Chapter 7: Groups of Symmetry

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LEARNING OUTCOMES FOR THE LECTURE

By the end of this lecture, students will be able to:

- ♣ state when two permutations are conjugate to each other
- ♣ conjugate a permutation
- ♣ prove that conjugate permutations form an equivalence class in S_n
- ♣ determine if a permutation is even or odd
- ♣ define the alternating group
- △ determine the parity of a permutation

Conjugation of Permutations

Definition (7.7.1)

Two permutations σ and τ in S_n are conjugate in S_n if we can find $\gamma \in S_n$ such that $\gamma\sigma\gamma^{-1} = \tau$ and $\gamma^{-1}\tau\gamma = \sigma$.

Proposition (7.7.2)

Let σ and τ be permutations in S_n where σ has a decomposition as follows:
$\sigma = (a_1 \quad a_2 \quad \cdots \quad a_{k_1})(b_1 \quad b_2 \quad \cdots \quad b_{k_2})\cdots$ *then* $\tau\sigma\tau^{-1}$ *has cyclic decomposition*

$$\tau\sigma\tau^{-1} = (\tau(a_1) \quad \tau(a_2) \quad \cdots \quad \tau(a_{k_1}))(\tau(b_1) \quad \tau(b_2) \quad \cdots \quad \tau(b_{k_2}))\cdots$$

PROOF: Apply τ directly to the entries in σ.

e.g. $\sigma = (1 \quad 3 \quad 5 \quad 7)(2 \quad 4)(6 \quad 8)$ and
$\tau = (1 \quad 4 \quad 3 \quad 2)(6 \quad 7 \quad 8)$

$$
\begin{aligned}
\tau\sigma\tau^{-1} &= (\tau(1) \quad \tau(3) \quad \tau(5) \quad \tau(7))\,(\tau(2) \quad \tau(4))\,(\tau(6) \quad \tau(8)) \\
&= (4 \quad 2 \quad 5 \quad 8)(1 \quad 3)(7 \quad 6).
\end{aligned}
$$

= (1 4 3 2)(6 7 8)(1 3 5 7)(2 4)(6 8)(8 7 6)(2 3 4 1)

and

$$
\begin{aligned}
\sigma\tau\sigma^{-1} &= (\sigma(1) \quad \sigma(4) \quad \sigma(3) \quad \sigma(2))\,(\sigma(6) \quad \sigma(7) \quad \sigma(8)) \\
&= (3 \quad 2 \quad 5 \quad 4)(8 \quad 1 \quad 6).
\end{aligned}
$$

Proposition (7.7.3)

σ and τ are conjugate iff they have the same cyclic structure.

PROOF: We find γ such that $\gamma\sigma\gamma^{-1} = \tau$.
Write down σ and write downτ below σ
allowing cycles to correspond (not necessarily unique way).
Eg. Find γ by linking entries in these two rows

$$\sigma = (1 \quad 3 \quad 4)(5 \quad 2 \quad 7 \quad 8)(6 \quad 9)$$

$$\tau = (2 \quad 4 \quad 7)(8 \quad 9 \quad 1 \quad 3)(6 \quad 5)$$

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 9 & 4 & 7 & 8 & 6 & 1 & 3 & 5 \end{pmatrix} =$$

$$(1 \quad 2 \quad 9 \quad 5 \quad 8 \quad 3 \quad 4 \quad 7).$$

$\gamma\sigma\gamma^{-1} = \tau$.
Similarly, $\delta\tau\delta^{-1} = \sigma$ where $\delta = \gamma^{-1}$.

Proposition (7.7.4)

On S_n define $\equiv$ as follows: $\sigma \equiv \tau$ iff σ and τ are conjugate in S_n. Then $\equiv$ is an equivalence relation on S_n.

PROOF σ and τ are conjugate in S_n iff $\exists \quad \gamma$ such that $\gamma\sigma\gamma^{-1} = \tau$.

1. $\sigma = e\sigma e$, identity permutation.
2. $\sigma \equiv \tau \quad \Rightarrow \quad \exists\gamma$ such that $\gamma\sigma\gamma^{-1} = \tau$
   $\Rightarrow \quad \sigma = \gamma^{-1}\tau\gamma \quad \Rightarrow \quad \sigma \equiv \tau$.
3. $\sigma \equiv \tau$ and $\tau \equiv \delta$
   $\Rightarrow \quad \gamma_1\sigma\gamma_1^{-1} = \tau$ and $\gamma_2\tau\gamma_2^{-1} = \delta$
   $\Rightarrow \quad (\gamma_2\gamma_1)\sigma(\gamma_2\gamma_1)^{-1} = \delta$, for $\gamma_1, \gamma_2 \in S_n$.

$\therefore \quad \equiv$ is an equivalence relation on S_n.

Alternating Group A_n

Definition (7.8.1)

*A permutation $\sigma \in S_n$ is called **even** or **odd** if can be written in some way as a product of an even or odd number of transpositions, respectively. The set of all even permutations is the set called A_n. We note that the identity element is even and in A_n. We refer to the eveness or oddness of $\sigma \in S_n$ as its parity, A_n is called the alternating group.*

Example (7.8.2)

Determine the parity of $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 6 & 1 & 7 & 8 & 2 & 9 & 3 \end{pmatrix}$.

$$\begin{aligned} \sigma &= (1\ \ 5\ \ 7\ \ 2\ \ 4)(3\ \ 6\ \ 8\ \ 9) \\ &= (1\ \ 4)(1\ \ 2)(1\ \ 7)(1\ \ 5)(3\ \ 9)(3\ \ 8)(3\ \ 6). \end{aligned}$$

There are 7 transpositions in this factorisation and the permutation is odd.

Theorem (7.8.3)

If $n \geq 2$, the set A_n has the following properties.

1. *e is in A_n and if σ and τ are in A_n then so are σ^{-1}, τ^{-1} and $\sigma\tau$.*
2. $|A_n| = \frac{1}{2}n!$

PROOF:

1. If σ^{-1} is not even then it is odd and so $\sigma^{-1}\sigma$ will have an odd number of transpositions. Thus e is odd, contradiction. Therefore σ^{-1}, τ^{-1} are even if σ, τ are even.
$\sigma\tau$ is a product of even number of ~~permutations~~ transpositions and this is even.

2. Let B_n be the set of all odd permutations. Let $\tau \notin A_n$ where $\tau = (a\ b)$ is a transposition, $\tau^{-1} = \tau \in S_n$. Define $\theta : A_n \to B_n$ by $\theta(\sigma) = \tau\sigma \quad \forall \sigma \in A_n$.

Show θ is a bijection and then $|A_n| = |B_n| = \frac{1}{2}n!$

$\sigma_1 = \sigma_2 \iff \tau\sigma_1 = \tau\sigma_2 \iff \theta(\sigma_1) = \theta(\sigma_2)$ $(\theta W.D.1-1)$

$\gamma \in B_n \Rightarrow \tau\gamma = \tau^{-1}\gamma \in A_n$

$\theta(\tau^{-1}\gamma) = \tau\tau^{-1}\gamma = \gamma$

Thus θ is onto.