DIFFERENTIAL TOPOLOGY - LECTURE 1

1. Introduction

The subject of differential topology, as the name suggests, is the study of spaces using the methods of calculus. This entails understanding which properties of spaces are invariant under a suitable notion of equivalence (diffeomorphism). Often, a tremendous amount of algebraic topology is involved in understanding (and defining) the invariants and finding solutions to problems in differential topology.

A class of objects that yield to methods of calculus are the so-called (smooth) manifolds. Examples of such objects first arose as solutions to equations, spaces with group actions and their quotients, etc. Although a formal definition of a topological space was only given around 1906 (Frechet) and 1914 (Hausdorff), mathematicians worked with an intuitive understanding of what a smooth manifold meant. The now accepted definition of a smooth manifold took some time to take shape. The reader interested in this history can have a look at the following wonderful references :

(1) http://www.quantum-gravitation.de/media/3a2a81c0493b7f72ffff8061ffffff0.pdf
(2) The history of algebraic and differential topology, by J Dieudonne.
(3) Algebraic Topology, by S Lefschetz. Here, 9 different definitions of a manifold appear.
(4) Foundations of Differential geometry, by Veblen and Whitehead. This more or less gave what is now the standard definition of a smooth manifold.

The first two references[1] above takes the reader through an interesting journey of the initial development of the subject of differential topology.

Our plan is to study some basic properties of manifolds inside the euclidean space. There will be some exercises at the end of each note. It is important to work these out or at the very least be familiar with the statements. These will be used in the sequel, often implicitly. The topics and material are mostly borrowed from the wonderful book *Differential Topology* by V Guillemin and A Pollack which we shall closely follow.

2. Review of Calculus of several variables

As we shall be dealing with several variable calculus throughout the course, we shall begin by recalling the basic definitions from several variable calculus and fixing notations. We shall state some basic theorems (without proofs).

As usual $\mathbb{R}^n$ will denote the euclidean n-space and the functions

$$x_i : \mathbb{R}^n \longrightarrow \mathbb{R}$$

$1 \leq i \leq n$, defined by

$$x_i(a_1, \ldots, a_n) = a_i$$

will be called the i-th coordinate function on $\mathbb{R}^n$. $\mathbb{S}^n$ is the unit sphere of length 1 vectors in $\mathbb{R}^{n+1}$ and $\mathbb{D}^n$ (sometimes $\mathbb{B}^n$) the unit ball of vectors of length at most 1 in $\mathbb{R}^n$.

[1]The first reference above appears in the book "History of Topology" edited by I. M. James.

Suppose $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a function and $x \in U$. Given a vector $v \in \mathbb{R}^n$ the change in the function f along the line segment from x to $x + v$ may be measured by the limit

$$\lim_{h\to 0} \frac{f(x+hv) - f(x)}{h}.$$

Observe that for small h, $x + hv \in U$. If the above limit exists it is denoted by

$$f'(x; v) = \lim_{h\to 0} \frac{f(x+hv) - f(x)}{h}$$

and is called the *directional derivative* of f at x in the direction v. Note that $f'(x; v) \in \mathbb{R}^m$ (if it exists). The directional derivative can therefore be thought of as a (partial) function

$$f'(x; -) : \mathbb{R}^n \longrightarrow \mathbb{R}^m. \tag{2.0.1}$$

The above function is defined on a (non-empty) subset of $\mathbb{R}^n$ and may not be globally defined. Note that the zero vector 0 is always in the domain of definition of $f'(x; -)$.

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ may be written in coordinates as $f = (f_1, \ldots, f_m)$ where $f_i = x_i \circ f$. We may now write

$$\lim_{h\to 0} \frac{f(x+hv) - f(x)}{h} = \lim_{h\to 0} \left(\frac{f_1(x+hv) - f_1(x)}{h}, \ldots, \frac{f_m(x+hv) - f_m(x)}{h} \right)$$

Thus the directional derivative $f'(x; v)$ exists if and only if each $f_i'(x; v)$ $(i = 1, \ldots, m)$ exists and in this case we have

$$f'(x; v) = (f_1'(x; v), \ldots, f_m'(x; v)). \tag{2.0.2}$$

If $m = 1$, then the directional derivative $f'(x; e_i)$ is traditionally denoted[2] by either of

$$\left.\frac{\partial f}{\partial x_i}\right|_x ; \quad \frac{\partial f}{\partial x_i}(x),$$

and is called the *partial derivative* of f with respect to the coordinate function x_i. Here e_i is the standard i-th basis vector of $\mathbb{R}^n$.

As we noted above, the directional derivative function is usually only defined on a subset of $\mathbb{R}^n$. If the function f is differentiable, then the directional derivative is globally defined.

We now recall the definition of a differentiable function. As before, let $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a function and $x \in U$. We say that f is differentiable at x if there exists a $\mathbb{R}$-linear transformation

$$T_x : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

such that

$$f(x+v) - f(x) - T_x(v) = ||v|| E_x(v) \tag{2.0.3}$$

where E_x is a function defined in a neighborhood of 0 and $E_x(v) \to 0$ as $v \to 0$. Thus we have

$$\lim_{v\to 0} \frac{f(x+v) - f(x) - T_x(v)}{||v||} = 0.$$

Equation (2.0.3) is called a first order Taylor's formula for f. The linear transformation T_x is called the (in anticipation of its uniqueness) derivative of f at x and henceforth will be denoted by df_x.

Now assume that f is differentiable at x and let $u \in \mathbb{R}^n$. Then setting $v = hu$ we have

$$f(x+hu) - f(x) = hT_x(u) + |h|||v||E_x(hu).$$

[2]The notation ∂ for the partial derivative goes back to Marquis de Condorcet, about 1770.

Hence, dividing by h and letting $h \to 0$ we have

$$T_x(u) = f'(x;u).$$

Thus, if f is differentiable at x, then directional derivative $f'(x;u)$ exists for all $u \in \mathbb{R}^n$ and

$$f'(x;u) = T_x(u).$$

This in particular proves that the derivative $T_x = df_x$ is unique. It is now easy to describe the derivative df_x which is a linear map

$$df_x : \mathbb{R}^n \longrightarrow \mathbb{R}^m.$$

The matrix of df_x can be computed by understanding the action of df_x on the standard basis vectors $e_j \in \mathbb{R}^n$. We note that, by equation (2.0.2), we have

$$\begin{aligned} df_x(e_j) = f'(x;e_j) &= \left(\tfrac{\partial f_1(x)}{\partial x_j}, \ldots, \tfrac{\partial f_m(x)}{\partial x_j}\right) \\ &= \textstyle\sum_1^m \tfrac{\partial f_i(x)}{\partial x_j} e_i \end{aligned}$$

Hence the matrix of df_x in the standard bases is

$$Jf(x) = \left(\frac{\partial f_i(x)}{\partial x_j}\right).$$

and is called the *Jacobian matrix* of f at x.

The *chain rule* for the derivative of the composition of two differentiable maps $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow V \underset{\text{open}}{\subseteq} \mathbb{R}^m$ and $g : V \underset{\text{open}}{\subseteq} \mathbb{R}^m \longrightarrow \mathbb{R}^\ell$ states that

$$d(g \circ f)_x = dg_{f(x)} \circ df_x$$

and in terms of the Jacobian matrices we have

$$J(g \circ f)(x) = Jg(f(x))Jf(x)$$

where the product on the right is a matrix product. It is therefore implicit (in the statement of the chain rule) that the composition of two differentiable functions is differentiable.

A function $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^1$ is said to be *smooth* (or sometimes C^∞) if the iterated partial derivatives

$$\frac{\partial^k f(x)}{\partial x_{i_1} \cdots \partial x_{i_k}}$$

exist and are continuous for all $k = 1, 2, 3, \ldots$. A function $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be smooth if each component function f_i of f is smooth. Throughout these notes all functions will be assumed to be smooth.

Remark 2.1. Let $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a a function. If f is continuous we say that f is a C^0 function on U. If f is differentiable, then df_x exists for each $x \in U$ and

$$df_x : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

is a linear map. This gives rise to a function

$$Df : U \longrightarrow \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m).$$

defined by $x \mapsto df_x$. The continuity of Df is equivalent to the continuity of each partial derivative

$$\frac{\partial f_i}{\partial x_j}$$

$1 \leq i \leq m, 1 \leq j \leq n$. The function f is said to be C^1 if Df exists and is continuous. If the function Df is differentiable we may then talk of $D(Df) = D^2 f$ which is now a map

$$D(Df) : U \longrightarrow \mathrm{Hom}(\mathbb{R}^n, \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^m)).$$

A moments thought tells us that the coordinate functions of $D^2 f$ are the mixed (second order) partial derivatives

$$\frac{\partial^2 f_i}{\partial x_j^2}.$$

Thus $D^2 f$ is continuous if and only if the second order partial derivatives of f are continuous. We say f is of class C^2 on U if $D^2 f$ exists and is continuous. Iterating this process, we say that f is of class C^k on U if $D^k f$ exists and is continuous. Finally, the function f is of class C^∞ (or smooth) on U if it is of class C^k for each $k = 1, 2, \ldots$. This is easily seen to be equivalent to the definition of smoothness given above.

A smooth function $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow V \underset{\text{open}}{\subseteq} \mathbb{R}^n$ is said to be a *diffeomorphism* if f is $1-1$, onto and f^{-1} is smooth. In this case we say that U and V are *diffeomorphic*. Observe that, by the chain rule, df_x is a linear isomorphism for each $x \in U$.

A fundamental result which will be used often is the Inverse function theorem. We state it without proof.

Theorem 2.2. (Inverse function theorem) Let $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a smooth function. Assume that df_x is invertible as a linear transformation (equivalently, the Jacobian matrix $J(f(x)$ is invertible), then there exist open sets $U_1 \subset U$ and $V_1 \subset f(U)$ with $x \in U_1$ such that

$$f/U_1 : U_1 \longrightarrow V_1$$

is a diffeomorphism (onto V_1). □

The importance of this theorem cannot be overstated. This theorem transforms alegebraic information of the derivative (invertibility of df_x) into a topological/analytical conclusion, namely, that f is $1-1$ in a neighborhood of x and the smoothness of the inverse (restricted to that neighborhood). We shall have occasion to understand what conclusions can be drawn from other (algebraic) restriction(s) on the derivative.

At this point we recall the definition of a local diffeomorphism. Given a smooth function $f : U \underset{\text{open}}{\subseteq} \mathbb{R}^n \longrightarrow \mathbb{R}^n$ we say that f is a *local diffeomorphism* at x if f maps a neighborhood of x diffeomorphically onto a neighborhood of $f(x)$. The map f itself will be called a local diffeomorphism if it is a local diffeomorphism at each $x \in U$. With this notion, the Inverse function theorem simply states that if df_x is invertible, then f is a local diffeomorphism at x. Observe that every diffeomorphism is a local diffeomorphism.

One now extends the definition of smoothness of maps to maps that are defined on subsets[3] that are not necessarily open. Given $X \subseteq \mathbb{R}^N$ and a function

$$f : X \longrightarrow \mathbb{R}^m$$

we say that f is smooth if for every $x \in X$ there exists an open set $U \subseteq \mathbb{R}^N$, $x \in U$, and a smooth function

$$F : U \longrightarrow \mathbb{R}^m$$

[3]When one talks of smooth maps between subsets of euclidean spaces it is implicitly assumed that the subsets have the subspace topology.

with $F/(U \cap X) = f$. That is, f must locally be the restriction of a smooth function defined on an open set in $\mathbb{R}^N$. As an example, the function

$$f : S^1 \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$$

given by $f(x, y) = x$ is smooth. It is the restriction of the projection which is smooth.
A function

$$f : X \subseteq \mathbb{R}^N \longrightarrow Y \subseteq \mathbb{R}^M$$

is said to be a diffeomorphism if f is bijective, smooth with f^{-1} also smooth. The subsets X and Y are then said to be *diffeomorphic*. The notion of a local diffeomorphism between subsets of euclidean spaces may be similarly defined.

Example 2.3. Suppose that we have composable smooth functions

$$X \subseteq \mathbb{R}^N \xrightarrow{f} Y \subseteq \mathbb{R}^M \xrightarrow{g} Z \subseteq \mathbb{R}^L$$

defined on subsets of euclidean spaces. Then the composition $g \circ f$ is also smooth. To see this, given $x \in X$ find open sets U and V about x and $f(x)$ respectively and smooth functions

$$F : U \longrightarrow \mathbb{R}^M; \quad G : V \longrightarrow \mathbb{R}^L$$

that extend f and g respectively. We may assume that $F(U) \subseteq V$. Then clearly

$$(G \circ F)/(U \cap X) = g \circ f$$

and hence $g \circ f$ is smooth.

Thus the composition of two smooth functions is smooth. Here is another example.

Example 2.4. Let $f : X \subseteq \mathbb{R}^N \longrightarrow \mathbb{R}^M$ be a smooth function. The set

$$Z = \text{graph}(f) = \{(x, f(x)) \ : \ x \in X\} \subseteq \mathbb{R}^N \times \mathbb{R}^M$$

is called the graph of f. We claim that the function

$$F : X \longrightarrow Z$$

defined by $F(x) = (x, f(x))$ is a diffeomorphism. It is clear that F is bijective. We shall check that both F and F^{-1} are smooth. The smoothness of F can be checked as follows. We first observe that the diagonal map

$$\triangle : X \longrightarrow X \times X; \quad \triangle(x) = (x, x)$$

is smooth. This is because it is the restriction of the diagonal map of the euclidean space which we know is smooth. Next suppose we have smooth maps $h_i : X_i \longrightarrow Y_i$, $i = 1, 2$ between subsets of euclidean spaces. We claim that the map

$$h_1 \times h_2 : X_1 \times X_2 \longrightarrow Y_1 \times Y_2$$

defined by

$$h_1 \times h_2(x, y) = (h_1(x), h_2(y))$$

is smooth. This is because the map $h_1 \times h_2$ is locally the restriction of product of two smooth functions defined on open subsets of the euclidean space (use the definition of smoothness of h_1 and h_2). Finally, F is the composition

$$F = (id \times f) \circ \triangle : X \longrightarrow X \times X \longrightarrow Z$$

of two smooth functions and hence is smooth. The inverse function F^{-1} is evidently just the projection to the first factor and hence is smooth. Thus F is indeed a diffeomorphism.

Conventions. Throughout our course of discussions a function/map will always mean a smooth function/map. There should be no confusion about this.

Here are some exercises.

Exercise 2.5. Picturise the set

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, \Sigma_i x_i^2 = 1, \Sigma x_i = 0\}.$$

Exercise 2.6. Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be the function

$$g(x) = ||x|| = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Discuss the differentiability of the functions $g(x)$ and $h(x) = ||x||^2$. Compute dg_p and dh_p at points p where they are defined.

Exercise 2.7. Consider the function

$$f(x,y) = \begin{cases} 0 & y \leq 0 \text{ or } y \geq x^2 \\ \left[\frac{y}{x^2}\left(1 - \frac{y}{x^2}\right)\right]^2 & 0 < y < x^2 \end{cases}$$

Show that f is continuous at all points other than $(0,0)$. Show that $f'((0,0); v)$ exists for all $v \in \mathbb{R}^2$.

Exercise 2.8. Show that the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x^2} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

is smooth.

Exercise 2.9. Construct a diffeomorphism $f : \mathrm{Int}(\mathbb{D}^n) \longrightarrow \mathbb{R}^n$. This raises several questions. The most general question would be : what are the open subsets of $\mathbb{R}^n$ that are diffeomorphic to $\mathbb{R}^n$. One could also ask if two open subsets of $\mathbb{R}^n$ are homeomorphic, then are they are diffeomorphic? The answers to these questions involves understanding some beautiful and deep mathematics. One can show that certain well behaved open subsets of $\mathbb{R}^n$ are indeed diffeomorphic to $\mathbb{R}^n$. For example, it is a theorem that every open star shaped subset of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$. In particular, every convex open subset of $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$.

Exercise 2.10. Let $\triangle^n$ and A_n be the sets

$$\triangle^n = \{(t_1, \ldots, t_{n+1}) \,:\, \Sigma t_i = 1, t_i \geq 0\}$$
$$A_n = \{(x_1, \ldots, x_n) \,:\, 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq 1\}.$$

Construct a diffeomorphism $f : A_n \longrightarrow \triangle^n$. It might be helpful to draw the spaces for $n = 1, 2$. $\triangle^n$ is called the standard n-simplex.

Exercise 2.11. Let $f : U \longrightarrow V$ be a diffeomorphism where U is open in $\mathbb{R}^n$ and V open in $\mathbb{R}^m$. Show that $n = m$. Give an example to show that the conclusion is not true if the "open" condition is dropped.

Exercise 2.12. If $k \leq \ell$, we can consider $\mathbb{R}^k$ to be the subset $\{(a_1, \ldots, a_k, 0, \ldots, 0)\}$ in $\mathbb{R}^\ell$. Show that smooth functions on $\mathbb{R}^k$ considered as a subset of $\mathbb{R}^\ell$ are the same as usual.

Exercise 2.13. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a smooth map and

$$U = \{p \in \mathbb{R}^n \,:\, df_p \text{ is invertible}\}.$$

Show that U is open.

Exercise 2.14. Suppose that $Z \subseteq X \subseteq \mathbb{R}^N$. Show that the restriction to Z of any smooth function on X is a smooth function on Z.

Exercise 2.15. Show that if $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is a local diffeomorphism, then $m = n$. Thus $\mathbb{R}^m$ and $\mathbb{R}^n$ are diffeomorphic if and only if $m = n$. This statement with diffeomorphism replaced by homeomorphism is harder to prove.

Exercise 2.16. Define the notion of a local diffeomorphism between two subsets of euclidean spaces. Show that a local diffeomorphism is an open map.

Exercise 2.17. Show that a smooth map $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a local diffeomorphism if and only if it is a diffeomorphism onto its image. Is this true for a smooth map $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$?

Exercise 2.18. Let

$$U = \mathbb{S}^n - (0, \ldots, 0, 1) \quad V = \mathbb{S}^n - (0, \ldots, 0, -1)$$

and

$$\varphi(x_1, , \ldots, x_n, x_{n+1}) = \frac{(x_1, \ldots, x_n)}{1 - x_{n+1}}; \quad \psi(x_1, , \ldots, x_n, x_{n+1}) = \frac{(x_1, \ldots, x_n)}{1 + x_{n+1}}$$

Check that φ and ψ are diffeomorphisms onto $\mathbb{R}^n$. They are called the *stereographic projections.*

Exercise 2.19. Let X denote the subspace

$$X = \{[0,1] \times \{y\} \cup \{x\} \times [0,1] \,:\, x, y \in \{0,1\}\}$$

of $\mathbb{R}^2$. Show that no neighborhood of $(0,0) \in X$ is diffeomorphic to an open set in $\mathbb{R}$.

Exercise 2.20. Let $U \subseteq \mathbb{R}^n$ be an open set. Fix a point $x \in U$ and a vector $v \in \mathbb{R}^n$. We can now construct a function

$$v(x; -) : \text{smooth in a neighborhood of } x \longrightarrow \mathbb{R} \tag{2.20.1}$$

by setting

$$v(x; f) = f'(x; v)$$

where f is a smooth (real valued) function defined in a neighborhood of x. The domain of $v(x; -)$ consists of the set of smooth functions that are defined in some (varying) neighborhood of x. This gives another way of looking at the directional derivative function in (2.0.1) where the function f was fixed and v was varying. Thus the vectors $v \in \mathbb{R}^n$ "operate" on smooth functions defined in a neighborhood of a point to produce real numbers. Check the following.

(1) If f, g are two smooth functions that are defined in some (possibly different) neighborhood of x and such that they agree in a (smaller) neighborhood of x, then

$$v(x; f) = v(x; g)$$

for all $v \in \mathbb{R}^n$.

(2) If $v, u \in \mathbb{R}^n$, then

$$(v + u)(x; f) = v(x; f) + u(x; f), \quad (av)(x; f) = a \cdot v(x; f)$$
$$v(x; f + ag) = v(x; f) + a \cdot v(x; g)$$
$$v(x; f \cdot g) = f(x) \cdot v(x; g) + g(x) \cdot v(x; f)$$

for all $a \in \mathbb{R}$ and all smooth functions f, g defined in a neighborhood of x. The domains of the various functions involved are suitably chosen. The last property above is called the *derivation* property of $v(x; -)$.