Linear Algebra Exam

Question 1 The linear operator $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$ is given by the matrix

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 1 & -2 \\ 3 & -2 & 0 \end{pmatrix}$$

in the standard basis. Find the matrix B of $\mathcal{A}$ in the basis $\{(1,0,1),(1,1,0),(0,1,1)\}$.

[10]

Question 2 Prove that the characteristic polynomial of a linear operator does not depend on the choice of a basis.

[10]

Question 3 Determine whether the matrix

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 2 & 1 & -1 \\ 2 & 0 & 0 \end{pmatrix}$$

is diagonalizable, and if yes, find a diagonal matrix D and a matrix T such that $D = T^{-1}AT$.

[10]

Question 4 From the Cauchy-Bunyakowski inequality deduce that for any vectors x, y of an inner product space, $||x+y|| \le ||x|| + ||y||$.

[10]

Question 5 Using the Gram-Schmidt process, transform the basis $\{(0,1,1),(1,0,1),(1,1,0)\}$ of $\mathbb{R}^3$ into an orthonormal basis.

[10]

Question 6 Find a system of linear equations whose solution space is the subspace $\langle a_1, a_2, a_3 a4 \subseteq \mathbb{R}^5$, where

$a_1 = (2,1,-1,0,1), a_2 = (-1,1,-2,-1,0), a_3 = (2,0,1,0,-1)$
a4 = (3, 2, -2, -1, 0)

[10]