Analysis IV

Lecture 1

January 19, 2022

Let (X, d) be a metric space. For any $r > 0$, $x \in X$. Let $B_r(x) := \{y \in X | d(x, y) < r\}$. $U \subseteq X$ is called open if for each $x \in U$, $\exists r > 0$ such that $B_r(x) \subseteq U$. $F \subseteq X$ is called closed if its complement F^C is open.
Examples: $B_r(x)$ are open, any Singleton subset $\{x\}$ is closed.
A subset $F \subseteq X$ is called compact if every open cover of F has a finite subcover.
TFAE:

1. $F \subseteq X$ is compact
2. Every sequence in $F \subseteq X$ has a convergent subsequence in F. (The limit of every sequence in F is contained in F)
3. Any collection of closed sets with finite intersection property (fip) that is, the intersection of any finite subcollection is non-empty, has non-empty intersection.

Any compact set is closed. X is compact

Consider the Euclidean space $\mathbb{R}^d$ with the Euclidean metric. Here, $F \subseteq X$ is compact if and only if F is closed and bounded. In general metric spaces, only the forward direction holds.

A metric space (X, d) is called separable if X contains a countable dense($A \subseteq X$ is called dense if $\overline{A} = X$) subset. For instance, $X = \mathbb{R}$, $A = \mathbb{Q}$, $\overline{A} = \mathbb{R}$

Proposition: Any compact metric space is separable.

Proof. Let X be a compact metric space. Let $n \geqslant 0$. Then, $\{B_{\frac{1}{n}}(x)\}_{x \in X}$ is an open cover of X. Since X is compact, $\exists$ $x_{n,1}, \ldots, x_{n,k_n}$ in X such that $X \subseteq \bigcup_{i=1}^{n} B_{\frac{1}{n}}(x_{n,k_i})$. Let $F_n = \{x_{n,1}, \ldots, x_{n,k_n}\}$. Then, $F = \bigcup F_n$ is a countable set. Let $y \in X$ be arbitrary. For all $n \in \mathbb{N}$, let $r_n \in F_n$ such that $y \in B_{\frac{1}{n}}(r_n)$. Then, $(r_i) \subseteq F$ converges to y. $\square$

Examples: (Compact metric spaces) $X = [0, 1]$, $X = \{0, 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots\}$, $[0, 1]^n$

A metric space X is called complete if every Cauchy sequence converges.
Example: $\mathbb{R}^d, [a, b]$ are complete metric spaces. $\mathbb{Q}$ is not complete.
Exercise: A subset F of a complete metric space is complete (wrt the subspace metric) iff F is closed.

Proposition: Every compact metric space is complete.

Proof. Let (x_n) be a Cauchy sequence in a compact metric space X. (x_n) has a subsequence (x_{k_n}) such that $x_{k_n} \to x \in X$.

Choose $N \in \mathbb{N}$ such that For $\varepsilon > 0$, $\exists N$ such that $d(x_{k_n}, x) < \varepsilon$ for all $n \geq N$ and $d(x_n, x_m) < \varepsilon$ for all $n, m \geq N$. For $n \geq N, k_n \geq N$ $d(x_n, x) \leq d(x_n, x_{k_n}) + d(x_{k_n}, x) \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. This implies that $x_n \to x$ □

Function Spaces:

Let X be a compact metric space. Let $C(X) := \{f : X \to \mathbb{C} | f \text{ is continuous}\}$ and $C_{\mathbb{R}}(X) := \{f : X \to \mathbb{R} | f \text{ is continuous}\}$. Let $f \in C_{\mathbb{R}}(X)$. One can show that $\exists a \in X$ such that $f(a) \geqslant f(x)$ for all $x \in X$. That is, $\sup_{x \in X} f(x)$ is attained. If $f \in C(X)$, then $|f| \in C_{\mathbb{R}}(X)$.

Convergence in $C(X)$: $(f_n) \subseteq C(X)$. Pointwise convergence corresponds to $f_n(x)$ converging for all $n \in \mathbb{N}$. Limit need not be continuous. Uniform convergence: f_n converges uniformly. In this case, limits are continuous functions. For general metric spaces, (not necessarily compact), uniform convergence is required only on compact subspaces and not on the whole space. In our study, the whole space X is compact.

Metric on $C(X)$: Let $f, g \in C(X)$. Define $d_\infty(f, g) := \sup_{x \in X} |f(x) - g(x)|$. d is a metric on $C(X)$. $f_n \to f$ uniformly iff $d_\infty(f_n, f) = 0$. d_∞ is called the uniform/sup/infinite/L_∞ metric. $(C(X), d_\infty)$ - One can find a sequence of continuous complex functions with no convergent subsequence. However, $C(X)$ is a complete separable metric space. How does one classify compact subsets $E \subseteq C(X)$?