Assignment 1 BSE666

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Question 1(e)

The fixed-point iteration converges to the unique solution for all positive initial guesses. For non-positive initial guesses, the logarithm is undefined in the real domain, and the method does not converge in the real domain. Even for values that are very close to 0 we observe that the values become imaginary.

Question 1(f)

We aim to prove that

$$\frac{e^{(n+1)}}{e^{(n)}} \approx |F'(Q^*)|$$

The fixed-point iteration is given by

$$Q^{n+1} = F(Q^n),$$

and the true solution satisfies

$$Q^* = F(Q^*).$$

Define the error at the n-th iteration as

$$e^{(n)} = |Q^n - Q^*|.$$

Using the first-order Taylor series expansion of $F(Q^n)$ about Q^*,

$$F(Q^n) \approx F(Q^*) + F'(Q^*)(Q^n - Q^*).$$

Since $F(Q^*) = Q^*$, we obtain

$$Q^{n+1} - Q^* = F'(Q^*)(Q^n - Q^*).$$

Taking absolute values,

$$\frac{|Q^{n+1} - Q^*|}{|Q^n - Q^*|} \approx |F'(Q^*)|.$$

Thus,

$$\frac{e^{(n+1)}}{e^{(n)}} \approx |F'(Q^*)|.$$

Convergence Behavior

- If $|F'(Q^*)| < 1$: the fixed point is stable and the iteration converges.
- If $|F'(Q^*)| = 1$: the error remains nearly constant, leading to very slow or oscillatory convergence.
- If $|F'(Q^*)| > 1$: the fixed point is unstable and the iteration diverges.

Question 2

(a) Analytical Solution

Consider the ODE

$$\frac{dT}{dt} + T = 0, \qquad T(0) = 1, \qquad 0 \le t \le 1.$$

Separating variables,

$$\frac{1}{T}\,dT = -dt.$$

Integrating,

$$\int \ln T = \int -t,$$

which gives

$$\ln T = -t.$$

Hence,

$$T(t) = e^{-t}.$$

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(b) Time Discretization

Let

$$\Delta t = 0.1,\ 0.05,\ 0.01, \qquad t^n = n\Delta t, \qquad N = \frac{1}{\Delta t}.$$

Explicit Euler Method

The scheme is

$$\frac{T^{n+1} - T^n}{\Delta t} + T^n = 0,$$

which gives

$$T^{n+1} = T^n(1 - \Delta t).$$

Implicit Euler Method

The scheme is

$$\frac{T^{n+1} - T^n}{\Delta t} + T^{n+1} = 0,$$

which yields

$$T^{n+1} = \frac{T^n}{1 + \Delta t}.$$

Crank–Nicolson Method

The scheme is

$$\frac{T^{n+1} - T^n}{\Delta t} + \frac{T^n + T^{n+1}}{2} = 0.$$

Rearranging,

$$T^{n+1}(2 + \Delta t) = T^n(2 - \Delta t),$$

or

$$\frac{T^{n+1}}{T^n} = \frac{2 - \Delta t}{2 + \Delta t} = \frac{1 - \Delta t/2}{1 + \Delta t/2}.$$

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(c) Error Analysis

From the exact solution,

$$\frac{T^{n+1}}{T^n} = \frac{e^{-t^{n+1}}}{e^{-t^n}} = e^{-\Delta t} = 1 - \Delta t + \frac{(\Delta t)^2}{2} - \frac{(\Delta t)^3}{6} + \cdots$$

Explicit Euler

$$\frac{T^{n+1}}{T^n} = 1 - \Delta t.$$

Error term up to third order:

$$E_{\text{explicit}} = \left|(1 - \Delta t) - \left(1 - \Delta t + \frac{(\Delta t)^2}{2} - \frac{(\Delta t)^3}{6}\right)\right| = \left|\frac{(\Delta t)^2}{2} - \frac{(\Delta t)^3}{6}\right|.$$

Implicit Euler

$$\frac{T^{n+1}}{T^n} = \frac{1}{1 + \Delta t}.$$

Using Taylor expansion,

$$\frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots,$$

hence

$$\frac{T^{n+1}}{T^n} = 1 - \Delta t + (\Delta t)^2 - (\Delta t)^3 + \cdots$$

Error term:

$$E_{\text{implicit}} = \left| \frac{(\Delta t)^2}{2} - \frac{5(\Delta t)^3}{6} \right|.$$

Crank–Nicolson

$$\frac{T^{n+1}}{T^n} = \frac{1 - \Delta t/2}{1 + \Delta t/2}.$$

Using Taylor series,

$$\frac{T^{n+1}}{T^n} = 1 - \Delta t + \frac{(\Delta t)^2}{2} - \frac{(\Delta t)^3}{4} + \cdots$$

Error term:

$$E_{\text{CN}} = \left| \frac{(\Delta t)^2}{12} \right|.$$

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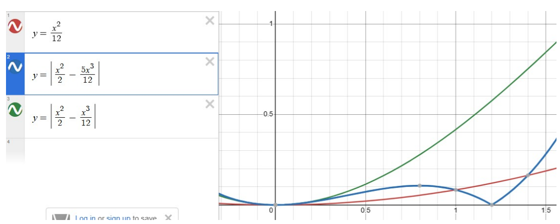
Conclusion

The Crank–Nicolson (CN) method is likely to be the most accurate among the three methods. From the Taylor expansion up to third order, we see that the error term for the CN method is smaller than those of the explicit and implicit methods for all real values of Δt, except for a small interval $\Delta t \in [1, 1.4]$. As we include higher-order terms in the Taylor expansion, this interval reduces further. From this we can say that CN is generally a more accurate method



Question 3(a)

The governing equation is

$$\frac{dh}{dt} + \sqrt{h} = q(t), \qquad h(0) = H.$$

For the tank-emptying case, there is no inflow:

$$q(t) = 0.$$

Hence,

$$\frac{dh}{dt} + \sqrt{h} = 0.$$

Separating variables,

$$\int_H^{h(t)} \frac{dh}{\sqrt{h}} = -\int_0^t dt.$$

Evaluating the integrals,

$$2\sqrt{h(t)} - 2\sqrt{H} = -t.$$

Rearranging,

$$2\sqrt{h(t)} = 2\sqrt{H} - t,$$

$$\sqrt{h(t)} = \sqrt{H} - \frac{t}{2}.$$

For $H = 2$,

$$h(t) = \left(\sqrt{2} - \frac{t}{2}\right)^2.$$

Expanding,

$$h(t) = \frac{t^2}{4} - \sqrt{2}\,t + 2,$$

valid for

$$0 \le t \le 2\sqrt{2}.$$

Question 3(b): Choice of Time Step

The governing equation behaves as

$$\frac{dh}{dt} \sim \sqrt{h}.$$

At $h \approx H$,

$$\frac{H}{\tau} \sim \sqrt{H} \quad \Rightarrow \quad \tau \sim \sqrt{H}.$$

For $H = 2$,

$$\tau \sim \sqrt{2} \approx 1.4.$$

To resolve the dynamics accurately, we select

$$\Delta t \le 0.01\,\tau \approx 0.014.$$

Explicit

$$\frac{h^{n+1} - h^n}{\Delta t} = -\sqrt{h^n},$$

which gives

$$h^{n+1} = h^n - \Delta t\,\sqrt{h^n}.$$

Implicit

$$\frac{h^{n+1} - h^n}{\Delta t} = -\sqrt{h^{n+1}},$$

which leads to

$$h^{n+1} + \Delta t \sqrt{h^{n+1}} = h^n.$$

Let

$$y = \sqrt{h^{n+1}},$$

then

$$y^2 + \Delta t\, y - h^n = 0,$$

which can be solved at each time step.