Please submitted by 20 Jan. 2026, 11:59pm.

- Attempt all problems. There is no penalty for submitting incorrect or partially correct solutions.
- Include all codes, and a readme file containing instructions on how to run the code for each problem. Preferably, use a single script file per question.
- You may form groups of 1-3 for solving the assignments, mention the names and roll numbers of group members in the readme file.
- However, plagiarism across groups or solutions solved via AI will result in serious penalties, such as reporting to SSAC.

$\boxed{5}$ 1. Consider the formula for the pressure drop $\Delta p$ (measured in mmHg) across a tube in terms of the flow rate $Q$ (specified in mL/s):

$$\frac{\Delta p}{Q} = A\log(Q) + B.$$

where the parameters $A = 0.3$ mmHg s/mL and $B = 2.5$ mmHg s/mL. Consider $\Delta P = 15$ mmHg and answer the following questions.

(a) Plot the function $G(Q) = \frac{\Delta p}{Q} - A\log(Q) - B$ for $Q$ in range $(0.1, 10)$.

(b) Read the documentation of fzero and find the root $Q^\star$ of $G(Q)$ using fzero in the range $(0.1, 10)$.

(c) Find the solution using the fixed-point method starting at $Q^{(1)} = 1$ using the iterations $Q^{(n+1)} = \frac{\Delta p}{A\log(Q^{(n)})+B}$ for $n = 1, \ldots, 6$.

(d) Plot $|Q^n - Q^\star|$ vs. $n$ where the y-axis should be in log-scale (use semilogy in MATLAB). You should observe a straight line.

(e) Is there a starting point for which the method does not converge?

(f) Consider the fixed point iterations $Q^{(n+1)} = F(Q^n)$ for which the error at the $n$-th iteration is given by $e^{(n)} = \left|Q^{(n)} - Q^\star\right|$. Using first-order Taylor series expansion, show that

$$\frac{e^{(n+1)}}{e^{(n)}} \approx \left|F'(Q^\star)\right|$$

and comment on what might happen if $|F'(Q^\star)|$ is large.

$\boxed{6}$ 2. Consider the ordinary differential equation (ODE):

$$\frac{dT}{dt} + T = 0$$

where the initial condition is given as $T(0) = 1$. Answer the following questions:

(a) Solve the ODE analytically to find $T_\star(t)$ for $0 \le t \le 1$.

(b) Let us discretize the time with $\Delta t = 0.1, 0.05, 0.01$ so that $t^n = n\Delta t$ for $0 \le n \le N$ where $N = \frac{1}{\Delta t}$, solve the equation using the following time-marching methods:

(Explicit Euler) $$\frac{T^{(n+1)} - T^{(n)}}{\Delta t} + T^{(n)} = 0$$

(Implicit Euler) $$\frac{T^{(n+1)} - T^{(n)}}{\Delta t} + T^{(n+1)} = 0$$

(Implicit Crank-Nicolson) $$\frac{T^{(n+1)} - T^{(n)}}{\Delta t} + \frac{T^{(n)} + T^{(n+1)}}{2} = 0$$

For each of the cases, calculate the RMSE $= \sqrt{\frac{1}{N+1}\sum_{n=0}^{N}(T^{(n)} - T_\star(t^n))^2}$. Which of these method is the most accurate for this problem?

(c) Note that

$$\frac{T^{(n+1)}}{T^{(n)}} \approx \frac{e^{-\Delta t(n+1)}}{e^{-\Delta t n}} \approx 1 - \Delta t + \frac{(\Delta t)^2}{2} - \frac{(\Delta t)^3}{6} + \dots$$

Write the updates for the three approaches in the form of $\frac{T^{(n+1)}}{T^{(n)}} \approx$ polynomial in $\Delta t$. For a given (small) $\Delta t$, compare how well the three approaches (explicit, implicit, and implicit CN) approximate the right-hand side and why the third approach (implicit CN) is likely to be the better than the other two (explicit and implicit).

$\boxed{4}$ 3. The following mathematical model is obtained during the analysis of a tank-and-tube system:

$$\frac{dh}{dt} + \sqrt{h} = q(t), \qquad h(t) > 0, h(t=0) = H$$

To study the tank emptying process, let us consider the simpler case of $q(t) = 0$ and set $H = 2$ units. Answer the following:

(a) Solve the tank emptying problem analytically and provide the expression for $h(t)$.

(b) Solve the differential equation using time-marching by using the following two approximations of the square root term:

$$\text{(i)} \qquad \sqrt{h} = \sqrt{h^{(n)}}$$
$$\text{(ii)} \qquad \sqrt{h} = \sqrt{h^{(n+1)}}$$

where we have discretized the time as $t = (\Delta t)n$ for $0 \le t \le 2\sqrt{H}$. Select the time step judiciously and justify your choice (by comparing it with the time constant). Tabulate the data obtained for each method and compare with the analytical. Stop when $h(t) < H/2$ and specify the time required for the water level in the tank to fall just below $H/2$.