# METHODS A - MATHEMATICAL METHODS AND MODELLING

## APPM2021A/APPM2022A

Course Notes
by
Authors

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# Contents

# Course brief and outline 2023

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## Venues and lecture times

Please check the noticeboard for lecture venues and times.

## Course background and purpose

The aim of this course is to cover the basic theory of integration of ordinary differential and difference equations. We will make the vocabulary *(integration, differential equation and difference equation)* more transparent in subsequent chapters.

Differential and difference equations are widely used in the mathematical modelling of real systems. A mathematical model is merely a mathematical construct designed to study real-world systems. The main purpose of a mathematical model is to explain and make predictions about the system studied and analyse the effects of various changes on the system.

The modelling process can roughly be separated into four phases. The first phase is the formulation of the model. In our case, we are interested in models which are given in terms of differential or difference equations. The second phase is the analysis of the model. This phase consists of finding the solution of the model (difference or differential equation) obtained in the first phase. The third phase is the interpretation of the solution: the predictions of the solution and their explanations are given. The last phase i.e. the fourth phase is the validation of the predictions of the model with experimental data. If should be obvious that the model that agrees the most with experimental data will be preferred.

There are three main methods for solving mathematical models: analytical, numerical and qualitative. The three approaches complement each other. In this course, we will focus mainly on analytical techniques. These are basically recipes to obtain solutions of differential and difference equations in terms of elementary functions.

## Examples of mathematical models.

1. Bank loan model - Assume that you borrowed an amount of $D$ Rand from a bank at an interest rate of $r\%$ per annum and that you agreed to settle your debt by paying an amount of $p$ Rand per annum for $n$ years. What is your annual payment? The mathematical model that governs this transaction is the following

$$d_{k+1} = d_k + r d_k - p, \quad d_0 = D, \quad d_n = 0,$$

   where $d_k$ is the amount owed after $k$ years. It is a difference equation.

2. Newton's equations of motion - This model is used to predict the motion of objects acted on by external forces. In the case of an object of mass $m$ moving under the action of a force $F$, the motion is described by

$$m\dot{r} = F(t, r, \dot{r}),$$

   where $r$ is the position vector of the object.

3. Finance: Black-Scholes model - 1973

$$u_i + \frac{1}{2}A^2 x^2 u_{xx} + Bxu_x - Cu = 0.$$

4. Fluid Mechanics: Navier-Stokes equation

$$\rho\frac{Du}{Df} = -\nabla p + \rho g + \mu\nabla^2 u; \quad \nabla \cdot u = 0$$

# Course content and outcomes

## Course content.

1. Integration techniques: we summarise some important integration methods that will be important for the solution of differential equations.
2. Methods for finding solutions of scalar ordinary differential equations. We will focus mainly on first- and second-order ordinary differential equations.
3. Introduction to discrete equations.
4. Introduction to systems of equations.
5. Solutions to linear systems of ordinary differential and difference equations. Outcomes

## Chapter one - Review of integration techniques.

This chapter reviews some of the important integration techniques. After completing this chapter you should be able to perform the following:

1. Calculate the differential of a given function.
2. Simplify an integral using a change of variable.
3. Apply the method of integration by parts to simplify an integral.
4. Recognise a rational function.
5. Use the method of partial fractions to integrate rational functions.
6. Recognise the common integrals listed in Table 1.5

## Chapter two - Scalar $n$ th-order ordinary differential equations.

This chapter introduces the concept of a differential equation, and introduces some elementary solution techniques. After completing this chapter you should be able to:

1. Classify a differential equation as ordinary or partial.
2. Determine the order of a given differential equation.
3. Verify that a given function is a solution to a differential equation.
4. Determine the degree of an ordinary differential equation.

5. Integrate seperable first order odes.

6. Apply a change of variables to an ordinary differential equation.

7. Verify that an ordinary differential equation is exact.

8. Solve exact ordinary differential equations by direct integration.

9. Show that a given function is an integrating factor for a given ordinary differential equation.

10. Apply the method of variation of parameters to first-order linear differential equations.

11. Apply a given change of variables to re-write the following equations in linear form

    - The bernoulli equation
    - The Ricatti equation
    - The Abel equation of the second kind

12. Use the "D-operator" method to solve general $n$-th order odes.

13. Solve linear $n$-th order homogenous odes with constant coefficients by assuming a solution in the form $y = e^{\lambda x}$.

14. Use the method of undetermined coefficients to find a particular solution to non-homogenous constant coefficient linear odes.

15. Apply the method of variation of parameters to second-order linear odes.

## Chapter three - Difference equations.

This chapter introduces the discrete counterpart of the differential equation - the difference equation. We also discuss techniques for solving difference equations that are similar to the techniques for differential equations. After completing this chapter you should be able to :

1. Determine the order of a difference equation.

2. Determine whether a given difference equation is homogenous.

3. Classify a difference equation as linear or non-linear.

4. Solve linear homogenous difference equations by assuming a solution in the form $y_k = \lambda^k$.

5. Use an analogue of the method of undetermined coefficients to calculate a particular solution to non-homogenous constant coefficient linear difference equations.

## Chapter four - Solutions to Linear Systems.

This chapter introduces the important concept of a system of differential equations. We then examine the linear case in detail, and build a powerful algebraic technique for their solution. After completing this chapter you should be able to :

1. Distinguish between continuous and discrete systems.
2. Classify a dynamical system as linear or non-linear.
3. Reduce a higher order ordinary differential equations to a system of first order ordinary differential equations.
4. Rewrite a system of first order linear ordinary differential equations in algebraic notation.
5. Determine the eigenvalues and eigenvectors of a real matrix.
6. Diagonalise a given real matrix.
7. Solve a linear first order system of differential equations by diagonalisation.
8. Define what is meant by the matrix exponential $e^{At}$.
9. Show that the matrix exponential $e^{At}$ is a solution to the matrix differential equation $\dot{X} = A\mathbf{X}$
10. Show that for any constant vector $u, e^{At}u$ is a solution to the differential equation $\dot{x} = Ax$.
11. Calculate the matrix exponential $e^{\mathrm{At}}$ using diagonalisation when $A$ has distinct eigenvalues.
12. Compute the matrix exponential from the series definition.
13. Define what is meant by a fundamental matrix solution of the system $\dot{x} = Ax$.
14. Compute a fundamental matrix solution of the system $\dot{x} = Ax$ when $A$ has distinct eigenvalues.
15. Compute a fundamental matrix solution of the system $\dot{x} = Ax$ when $A$ has repeated eigenvalues.

## Assessment

Your marks for assignments, class test, spot tests/quizzes and the final examination will be combined as follows to yield your final mark for the topic:

- 30% Examination
- 60% Three Class tests
- 10% Assignments

*If you are absent from a test you will get a zero mark for the test unless you produce a valid doctors certificate.*

If you bring a valid doctors certificate you will be excused from the test and a deferred test will be arranged for you. The certificates will be sent to the Faculty office and the doctor may be contacted to verify the certificate.

## Reference

**R Bronson** 1993 Theory and problems of differential equations (McGraw-Hill: New York)

**D G Zill, M R Cullen** Differential Equations with Boundary Value Problems (Thomson - Brooks/ColeMcGraw-Hill: New York)

# Chapter 1

# Scalar $n$ th-order ordinary differential equations

## 1.1 Preliminaries

Throughout this chapter, we will consider *scalar* i.e. single differential equations. We concentrate on ordinary differential equations (ODEs). A special emphasis will be put on some integration techniques. We begin with some useful notation and definitions. Then will revise some important basic integration techniques.

**NOTATION** 1.1**.**

$$\begin{aligned} y' &= \frac{dy}{dx} \text{ First order differential equation} \\ y'' &= \frac{d^2y}{dx^2} \\ y''' = y^{(3)} &= \frac{d^3y}{dx^3} \\ y^{(n)} &= \frac{d^ny}{dx^n} \end{aligned}$$

We usually try to write mathematics in the most readable, compact form, so we prefer $y^{(7)}$ to the rather clumsy $y'''''''$. The order of a differential equation is given by the highest derivative involved in the equation.

**DEFINITION** 1.1 (Ordinary differential equation). Let $x$ be the independent variable, $y$ the dependent variable and $y', y'', \ldots, y^{(n)}$ represent successive derivatives of $y$ with respect to $x$. Then any equation which involves at least one of these derivatives is called an *ordinary differential equation* or *ODE*. Therefore,

- A $1st$-order differential equation is derived from a function having 1 arbitrary constant.

- A $2nd$-order differential equation is derived from a function having 2 arbitrary constants.

- An $nth$-order differential equation is derived from a function having $n$ arbitrary constants.

**EXAMPLE 1.1.**
(1) $y' + 3x^2y^3 + C = 0$, where $C$ is a constant. An equation of the first order
(2) $y'' + a(x)y' + b(x)y = c(x)$, where $a, b$ and $c$ are functions of $x$ only. An equation of the second order.

**EXERCISE 1.1.** Is $y' + y^2 \geq x^2$ an ordinary differential equation? Why? It involves just the independent variable $x$ and independent variable $y'$

**DEFINITION 1.2.** A scalar partial differential equation or PDE is an equation involving two or more independent variables, a dependent variable and its partial derivatives.

**EXAMPLE 1.2.**
(1) $x^2\frac{\partial u}{\partial x} + 3y\frac{\partial u}{\partial y} = u$.
(2) $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$ (heat equation).
(3) $\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$ (wave equation).

We reiterate that we will not deal with the integration of partial differential equations.

**DEFINITION 1.3.** The order of a differential equation is the order of the highest derivative involved.

**EXAMPLE 1.3.**
(i) $\frac{d^2y}{dx^2} + x^2\left(\frac{dy}{dx}\right)^3 + y^4 = 0$ is of order 2 .
(ii) Any relation of the form $F\left(x, y, y', y'', \ldots, y^{(n)}\right) = 0, n \geq 1$, is an ODE of order $n$.

**DEFINITION 1.4.** A solution to a differential equation is any function that upon substitution into the differential equation yields an identity.

**EXAMPLE 1.4.** The function $y = x^2$ is a solution to the ODE $y' = 2x$. Why?

## 1.2 Origins of ordinary differential equations

**(a) As mathematical models**

For example, Newton's second law for a particle moving in an external force field gives rise to a first-order equation of the form $y' = f(t, y)$ where $y(t)$ is the particle velocity at time $t$ and second-order ODE of the form $y'' = f\left(t, y, y'\right)$, where $y(t)$ is the position of the object at time $t$. $f$ is the net force on the particle.

**A falling object**

Consider an object falling down from height $y_0$. Let $v(t)$ be its velocity at a time $t$. According to Newton's second law,

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = F_{\text{net}}\,. \tag{1.1}$$

or in terms of distance $y(t)$ covered,

$$m\frac{d^2y}{dt^2} = F_{\text{net}} \tag{1.2}$$

If $F_{\text{net}}$ is a constant, say $F_{\text{net}} = mg$, then with the positive direction taken to be in the downward direction,

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g \tag{1.3}$$

where $g$ is the acceleration due to gravity.

In real life, however, the particle will experience some resistive force due to air.
If the resistive force is assumed proportional to the particle velocity, say $F_r = hv$, the model becomes

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - kv, \quad k = \frac{h}{m} \tag{1.4}$$

The constant $k$ is the resistance per unit mass which depends on the size and shape of the object.
If, however, the resistive force is assumed proportional to the square of particle velocity, say $F_r = hv^2$, the model becomes

$$\frac{\mathrm{d}v}{\mathrm{d}t} = g - kv^2, \quad k = \frac{h}{m} \tag{1.5}$$

Equations (1.4) and (1.5) are differential equations of motion of the falling object.

If $v(t)$ reaches a limiting velocity (terminal velocity) that is constant, say, $v_1$, for which acceleration is zero, then in (1.4), we have

$$v_l = \frac{g}{k}$$

while in (1.5), we have

$$v_j = \sqrt{\frac{g}{k}}.$$

Given that the particle begins to fall from rest, students should be able to show that the solutions to (1.4) and (1.5) are respectively

$$v(t) = v_l\left(1 - e^{-kt}\right), \tag{1.6}$$

$$v(t) = \frac{v_l\left(e^{\frac{2gt}{v_l}} - 1\right)}{e^{\frac{2gt}{v_l}} + 1}. \tag{1.7}$$

Show that as $t \longrightarrow \infty$ in (1.6) and (1.7), v(t) $\longrightarrow v_l$.

Show also that when there are no air resistance, equations (1.6) and (1.7) reduce to

$$v(t) = gt.$$

**(b) Mathematically**

Let us now illustrate in a simple way, how differential equations arise here. We resort to the geometry of plane curves. The equation

$$f(x, y, C) = 0 \tag{1.8}$$

where $C$ is an arbitrary constant and represents a family of curves in which each curve corresponds to a given value of $C$. Fix $C$ for a moment and differentiate (1.8) with respect to $x$ remembering that $y$ depends on $x$. This gives

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} y' = 0. \tag{1.9}$$

In general (1.9) will involve $C$. Let us assume that $\partial f / \partial y \neq 0$. By eliminating $C$ between (1.8) and (1.9), we obtain an equation involving $x, y$ and $y'$, say

$$F\left(x, y, y'\right) = 0, \tag{1.10}$$

which is an ODE of the first order.

Geometrically, the ODE (1.10) implies that at any chosen point $(x, y)$, the derivative $y'$ has a certain value(s). A property of the gradient or slope of any curve of the family (1.8) that passes through the selected point $(x, y)$ is thus characterised.

**EXAMPLE 1.5.**

(1) $y = x^2 + C$ is a family of equal parabolas. Differentiating both sides with respect to $x$

yields $y' = 2x$. The constant $C$ has disappeared. So this is the differential equation of the family of parabolas. What does it express geometrically?

(2) $y = Cx^2$ is a family of parabolas. Differentiate both sides with respect to $x$ to get $y' = 2Cx$ which involves $C$. Eliminate $C$ using $y = Cx^2$ to obtain the differential equation $y' = 2y/x$. Geometrically, what does this imply?

If (1.10) is polynomial in $y'$, the highest power of $y'$ is referred to as the degree of the equation.

**EXAMPLE 1.6.** The equation

$$\left(\frac{dy}{dx}\right)^2 + xy^3 + y^2 + y = 0$$

is of degree 2 . Similarly, the degree of an ODE of any order is the highest power of the highest derivative.

**EXAMPLE 1.7.** The equation

$$\left(\frac{d^2y}{dx^2}\right)^3 + x\left(\frac{dy}{dx}\right)^4 + y = 0$$

is of degree 3 .

## 1.3 Integration

Consider a general $n$ th-order ODE

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0, \quad y^{(i)} \equiv \frac{d^i y}{dx^i}.$$

This equation is said to be in implicit form. The explicit or normal form is

$$y^{(n)} = G\left(x, y, y', \ldots, y^{(n-1)}\right).$$

To integrate or solve a differential equation of order $n$ is to find all relations $f(x, y) = 0$ such that the values of $y, y', \ldots, y^{(n)}$ obtained from $f(x, y) = 0$ in terms of $x$ satisfy the differential equation identically.

When an infinite set of solutions is grouped as

$$f\left(x, y, C_1, \ldots, C_n\right) = 0$$

involving $n$ arbitrary constants $C_1, \ldots, C_n$, it is the general solution. A particular solution is one that is obtained from the general solution by giving particular values to the arbitrary constants $C_i$ s.

If for an $n$ th-order ODE, the values $y(x_0), \dots, y^{(n-1)}(x_0)$ are specified at a given point $x_0$, we say that we have an initial value problem.

**EXAMPLE 1.8.** The equation $y' = 1$ implies $dy = dx$. By integrating both sides, we obtain that $y = x + C$ is the general solution. It describes a family of straight line parallel to each other (in order to probe this plot the solution curve for different values of $C$ ). If we set $C = 0$, we get a particular solution. Note that the general solution of a first-order ODE has one arbitrary constant.

The pair consisting of the equation $y' = 1$ and $y(0) - 1$ is an initial value problem since there is an initial condition, viz, $y(0) = 1$, i.e. $y(0) = 1$ into $y = x + C$ yields $C = 1$. So the solution to the initial value problem is $y = x + 1$.

Also, ODEs may have one solution or no real solution.

**EXAMPLE 1.9.** The ODE $|y'| + |y| = 0$ admits the trivial solution $y = 0$ whereas $(y')^2 + 1 = 0$ has no real solution although solutions do exist in the complex plane.

**DEFINITION** 1.5 (Singular solutions). Some differential equations admit solutions which cannot be obtained from the general solution by assigning given values to the arbitrary constants. Such solutions are termed singular.

**EXAMPLE 1.10.** Consider the first-order ODE $(y')^2 - xy' + y = 0$. Upon differentiation with respect to $x$, it yields $(2y' - x)y'' = 0$, i.e. $y'' = 0$ or $y' = x/2$. Hence $y = Ax + B$ or $y = x^2/4 + C$. Substituting these solutions into the original equation, we find that $B = -A^2$ and $C = 0$.
$y = Ax - A^2$ is the general solution whereas $y = x^2/4$ is a singular solution. It cannot be obtained from the general solution: it is quadratic contrar y to the general solution which is linear.

## 1.4 Review of Integration Techniques

In this preliminary chapter, we review some of the major techniques of integration, comprising *substitution methods, integration by parts, integration rational functions*

### 1.4.1 Differential of a function

Consider a function $u(x_1, \ldots, x_n)$. Its differential $du$ is

$$du = \frac{\partial u}{\partial x_1} dx_1 + \cdots + \frac{\partial u}{\partial x_n} dx_n \tag{1.11}$$

**EXAMPLE 1.11.**

$$\begin{aligned} u(x,y) &= x^2 + y^2 \\ du &= \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial x} dy \\ &= 2x dx + 2y dy \end{aligned} \tag{1.12}$$

**EXERCISE 1.1.**

Find the differentials of the following functions:
(1) $u = C$, where $C$ is a constant,
(2) $u(x,y) = \sqrt{x^2 + y^2}$,
(3) $u(x,y) = \sin(xy)$,
(4) $u(x) = f(g(x))$, where $f$ and $g$ are two real functions of $x$.

### 1.4.2 Substitution (change of variable in an integral)

In some cases, an integral can be converted into a more manageable form by just changing variables. If the integrand can be written in the form $f(g(x))g'(x)$, then we may make the substitution $u = g(x)$ (which implies $du = g'(x)dx$ ) and integrate as follows:

$$\int f(g(x))g'(x)dx = \int f(u)du. \tag{1.13}$$

If the function $f(u)$ has an easily identifiable integral, then all is well. If not, another substitution or integration method may be required. The common choices for $g(x)$ are arguments of trigonometric, hyperbolic functions or functions raised to powers.

**EXAMPLE 1.12.** To evaluate the integral

$$\int 12(3\sin^2(x+1))^{1/4} \sin(x) \cos(x)\, dx$$

we may consider choosing

$$u = 3\sin^2 x + 1$$

which implies

$$du = 6\sin(x) \cos(x)\, dx.$$

Then

$$\begin{aligned}
&\int 12\left(3\sin^2 x+1\right)^{1/4} \sin(x) \cos(x)\, dx \\
&= \int 2\left(3\sin^2 x+1\right)^{1/4} (6\sin(x) \cos(x)\, dx) \\
&= \int 2u^{1/4} du \\
&= 2\left(\frac{4}{5}u^{5/4}\right)+C \\
&= \frac{8}{5}\left(3\sin^2 x+1\right)^{5/4}+C_r
\end{aligned}$$

where $C$ is a constant of integration.

### 1.4.3 Integration by parts

Let $u(x)$ and $v(x)$ be two differentiable functions. By the Leibniz rule for differentiating products, we have

$$[u(x)v(x)]' = u'(x)v(x) + u(x)v'(x), \tag{1.14}$$

i.e.

$$u(x)v'(x) = [u(x)v(x)]' - u'(x)v(x). \tag{1.15}$$

This yields upon integration

$$\int u(x)v'(x)dx = u(x)v(x) - \int u'(x)v(x)dx. \tag{1.16}$$

In the case of the definite integral, we have

$$\int_a^b u(x)v'(x)dx = [u(x)v(x)]_{x=a}^{x=b} - \int_a^b u'(x)v(x)dx. \tag{1.17}$$

Integration by parts is useful in getting rid of a part of the integral that makes it difficult to do. The annoying part of the integral is often chosen to be $u(x)$.

**EXAMPLE 1.13.** For the function $f(x) = x^3 \mathrm{e}^{x^2}$, we notice that it could be integrated by substitution if the $x^3$ were only $x$. If we choose $u(x) = x^2$ and $v'(x) = xe^{x^2}$, then we have

$$\int x^3 e^{x^2} dx = \frac{1}{2}x^2 e^{x^2} - \int xe^{x^2} dx$$

which implies

$$\int x^3 e^{x^2} dx = \frac{1}{2}x^2 \mathrm{e}^{x^2} - \frac{1}{2}\mathrm{e}^{x^2} + C = \frac{1}{2}\mathrm{e}^{x^2}\left(x^2-1\right) + C$$

where $C$ is a constant of integration.

**EXERCISE 1.2.**
(1) Integrate $f(x) = x^2 \sin(x)$ and $g(x) = \ln(x)$ by parts.
(2) Let

$$L_m = \int x^{\mathrm{m}} \sin(x)\, dx,$$

where $m$ is a natural number. Find a relation between $I_m$ and $I_{m-2}$ (hint: use integration by parts).

### 1.4.4 Integrating rational functions

**DEFINITION** 1.1 (Rational function). A rational function is a function that can be expressed as the ratio of two polynomials.

**EXAMPLE 1.14.**

$$R(x) = \frac{3x+2}{2x^2+x-3} \tag{1.18}$$

To integrate a rational function we use the *method of partial fractions* to split the fraction into easily integrable pieces. It is the reverse process of finding the common denominator in the addition of fractions.

Consider the operation

$$\frac{3}{2x-1} + \frac{-2}{x+4} = \frac{3}{(2x-1)}\frac{(x+4)}{(x+4)} + \frac{-2}{(x+4)}\frac{(2x-1)}{(2x-1)} = \frac{-x+14}{2x^2+7x-4}$$

In this operation, we proceeded from left to right. The method of partial fractions is concerned with the reverse operation: given a rational function $R(x) = P(x)/Q(x)$ with the degree of $P$ less than that of $Q$, split it into simple fractions of the form

$$\frac{A}{(ax+b)^r}, \frac{Bx+C}{\left(\alpha x^2+\beta x+\gamma\right)^5}, \tag{1.19}$$

where $A, B, C$ and $a, b, \alpha, \beta, \gamma$ are constants, $r$ and $s$ are natural numbers. The main task is to find $A, B$ and $C$ since the other parameters stem from the form of $Q(x)$.

We will use an example to illustrate the method.

**EXAMPLE 1.15.** Let us use the method of partial fractions to break up

$$\frac{x+6}{(3x-2)(x+1)} \equiv \frac{A}{3x-2} + \frac{B}{x+1}.$$

We wish to solve for $A$ and $B$ so that the identity holds. To do this, find a common denominator on the right and then set the numerators equal:

$$\frac{x+6}{(3x-2)(x+1)} = \frac{A(x+1)+B(3x-2)}{(3x-2)(x+1)}$$

which implies $x+6 = Ax + A + 3Bx - 2B$. Setting the coefficients of like powers of $x$ equal to zero, we get $A = -4$ and $B = -1$. So we find

$$\frac{x+6}{(3x-2)(x+1)} = \frac{4}{3x-2} - \frac{1}{x+1}. \tag{1.20}$$

**REMARK 1.1.** If the polynomial in the denominator does not factor out completely into linear terms over real numbers as in

$$\frac{5}{\left(x^2+1\right)(x+4)} \tag{1.21}$$

then we use the partial fraction decomposition

$$\frac{5}{\left(x^2+1\right)(x+4)} = \frac{Ax+B}{x^2+1} + \frac{C}{x+4}. \tag{1.22}$$

Notice the linear polynomial in the numerator of the fraction with the irreducible quadratic in the denominator. The remainder of the process is the same.

In summary, the method of partial fractions consists of the following steps

- Find a common denominator on the right.
- Set the numerators equal.
- Set the coefficients of the different powers of $x$ equal.
- Solve the system of equations obtained for the unknown coefficients.

**REMARK 1.2.** Note that the process of fraction decomposition is designed for fractions with a polynomial of a lesser degree in the numerator than the denominator. For problems where the opposite is true, long (or synthetic or Euclidean) division must be carried out first, and then the method of partial fractions used on the remainder term if needed.

### 1.4.5 Common integrals

Below we give a table of common integrals. Other integrals can be found in appropriate textbooks. We conclude this chapter with an important remark about notation.

**NOTATION 1.2.** In calculus, the symbol $\int f(x)dx$ of the *indefinite integral* stands for all the possible integrals of $f(x)$ i.e. all functions with derivatives equal to $f(x)$. But in the theory of differential equations, a different interpretation is widely used. Namely, $\int f(x)dx$ is identified with any particular integral of the function $f(x)$. This is the interpretation we shall use throughout the forthcoming chapters.

| $f(u)$ | $\int f(u)du$ |
|---|---|
| $u^n$, $n \neq -1$ | $u^{n+1}/(n+1)+C$ |
| $u^{-1}$ | $\ln(\lvert u\rvert)+C$ |
| $e^u$ | $e^u+C$ |
| $a^u$ | $a^u/\ln(a)+C$ |
| $\sin(u)$ | $-\cos(u)+C$ |
| $\cos(u)$ | $\sin(u)+C$ |
| $\sec(u)$ | $\ln(\lvert\sec(u)+\tan(u)\rvert)+C$ |
| $\csc(u)$ | $\ln(\lvert\csc(u)-\cot(u)\rvert)+C$ |
| $\sec(u)$ | $\tan(u)+C$ |
| $(\csc(u))^2$ | $-\cot(u)+C$ |
| $\sec(u)\tan(u)$ | $\sec(u)+C$ |
| $\csc(u)\cot(u)$ | $-\csc(u)+C$ |
| $\tan(u)$ | $\ln(\lvert\sec(u)\rvert)+C$ |
| $\cot(u)$ | $\ln(\lvert\sin(u)\rvert)+C$ |
| $\left(a^2+u^2\right)^{-1}$ | $a^{-1}\arctan u/a+C$ |
| $\left(a^2-u^2\right)^{-1/2}$ | $\arcsin u/a$ |

Table 1.1

# Chapter 2

# First order ordinary differential equations

## 2.1 Variables separable equations

They assume the form

$$f(y)y' + g(x) = 0,$$

where $f$ and $g$ are arbitrary functions. In order to integrate these equations, first rewrite them in the form

$$f(y)dy + g(x)dx = 0$$

and then integrate to obtain

$$\int f(y)dy + \int g(x)dx = C_r$$

where $C$ is an arbitrary constant.

Some equations do not present themselves readily in a separable form. Sometimes we need to perform a change of variables to uncover a separable form. Below we discuss few examples.

**Equations reducible to separable form (Homogeneous equations)**
Consider the first-order ODE

$$y' = \frac{M(x,y)}{N(x,y)},$$

where $M$ and $N$ are arbitrary functions of the same degree, say $n$. That is,

$$M(x,y) = x^n f(y/x), \quad N(x,y) = x^n g(y/x).$$

The original equation becomes

$$y' = \frac{f(y/x)}{g(y/x)} \equiv h(y/x).$$

This equation is known as a *homogeneous* first-order ODE. Clearly, as it stands, this equation is not variables separable. By changing the dependent variable to $v = y/x$ and keeping the independent the same, we obtain after a few manipulations the transformed equation

$$\frac{dv}{h(v)-v} = \frac{dx}{x},$$

which is variables separable.

**EXERCISE 2.1** Consider the change $X = kx, Y = ky$. Show that in these variables, the original equation reads

$$\frac{dY}{dX} = h(Y/X)$$

**EXAMPLE 2.1**
(1) Consider the ODE

$$\frac{dy}{dx} = \frac{xy}{x^2+y^2} = \frac{y/x}{1+y^2/x^2}.$$

Let $v = y/x$, then the original equation becomes

$$v + xv' = \frac{v}{1+v^2}$$

i.e.

$$\frac{1+v^2}{v^3}dv = -\frac{dx}{x}.$$

(2) $y' = \frac{ax+by+c}{dx+ey+f}$, where $a, \ldots, f$ are constants satisfying $ae - db \neq 0$. Then make the change

$$Y = ax + by + c, \quad X = dx + ey + f.$$

Why is this coordinates transformation invertible?
Since $dY = adx + bdy$ and $dX = ddx + edy$, we have in the new variables

$$\frac{dY}{dX} = \frac{a+by'}{d+ey'} = \frac{a+b(Y/X)}{d+e(Y/X)}.$$

This last equation is homogeneous and can be solved using the method described in the exercise above.
(3) Solve the same equation when $ae - db = 0$ (hint: use the fact that $(d, e) = \lambda(a, b)$, where $\lambda$ is a constant and make the change of variable $Y = ax+$ by).

## 2.2 Exact equations

Consider a first-order *ODE* in the form

$$y' = -\frac{M(x,y)}{N(x,y)} \tag{2.1}$$

or equivalently (verify!)

$$M(x,y)dx + N(x,y)dy = 0. \tag{2.2}$$

Equation (2.2) is said to be exact if its left-hand side is equal to $du(x,y)$, for some function $u(x,y)$. Then since $du(x,y) = 0$ the solution will be $u(x,y) = C$ where $C$ is an arbitrary constant. Now

$$M(x,y)dx + N(x,y)dy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0 \tag{2.3}$$

implies that (2.1) is exact if and only if there is a function $u(x,y)$ such that

$$M = \frac{\partial u}{\partial x'}, \quad N = \frac{\partial u}{\partial y'}, \tag{2.4}$$

If $u$ is twice differentiable, then

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

i.e.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

This last equation provides a practical criteria for (2.1) to be exact. Often we write this as $M_y = N_x$ in the subscript notation.

**EXAMPLE 2.2.** Consider the equation

$$xdx + ydy = 0.$$

For this equation, $M(x,y) = x$ and $N(x,y) = y$. It can be checked that $M_y = 0 = N_x$. Thus our equation is exact and we can take $u(x,y) = x^2/2 + y^2/2$ (check!).

**REMARK 2.1.** Note that $u(x,y)$ is not unique in the discussion above. Indeed, adding a constant to $u$ does not violate (2.4).

**EXERCISE 2.2.**

(1) Check that the following equation is exact and find its solution.

$$\left(\frac{1}{x} - \frac{y^2}{x^2}\right)dx + \frac{2y}{x}dy = 0.$$

(2) In the equation below, find $N(x,y)$ such that its exact. Then give the solution of the equation obtained.

$$\left(x^2+y^2\right)dx+N(x,y)dy=0$$

(3) Is the equation

$$y^2dx+x^2dy=0$$

exact? Justify your answer.

## 2.3 Inexact equations that can be made Exact

Suppose that

$$M(x,y)dx+N(x,y)dy=0$$

is not exact, i.e. $M_y \neq N_x$. If we can make it exact by multiplying it by a function $I(x,y)$, then $I(x,y)$ is an integrating factor. So an integrating factor is a solution of

$$\frac{\partial}{\partial y}(IM)=\frac{\partial}{\partial x}(IN). \tag{2.5}$$

Expanding (2.5), we find that

$$M\frac{\partial I}{\partial y}-N\frac{\partial I}{\partial x}=I\left(N_x-M_y\right) \tag{2.6}$$

which is a first-order PDE. Associate with this PDE the system

$$\frac{dy}{M}=\frac{dI}{I\left(N_x-M_y\right)}=\frac{dx}{-N}. \tag{2.7}$$

Solve the first or the second equation of the system (2.7) for $I$. Quite often, this is not an easy task!

**EXAMPLE 2.3.** The equation $xdy-ydx=0$ is not exact as $M_y=-1\neq 1=N_x$. In order to find an integrating factor, use the second equation of (2.7) i.e.

$$\frac{dI}{I(2)}=\frac{dx}{-x}.$$

This is a variables separable equation whose solution is $I=A/x^2$, where $A$ is an arbitrary constant. We choose for instance $A=1$, i.e. $I=1/x^2$. Thus

$$\frac{xdy-ydx}{x^2}=d(y/x)=0.$$

The solution to the initial equation is then $y/x=C$. This solution describes a family of straight lines.

**EXERCISE 2.3.** Show that $e^{\int a(x)dx}$ is an integrating factor of

$$(b(x)-a(x)y)dx-dy=0,$$

where $a$ and $b$ are functions of $x$.

## 2.4 Linear equations

The most general scalar first-order linear equation is:

$$y'+a(x)y=b(x). \tag{2.8}$$

where $a$ and $b$ are functions defined on an interval of $\mathbb{R}$. A first-order ODE which is not of this form is said to be *nonlinear.*

If $b\equiv 0$, then (2.8) is referred to as a homogeneous equation.
If $b\neq 0$, then (2.8) is said to be non-homogeneous.
In order to solve (2.8) we use the variation of parameters method. First, we solve the associated homogeneous equation which is variables separable to obtain

$$y=\mathrm{Ce}^{-\int a(x)dx},$$

where $C$ is an arbitrary constant. To solve (2.8), we assume that

$$y=y_1v(x), \tag{2.9}$$

where $y_1=e^{-\int a(x)dx}$ is a particular solution of the associated homogeneous equation. Now differentiate (2.9) and substitute the result into (2.8) to find that: $v(x)$ must satisfy

$$v'=b(x)e^{\int a(x)dx}$$

and thus

$$v=\int b(x)e^{\int a(x)dx}dx+A$$

where $A$ is an arbitrary constant, hence the general solution to (2.8) is

$$y=e^{-\int A(x)dx}\left(\int b(x)e^{\int a(x)dx}dx+A\right)$$

**EXERCISE 2.4.** Use the integrating factor method to solve (2.8).
Verify that if $y_1$ and $y_2$ are two solutions of a homogeneous linear equation, so is $\alpha y_1+\beta y_2$, where $\alpha$ and $\beta$ are constants.

## 2.5 Solutions by transformations

### 2.5.1 Homogeneous equation

### 2.5.2 Bernoulli equation (nonlinear equation)

The equation

$$y' + a(x)y = b(x)y^r, \quad r \neq 0,1, \quad r \in \mathbb{R} \tag{2.10}$$

is named the Bernoulli equation after its discoverer Jacques Bernoulli (1654-1705). The integration procedure for this equation was given by Leibniz in 1696 and will. be discussed below.

Divide (2.10) through by $y^r$ :

$$\frac{y'}{y^r} + a(x)y^{1-r} = b(x) \tag{2.11}$$

Note that the first term of (2.11) can be rewritten as

$$\frac{y'}{y^r} = \frac{d}{dx}\left(\frac{y^{1-r}}{1-r}\right).$$

Then (2.10) becomes

$$\frac{d}{dx}\left(\frac{y^{1-r}}{1-r}\right) + a(x)y^{1-r} = b(x).$$

This suggests the change of variable $z = y^{1-r}$. In the new variable, (2.10) reads

$$\frac{1}{1-r}z' + a(x)z = b(x) \tag{2.12}$$

which is linear.

**EXERCISE 2.5.** Directly make the change of variable $z = y^{1-r}$ in (2.10) and verify that you obtain (2.12).

### 2.5.3 Riccati equation

J. Francesco Riccati (1676 – 1754) in 1724 investigated the equation

$$y' = ay^2 + bx^m,$$

where $a \neq 0$, $b$ and $m$ are constants. In 1763, J. d'Alembert (1717-1783) investigated the more general equation

$$y' = a(x)y^2 + b(x)y + c(x), \quad a \neq 0 \tag{2.13}$$

which is referred to in the literature as the Riccati equation although Riccati-d'Alembert would be more appropriate! This equation enjoys some interesting properties. Note that if $a = 0$, (2.13) is linear, and if $c = 0$, (2.13) is a particular case of the Bernoulli equation. From now on we assume that $ac \neq 0$.

If $y_1(x)$ is a particular solution of (2.13), then perform the substitution

$$y = y_1 + \frac{1}{v}. \tag{2.14}$$

The equation becomes linear in $v$ :

$$v' + \left(2a(x)y_1 + b(x)\right)v = -a(x). \tag{2.15}$$

Equation (2.15) is a linear first-order ODE and its solution is

$$v = e^{-\int (2a(x)y_1 + b(x))\,dx}\left[K - \int a(x)e^{\int (2a(x)y_1+b(x))\,dx}\,dx\right], \tag{2.16}$$

where $K$ is an arbitrary constant.

We now deal with another first-order nonlinear ODE.

### 2.5.4 Abel equation of the second kind

The following equation

$$yy' + a(x)y = b(x) \tag{2.17}$$

appears in different fields of applied science for example Fluid Mechanics and General Relativity. It is named the Abel equation of the second kind. It is solvable only for certain forms of $a(x)$ and $b(x)$. Consider for instance the equation

$$yy' + Ax^{-2}y = Bx^{-3}. \tag{2.18}$$

In (2.18), perform the change $y = u/x$. Then $y' = u'/x - u/x^2$. Substituting this into (2.18), we obtain

$$\frac{u}{x}\left(\frac{u'}{x} - \frac{u}{x^2}\right) + Ax^{-2}\left(\frac{u}{x}\right) = Bx^{-3}. \tag{2.19}$$

Multiply (2.19) through by $x^3$ and deduce

$$xuu' - u^2 + Au = B, \tag{2.20}$$

i.e.,

$$\frac{udu}{u^2 - Au + B} = \frac{dx}{x}. \tag{2.21}$$

which is variable separable.

**EXERCISE 2.6.**
(1) Complete the integration of (2.21) (hint; consider the cases $A^2 - 4B$ is equal to, less than, and greater than 0)
(2) Consider the Abel equation

$$yy' + Ax^{k-1}y + Bx^{2k-1} = 0, \tag{2.22}$$

where $A, B$ and $k$ are constants. In (2.22), make the change $y = x^k u$ and show that (2.22), in term of $u$, is variables separable. Let $k = -1$ and recover (2.21).

## 2.6 Existence and uniqueness of solutions of first order Ordinary differential equation

Many differential equations cannot be solved by the standard and straightforward methods discussed so far, where solutions are obtained in terms of elementary functions like $e^x$, $\cos(x)$, $\sin(x)$, etc. There are many further techniques and tricks out there for finding explicit solutions of first order ordinary differential equations. However, if you write down an ordinary differential equation offhand at random, you have a slim chance of being able to integrate it to obtain solutions in terms of elementary functions, even if you use all of the tricks known to mathematicians.

It should, however, be noted that the mere absence of an explicit solution does not imply that no such solution exists. We would therefore still like to have a general criterion to determine if a given ordinary differential equation has a solution and also what additional conditions are required to ensure that the solution to the ordinary differential equation is unique.

Consider the initial value problem for the general first order ordinary differential equation given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(x, y), \quad y(x_0) = y_0. \tag{2.23}$$

So far, our experience is that solutions of first order ordinary differential equations are unique once we specify the value of the unknown function at a point.

For this general first order ordinary differential equations of the form (2.23), our interest is in answering the following questions:

1. Under what condition can we be sure that a solution to (2.23) exist? That is, among all the solutions to $\mathrm{d}y/\mathrm{d}x = F(x, y)$, do any of the solutions pass through the point $(x_0, y_0)$?

2. Under what condition can we be sure that there is precisely one and only one solution (unique solution) to (2.23) ?

An important theorem by Charles Emile Picard says that, under fairly mild assumption on $F(x, y)$, initial value problems have unique solutions, at least locally. Before stating the theorem, here is what we mean by a solution of a differential equation.

**DEFINITION 2.1** Let $I$ be an interval and $x_0 \in I$. We say that a differentiable function $y : I \to \mathbb{R}$ is a solution of (2.54) in the interval $I$ if $\frac{dy}{dx} = F(x, y)$ for all $x \in I$ and $y(x_0) = y_0$.

**THEOREM 2.1.** **(Existence)** Consider the interval $I_T = [x_0 - T, x_0 + T]$ and $B_d = [y_0 - e, y_0 + e]$ for positive, real

numbers $T$, e. Suppose that $F : I_T \times B_d \to \mathbb{R}$ is continuous. Then there is a d such that the initial value problem (2.23) has a solution in the interval $I_d = [x_0 - d, x_0 + d]$.

**THEOREM 2.2.** **(Uniqueness)** Suppose that $F : I_T \times B_d \to \mathbb{R}$ is continuous and that its partial derivative $\frac{\partial F}{\partial y} : I_T \times B_d \to \mathbb{R}$ is also continuous. Then the solution to the initial value problem (2.23) described in Theorem (2.1) in the interval $I_d = [x_0 - d, x_0 + d]$ is unique.

Since $F(x, y)$ and $\frac{\partial F}{\partial y}(x, y)$ are continuous in the closed and bounded domain $I_T \times B_d$, they are necessarily bounded there. This means that there exists a constant $K$ and $L$ such that

$$|F(x, y)| \leq K, \text{ for all } x, y \in I_T \times B_d. \tag{2.24}$$

and

$$\left|\frac{\partial F}{\partial y}(x, y)\right| \leq L \quad \text{for all} \quad x, y \in I_T \times B_d \tag{2.25}$$

The condition that $\frac{\partial F}{\partial y}$ exists and is continuous, implied by equation (2.25) can be replaced by the weaker condition that $F(x, y)$ satisfies the Lipschitz condition with respect to its

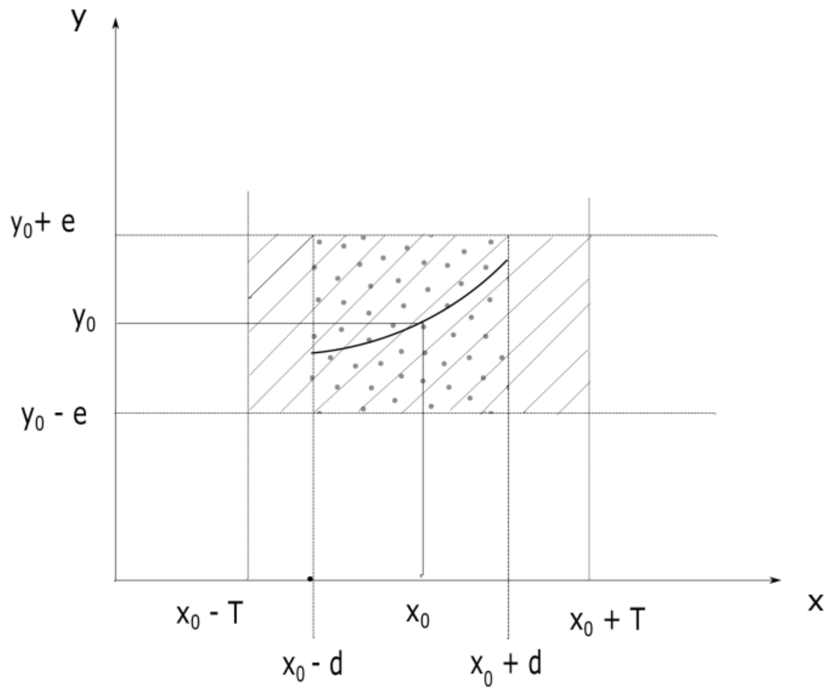


Figure 2.1: A picture illustrating the existence and uniqueness theorem

second argument. Thus, instead of the continuity of $\frac{\partial F}{\partial y}$, we require

$$\left|F\left(x, y_1\right) - F\left(x, y_2\right)\right| \leq L\left|y_1 - y_2\right| \text{ for all } x, y \in I_T \times B_d \tag{2.26}$$

If $\frac{\partial F}{\partial y}$ exists and is bounded, then it necessarily satisfies the Lipschitz condition. On the other hand, a function $F(x, y)$ may be Lipschitz continuous but $\frac{\partial F}{\partial y}$ may not exist.

For example, the function defined by

$$F(x, y) = x^2|y|, \quad |x| \leq 1, \quad |y| \leq 1$$

is Lipschitz continuous in $y$ but $\frac{\partial F}{\partial y}$ does not exist at $(x, 0)$. (Students to prove that this is true)

Let us now gain more understanding on how the existence and uniqueness theorem works. In Figure (2.1), the diagonally shaded region is the region $I_T \times B_d$ where the functions $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ are continuous. The point $\left(x_0, y_0\right)$ is contained in this region and the graph of the solution we seek has got to pass though this point, since at point $x_0$, the solution $y(x)$ satisfies $y\left(x_0\right) = y_0$. The region for the smaller interval $I_d$, which is $I_d \times B_d$, is, however, the region where we expect to have a unique solution to the initial value problem.

When faced with the statement of an unfamiliar theorem, certain questions come up in the mind of a mathematician. Questions like:

1. Is the meaning of each of the terms and conditions in the theorem clear?Are the stated conditions just sufficient and not necessary?

2. What then happens when these terms and conditions are violated

3. Can one give examples of initial value problems which do not satisfy the conditions of the existence and uniqueness theorem, for which there are no solutions or for which there are infinitely many solutions

These questions are addressed as follows:

1. Although Picard's theorem guarantees the existence of a solution for a large class of initial value problems, it is not always possible to find an explicit exact form of the solution in terms of elementary functions like $e^x$, $\sin(x)$, $\cos(x)$ etc. For example, the initial value problem
$$\frac{dy}{dx} = \sin(xy), \quad y(0) = 1$$
satisfies the condition of Picard's existence and uniqueness theorem, but no exact solution exist. Solutions may be obtained numerically.

2. If $F(x, y)$ is not continuous, then the initial value problem (2.23) may not have a solution. Consider for example, the function
$$F(x, y) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}, \quad y(0) = 0 \tag{2.27}$$
With $F(x, y)$ defined in (2.58), the initial value problem (2.54) is solved to obtain
$$y(x) = \begin{cases} x & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases} \tag{2.28}$$
Function $y(x)$ as shown in Figure (i) is not differentiable at $x = 0$. It therefore does not qualify as a solution.

3. If $F(x, y)$ is not continuous, the initial value problem (2.23) may have a u− nique solution. Consider for example, the initial value problem
$$\frac{dy}{dx} = \frac{1}{x^{\frac{1}{2}}}, \quad y(0) = 1. \tag{2.29}$$
with solution
$$y(x) = 2x^{\frac{1}{2}} + 1$$
Even though $F(x, y) = 1/x^{\frac{1}{2}}$ is discontinuous at $x = 0$,(**??**) has a unique solution over the entire domain of real numbers. That is, the interval $I_d$ on which the solution $y(x)$ is defined is $I_d = (-\infty, +\infty)$.

Note that the existence and uniqueness theorem gives a sufficient but not necessary condition for there to be a (unique) solution.

4. If $F(x, y)$ does not have a continuous first order partial derivative, (2.23) may have more than one solution. For example, the initial value problem

$$\frac{dy}{dx} = y^{\frac{1}{3}}, \quad y(0) = 0. \tag{2.30}$$

has more than one solution. One solution is the constant function $y(x) = 0$ for which the graph lies on the $x$-axis. A second solution is found by integrating (2.30) to obtain

$$y(x) = \left(\frac{2}{3}x\right)^{\frac{3}{2}},$$

where $x$ is non-negative for $y(x) \in \mathbf{R}$. Noting that $y(x-c)$ is a solution of (2.30) whenever $y(x)$ is, (2.30) is solved by

$$y(x) = \begin{cases} \left(\frac{2}{3}(x-c)\right)^{\frac{3}{2}} & \text{for } x \geq c \\ 0 & \text{for } x < c \end{cases} \tag{2.31}$$

for any $c \geq 0$. This is shown in Figure (ii).

5. Picard's theorem guarantees a solution for $x$ close to $x_0$, but this solution may or may not exist for all $x$. In (iii), the solution did exist for all $x \in \mathbb{R}$. However, the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2, \quad y(0) = 1 \tag{2.32}$$

has the solution

$$y(x) = \frac{1}{1-x^2} \tag{2.33}$$

which tends to infinity as $x \to \pm 1$. In fact, if solution $y(x)$ is considered as a solution of the differential equation $y' = 2xy^2$, the interval of definition could be taken to be any interval over which $y(x)$ is defined and differentiable. As shown in Figure (iii), the largest interval on which $y(x) = \frac{1}{1-x^2}$ is a solution are $-\infty < x < -1, -1 < x < 1$ and $1 < x < \infty$.

However, a solution of the initial value problem (2.32) in the sense of definition (2.6) has interval $I_d$ given as $I_d = -1 < x < 1$.

6. Very straightforward looking initial value problems may have no solution, for example, consider the initial value problem

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + y(x) = x, \quad y(0) = 1 \tag{2.34}$$

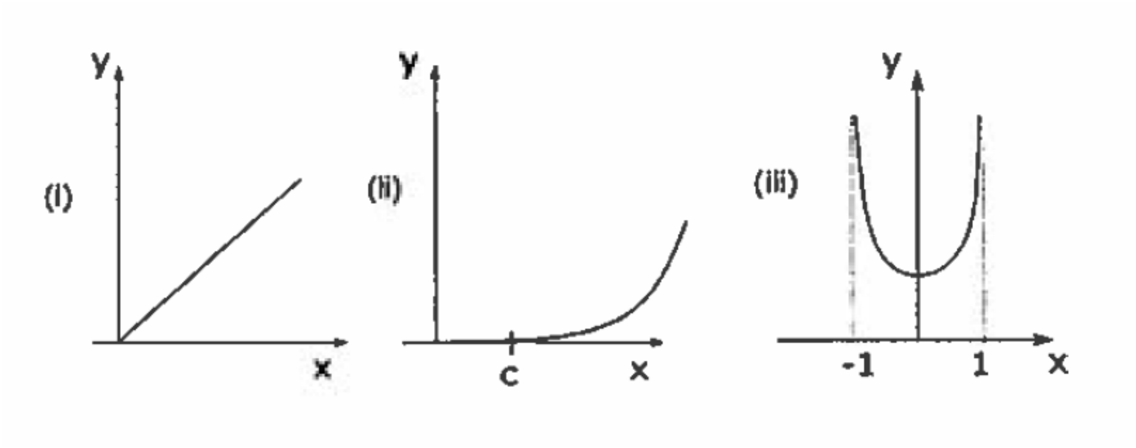


Figure 2.2: Graphs illustrating stated remarks. solution $y(x)$ given in (2.28) is shown in graph (i), solution $y(x)$ given in (2.31) is shown in graph (ii), solution $y(x)$ given in (2.33) is shown in graph (iii)

By setting $t = 0$ on both sides of (2.34), we can obtain $y(0) = 0$ which is not consistent with the initial condition in (2.34).

Students should be able to apply Piccard's theorem to deduce that (2.34) indeed has no solution.

**EXERCISE 2.7.** Show that each of the following initial value problems has two distinct solutions

(a) $\frac{dy}{dx} = |y|^{\frac{1}{2}}, \quad y(0) = 0$

(b) $y\frac{dy}{dx} = x, \quad y(0) = 0$

Discuss in each case whether or not the hypothesis of Piccard's theorem are satisfied.

**EXERCISE 2.8.** Do the following initial value problems have unique solutions?

(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = x\sqrt{y-3}, \quad y(4) = 3$

(b) $y\frac{dy}{dx} = e^x \cos(y), \quad y(0) = \pi$

Discuss in each case whether or not the hypothesis of Piccard's theorem are satisfied.

**EXERCISE 2.9.** Discuss using the existence and uniqueness theorem for first order ordinary differential equations whether or not the solution to the following differential equations are unique.

(a) $\frac{dy}{dx} = x - y + 1, \quad y(1) = 2$

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2, \quad y(0) = 0$

(c) $\frac{dy}{dx} = xy - \sin(y), \quad y(0) = 2$

(d) $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{y} + 1, \quad y(0) = 0, \quad x \in [0, 1]$

(e) $\frac{dy}{dx} = x|y|, \quad y(1) = 0$
(f) $\frac{\mathrm{d}y}{\mathrm{d}x} = y^{\frac{1}{3}} + x, \quad y(1) = 1$

**EXERCISE 2.10.** Discuss the existence and uniqueness of the solution to the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2y}{x}, \quad y(x_0) = y_0$$

# Chapter 3

# Second order ordinary differential equations

## 3.1 Basic properties

The most general scalar linear $n^{th}$-order ODE is

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = b(x), \tag{3.1}$$

where the $a_i s$ and $b$ are functions of $x$.

Introduce the operator $D = \frac{d}{dx}$ such that

$$Dy = \frac{dy}{dx}, \quad D^2y = D(Dy) = D\frac{dy}{dx} = \frac{d^2y}{dx^2}, \quad \ldots, D^n y = \frac{d^n y}{dx^n},$$

Then (3.1) becomes

$$a_n(x)D^n y + a_{n-1}(x)D^{n-1}y + \cdots + a_1(x)Dy + a_0(x)y = b(x) \tag{3.2}$$

or equivalently

$$\left(a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)\right)y = b(x) \tag{3.3}$$

Let

$$L_n = \left(a_{n1}(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x)\right).$$

Then (3.3) reads

$$L_n y = b(x) \tag{3.4}$$

Since (3.1) is of order $n$, its general solution contains $n$ arbitrary constants. Suppose that $y_p$ is any particular solution of (3.1). Then

$$L_n y_p = b(x)$$

and for any solution $y$ of (3.1), we have

$$L_n\left(y - y_p\right) = L_N y - L_n y_p = b(x) - b(x) = 0.$$

If $u = y - y_P$ then

$$L_n u = 0. \tag{3.5}$$

The reduced equation (3.5) is called the homogeneous equation associated with (3.1).

**REMARK 3.1.** From the discussion above we infer that in order to solve (3.1), we need to know a particular solution and the general solution of the associated homogeneous equation.

If $u_1, u_2, \ldots, u_n$ are $n$ linearly independent solutions of (3.5) and $C_1, C_2, \ldots, C_n$ are $n$ arbitrary constants, then

$$L_n\left(C_1 u_1 + C_2 u_2 + \cdots + C_n u_n\right) = C_1 L_n u_1 + C_2 L_n u_2 + \cdots + C_n L_N u_n = 0.$$

That is, $C_1 u_1 + C_2 u_2 + \cdots + C_n u_n$ is also a solution of (3.5). Moreover, since it has $n$ arbitrary constants $C_1, \ldots, C_n$, it is the general solution of (3.5). Any set of $n$ linearly independent solutions of (3.5) is called a fundamental set of solutions.

The original equation (3.1) has the general solution

$$y = C_1 u_1 + C_2 u_2 + \cdots + C_n u_n + y_P.$$

Note that $y_p$ has no arbitrary constants in it.

The general solution of (3.5) i.e. $C_1 u_1 + C_2 u_2 + \cdots + C_n u_n$ is called the *complementary function and* $y_p$ *is a particular integral.*

## 3.2 Scalar linear $n$ th-order ODEs with constant coefficients (revision)

Here we assume that the coefficients $a, s$ appearing in (3.1) are all constants and $b$ only is allowed to depend on $x$. We also suppose that $a_n = 1$. This can always be done by dividing (3.1) through by $a_n$.

In this section, we will describe the integration procedure for the resulting constant coefficient ODEs.

### 3.2.1 Homogeneous equations

Consider the homogeneous equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0 y = 0, \tag{3.6}$$

where we have used the abbreviation $y^{(i)} = \frac{d^i y}{dx^i}$.

Let $y = e^{\lambda x}$ be a solution of (3.6). Then

$$y' = \lambda e^{\lambda x}, y'' = \lambda^2 e^{\lambda x}, \ldots, y^{(n)} = \lambda^n e^{\lambda x}. \tag{3.7}$$

Substitute (3.7) into (3.6). We obtain after cancelling $e^{\lambda x}$

$$\lambda^n + a_{n-1}\lambda^{n-1} + a_{n-2}\lambda^{n-2} + \cdots + a_0 = 0 \tag{3.8}$$

which is called the characteristic or auxiliary equation. If all the $n$ roots of (3.8) are distinct, then $n$ linearly independent solutions (fundamental set of solutions) are

$$y_1 = e^{\lambda_1 x}, y_2 = e^{\lambda_2 x}, \ldots, y_n = e^{\lambda_n x}{}_+$$

A linear combination (principle of linearity or superposition) of these $n$ solutions provide the general solution to the homogeneous equation (3.6), i.e.,

$$\begin{aligned} y &= C_1 y_1 + C_2 y_2 + \cdots + C_n y_n \\ &= C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_n e^{\lambda_n x}, \end{aligned}$$

where the $C_t$ are arbitrary constants.

If some roots are equal, say $\lambda_1 = \lambda_2$ and the rest distinct, then

$$y = (C_1 + C_2 x)\, e^{\lambda_1 x} + C_3 e^{\lambda_3 x} + \cdots + C_n e^{\lambda_n x}$$

is the general solution to (3.6).

In general, if $\lambda$ is a root of multiplicity $p$ greater than 1, it will induce the following $p$ linearly independent solutions

$$x^i e^{\lambda x}, \quad i = 0, \ldots, p-1.$$

If there are complex conjugate roots, say $\lambda_1 = u + iv, \lambda_2 = u - iv, t = \sqrt{-1}$ and the remaining roots are distinct, then

$$\begin{aligned} y &= C_1 e^{(u+iv)x} + C_2 e^{(u-iv)x} + C_3 e^{\lambda_3 x} + \cdots + C_n e^{\lambda_n x} \\ &= e^{ux}\left(C_1 e^{ivx} + C_2 e^{-ivx}\right) + C_3 e^{\lambda_3 x} + \cdots + C_n e^{\lambda_n x} \end{aligned}$$

In real form

$$y = e^{ux}\left(\bar{C}_1 \cos(vx) + \bar{C}_2 \sin(vx)\right) + C_3 e^{\lambda_3 x} + \cdots + C_n e^{\lambda_n x}$$

is the general solution to the homogeneous equation.

**EXAMPLE** 3.1
(1) $y'' - y' - 2y = 0$.
(2) $y'' - 2y' + y = 0$.
(3) $y'' + \omega^2 y = 0$.
(4) $y''' + y = 0$. Give the solution in both complex and real forms.

### 3.2.2 Method of Undetermined Coefficients.

The general solution of a nonhomogeneous linear ODE, i.e. if $b \neq 0$, is obtained by firstly deducing the general solution of the associated homogeneous equation. The next task is to find a particular solution of the non-homogeneous equation (recall that $y = y_h + y_p$ where $y_h$ is the general solution of the associated homogeneous equation and $y_p$ is a particular solution of the non-homogeneous equation ). This step may not be straightforward.

We are going to describe a method enabling the construction of a particular solution. The underlying idea behind this method is to try a particular solution which looks like the right-hand side, i.e. $b(x)$. This method works for constant coefficient linear equations only.

- If $b(x) = k_p x^{y^r} + k_{p-1} x^{y-1} + \cdots + k_1 x + k_0$, look for a particular solution which is also a polynomial of degree $p$ of the form

$$y_p = A_p x^p + A_{p-1} x^{p-1} + \cdots + A_1 x + A_0.$$

- If $b(x) = ke^{\omega x}$, try a particular solution of the form

$$y_p = Ae^{\omega x}.$$

- If $b(x) = k\sin(\omega x)$ or $b(x) = k\cos(\omega x)$, try

$$y_p = A\cos(\omega x) + B\sin(\omega x), \quad y_p^* = Ae^{i\omega x}$$

- If $b(x)$ is a sum of the forms given above, try a particular solution which is a sum of the above forms.

- If $b(x)$ is a product of the forms given above, try a particular solution which is a product of the above forms.

**EXAMPLE 3.2**
(1) $y'' + y = 2x^2$.
(2) $y'' - 3y' + 2y = 4e^{3x}$.
(3) $y'' - y' - 2y = 3\sin(x)$.
(4) $y'' - 3y' + 2y = 4x + e^{3x}$.
(5) Use the complex method to solve $y'' - y' - 2y = 5\cos(x)$. The following examples illustrate how one handles the situation if the RHS of the equation is contained in $y_h$.
(6) $y'' - 3y' + 2y = 5x + e^{x}$.
(7) $y'' - 2y' + y = e^{x} + x$.

Thus far we have used an approach in the search for particular solutions of non-homogeneous linear constant coefficient equations. Now we are going to give a more general way of finding particular solutions of ODEs which need not be constant coefficient.

## 3.3 Method of variation of parameters

For the sake of simplicity we will restrict all considerations to scalar second-order linear ODEs although the same techniques apply to higher order ODEs as well.

Consider the second-order linear ODE

$$y'' + a(x)y' + b(x)y = c(x) \tag{3.9}$$

in which $a, b$ and $c$ are continuous functions of $x$ on an interval I over which (3.9) is defined. Suppose that $y_1$ and $y_2$ are two linearly independent solutions of the homogeneous equation associated with (3.9). Then by the principle of superposition

$$y_h = C_1 y_1 + C_2 y_2$$

where $C_1$ and $C_2$ are arbitrary constants. The general solution of (2.41) is

$$y = y_h + y_p$$

where $y_p$ is a particular solution of (3.9). Now assume that

$$y_p = u(x)y_1 + v(x)y_2$$

Then

$$y_p' = u' y_1 + u y_1' + v' y_2 + v y_2'.$$

In order to avoid the appearance of second derivatives of $u$ and $v$ in $y_p''$, we set

$$u'y_1 + v'y_2 = 0. \tag{3.10}$$

So

$$y_p' = uy_1' + vy_2'$$

and

$$y_p'' = u'y_1' + v'y_2' + uy_1'' + vy_2''.$$

Substitute the above expressions of $y_p, y_j'$ and $y_p''$ into (3.9):

$$u\left(y_1'' + ay_1' + by_1\right) + v\left(y_2'' + ay_2' + by_2\right) + u'y_1' + v'y_2' = c(x).$$

The expressions within the parentheses vanish (why?) and we get

$$u'y_1' + v'y_2' = c(x). \tag{3.11}$$

Solving the system (3.10)-(3.11) for $u'$ and $v'$, we get

$$u' = -\frac{y_2 c(x)}{W}, \quad v' = \frac{y_1 c(x)}{W}, \tag{3.12}$$

where $W = y_1 y_2' - y_2 y_1' \neq 0$ (why?). Hence by integrating (3.12), we can choose

$$y_p = -y_1 \int \frac{y_2 c(x)}{W} dx + y_2 \int \frac{y_1 c(x)}{W} dx.$$

Finally, the general solution to (3.9) is

$$y = C_1 y_1 + C_2 y_2 - y_1 \int \frac{y_2 c(x)}{W} dx + y_2 \int \frac{y_1 c(x)}{W} dx. \tag{3.13}$$

**EXAMPLE 3.3** Prove that $W$ satisfies

$$W' + a(x)W = 0$$

and deduce that

$$W = W(x_0)\, e^{-\int_{x_0}^{x} a(t)\,dt}.$$

## 3.4 Reduction of order

In this section we will consider a set of second order ordinary differential equations that can be reduced to first order ordinary differential equations. The resulting first order ordinary differential equations can be analysed and it can be determined which method of solution from Chapter ... can be used to solve them. Then the solution to the to the first order differential equation can be used to find the solution to the original second order differential equation.

Suppose we know one independent solution, say $y_1(x)$, of the homogeneous equation associated with (3.9). The second linearly independent solution can be obtained by setting $y_2 = u(x)y_1$.

Show that $u(x)$ satisfies the linear second order ordinary differential equation

$$u'' + \left(\frac{2y_1'}{y_1} + a(x)\right)u' = 0. \tag{3.14}$$

The form of (3.14) allows for a reduction of order. Thus, in order to solve (3.14), set

$$p = u' \tag{3.15}$$

and deduce that (3.14) reduces to

$$p' + \left(\frac{2y_1'}{y_1} + a(x)\right)p = 0 \tag{3.16}$$

Solve (3.16) for $p(x)$, then substitute the expression for $p(x)$ into (3.15). Integrate the resulting equation (3.15) to obtain $u(x)$. The second independent solution $y_2(x)$ can then be determine.

Let $y = u(x)e^{-\int \frac{a(x)}{2}dx}$. Substitute this into the homogeneous equation and find the differential equation that $u$ satisfies.

### 3.4.1 Euler or Cauchy equation

Consider the equation

$$x^2y'' + axy' + by = c(x), \tag{3.17}$$

in which $a$ and $b$ are constants and $c(x)$ is a function of $x$. Make the change of variable $x = e^t$ and obtain

$$\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = c\left(e^t\right).$$

Use the method of variation of parameters or apply directly (3.13) to solve this last equation. One can also use the method of undetermined coefficients.

### 3.4.2 Equations explicitly independent of $x$

Consider the equation

$$y'' = h\left(y, y'\right). \tag{3.18}$$

Note that by virtue of the chain rule,

$$y'' = \frac{dy'}{dy}\frac{dy}{dx} = y'\frac{dy'}{dy}.$$

So (3.18) becomes

$$y'\frac{dy'}{dy} = h\left(y, y'\right). \tag{3.19}$$

Now let $u = y'$. Then (3.19) reads

$$u\frac{du}{dy} = h(y,u) \tag{3.20}$$

which is a first-order ODE. Suppose that (3.20) has the general solution $y' = u = S(y,C)$, where $C$ is an arbitrary constant. Then $\frac{dy}{S(y,C)} = dx$ which is variable separable and has general solution $\int \frac{dy}{S(y,C)} = x + K$, where $K$ is a further constant of integration.

### 3.4.3 Equations explicitly independent of $y$

The most general second-order ODE not containing y explicitly is

$$y'' = g\left(x, y'\right). \tag{3.21}$$

This equation can be rewritten as

$$\frac{dy'}{dx} = g\left(x, y'\right).$$

Make the change $u = y'$ and obtain the first-order ODE

$$u' = g(x,u).$$

From here on, proceed as in the previous section.

## 3.5 Laplace transforms

In the previous sections, we solved problems with constant coefficients and problems with variable coefficients. The one assumption that was made was that the forcing/driving functions were continuous in the domain of real numbers. However there are problems in mathematical modelling that require the the forcing/driving function to be discontinuous at a point and there be a pecewise functions. It becomes difficult to solve such problems using methods previously discussed. In this section you will learn the method of Laplace transforms which simpifies the process of determining solutions to differential equations with piecewise forcing/driving functions.

The schematic below shows how differential equations can be solved using the Laplace

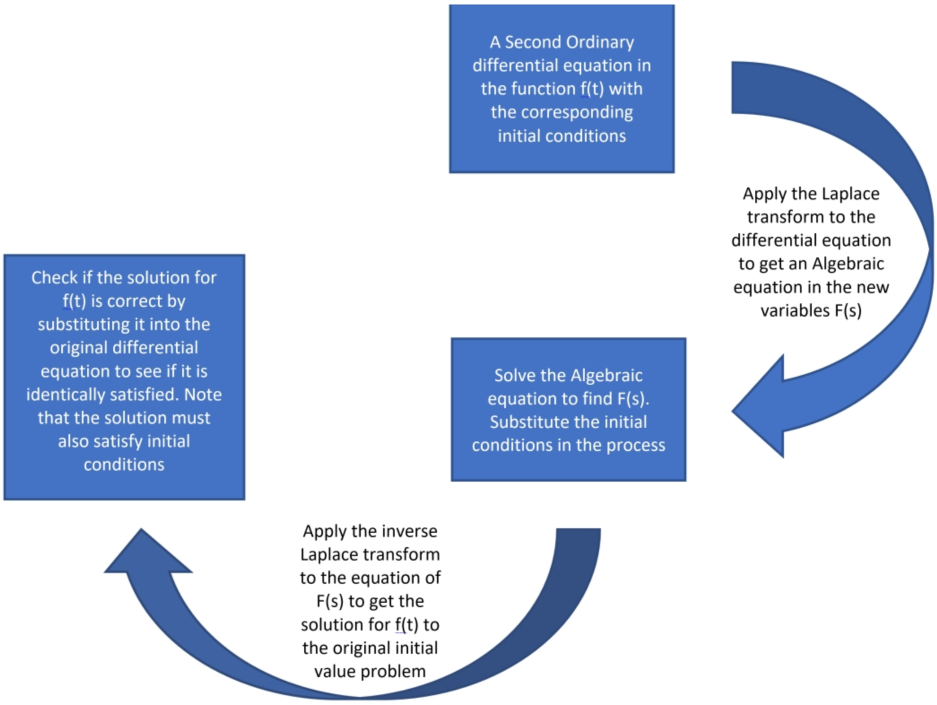


Figure 3.1: Schematic representation of the process of solving an initial value problem with the Laplace transform method of solution

transform:

**REMARK 3.1** We note that while the Laplace transform method of solution is introduced in this chapter when solving second order ordinary differential equations, the method can also solve first order and higher order ordinary differential equations with constant coefficients or with variable coefficients.

Now, we need to first define a few concepts and learn a few techniques that will allow us to be able to ultimately solve an ordinary differential equation using Laplace transform.

### 3.5.1 Laplace and inverse Laplace transforms of functions

**DEFINITION 3.1** A Laplace transform, denoted by the symbol $\mathscr{L}$, of a function $y(t)$ defined as

$$\mathscr{L}\{y(t)\} = \int_0^\infty e^{-st} y(t)dt = Y(s), \tag{3.22}$$

exists provided the integral in (3.22) converges.

There are two sufficient conditions for teh existence of a Laplace transform of a function, which are:

1. The function y(t) has to be piecewise continuous on the domain $[0,\infty)$ and
2. The function y(t) has to be of exponential order.

**DEFINITION 3.2** A function y(t) is piecewise continuous on the domain $[0,\infty)$ provided the function has finite number of discontinuities in the specifies domain and the function $y(t)$ is continuous everywher between the specified discontinuities.

**DEFINITION 3.3** A function is of exponential order $c$ if there exists constants $c, M > 0$ and $T > 0$ such that $|y(t)| \leq Me^{ct}$ for all $t > T$.

An important property of the Laplace transform is that it is a linear transform such that:

$$\begin{aligned} \mathscr{L}\{\alpha y(t)+\beta f(t)\} &= \mathscr{L}\{\alpha y(t)\}+\mathscr{L}\{\beta f(t)\} && (3.23)\\ &= \alpha\mathscr{L}\{y(t)\}+\beta\mathscr{L}\{f(t)\} && (3.24)\\ &= \alpha Y(s)+\beta F(s), && (3.25) \end{aligned}$$

where $\alpha$ and $\beta$ are constants.

**EXERCISE 3.1** Show that the result in equation (3.23) is true.

**EXAMPLE 3.4** Consider a function $y(t) = 1$. The Laplace transform of 1 is

$$\begin{aligned} \mathscr{L}\{1\} &= \int_0^{\infty} e^{-st} y(t)dt && (3.26)\\ &= \int_0^{\infty} e^{-st} dt && (3.27)\\ &= \frac{e^{-st}}{-s}\bigg|_0^{\infty} && (3.28)\\ &= \frac{1}{-s}\Big(0-1\Big) && (3.29)\\ &= \frac{1}{s} && (3.30) \end{aligned}$$

The Laplace transform in Example 3.4 is one of the many important basic transforms. Next, is a table of Laplace transforms of basic functions:

| Laplace transforms of basic functions | |
|---|---|
| $a)\mathscr{L}\{1\} = \frac{1}{s},$ | $b)\mathscr{L}\{e^{at}\} = \frac{1}{s-a},$ |
| $c)\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = 1,2,3,...$ | |
| $d)\mathscr{L}\{\sin(kt)\} = \frac{k}{s^2+k^2},$ | $e)\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2},$ |
| $f)\mathscr{L}\{\sinh(kt)\} = \frac{k}{s^2-k^2},$ | $g)\mathscr{L}\{\cosh(kt)\} = \frac{s}{s^2-k^2},$ |

**EXERCISE 3.2** Show that the solutions given in Table... are correct.

**DEFINITION 3.4** An Inverse Laplace transform is an operation which when applied to $Y(s)$ gives the original function $y(t)$:

$$\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1. \tag{3.31}$$

Below, we compute a table of inverse Laplace transforms of basic fuctions. We note that

| Inverse Laplace transforms of basic functions | |
|---|---|
| $a)\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1,$ | $b)\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at},$ |
| $c)\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n, \quad n = 1,2,3,...$ | |
| $d)\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt),$ | $e)\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt),$ |
| $f)\mathscr{L}^{-1}\left\{\frac{k}{s^2-k^2}\right\} = \sinh(kt),$ | $g)\mathscr{L}^{-1}\left\{\frac{s}{s^2-k^2}\right\} = \cosh(kt),$ |

while we have not computed the inverse Laplace transform explicitly, the definition provided as well as the table of inverse Laplace transforms of basic functions give sufficient information to compute inverse Laplace transforms.

### 3.5.2 Laplace transform of derivatives

**DEFINITION 3.5** If $y, y', y'', y''', y^{(n-1)}$ are continuous on the domain $[0, \infty)$ and of exponential order and the n-th order derivative $y^{(n)}$ is piecewise continuous on the same domain, then the Laplace transform of the $n-th$ derivative for $y(t)$ is

$$\mathscr{L}\{y^{(n)}(t)\} = s^n Y(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \ldots - y^{(n-1)}(0), \tag{3.32}$$

where $Y(s) = \mathscr{L}\{y(t)\}$.

**EXAMPLE 3.5**

$$\mathscr{L}\{y'(t)\} = \int_0^\infty e^{-st}y'(t)dt \tag{3.33}$$

$$= e^{-st}y(t)\Big|_0^\infty + s\int_0^\infty e^{-st}y(t)dt \tag{3.34}$$

$$= -y(0) + s\mathscr{L}\{y(t)\} \tag{3.35}$$

$$= sY(s) - y(0). \tag{3.36}$$

**EXERCISE 3.3**

1. Calculate the following transforms:

   a) $\mathscr{L}\{f''(t)\}$

   b) $\mathscr{L}\{g''(t)\}$

2. Calculate the solution for the following initial value problem using the Laplace and the inverse Laplace transforms:

   a) $y''(t) + 5y'(t) + 4y(t) = 0, \quad y(0) = 1, \quad y'(0) = 0$

   b) $y''(t) - 3y'(t) + 2y(t) = e^{-4t}, \quad y(0) = 1, \quad y'(0) = 5$

### 3.5.3 Translation on the s-axis

**DEFINITION 3.6** Translation on the s-axis occurs when the function we apply the Laplace transform to is multiplied by an exponential function to give

$$\mathscr{L}\{e^{at}f(t)\} = \int_0^\infty e^{-st}e^{at}f(t)dt \tag{3.37}$$

$$= \int_0^\infty e^{-(s+a)t}f(t)dt \tag{3.38}$$

$$= F(s-a) \tag{3.39}$$

If the translation is to the right then the constant $a > 0$, while if the translation is to the left then the constant $a < 0$.
Applying the inverse Laplace transform to $F(s-a)$ gives:

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t). \tag{3.40}$$

**EXERCISE 3.4**

1. Evaluate the following transforms:
   a) $\mathcal{L}\{e^{5t} t^3\}$
   b) $\mathcal{L}^{-1}\left\{\frac{2s+5}{(s+3)^2}\right\}$
2. Solve the following initial value problems
   a) $y'' - 6y' + 9y = t^2 e^{3t}, \quad y(0) = 2, \quad y'(0) = 17$
   b) $y'' + 4y' + 6y = 1 + e^{-t}, \quad y(0) = 0, \quad y'(0) = 0.$

### 3.5.4 Translation on the t-axis

**DEFINITION 3.7** If $F(s) = \mathcal{L} f(t)$ and $a > 0$, then

$$\mathcal{L} f(t-a)\mathcal{H}(t-a) = e^{-as} F(s), \tag{3.41}$$

where

$$\mathcal{H}(t-a) = \begin{cases} 0, & 0 \leq t < a, \\ 1, & t \geq a. \end{cases} \tag{3.42}$$

The heaviside function $\mathcal{H}(t-a)$ is used to affirm that the piecewise function in the respective problem can be expressed simply without the use of brackets.

**EXERCISE 3.5**

1. Evaluate the following transforms:
   a) $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s-4}\right\}$
   b) $\mathcal{L}^{-1}\left\{\frac{se^{-\frac{\pi s}{2}}}{s^2+9}\right\}$
2. Solve the following initial value problem
   a) $y'' - 5y' + 6y = \mathcal{H}(t-1), \quad y(0) = 0, \quad y'(0) = 1$
   b) $y'' + 4y = f(t), \quad y(0) = 0, \quad y'(0) = -1,$
      where

$$f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ 0, & t \geq 1. \end{cases}$$

# Chapter 4

# Difference Equations

In some sense, difference equations are discrete counterparts of differential equations. The techniques used to solve them are sometimes similar to those used for differential equations.

## 4.1 Motivation

Suppose that you take a loan from a bank for a certain amount $D$ and that the bank charges interest at a rate of $r$ per month. What should your monthly payment $p$ (at the end of each month) be if you wish to settle your debt in n months?

If $d(k)$ is the total debt at the beginning of the $k$ th month, then

$$d(k+1) = d(k) + rd(k) - p = (r+1)d(k) - p, \tag{4.1}$$

where $d(0) = D$. Equation (4.1) is an example of a difference equation. In order to find the monthly payment $p$ we set $d(n) = 0$ since you wish to settle your debt in $n$ months.

Let $q = 1 + r$, then

$$d(k+1) = qd(k) - p = q^2 d(k-1) - (1+q)p = \cdots = q^{k+1}d(0) - \left(1 + q + \cdots + q^k\right)p$$

Thus

$$d(k+1) = q^{k+1}D - \frac{1-q^{k+1}}{1-q}p.$$

By letting $k = n-1$, we find

$$d(n) = q^n D - \frac{1-q^n}{1-q}p$$

and $d(n) = 0$ gives

$$p = \frac{rD}{1-(1+i)^{-n}}$$

## 4.2 Definitions

Consider a mapping

$$y : \mathbb{Z} \to \mathbb{F},$$

where $\mathbb{Z}$ is the ring of integers and $\mathbb{F}$ is the field of real or complex numbers. $y$ is called a sequence of real (respectively complex) numbers.

**NOTATION** 4.1(*k* **th term).** The number $y(k)$ is called the $k$ th term of the sequence and is denoted by $y_k$.

**DEFINITION** 4.1 **(Difference equation).** Any equation relating the terms $y_k, y_{k+1}, \ldots$ of a sequence $(y_j)$ is called a difference equation.

**EXAMPLE 4.1.** The following are examples of difference equations.

(1) $y_{k+n} = y_k y_{k+n-1}$,

(2) $y_{k+1} = 2y_k^2$

(3) $y_{k+1} = 3y_k$.

**DEFINITION** 4.2 **(Order of a difference equation).** The order of a difference equation is the difference between the highest and the lowest indices appearing in the equation.

**EXAMPLE 4.2.** The difference equation

$$y_{k+3} + k y_{k+1}^2 + 3y_{k-1} = 0 \tag{4.2}$$

has order $(k+3) - (k-1) = 4$.

## 4.3 Linear difference equations

A linear difference equation of order $n$ has the form

$$a_n(k) y_{k+n} + a_{n-1}(k) y_{k+n-1} + \cdots + a_0(k) y_k = g(k), \tag{4.3}$$

where the $a_I s$ and $g$ are sequences i.e. functions defined on $Z$.

**DEFINITION** 4.3 **(Homogenous difference equation).** If $g(k) = 0$ then (4.3) is a homogeneous difference equation. It is non-homogeneous otherwise.

In order to obtain the general solution of (4.3), first solve the homogeneous equation. Then find a particular solution of the non-homogeneous equation. The general solution is the sum of the general solution of the homogeneous equation and a particular solution to the non-homogeneous equation.

The *principle of superposition* holds for homogeneous linear difference equations ie. any linear combination of solutions to a homogeneous linear difference equation is also a solution. If there are $n$ linearly independent solutions to a homogeneous linear difference equation of order $n$, this forms a fundamental set of solutions.

**EXAMPLE 4.3.** Consider the first-order linear difference equation (geometric model)

$$y_{k+1} = a y_k, \quad a = \text{ constant.} \tag{4.4}$$

This has solution $y_k = Ca^k$, where $C$ is an arbitrary constant, since $y_{k+1} = Ca^{k+1} = aCa^k = ay_k$. So, for each value of $C$, we have a solution. Thus there is a one parameter family of solutions.

A difference equation can have one solution, for example the equation

$$y_{k+1}^2 + y_k^2 = 0 \tag{4.5}$$

has the trivial solution $y_k = 0$ as the only real solution.
It is also possible for no real solution to exist, consider the equation

$$y_{k+1}^2 + 1 = 0 \tag{4.6}$$

Consider the linear difference equation with constant coefficients

$$b_n y_{k+n} + b_{n-1} y_{k+n-1} + \cdots + b_1 y_{k+1} + b_0 y_k = f(k). \tag{4.7}$$

where the $b_i s$ are constants and $f(k)$ is a function. Introduce the operator $E$ such that

$$E y_k = y_{k+1}, \quad E^2 y_k = E\left(E y_k\right) = E\left(y_{k+1}\right) = y_{k+2} \tag{4.8}$$

and in general

$$E^s y_k = y_{k+s} \tag{4.9}$$

We can then write (4.7) as

$$P(E) y_k \equiv \left(b_n E^n + b_{n-1} E^{n-1} + \cdots + b_1 E + b_0\right) y_k = f(k). \tag{4.10}$$

For $y_k = \lambda^k$ to be a solution of the homogeneous equation associated with (3.7), it is necessary and sufficient that

$$P(\lambda) \equiv b_n \lambda^n + b_{n-1} \lambda^{n-1} + \cdots + b_1 \lambda + b_0 = 0. \tag{4.11}$$

Equation (4.11) is called the characteristic or auxiliary equation of the homogeneous equation associated with (4.7).
If all roots of (4.11) are real and distinct, then

$$y_k^h = C_1\lambda_1^k + C_2\lambda_2^k + \cdots + C_n\lambda_n^k \tag{4.12}$$

is the general solution of the homogeneous equation. If some roots are equal, say $\lambda_1 = \lambda_2$ and the rest are distinct,

$$y_k^h = (C_1 + C_2 k)\,\lambda_1^k + C_3\lambda_3^k + \cdots + C_n\lambda_n^k. \tag{4.13}$$

In general, if there is a root $\lambda$ of multiplicity $p$, the contribution of this root in $y_k^h$ will be

$$\left(C_1 + C_2 k + \cdots + C_p k^{p-1}\right)\lambda^k. \tag{4.14}$$

If some roots are complex, say $\lambda_1 = p - iq, \lambda_2 = p + iq$, then

$$y_k^h = C_1(p+iq)^k + C_2(p-iq)^k + C_3\lambda_3^k + \cdots + C_n\lambda_n^k \tag{4.15}$$

in the complex domain and in the real domain

$$y_k^h = \left(\bar{C}_1\cos(k\theta) + \bar{C}_2\sin(k\theta)\right)\rho^k + C_3\lambda_3^k + \cdots + C_n\lambda_{n'}^k \tag{4.16}$$

where $(p,q) = \rho(\cos(\theta), \sin(\theta))$ and $\rho = \sqrt{p^2+q^2}$.
**EXAMPLE 4.4.** The following examples illustrate the various situations that can arise

(1) Distinct roots: Consider the linear difference equation

$$y_{k+2} - y_{k+1} - 2y_k = 0. \tag{4.17}$$

This equation can be rewritten as

$$\left(E^2 - E - 2\right)y_k = 0. \tag{4.18}$$

The associated characteristic equation is

$$\lambda^2 - \lambda - 2 = 0, \tag{4.19}$$

so $\lambda = -1$ or $\lambda = 2$ and the general solution is thus

$$y_k = (-1)^k C_1 + 2^k C_2. \tag{4.20}$$

(2) Repeated roots: Consider the linear difference equation

$$y_{k+2} - 6y_{k+1} + 3^2 y_k = 0 \tag{4.21}$$

In terms of the $E$ operator, this equation is

$$\left(E^2 - 6E + 3^2\right) y_k = 0 \tag{4.22}$$

factorising, we obtain

$$(E-3)^2 y_k = 0. \tag{4.23}$$

So that the characteristic equation is given by

$$(\lambda - 3)^2 = 0, \tag{4.24}$$

and thus $\lambda = 3$ is a root with multiplicity two and the general solution is given by

$$y_k = (C_1 + C_2 k)\, 3^k. \tag{4.25}$$

(3) Complex roots: Consider the linear difference equation

$$y_{k+2} + 9y_k = 0. \tag{4.26}$$

This equation can be written as

$$\left(E^2 + 9\right) y_k = 0 \tag{4.27}$$

The characteristic equation is

$$\lambda^2 + 9 = 0, \tag{4.28}$$

with complex roots

$$\lambda = 3i \text{ or } \lambda = -3i \tag{4.29}$$

In the complex domain,

$$y_k = C_1(3i)^k + C_2(-3i)^k, \tag{4.30}$$

where $C_1$ and $C_2$ are arbitrary constants. To obtain $y_k$ in the real form, we rewrite it as

$$y_k = 3^k \left(C_1 e^{i\frac{\pi}{2}k} + C_2 e^{-i\frac{\pi}{2}k}\right)$$

In order to find a particular solution to the non-homogeneous equation (4.7) use the method of undetermined coefficients that consists of trying a particular solution which looks like $f(k)$. We will be concerned with $f's$ which are polynomial in $k$ or of the form $Aa^k$ or sum a of these.

- If $f(k)$ is polynomial in $k$, try a particular solution which is also polynomial and of the same order as $f$.
- If $f(k) = Aa^k$, look for $y_k^p$ in the form $y_k^p = Ba^k$
- If $f(k)$ is a combination of the above forms, look for a particular solution in the same form.

**REMARK 4.1.** Note that there is no analogue of a variation of parameters method for linear difference equation.

**EXAMPLE 4.5.** Consider the equation

$$2y_{k+2} + 5y_{k+1} + 2y_k = k \tag{4.31}$$

The solution of the associated homogeneous equation is

$$y_k^h = C_1\left(-\frac{1}{2}\right)^k + (-2)^k C_{2+} \tag{4.32}$$

Look for a particular solution in the form $y_k^p = Ak + B$. Then

$$y_{k+1}^p = A(k+1) + B \tag{4.33}$$

$$y_{k+2}^p = A(k+2) + B \tag{4.34}$$

After substituting $y_k^p, y_{k+1}^p$ and $y_{k+2}^p$, we obtain $A = 1/9$ and $B = -1/9$. The general solution is

$$y_k = y_k^h + y_k^p = C_1\left(-\frac{1}{2}\right)^k + (-2)^k C_2 + \frac{1}{9}k - \frac{1}{9}.$$

**EXERCISE 4.1.** Solve the following linear difference equations by first finding solutions of the associated homogeneous equations and then particular solutions using the method of undetermined coefficients.

1. $y_{k+2} + 2y_{k+1} + y_k \equiv 2\left(3^k\right)_.$
   (Answer. $y_k^h = (C_1 + C_2 k)(-1)^k, y_k^p = 3^k/8.$)

2. $y_{k+2} + y_k = 2^k.$
   (Answer. $y_k^h = C_1 \cos(()\,k\pi/2) + C_2 \sin(()\,k\pi/2), y_k^p = 2^k/5.$)

3. $y_{k+1} - ay_k = b$, where $a$ and $b$ are constants.
(Hint consider the cases $a = 1$ and $a \neq 1$. For $a = 1$, try $y_k^p = Ak$ and for $a \neq 1$, try $y_k^p = B$.)

# Chapter 5

# Introduction to systems

## 5.1 Some definitions

**NOTATION 5.1.** Vectors will be denoted by lowercase boldface letters $\mathbf{x}, \mathbf{y}, \boldsymbol{\phi}$. Compare this with $x, y, \phi$. Some authors use underline $\underline{x}$ or squiggles $\underset{\sim}{x}$. For handwritten formulae, squiggles are probably best.

**NOTATION 5.2.** Matrices will be denoted by uppercase boldface letters $\mathbf{A}, \mathbf{B}, \boldsymbol{\Lambda}$

**NOTATION 5.3.** Scalars will be denoted by upper or lowercase letters, without boldface, bars or squiggles $x, a, N$. For handwritten formulae, confusion could arise as to whether we mean a matrix $A$ or a scalar $A$. These situations are avoided by simply not using the same letter to denote scalars and matrices.

### 5.1.1 System.

A group of components which interact and operate in a dependent way, e.g., genes, populations, cells are life systems; solids, liquids, gases are physical or chemical systems; nations or communities are social systems and companies, and producers are economic systems.

### 5.1.2 Dynamic system.

A system that depends on time, e.g., launching a satellite, solar-heating system, transference of genes, family system. It is important to bear in mind the time scale on which changes within the system occur in relation to the system itself. On a time scale of 50 years the family system may appear constant, yet on a scale of say 200 years changes may become evident.

### 5.1.3 Continuous system.

A system that depends continuously on time, e.g., heat exchange between bodies. Such systems are usually modelled by differential equations.

### 5.1.4 Discrete system.

A system that depends discretely on time, e.g., seasonal insect populations or monthly/yearly budgets. Such systems are usually described by difference equations. Note that systems that require a mixture of differential and difference equations also exist.

### 5.1.5 State variables.

Dependent variables that totally describe the state of a system at a given time.

### 5.1.6 State vector.

A vector representation of the state of a system at a given time. If $x_i, i = 1, 2, \ldots, n$ are the state variables for a system, then the state vector is written as $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ with $x \in \mathbb{R}^n$

### 5.1.7 Parameters.

Characteristic values associated with a system. Parameters may be constant in time or time-dependent and are usually measured experimentally or "guessed" by local experts.

### 5.1.8 Linear systems.

Systems that can be represented (i.e., modelled) by linear equations (i.e., equations with no product or power terms of the dependent variables and its derivatives). It is worth studying linear systems in depth because

- they are simple and there is a compact way of representing them;
- there is a large body of linear algebra theory to rely on;
- they are easier (than non-linear systems) to implement numerically.

### 5.1.9 First-order differential equations.

Differential equations involving only first derivatives of variables and the variables themselves.

## 5.2 Higher-Order ODEs

Any ODE of order higher than one can be reduced to a system of first-order ODEs:

$$x^{(n)} = F\left(t, x, \dot{x}, \dots, x^{(n-1)}\right), \quad n > 1, \tag{5.1}$$

where $x^{(i)} = d^i x / dt^i$. Define new set of state variables ( $n$ of them)

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \\ &\vdots \\ x_{n-1} &= x^{(n-2)} \\ x_n &= x^{(n-1)}. \end{aligned}$$

Then (5.1) becomes

$$\frac{dx_n}{dt} = F(t, x_1, x_2, \dots, x_n)$$

and we have a system of $n$ first-order d.e.s

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= F(t, x_1, x_2, \dots, x_n). \end{aligned} \tag{5.2}$$

Equation (5.2) is equivalent to (5.1).
Similarly, any ordinary difference equation of order higher than one can be represented by a system of $n$ first-order difference equations:

$$x_{k+n} = F(k, x_k, x_{k+1}, \dots, x_{k+n-1}) \quad n > 1, \tag{5.3}$$

Define

$$\begin{aligned} u_k &= x_k \\ v_k &= x_{k+1} \\ w_k &= x_{k+2} \\ &\vdots \\ z_k &= x_{k+n-1}. \end{aligned}$$

Then (5.3) becomes

$$z_{k+1} = F(k, u_k, v_k, \dots, z_k),$$

and we have $n$ first-order equations.

$$\begin{aligned} u_{k+1} &= v_k \\ v_{k+1} &= w_k \\ &\vdots \\ z_{k+1} &= F(k, u_k, v_k, \dots, z_k). \end{aligned} \tag{5.4}$$

Equation (5.4) is equivalent to (5.3).

## 5.3 Motivation for algebraic notation

The system of equations

$$\begin{aligned} \dot{x}_1 &= 2x_1 + 3x_2 + x_3 \\ \dot{x}_2 &= -x_2 + x_3 \\ \dot{x}_3 &= 5x_1 + 7x_3 \end{aligned}$$

can neatly and compactly be expressed as the system

$$\dot{x} = Ax$$

$$\text{where } x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \text{ and } A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 1 \\ 5 & 0 & 7 \end{pmatrix} \in \mathbb{R}^{3\times 3}$$

Algebraic notation is compact, we can replace many equations by a single equation.

## 5.4 General system of equations

Any system of first-order ODEs can be written in the form

$$\dot{x}(t) = A(t)x(t) + g(x(t), t) + f(t),$$

where $x : \mathbb{R} \to \mathbb{R}^n$ is the vector-valued state function,
$A(t)x(t) \in \mathbb{R}^n$ is the linear term if any; $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ is a matrix-valued function,

$g : \mathbb{R}^{n\times n} \times \mathbb{R} \to \mathbb{R}^n$ represents the nonlinear terms if any,
$f : \mathbb{R} \to \mathbb{R}^n$ is the term involving the independent variable (often time).

- If $g \equiv 0$, the system is linear otherwise it is nonlinear.

- If $g \equiv 0$ and $f = 0$, then the system is a homogeneous linear system.
- If $g \equiv 0$ and $f \neq 0$, then the system is a non-homogeneous linear system.
- If $A$ is independent of $t$ and $g \equiv f \equiv 0$, then the system is a linear homogeneous system with constant coefficients.
- If the system is not explicitly dependent on $t$, then it is said to be autonomous otherwise it is non-autonomous.

Any system of first-order ordinary difference equations can be written in the form

$$x_{k+1} = A(k)x_k + g(x_k, k) + f(k).$$

Similar terminology as above applies in this case too.

## 5.5 Linear Algebra

The general system of $m$ linear algebraic equations in $n$ unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= y_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= y_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= y_m. \end{aligned}$$

In summation notation this is:

$$\sum_{j=1}^{n} a_{ij}x_i = y_i, \text{ for } i = 1, 2, \ldots, m$$

In matrix notation this system is:

$$Ax = y, \quad A \in \mathbb{R}^{m\times n}, x \in \mathbb{R}^n, y \in \mathbb{R}^m.$$

### 5.5.1 Square matrix.

a square matrix is a matrix with same number of column$s$ and rows

### 5.5.2 Diagonal matrix.

A diagonal matrix is a square matrix in which all elements except the diagonal elements are zero. (Diagonal elements are the elements $a_{ii}$ for $i = 1, 2, \ldots, n$). It is often useful to write a diagonal matrix using the notation $\operatorname{diag}(a_1, a_2, \ldots, a_n)$

### 5.5.3 Matrix addition.

If $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times n}$ then we define the matrix $A+B \in \mathbb{R}^{m\times n}$ to be $\left[c_{ij}\right]$
where $c_{ij} = a_{ij} + b_{ij}, A = \left[a_{ij}\right], B = \left[b_{ij}\right]$

**Properties.**

- $A+B = B+A$
- $A+(B+C) = (A+B)+C$

### 5.5.4 Zero matrix.

A zero matrix $B \in \mathbb{R}^{n\times m}$ is a matrix such that $A+B = B+A = A$ for any choice of matrix $A \in \mathbb{R}^{n\times m}$

### 5.5.5 Matrix multiplication.

If $A = \left[a_{ij}\right] \in \mathbb{R}^{m\times n}, B = \left[b_{jk}\right] \in \mathbf{R}^{n\times p}$, then we define the matrix product $AB = [c_{ik}] \in \mathbb{R}^{m\times p}$ where $c_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk}$. (NB: in general $AB \neq BA$.

### 5.5.6 The identity matrix.

The identity matrix is denoted $I_n$ or simply I;

**Properties.**

- $AI = IA = A$
- $I = \operatorname{diag}(1, 1, \ldots, 1)$

### 5.5.7 Matrix transpose.

If $A = \left[a_{ij}\right] \in \mathbb{R}^{m\times n}$, then the transpose of $A$, written $A^T$,is $A^T = \left[a_{ji}\right] \in \mathbb{R}^{n\times m\prime}$ (interchange rows and columns.)

### 5.5.8 Symmetric matrices.

A square matrix $A \in \mathbb{R}^{n\times n}$ is symmetric if $A^T = A$

### 5.5.9 Matrix inverse.

If $A \in \mathbb{R}^{n \times n}$ and there exists a matrix $B \in \mathbb{R}^{n \times n}$ such that $BA = AB = I$ then $B$ is called the inverse of $A$ and is denoted $A^{-1}$

- Any matrix that has an inverse is called invertible or non-singular
- If a matrix has an inverse, then the inverse is unique.
- If $A$ and $B$ are nonsingular, then $AB$ is nonsingular and $(AB)^{-1} = B^{-1}A^{-1}$
- If $A$ is nonsingular, then $\left(A^T\right)^{-1} = \left(A^{-1}\right)^T$

### 5.5.10 Scalar multiplication.

If $A = \left[a_{ij}\right] \in \mathbb{R}^{m \times n}$ and $r$ any scalar, then $rA = \left[b_{ij}\right] \in \mathbb{R}^{m \times n}$, where $b_{ij} = ra_{ij}, 1 \le i \le m, 1 \le j \le n$($r$ multiplies each element of A).

If $A, B, C$ are the appropriate sizes then the following properties hold:

- $A(BC) = (AB)C$
- $A(B + C) = (AB) + (AC)$
- $(A + B)C = (AC) + (BC)$

### 5.5.11 Determinants.

The determinant of a square matrix $A \in \mathbb{R}^{n \times n}$, denoted by $\det(A)$ or $|A|$ is defined recursively as follows

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

**Properties:**

(a) $|A| = \left|A^T\right|$
(b) $|A| = 0$ if any two rows or columns of $A$ are equal
(c) $|A| = 0$ if a row or column of $A$ consists entirely of zeroes
(d) If $B$ results from $A$ by interchanging two rows or columns of $A$, then $|B| = -|A|$
(e) If $B$ results from $A$ by multiplying a row or column of $A$ by a scalar $c$, then $|B| = c|A|$

(f) $|AB| = |A||B|$
(g) If any multiple of one row or column is added to another row (respectively column), element by element, the value of the determinant is unchanged
(h) If $A$ is upper or lower triangular, then $|A| = a_{11}a_{22}a_{33}\dots a_{nn}$
(i) If $A$ is nonsingular (i.e., invertible) then $|A| \neq 0$ and $\left|A^{-1}\right| = 1/|A|$.

**THEOREM 5.1.** A matrix $A$ is nonsingular if and only if $|A| \neq 0$.

**THEOREM 5.2.** If $A$ is an $n \times n$ matrix, the homogeneous system $Ax = 0$ has a nontrivial (nonzero) solution if and only if $A$ is singular.

### 5.5.12 Differentiation and integration.

If $A(t) = \left[a_{ij}(t)\right]$, where the $a_{ij}$ are functions of $t$, then $\frac{d}{di}A(t) = \left[\frac{d}{dt}a_{ij}(t)\right]$, i.e., $\dot{A}(t) = \left[\dot{a}_{ij}(t)\right]$ and $\int_{t_0}^{t_1} A(t)dt = \left[\int_{t_0}^{t_1} a_{ij}(t)dt\right]$

### 5.5.13 Linear dependence.

A set of vectors in $\mathbb{R}^n$, $\{x_1, x_2, \dots, x_n\}$, is said to be a linearly dependent set of vectors if there exist constants $c_1, c_2, \dots, c_k$, not all zero, such that

$$c_1x_1 + c_2x_2 + \dots + c_kx_k = 0 \tag{5.5}$$

If the only solution to (5.5) is for all the constants $c_i (i = 1, 2, \dots, k)$ to be zero, then the set of vectors is said to be linearly independent.

### 5.5.14 Vector space.

A set of vectors that obey certain properties of multiplication, addition etc., e.g., $\mathbb{R}^n$ is a vector space.

### 5.5.15 Basis of a vector space.

A set of linearly independent vectors in the vector space with the property that any vector in the vector space is a linear combination of the vectors is the basis.

### 5.5.16 Standard basis.

That basis containing unit vectors that are orthogonal. The standard basis of $\mathbb{R}^n$ is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \right\}$$

**THEOREM 5.3.** The vectors $a_1, a_2, a_3, \dots, a_n$ comprising the columns of $A \in \mathbb{R}^{n\times n}$ are linearly independent if and only if the matrix $A$ is non-singular (i.e., invertible $\Rightarrow |A| \neq 0$ ).
**PROOF.**
A linear combination of the vectors $a_1, a_2, \dots, a_n$ with respective weights $x_1, x_2, \dots, x_n$ can be represented as $Ax$ where $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$
By Theorem 5.2 $Ax = 0$ has a nonzero solution iff $A$ is singular. Thus $Ax = 0$ has only the zero solution iff $A$ is nonsingular.

### 5.5.17 Linear transformations.

A linear function $T : V \to V$ that maps a vector space $V$ to itself, with the property that for any vectors $x$ and $y$ in $V$

$$T(\alpha x + \beta y) = \alpha T x + \beta T y, \text{ for arbitrary scalars } \alpha \text{ and } \beta$$

is called a *linear transformation.*

The above definition for a linear transformation is equivalent to showing that the linear function satisfies

$$T(x + y) = T(x) + T(y)$$
$$T(\alpha x) = \alpha T(x).$$

### 5.5.18 Change of basis on a vector representation.

If $x$ is given w.r.t. the standard basis $e_1, e_2, \dots, e_n$, then $x = x_1 e_1 + \ldots + x_n e_n = I_n \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$.
To obtain the representation of $x$ w.r.t. a new basis $p_1, p_2, \dots, p_n$, we need $y_1, \dots, y_n$ such that $\boldsymbol{x} = y_1 p_1 + \cdots + y_n \boldsymbol{p}_n = \left(\boldsymbol{p}_1 \boldsymbol{p}_2 \ldots \boldsymbol{p}_n\right) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \boldsymbol{P} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, say.

$P$ has columns formed by the basis vectors $p_1, \ldots, p_n$ (which are linearly independent $\Rightarrow P$ is nonsingular $\Rightarrow P^{-1}$ exists.

Thus $\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = P^{-1}x$ or $y = P^{-1}x$

**EXERCISE 5.1** Show that the vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ w.r.t. the standard basis, is represented as $\begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix}$ w.r.t. the basis $\left\{ \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$.

### 5.5.19 Matrix representation of a linear transformation.

**THEOREM 5.4.** Every linear transformation in $\mathbb{R}''$, say, $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ can be expressed by matrix multiplication, such that $T(x) = Ax$, where $x \in \mathbb{R}^n$ is with respect to the standard basis $e_1, e_2, \ldots, e_n$.

For example, consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x_1, x_2) = (x_2, x_1 - x_2, 2x_1 + x_2)$.

First, we show that $T$ is a linear transformation and as such, by theorem (5.4) must have a matrix representation.

For $x, y \in \mathbb{R}^2$,

$$\begin{aligned} T(x+y) &= T\left(\left[x_1 + y_1, x_2 + y_2\right]\right) = \left(x_2 + y_2, x_1 + y_1 - x_2 - y_2, 2\left(x_1 + y_1\right) + \left(x_2 + y_2\right)\right) \\ &= (x_2, x_1 - x_2, 2x_1 + x_2) + \left(y_2, y_1 - y_2, 2y_1 + y_2\right) \\ &= T(x) + T(y) \end{aligned}$$

Thus vector addition is preserved.

Now,

$$\begin{aligned} T(\alpha x) &= T\left([\alpha x_1, \alpha x_2]\right) = (\alpha x_2, \alpha x_1 - \alpha x_2, 2\alpha x_1 + \alpha x_2) \\ &= \alpha\left(x_2, x_1 - x_2, 2x_1 - x_2\right) \\ &= \alpha T(\boldsymbol{x}) \end{aligned}$$

which shows that scalar multiplication by a vector is preserved by the transformation.

Hence, $T$ is a linear transformation and therefore has a matrix representation with respect to the standard basis $e_1, e_2$ spanning vector $x \in \mathbb{R}^2$.

That is, $T(\boldsymbol{x}) = \begin{pmatrix} x_2 \\ x_1 - x_2 \\ 2x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A\boldsymbol{x}.$

Thus matrix $A$ is the transformation matrix with respect to the standard basis for the linear map $T$.

**EXAMPLE 5.1.** Show that the transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x,y,z) = (x+y, x-y, 3z)$ is linear. Then obtain the matrix representation of this linear transformation with respect to the standard basis $e_1, e_2, e_3$.

A theorem that highlights the relationship between bases in $\mathbb{R}^n$ and the matrix representation of linear transformation in $\mathbb{R}^n$ is given below.

**THEOREM 5.5.** Let $T:\mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation, and let $P = \{p_1, p_2, \dots p_n\}$ be basis for $\mathbb{R}''$. For any vector $x \in \mathbb{R}^n$, the vector $T(\boldsymbol{x}) = Ax$ is uniquely determined by the vectors $T(p_1), T(p_2) \dots T(p_n)$.

Before proving, note that the last sentence of the theorem implies that matrix $A$ is uniquely determined by the vectors $T(p_1), T(p_2) \dots T(p_n)$.

### 5.5.20 Change of basis on a matrix representation of a linear transformation.

Suppose that a particular linear transformation has a matrix representation $A$ w.r.t. the standard basis: $e_1, e_2, \dots, e_n$. Then we have $A\boldsymbol{x} = z$. We wish to find the matrix representation of the linear transformation $B$ w.r.t. a new basis $\boldsymbol{p}_1, \boldsymbol{p}_2, \dots, \boldsymbol{p}_n$. Let $\boldsymbol{P} = (\boldsymbol{p}_1 \boldsymbol{p}_2 \dots \boldsymbol{p}_n)$. From the previous result $x$ would have the representation $y = P^{-1}x$ w.r.t. the basis $p_1, p_2, \dots, p_n$ and $z$ would have the representation $w = P^{-1}z$ say w.r.t. the basis $p_1, p_2 \dots p_n$. B represents the same linear transformation as $A$, so we need

$$\begin{gathered} By = w \text{ or} \\ B\left(P^{-1}x\right) = P^{-1}z, \\ \text{but } z = Ax \\ \text{so } BP^{-1}x = P^{-1}(Ax) = P^{-1}Ax \\ \Rightarrow BP^{-1} = P^{-1}A \\ \Rightarrow B = P^{-1}AP. \end{gathered}$$

**EXAMPLE 5.2.**

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(w.r.t. standard basis) rotates a vector clockwise by 90 degrees. Show that the representation of this rotation (i.e., linear transformation) w.r.t. the basis defined by

$$P = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

is

$$B = \begin{pmatrix} 0.5 & -0.5 \\ 2.5 & -0.5 \end{pmatrix}$$

### 5.5.21 Eigenvalues and eigenvectors.

Let $A \in \mathbb{R}^{n \times n}$. If there exists $x \in \mathbb{R}^n$ s.t. $x \neq 0$ and $Ax = \lambda x$ for $\lambda$ scalar, then $\lambda$ is an eigenvalue of $A$, with corresponding eigenvector $x$. 'Eigen' is often replaced by 'proper', 'characteristic' or 'latent' in textbooks and research papers. Geometrically $x$ is a direction along which the linear transformation $A$ expands or contracts, but does not rotate. Such directions have important physical significance for many systems in physics, mechanics, biology, chemical engineering etc.

### 5.5.22 Characteristic polynomial of a matrix.

The characteristic polynomial of $A \in \mathbb{R}^{n \times n}$ is the polynomial of degree $n$ in $\lambda$ obtained by expanding $|A - \lambda I_n|$. The roots of the characteristic polynomials are the eigenvalues of $A$. (Roots may be complex and may be repeated.)

### 5.5.23 Fundamental theorem of algebra.

Every polynomial of degree $n > 1$ has at least one root and can be decomposed into first degree factors. (Thus, every linear transformation has at least one eigenvalue and eigenvector.)

### 5.5.24 Eigenvalues and eigenvectors when the basis is changed.

Eigenvalues are independent of the basis used. If $Ax = \lambda x$ and $B = P^{-1}AP$ is the representation of $A$ w.r.t. the new basis defined by $P$, then we have

$$\begin{aligned} & Ax = \lambda x & \\ \Rightarrow P^{-1}(Ax) & = P^{-1}\lambda x \mid & \text{but } x = Pz \\ \Rightarrow P^{-1}APz = P^{-1}\lambda Pz & & \left(z = P^{-1}x\right) \\ & \Rightarrow Bz = \lambda P^{-1}Pz & \\ & \Rightarrow Bz = \lambda z & \end{aligned}$$

$\Rightarrow z = P^{-1}x$ (the representation of $x$ w.r.t. the basis defined by $P$ ) is an eigenvector of $B$ (the representation of $A$ w.r.t. the basis defined by $P$ ), with eigenvalue $\lambda$.

**EXERCISE 5.2.** Show that if $x$ is an eigenvector of $A$, then so is $\alpha x$ for some scalar $\alpha$.

**THEOREM 5.6.** If $A \in \mathbb{R}^{n\times n}$ has distinct eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_m$ with corresponding eigenvectors $u_1, u_2, \ldots, u_m, (m \leq n)$, then the set of vectors $u_1, u_2, \ldots u_m$ is linearly independent.

**THEOREM 5.7.** Any square matrix with all its eigenvalues distinct can be put into diagonal form by a change of basis. More specifically, if $A \in \mathbb{R}^{n\times n}$ with $n$ distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $u_1, u_2, \ldots, u_n$, then

$$\Lambda = M^{-1}AM$$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0\ldots & 0 \\ 0 & \lambda_2\ldots 0 & \\ \vdots & & \\ 0 & 0\ldots\lambda_n & \end{bmatrix}$$

and $M = [u_1 u_2 \ldots u_n]$. Thus, the linear transformation with representation $A$ w.r.t. the standard basis, has the form of a diagonal matrix w.r.t. an eigenvector basis.

**EXERCISE** 5.3**.**
(1) Diagonalise the matrix $\begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$. Answer $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.
(2) Solve the following systems using diagonalisation:

a

$$\begin{aligned} \dot{x}_1 &= 2x_1 + x_2; \quad x_1(0) = 2 \\ \dot{x}_2 &= 2x_1 + 3x_2; \quad x_2(0) = 1. \end{aligned}$$

b

$$\begin{aligned} x_{k+1} &= x_k - 2y_k; \quad x_0 = 1 \\ y_{k+1} &= -x_k + 2y_k; \quad y_0 = -1. \end{aligned}$$

# Chapter 6

# Solutions to linear systems

## 6.1 Homogeneous, constant coefficients, linear systems

There are various methods for solving systems of the form $\dot{x} = Ax, x(t_0)$ given with $A$ constant with respect to time (the independent variable). We consider three commonly used methods.

### 6.1.1 Decoupling system (diagonalisation of $A$ ).

If $A \in \mathbb{R}^{n\times n}$ and has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then we have seen that $A$ can be diagonalised by changing from the standard basis to an eigenvector basis.

### 6.1.2 Existence and uniqueness theorem.

**THEOREM 6.1.** Suppose we have a linear system $\dot{x} = Ax, x(t_0) = x_0$ with $A$ a constant matrix having $n$ distinct eigenvalues $\lambda_1, \ldots \lambda_n$, with corresponding eigenvectors $v_1, \ldots v_n$. Then

$$\boldsymbol{x}(t) = \alpha_1 v_1 e^{\lambda_1(t-t_0)} + \alpha_2 v_2 e^{\lambda_2(t-t_0)} + \cdots + \alpha_n v_n e^{\lambda_n(t-t_0)},$$

where

$$x_0 = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n.$$

**EXISTENCE.** Differentiate $x(t)$ to see if the equation for the solution satisfies $\dot{x} = Ax$

**UNIQUENESS.** Assume that we have two solutions

$$x_1(t) = \alpha_1 v_1 e^{\lambda_1(t-t_0)} + \alpha_2 v_2 e^{\lambda_2(t-t_0)} + \cdots + \alpha_n v_n e^{\lambda_n(t-t_0)},$$

and

$$x_2(t) = \beta_1 v_1 e^{\lambda_1(t-t_0)} + \beta_2 v_2 e^{\lambda_2(t-t_0)} + \cdots + \beta_n v_n e^{\lambda_n(t-t_0)}.$$

Since the initial condition are the same i.e. $x_1(t_0) = x_2(t_0)$, we have

$$\left(\alpha_1 - \beta_1\right) v_1 + \cdots + \left(\alpha_n - \beta_n\right) v_n = 0.$$

This implies that $\alpha_i = \beta_i, i = 1, \ldots, n$ since the vectors $v_i$ are linearly independent. Hence $x_1 = x_2$.

**EXAMPLE 6.1.** Solve the following systems by using the above Theorem.

a

$$\begin{aligned} \dot{x}_1 &= x_1 + 12x_2, x_1(0) = 0 \\ \dot{x}_2 &= 3x_1 + x_2, x_2(0) = 1. \end{aligned}$$

b

$$\begin{aligned} \dot{x}_1 &= x_2, x_1(0) = 1 \\ \dot{x}_2 &= -x_1, x_2(0) = 1. \end{aligned}$$

### 6.1.3 Exponential matrix.

Recall that the solution to the Malthusian growth type equation $dn/dt = kn, n(0) = n_0$ is $n(t) = n_0 e^{kt}$. We can extend this approach to systems. For $\alpha$ a scalar, we have

$$e^{\alpha} = 1 + \alpha + \frac{\alpha^2}{2!} + \cdots = \sum_{r=0}^{\infty} \frac{\alpha^r}{r!},$$

so that we can formally write

$$e^{A} = I_n + A + \frac{A^2}{2!} + \cdots = \sum_{r=0}^{\infty} A^r / r!,$$

where $A^0 = I_n$.
Let us assume that $A$ has $n$ distinct eigenvalues $\lambda_1, \ldots, \lambda_n$, with corresponding eigenvectors $v_1, \ldots, v_n$. Also let $M$ denote the matrix defining the eigenvector basis $M = [v_1 \ldots v_n]$. Then $\Lambda = M^{-1} AM$ implies that $A = \boldsymbol{M \Lambda M^{-1}}$,
where

$$\boldsymbol{\Lambda} = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Thus

$$\begin{aligned}
e^{At} &= \sum_{r=0}^{\infty} \left(M\Lambda M^{-1}t\right)^r / r! \\
&= \sum_{r=0}^{\infty} M\Lambda^r M^{-1} t^r / r! \\
&= M \sum_{r=0}^{\infty} \Lambda^r t^r / r! M^{-1} \\
&= M \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} M^{-1}.
\end{aligned}$$

Thus, $e^{At}$ is well-defined.

**LEMMA 6.1.** The matrix $e^{At}$ satisfies the matrix differential equation $\boldsymbol{X} = A\mathbf{X}$ which is not to be confused with the vector differential equation $\dot{x} = A\dot{x}$.

**PROOF.**

Substitute $e^{At}$ into the matrix d.e. and show that $e^{At}$ is a solution to the equation $X = AX$.

$$\begin{aligned}
&\frac{d}{dt} e^{At} = \frac{d}{dt} M \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} \boldsymbol{M}^{-1} \\
&= M \begin{pmatrix} \lambda_1 e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & \lambda_n e^{\lambda_n t} \end{pmatrix} M^{-1} \\
&= M \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \boldsymbol{M}^{-1} \boldsymbol{M} \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix} \boldsymbol{M}^{-1} \\
&= Ae^{At}.
\end{aligned}$$

Hence $e^{At}$ satisfies $\dot{X} = AX$.

**LEMMA 6.2.** For any constant vector $u \in \mathbb{R}^n$, $e^{At}u$ is a solution of $\dot{x} = Ax$. **PROOF.** Exercise

If $x(t) = e^{A(t-t_0)}x_0$, then $x(t)$ satisfies the vector differential equation $\dot{x} = Ax$ and $x(t_0) = I_n x_0 = x_0$ as required.

Hence $x(t) = e^{A(t-t_0)}x_0$ is the solution to $\dot{x} = Ax, x(t_0) = x_0$.

**EXAMPLE 6.2.**

The matrix $\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ has eigenvalues 0 and 1 with corresponding eigenvectors $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, say.

Thus if $\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ then $M = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $M^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$

This implies

$$\begin{aligned} e^{At} &= M \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} M^{-1} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^t & 0 \\ \left(e^t - 1\right) & 1 \end{pmatrix}. \end{aligned}$$

To solve the system $\dot{x} = Ax, x(t_0) = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$,

$$\begin{aligned} x(t) &= e^{A(t-t_0)} x(t_0) \\ &= \begin{pmatrix} e^{(t-t_0)} & 0 \\ \left(e^{(t-t_0)} - 1\right) & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ &= \begin{pmatrix} \alpha e^{(t-t_0)} \\ \alpha\left(e^{(t-t_0)} - 1\right) + \beta \end{pmatrix}. \end{aligned}$$

Check: When $t = t_0, x = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ as required.

**REMARK 6.1.** $e^{At}$ is called the transition matrix of the system - it tells us how to move from $x(t_0)$ to $x(t)$.

**REMARK 6.2.** If $A$ is diagonal with entries $\lambda_i, e^{At} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix}$, i.e. $M = I_n$.

**REMARK 6.3.** For any invertible matrix $Q, e^{Q^{-1}AQ} = Q^{-1} e^A Q$.

**REMARK 6.4.** Sometimes it is easier to compute $e^{At}$ from the series definition of the exponential function. For example, if $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $A^2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $A^r = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

Therefore

$$\begin{aligned}
e^{At} &= \sum_{r=0}^{\infty} (At)^r / r! \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} t + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} t^2/2! + \cdots \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \left(e^t - 1\right)
\end{aligned}$$

**REMARK 6.5.** Further $e^0 = I; \left(e^A\right)^{-1} = e^{-A}$; if $AB = BA$ then $e^A e^B = e^{A+B}$

**REMARK 6.6.** $\frac{d}{d}\left(e^{At}\right) = Ae^{tA} = e^{tA}A$ and $\left(\frac{d}{dt}\right)^m e^{tA} = A^m e^{tA}; e^{(t_1+t_2)A} = e^{t_1 A + i_2 A}$; $\left(e^{tA}\right)^{-1} = e^{-tA}$.

## 6.2 Fundamental Matrix Solution

In Section 5.1.3 we discussed a method for determining $e^{At}$. This method only worked for matrices with distinct eigenvalues. In this section, we deal with a more general framework for finding solutions to $\dot{x} = Ax$ which works for distinct as well as repeated eigenvalues.

A fundamental matrix solution (FMS) of the system $\dot{x} = Ax$ is a matrix $X$ (t) which satisfies the corresponding matrix differential equation $\dot{X} = AX$ and also has $|X(t_0)| \neq 0$ (i.e. a non-singular solution at $t = t_0$ ).

**LEMMA 6.3.**
If $X(t)$ is a fundamental matrix solution of the system $\dot{x} = Ax$ then $|X(t)| \neq 0 \forall t$.

**PROOF.** Since $X(t)$ is a FMS, $\dot{X} = AX$ and $|X(t_0)| \neq 0$. Suppose there exists $\tau \neq t_0$ such that $|\boldsymbol{X}(\tau)| = 0$. Since $|\boldsymbol{X}(\tau)| = 0$, there exists $\boldsymbol{w} \neq \boldsymbol{0}$ such that $X(\tau)w = 0$. Let $x(t) = X(t)w$. Then

$$\begin{aligned}
\frac{d}{dt}x &= \frac{d}{dt}X(t)w \\
&= AX(t)w \\
&= Ax(t).
\end{aligned}$$

Thus $x(t) = X(t)w$ satisfies $\dot{x} = Ax$. But $x(\tau)$ is just $X(\tau)w = 0$. Hence $x(t) = 0$ for all $t$ and in particular $x(t_0) = 0$ which means that $X(t_0)w = 0$ and since $w \neq 0$, $|X(t_0)| = 0$ which is a contradiction and proves the result.

**THEOREM 6.2.** Let $X(t)$ be any FMS of $\dot{x} = Ax$. Then

$$e^{A(t-t_0)} = X(t)X^{-1}(t_0).$$

**PROOF.**

1. $X(t)$ is a FMS of $\dot{x} = Ax$. Hence $\dot{X} = AX$ and $|X(t_0)| \neq 0$ and thus $\boldsymbol{X}^{-1}(t_0)$ exists.

2. $e^{A(t-t_0)}$ is a FMS of $\dot{x} = Ax$ since $\frac{d}{dt}e^{A(t-t_0)} = Ae^{A(t-t_0)}$ and $\left|e^{A(t_0-t_0)}\right| = |I_n| = 1 \neq 0$.

3. If $Y$ and $Z$ are both FMS of $\dot{x} = Ax$, then there exists a constant matrix $C$ such that $Y = ZC$. (Prove!)

4. By (1), (2),(3), there exists a constant matrix $C$ such that $e^{A(t-t_0)} = X(t)C$. At $t = t_0, e^{A(t_0-t_0)} = I_n = X(t_0)C$. By (i) this implies that $C = X^{-1}(t_0)$. Thus $e^{A(t-t_0)} = X(t)X^{-1}(t_0)$.

The problem has been transferred from finding $e^{A(t-t_0)}$ to finding $X(t)$, a FMS of $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$.

### 6.2.1 $A$ has distinct eigenvalues.

If $A$ has distinct eigenvalues with corresponding eigenvectors $v_1, v_2, \ldots, v_n$, then a FMS is given by

$$\boldsymbol{X}(t) = \begin{pmatrix} e^{\lambda_1(t-t_0)}v_1 & e^{\lambda_2(t-t_0)}v_2 & \cdots & e^{\lambda_n(t-t_0)}v_n \end{pmatrix}$$

This is because (1) distinct eigenvalues imply linearly independent eigenvectors imply linearly independent columns at $t = t_0$ imply $X(t_0)$ is invertible and (2) the matrix as given satisfies the matrix differential equation $\dot{X} = AX$ :

$$\begin{aligned}
&\frac{d}{dt}\left(e^{\lambda_1(t-t_0)}v_1 \quad \cdots \quad e^{\lambda_1(t-t_0)}v_n\right) \\
&= \left(\lambda_1 e^{\lambda_1(t-t_0)}v_1 \quad \cdots \quad \lambda_n e^{\lambda_n(t-t_0)}v_n\right) \\
&= A\left(e^{\lambda_1(t-t_0)}v_1 \quad \cdots \quad e^{\lambda_n(t-t_0)}v_n\right) \\
&\text{since } Av_i = \lambda_i v_i\, i = 1, \ldots, n \\
&= AX
\end{aligned}$$

**EXAMPLE 6.3.** Calculate $e^{At}$ by first forming a FMS $X(t)$, where $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$. As a check, calculate $e^{At}$ from $M\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}M^{-1}$.

### 6.2.2 $A$ has repeated eigenvalues.

If $A$ has repeated eigenvalues, we need another technique to form a fundamental matrix solution - we need to be able to produce $n$ linearly independent columns to ensure that $|\mathbf{X}(t_0)| \neq 0$.

(i) $e^{-\lambda I_n t} = I_n \begin{pmatrix} e^{-\lambda t} & & 0 \\ & \ddots & \\ 0 & & e^{-\lambda t} \end{pmatrix} I_n^{-1} = e^{-\lambda i} I_n$. Thus

$$e^{A(t-t_0)} = e^{(A-\lambda I_n)(t-t_0)} e^{\lambda(1-t_0)}.$$

We now show that this holds for all $t$. If we set $X(t) = e^{A(t-t_0)} - e^{(A-\lambda I_n)(t-t_0)} e^{\lambda(t-t_0)}$, then $X(t_0) = I_n - I_n = O(O$ is the zero matrix).
Also

$$\begin{aligned} \frac{d}{dt} X(t) =& A e^{A(t-t_0)} - (A-\lambda I_n) e^{\lambda(t-t_0)} e^{(A-\lambda I_n)(t-t_0)} \\ & - \lambda e^{\lambda(t-t_0)} e^{(A-\lambda I_n)(t-t_0)} \\ =& A\left(e^{A(t-t_0)} - e^{(A-\lambda I_n)(t-t_0)_e \lambda(t-t_0)}\right) \\ =& AX(t). \end{aligned}$$

Hence $X(t) = 0$ for all $t$ which means that

$$e^{A(t-t_0)} = e^{\lambda(t-t_0)} e^{(A-\lambda I_n)(t-t_0)} \forall t.$$

(ii) $e^{\boldsymbol{A}(t-t_0)}$ and $e^{\boldsymbol{A}(t-t_0)} v$ for some constant vector $v$ are always solutions to the matrix and vector d.e.s. The problem is to find $e^{A(t-t_0)} v$ and by using the above equation

$$e^{A(t-t_0)} v = e^{\lambda(t-t_0)} e^{(A-\lambda I_n)(t-t_0)} v \forall t.$$

The expansion of $e^{(A-\lambda I_n)(t-t_0)} v$ is finite provided $(A-\lambda I_n)^n v = 0$ for some integer $m$ (i.e. the series terminates after $m$ terms). In this case

$$\begin{aligned} e^{(A-\lambda I_n)(t-t_0)} v =& v + (A-\lambda I_n)(t-t_0) v + \frac{1}{2!}(A-\lambda I_n)^2 \langle t-t_0)^2 v + \cdots \\ & + \frac{1}{(m-1)!}(A-\lambda I_n)^{m-1}(t-t_0)^{m-1} v \end{aligned}$$

Thus if there exists an integer $m$ such that $(A-\lambda I_n)^m v = 0$ and $(A-\lambda I_n)^{m-1} v \neq 0$ , then

$$e^{A(t-t_0)_v} = e^{\lambda(t-t_0)}\left[v + (A-\lambda I_n)(t-t_0) v + \cdots + \frac{1}{(m-1)!}(A-\lambda I_n)^{m-1}(t-t_0)^{m-1} v\right]$$

and we have a procedure for forming $e^{A(t-t_0)} v$ which is another linearly independent solution to the vector d.e. $\dot{x} = Ax$ (i.e. another column of a FMS).

**EXAMPLE 6.4.** Find three linearly independent solutions of

$$\dot{x} = A\boldsymbol{x}$$

where $A$ is given by

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

you may assume that $t_0 = 0$.

**REMARK 6.7.** If one found $v$ such that $(A - \lambda I_n)^2 v = 0$ and $(A - \lambda I_n) v \neq 0$ and still need further linearly independent solutions, then one can solve for $v$ such that $(A - \lambda I_n)^3 v = 0$ and $(A - \lambda I_n)^2 v \neq 0$

## 6.3 Homogeneous, Variable Coefficient, Linear Systems

We extend the idea of a FMS from the constant $A$ case to the time-dependent $A(t)$ case with the following theorem.

**THEOREM 6.3.** If $\boldsymbol{A}(t)$ is an $n \times n$ matrix whose elements are continuous functions of time $\left(t \in \left[t_0, t_f\right]\right)$, then there is a unique solution $x(t)$ of $\dot{x} = A(t)x$ which is defined on $\left[t_0, t_f\right]$ and takes the value $x_0$ at $t_0$.

**PROOF.** Let $\phi_i\,(t, t_0)$ be the solution of the system $\dot{x} = Ax$ with $x\,(t_0) = e_i$ (the ith standard basis vector). Define the matrix

$$\Phi\,(t, t_0) = \left[\phi_1 \ldots \phi_n\right].$$

$\Phi\,(t, t_0)$ is called the transition matrix of the system $\dot{x} = A(t)x.\Phi\,(t, t_0)$ satisfies the matrix d.e. (i.e. $\dot{\Phi} = A\Phi$ ) and also $\Phi\,(t_0, t_0) = [e_1 \ldots e_n] = I_n$ which implies $|\Phi\,(t_0, t_0)| \neq 0$.

**THEOREM 6.4.** The solution of $\dot{x} = A(t)x, x\,(t_0) = x_0$ is given by $x(t) = \Phi\,(t, t_0)\,x_0$.

**PROOF.**
(i) $\frac{d}{dt}\Phi\,(t, t_0)\,x_0 = A\Phi\,(t, t_0)\,x_0 = Ax(t)$,
(ii) $\Phi\,(t_0, t_0)\,x_0 = I_n x_0 = x_0$. By theorem 1 the solution is unique.

**REMARK 6.8.** In spite of Theorem 5.3 and 5.4 we cannot always find $\Phi\,(t, t_0)$ for arbitrary $A(t)$. The following theorem gives a special case for which we can find $\Phi\,(t, t_0)$.

**THEOREM 6.5.** If $A(t)\left(\int_{i_0}^{t} A(\tau)d\tau\right) = \left(\int_{t_0}^{t} A(\tau)d\tau\right) A(t)$ for all $t$, then the system $\dot{x} = A(t)x$ has

$$\Phi(t, t_0) = e^{\int_{t_0}^{t} A(\tau)d\tau}$$

as its transition matrix.

**PROOF.**

Write down the series expansion for the exponential. Then show that $\Phi(t, t_0)$ as given satisfies the d.e. and initial condition.

**COROLLARY 6.1.** If $A$ is constant, then $\Phi(t, t_0) = e^{A(t-t_0)}$

**PROOF.**

Use Theorem 6.5 and evaluate $\int_{t_0}^{t} A(\tau)d\tau$ for constant $A$.

**EXAMPLE 6.5.** Solve

$$\dot{x} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x, \quad x(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

## 6.4 Properties Of Transition Matrix

(1) $\Phi$ is always invertible (i.e. $|\Phi(t, t_0)| \neq 0$ for $t_0 \leq t < \infty$ )
(2) $\Phi(t, t_0) = \Phi(t, t_1)\Phi(t_1, t_0)$
(3) $\Phi^{-1}(t, t_0) = \Phi(t_0, t)$

The proofs of the above although straightforward are not given here!

## 6.5 General Non-Homogeneous Linear Systems

**THEOREM 6.6.** If $\Phi(t, t_0)$ is the transition matrix for the system $\dot{x} = A(t)x$, then the unique solution of $\dot{x} = A(t)x + f(t)$ with $x(t_0) = x_0$, where $f(t)$ is continuous on $[t_0, t_f]$, is

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau) f(\tau)d\tau.$$

Thus, once the transition matrix $\Phi$ is known, one can calculate the complete solution.

**PROOF.**

(1) Check that the initial conditions are satisfied.
(2) Check that $x(t)$ as given satisfies the differential equation $\dot{x} = A(t)x + f(t)$. (Use Fundamental Theorem of Calculus.)
(3) Uniqueness: Suppose there exist two solutions $x_1$ and $x_2$. Set $z = x_1 - x_2$ and show that $z(t) = 0 \forall t \in [t_0, t_f]$.

**COROLLARY 6.2.** If $A$ is a constant matrix, then

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)} f(\tau)\,d\tau$$

**EXAMPLE 6.6.** Solve

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} x(t) + e^t \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ x(0) &= \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

## 6.6 Discrete Systems (Equivalent Theory)

### 6.6.1 Homogeneous, constant coefficients, linear systems.

To solve the system $x_{k+1} = Ax_k$, where $x_0$ is given, one could use the decoupling technique (provided $A$ has distinct eigenvalues) illustrated earlier. Alternatively, provided $A$ has distinct eigenvalues, we have the discrete version of the existence and uniqueness theorem.

**THEOREM 6.7.**
Suppose $A$ has distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ with corresponding eigenvectors $v_1, \ldots, v_n$. Also assume that $A$ is invertible, if not we could have problems with initial conditions. Then the system $x_{k+1} = Ax_k$, with $x_0$ given, has the unique solution

$$x_k = \alpha_1 \lambda_1^k v_1 + \cdots + \alpha_n \lambda_1^k v_n,$$

where

$$x_0 = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

**PROOF.**

The formula $x_k = \sum_{i=1}^{n} \alpha_i \lambda_i^k v_i$ satisfies the difference equation

$$\begin{aligned} Ax_k &= A\sum_{i=1}^{n} \alpha_i \lambda_i^k v_i \\ &= \sum_{i=1}^{n} \alpha_i \lambda_i^k A v_i \\ &= \sum_{i=1}^{n} \alpha_i \lambda_i^{k+1} v_i \\ &= x_{k+1}. \end{aligned}$$

The eigenvectors of $A$ form a basis (i.e. are linearly independent) and thus $\alpha_1 v_1 + \cdots + \alpha_n v_n = x_0$ has a unique solution for $\alpha_1, \ldots, \alpha_n$.

Another alternative is to solve $x_{k+1} = Ax_k$ by finding the transition matrix $A^k$. The discrete system $x_{k+1} = Ax_k$ is completely solved by $x_k = A^k x_0$.

If $A$ has distinct eigenvalues and $M$ is the matrix whose columns are the eigenvectors, then

$$A^k = M\Lambda^k M^{-1} = M \begin{pmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{pmatrix} M^{-1}$$

( $A^k$ is not always invertible, e.g. if $\lambda_i = 0$ for some $i$ ) or we can (as a further alternative) construct fundamental matrix solutions. A FMS is defined as a matrix $X_k$ satisfying the corresponding matrix difference equation $X_{k+1} = AX_k$ with $|X_0| \neq 0$. (if $X_k$ is a FMS, then $|X_k| \neq 0 \forall k$.)

If $A$ has distinct eigenvalues the a FMS is obtained by

$$\mathbf{X}_k = \left[\lambda_1^k v_1 \ldots \lambda_n^k v_n\right].$$

If $A$ has repeated eigenvalues then linearly independent columns of a FMS are obtained in a similar fashion to those in the continuous case:
If $v$ satisfies $(A - \lambda I_n)^2 v = 0$ and $(A - \lambda I_n) v \neq 0$, then

$$A^k v = k\lambda^{k-1} Av - (k-1)\lambda^k v,$$

which is easy to compute.

Either way, once we have a FMS $X_k$, then $A^k = X_k X_0^{-1}$ and $x_k = A^k x_0 = X_k X_0^{-1} x_0$.

### 6.6.2 Homogeneous, variable coefficients, linear systems.

Consider the system $x_{k+1} = A_k x_k$ with $|A_k| \neq 0$ and $x_{k_0}$ known. As in the continuous case there is a theorem which asserts the existence of a unique solution satisfying any given initial condition. Thus we are able to compute $\Phi(k, k_0)$ as that matrix with the $i$ th column being the solution of $x_{k+1} = A_k x_k$ with $x_{k_0} = e_i$, the $i$ th standard basis vector.

**THEOREM 6.8** The solution to $x_{k+1} = A_k x_k, x_{k_0}$ known (often $x_0 = x_{k_0}$ ) is

$$x_k = \Phi(k, k_0)\, x_{k_0}.$$

If $A$ is constant, then $\Phi(k, k_0) = A^{k-k_0}$. There is no equivalent special case as in the continuous case for finding $\Phi(k, k_0)$ in general.

### 6.6.3 General non-homogeneous linear systems.

Consider the more general equation

$$x_{k+1} = A_k x_k + f_{k'} \quad \text{with } x_0 = x_{k_0} \text{ known.}$$

**THEOREM 6.9**. The solution to the above system is given by

$$x_k = \Phi(k, k_0)\, x_{k_0} + \sum_{l=k_0}^{k-1} \Phi(k, l+1) f_l,$$

where $\Phi(k, k_0)$ is a solution to $x_{k+1} = A_k x_k$.

**PROOF.** For $k = k_0$ we interpret the sum as zero. Then

$$x_k = \Phi(k_0, k_0)\, x_{k_0} = I_n x_{k_0} = x_{k_0}.$$

Also,

$$\begin{aligned} x_{k+1} &= \Phi(k+1, k_0)\, x_{k_0} + \sum_{l=k_0}^{k} \Phi(k+1, l+1) f_l \\ &= A_k \Phi(k, k_0)\, x_{k_0} + \sum_{l=k_0}^{k-1} \Phi(k, l+1) f_l + \Phi(k+1, k+1) f_k \\ &= A_k x_k + f_k \end{aligned}$$

as required.

**COROLLARY 6.3.** If $A$ does not depend on $k$, then the solution to $x_{k+1} = A x_k + f_k, x_{k_0}$ known, is

$$x_k = A^{k-k_0} x_{k_0} + \sum_{l=k_0}^{k-1} A^{k-1-1} f_l.$$

**EXERCISE 6.1.** Find an expression for the general solution to the system $x_{k+1} = A x_k + b$, where $A$ is a constant matrix and $b$ a constant vector.

# Chapter 7

# Power Series Solution of linear ordinary differential equations

# Chapter 8

# Tutorials

## 8.1 Tutorial 1 - Integration Techniques

**EXERCISE .1.** Find the differentials of the following functions.

(1) $u = \sqrt{1+x^2}$
(2) $u = \sqrt{1+x^3}$
(3) $u = \sin^3(2x)$
(4) $u = \cos(()\,2x+3)$
(5) $u = \left(x^2+y^2\right)^{-3/2}$
(6) $u = \exp\left[\sin\left(xy\right)\right]$
(7) $u = \frac{\sin(x)}{1+\cos(x)}$

**EXERCISE .2.** Evaluate the following integrals using appropriate substitutions.
(1) $\int 2x\left(x^2+1\right)^{1/2} dx$
(2) $\int \cos^3 x \sin(x)\, dx$
(3) $\int \sin^3 2x \cos(2x)\, dx$
(4) $\int x^2 \mathrm{e}^{x^3}\, dx$

**EXERCISE .3.** Calculate the following integrals using integration by parts.
(1) $\int x^2 e^{ax} dx$
(2) $\int e^x \sin(x)\, dx$
(3) $\int \mathrm{e}^{nx} \sin(bx)\, dx$
(4) $\int \mathrm{e}^{ax} \cos(bx)\, dx$
(5) $\int xe^x \cos(x)\, dx$
(6) $\int x\mathrm{e}^x \sin(x)\, dx$

**EXERCISE 4 .**

Recall that the formula for the volume of a sphere of radius $r$ is $V(r) = \frac{4}{3}\pi r^3$

Assuming that the error on the radius is $\Delta r$, give an approximation of the error on the volume (hint: use differentials to approximate errors).

The kinetic energy of a particle of constant mass $m$ moving at the speed $v$ is given by

$$K = \frac{1}{2}mv^2.$$

Find the error on the kinetic energy as a function of the error on the speed $\Delta v$ (hint: use differentials to approximate errors).

**EXERCISE .5.** Integrate the following rational functions by using suitable partial fraction decompositions.
(1) $\frac{-2x+2}{(x^2+2x+2)x^2}$
(2) $\frac{x^2+4x+4}{(x+1)^2(x^2+3x+3)}$
(3) $\frac{3x+5}{(x-2)(x-3)(x-5)}$
(4) $\frac{3x^2-5x+6}{(x-1)^3}$
(5) $\frac{2x-3}{(x-1)(x-2)}$
(6) $\frac{x^2+4x+4}{(x+1)^2(x^2+3x+3)}$
(7) $\frac{-x^2+4x+9}{(x^2-1)(x+2)}$
(8) $\frac{x^2+1}{x^2-1}$
(9) $\frac{2x^4+1}{(x+1)(x-1)}$
(10) $\frac{1}{x^3+a^3}$, $a \equiv$ constant

## 8.2 Tutorial 2 - Ordinary differential equations

**EXERCISE .6.** Find the order and degree (if any) of the ordinary differential equations
(1) $y' + 2x = \cos(y)'$, where $y' = \frac{dy}{dx'}$
(2) $\sqrt[3]{\left(y'''\right)^2} = \sqrt{1 + \left(y'\right)^2}$, where $y''' = \frac{d^3y}{dx^3}$.

**EXERCISE .7.** Find the differential equation whose general solution is

$$y = C_1 e^x + C_2 e^{-x} + x$$

where $C_1$ and $C_2$ are arbitrary constants.

**EXERCISE .8.** Solve the following ordinary differential equations:
(1) $\left(1 + x + xy^2\right) dy + \left(y + y^3\right) dx = 0,$

(2) $y' + \frac{3}{x} y = \frac{\alpha}{x^4}$, $\alpha$ is a constant,
(3) $y' + \frac{y}{x} = x^n y^{1/3}$, $n$ is a real number,
(4) $x\left(y - x^2\right) y' = y^2$,
(5) $(x+y)dy + (x-y)dx = 0$,
(6) $y' = \frac{x-2y+4}{2x+y-2}$.
(7) $(4x+6y+4)dx - (2x+3y+3)dy = 0$,
(8) $\left(6xy^2 - 3x^2\right) dx + \left(6x^2 y + 3y^2 - 4\right) dy = 0$,
(9) $xdx + ydy = 2\sqrt{x^2+y^2}\, ydy$,
(10) $\frac{d}{dx}\left(y' + \frac{2}{x} y\right) = x$,
(11) $\left(\frac{d}{dx} + x\right)\left(y' + xy\right) = \mathrm{e}^{-x^2/2}$. (Hint: define a new dependent variable $u = y' + xy$).

**EXERCISE .9.** Find two solutions of the ordinary differential equation

$$\left(y'\right)^2 + xy - 2x^2 = 0$$

**EXERCISE .10.** Consider the differential equation

$$y' = 2x^{-2} y^2 + 3 \tag{8.1}$$

Perform the change of the independent and dependent variables

$$\bar{x} = \mathrm{e}^a x, \quad \bar{y} = \mathrm{e}^n y,$$

where $a$ is a constant. Show that the form of the equation (.1) does not change in the new variables.

Now define a new dependent variable $z = y/x$. Show that $z$ satisfies the ordinary differential equation:

$$xz' = 2z^2 - z + 3. \tag{8.2}$$

Solve (6.2) using separation of variables. What is the solution of (6.1)?

## 8.3 Tutorial 3 - Ordinary differential equations

**EXERCISE .11.** Solve the following differential equations.
(1) $y'' - y' + y = x$,
(2) $y'' - y' + y = e^{2x} + e^x$,
(3) $y'' + 2y' + y = e^{-x} + xe^{-x}$,
(4) $y'' + 9y = \sin(3x)$
(5) $4y'' + 36y = 2x + e^x$,
(6) $y^{(4)} + 2y'' + y = 1 + x^2$, $\left(y^{(4)} = \frac{d^4 y}{dx^4}\right)$,

(7) $y'' - 4y = \cos^2 x$,
(8) $y'' + y' + y = (1 + \sin(x))^2$,
(9) $y'' + 4y = x\sin(x)$,
(10) $y''' - y'' - 4y' + 4y = e^{2x}$, $\left(y''' = \frac{d^3y}{dx^3}\right)$,
(11) $y^{(4)} + y = x\sin(x)$, $\left(y^{(4)} = \frac{d^4y}{dx^4}\right)$,
(12) $y'' - 2y' + 2y = 3e^x\cos(x)$.
(13) Find the general solution of $y'' + y' + 2y = \cos(x)$ by solving $y'' + y' + 2y = e^{ix}$, $i = \sqrt{-1}$. (Hint: recall that $\mathrm{Re}\, e^{ix} = \cos(x)$.)

**EXERCISE .12.** Use the method of variation of parameters to solve the following linear ordinary differential equations.
(1) $y'' + y = \sec(x)$, ( recall that $\sec(x) = 1/\cos(x)$ )
(2) $y'' - 4y' + 5y = e^{2x}/\sin(x)$,
(3) $y'' - 6y' + 9y = e^{3x}/x^2$.

**EXERCISE .13.** Solve the following Euler (or Cauchy) equations.
(1) $x^2y'' + 3xy' + y = 0$
(2) $x^2y'' - 2y = x$,
(3) $2(3x+1)^2y'' + 12(3x+1)y' - 36y = 3(3x+1)^3$. (Hint: let $Y = y, X = 3x+1$ and rewrite the equation in the new variables.)

**EXERCISE .14.** Solve the following second-order ordinary equations by first reducing them to first-order ordinary differential equations.

1. $xy'' = 3y'$,
2. $yy'' - y'^2 = 0$.
3. If $y_1(x)$ is a particular solution of $y'' + f(x)y' + g(x)y = 0$, show that another solution is $y_2 = v(x)y_1(x)$, where $v(x)$ is obtained by solving the first-order ordinary differential equation in $z = v'(x)$
$$y_1 v'' + \left(2y_1' + f y_1\right) v' = 0$$
Solve this equation and write down $y_2$ explicitly.
4. Using $y_1 = e^{x^2}$ solve the linear second-order ordinary differential equation
$$y'' - 4xy' + \left(4x^2 - 2\right) y = 0.$$

**EXERCISE .15.** The Euler or Cauchy equation of second-order is $x^2y'' + axy' + by = 0$, where $a$ and $b$ are constants.

1. Show that $y = x^m$ is a solution of the Euler (or Cauchy) equation if and only if $m$ is a root of the auxilary equation

$$m^2 + (a-1)m + b = 0.$$

2. If $m_1$ and $m_2$ are real roots, the general solution is

$$y = C_1 x^{m_1} + C_2 x^{m_2}. \text{ Why?}$$

3. If $m_1 = m_2 (= m)$, use variation of parameters to show that $y_2 = x^m \ln(x)$ is another solution (apart from $y_1 = x^m$ ). Hence conclude that the general solution of the Euler equation in this case is

$$y = (C_1 + C_2 \ln(x))\, x^m$$

**EXERCISE .16.** Solve the following systems by the method of elimination.

(1)

$$\begin{aligned} \frac{dx}{dt} &= 2x + y \\ \frac{dy}{dt} &= 4x + 3y + 5\cos(t) \end{aligned}$$

(2)

$$\begin{aligned} \frac{dx}{dt} + 2y + 3x &= 0 \\ \frac{dy}{dt} + 3x - 2y &= 0 \end{aligned}$$

(3)

$$\begin{aligned} \frac{dx}{dt} &= y - x, \\ \frac{dy}{dt} &= -x - 3y. \end{aligned}$$

(4)

$$\begin{aligned} \frac{dx}{dt} + y &= 0 \\ \frac{dx}{dt} - \frac{dy}{dt} &= 3x + y \end{aligned}$$

(5)

$$\begin{aligned} 4\frac{dx}{dt} - \frac{dy}{dt} + 3x &= \sin(t) \\ \frac{dx}{dt} + y &= \cos(t). \end{aligned}$$

(6)

$$\begin{aligned} 2\frac{dx}{dt} + \frac{dy}{dt} - 2x - 2y &= 5e^t \\ \frac{dx}{dt} + \frac{dy}{dt} + 4x + 2y &= 5e^{-t} \end{aligned}$$

## 8.4 Tutorial 4 - Difference equations

**EXERCISE .17.** Solve the following first-order difference equations:
(1) $y_{k+1} = 2y_k$ given that $y_1 = 5$.
(2) $y_{k+1} = y_k + k$
(3) $y_{k+1} = \frac{1}{2}\left(1 + y_k\right)$.
(4) $y_{k+1} = y_k / \left(1 + y_k\right)$ given that $y_0 = 1$.
(5) $y_{k+1} + (k+2)\,y_k = (k+2)\,!$ (the symbol '!' stands for factorial).

**EXERCISE .18.**Use the method of undetermined coefficients to solve:
(1) $y_{k+2} - 3y_{k+1} + 2y_k = k$.
(2) $y_{k+2} - 2y_{k+1} + y_k = 2^k$.
(3) $y_{k+2} - 4y_k = 2$.
(4) $y_{k+2} + y_{k+1} + y_k = e^k$.
(5) $y_{k+2} + y_k = \cos(3k)$.
(6) $y_{k+2} + y_k = \cos\left(\frac{\pi}{2}k\right)$.
(7) $(E-2)^2(E-1)\,y_k = k2^k$, where $E$ is the shift operator.

**EXERCISE . 19.** Consider the $n$ th-order constant coefficients linear difference equation

$$\left(\sum_{i=0}^{n} b_i E^i\right) y_k \equiv f(E)\,y_k = a^k g(k),$$

where $b_i, a \neq 1$ are constants.

(1) Let $y_k = a^k v_k$. Prove that the equation reduces to

$$f(aE)\,v_k = g(k).$$

(2) Solve $(E-3)(E-1)\,y_k = k3^k$ by setting $y_k = 3^k v_k$.
(3) Solve by the method of elimination the following system:
(4)

$$x_{k+1} + 2y_{k+1} - x_k = 0, \quad y_{k+1} - 2x_k - y_k = a^k,$$

where $a$ is a constant not equal to 1 .

## 8.5 Tutorial 5 - Matrix algebra

**EXERCISE .20.**

(1) Find the determinant of the matrix

$$A = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ 1 & 0 & 2 & 1 \\ 4 & 2 & 5 & 0 \end{pmatrix}.$$

(2) Would $Ax = 0$ have a nonzero solution?
(3) Are the columns of $A$ linearly independent vectors?

**EXERCISE .21.**
(1) Find suitable eigenvectors for the corresponding eigenvalues of the matrix

$$B = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$$

(Note that a matrix with real elements may have complex conjugate eigenvalues and vectors.)

(2) Diagonalise the matrix

$$C = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$$

(3) Find the eigenvalues and corresponding eigenvectors for the matrix

$$D = \begin{pmatrix} 3 & 3 \\ 2 & 4 \end{pmatrix}$$

**EXERCISE .22.**

(1) Derive the general formula for the inverse of a nonsingular $2 \times 2$ matrix.
(2) Consider a $2 \times 2$ diagonal matrix and show that the diagonal elements are indeed the eigenvalues. Extend the result to an $n \times n$ matrix.

**EXERCISE .23.**

(1) Write the system

$$\dot{x}_1 = x_2 - 5x_1, \quad \dot{x}_2 = x_2 + f(t) + C, C \text{ a constant}$$

in matrix/vector notation.

(2) Write the equation

$$\frac{d^4 x}{dt^4} = 3x + \frac{dx}{dt} + t^2 \frac{d^2 x}{dt^2} - t \frac{d^3 x}{dt^3} + 5$$

as a system of first-order equations. Then write this linear system of equations in the form

$$\dot{\boldsymbol{y}} = \boldsymbol{A}(t)\boldsymbol{y} + \boldsymbol{u},$$

where $\boldsymbol{u}$ is a constant vector. Is this system autonomous?

**EXERCISE .24.** If $A = \begin{pmatrix} 1 & t \\ 2t & e^t \end{pmatrix}$, calculate $\frac{dA}{dt}$, $\int A(t)dt$ and $\int_1^2 A(t)dt$.

## 8.6 Tutorial 6 - Linear Systems

**EXERCISE .25.** Solve the system

$$\dot{x}_1 = 3x_1 + 3x_2$$
$$\dot{x}_2 = 2x_1 + 4x_2$$

subject to $x_1(0) = 0, x_2(0) = 5$ by

(1) diagonalisation of $A$ (express the system as $\dot{x} = Ax$ ),
(2) using existence and uniqueness theorem and
(3) calculating $e^{tA}$ in two ways.

**EXERCISE .26.** Consider the second-order linear difference equation

$$f_{n+2} = 5f_{n+1} - 4f_n, n \geq 0. \tag{8.3}$$

(1) Solve (6.3) using the theory of scalar linear discrete equations with constant coefficients.
(2) Express (6.3) as a system of two first-order difference equations and use diaginalisation to solve it.
(3) Use existence and uniqueness theorem to solve (6.3).
(4) After rewriting (6.3) in the form $v_{n+1} = Av_n$, calculate $A^k$ in two ways and deduce the general solution of (6.3).

**EXERCISE .27.** If $\dot{x} = Ax$, where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

(1) Find $e^{iA}$.

(2) If $x_{k+1} = Ax_k$, find $A^k$.

**EXERCISE .29.**

(1) If $A = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$, find $e^{tA}$ and $A^k$.

(2) If $\dot{x} = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix} x, x(0) = (1,0)^T$, solve the system by first calculating the transition matrix $\Phi(t, t_0)$.

(3) If $x_{k+1} = \begin{pmatrix} 0 & k \\ 1 & 0 \end{pmatrix} x_k, x_0 = (2,1)^T$, solve the system by forward recursion.

(4) Solve the system

$$\dot{x} = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t, \quad x(0) = (1,1)^T$$

(5) Solve the system

$$\dot{x}_1 = 3x_1 + 3x_2 + 6, x_1(0) = 0,$$
$$\dot{x}_2 = 2x_1 + 4x_2 + 6, x_2(0) = 5.$$

(6) Solve the system

$$z_{k+1} = \begin{pmatrix} 4 & 1 \\ 0 & 1 \end{pmatrix} z_k + 4^k \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad z_0 = (0,1)^T.$$

(7) Solve the system

$$\dot{z} = \begin{pmatrix} 5 & -3 \\ 3 & -1 \end{pmatrix} z + \begin{pmatrix} t \\ 7 \end{pmatrix} e^{2t}, \quad z(0) = (1/2, 7)^T.$$