BSE 666 Assignment 2

Q1] Taylor Series Approximation

$f(x) = x^4 - 3x^3 + 2x^2 + 7$, $x_0 = 2$ base point

estimate $f(3)$

Ans : $f(x) = x^4 - 3x^3 + 2x^2 + 7$ | $f(x_0 = 2) = 7$

$f'(x) = 4x^3 - 9x^2 + 4x$ | $f'(x_0 = 2) = 4$

$f''(x) = 12x^2 - 18x + 4$ | $f''(x_0 = 2) = 16$

Taylor expansion about $x_0 = 2$:

$$f(x) \simeq f(2) + f'(2)\cdot(x-2) + \frac{f''(2)}{2!}(x-2)^2 \cdots$$

0^{th} order : $f(x) \simeq f(2)$

1^{st} order : $f(x) \simeq f(2) + f'(2)\cdot(x-2)$

2^{nd} order : $f(x) \simeq f(2) + f'(2)\cdot(x-2) + \frac{f''(2)}{2!}(x-2)^2$

$\therefore$ 0^{th} order $f(3)$ estimate $= f(2) = 7$

1^{st} order $f(3)$ estimate $= f(2) + f'(2)\cdot(1) = 11$

2^{nd} order $f(3)$ estimate $= f(2) + f'(2) + \frac{f''(2)}{2} = 19$

exact $f(3) = 25$

Approximation	f(3) estimate	f(3) exact	% error
zeroth	7	25	72%
Linear	11	25	56%
Quadratic	19	25	24%

Derivative Appoximations

0^{th} order $f(x) \simeq f(2)$ $\therefore$ Derivative $f'(x) \simeq 0$

1^{st} order $f(x) \simeq f(2) + f'(2)\cdot(x-2)$ $\therefore$ Derivative $f'(x) \simeq f'(2)$

2^{nd} order $f(x) \simeq f(2) + f'(2)\cdot(x-2) + \frac{f''(2)}{2!}(x-2)$ $\therefore$ Derivative $f'(x) \simeq f'(2) + f''(2)\cdot(x-2)$

exact $f'(3) = 4(3)^3 - 9(3)^2 + 4(3) = 39$

Approximation	$f'(3)$ estimate	exact $f'(3)$	% error
zeroth	0	39	100
Linear	$f'(2) = 4$	39	89.7%
Quadratic	$f'(2) + f''(2)\cdot(3-2) = 20$	39	48.7%

Q2) Newton Raphson method, initial (1, 1, 1), 4 iterations

$f_1 : x_1 - x_2 - x_3 = 0$

$f_2 : 0 + x_2^2 - x_3^2 = 0$

$f_3 : x_1^2 + x_2^2 + 2x_3^2 = 7$

Ans: Let $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $\bar{F}(\bar{x}) = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$

Jacobian $J(\bar{x}) = \frac{\partial F}{\partial \bar{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2x_2 & -2x_3 \\ 2x_1 & 2x_2 & 4x_3 \end{bmatrix}$

$J(x^{(k)}) \cdot \Delta x^{(k)} = F(x^{(k)})$, $x^{(k+1)} = x^{(k)} - \Delta x^{(k)}$

$\boxed{0^{th} \text{ iter}}$: $x^{(0)} = (1, 1, 1)$

$F(x^{(0)}) = \begin{bmatrix} 1-1-1 \\ 0+1-1 \\ 1+1+2(1)^2-7 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}$ $\|F\|_{rms} = \sqrt{\frac{f_1^2 + f_2^2 + f_3^2}{3}}$

$= 1.825$

$J(x^{(0)}) = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & -2 \\ 2 & 2 & 4 \end{bmatrix}$

Solve $J \cdot \Delta x = F$: $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2 & -2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \Delta x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix}$

get $\Delta x^{(0)} = \begin{bmatrix} -1.2 \\ -0.1 \\ -0.1 \end{bmatrix}$ update $x^{(1)} = x^0 - \Delta x^{(0)} = \begin{bmatrix} 2.2 \\ 1.1 \\ 1.1 \end{bmatrix}$

$\boxed{\text{Iter}(1)}$: $x^{(1)} = \begin{bmatrix} 2.2 \\ 1.1 \\ 1.1 \end{bmatrix}$ $F(x^{(1)}) = \begin{bmatrix} 0 \\ 0 \\ 1.47 \end{bmatrix}$ $J(x^{(1)}) = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2.2 & -2.2 \\ 4.4 & 2.2 & 4.4 \end{bmatrix}$

solve $J \Delta x^{(1)} = F$: $\begin{bmatrix} 1 & -1 & -1 \\ 0 & 2.2 & -2.2 \\ 4.4 & 2.2 & 4.4 \end{bmatrix} \begin{bmatrix} \Delta x_1^{(1)} \\ \Delta x_2^{(1)} \\ \Delta x_3^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1.47 \end{bmatrix}$

get $\Delta x^{(1)} = \begin{bmatrix} 0.1909 \\ 0.0954 \\ 0.0954 \end{bmatrix}$ $\therefore x^{(2)} = x^{(1)} - \Delta x^{(1)}$

$$= \begin{bmatrix} 2.0090 \\ 1.0045 \\ 1.0045 \end{bmatrix}$$

$\boxed{\text{Iter}(2)}$: $x^{(2)} = \begin{bmatrix} 2.0090 \\ 1.0045 \\ 1.0045 \end{bmatrix}$ $F(x^{(2)}) = \begin{bmatrix} 0 \\ 0 \\ 0.0637 \end{bmatrix}$ $J(x^{(2)}) = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 2.009 & -2.009 \\ 4.0181 & 2.009 & 4.0181 \end{bmatrix}$

$J \Delta x^{(2)} = F$ $\therefore$ $\Delta x^{(2)} = \begin{bmatrix} 0.0090 \\ 0.0045 \\ 0.0045 \end{bmatrix}$

$x^{(3)} = x^{(2)} - \Delta x^{(2)} = \begin{bmatrix} 2.000 \\ 1.000 \\ 1.000 \end{bmatrix}$

iter	x1	x2	x3	f1	f2	f3	RMS(F)	dx1	dx2	dx3	RMS(dX)	x1new=x1-dx1	x2new=x2-dx2	x3new=x3-dx3
0	1	1	1	-1	0	-3	1.825741858	-1.2	-0.1	-0.1	0.697614985	2.2	1.1	1.1
1	2.2	1.1	1.1	0	0	1.47	0.848704896	0.190909091	0.095454545	0.095454545	0.134993113	2.009090909	1.004545455	1.004545
2	2.009091	1.004545	1.004545	0	0	0.063780992	0.036823973	0.009070341	0.004535171	0.004535171	0.0064137	2.000020568	1.000010284	1.00001
3	2.000021	1.00001	1.00001	0	0	0.000143974	8.31237E-05	2.05676E-05	1.02838E-05	1.02838E-05	1.45435E-05	2	1	1
4	2	1	1	0	0	7.40294E-10	4.27409E-10							

P	Q	R	S	T	U	V	W	X	Y	Z	AA
	JACOBIAN iter 0				**F iter0**		**J INVERSE iter0**				**dX=J.inv*F iter0**
	1	-1	-1		-1		0.6	0.1	0.2		-1.2
	0	2	-2		0		-0.2	0.3	0.1		-0.1
	2	2	4		-3		-0.2	-0.2	0.1		-0.1
	JACOBIAN ITER 1				**F iter1**		**J INVERSE iter1**				**dX=J.inv*F iter1**
	1	-1	-1		0		0.428571	0.064935	0.12987		0.190909091
	0	2.2	-2.2		0		-0.28571	0.25974	0.064935		0.095454545
	4.4	2.2	4.4		1.47		-0.28571	-0.19481	0.064935		0.095454545
	JACOBIAN ITER 2				**F iter2**		**J INVERSE iter2**				**dX=J.inv*F iter2**
	1	-1	-1		0		0.428571	0.071105	0.142211		0.009070341
	0	2.009091	-2.00909		0		-0.28571	0.284421	0.071105		0.004535171
	4.018182	2.009091	4.018182		0.063781		-0.28571	-0.21332	0.071105		0.004535171
	JACOBIAN ITER 3				**F iter3**		**J INVERSE iter3**				**dX=J.inv*F iter3**
	1	-1	-1		0		0.428571	0.071428	0.142856		2.05676E-05
	0	2.000021	-2.00002		0		-0.28571	0.285711	0.071428		1.02838E-05
	4.000041	2.000021	4.000041		0.000144		-0.28571	-0.21428	0.071428		1.02838E-05

Q3) $\frac{d^2T}{dx^2} - 2T^4 = 0 \quad T(0) = 1 \quad \frac{dT}{dx}(x=1) = 0$

5 equally spaced grid points on $0 \le x \le 1$

$x_1 = 0, \; x_2 = 0.25, \; x_3 = 0.5, \; x_4 = 0.75, \; x_5 = 1$

step $h = \frac{1-0}{5-1} = 0.25, \quad \frac{1}{h^2} = 16$

unknown temperatures $T_1, \dots T_5$

boundary conditions $T_1 = T(0) = 1$

neumann insulated tip $\left.\frac{dT}{dx}\right|_{x=1} = 0$

backward difference at $x_5 = 1$: $\frac{T_5 - T_4}{h} = 0 \quad \therefore T_5 = T_4$

3 independent unknowns T_2, T_3, T_4

discretize ODE:

$$\frac{T_{i+1} - 2T_i + T_{i-1}}{h^2} - 2T_i^4 = 0 \quad \text{with } \frac{1}{h^2} = 16$$

$i = \{2, 3, 4\}$

$\therefore$ node 2: $16(T_3 - 2T_2 + T_1) - 2T_2^4 = 0$

node 3: $16(T_4 - 2T_3 + T_2) - 2T_3^4 = 0$

node 4: $16(T_5 - 2T_4 + T_3) - 2T_4^4 = 0$

but $T_5 = T_4 \; \therefore \Rightarrow 16(T_3 - T_4) - 2T_4^4 = 0$

non linear system

linearize (1st order TSA)

put $\rightarrow \left(T_i^{(k+1)}\right)^4 \approx \left(T_i^{(k)}\right)^4 + 4\left(T_i^{(k)}\right)^3\left(T_i^{(k+1)} - T_i^{(k)}\right)$ in T_i^4

$\therefore$ node 2: $-16T_1 - 6\left(T_2^{(k)}\right)^4$

node 3: $-6\left(T_3^{(k)}\right)^4$

node 4: $-6\left(T_4^{(k)}\right)^4$

For node i $\dfrac{T_{i+1} - 2T_i + T_{i-1}}{h^2} - 2T_i^4 = 0$

with $\dfrac{1}{h^2} = 16$ $16(T_{i+1} - 2T_i + T_{i-1}) - 2T_i^4 = 0$

non linear term $\left(T_i^{(k+1)}\right)^4$

TSA 1st order:

$$\left(T_i^{(k+1)}\right)^4 \simeq \left(T_i^{(k)}\right)^4 + 4\left(T_i^{(k)}\right)^3\left(T_i^{(k+1)} - T_i^{(k)}\right)$$

$$\simeq 4\left(T_i^{(k)}\right)^3 T_i^{(k+1)} - 3\left(T_i^{(k)}\right)^4$$

plug : $-2\left(T_i^{(k+1)}\right)^4 \simeq -2\left(4\left(T_i^{(k)}\right)^3 T_i^{(k+1)} - 3\left(T_i^{(k)}\right)^4\right)$

$$= -8\left(T_i^{(k)}\right)^3 T_i^{(k+1)} + 6\left(T_i^{(k)}\right)^4$$

$\therefore 16T_{i+1}^{(k+1)} - 32T_i^{(k+1)} + 16T_{i-1}^{(k+1)} - 8\left(T_i^{(k)}\right)^3 T_i^{(k+1)} + 6\left(T_i^{(k)}\right)^4 = 0$

$\therefore 16T_{i+1}^{(k+1)} + 16T_{i-1}^{(k+1)} + \left(-32 - 8\left(T_i^{(k)}\right)^3\right) T_i^{(k+1)} = -6\left(T_i^{k}\right)^4$

node 2 : uses $T_1 = 1$ put $i = 2$

$\therefore \left(-32 - 8\left(T_2^{k}\right)^3\right) T_2^{(k+1)} + 16T_3^{(k+1)} = -16T_1 - 6\left(T_2^{(k)}\right)^4$

node 3 : $-6\left(T_3^{k}\right)^4$

node 4 : $-6\left(T_4^{k}\right)^4$

Matrix form : $A^{(k)} \cdot T^{(k+1)} = b^{(k)}$

let $T^{(k+1)} = \begin{bmatrix} T_2^{(k+1)} \\ T_3^{(k+1)} \\ T_4^{(k+1)} \end{bmatrix}$

$$\underbrace{A^{(k)} = \begin{bmatrix} -32-8(T_2^{k})^3 & 16 & 0 \\ 16 & -32-8(T_3^{(k)})^3 & 16 \\ 0 & 16 & -16-8(T_4^{(k)})^3 \end{bmatrix}}_{A^{(k)}} \underbrace{\begin{bmatrix} T_2^{(k+1)} \\ T_3^{(k+1)} \\ T_4^{(k+1)} \end{bmatrix}}_{T^{(k+1)}} = \underbrace{\begin{bmatrix} -16T_1-6(T_2^{k})^4 \\ -6(T_3^{k})^4 \\ -6(T_4^{k})^4 \end{bmatrix}}_{b^{(k)}}$$

$T_1 = 1$ and $T_5^{(k+1)} = T_4^{(k+1)}$.

initial : $T_1 = 1$, $T_2^{(0)} = 1 = T_3^{(0)} = T_4^{(0)} = T_5^{(0)}$

E2 =TRANSPOSE(MMULT(P2:R4, N2:N4))

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
1	**iter**	**T2_guess**	**T3_guess**	**T4_guess**	**T2_new**	**T3_new**	**T4_new**	**RMS_change**		**MATRIX A**				**vector b**		**inverse A**		
2	0	1	1	1	0.877907	0.819767	0.796512	0.172045		-40	16	0		-22		-0.03198	-0.01744	-0.01163
3	1	0.877907	0.819767	0.796512	0.854118	0.77444	0.738727	0.04457		16	-40	16		-6		-0.01744	-0.0436	-0.02907
4	2	0.854118	0.77444	0.738727	0.852908	0.771963	0.735406	0.002492		0	16	-24		-6		-0.01163	-0.02907	-0.06105
5	3	0.852908	0.771963	0.735406	0.852904	0.771956	0.735397	7.22E-06		**MATRIX A iter 1**				**vector b iter1**		**inverse A iter1**		
6						**T_new=A.inv*b**				-37.413	16	0		-19.56406201		-0.03762	-0.02547	-0.02033
7										16	-36.4072	16		-2.709654466		-0.02547	-0.05955	-0.04754
8										0	16	-20.0427		-2.415014437		-0.02033	-0.04754	-0.08785
9										**MATRIX A iter 2**				**vector b iter2**		**inverse A iter2**		
10										-36.9848	16	0		-19.1931797		-0.03913	-0.02795	-0.02326
11										16	-35.7158	16		-2.158252282		-0.02795	-0.06461	-0.05377
12										0	16	-19.2251		-1.786848613		-0.02326	-0.05377	-0.09676
13										**MATRIX A iter 3**				**vector b iter3**		**inverse A iter3**		
14										-36.9636	16	0		-19.17511435		-0.03921	-0.02809	-0.02343
15										16	-35.6803	16		-2.130770765		-0.02809	-0.0649	-0.05413
16										0	16	-19.1818		-1.754932451		-0.02343	-0.05413	-0.09729
17																		

Q4) $h_1(t)$ $h_2(t)$ = water levels

A_1, A_2 = areas

$f_1(t)$ $f_2(t)$ = external inflows

$q_{12}(t)$ = tank $1 \to 2$, $q_{20}(t)$ = tank $2 \to$ out

mass balance :

$$\therefore A_1 \frac{dh_1}{dt} = f_1(t) - q_{12}(t)$$

$$A_2 \frac{dh_2}{dt} = f_2(t) + q_{12}(t) - q_{20}(t)$$

tube flow model

$q_{12} = C \cdot \sqrt{h_1 - h_2}$ with loss coefficient C, $h_1 \geq h_2$

if $f_1 = f_2 = 0$ $q_{20} = 0$ $A_1 = A_2 = A$ then

$$A \frac{dh_1}{dt} = -q_{12} \qquad A \frac{dh_2}{dt} = +q_{12} = C \cdot \sqrt{h_1 - h_2}$$

$\therefore$ coupled system : $\dfrac{dh_1}{dt} = \dfrac{-C}{A}\sqrt{h_1 - h_2}$

$$\frac{dh_2}{dt} = \frac{C}{A}\sqrt{h_1 - h_2}$$

steady state $q_{12} = 0 \Rightarrow h_1 = h_2$

total volume conserved $\therefore (h_1 + h_2)$ constant

$$\therefore h_1(\infty) = h_2(\infty) = \frac{H_1 + H_2}{2}$$

let t^n, $t^{n+1} = t^n + \Delta t$

implicit: $h_1^{n+1} = h_1^n - \frac{\Delta t}{A} q_{12}^{n+1}$

$$h_2^{n+1} = h_2^n + \frac{\Delta t}{A} q_{12}^{n+1}$$

where $q_{12}^{n+1} = c\sqrt{h_1^{n+1} - h_2^{n+1}}$ → non linear

let $\alpha = \frac{\Delta t}{A}$ $\quad \Delta h^{n+1} = h_1^{n+1} - h_2^{n+1}$

$\therefore h_1^{n+1} = h_1^n - \alpha C\sqrt{\Delta h^{n+1}}$, $h_2^{n+1} = h_2^n + \alpha C\sqrt{\Delta h^{n+1}}$

linearization about a guessed estimate 1st order TSA

$k = 0, 1, 2 \ldots$

guess $\left(h_1^{n+1(k)}, h_2^{n+1(k)}\right)$ $\Delta h^{(k)} = h_1^{n+1(k)} - h_2^{n+1(k)}$

$$q^{(k)} = C\sqrt{\Delta h^{(k)}}$$

$$\frac{dq}{d(\Delta h)} = \frac{C}{2\sqrt{\Delta h}}$$

1st order TSA about $\Delta h^{(k)}$:

$$q^{(k+1)} \approx q^{(k)} + d^{(k)} \cdot \left(\Delta h^{(k+1)} - \Delta h^{(k)}\right)$$

where $d^{(k)} \equiv \frac{C}{2\sqrt{\Delta h^{(k)}}}$ $\quad$ $q^{(k+1)}$ plug into

$$h_1^{n+1(k+1)} + \alpha q^{(k+1)} = h_1^n$$

$$h_2^{n+1(k+1)} - \alpha q^{(k+1)} = h_2^n$$

$$\begin{bmatrix} 1+\alpha d^{(k)} & -\alpha d^{(k)} \\ -\alpha d^{(k)} & 1+\alpha d^{(k)} \end{bmatrix} \begin{bmatrix} h_1^{n+1\,(k+1)} \\ h_2^{n+1\,(k+1)} \end{bmatrix} = \begin{bmatrix} h_1^n - \alpha q^{(k)} + \alpha d^{(k)} \Delta h^{(k)} \\ h_2^n + \alpha q^{(k)} - \alpha d^{(k)} \cdot \Delta h^{(k)} \end{bmatrix}$$

Algorithm

input H_1 H_2 A C Δt final t_{max} tolerance ϵ

set $n=0$ $h_1^0 = H_1$ $h_2^0 = H_2$

for each $n \longrightarrow n+1$ timestep

$h_1^{n+1,(0)} = h_1^n$ $h_2^{n+1,(0)} = h_2^n$

for $k = 0, 1, 2 \ldots$

$$\Delta h^{(k)} = h_1^{n+1\,(k)} - h_2^{n+1\,(k)}$$

$$q^{(k)} = C\sqrt{\Delta h^{(k)}}$$

$$d^{(k)} = \frac{C}{2\sqrt{\Delta h^{(k)}}}$$

solve for $h_1^{n+1,(k+1)}$, $h_2^{n+1\,(k+1)}$ using above matrix

convergence check $\sqrt{(h_1^{k+1} - h_1^k)^2 + (h_2^{k+1} - h_2^k)^2} < \epsilon$

set $h_1^{n+1} = h_1^{n+1,(k+1)}$

$t^{n+1} = t^n + \Delta t$

choose Δt : such that $\left\| \frac{h_1^{n+1} - h_1^n}{h_1^n} \right\|$ is small