Tutorial Solutions Ch1

Multivariable Calculus (University of the Witwatersrand, Johannesburg)

Chapter 1, Part 2: Vector Analysis

1. (a) $\nabla f = \begin{pmatrix} \frac{2x_1}{x_1^2+x_2^2} \\ \frac{2x_2}{x_1^2+x_2^2} \end{pmatrix}$ (b) $\nabla f = \begin{pmatrix} -x_2 e^{x_1 x_2} + \cos x_1 - x_2 \\ -x_1 e^{x_1 x_2} - \cos x_1 - x_2 \\ 2x_3 \end{pmatrix}$

2. (a)

$$\mathbf{G} = \nabla \times \mathbf{F} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix} \times \begin{pmatrix} x_1 e^{x_2} \\ x_2 \sin x_3 \\ x_1 x_2 x_3 \end{pmatrix} = \begin{pmatrix} x_1 x_3 - x_2 \cos x_3 \\ 0 - x_2 x_3 \\ 0 - x_1 e^{x_2} \end{pmatrix}$$

and

$$\nabla \cdot \mathbf{F} = e^{x_2} + \sin x_3 + x_1 x_2.$$

(b) $\nabla \cdot \mathbf{G} = \nabla \cdot (\nabla \times \mathbf{F}) = 0.$

3.

$$\nabla \times \mathbf{F} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix} \times \begin{pmatrix} x_1 x_3 \\ e^{x_2} \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 1 - 0 \\ x_1 - 1 \\ 0 - 0 \end{pmatrix}$$

and $\nabla \cdot \mathbf{F} = x_3 + e^{x_2}$. Next,

$$\begin{aligned} \nabla \cdot (\nabla \times \mathbf{F}) &= 0, \quad \mathbf{F} \cdot (\nabla \times \mathbf{F}) = x_1 x_3 + (x_1 - 1) e^{x_2} \\ \nabla \times (\nabla \times \mathbf{F}) &= \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix} \times \begin{pmatrix} x_1 x_3 \\ e^{x_2} \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} 1 - 0 \\ x_1 - 1 \\ 0 - 0 \end{pmatrix}. \end{aligned}$$

4. (i)

$$\nabla \cdot \nabla u = \nabla \cdot \begin{pmatrix} 1 \\ -1 + e^{x_3} \cos x_2 \\ e^{x_3} \sin x_2 \\ -3 \end{pmatrix} = -e^{x_3} \sin x_2 + e^{x_3} \sin x_2 = 0$$

so u is harmonic.

(ii)

$$\nabla \cdot \nabla u = \nabla \cdot \begin{pmatrix} 6x_1x_3 + 2x_3 \\ -2 \\ 3x_1^2 + 2x_1 - 3x_3^2 \end{pmatrix} = 6x_3 - 6x_3 = 0,$$

so u is harmonic.

(iii)

$$\nabla \cdot \nabla u = \nabla \cdot \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} = 2 + 2 = 4 \neq 0$$

so u is not harmonic.

5. Straight forward but tedious.

6. (a)

$$\begin{aligned} & f_h(y+k) - f_h(y) \\ = \quad & (\psi(x+h, y+k) - \psi(x, y+k)) - (\psi(x+h, y) - \psi(x, y)) \\ & g_k(x+h) - g_k(x) \\ = \quad & (\psi(x+h, y+k) - \psi(x+h, y)) - (\psi(x, y+k) - \psi(x, y)). \end{aligned}$$

(b) The first Mean Value Theorem says

$$f_h(y+k) = f_h(y) + ((y+k) - y) f_h'(y+K_1)$$

where $y+K_1$ lies between y and $y+k$, i.e. K_1 lies between 0 and k. The 1st MVT also says

$$g_k(x+h) = g_k(x) + ((x+h) - x) g_k'(x+H_1)$$

where $x+H_1$ lies between x and $x+h$, i.e. H_1 lies between 0 and h.

(c) Fom what we found in (a) and (b) we have:

$$k f_h'(y+K_1) = f_h(y+k) - f_h(y) = g_k(x+h) - g_k(x) = h g_k'(x+H_1).$$

(d)

$$\begin{aligned} k\frac{\partial\psi}{\partial y}\Bigg|_{(x,\,y+K_1)}^{(x+h,\,y+K_1)} &= k\left(\frac{\partial\psi}{\partial y}(x+h, y+K_1) - \frac{\partial\psi}{\partial y}(x, y+K_1)\right) \\ &= k f_h'(y+K_1) = h g_k'(x+H_1) \\ &= h\left(\frac{\partial\psi}{\partial x}(x+H_1, y+k) - \frac{\partial\psi}{\partial x}(x+H_1, y)\right) \\ &= h\frac{\partial\psi}{\partial x}\Bigg|_{(x+H_1,\,y)}^{(x+H_1,\,y+k)} \end{aligned}$$

(e) As $p(h) - p(0) = hp'(H_2)$ where H_2 lies between 0 and h, we have

$$\begin{aligned} \frac{\partial\psi}{\partial y}(x+h, y+K_1) - \frac{\partial\psi}{\partial y}(x, y+K_1) &= hp'(H_2) \\ &= h\frac{\partial^2\psi}{\partial x\partial y}(x+H_2, y+K_1). \end{aligned}$$

Setting $q(t) = \frac{\partial \psi}{\partial x}(x + H_1, y + t)$, we get $q(k) - q(0) = kq'(K_2)$ where K_2 lies between 0 and k, and thus

$$\begin{aligned}
\frac{\partial \psi}{\partial y}(x + H_1, y + k) - \frac{\partial \psi}{\partial y}(x + H_1, y) &= kq'(K_2) \\
&= k\frac{\partial^2 \psi}{\partial y \partial x}(x + H_1, y + K_2).
\end{aligned}$$

(f) So, from (d) and (e) we get:

$$kh\frac{\partial^2 \psi}{\partial x \partial y}(x + H_2, y + K_1) = kh\frac{\partial^2 \psi}{\partial y \partial x}(x + H_1, y + K_2).$$

Dividing by hk and letting $h, k \to 0$ we get that

$$\frac{\partial^2 \psi}{\partial x \partial y}(x, y) \leftarrow \frac{\partial^2 \psi}{\partial x \partial y}(x+H_2, y+K_1) = \frac{\partial^2 \psi}{\partial y \partial x}(x+H_1, y+K_2) \to \frac{\partial^2 \psi}{\partial y \partial x}(x, y).$$

7.

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{F}) &= \nabla \cdot \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{pmatrix} \\
&= \frac{\partial}{\partial x_1}\left(\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2}\right) \\
&= 0
\end{aligned}$$

by Theorem 1.2.6.

8.

$$\nabla^2 \quad = \quad \nabla \cdot \nabla(\phi\psi) = \nabla \cdot (\phi\nabla\psi + \psi\nabla\phi)$$

$$
\begin{aligned}
&= \nabla\phi\cdot\nabla\psi + \phi\nabla^2\psi + \nabla\psi\cdot\nabla\phi + \psi\nabla^2\phi \\
&= 2\nabla\phi\cdot\nabla\psi
\end{aligned}
$$

since $\nabla^2\phi = \nabla^2\psi = 0$.

9.

$$\nabla^2 f^2 = \nabla\cdot\nabla f^2 = \nabla\cdot(2f\nabla f) = 2\nabla f\cdot\nabla f + 2f\nabla^2 f = 2\|\nabla f\|^2$$

since $\nabla^2 f = 0$.

10. By Theorem 1.2.5(f) we have $\nabla\times(f\nabla g) = \nabla f\times\nabla g + f\nabla\times\nabla g = \nabla f\times\nabla g$ since $\nabla\times\nabla g = 0$ by Theorem 1.2.7(1).

11. $\nabla\times(g\nabla f) = \nabla g\times\nabla f$ and $\nabla\times(f\nabla g) = \nabla f\times\nabla g = -\nabla g\times\nabla f$. Thus $\nabla\times(g\nabla f) = \nabla\times(f\nabla g)$ iff $\nabla f\times\nabla g = 0$, i.e. iff ∇f and ∇g are *parallel.*

12. $\nabla^2 f = 2b + 4 - c\cos x = 0\ \forall x, y \in \mathbb{R}$. Thus $c = 0$ and $b = -2$ and a can take any real value.

13. If f is harmonic then

$$\nabla\cdot(f\nabla f) = \nabla f\cdot\nabla f + f\nabla^2 f = \|\nabla f\|^2,$$

so that proves the one direction. On the other hand, if $\nabla\cdot(f\nabla f) = \|\nabla f\|^2 = \nabla f\cdot\nabla f$ then $f\nabla^2 f \equiv 0$, so $f = 0$ or $\nabla^2 f = 0$. If $f(\mathbf{x}) \neq 0$ then $\nabla^2 f = 0$ while if $f = 0$ on a region Ω then f is constant on Ω so $\nabla^2 f = 0$ on Ω. The remaining parts can be handled by using the fact that f and $\nabla^2 f$ are continuous.