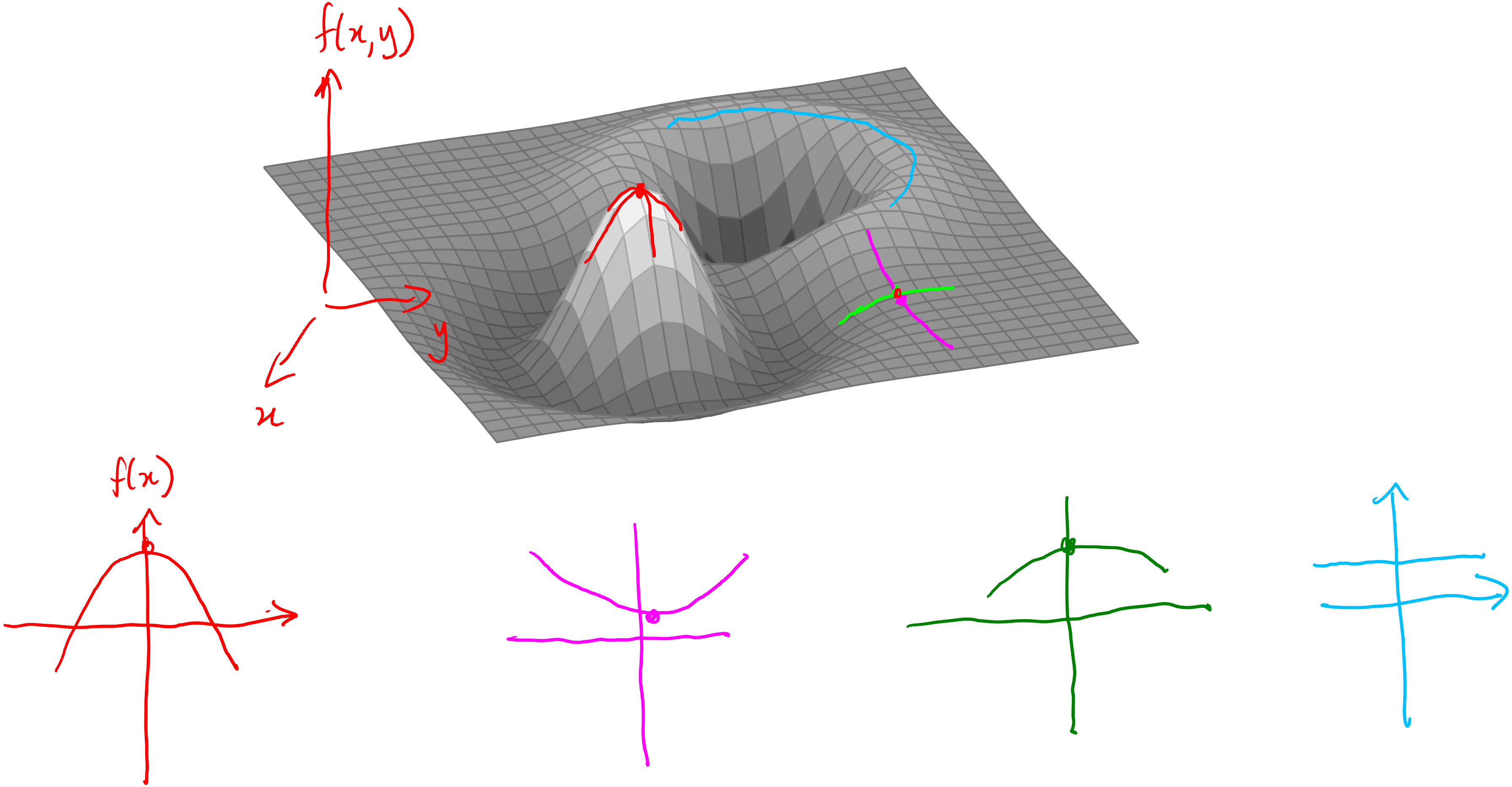
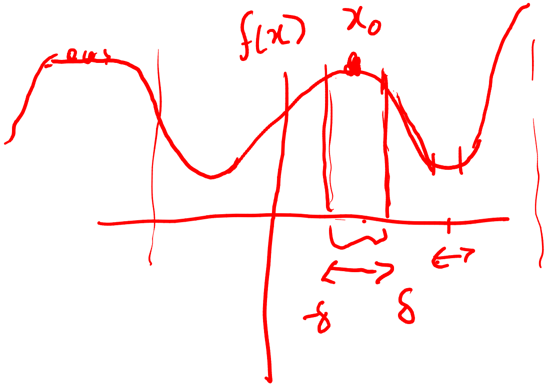
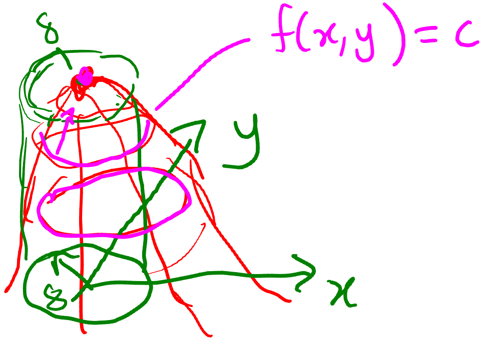
Example.

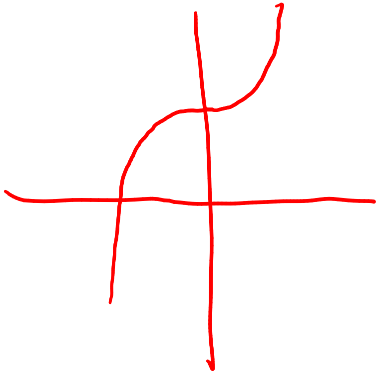


Definition (1.6.1)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$, we say that f has:

1. a local maximum at $\mathbf{x}_0$ if there exists $\delta > 0$ such that $\|\mathbf{x} - \mathbf{x}_0\| < \delta \implies f(\mathbf{x}) \leq f(\mathbf{x}_0)$;
2. a strict local maximum at $\mathbf{x}_0$ if there exists $\delta > 0$ such that $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta \implies f(\mathbf{x}) < f(\mathbf{x}_0)$;
3. a local minimum at $\mathbf{x}_0$ if there exists $\delta > 0$ such that $\|\mathbf{x} - \mathbf{x}_0\| < \delta \implies f(\mathbf{x}) \geq f(\mathbf{x}_0)$;
4. a strict local minimum at $\mathbf{x}_0$ if there exists $\delta > 0$ such that $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta \implies f(\mathbf{x}) > f(\mathbf{x}_0)$;
5. a **critical point** at $\mathbf{x}_0$ if $\nabla f(\mathbf{x}_0) = \mathbf{0}$;
6. a **saddle point** at $\mathbf{x}_0$ if $\mathbf{x}_0$ is a critical point which is neither a local maximum nor a local minimum.







Multivariable Calculus

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1.6 Maxima and Minima
(Part 2)

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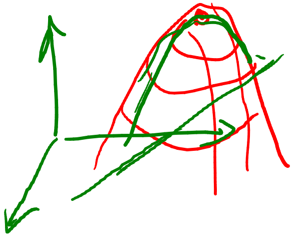
Definition (1.6.1)**.** Let $f : \mathbb{R}^n \to \mathbb{R}$, we say that f has:

1. a local maximum at $\mathbf{x}_0$ if there exists $\delta > 0$ such that $\|\mathbf{x} - \mathbf{x}_0\| < \delta \implies f(\mathbf{x}) \leq f(\mathbf{x}_0)$;
2. a strict local maximum at $\mathbf{x}_0$ if there exists $\delta > 0$ such that $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta \implies f(\mathbf{x}) < f(\mathbf{x}_0)$;
3. a local minimum at $\mathbf{x}_0$ if there exists $\delta > 0$ such that $\|\mathbf{x} - \mathbf{x}_0\| < \delta \implies f(\mathbf{x}) \geq f(\mathbf{x}_0)$;
4. a strict local minimum at $\mathbf{x}_0$ if there exists $\delta > 0$ such that $0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta \implies f(\mathbf{x}) > f(\mathbf{x}_0)$;
5. a **critical point** at $\mathbf{x}_0$ if $\nabla f(\mathbf{x}_0) = \mathbf{0}$;
6. a **saddle point** at $\mathbf{x}_0$ if $\mathbf{x}_0$ is a critical point which is neither a local maximum nor a local minimum.

Theorem (1.6.2)**.** If $f : \mathbb{R}^n \to \mathbb{R}$ has a local maximum or a local minimum at $\mathbf{x}_0$, then $\mathbf{x}_0$ is a critical point of f.

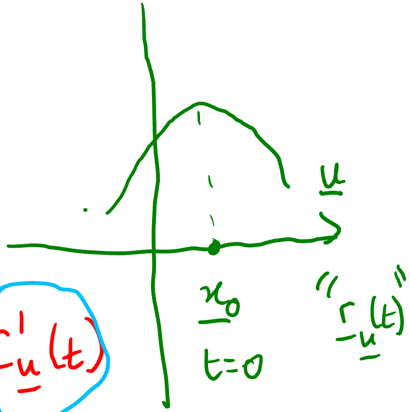
Proof. Let $\underline{r}_{\underline{u}}(t) = \underline{x}_0 + t\,\underline{u}$ be a straight line through $\underline{x}_0$, where $t \in \mathbb{R}$ and $\underline{u}$ is a unit vector.



Consider $f(\underline{r}_{\underline{u}}(t))$. $f(\underline{r}_{\underline{u}}(t))$ has a local maximum (minimum) at $t=0$ (i.e. $\underline{x}_0 = 0$) when $f(\underline{x})$ has a local maximum (minimum) at $\underline{x}_0$.

Assume $f(\underline{x})$ has a local maximum (minimum) at $\underline{x}_0$. Then

$$\frac{d}{dt} f(\underline{r}_{\underline{u}}(t))\Big|_{t=0} = 0, \quad \text{but} \quad \frac{d}{dt} f(\underline{r}_{\underline{u}}(t)) = \nabla f(\underline{r}_{\underline{u}}(t)) \cdot \underline{r}'_{\underline{u}}(t)$$



so that $$0 = \nabla f(\underline{r}_{\underline{u}}(t))\Big|_{t=0} \cdot \underline{u}.$$ Since $\underline{u}$ was arbitrary

$$0 = \nabla f(\underline{r}_{\underline{u}}(t))\Big|_{t=0} \cdot \underline{u}$$ for all unit vectors $\underline{u}$

□

Theorem (1.6.2)**.** If $f : \mathbb{R}^n \to \mathbb{R}$ has a local maximum or a local minimum at $\mathbf{x}_0$, then $\mathbf{x}_0$ is a critical point of f.

Proof. (continued) □

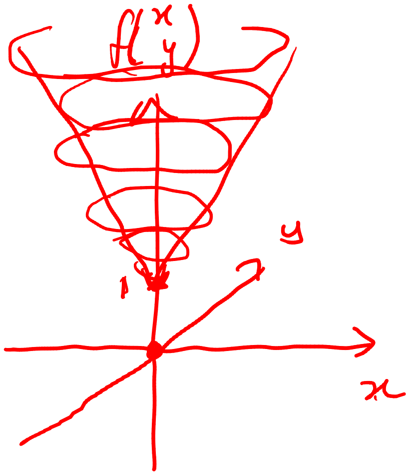
$$0 = \nabla f(\underline{r}_{\underline{u}}(t))\big|_{t=0} \cdot \underline{u} \quad \text{for all unit vectors } \underline{u}$$

Suppose $\boxed{\nabla f(\underline{r}_{\underline{u}}(t))\big|_{t=0} \neq \underline{0}}$. Take $\underline{u} = \dfrac{\nabla f(\underline{r}_{\underline{u}}(t))\big|_{t=0}}{\|\nabla f(\underline{r}_{\underline{u}}(t))\big|_{t=0}\|} = \dfrac{\nabla f(\underline{x}_0)}{\|\nabla f(\underline{x}_0)\|}$)

then $0 = \nabla f(\underline{x}_0) \cdot \dfrac{\nabla f(\underline{x}_0)}{\|\nabla f(\underline{x}_0)\|} = \|\nabla f(\underline{x}_0)\| \neq 0$, a contradiction.

Thus $\nabla f(\underline{r}_{\underline{u}}(t))\big|_{t=0} = \nabla f(\underline{x}_0) = \underline{0}$; and $\underline{x}_0$ is a critical point of f.

Example. Let $\underbrace{f\begin{pmatrix} x \\ y \end{pmatrix}}_{(r^2-1)} = x^2 + y^2 + 1$. Find the minimum by inspection. Verify that Theorem 1.6.2 holds at this point.



The local minimum (also global) of $f\begin{pmatrix} x \\ y \end{pmatrix}$ is 1

at $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$\nabla f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$ $\qquad \nabla f\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

and therefore $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is a critical point.

Theorem (1.6.3 Several variable Taylor Theorem)**.**
Let $f : \mathbb{R}^n \to \mathbb{R}$, then

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + df[\mathbf{x};\mathbf{h}] + R(\mathbf{x};\mathbf{h})$$

where $R(\mathbf{x};\mathbf{h})/\|\mathbf{h}\| \to 0$ as $\|\mathbf{h}\| \to 0$.

Proof: omitted.

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1.6 Maxima and Minima
(Part 3)

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Theorem (1.6.4)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $\nabla f(\mathbf{x}_0) = \mathbf{0}$. Define the **discriminant** of f at $\mathbf{x}_0$ by

$$\Delta = \frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}_0)\frac{\partial^2 f}{\partial x_2^2}(\mathbf{x}_0) - \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x}_0)\right]^2.$$

1. If $\Delta > 0$ and $\frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}_0) > 0$, then f has a strict local minimum at $\mathbf{x}_0$.
2. If $\Delta > 0$ and $\frac{\partial^2 f}{\partial x_2^2}(\mathbf{x}_0) > 0$, then f has a strict local minimum at $\mathbf{x}_0$.

equivalent

3. If $\Delta > 0$ and $\frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}_0) < 0$, then f has a strict local maximum at $\mathbf{x}_0$.
4. If $\Delta > 0$ and $\frac{\partial^2 f}{\partial x_2^2}(\mathbf{x}_0) < 0$, then f has a strict local maximum at $\mathbf{x}_0$.

equivalent

5. If $\Delta < 0$ then f has a saddle point at $\mathbf{x}_0$.

Hessian matrix of f at $\underset{\sim}{x}_0$: $H_f(\underset{\sim}{x}_0) = \left.\begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{pmatrix}\right|_{\underset{\sim}{x} = \underset{\sim}{x}_0}$

$\Delta = \det(H_f(\underset{\sim}{x}_0))$.

Example. Let $f(x,y) = y^2 + x^2 + 2x + 2y$. Find the critical points and classify them.

Critical points : $\nabla f(x,y) = 0$, i.e $\begin{pmatrix} 2x+2 \\ 2y+2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} x=-1 \\ y=-1 \end{matrix}$

There is only one critical point: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

$\frac{\partial^2 f}{\partial x^2} = 2 \qquad \left.\frac{\partial^2 f}{\partial x^2}\right|_{\substack{x=-1 \\ y=-1}} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0 \qquad \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{\substack{x=-1 \\ y=-1}} = 0$

$\frac{\partial^2 f}{\partial y^2} = 2 \qquad \left.\frac{\partial^2 f}{\partial y^2}\right|_{\substack{x=-1 \\ y=-1}} = 2 \qquad \Delta = \left.\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2\right)\right|_{\substack{x=-1 \\ y=-1}} = 2 \cdot 2 - 0 = 4$

Since $\Delta > 0$ and $\left.\frac{\partial^2 f}{\partial x^2}\right|_{\substack{x=-1 \\ y=-1}} = 2 > 0$, f has a strict local minimum at $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$.

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1.6 Maxima and Minima (Part 4)

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Example. Let $f(x, y) = (2 + \cos y)(3 + \sin x)$. Find the critical points and classify them.

Critical points: $\nabla f(x,y) = \underline{0}$

$$\nabla f(x,y) = \begin{pmatrix} (2+\cos y)\cos x \\ -(3+\sin x)\sin y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{ll} \cos y \in [-1,1] & \overbrace{(2+\cos y)}^{\in [1,3]}\cos x = 0 \\ \sin y \in [-1,1] & -\underbrace{(3+\sin x)}_{\in [2,4]}\sin y = 0. \end{array}\right\} \Rightarrow \left.\begin{array}{l} \cos x = 0 \\ \sin y = 0 \end{array}\right\} \Rightarrow \begin{array}{ll} x = \frac{\pi}{2} + n\pi & n \in \mathbb{Z} \\ y = m\pi & m \in \mathbb{Z} \end{array}$$

So the critical points are $\begin{pmatrix} \pi/2 + n\pi \\ m\pi \end{pmatrix}$ (infinitely many).

So the critical points are $\underset{\sim}{x}_{m,n} = \begin{pmatrix} \pi/2 + n\pi \\ m\pi \end{pmatrix}$ (infinitely many).

$f(x,y) = (2+\cos y)(3+\sin x)$

$\frac{\partial f}{\partial x} = (2+\cos y)\cos x$

$\frac{\partial f}{\partial y} = -(3+\sin x)\sin y$

$\cos x = 0 \quad \sin y = 0$

$\cos y = \cos(m\pi) = (-1)^m$

$\sin x = \sin\left(\frac{\pi}{2}+n\pi\right) = (-1)^n$

Classification:

$\frac{\partial^2 f}{\partial x^2} = -(2+\cos y)\sin x \qquad \frac{\partial^2 f}{\partial y^2} = -(3+\sin x)\cos y$

$\frac{\partial^2 f}{\partial x \partial y} = -\sin y \cos x$

$\left.\frac{\partial^2 f}{\partial x^2}\right|_{\underset{\sim}{x}_{m,n}} = -(2+(-1)^m)(-1)^n$

$\left.\frac{\partial^2 f}{\partial y^2}\right|_{\underset{\sim}{x}_{m,n}} = -(3+(-1)^n)(-1)^m \qquad \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{\underset{\sim}{x}_{m,n}} = 0$

$\Delta = +\underbrace{(2+(-1)^m)}_{\geq 1}\underbrace{(3+(-1)^n)}_{\geq 2}(-1)^{m+n} - 0$

$$\left.\frac{\partial^2 f}{\partial x^2}\right|_{\underset{\sim}{x}_{m,n}} = -\left(2+(-1)^m\right)(-1)^n$$

$$\left.\frac{\partial^2 f}{\partial y^2}\right|_{\underset{\sim}{x}_{m,n}} = -\left(3+(-1)^n\right)(-1)^m \qquad \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{\underset{\sim}{x}_{m,n}} = 0$$

$$\Delta = +\overbrace{\left(2+(-1)^m\right)}^{\geq 1}\overbrace{\left(3+(-1)^n\right)}^{\geq 2}\underbrace{(-1)^{m+n}}$$

$$\begin{cases} > 0 & m+n \text{ is even} \\ < 0 & m+n \text{ is odd} \end{cases}$$

Saddle point

$m+n$ even, $\quad \left.\dfrac{\partial^2 f}{\partial x^2}\right|_{\underset{\sim}{x}_{m,n}} = -\overbrace{\left(2+(-1)^m\right)}^{\geq 0}(-1)^n \quad \begin{cases} < 0 & n \text{ even} \\ > 0 & n \text{ odd} \end{cases}$

$m+n$ odd $\Rightarrow$ $\underset{\sim}{x}_{m,n}$ is a sadde point of $f(x,y)$

~~$m+n$ even~~, $\begin{cases} n \text{ even} \\ m \text{ even} \end{cases}$ $\Rightarrow$ $\underset{\sim}{x}_{m,n}$ gives a local maximum of $f(\underset{\sim}{x}_{m,n})$

~~$m+n$ even~~, $\begin{cases} n \text{ odd} \\ m \text{ odd} \end{cases}$ $\Rightarrow$ $\underset{\sim}{x}_{m,n}$ gives a local minimum of $f(\underset{\sim}{x}_{m,n})$