Lecture-2

<u>A review of calculus</u>: Though the basic objects of study in this course are manifolds, these are locally Euclidean, i.e. every point has a neighbourhood $\simeq$ an open ball in $\mathbb{R}^n$. This allows us to do calculus on such abstract objects, using what are called chart maps and transition maps. We therefore recall first the <u>calculus on $\mathbb{R}^n$</u>: A function $f: \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} =: \lambda$$

exists; then we define the derivative f' of f at a to be λ, i.e. $f'(a) = \lambda$.

→ • The derivative of f at a represents the best linear approximation of f at a.

The above may be <u>rephrased</u> as: $f: \mathbb{R} \to \mathbb{R}$ is diff at a with derivative $\lambda = f'(a)$ if

$$\lim_{x \to a} \frac{f(x) - f(a) - \lambda \cdot (x - a)}{x - a} = 0.$$

→

- Since any $\mathbb{R}$-linear map $\phi:\mathbb{R}\to\mathbb{R}$ is of the form $\phi(x)=\alpha x$ for some $\alpha\in\mathbb{R}$ fixed, we may think of $\lambda\cdot(x-a)$ above as $\phi(x-a)$ for $\phi:\mathbb{R}\to\mathbb{R}$, $x\mapsto\lambda\cdot x$.

This leads & motivates us to define

Def^n: A map $f:\mathbb{R}^n\to\mathbb{R}^m$ is differentiable at $a\in\mathbb{R}^n$ if $\exists$ a linear map $\lambda:\mathbb{R}^n\to\mathbb{R}^m$ such that
$$\lim_{h\to 0}\frac{\|f(a+h)-f(a)-\lambda(h)\|}{\|h\|}=0$$
(Note that the norms on numerator & denom are in $\mathbb{R}^m$ & $\mathbb{R}^n$ resp.).

Exercise: 1. Let $f:\mathbb{R}^n\to\mathbb{R}^m$ be diff at $a\in\mathbb{R}^n$ and $T_i:\mathbb{R}^n\to\mathbb{R}^m$ be linear maps, $i=1,2$, such that
$$\lim_{h\to 0}\frac{\|f(a+h)-f(a)-T_i(h)\|}{\|h\|}=0$$
for $i=1,2$; then $T_1=T_2$.

2. For sufficiently small $x\in\mathbb{R}^n$ we have
$$f(a+x)=f(a)+\lambda(x)+\|x\|\,\epsilon(x),$$
where $\epsilon(x)$ satisfies $\lim_{x\to 0}\epsilon(x)=0$.

Notation :- The linear map λ as above, if it exists, is denoted by $Df(a)$ and is called the derivative of f at a.

So $Df(a)$ is a linear transformation $: \mathbb{R}^n \to \mathbb{R}^m$.

Exercise : For any $x \in \mathbb{R}^n$,

$$Df(a)(x) = \lim_{t \to 0} \frac{1}{t}\left[f(a+tx)-f(a)\right].$$

Computing $Df(a)$:- For this, we'll compute the matrix of $Df(a) \in M_{m\times n}(\mathbb{R})$, where we write $Df(a) = \left[C_1 \cdots C_n\right]$, C_i are the columns of $Df(a)$, and $C_i := Df(a)(e_i)$.

By the above exercise

$$Df(a)(e_j) = \lim_{t\to 0} \frac{1}{t}\left[f(a+te_j) - f(a)\right]$$

→ We can write $f : \mathbb{R}^n \to \mathbb{R}^m$ in terms of its coordinate functions $f = (f_1, \cdots, f_m)^t$, $f_i : \mathbb{R}^n \to \mathbb{R}$, $1 \le i \le m$; so $f = \sum_{j=1}^{m} f_j e_j'$,

$e_j' = (0,\cdots,0,\overset{j\text{th}}{1},0,\cdots 0)^t$.

Hence we have ⟶

$$f(a+te_j)-f(a) = \begin{pmatrix} f_1(a+te_j)-f_1(a) \\ \vdots \\ f_i(a+te_j)-f_i(a) \\ \vdots \\ f_m(a+te_j)-f_m(a) \end{pmatrix}$$

and $f_i(a+te_j)-f_i(a)$, for $a=\sum_{k=1}^{n} a_k e_k$

$$= f_i\big((a_1,\dots,a_{j-1},a_j+t,a_{j+1},\dots,a_n)^t\big)$$
$$- f_i\big((a_1,\dots,a_j,\dots,a_n)^t\big).$$ Thus, the

i^{th} row of the j^{th} column of $Df(a)$ is

$$\lim_{t\to 0} \frac{f_i((a+te_j))-f_i(a)}{t} = (ij)^{th} \text{ entry of}$$

the matrix of $Df(a)$.

Def^n (Partial derivatives): For $\varphi: \mathbb{R}^n \to \mathbb{R}$ and $a\in\mathbb{R}^n$, the partial derivative of f at $a=(a_1,\dots,a_n)^t$ with respect to x_i (the i^{th} coordinate) is the limit

$$\left.\frac{\partial\varphi}{\partial x_i}\right|_a = \lim_{t\to 0} \frac{\varphi((a_1,\dots,a_{i-1},a_i+t,a_{i+1},\dots,a_n)) - \varphi((a_1,\dots,a_n))}{t},$$

if it exists. We therefore have

$$Df(a) = (a_{ij}) \text{ where } a_{ij} = \frac{\partial f_i}{\partial x_j}\Big|_a .$$

This matrix is called the Jacobian matrix of f at a.

Examples :- Let $f: \mathbb{R}^n \to \mathbb{R}$ be a diff. func.ⁿ The derivative of f at $p \in \mathbb{R}^n$, $Df(p)$ is a $1\times n$ matrix, i.e. a row vector, called the gradient of f at p

$$(\nabla f)\big|_p = \left(\frac{\partial f}{\partial x_1}\Big|_p, \ldots, \frac{\partial f}{\partial x_n}\Big|_p\right).$$

The level sets of f are the sets $f^{-1}(c)$, $c \in \mathbb{R}$. For generic c, these give hyper-surfaces in $\mathbb{R}^n$, i.e. geom. objects having dim $n-1$, $\subseteq \mathbb{R}^n$. Since f is differentiable, for generic c, these objects are 'differentiable'.

To make sense of this, assume that $p \in S := f^{-1}(0)$ is such that →

$\nabla f|_p \neq 0$. Then $\nabla f|_p$ would constitute a normal at p to the hyper-surface $S \subseteq \mathbb{R}^n$. If $\nabla f|_p \neq 0$, we can define

$$T_p S = \{(b_1, \dots, b_n) \in \mathbb{R}^n \mid \langle \nabla f|_p, (b_1 - a_1, \dots b_n - a_n) \rangle = 0\}$$

where $p = (a_1, \dots, a_n)$.

$$= \left\{(b_1, \dots, b_n) \in \mathbb{R}^n \;\middle|\; \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\Big|_p \cdot (b_i - a_i) = 0\right\},$$

the tangent space to S at p.

<u>Remark</u>:- There one examples (?) of $f: \mathbb{R}^n \to \mathbb{R}^m$ such that $\frac{\partial f_i}{\partial x_j}\Big|_a$ exist $\forall\, i, j$, yet f is not continuous at a.

→ $f: \mathbb{R}^n \to \mathbb{R}^m$ is diff at $a \Rightarrow f$ is cont. at a.

<u>Thm</u>:- $Df(a)$ exists if all $\partial f_i / \partial x_j$ exist at all points in an open nbd of a <u>and</u> if each $\partial f_i / \partial x_j$ is continuous at a. ⊠

<u>Chain rule</u> :- If $f: U \to \mathbb{R}^m$ and $g: V \to \mathbb{R}^\ell$, where $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open and $f(U) \subseteq V$; f is diff at $a \in U$, g is diff at $f(a) \in V$, then the composite $g \circ f: U \to \mathbb{R}^\ell$ is diff. at a and its derivative at a is given by

$$D(g \circ f)(a) = Dg(f(a)) \circ Df(a)$$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "Df(a)"] \arrow[rr, bend right, "Dg(f(a)) \circ Df(a) = D(g \circ f)(a)"'] & \mathbb{R}^m \arrow[r, "Dg(f(a))"] & \mathbb{R}^\ell \end{tikzcd}$$

→ If $L: \mathbb{R}^n \to \mathbb{R}^m$ is an affine map, i.e. $\exists\ T: \mathbb{R}^n \to \mathbb{R}^m$ linear & $v \in \mathbb{R}^m$ with $L(x) = T(x) + v \ \forall\ x \in \mathbb{R}^n$; then L is differentiable and $DL(a) = T \ \forall\ a \in \mathbb{R}^n$.

→ $f, g: U \to \mathbb{R}^m$ be diff at $a \in \underset{\text{open}}{U} \subseteq \mathbb{R}^n$; $\alpha, \beta \in \mathbb{R}$, then $\alpha f + \beta g$ is diff at a.

→ $f, g: U \to \mathbb{R}$ diff at $a \in U \Rightarrow f \cdot g$ diff at a. If $f(x) \neq 0 \ \forall\ x \in U$, then $x \mapsto f(x)^{-1}$ is also diff at a. →

→ Hence any polynomial function $P : \mathbb{R}^n \to \mathbb{R}$ is differentiable at all points; a rational function $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = P(x)/Q(x)$ where P & Q are polynomials, is differentiable on the open set $U = \mathbb{R}^n - Z(Q)$, $Z(Q) = \{x \in \mathbb{R}^n \mid Q(x) = 0\}$.

Exercise :- 1. Identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$. Then $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid \det(A) \neq 0\}$ is open. Prove that $i : GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$, $X \longmapsto X^{-1}$ is differentiable at all points in $GL_n(\mathbb{R})$.

2. Consider $f : \mathbb{R}^2 \to \mathbb{R}$; $(x, y) \longmapsto \begin{cases} \dfrac{x|y|}{(x^2+y^2)^{1/2}}, & (x,y) \neq (0,0) \\ 0, & x = y = 0. \end{cases}$

show that f is not diff at $(0,0)$, both partial derivatives exist at $(0,0)$.

- show that restriction of f to every line passing through $(0,0)$ is diff.

—— x —— x —— x —— x ——