MATH2001–Basic Analysis Final Examination June 2013
Solutions

Time: 60 minutes   Total marks: 60 marks

SECTION A Multiple choice

Solutions:

1	2	3	4	5	6
B	D	A	E	E	C

SECTION B

Question 1 . . . . [11 marks]

(a) Let $f : \mathbb{R} \to \mathbb{R}$. Write down the definition of $\lim_{x\to\infty} f(x) = -2$. (3)

Answer: $\forall\, \varepsilon > 0\ \exists\, A(> 0)\ \forall\, x > A\ |f(x) + 2| < \varepsilon$.

(b) Prove from the definition that $\lim_{x\to\infty} \dfrac{1-3x^2}{x^2+3} + 1 = -2$. (8)

Solution: First calculate

$$\begin{aligned}\left|\frac{1-3x^2}{x^2+3} + 1 - (-2)\right| &= \left|\frac{1-3x^2}{x^2+3} + 3\right| \\ &= \left|\frac{1-3x^2+3(x^2+3)}{x^2+3}\right| \\ &= \left|\frac{1+9}{x^2+3}\right| \\ &= \frac{10}{x^2+3}.\end{aligned}$$

Now let $\varepsilon > 0$. Then

$$\begin{aligned}\left|\frac{1-3x^2}{x^2+3} + 1 - (-2)\right| < \varepsilon &\Leftrightarrow \frac{10}{x^2+3} < \varepsilon \\ &\Leftrightarrow x^2+3 > \frac{10}{\varepsilon}.\end{aligned}$$

Since $x^2+3 > x$ for all x, it follows for $x > \dfrac{10}{\varepsilon}$, i. e., $A = \dfrac{10}{\varepsilon}$ and $x > A$, that

$$\left|\frac{1-3x^2}{x^2+3} + 1 - (-2)\right| < \varepsilon.$$

Question 2 .... [6 marks]

Let $a \in \mathbb{R}$ and suppose that f is continuous at a and g continuous at $f(a)$. Prove that the function $g \circ f$ is continuous at a.

Proof. Let $\varepsilon > 0$. Since g is continuous at $f(a)$, there is $\eta > 0$ such that

$$|y - f(a)| < \eta \Rightarrow |g(y) - g(f(a))| < \varepsilon. \tag{1}$$

Since f is continuous at a, there is $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \eta. \tag{2}$$

Putting $y = f(x)$ in (1) it follows from (1) and (2) that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \eta \Rightarrow |g(f(x)) - g(f(a))| < \varepsilon$$

that is,

$$|x - a| < \delta \Rightarrow |(g \circ f)(x) - (g \circ f)(a)| < \varepsilon.$$

Hence $g \circ f$ is continuous at a.

Question 3 .... [9 marks]

Let $a \in \mathbb{R}$ and let f be a real function which is defined in a neighbourhood of a.

Show that f is continuous at a if and only if for each sequence (x_n) in $\mathrm{dom}(f)$ with $\lim\limits_{n\to\infty} x_n = a$ the sequence $f(x_n)$ satisfies $\lim\limits_{n\to\infty} f(x_n) = f(a)$.

Proof. $\Rightarrow$: Let (x_n) be a sequence in $\mathrm{dom}(f)$ with $\lim\limits_{n\to\infty} x_n = a$. We must show that $\lim\limits_{n\to\infty} f(x_n) = f(a)$. Hence let $\varepsilon > 0$. Since f is continuous at a, there is $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon. \tag{1}$$

Since $\lim\limits_{n\to\infty} x_n = a$, there is $K \in \mathbb{R}$ such that for $n > K$, $|x_n - a| < \delta$. But then, by (1), $|f(x_n) - f(a)| < \varepsilon$ for $n > K$.

$\Leftarrow$: (indirect proof) Assume that f is not continuous at a. Then

$$\exists\, \varepsilon > 0\ \forall\, \delta > 0\ (x \in \mathrm{dom}(f),\ |x - a| < \delta \Rightarrow |f(x) - f(a)| \geq \varepsilon).$$

In particular, for $\delta = \frac{1}{n}$, $n = 1, 2, \ldots$ we find $x_n \in \mathrm{dom}(f)$ such that $|x_n - a| < \dfrac{1}{n}$ and $|f(x_n) - f(a)| \geq \varepsilon$. But then $\lim\limits_{n\to\infty} x_n = a$, whereas $(f(x_n))$ does not converge to $f(a)$.

Question 4 .... [7 marks]

(a) State the Intermediate Value Theorem. (2)

Intermediate Value Theorem. Suppose that f is continuous on the closed interval $[a, b]$ with $f(a) \neq f(b)$. Then for any number k between $f(a)$ and $f(b)$ there exists a number c in the open interval (a, b) such that $f(c) = k$.

(b) Let $a < b$ and let f be a continuous function on $[a, b]$ such that $f([a, b]) \subset [a, b]$. Show that there is $x \in [a, b]$ such that $f(x) = x$. (3)

Solution. Let $g(x) = f(x) - x$. Then

$$g(a) = f(a) - a \geq a - a = 0 \text{ and } g(b) = f(b) - b \leq b - b = 0.$$

If $f(a) = a$ the statement holds for $x = a$, and if $f(b) = b$, the statement holds with $x = b$.

Otherwise, $g(b) < 0 < g(a)$, and by the Intermediate Value Theorem there is $x \in (a, b)$ such that $g(x) = 0$. This means that $f(x) = x$.

(c) Give an example of a noncontinuous function $f : [a, b] \to [a, b]$ such that $f(x) \neq x$ for all $x \in [a, b]$. (2)

Solution. Let $f(x) = b$ for $x \in [a, b)$ and $f(b) = a$.

Question 5 .... [5 marks]

Let (a_n) be a sequence of nonzero real numbers such that $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.

Prove that $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Proof. Let $L = \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$.

Choose $\varepsilon > 0$ such that $L + \varepsilon < 1$.

Then there is $K \in \mathbb{N}$ such that $\left|\frac{a_{n+1}}{a_n}\right| < L + \varepsilon$ for all $n \geq K$. Hence for $m > K$:

$$|a_m| = |a_K| \left|\frac{a_{K+1}}{a_K}\right| \cdot \cdots \cdot \left|\frac{a_m}{a_{m-1}}\right| < |a_K|(L + \varepsilon)^{m-K}. \qquad (*)$$

Since $\sum_{m=K}^{\infty} |a_K|((L + \varepsilon)^{m-K}$ is a convergent geometric series, it follows from (*) and the Comparison Test that $\sum_{m=K}^{\infty} a_m$ converges absolutely. Hence also $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Question 6 .... [5 marks]

Let $a_n = (5 + 4(-1)^n)3^{-n}$.

(a) Find $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. (3)

Solution. If n is even, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{3^{-(n+1)}}{9 \cdot 3^{-n}}\right| = \frac{1}{27},$$

and if n is odd, we have

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{9 \cdot 3^{-(n+1)}}{3^{-n}}\right| = 3.$$

Hence

$$\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 3.$$

(b) Does $\sum_{n=1}^{\infty} a_n$ converge? (2)

Justify your answer.

Answer. The series converges (absolutely).

Indeed, since $|a_n| \leq 9 \cdot 3^{-n}$, the series is dominated by a convergent geometric series.