Chapter 6

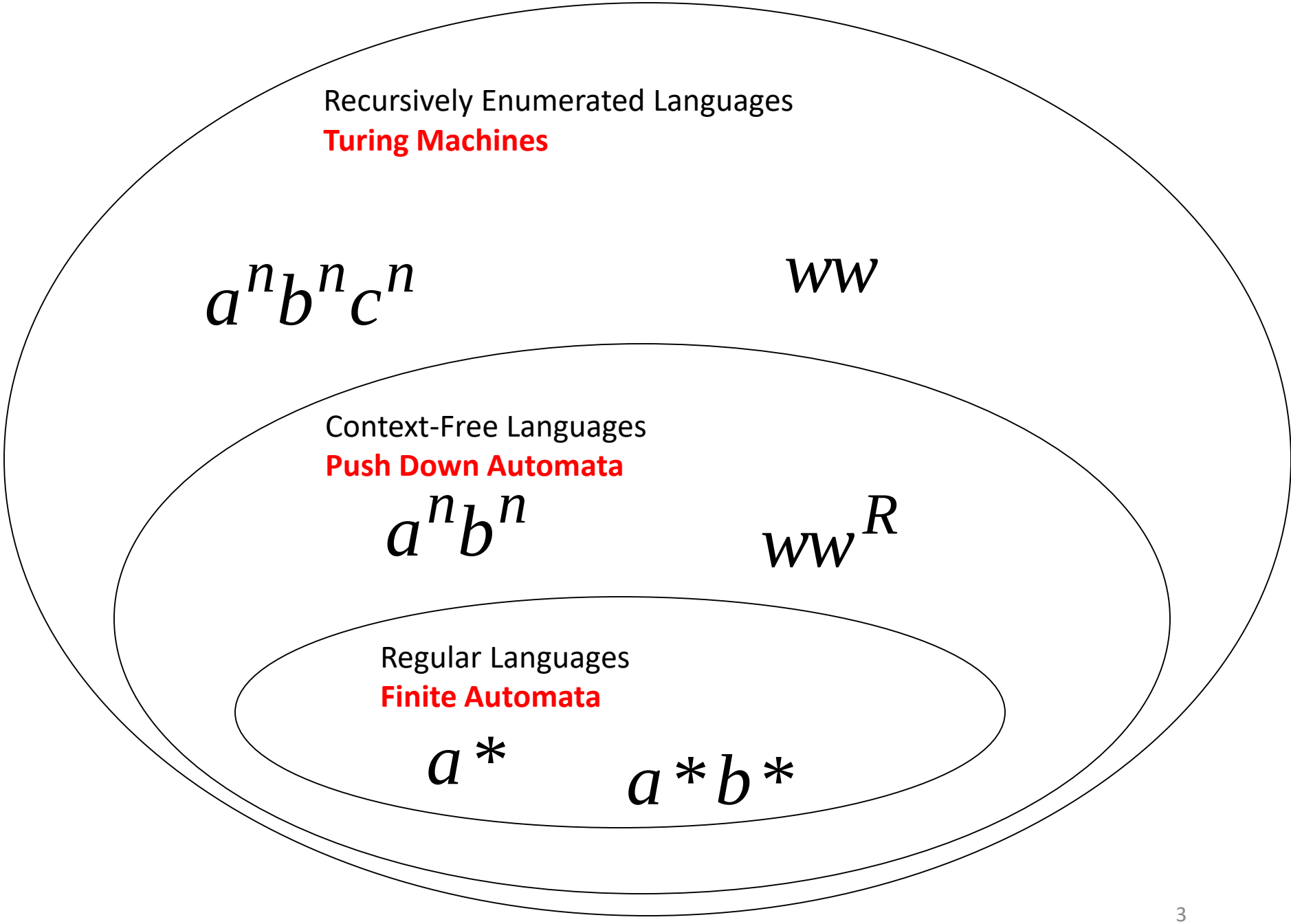
Turing Machine

Alan Turing -- 1912 – 1954

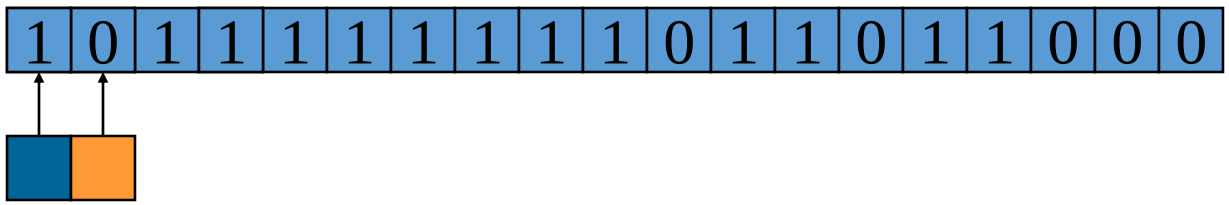
- Founder of computer science, mathematician, philosopher, codebreaker, strange visionary and a gay man before his time.

- Influential
  - development of computer sciences
  - provided an influential formalisation
    - of the concept of the algorithm
    - and computation with the
    - Turing Machine
    - Turing Test contribute to the debate of AI
      - Can machines think?

  - http://www.turing.org.uk/turing/


Recursively Enumerated Languages
Turing Machines
$a^n b^n c^n$
ww
Context-Free Languages
Push Down Automata
$a^n b^n$
ww^R
Regular Languages
Finite Automata
$a*$
$a*b*$

A Turing Machine

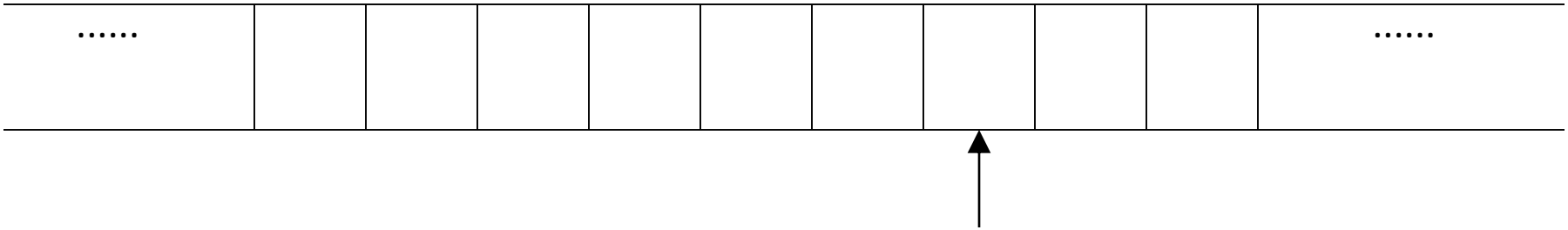


- A **TM** consists of an **infinite length tape**, on which input is provided as a finite sequence of symbols.

- A ***head*** reads the input tape.
- The TM starts at start state s_0.
- On reading an input symbol it optionally replaces it with another symbol, changes its internal state and ***moves one cell to the right or left***.

A Turing Machine…

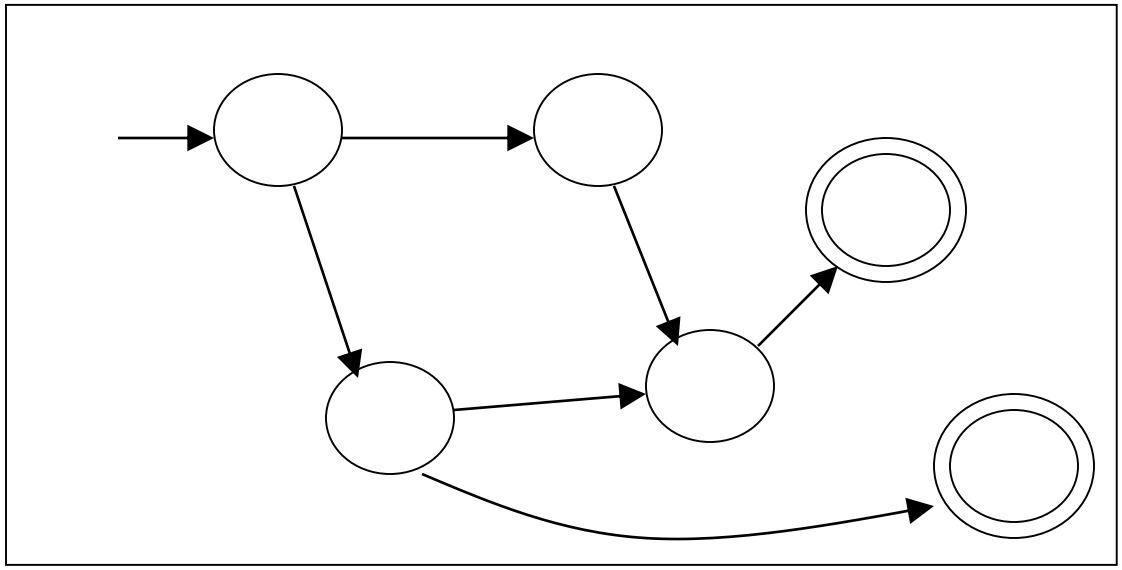
Tape - No boundaries -- infinite length



Read-Write head - The head moves **Left** or **Right**

- The head at each transition (time step):
  1. Reads a symbol
  2. Writes a symbol
  3. Moves Left or Right

Control Unit



Example:

Time 0

.....			a	b	a	c				.....

Time 1

.....			a	b	k	c				.....

1. Reads a

2. Writes k

3. Moves **Left**

Time 1

.....			a	b	k	c				.....

Time 2

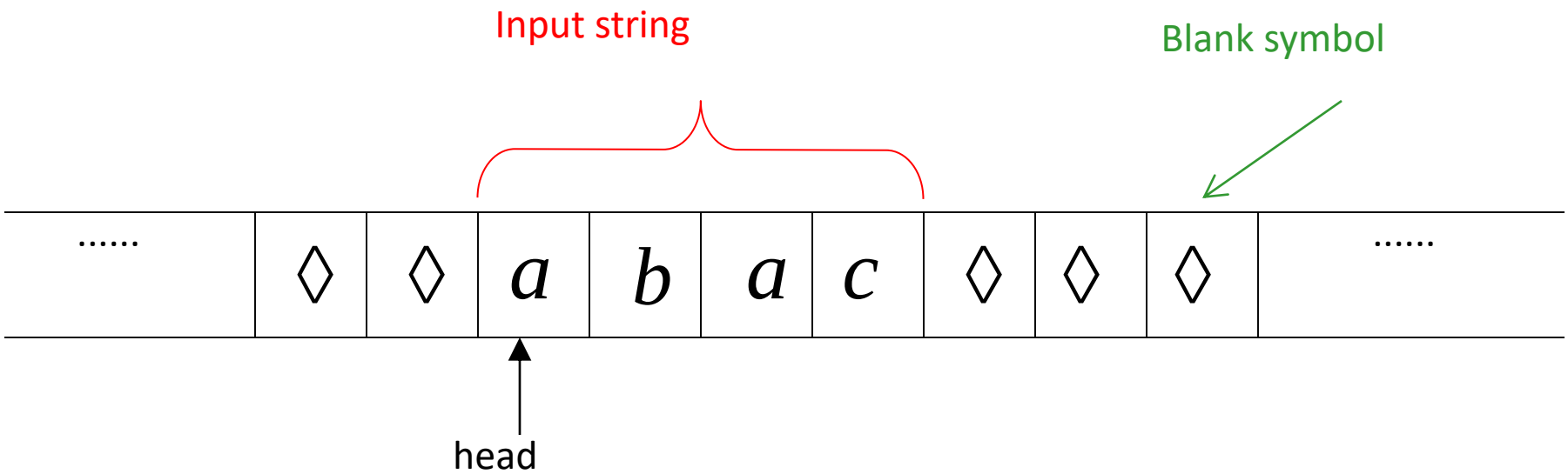
.....			a	f	k	c				.....

1. Reads b

2. Writes f

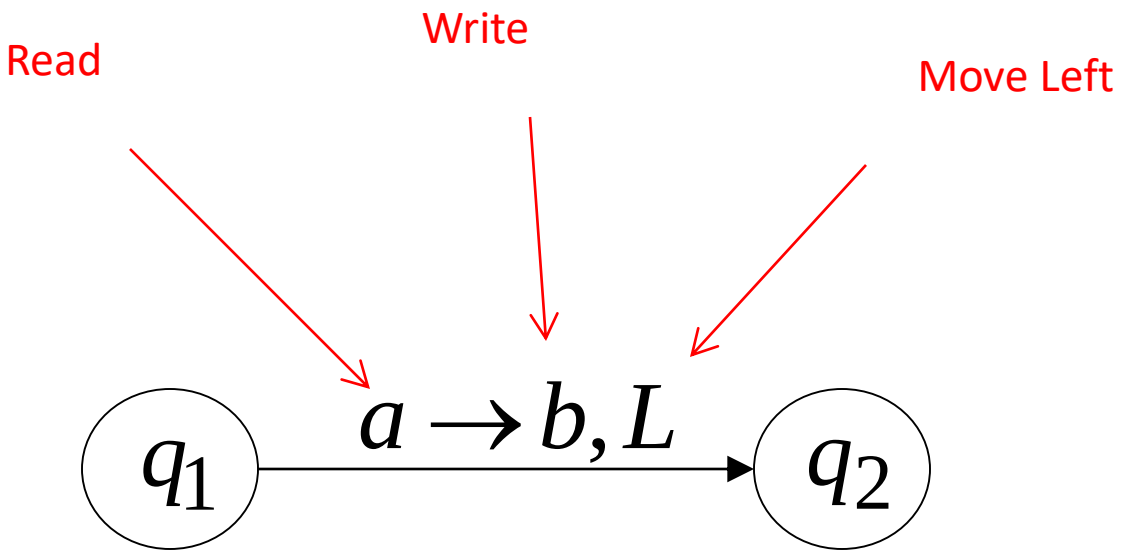
3. Moves Right

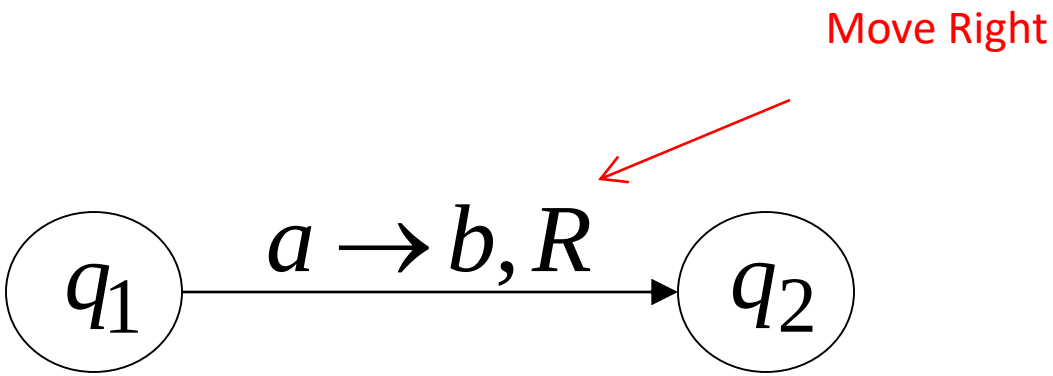
The Input String



Head starts at the leftmost position of the input string

States & Transitions





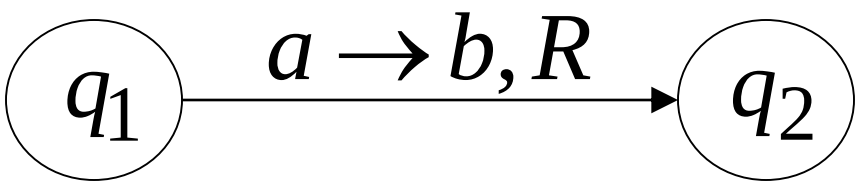
Example:

Time 1

.....	$\Diamond$	$\Diamond$	a	b	a	c	$\Diamond$	$\Diamond$	$\Diamond$	.....

q_1

current state



Time 1

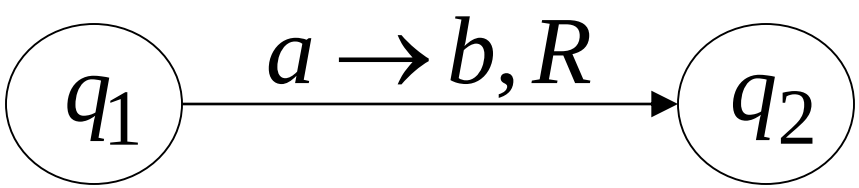
.....	$\Diamond$	$\Diamond$	a	b	a	c	$\Diamond$	$\Diamond$	$\Diamond$	.....

q_1

Time 2

.....	$\Diamond$	$\Diamond$	a	b	b	c	$\Diamond$	$\Diamond$	$\Diamond$	.....

q_2



Example:

Time 1

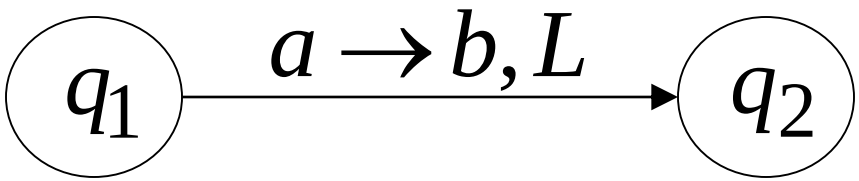
.....	$\Diamond$	$\Diamond$	a	b	a	c	$\Diamond$	$\Diamond$	$\Diamond$	.....

q_1

Time 2

.....	$\Diamond$	$\Diamond$	a	b	b	c	$\Diamond$	$\Diamond$	$\Diamond$	.....

q_2



Example:

Time 1

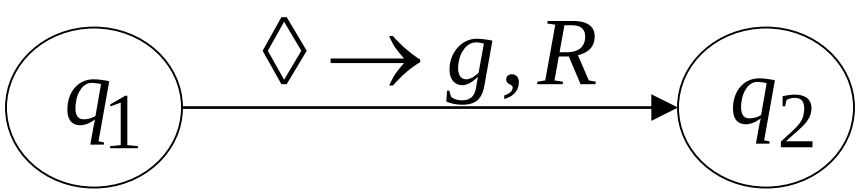
.....	$\Diamond$	$\Diamond$	a	b	a	c	$\Diamond$	$\Diamond$	$\Diamond$	.....

q_1

Time 2

.....	$\Diamond$	$\Diamond$	a	b	b	c	g	$\Diamond$	$\Diamond$	.....

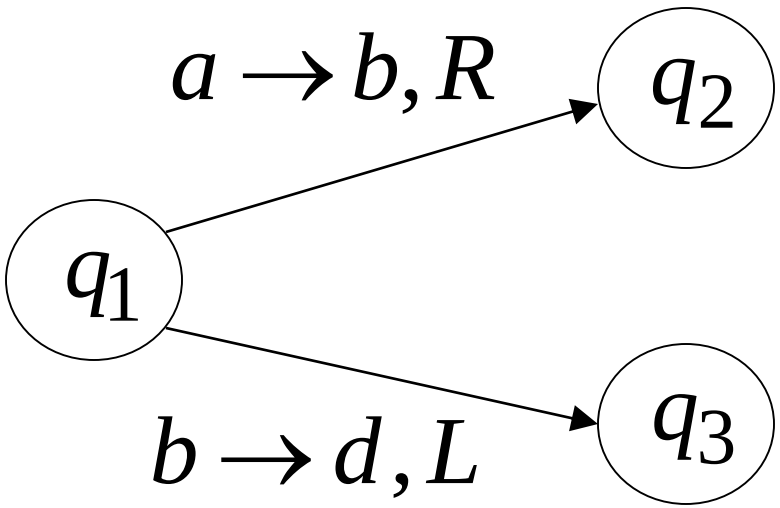
q_2



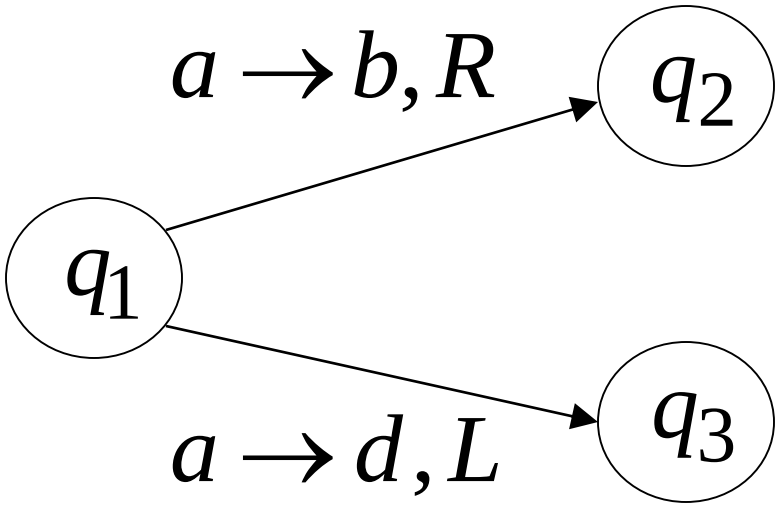
A Turing Machine…

- Turing Machines are deterministic

Allowed



Not Allowed



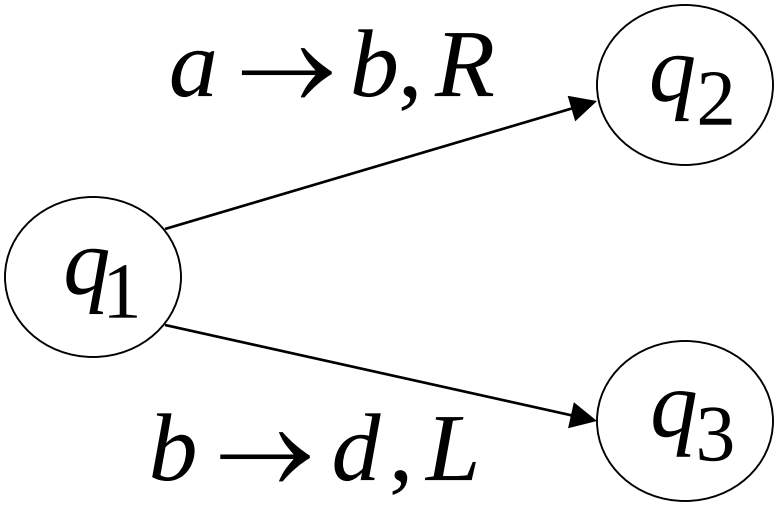
No lambda transitions allowed

Partial Transition Function

Example:

.....	$\Diamond$	$\Diamond$	a	b	a	c	$\Diamond$	$\Diamond$	$\Diamond$	.....

q_1



Allowed:

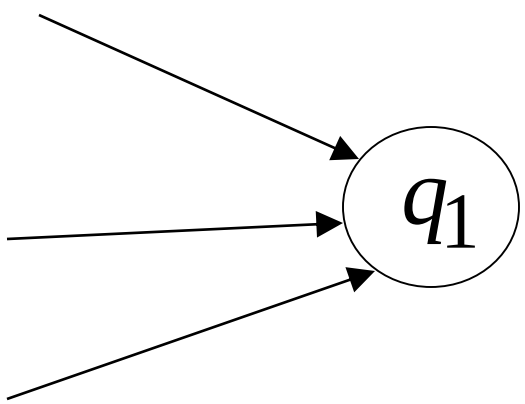
No transition
for input symbol c

Halting

- The machine ***halts*** in a state if there is no transition to follow.

.....	$\Diamond$	$\Diamond$	a	b	a	c	$\Diamond$	$\Diamond$	$\Diamond$	.....

q_1



No transition from q_1

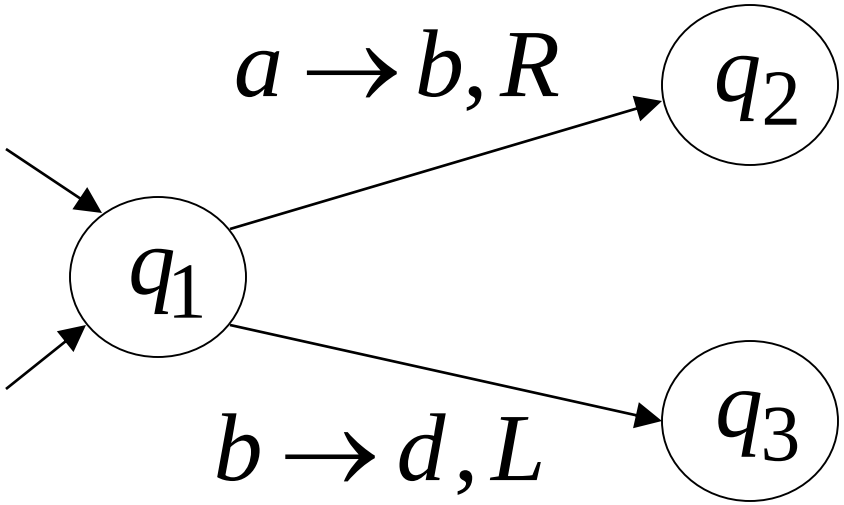
HALT!!!

Halting…

Example

……	$\Diamond$	$\Diamond$	a	b	a	c	$\Diamond$	$\Diamond$	$\Diamond$	……

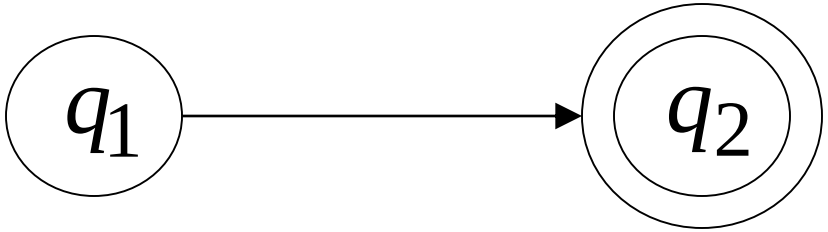
q_1



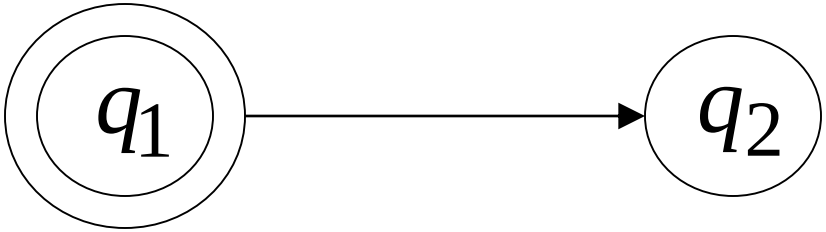
No possible transition from q_1 and symbol c

HALT!!!

Accepting States



Allowed



Not Allowed

- Accepting states ***have no outgoing transitions.***
- The machine ***halts*** and ***accepts***.

Acceptance…

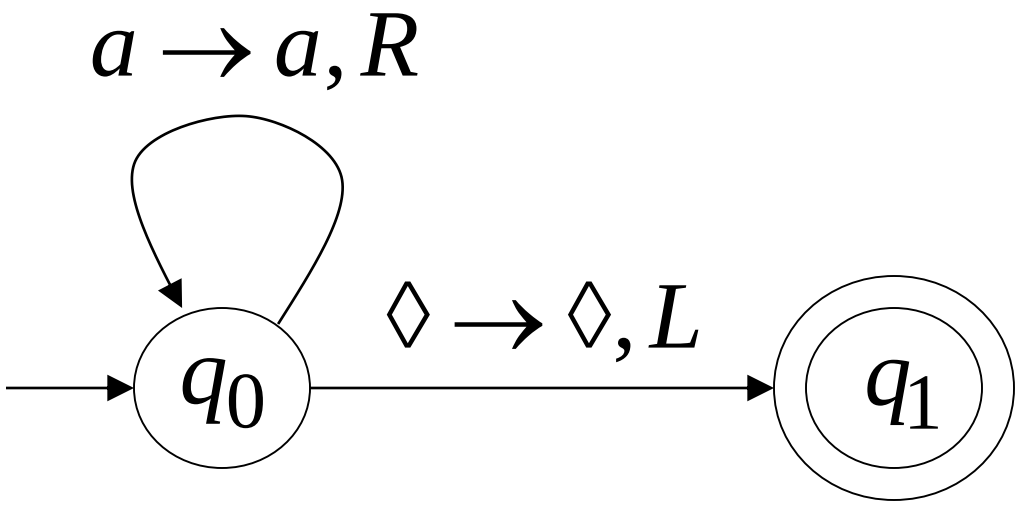
Accept Input string → If machine halts in an **accept state**

Reject Input string → If machine halts in a **non-accept state**
or
If machine enters an ***infinite loop***

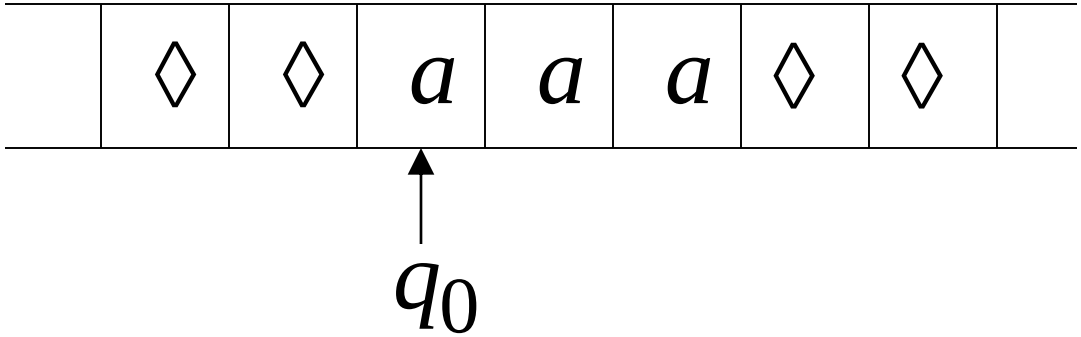
A Turing Machine… **Example**

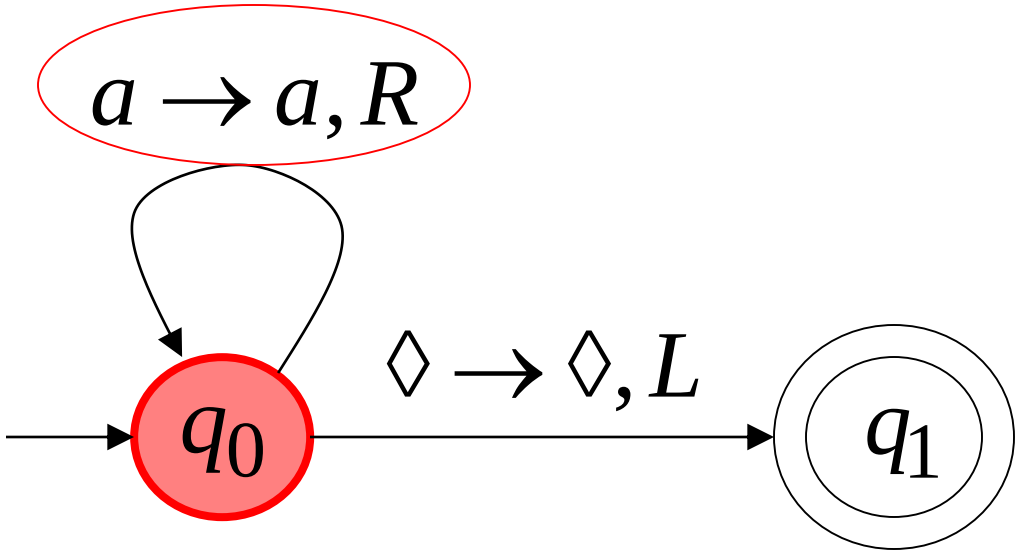
Input alphabet $\Sigma = \{a, b\}$

Accepts the language: a^*

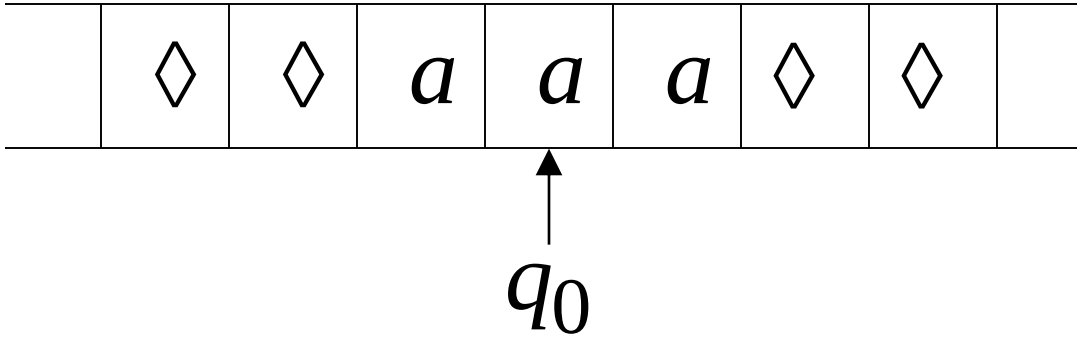


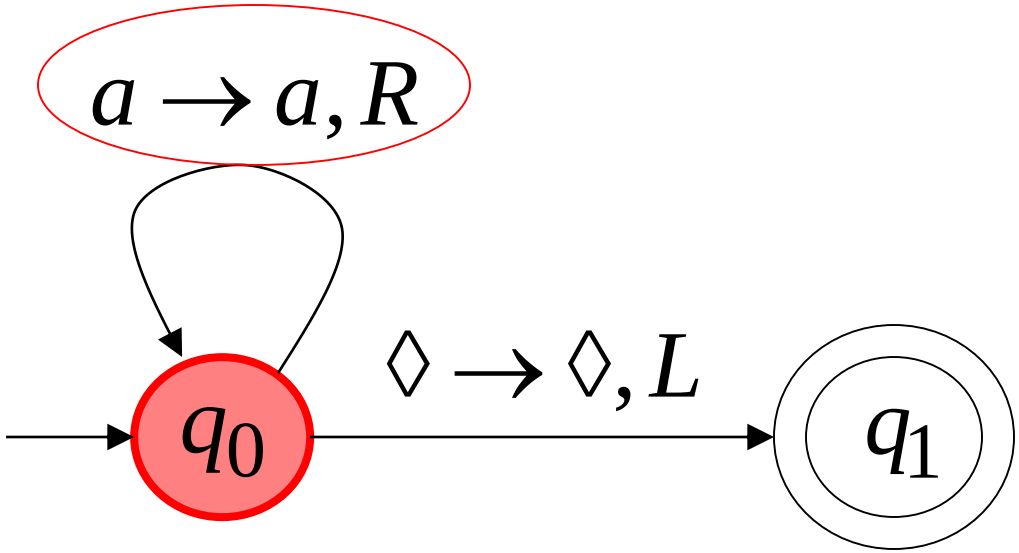
Time 0



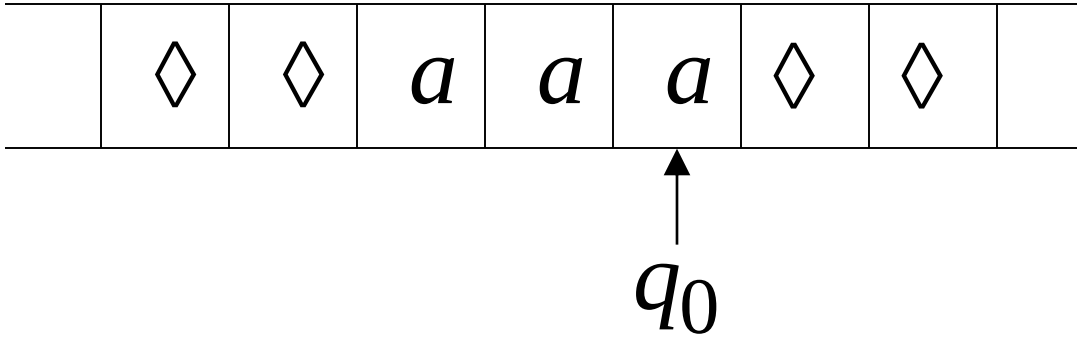


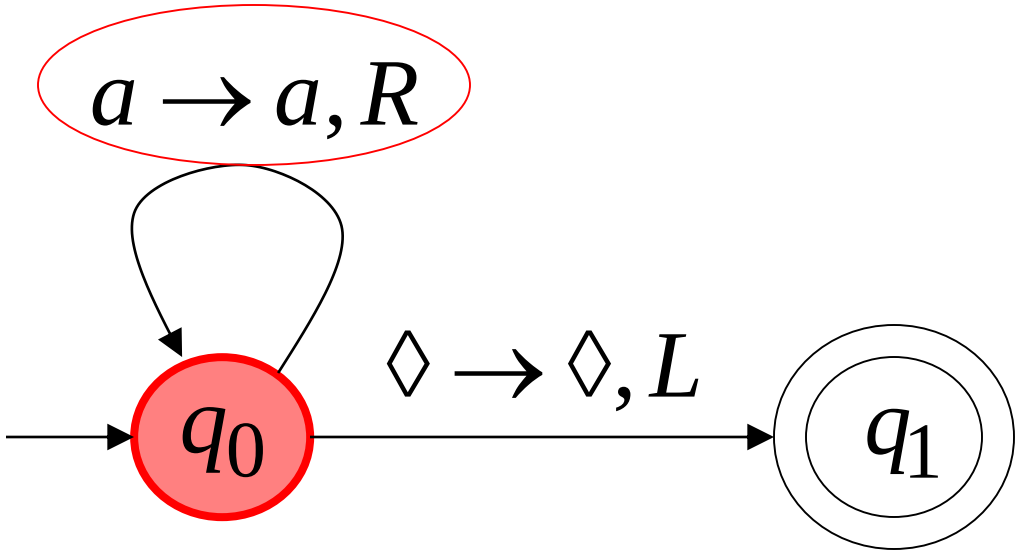
Time 1



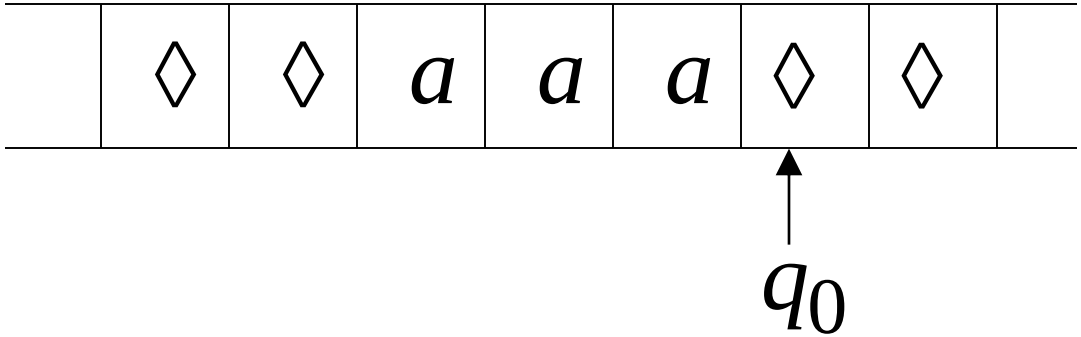


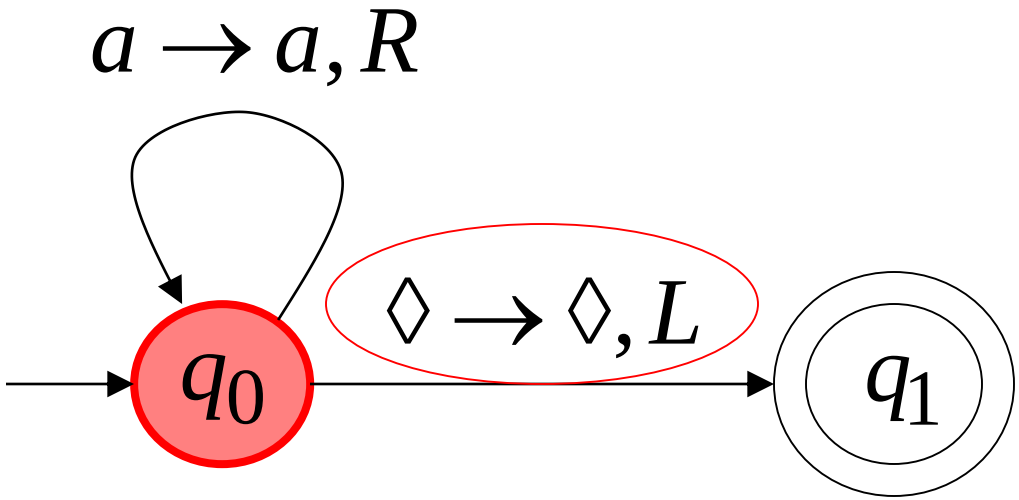
Time 2



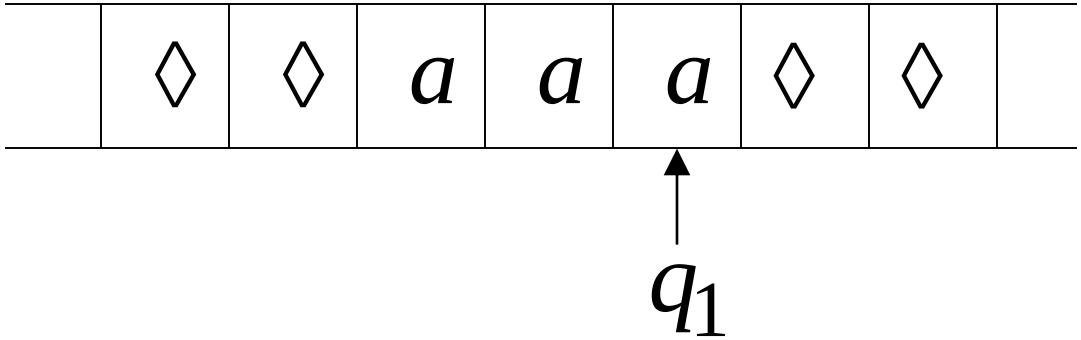


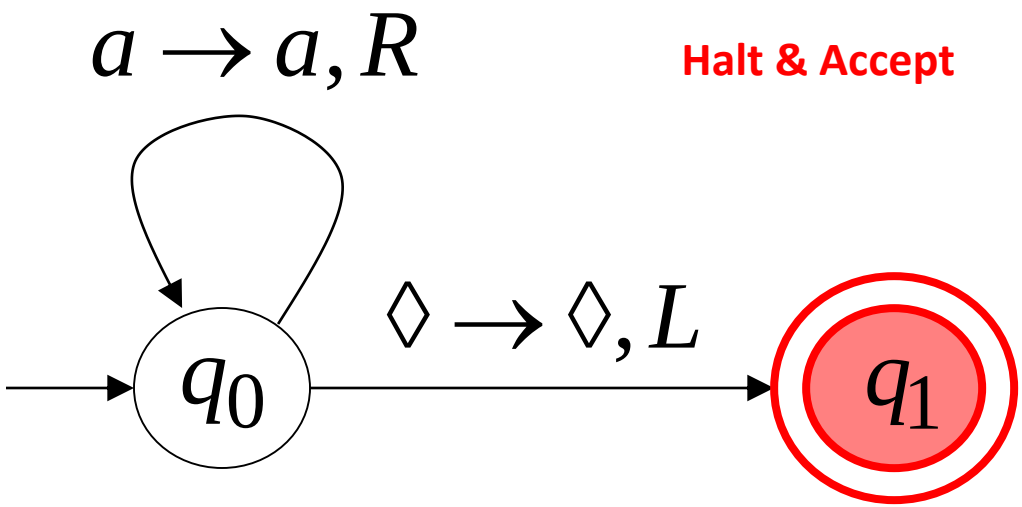
Time 3





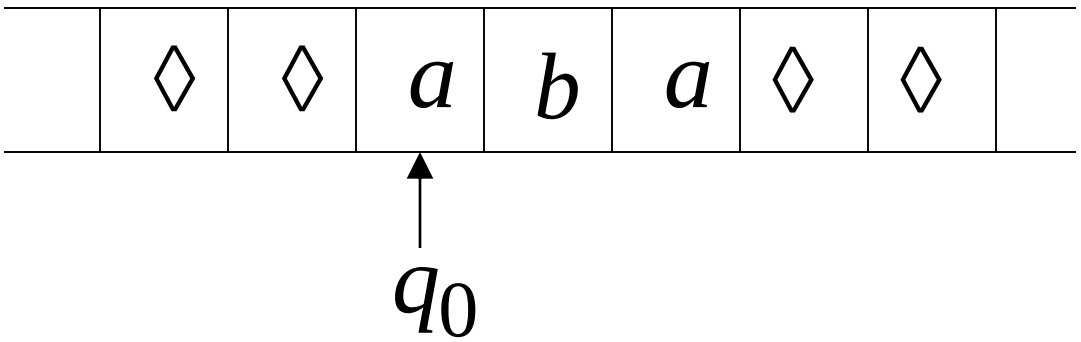
Time 4

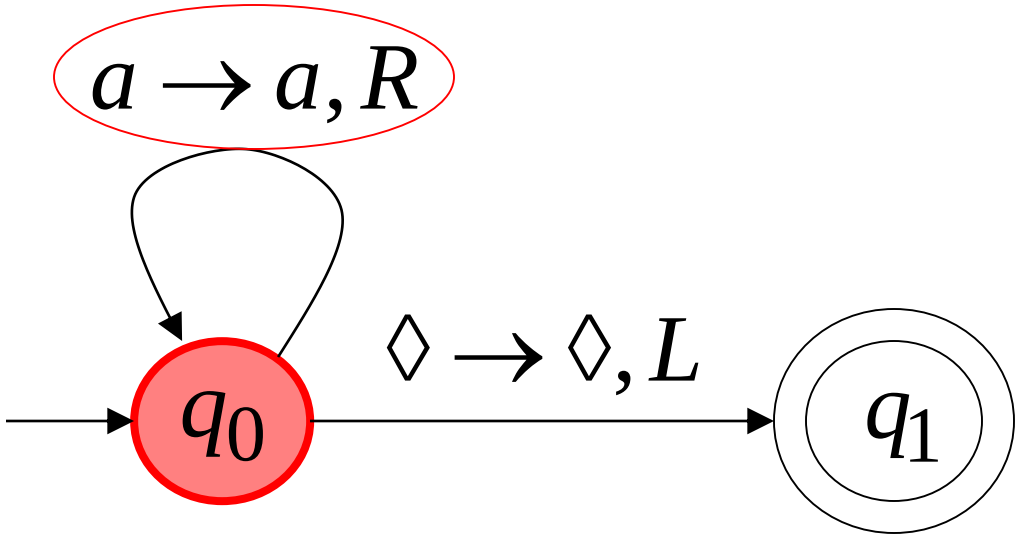

◊ ◊ a a a ◊ ◊
q_1


$a \rightarrow a, R$
Halt & Accept
$\Diamond \rightarrow \Diamond, L$
q_0
q_1

Rejection Example

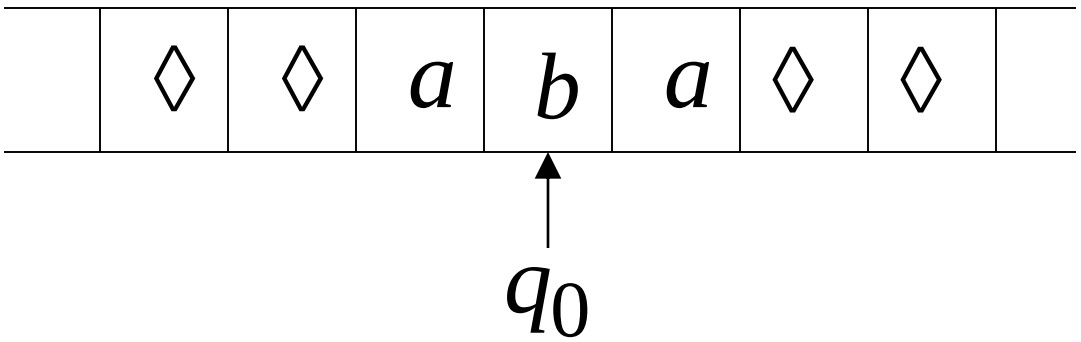
Time 0





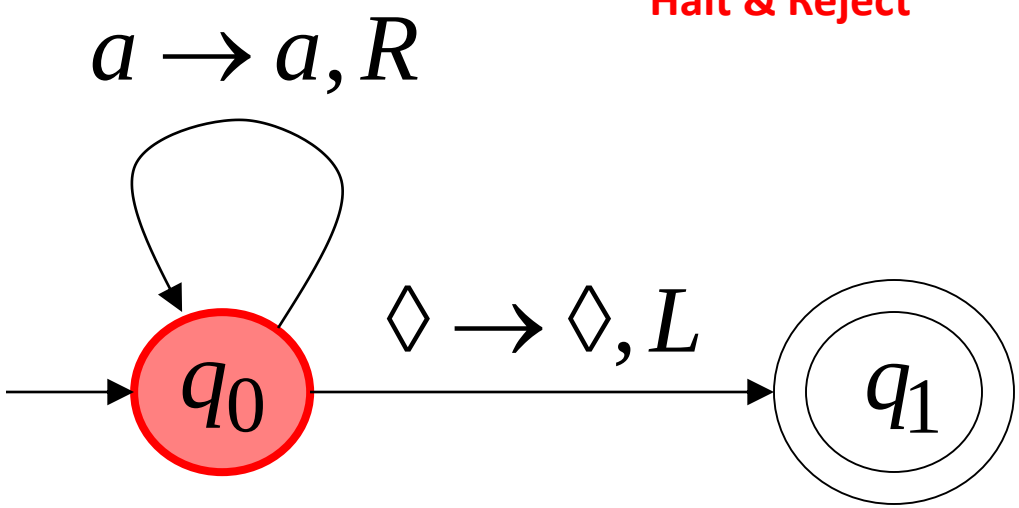
Rejection Example…

Time 1



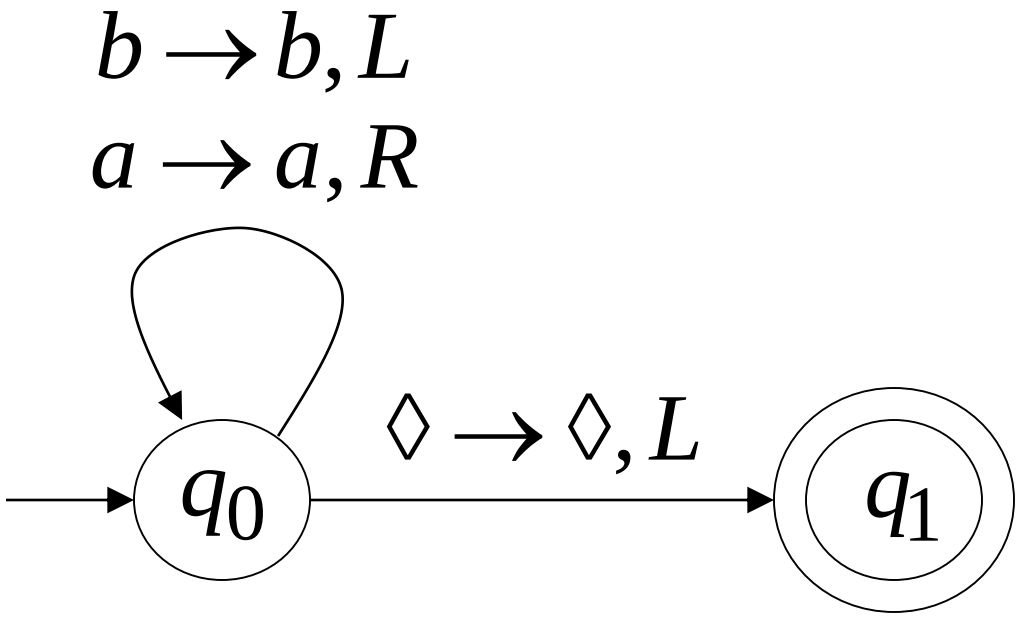
No possible Transition

Halt & Reject

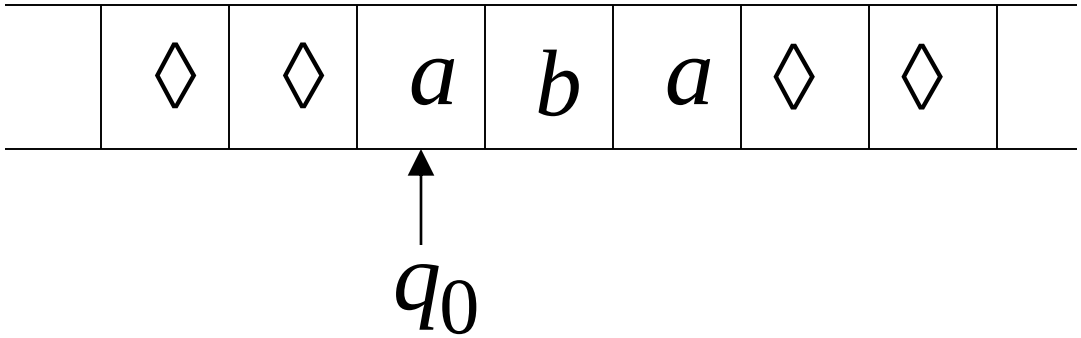


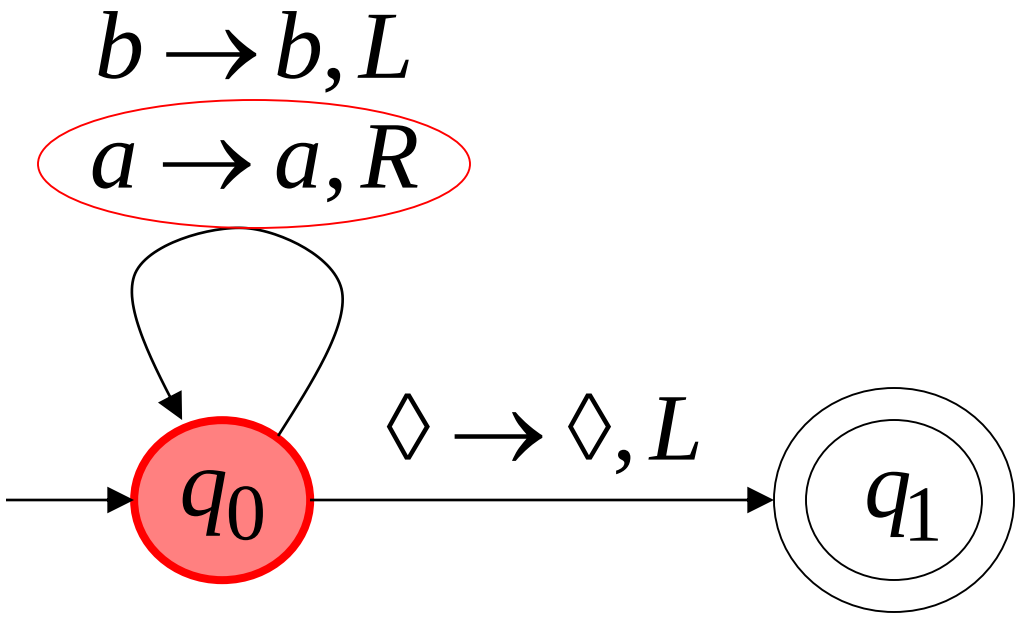
Infinite Loop Example

A Turing machine for language $a^* + b(a+b)^*$

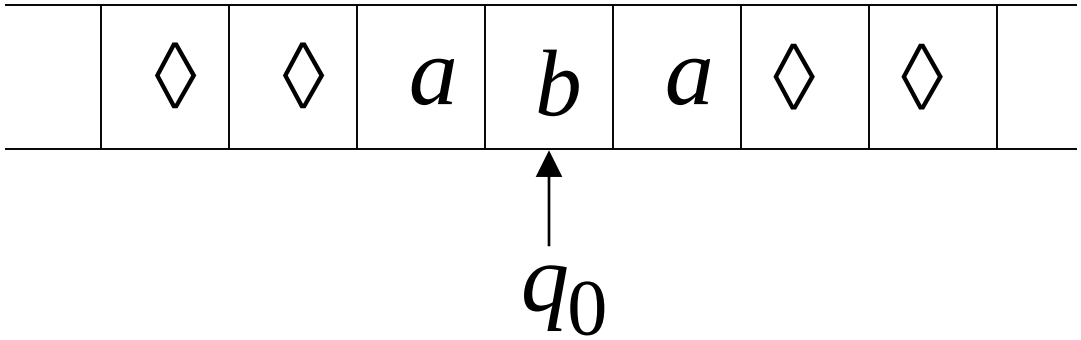


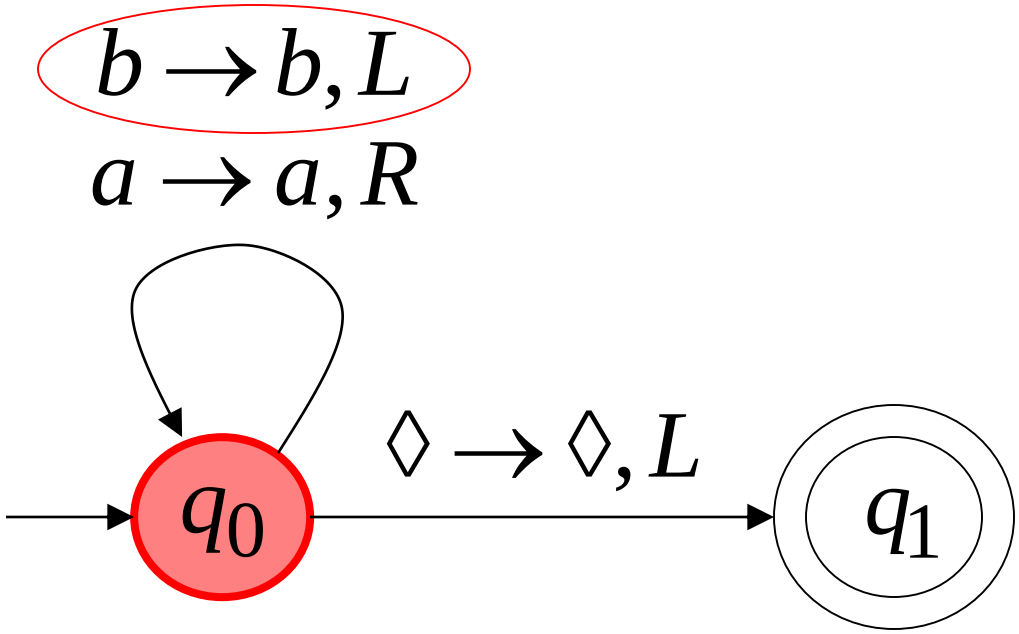
Time 0



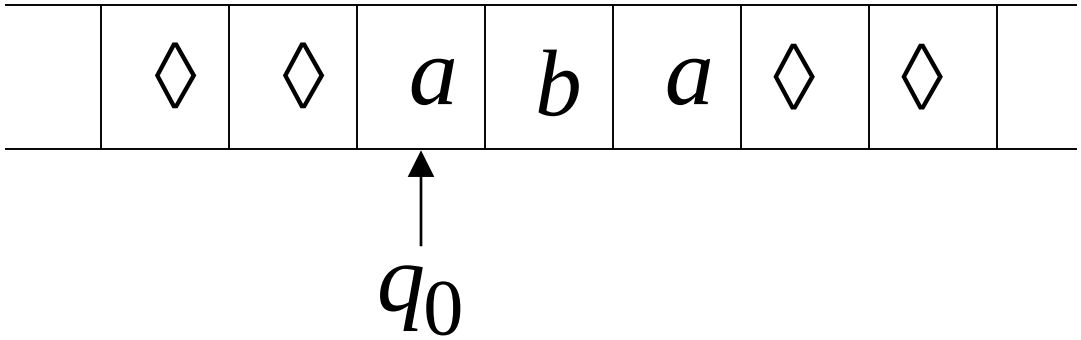


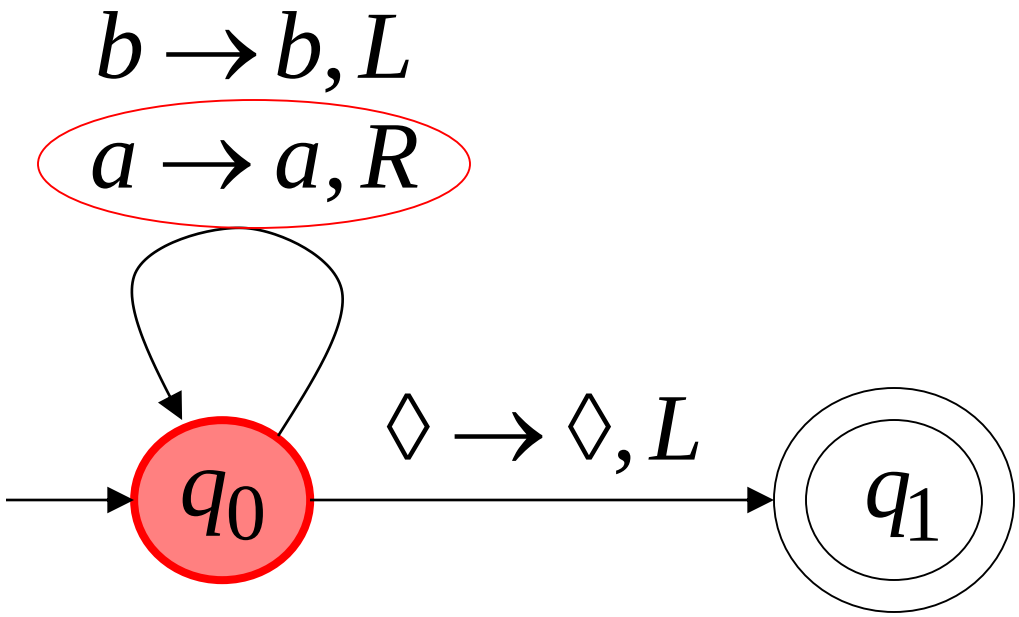
Time 1



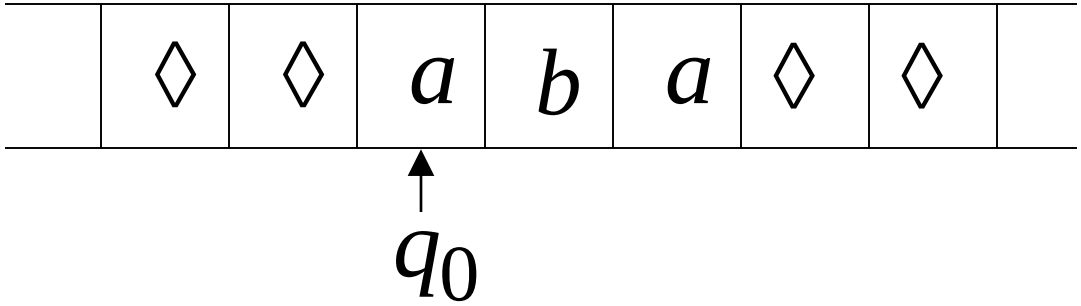


Time 2

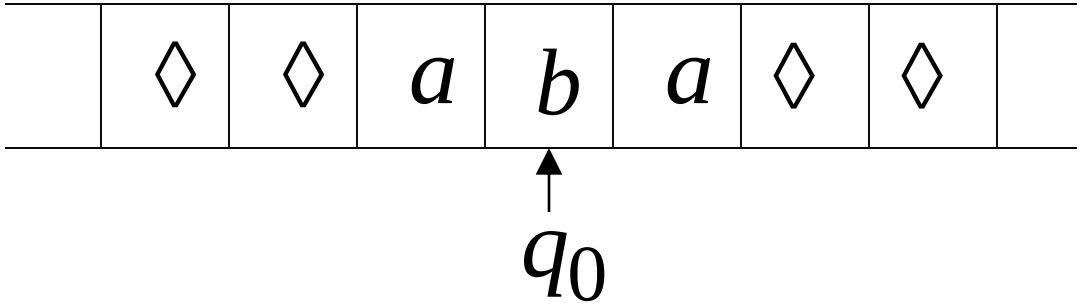




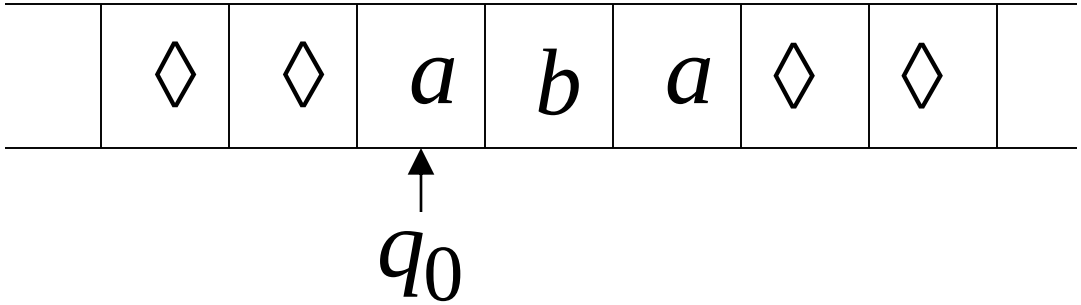
Time 2



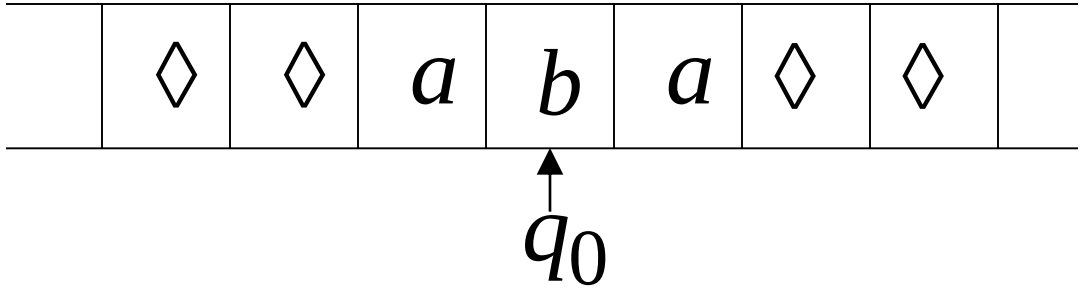
Time 3



Time 4



Time 5



Infinite loop

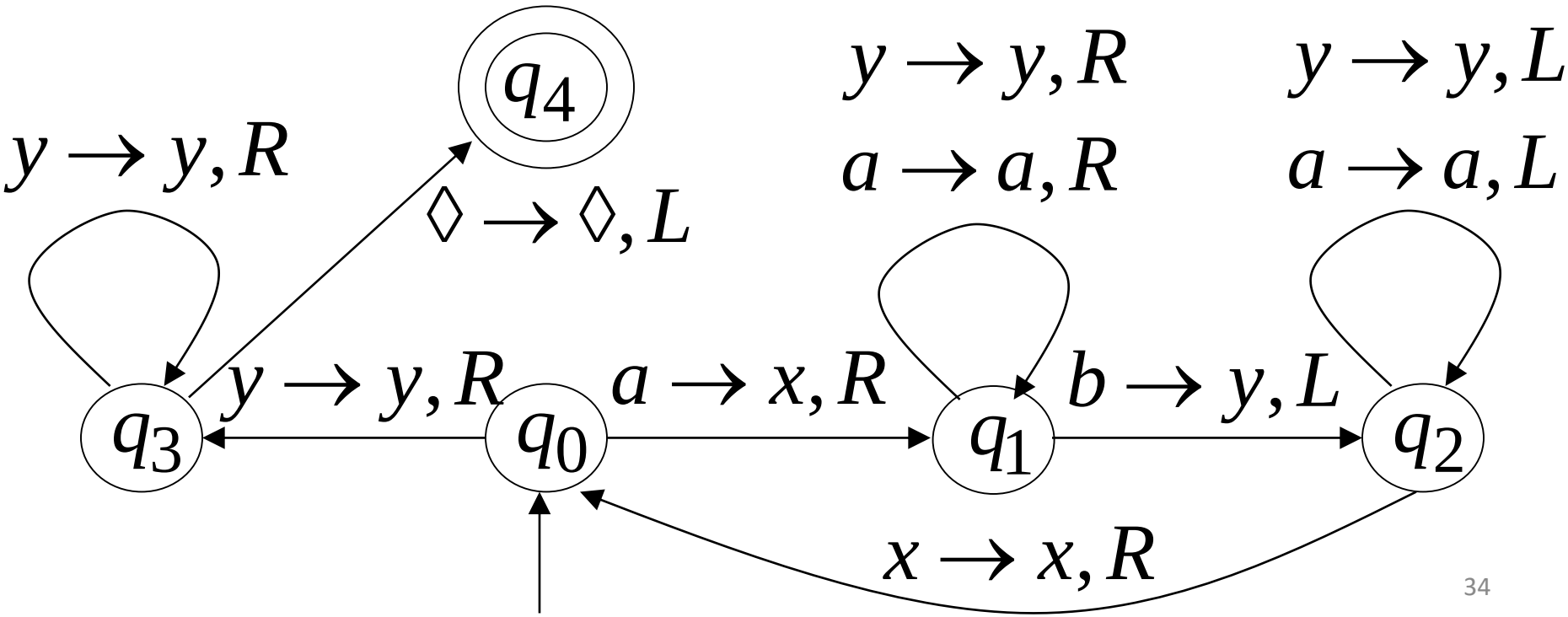
Infinite Loop….

Because of the infinite loop:

- The accepting state cannot be reached.
- The machine never halts .
- The input string is rejected.

A Turing Machine… **Example**

Turing machine for the language: $\{a^n b^n\}$ $n \geq 1$



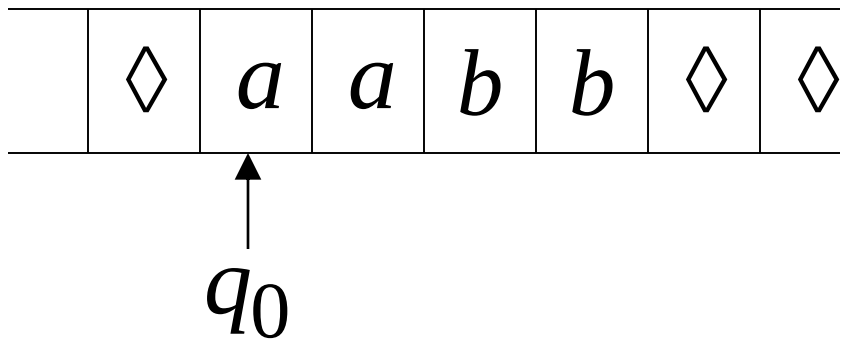
A Turing Machine… **Example**

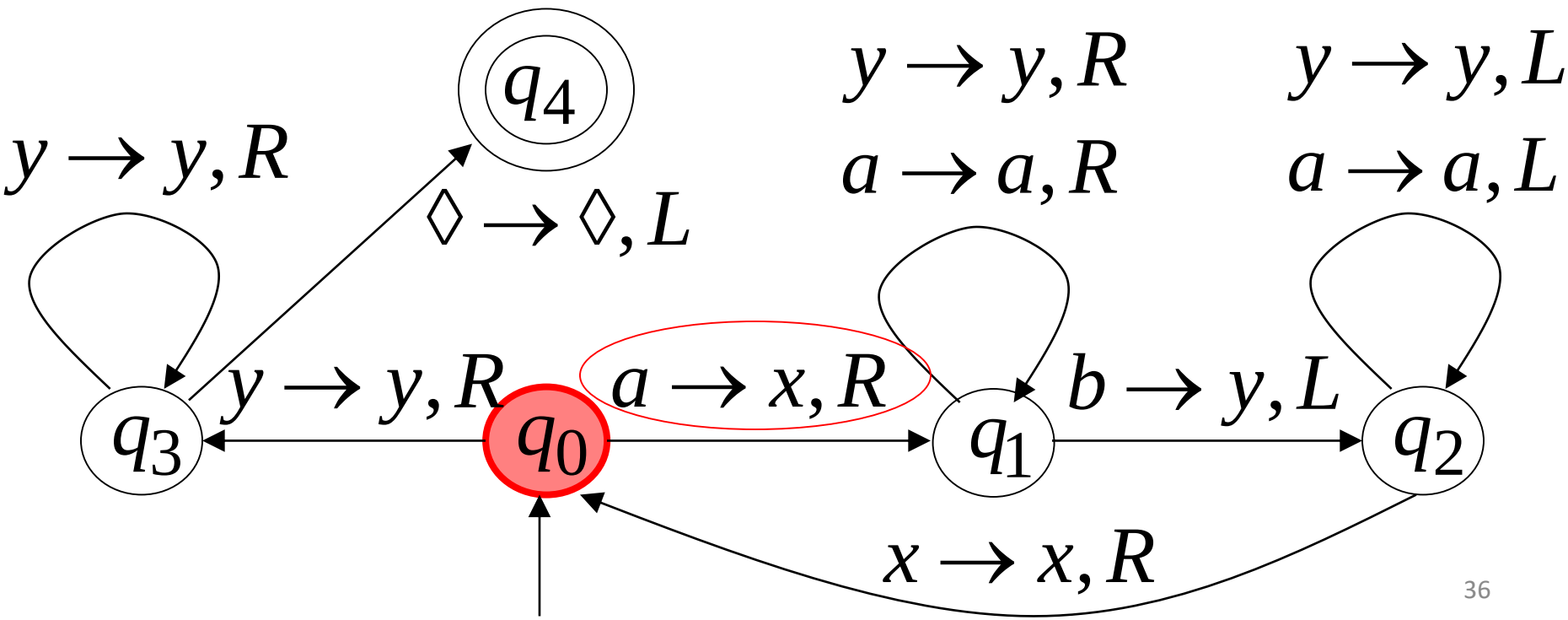
Basic Idea: Algorithm

Match a's with b's:
Repeat:
    replace leftmost a with x
    find leftmost b and replace it with y
Until there are no more a's or b's

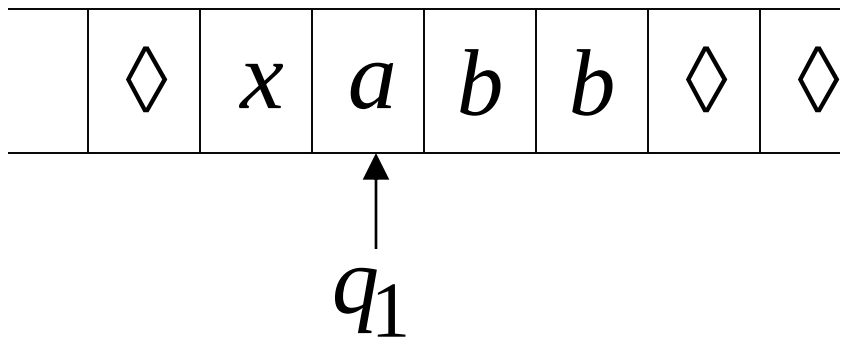
If there is a remaining a or b reject

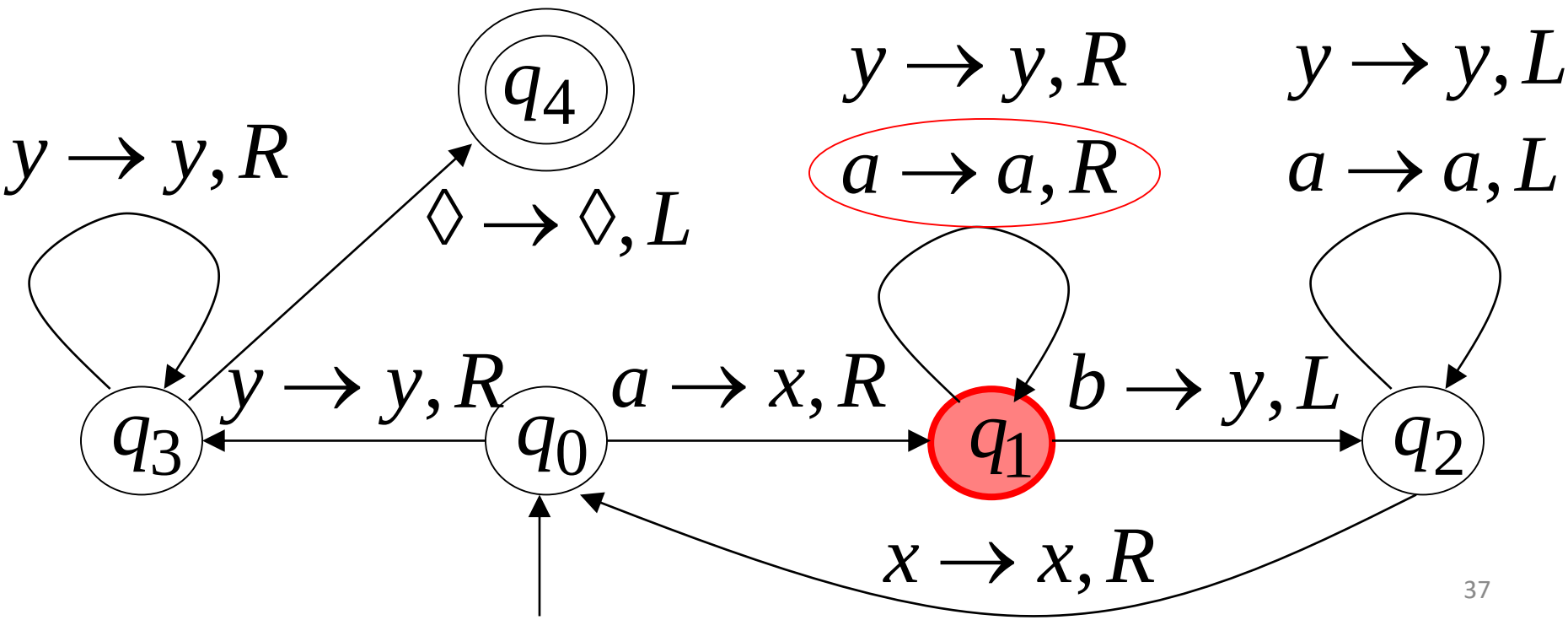
Time 0


◊ a a b b ◊ ◊
q_0

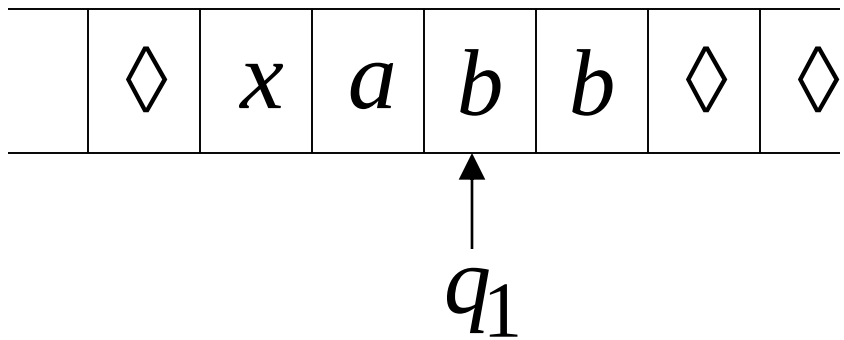

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

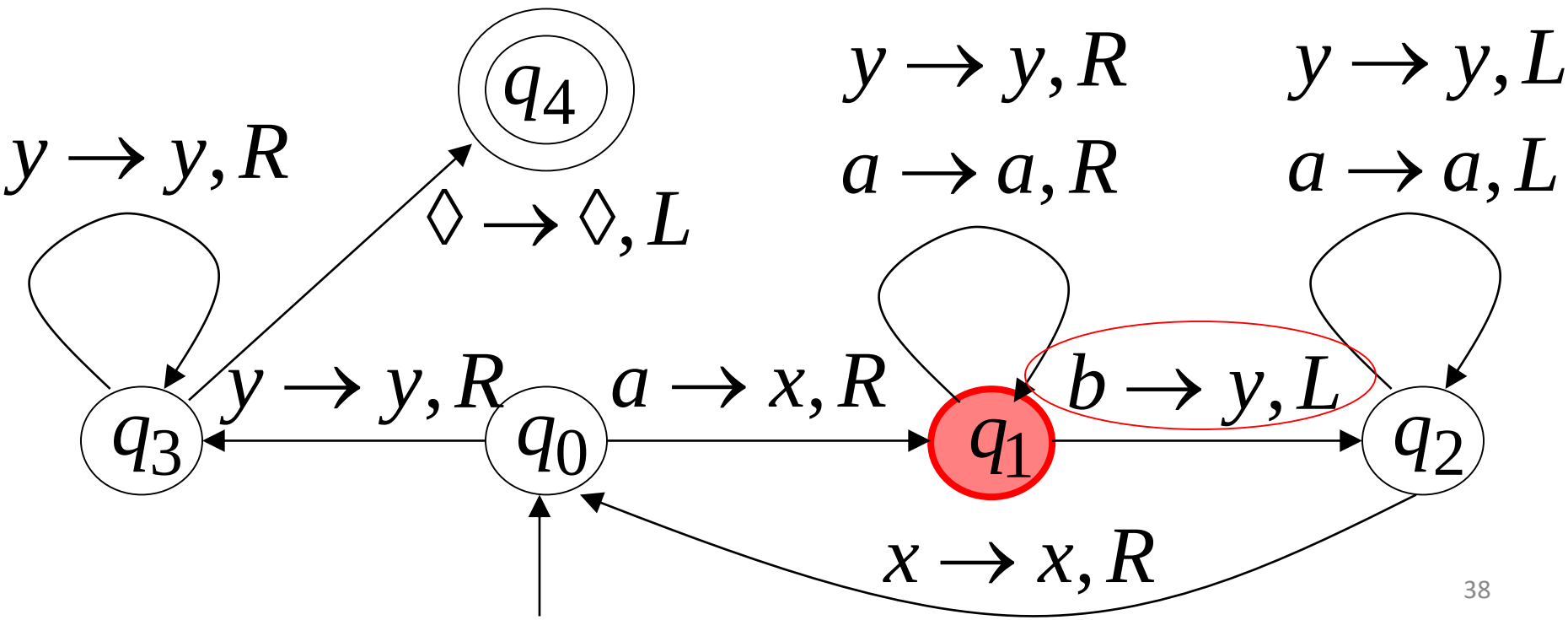
Time 1


◊ x a b b ◊ ◊
q_1

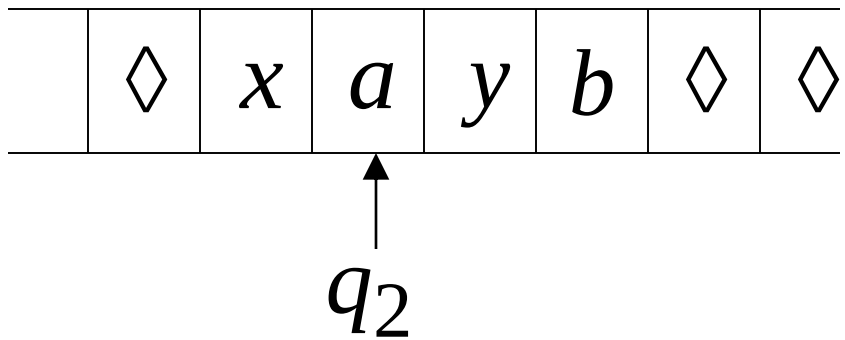

q_4
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

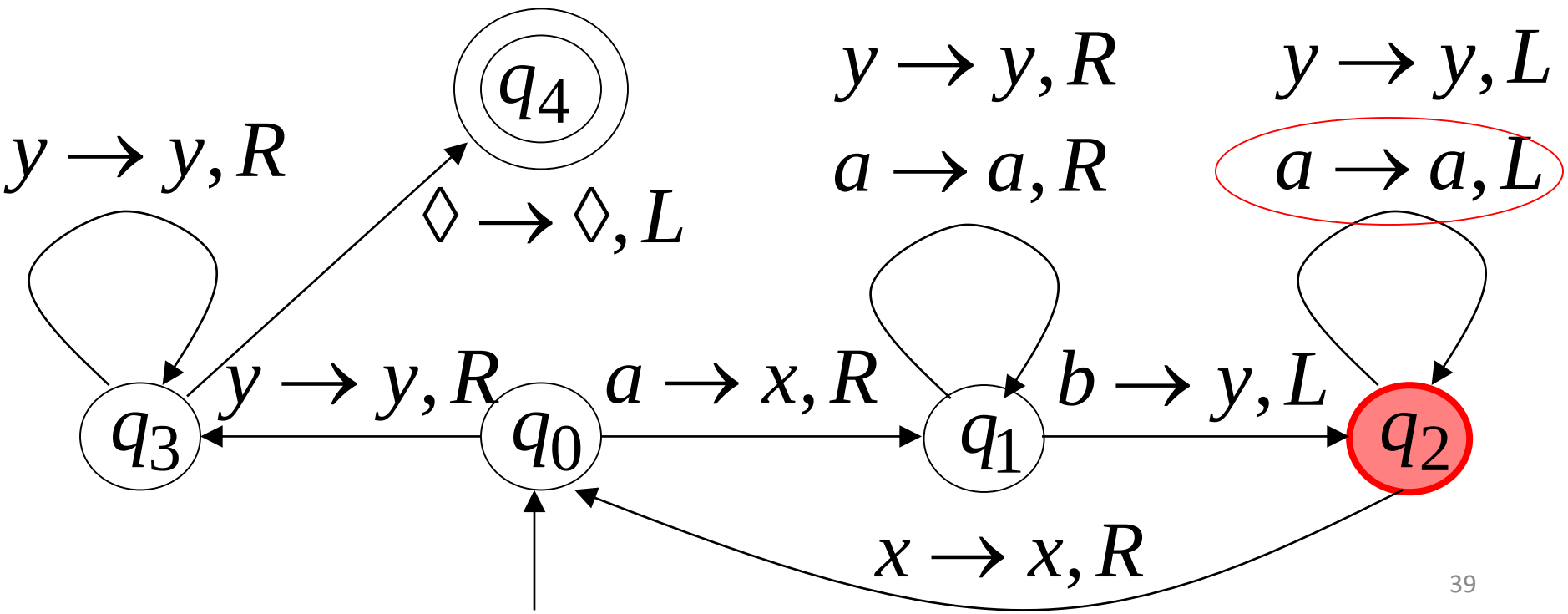
Time 2


◊ x a b b ◊ ◊
q_1

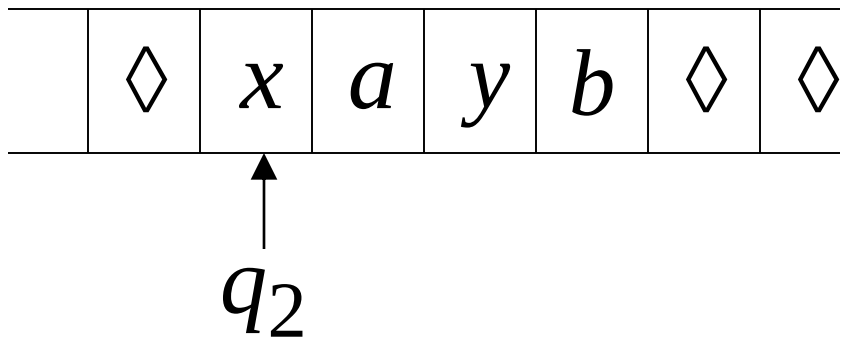

q_4
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

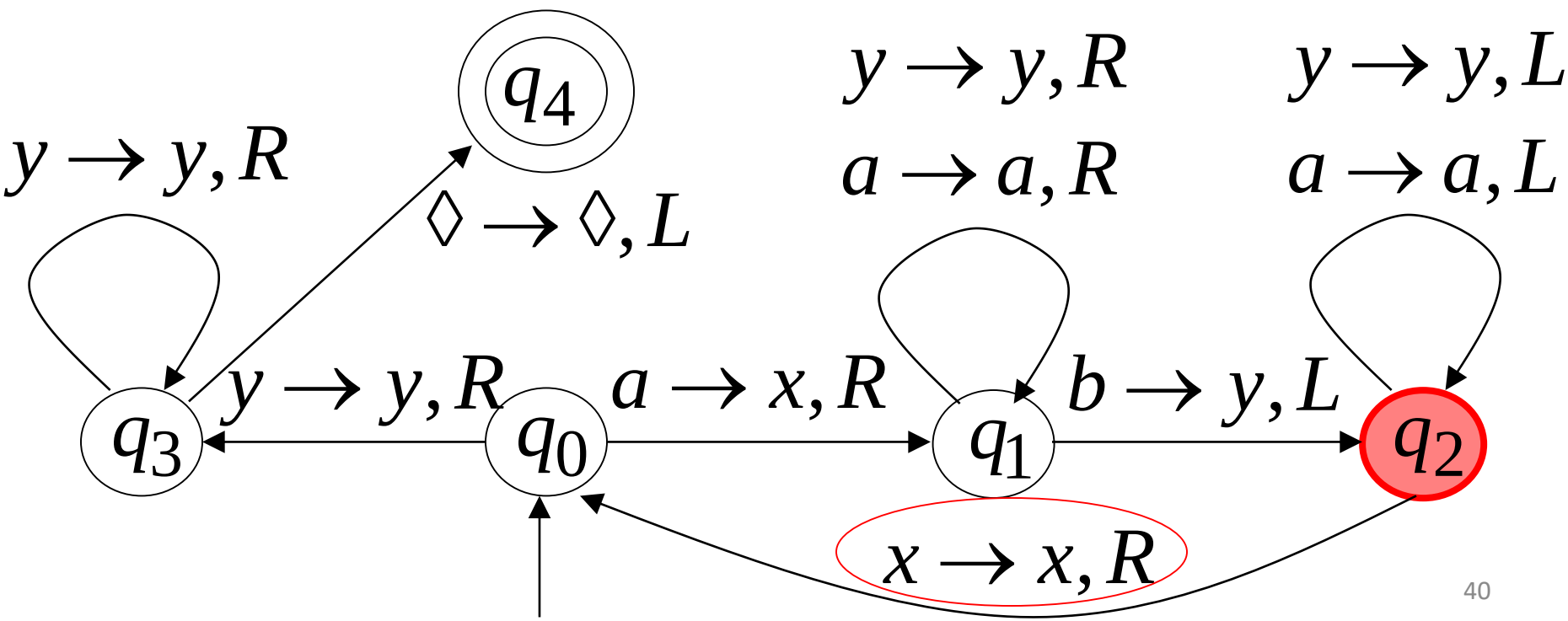
Time 3


◊ x a y b ◊ ◊
q_2

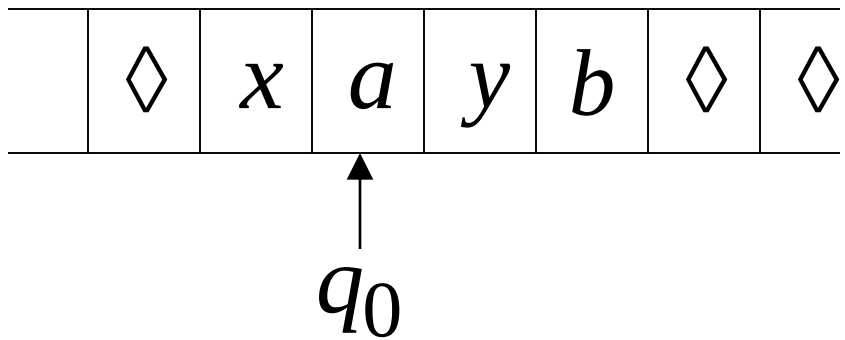

q_4
$y \to y, R$
$\lozenge \to \lozenge, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

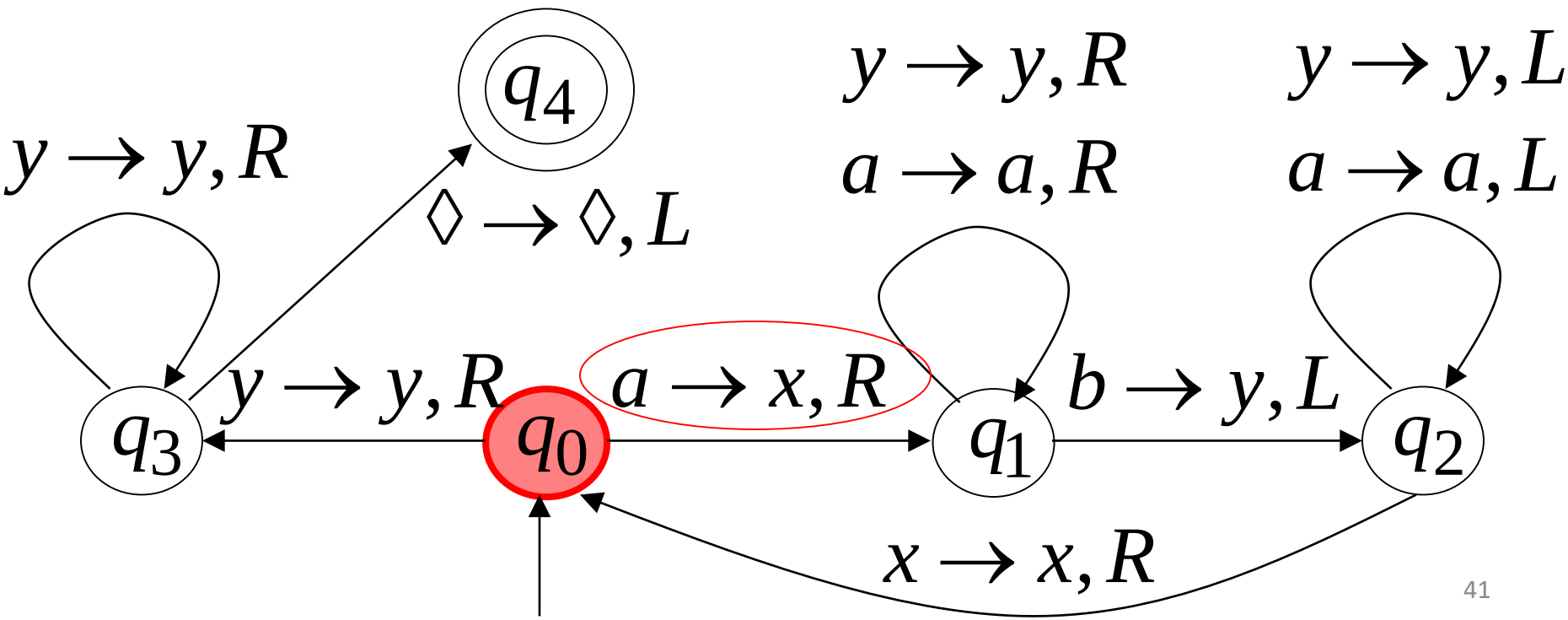
Time 4


$\Diamond$ x a y b $\Diamond$ $\Diamond$
q_2


q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

Time 5

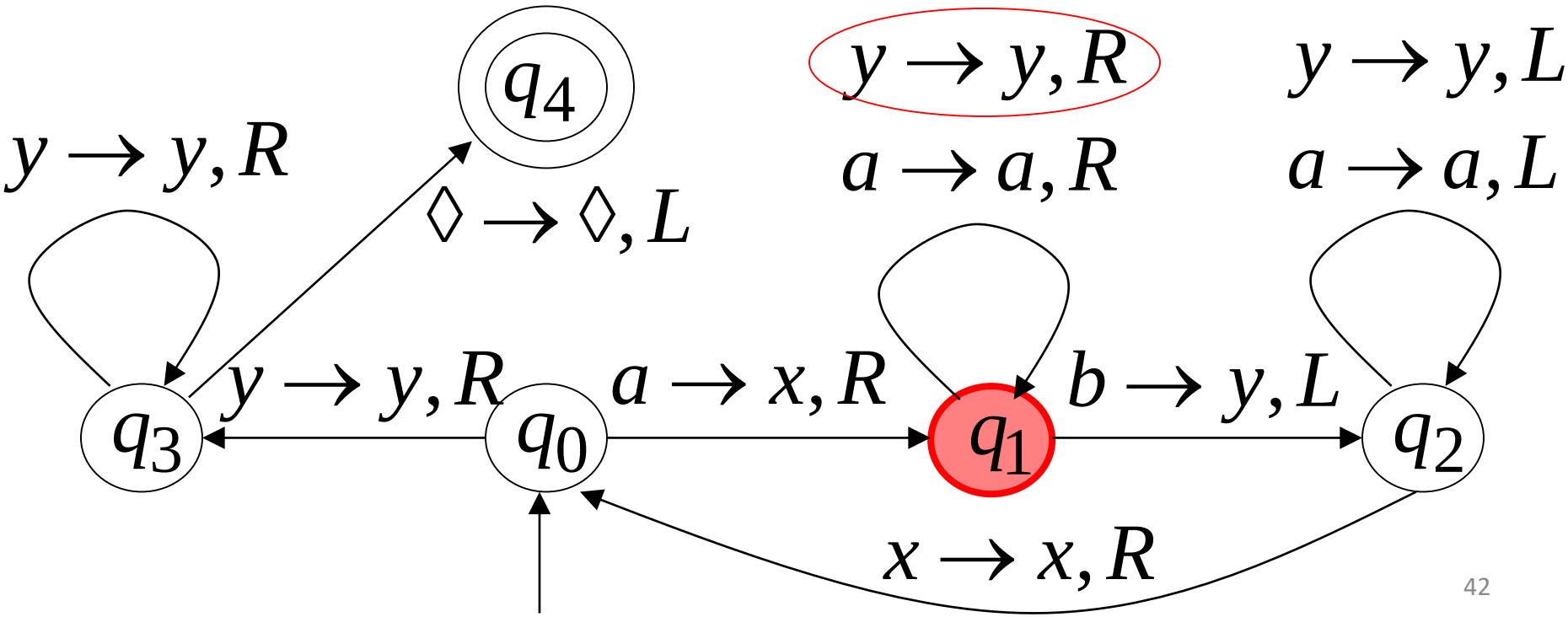

◊ x a y b ◊ ◊
q_0


q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

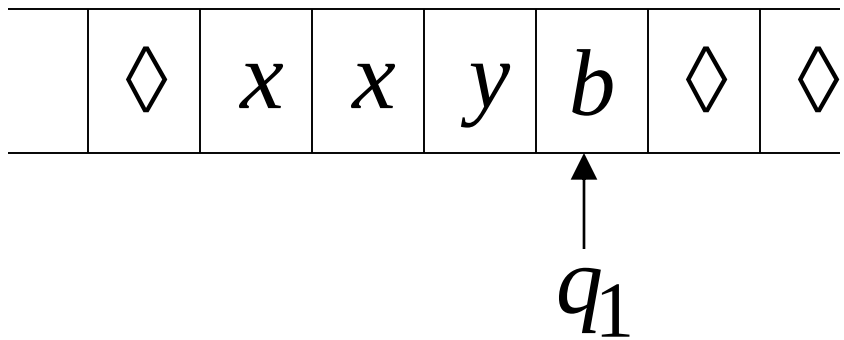
Time 6

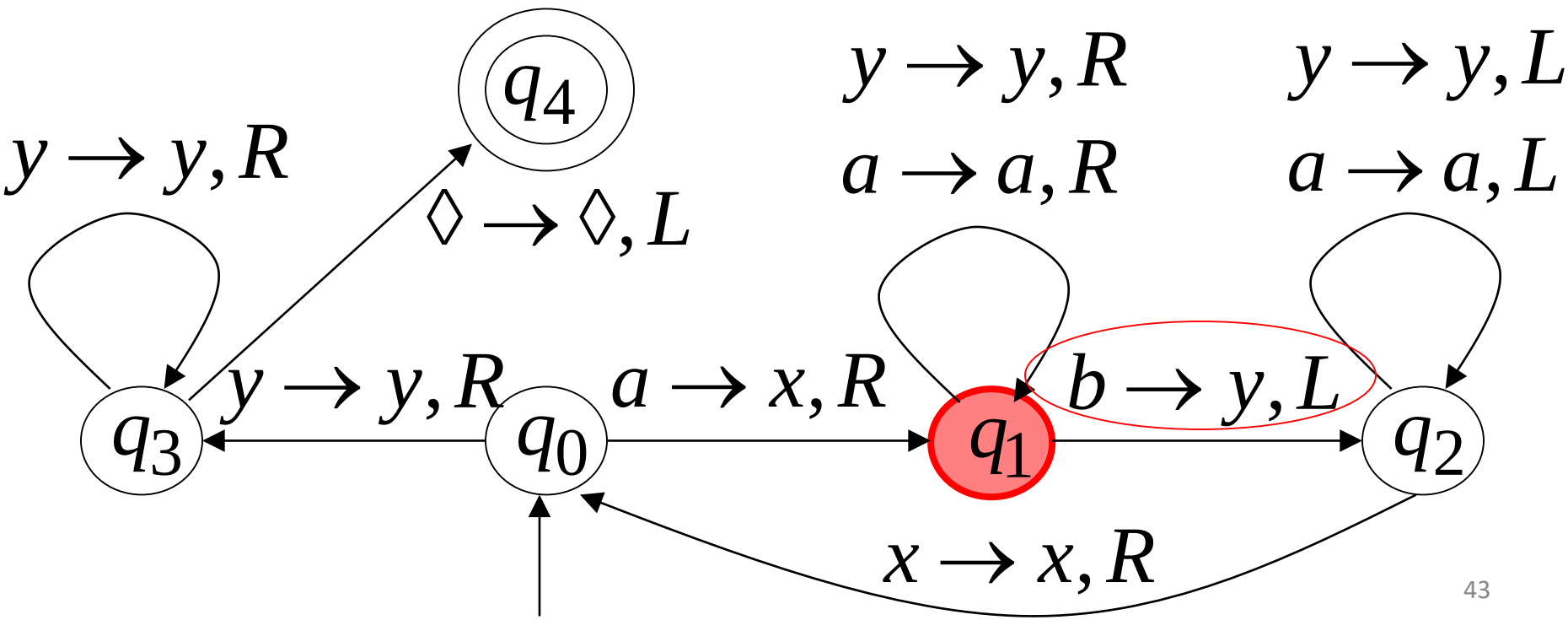
	$\Diamond$	x	x	y	b	$\Diamond$	$\Diamond$

q_1

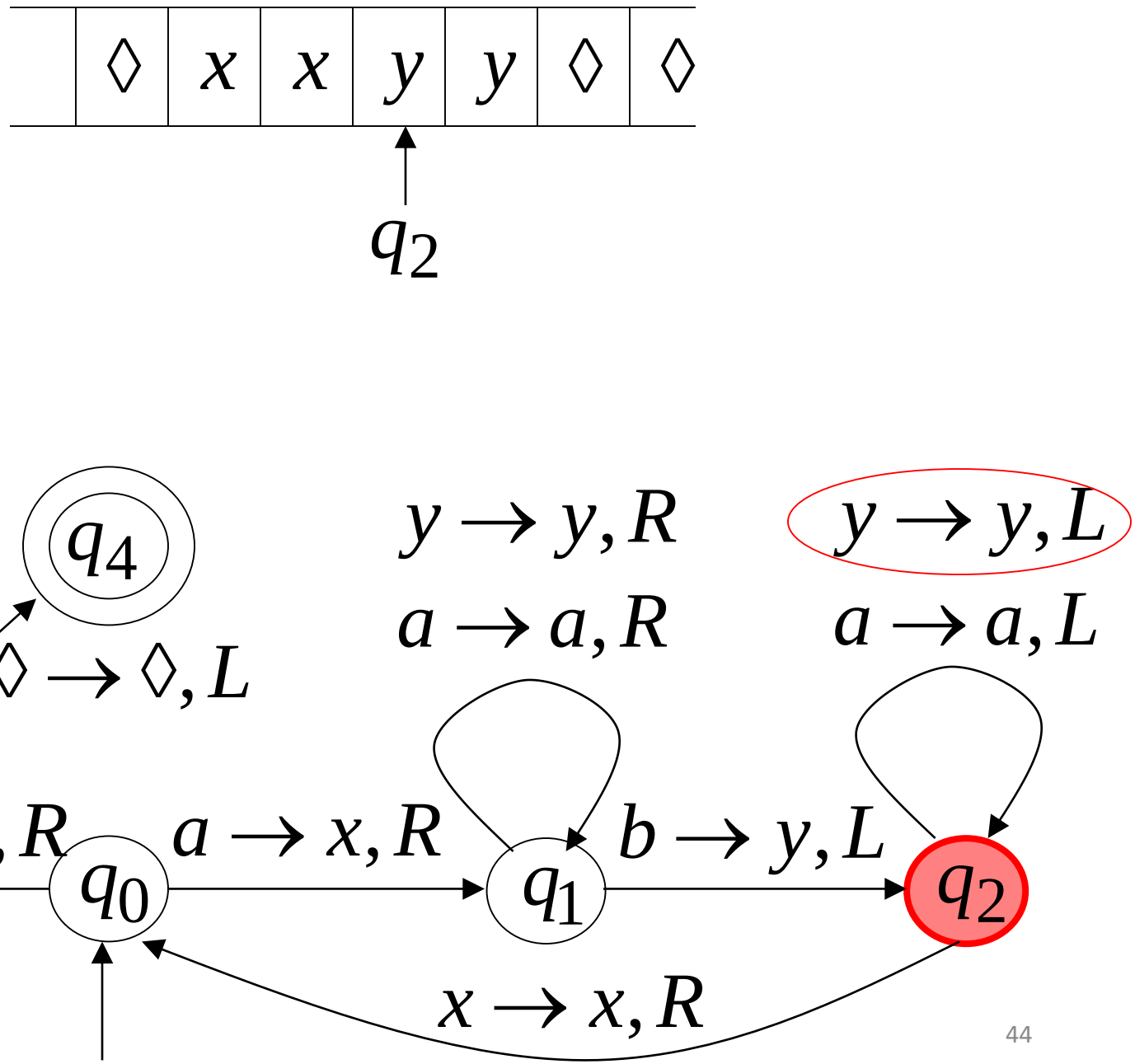


Time 7

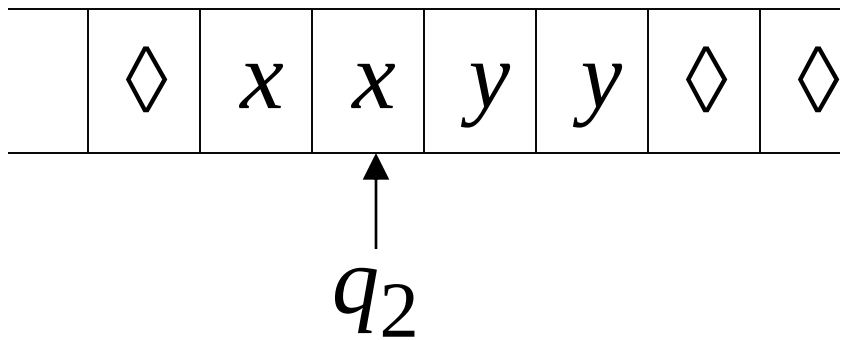

$\Diamond$ x x y b $\Diamond$ $\Diamond$
q_1

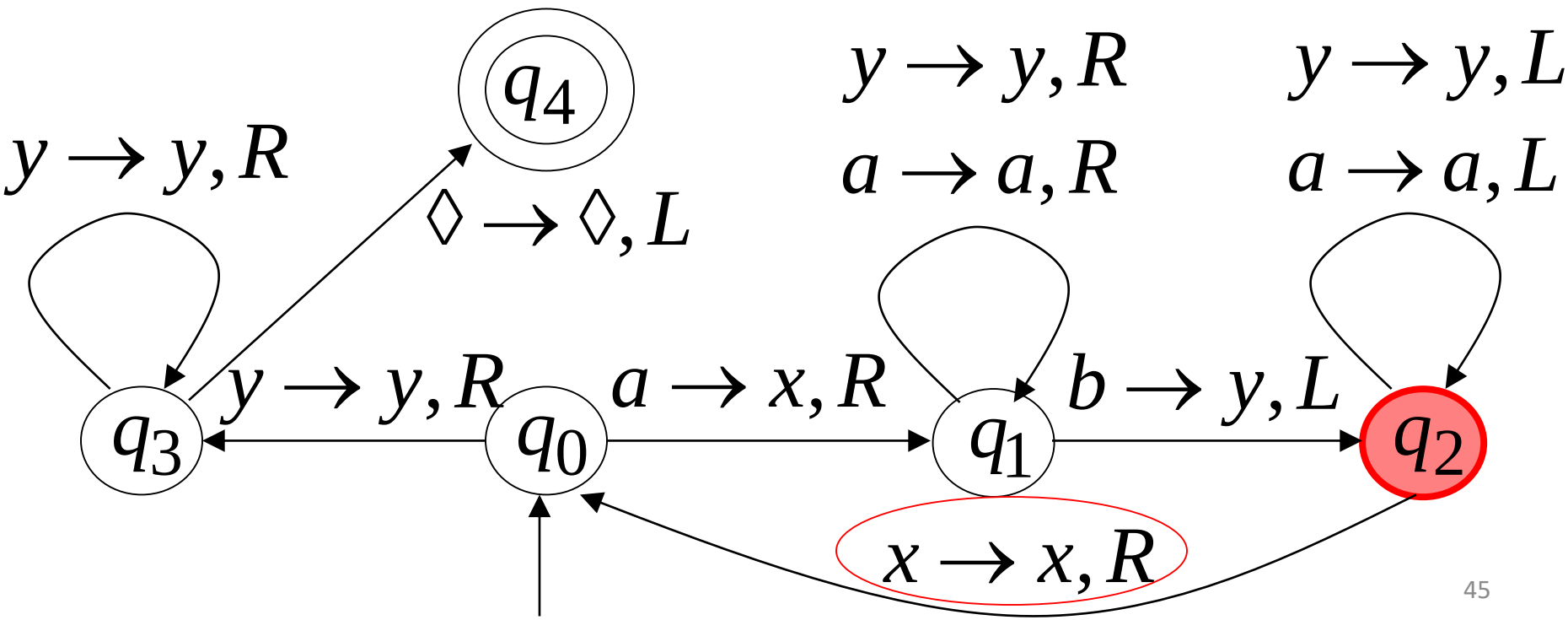

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

Time 8

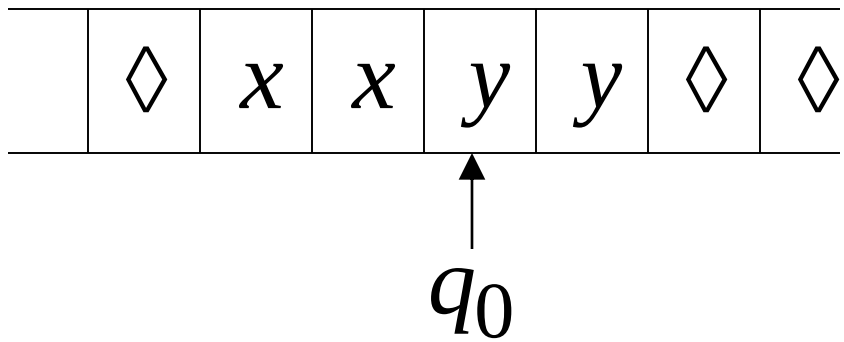

$\Diamond$ x x y y $\Diamond$ $\Diamond$
q_2
q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

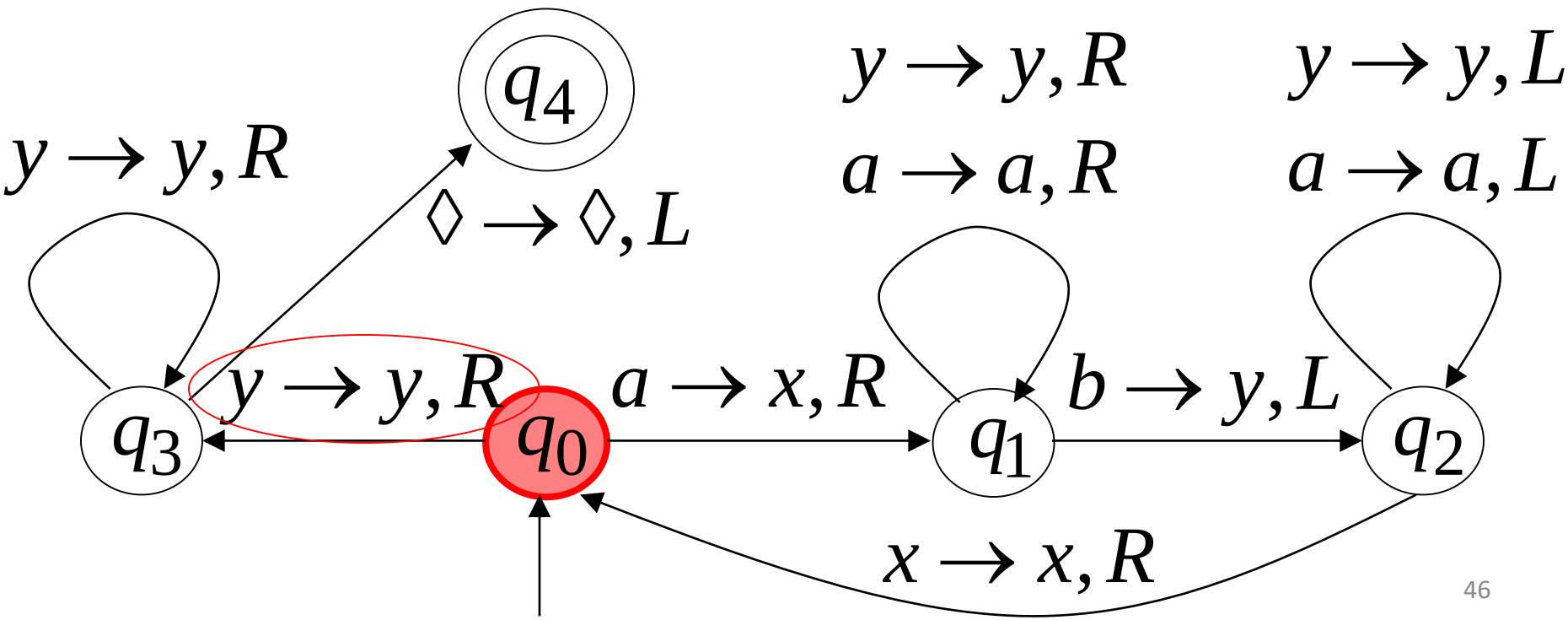
Time 9


$\Diamond$ x x y y $\Diamond$ $\Diamond$
q_2

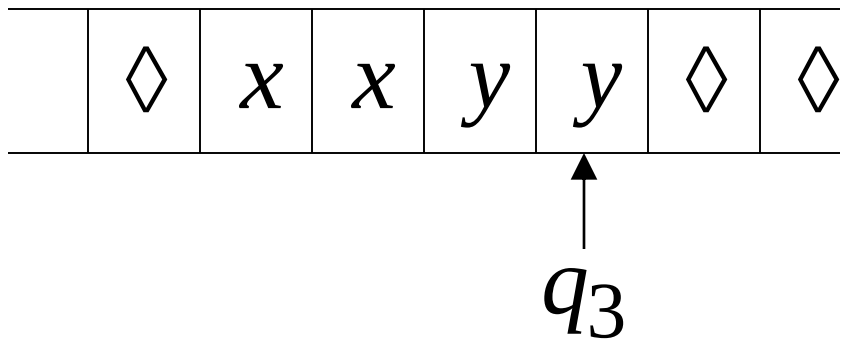

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

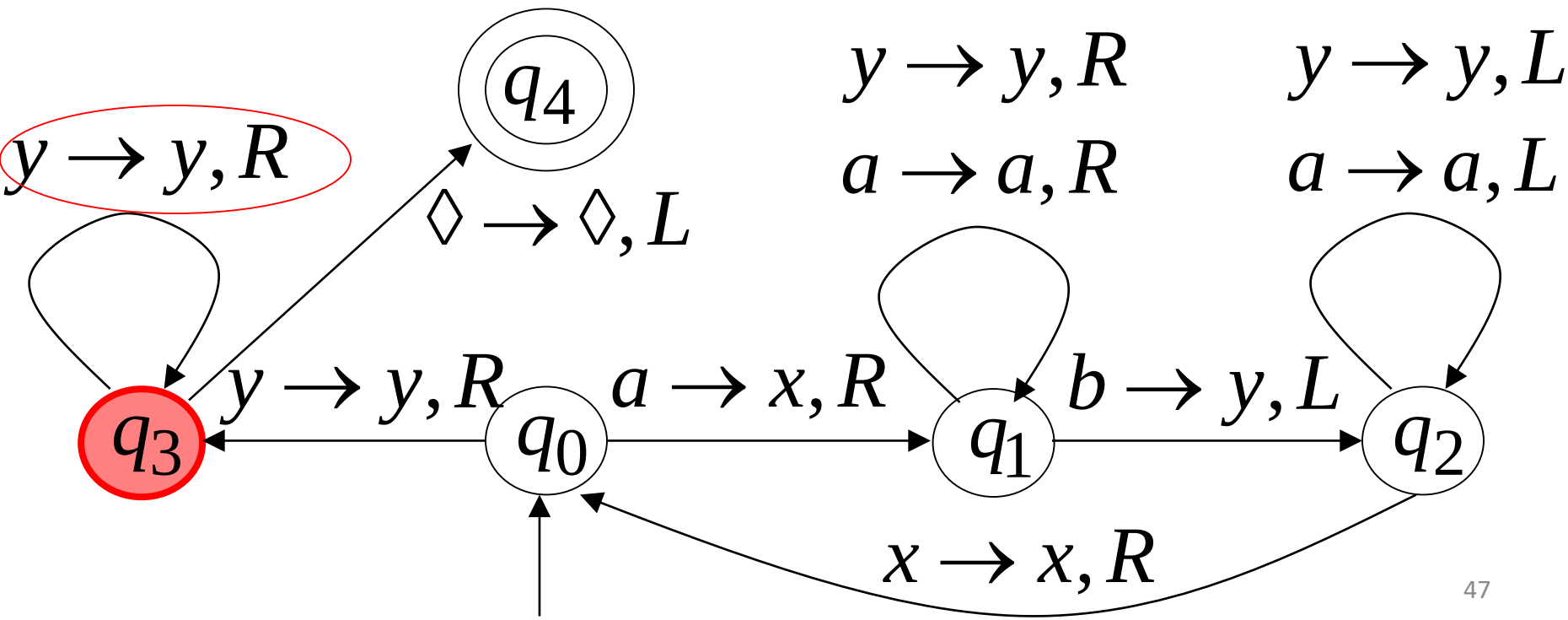
Time 10


$\Diamond$ x x y y $\Diamond$ $\Diamond$
q_0

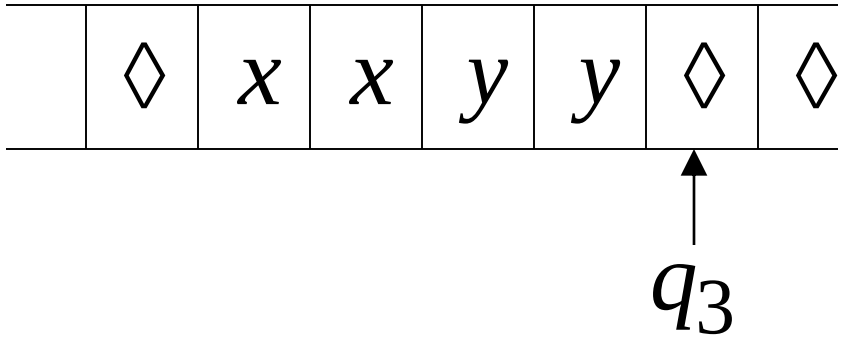

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

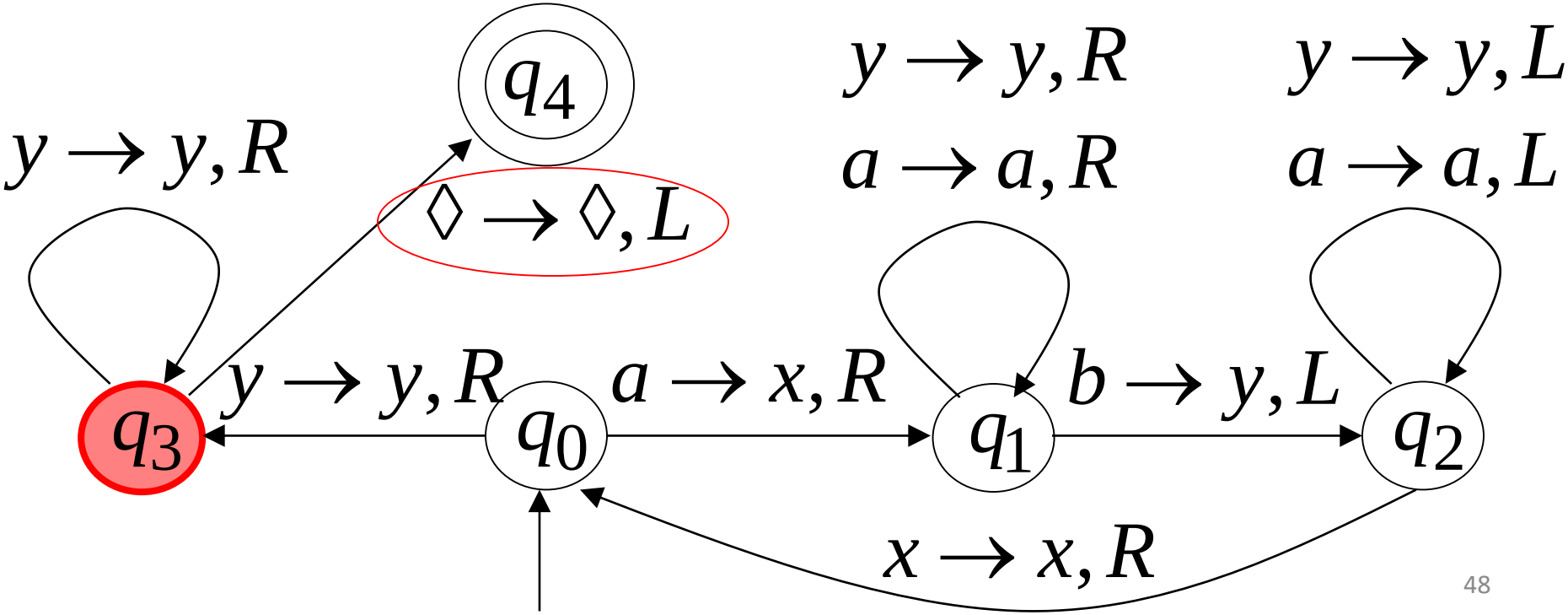
Time 11


◊ x x y y ◊ ◊
q_3


q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
q_3
$y \to y, R$
q_0
$a \to x, R$
q_1
$b \to y, L$
q_2
$x \to x, R$

Time 12


$\Diamond$ x x y y $\Diamond$ $\Diamond$
q_3

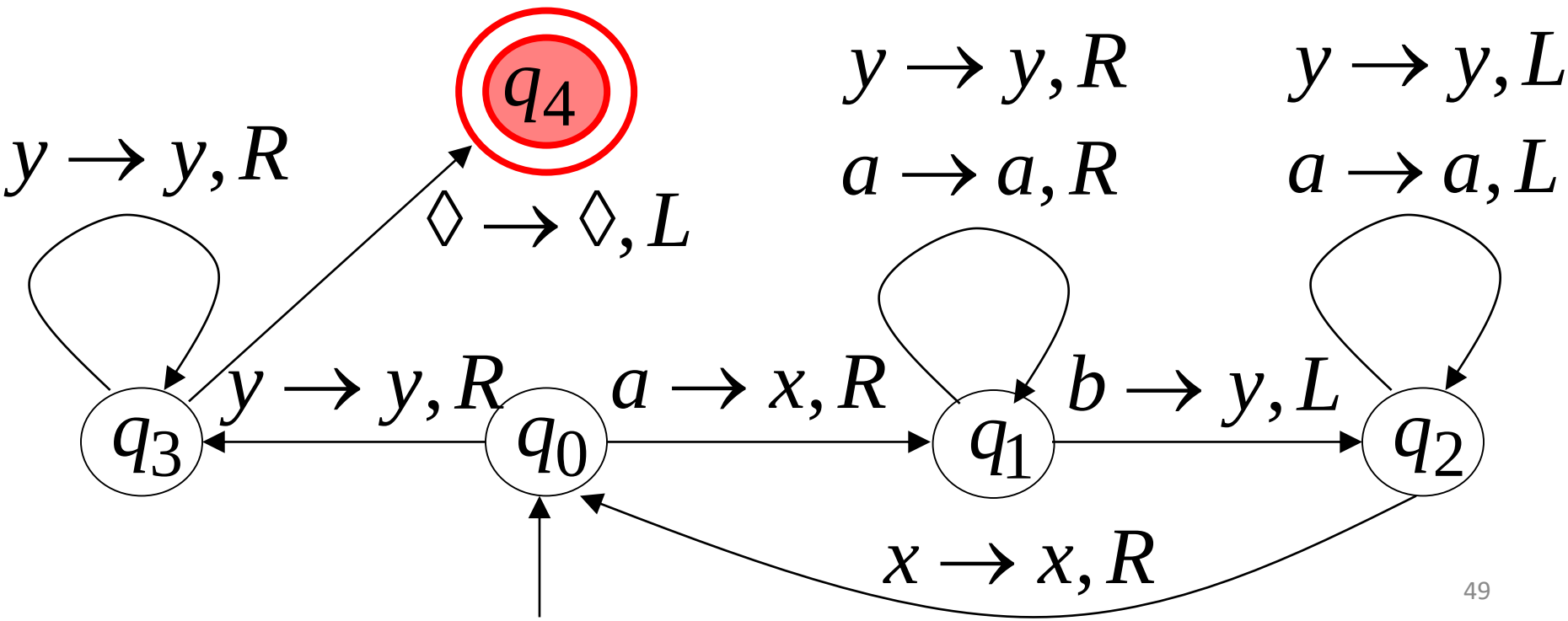

q_4
$y \to y, R$
$\Diamond \to \Diamond, L$
$y \to y, R$
$a \to a, R$
$y \to y, L$
$a \to a, L$
$y \to y, R$
$a \to x, R$
$b \to y, L$
q_3
q_0
q_1
q_2
$x \to x, R$

Time 13

	$\Diamond$	x	x	y	y	$\Diamond$	$\Diamond$

q_4

Halt & Accept



A Turing Machine… **Example**

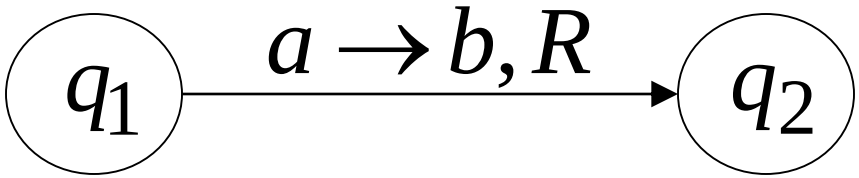
Observation:

If we modify the machine for the language $\{a^n b^n\}$

We can easily construct a machine for the language $\{a^n b^n c^n\}$

A Turing Machine: **Formal Definition**

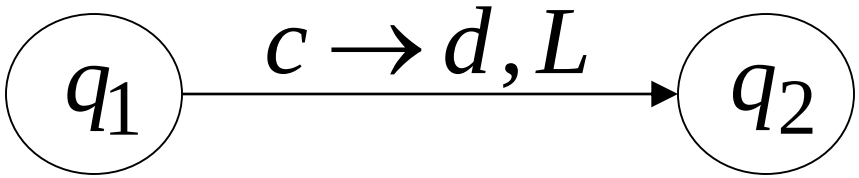
Transition Function



$$\delta(q_1, a) = (q_2, b, R)$$

A Turing Machine: **Formal Definition**

Transition Function



$$\delta(q_1, c) = (q_2, d, L)$$

A Turing Machine: **Formal Definition**

Input alphabet

Tape alphabet

States

$$M = (Q, \Sigma, \Gamma, \delta, q_0, \Diamond, F)$$

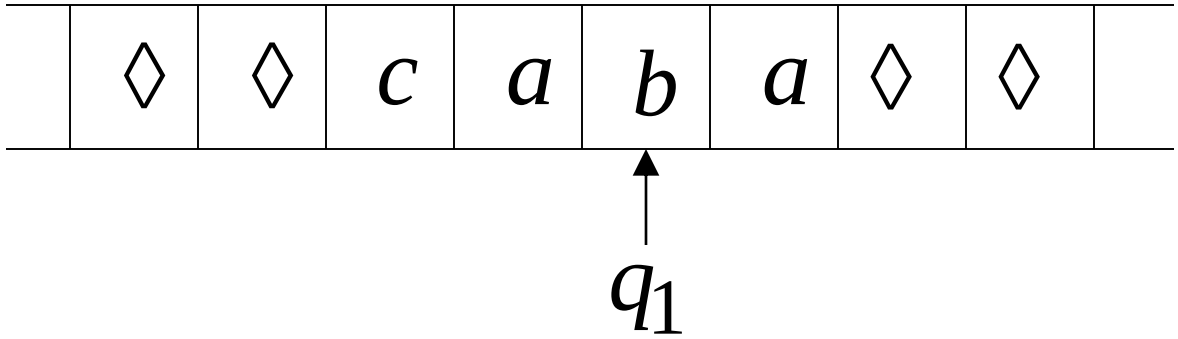
Transition function

Accept states

Initial state

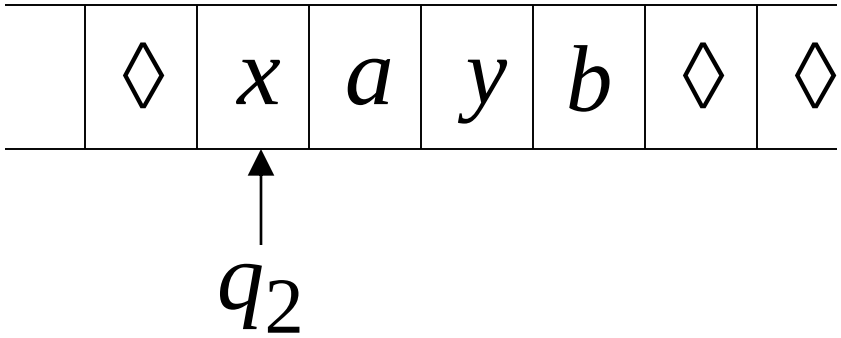
blank

A Turing Machine…**Configuration**

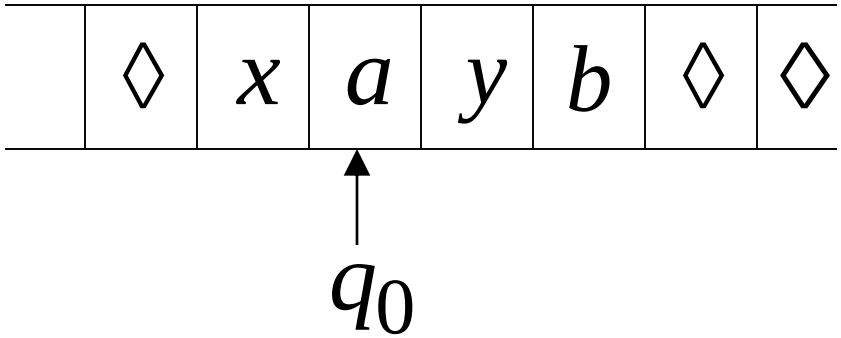


Instantaneous description: $ca\ q_1\ ba$

Time 4



Time 5

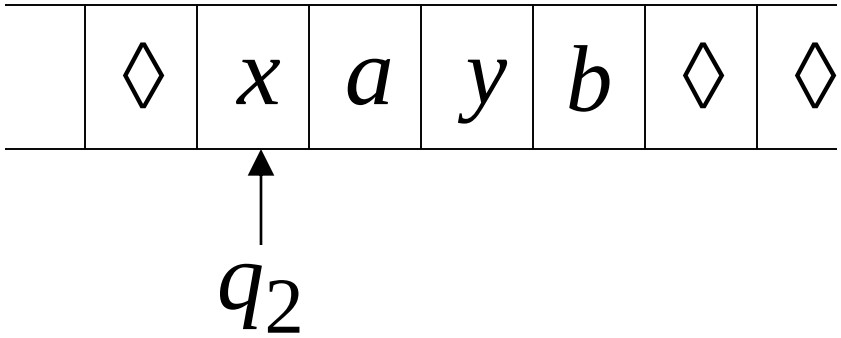


A move:

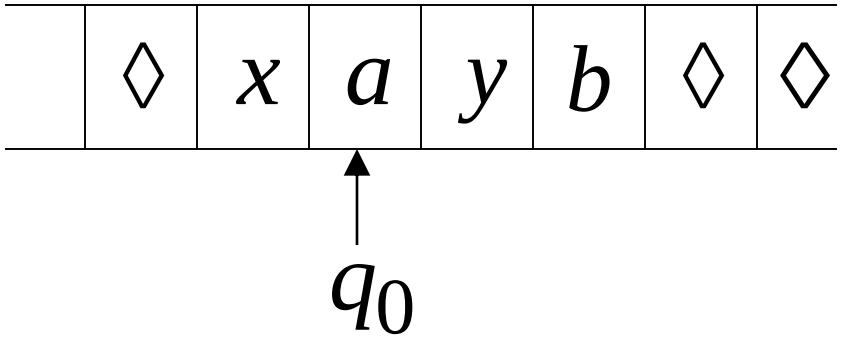
$$q_2\ xayb \succ x\ q_0\ ayb$$

(yields in one mode)

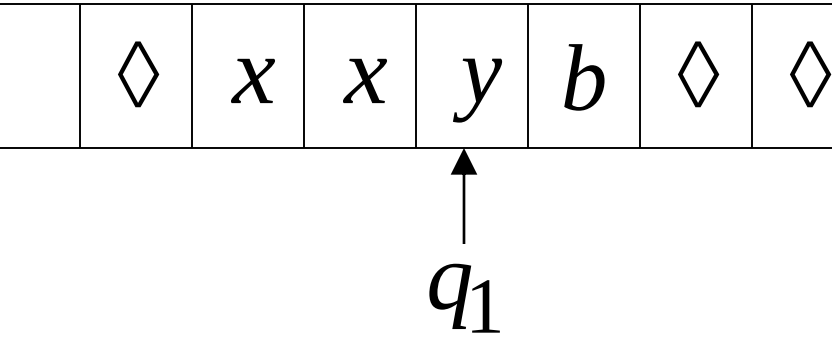
Time 4



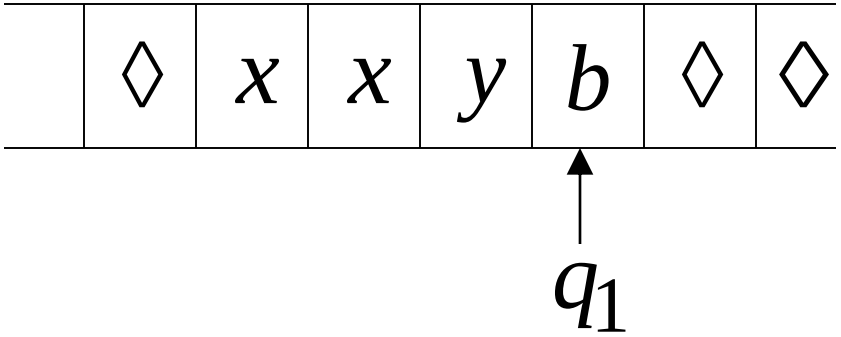
Time 5



Time 6



Time 7



A computation

$$q_2\ xayb \succ x\, q_0\ ayb \succ xx\, q_1\ yb \succ xxy\, q_1\ b$$

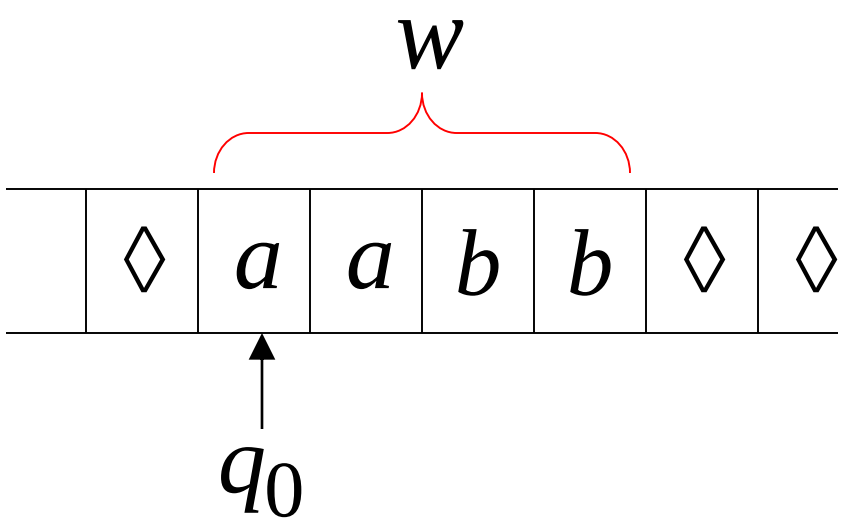
$$q_2\ xayb \succ x\, q_0\ ayb \succ xx\, q_1\ yb \succ xxy\, q_1\ b$$

Equivalent notation: $$q_2\ xayb \overset{*}{\succ} xxy\, q_1\ b$$

Initial configuration:

$$q_0 \; w$$

Input string



The Accepted Language

For any Turing Machine M

$$L(M) = \{ w : \quad q_0 \, w \succ^{*} x_1 \, q_f \, x_2 \}$$

Initial state

Accept state

The Accepted Language…

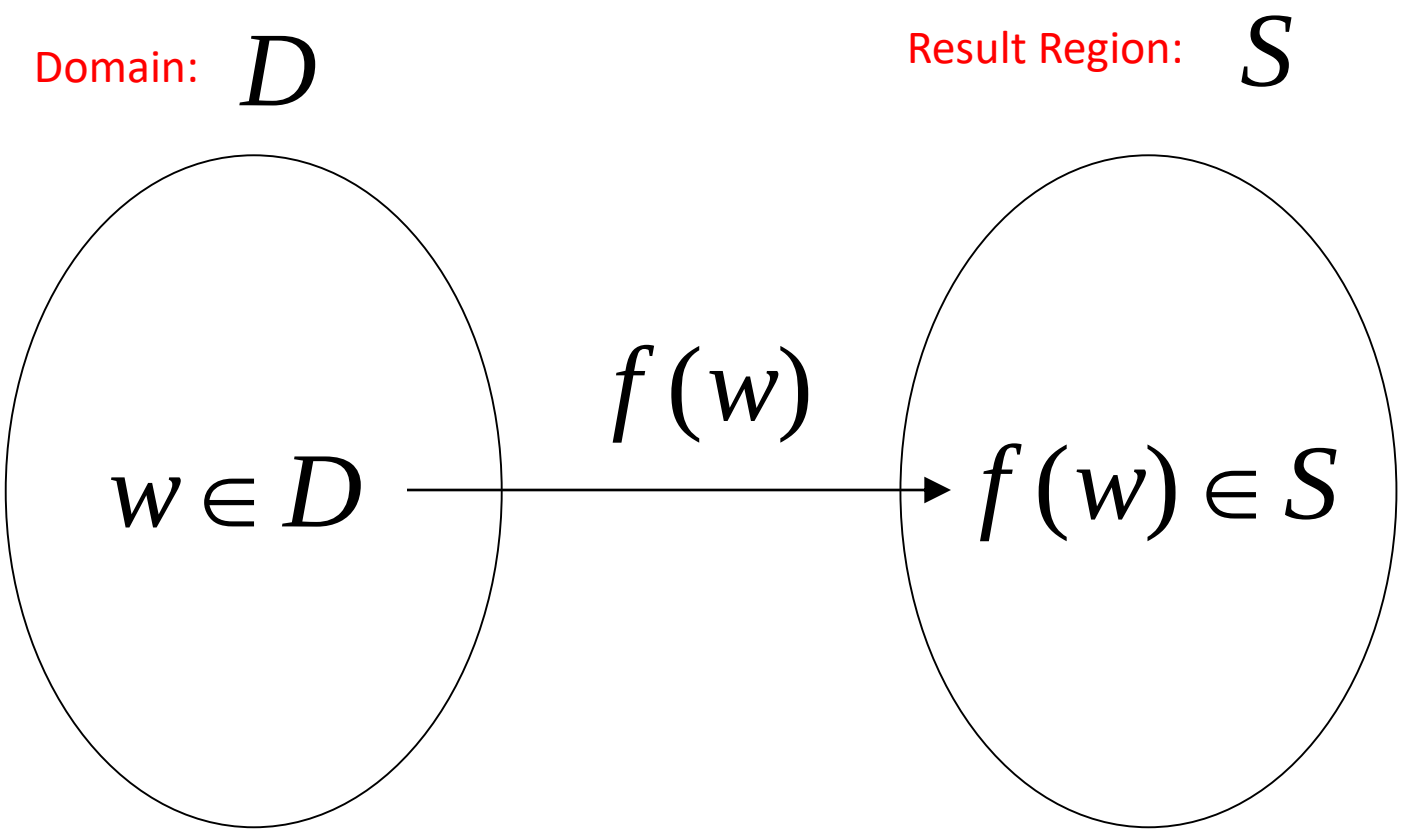
- If a language L is accepted by a Turing machine M then we say that L is:
  - Turing Recognizable

- Other names used:
  - Turing Acceptable
  - Recursively Enumerable

Turing Machine as Transducers:

Computing Functions with Turing Machines

- A function f(w) has:



Computing Functions with Turing Machines…

- A function may have many parameters:
- Example: Addition function

$$f(x, y) = x + y$$

Computing Functions with Turing Machines…

Integer Domain

Decimal:	5
Binary:	101
Unary:	11111

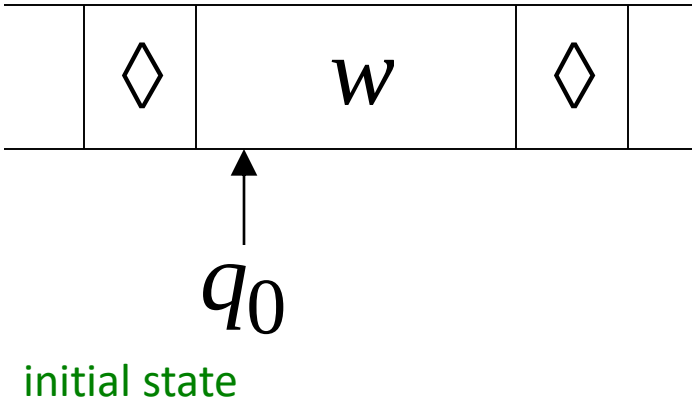
We prefer **unary** representation:

easier to manipulate with Turing machines

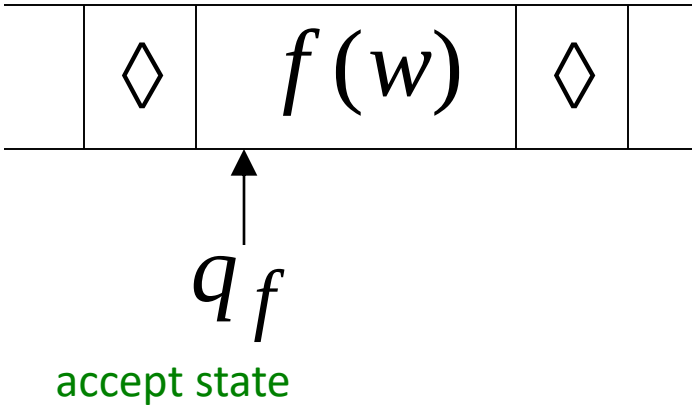
Computing Functions with Turing Machines…

Definition: A function ***f*** is computable if there is a Turing Machine ***M*** such that:

Initial configuration



Final configuration



For all $w \in D$ Domain

Computing Functions with Turing Machines…

In other words: A function ***f*** is computable if there is a Turing Machine ***M*** such that:

$$q_0\, w \overset{*}{\succ} q_f\, f(w)$$

Initial Configuration

Final Configuration

For all $w \in D$ Domain

Computing Functions with Turing Machines…

Example:

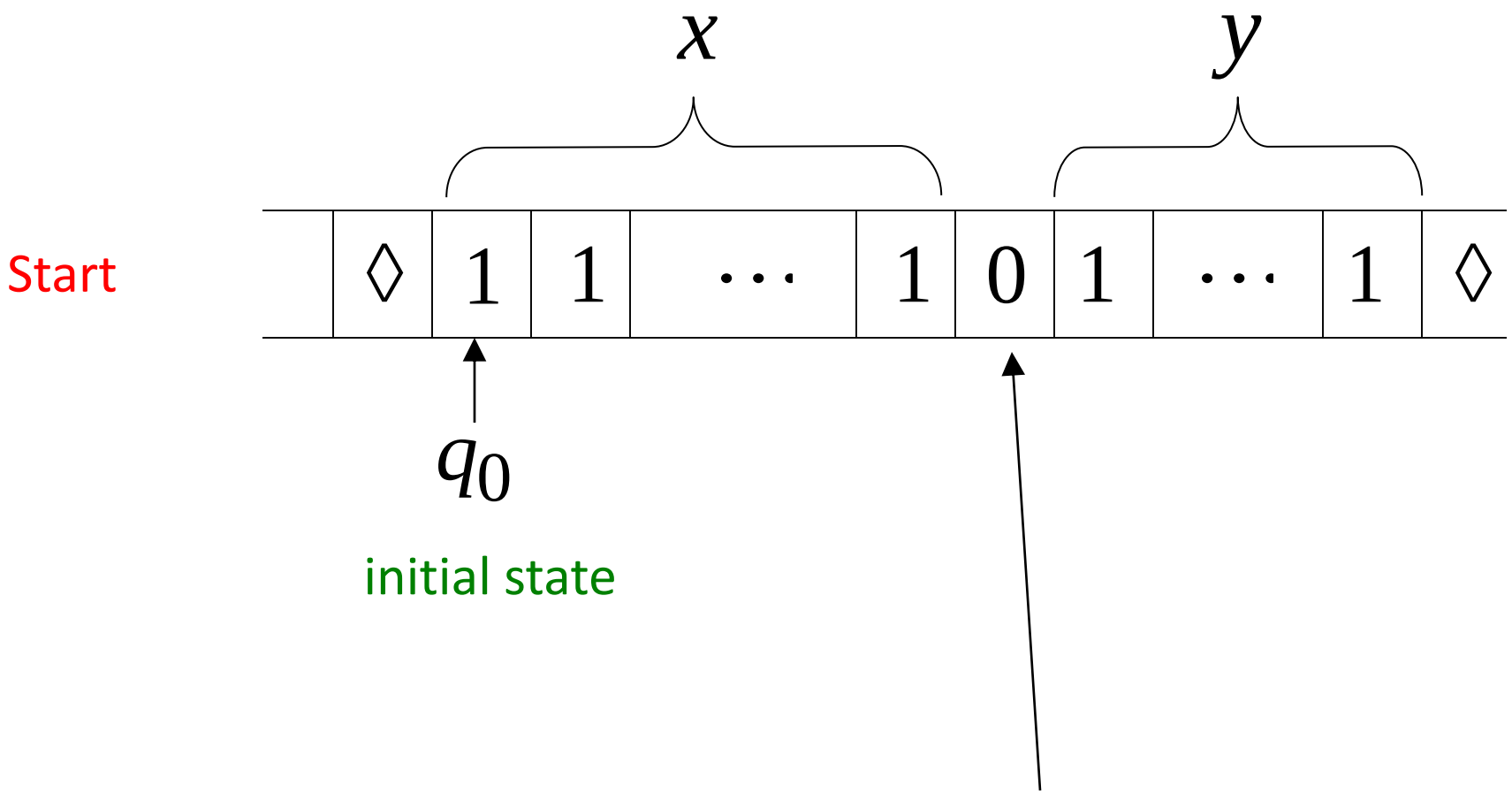
The function $f(x, y) = x + y$ is computable

x, y are integers

Turing Machine:

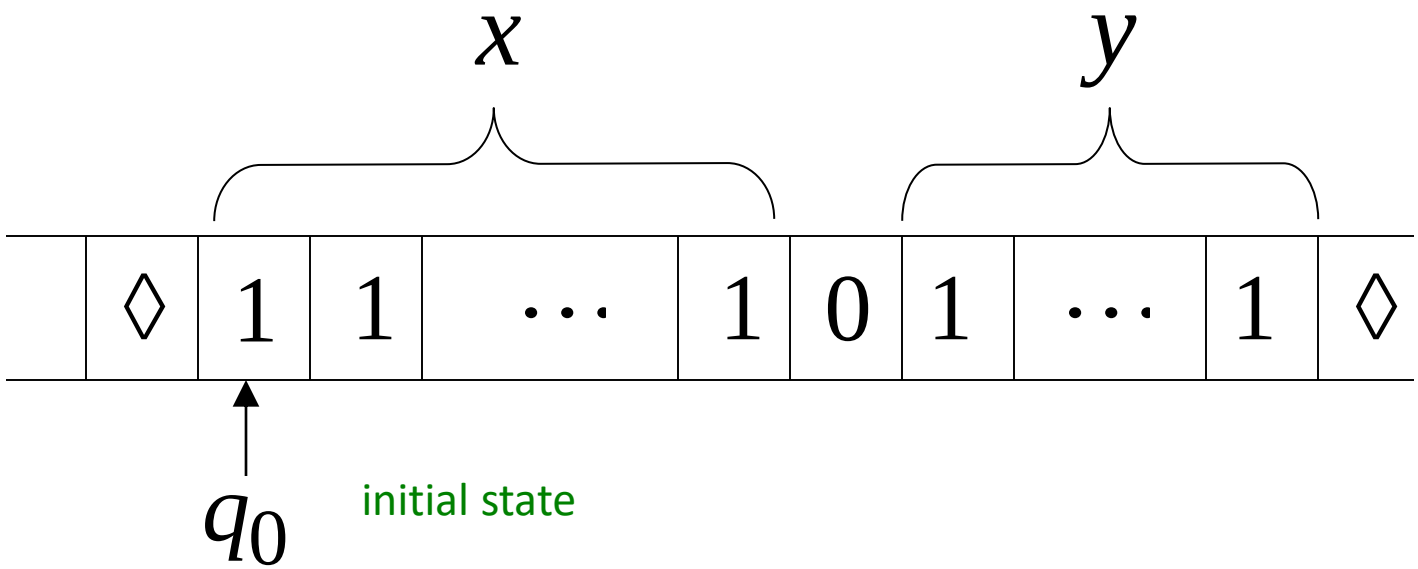
Input string: $x0y$ unary

Output string: $xy0$ unary

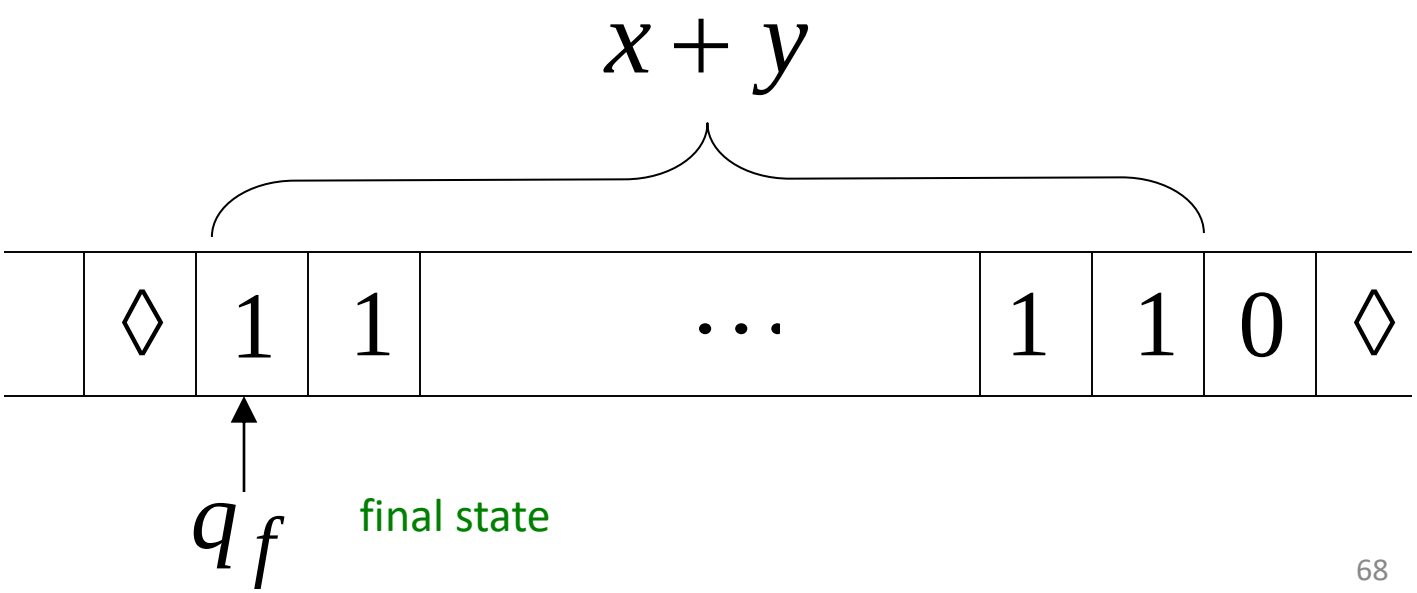


The 0 is the delimiter that separates the two numbers

Start

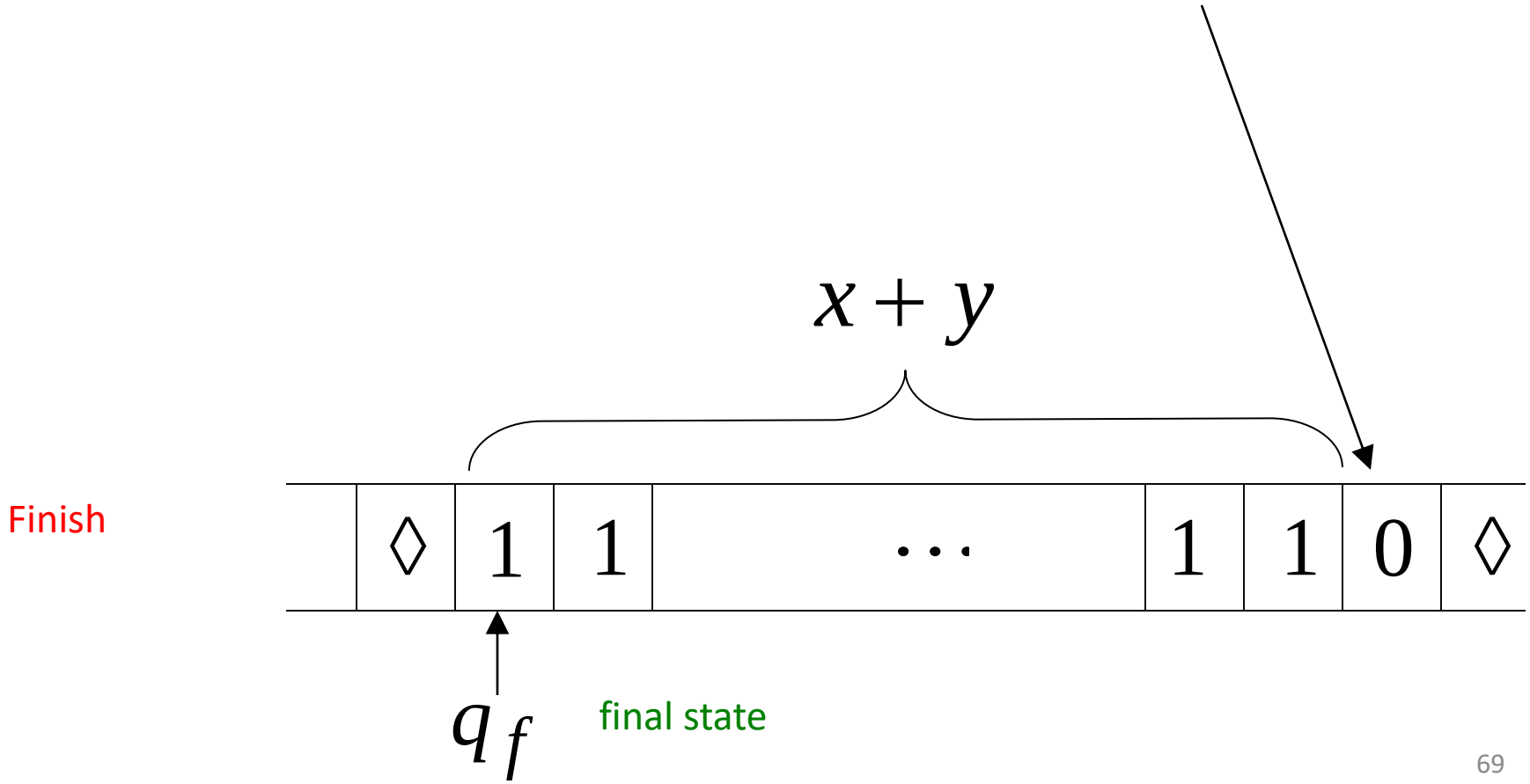


Finish



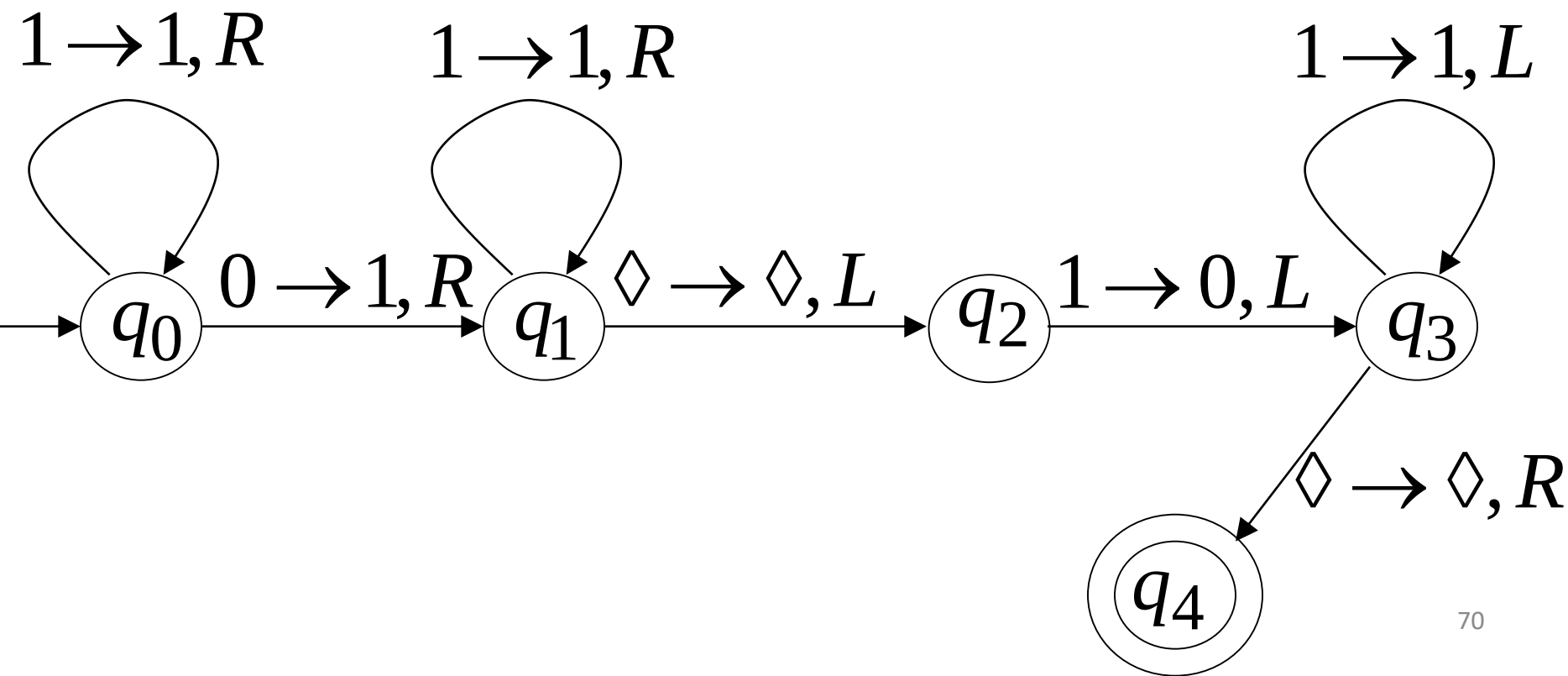
Computing Functions with Turing Machines…

The 0 here helps when we use the result for other operations



Computing Functions with Turing Machines…

Turing machine for function $f(x, y) = x + y$



Execution Example:

Time 0

$x = 11$ (=2)

$y = 11$ (=2)

		x			y	
$\Diamond$	1	1	0	1	1	$\Diamond$
	$\uparrow$					
	q_0					

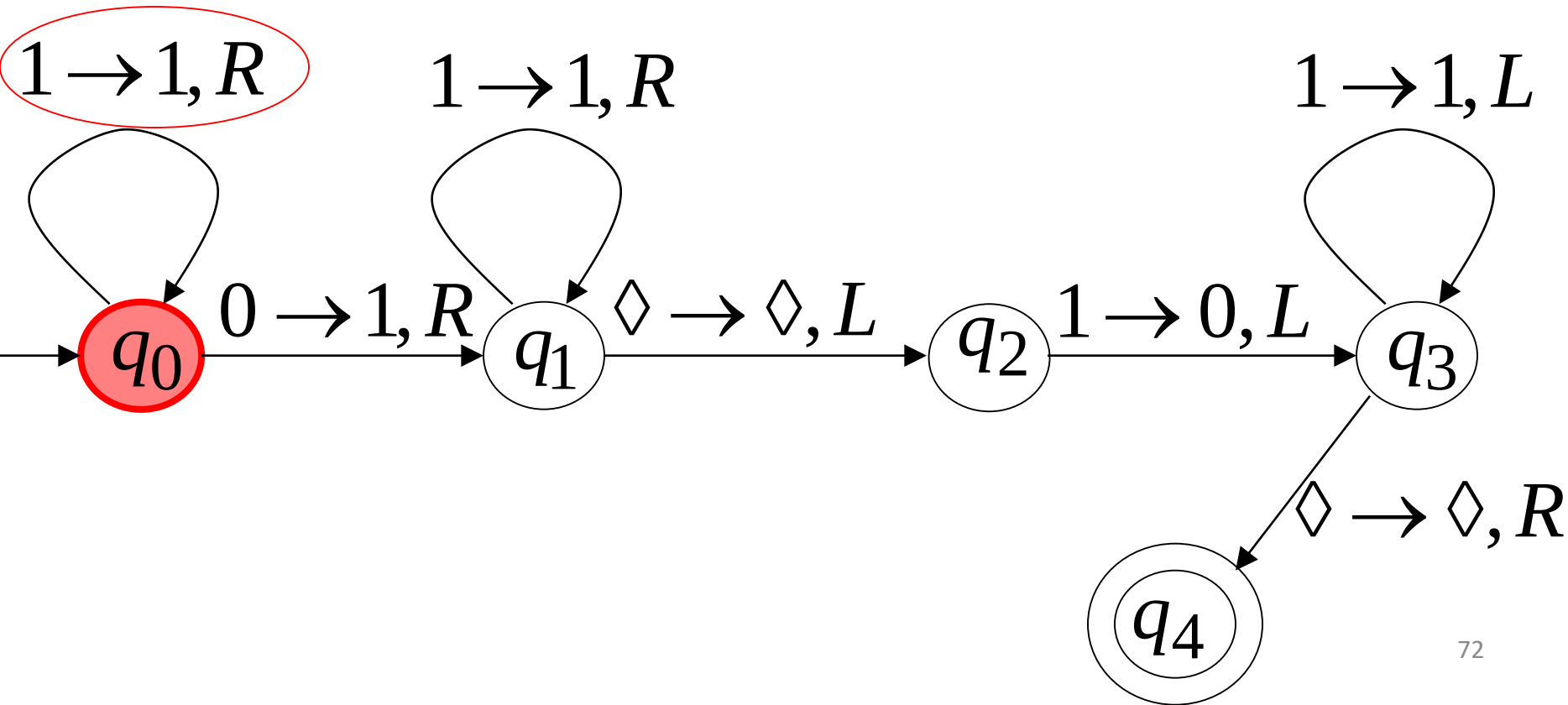
Final Result

		$x + y$				
$\Diamond$	1	1	1	1	0	$\Diamond$
	$\uparrow$					
	q_4					

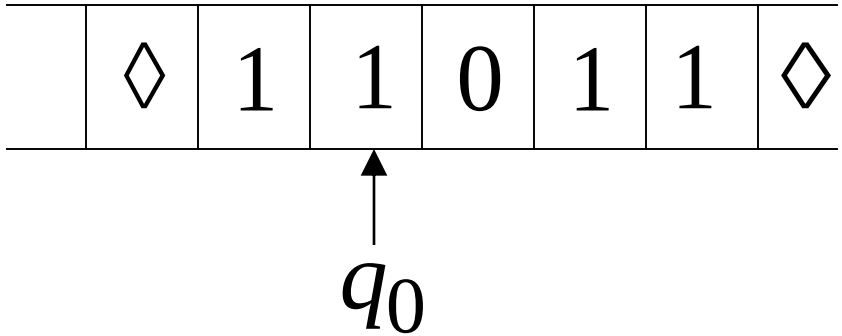
Time 0

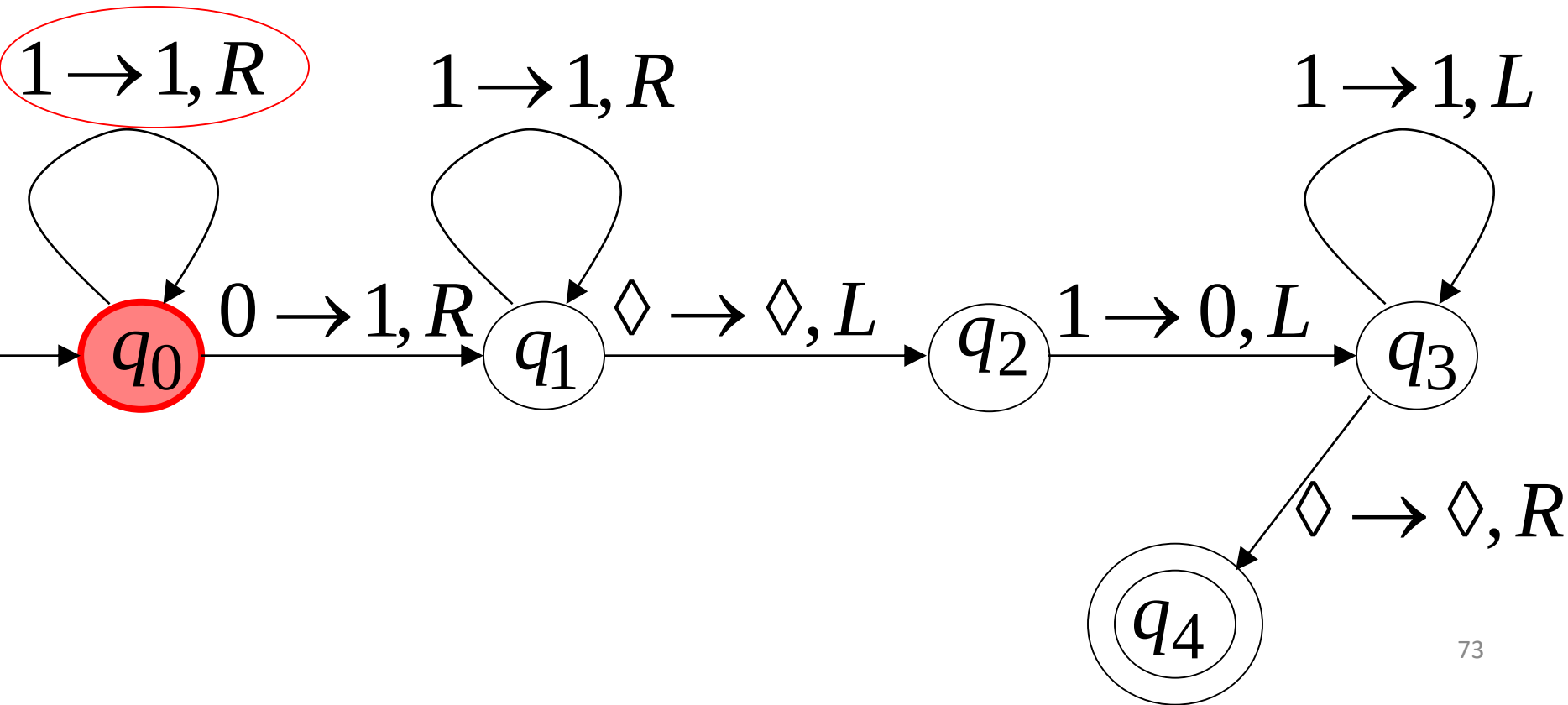
	$\Diamond$	1	1	0	1	1	$\Diamond$

q_0

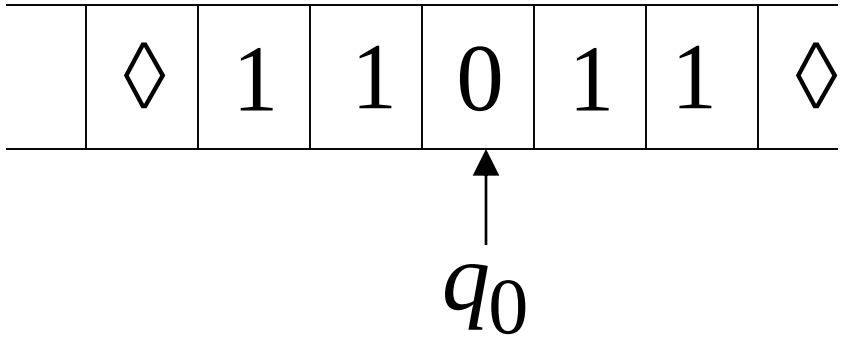


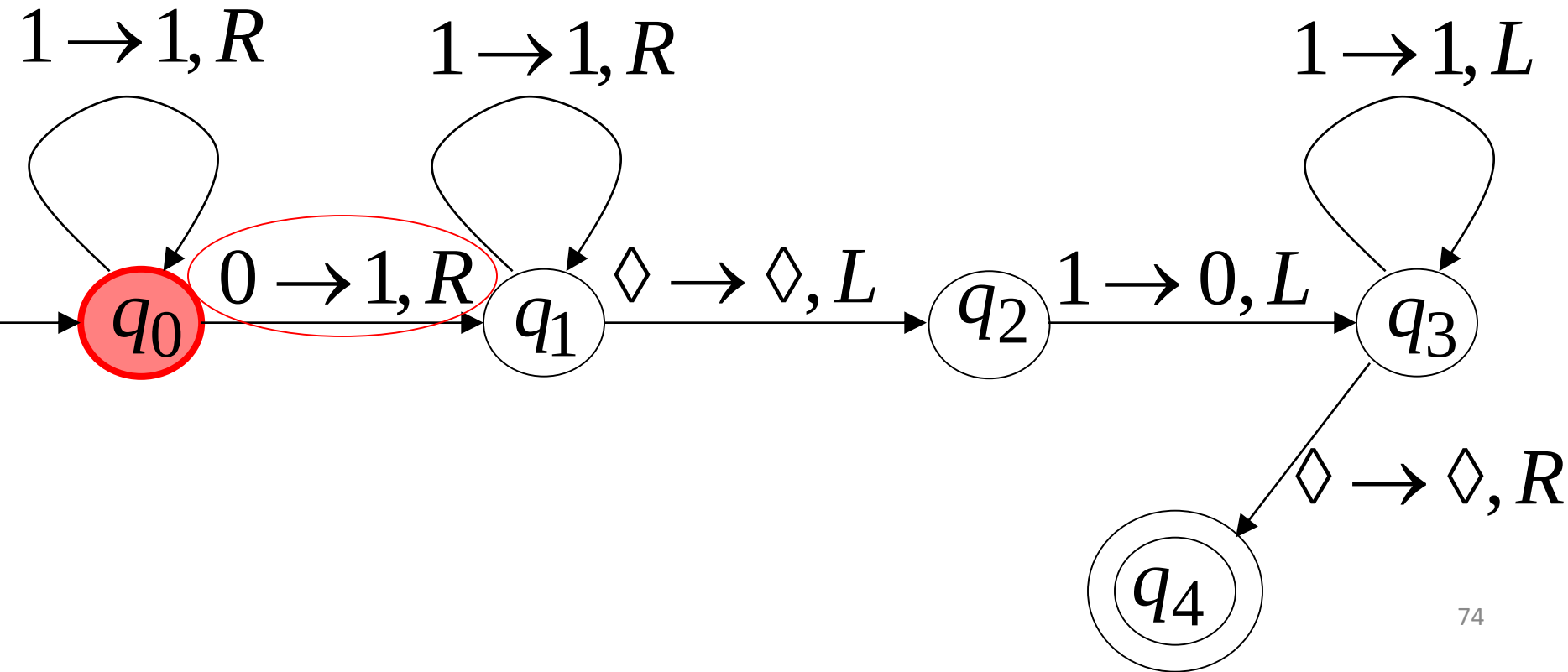
Time 1


◊ 1 1 0 1 1 ◊
q_0

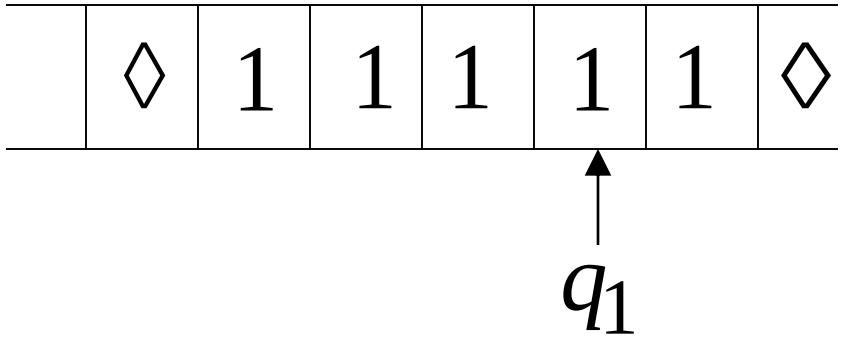

$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

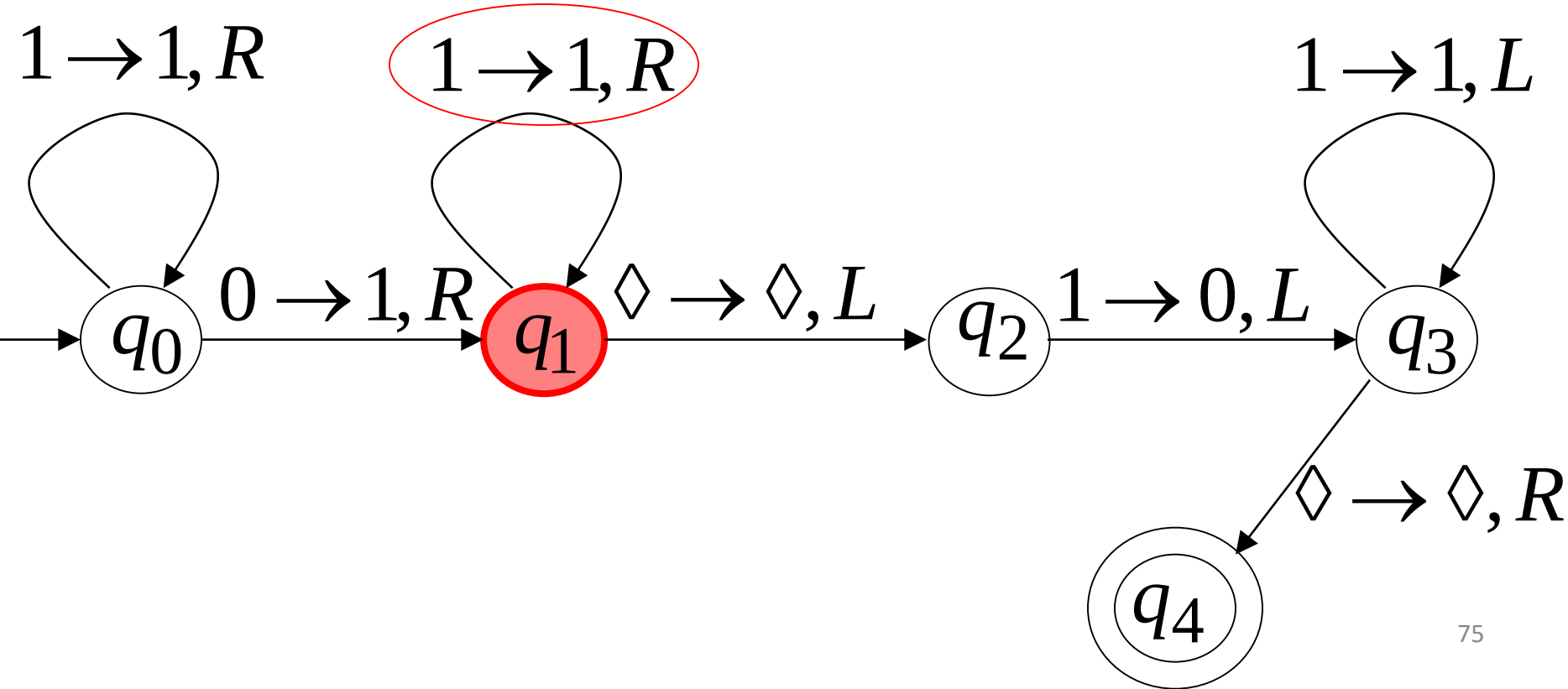
Time 2


◊ 1 1 0 1 1 ◊
q_0


$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 3

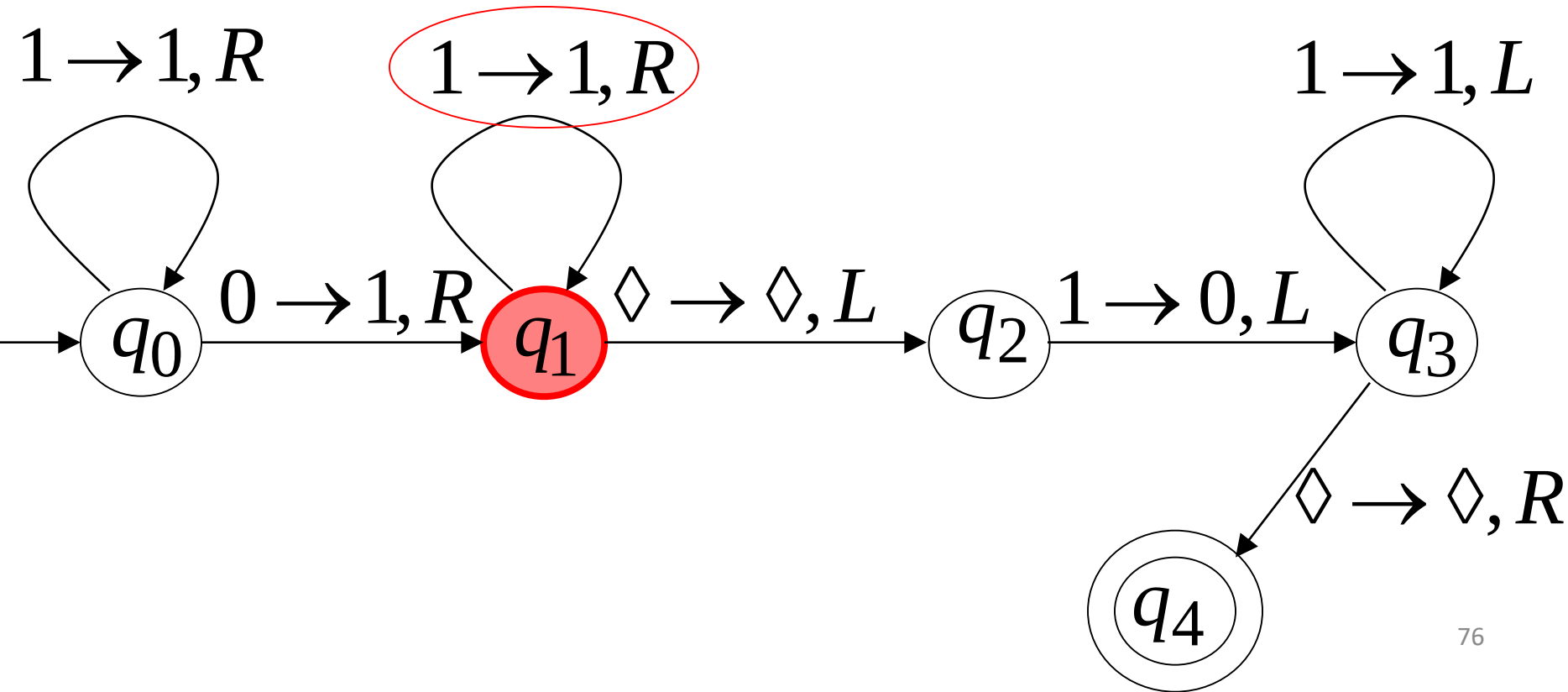

◊ 1 1 1 1 1 ◊
q_1


$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

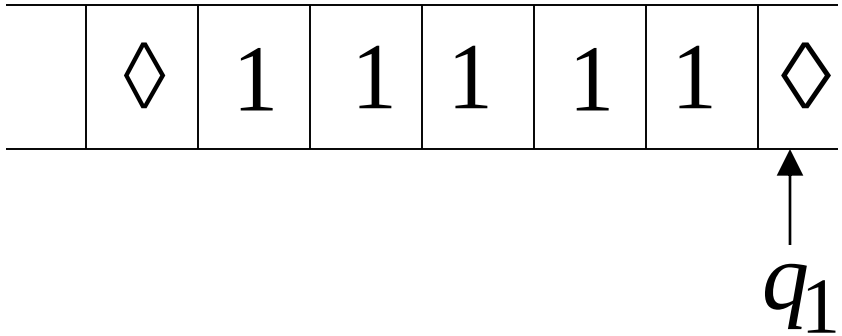
Time 4

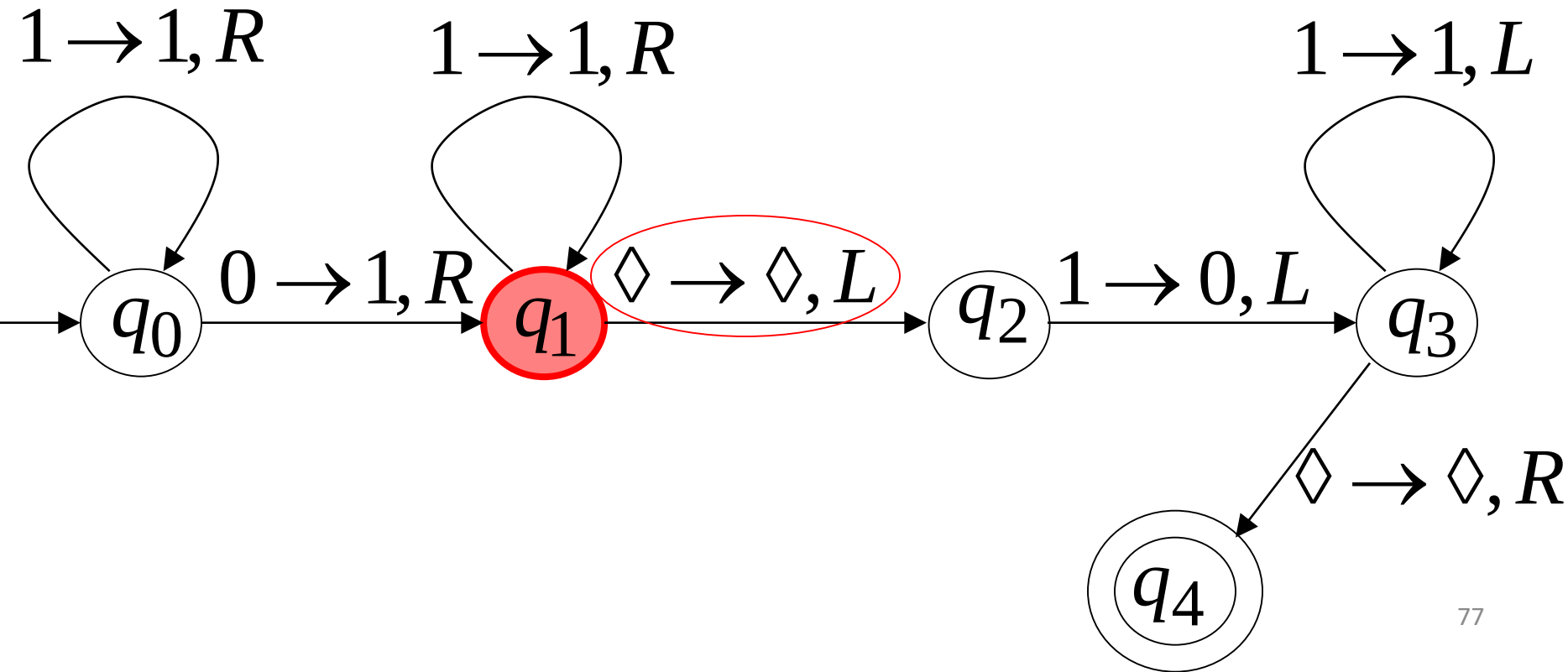
	$\Diamond$	1	1	1	1	1	$\Diamond$

q_1

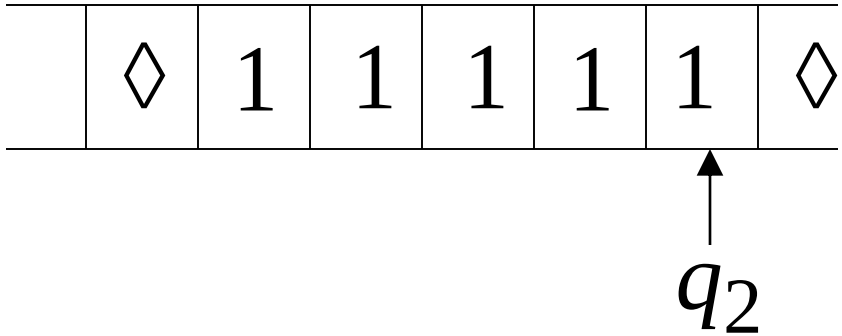


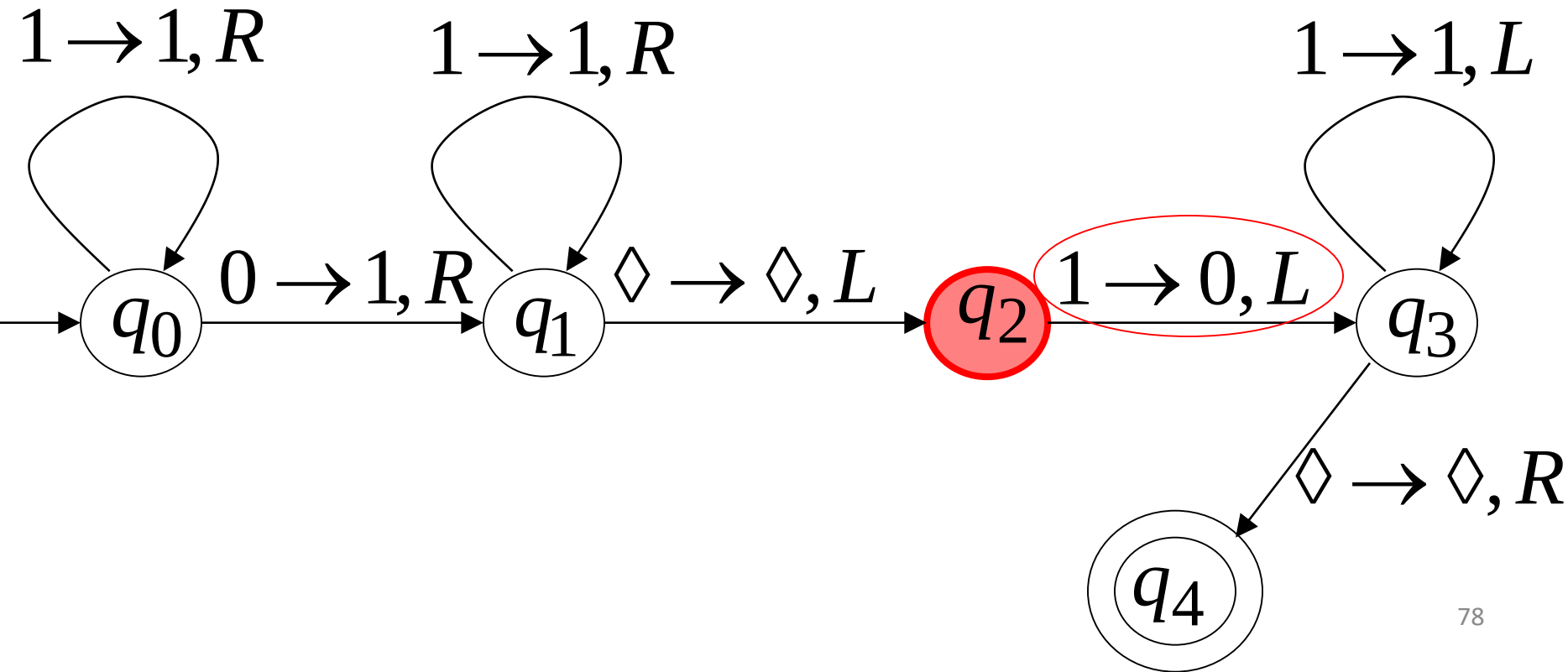
Time 5


◊ 1 1 1 1 1 ◊
q_1


$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 6

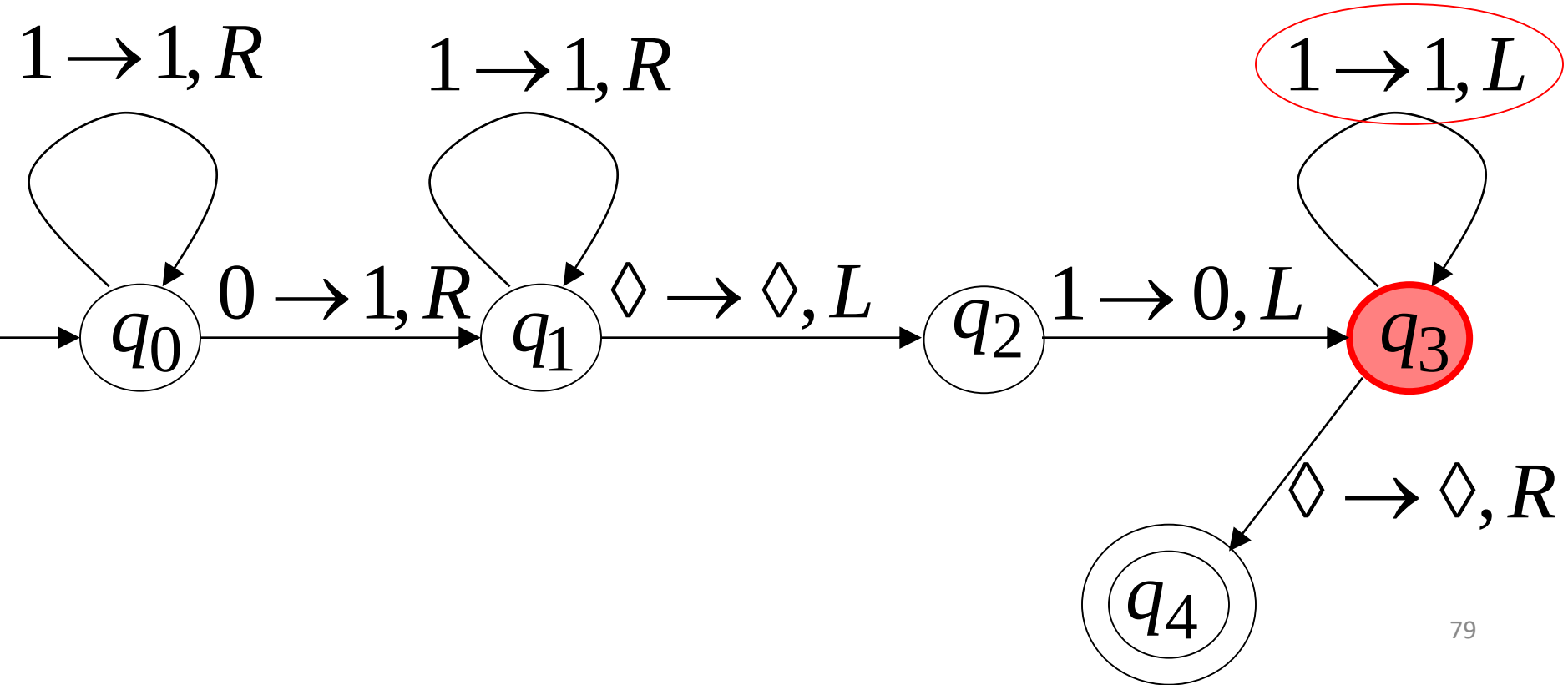

◊ 1 1 1 1 1 ◊
q_2


$1 \to 1, R$
$1 \to 1, R$
$1 \to 1, L$
$0 \to 1, R$
$\Diamond \to \Diamond, L$
$1 \to 0, L$
$\Diamond \to \Diamond, R$
q_0
q_1
q_2
q_3
q_4

Time 7

	$\Diamond$	1	1	1	1	0	$\Diamond$

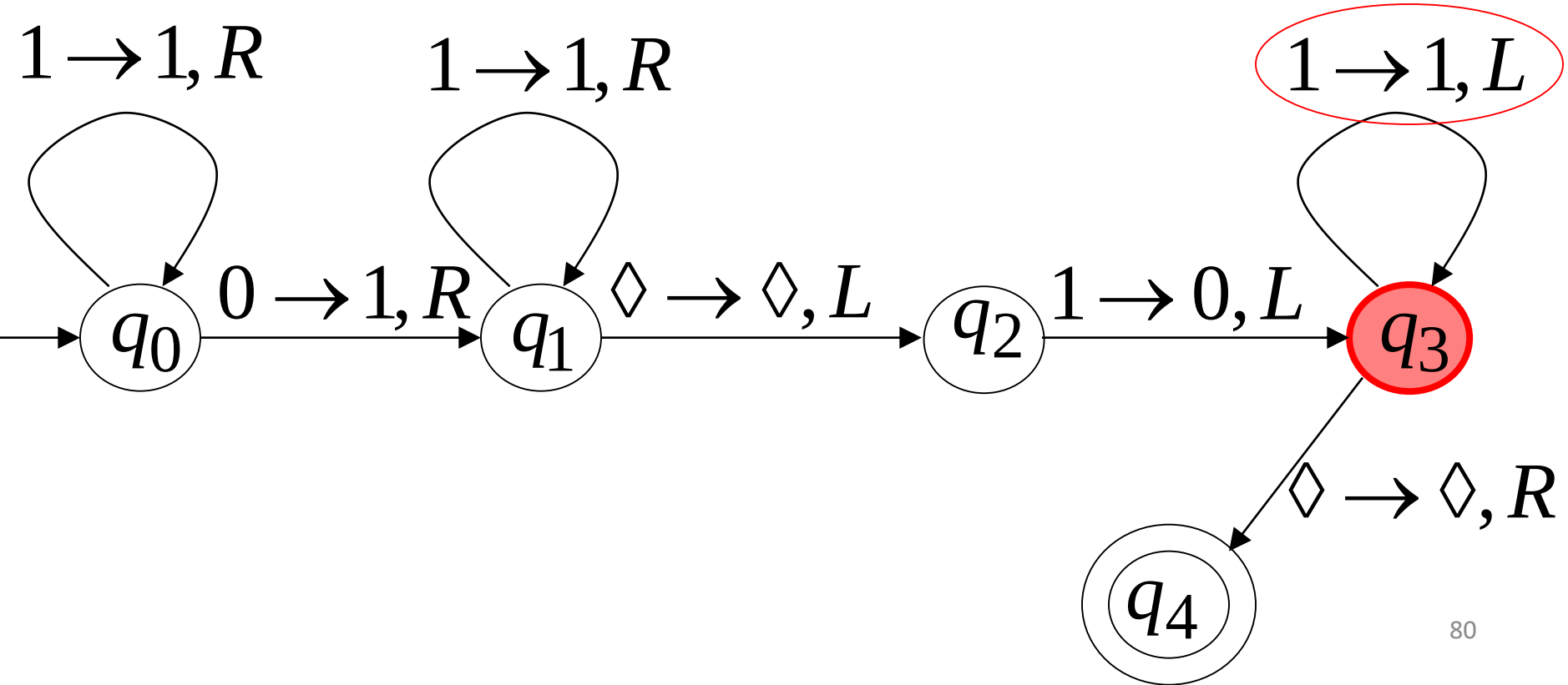
q_3



Time 8

	$\Diamond$	1	1	1	1	0	$\Diamond$

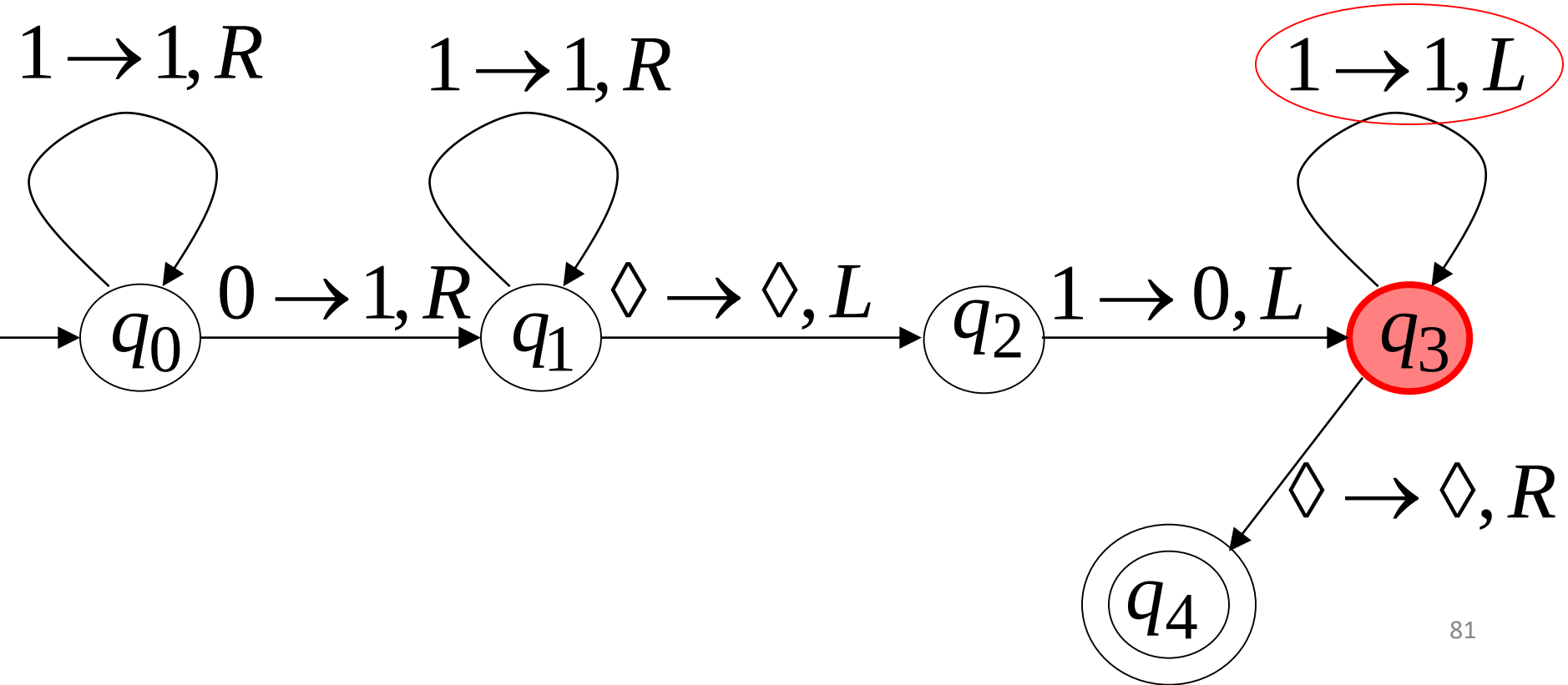
q_3



Time 9

	$\Diamond$	1	1	1	1	0	$\Diamond$

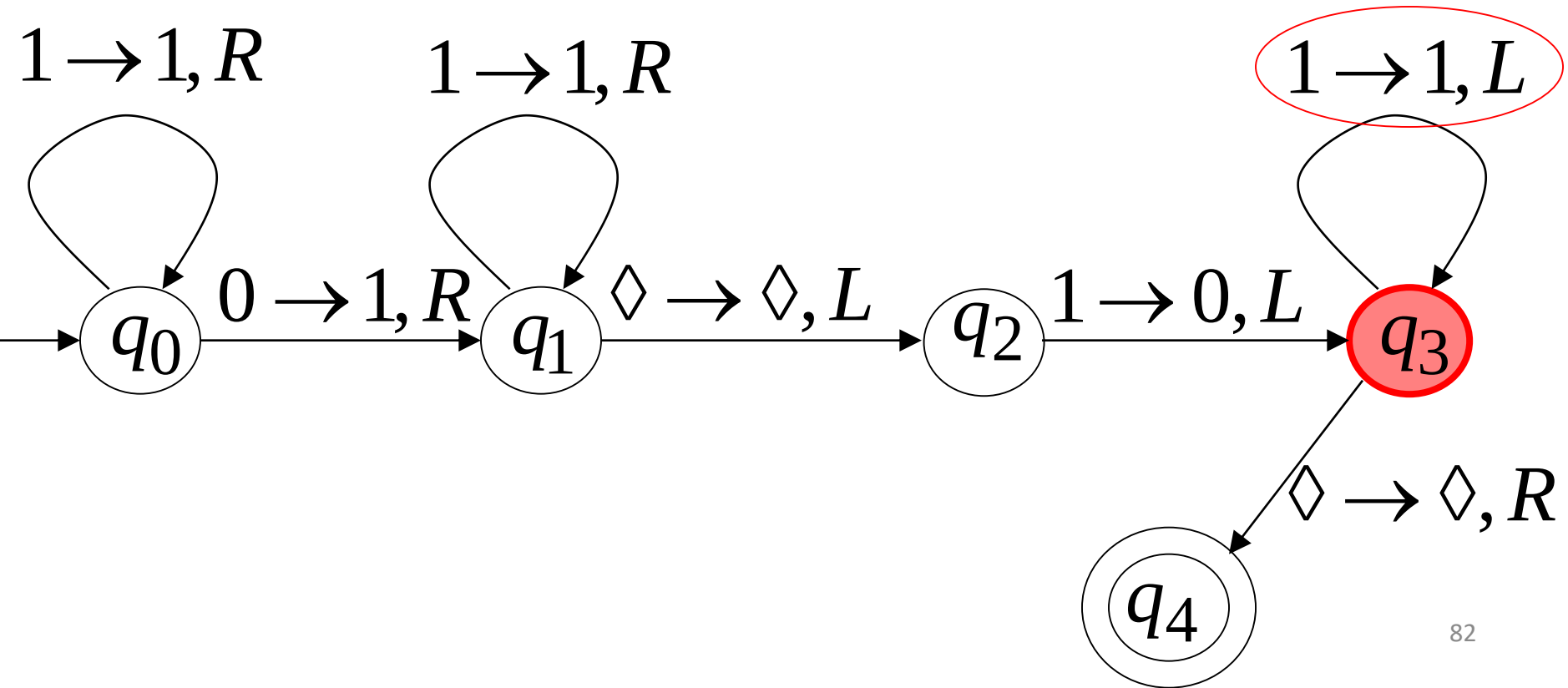
q_3



Time 10

	$\Diamond$	1	1	1	1	0	$\Diamond$

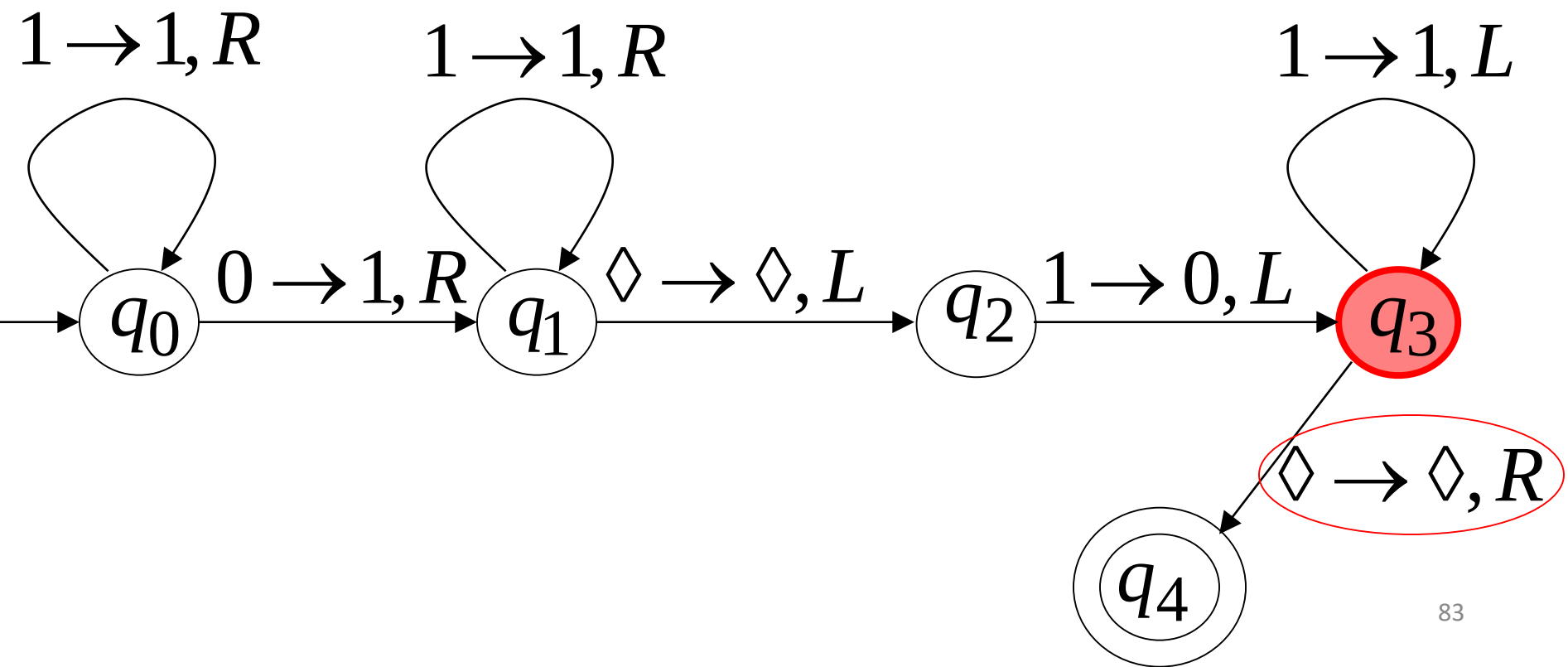
q_3



Time 11

	$\Diamond$	1	1	1	1	0	$\Diamond$

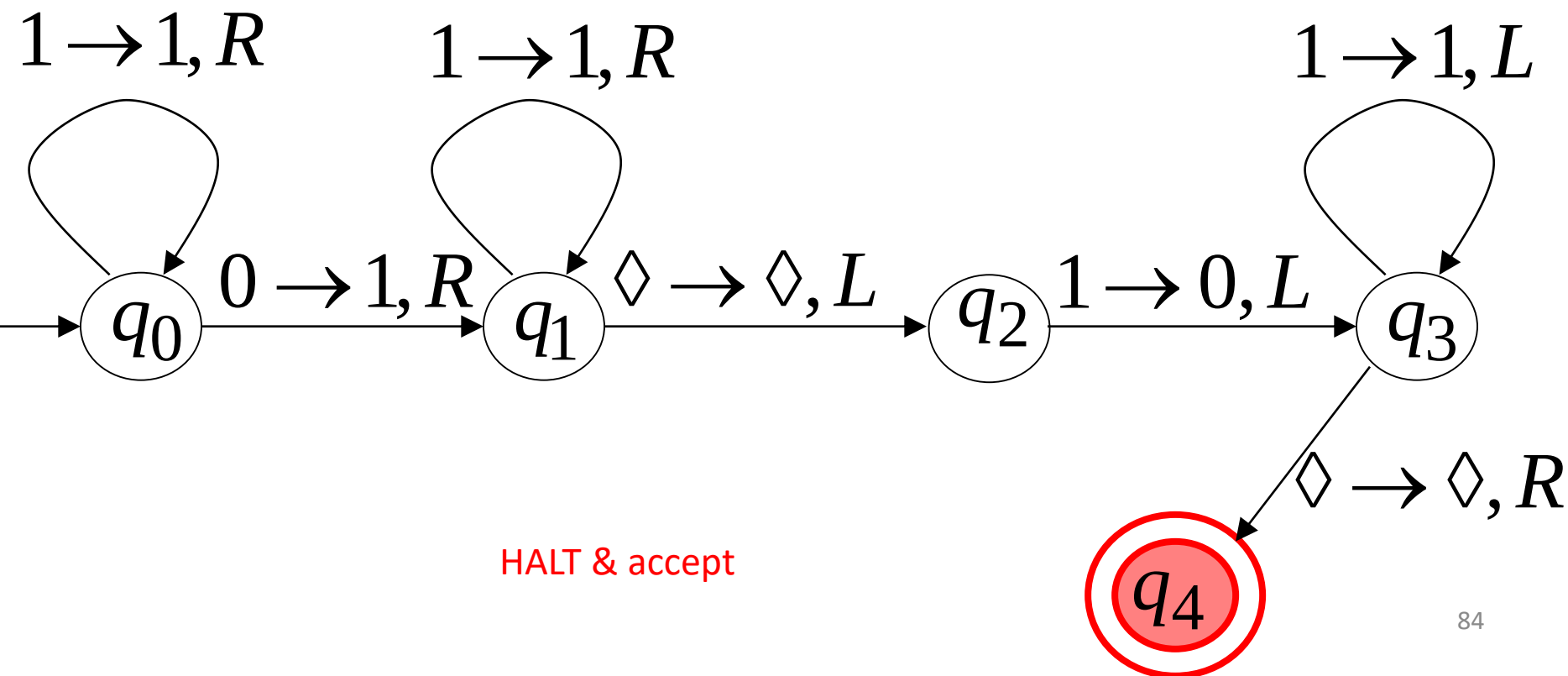
q_3



Time 12

	$\Diamond$	1	1	1	1	0	$\Diamond$

q_4



HALT & accept

Exercise

The function $f(x) = 2x$ is computable.

Design a Turing Machine that computes the above function.

Reading (Self Study)

- Turing Thesis
- Universal Turing Machine.

Group Assignment: *Due date: June 7, 2022*

- Form a ***group of five members*** and prepare a report **not more than five pages** on the following topics.

- **Complexity Theory**
  - ***Introduction***
  - ***Polynomial- Time Algorithm***
  - ***Non- Deterministic Polynomial Time Algorithm***
  - ***NP Problems***

- Copying from one another is prohibited and will nullify your mark.
- Prepare a presentation for evaluation.
- **Submission**: The section representatives will collect from all groups and submit a zipped file which contains all groups assignment report.