Basic Analysis 2015 — Solutions of Tutorials

Section 3.4

Tutorial 3.4.1 1. Consider the function

$$f(x) = \begin{cases} \dfrac{\lfloor x \rfloor}{x} & \text{if } x \neq 0, \\ -1 & \text{if } x = 0. \end{cases}$$

Investigate continuity from the left and the right at $x = 0$, $x = \pi$ and $x = 1$.

Solution. We note that

$$\lfloor x \rfloor = \begin{cases} -1 & \text{if } x \in (-1, 0), \\ 0 & \text{if } x \in (0, 1), \\ 1 & \text{if } x \in (1, 2), \\ 3 & \text{if } x \in (3, 4). \end{cases}$$

Hence

$$\begin{aligned}
\lim_{x} \to 0^- f(x) &= \lim_{x \to 0^-} \frac{-1}{x} \text{ does not exist,} \\
\lim_{x} \to 0^+ f(x) &= \lim_{x \to 0^+} \frac{0}{x} = 0 \neq -1 = f(0), \\
\lim_{x} \to 1^- f(x) &= \lim_{x \to 1^-} \frac{0}{x} = 0 \neq 1 = f(0), \\
\lim_{x} \to 1^+ f(x) &= \lim_{x \to 1^+} \frac{1}{x} = 1 = f(0), \\
\lim_{x} \to \pi^- f(x) &= \lim_{x \to \pi^-} \frac{3}{x} = \frac{3}{\pi} = f(\pi), \\
\lim_{x} \to \pi^+ f(x) &= \lim_{x \to \pi^+} \frac{3}{x} = \frac{3}{\pi} = f(\pi).
\end{aligned}$$

Hence f is not continuous from the left at 0, f is not continuous from the right at 0, f is not continuous from the left at 1, f is continuous from the right at 1, f is continuous from the left at π, and f is continuous from the right at π.

2. Let $f(x) = x \sin \frac{1}{x}$ for $x \neq 0$ and $g(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$ Show that $f(x) \to 0$ as $x \to 0$ and that $g(x) \to 0$ as $x \to 0$, but that $g(f(x))$ does not have a limit as $x \to 0$. Explain this behaviour.

Solution. From

$$0 \leq \left| x \sin \frac{1}{x} \right| \leq |x|, \quad x \neq 0,$$

and the Sandwhich Theorem it follows that $f(x) \to 0$ as $x \to 0$, whereas $g(x) \to 0$ as $x \to 0$ is trivial.

Let $\varepsilon = \frac{1}{2}$ and let $\delta > 0$. Then there is $n \in \mathbb{N}$ such that $n > \dfrac{1}{2\pi\delta}$.

Putting $x = \dfrac{1}{2\pi n}$ we have $x \in (0, \delta)$, and

putting $y = \dfrac{1}{\pi\left(2n + \frac{1}{2}\right)}$ we have $y \in (0, \delta)$.

Then

$$g(f(x)) = g(x \sin(2\pi n)) = g(0) = 1,$$

whereas

$$g(f(y)) = g\left(y \sin\left(\pi\left(2n + \frac{1}{2}\right)\right)\right) = g(y) = 0,$$

and for each $L \in \mathbb{R}$ we have

$$\begin{aligned} 2\varepsilon = 1 = |g(f(y)) - g(f(x))| &= |(g(f(y)) - L) - (g(f(x)) - L)| \\ &\le |g(f(y)) - L| + |g(f(x)) - L|, \end{aligned}$$

so that $|g(f(y)) - L| \ge \varepsilon$ or $|g(f(x)) - L| \ge \varepsilon$. By Tutorial 3.1.1, Question 2, it follows that $g\circ f$ does not tend to any limit as $x \to 0$.
The behaviour of these limits can be explained by the fact that $g(0)$ is never used when the limit of $g(x)$ as $x \to 0$ is investigated, but $g(0)$ matters for $g(f(x))$ as $x \to 0$.

3. Find the values of a and b which make the function

$$f(x) = \begin{cases} x - 1 & \text{if } x \le -2, \\ ax^2 + c & \text{if } -2 < x < 1, \\ x + 1 & \text{if } x \ge 1, \end{cases}$$

continuous at $x = -2$ and $x = 1$.

Solution. From Theorem 3.10 we can conclude that f is continuous from the left at $x = -2$ and continuous from the right at $x = 1$, and that

$$\lim_{x\to -2^+} f(x) = a(-2)^2 + c = 4a + c, \qquad \lim_{x\to 1^-} f(x) = a(1)^2 + c = a + c.$$

Henc f is continuous at $x = -2$ and at $x = 1$ if and only if

$$4a + c = f(-2) = -3, \quad \text{and} \quad a + c = f(1) = 2.$$

This holds if and only if $a = -\dfrac{5}{3}$ and $c = \dfrac{11}{3}$.

4. Prove that if $\lim\limits_{x\to 0^-} f(x)$ exists, then $\lim\limits_{x\to 0^+} f(-x) = \lim\limits_{x\to 0^-} f(x)$.

Proof. Let $L = \lim\limits_{x\to 0^-} f(x)$ and let $\varepsilon > 0$. Choose $\delta > 0$ such $|f(y) - L| < \varepsilon$ for $y \in (-\delta, 0)$. For $x \in (0, \delta)$ we have $-x \in (-\delta, 0)$, and therefore $|f(-x) - L| < \varepsilon$. This shows that $\lim\limits_{x\to 0^+} f(-x) = L$. □

5. Prove that exp is continuous. You may use the following steps.
(a) The inequality $\exp(x) \ge 1 + x$ is true for all $x \in \mathbb{R}$.

Proof. This inequality has been shown for $x \ge 0$ in Tutorial 2.2.1, Question 8(d). It is easy to see that the same proof holds if we consider $n \ge -x$. □

(b) $\lim\limits_{x\to 0^-} \exp(x) = 1$.

Proof. By Tutorial 2.2.1, Question 8(f), exp is strictly increasing, so that $\exp x < \exp(0) = 1$ for $x < 0$. Together with the inequality (a), $\lim_{xto0^-}(1 + x) = 1$ and the Sandwich Theorem we get $\lim_{xto0^-} \exp(x) = 1$. □

(c) $\lim\limits_{x\to 0^+} \exp(x) = 1$. **Hint.** Use tutorial problem 4.

Proof. By tutorial problem 4,

$$\lim_{x\to 0^+} \exp(-x) = \lim_{x\to 0^-} \exp(x) = 1.$$

Therefore, using Tutorial 2.2.1, Question 8(c) to arrive at

$$\exp(-x) = \frac{\exp(x - x)}{\exp(-x)} = \frac{1}{\exp(-x)}$$

it follows that

$$\lim_{x\to 0^+} \exp(x) = \lim_{x\to 0^+} \frac{1}{\exp(-x)} = \frac{1}{\lim_{x\to 0^+} \exp(-x)} = \frac{1}{1} = 1.$$

□

From (b), (c) and $\exp(0) = 1$ it now follows that exp is continuous at 0.
If now $a \in \mathbb{R}$ is arbitrary, then, in view of Tutorial 2.2.1, Question 8(c),

$$\lim_{x\to a} \exp(x) = \lim_{h\to 0} \exp(a + h) = \lim_{h\to 0} \exp(a) \exp(h) = \exp(a) \lim_{h\to 0} \exp(h) = \exp(a) \cdot 1 = \exp(a),$$

which proves the continuity of exp at a.