BMath-III; DG-2

Lecture-1

Prerequisites: DG-1 (curves & surfaces in $\mathbb{R}^3$), Multivariable calculus, Linear algebra (tensor products, eigenvalues etc).

Differential geometry aims at studying "differentiable" properties of geometric objects e.g. curves, surfaces or more generally manifolds, as one of its goals, by studying smoothness of functions between two such objects; by developing a Calculus; one needs to worry about "integrating" functions in a suitable sense, which we need for variety of reasons.

To make this somewhat more precise, given a smooth (i.e. without 'spikes') object, we study its shape by studying how it curves around various points.

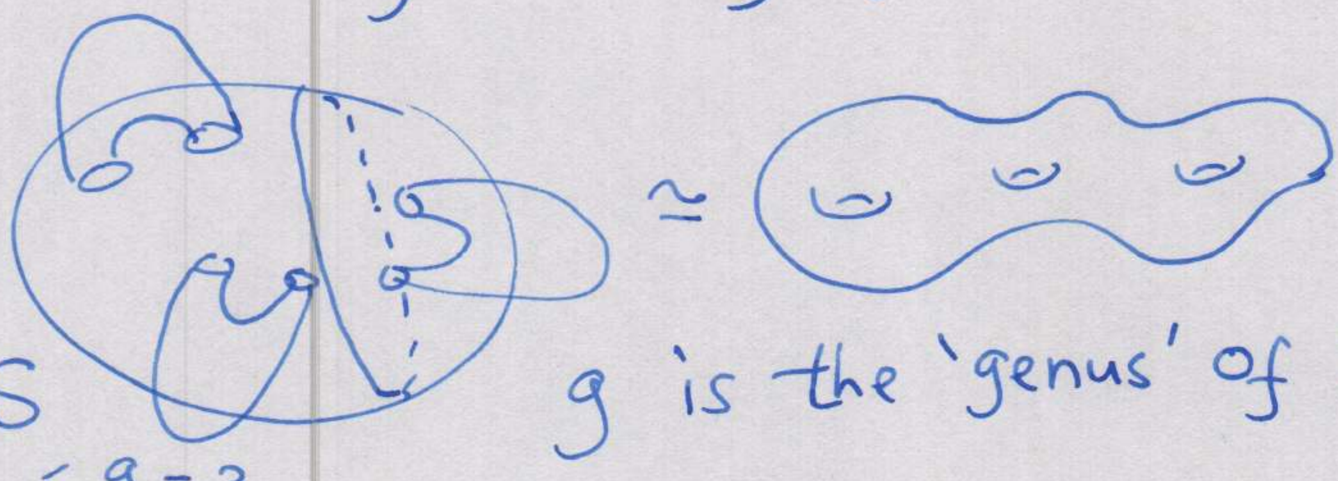
This job is simpler if we study such objects as subobjects of some Euclidean Space, i.e. embedded suitably in some $\mathbb{R}^n$, by talking of tangent spaces at points and how they

2. change while passing from one point to the other; leading to the notion of "normal curvature". This in one way of measuring how the given object curves around various points. Another way is to use the notion of curvature for 'plane' curves and consider curvatures of various curves on the surface, through a point, obtained by dissecting the surface by a normal plane at that point, containing a fixed tangent direction; this leads to "geodesic" curvature. One encounters more notions of curvatures for a surface and all contribute to determining the surface (locally). However, all these notions assume that the surface is contained in $\mathbb{R}^3$, i.e. we always have an embedding in picture. If $S \subseteq \mathbb{R}^3$ be a surface and assuming we have a reasonable function $K : S \to \mathbb{R}$ giving curvature $K(p)$ at $p \in S$, such that if $S =$ plane, then $K(p) = 0 \ \forall p$; $S = S^2$ then K is a constant function etc.

and assuming we have defined a suitable notion of $\int$ on S, then we obtain a fantastic <u>result</u>

$$\int_S K \, dA = 2\pi \chi(S)$$

Where dA is the area element, $\chi(S)$ is the "Euler characteristic" of S, where S is assumed compact. From topology one knows S orientable, is obtained from S^2 by attaching g- handles.



g is the 'genus' of S

and $\chi(S) = 2 - 2g$. The above result can be interpreted as giving the 'total curvature' of S in terms of a topological invariant of S!!

Just to remind you, topology does not regard a 'deformed' version of S as different, as long as the deformation does not tear S (only 'stretches' or 'bends'). This is why the above result is amazing; it says that even after any deformation, the total curvature of S remains same.

Going beyond: As you may have guessed, this course is about objects beyond curves & surfaces, i.e. (smooth) manifolds. One can ask at this point: Why study manifolds? Of course, an easy & defensive answer is that manifolds of higher dimensons occur in physics, e.g. in relativity theory & so on. These may be sources of geometric examples, however manifolds do occur in branches of math as well, perhaps not so 'geometrically'. Let us see some such examples:

- Let $M_{m,n}(\mathbb{R})$ be the space of all $m \times n$ matrices with entries from $\mathbb{R}$. We identify this with $\mathbb{R}^{mn}$ to topologize it. $\rightarrow$

Then, for k, the set

$$M_{m,n;k}(\mathbb{R}) = \{A \in M_{m,n}(\mathbb{R}) \mid \text{rank } k\}$$

is a C^∞-manifold (We'll discuss this later).

- <u>Grassmann manifolds</u>: You may have come across projective spaces $\mathbb{P}^n_{\mathbb{R}}$; this is the set of all 1-dim'l subspaces of $\mathbb{R}^{n+1}$, with a natural topology.

The Grassmann manifold $G_k(\mathbb{R}^n)$ consists of all k-dim'l subspaces of $\mathbb{R}^n$. We can give this a natural manifold structure.

So $\mathbb{P}^n_{\mathbb{R}} = G_1(\mathbb{R}^{n+1})$. We'll see that

$$\underline{M}^{M_{k,n}}_{\cancel{m,n;k}}(\mathbb{R}) \,/\, \sim \;\; \simeq \; G_k(\mathbb{R}^n), \text{ where}$$

for $x, y \in M_{k,n}(\mathbb{R}) = $ [illegible], $x \sim y$ if $\exists\, C \in GL_k(\mathbb{R})$ such that $X = CY$.

These two examples are extremely useful mathematically, yet they do not appear in nice, geometrically speaking, shape. However, by investigating geometric notions about such 'abstract' spaces, one arrives at concrete results that are useful.

→ Another class of examples comes from Lie groups, which occur naturally in Physics and Math, and what are called 'homogeneous spaces'.

When our surface is embedded in $\mathbb{R}^n$, we can talk of the normal at a point, since we have $\langle , \rangle$ on $\mathbb{R}^n$. How do we do this for manifolds as above? This is necessary in defining ideas of curvature etc. The idea of tangent space at a point is the first hurdle to cross; for a manifold M and $p \in M$, to define $T_p(M)$, the tangent space to →

7. to M at p. Once that is done, we can define an inner product on $T_p(M)$, say $\langle , \rangle_p$, for all $p \in M$ is a 'smooth' way. Using this, we can overcome some of the difficulties.

Using tangent spaces, we can talk of vector fields on M, or more generally, tensor fields on M. These will be defined intrinsically, without embedding M in any Euclidean space. Once we have overcome the technical hurdles and defined notion of curvature, one may ask if we have a Gauss-Bonnet like theorem; investigate relations with vector fields (on S^2, which has constant $\neq 0$ curvature, there is no nowhere vanishing vector field).

→ The topology of a manifold itself poses challenges. To give an example, →

8. Using curves on M, we may define a metric on M, natural enough to M. How is that related to the topology of M? Using curves on M, we may talk of geodesics on M ("straight" curves on M) and study those M whose any two points can be joined by a geodesic. Can we relate this property to M being complete, e.g. in the above metric? This is a fundamental result known as the Hopf-Rinow theorem, giving an affirmative answer in the case of M compact. One proves that when M is smooth, simply connected and has non positive 'sectional' curvature, M is diffeo to $\mathbb{R}^n$. This course will aim to prove such landmark results as well as the classical Stokes theorem. Initial parts will develop the technical machinery that will be needed $\rightsquigarrow$ Starting from next class!