Complex Analysis
Lecture-10

MA201 Mathematics III

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Learning Outcome of this Lecture

We learn

- Consequences/Applications of Cauchy's Integral Formulas
  - Cauchy's Estimate, Liouville's Theorem, Fundamental Theorem of Algebra, Gauss Mean Value Theorem
  - Maximum Modulus Theorem, Minimum Modulus Theorem, Schwarz Lemma
- Morera's Theorem

Consequences/ Applications of

Cauchy's Integral Formula and Cauchy's Integral Formula for n-th Derivative

Analytic $\Longrightarrow$ Infinitely Many Times Differentiable

Theorem

Let D be an open set in $\mathbb{C}$. If a function f is analytic in D then for each $n \in \mathbb{N}$, the n-th derivative $f^{(n)}$ of f exists and analytic in D.

Proof: By Cauchy's integral formula for derivatives, the above theorem follows.

Corollary

Let D be an open set in $\mathbb{C}$. If a function $f(z) = u(x,\ y) + i\ v(x,\ y)$ is analytic in D, then the component functions $u \equiv u(x,\ y)$ and $v \equiv v(x,\ y)$ have continuous partial derivatives of all orders at each point of D.

Proof: Follows from the above theorem.

Cauchy's Estimate

Theorem

Let $f(z)$ be analytic on and inside the circle $C : |z - z_0| = R$. Let $M = \max\{|f(z)| : |z - z_0| = R\}$. Then,

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!\, M}{R^n} \qquad \text{for} \qquad n = 1,\ 2, \cdots .$$

Proof: Worked out on the board.

Joseph Liouville (1809 - 1882)
French Mathematician

Liouville worked in a number of different fields in mathematics, including number theory, complex analysis, differential geometry and topology, but also mathematical physics and even astronomy.
He is remembered particularly for Liouville's theorem, a nowadays rather basic result in complex analysis.

Liouville's Theorem

Recall that a function $f(z)$ is said to be an entire function if $f(z)$ is analytic at every point of the complex plane $\mathbb{C}$.

Recall that a function $f : D \subseteq \mathbb{C} \to \mathbb{C}$ is said to be bounded in D if there exists $M > 0$ such that $|f(z)| < M$ for all $z \in D$.

Theorem

Liouville's Theorem: If f is entire and bounded in the complex plane $\mathbb{C}$, then $f(z)$ is a constant function in $\mathbb{C}$.

Proof: Worked out on the board.

Properties of Non-Constant Polynomials

Theorem

Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial of degree $n \geq 1$ with $a_n \neq 0$. Then there exists $R > 0$ such that

$$\frac{1}{2}|a_n||z|^n \leq |P(z)| \leq \frac{3}{2}|a_n||z|^n \qquad \text{for} \qquad |z| > R\,.$$

Proof:

Let $w = \dfrac{a_0}{z^n} + \dfrac{a_1}{z^{n-1}} + \cdots + \dfrac{a_{n-1}}{z}$.

Then, $P(z) = a_n z^n + w z^n = (a_n + w) z^n$.

$$|w| = \left| \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \cdots + \frac{a_{n-1}}{z} \right| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \cdots + \frac{|a_{n-1}|}{|z|}$$

There exists $R > 0$ (sufficiently large) such that

$$\frac{|a_k|}{|z|^{n-k}} < \frac{|a_n|}{2n} \qquad \text{for } k = 0, 1, \cdots (n-1) \text{ and for } |z| \geq R\,.$$

This implies $|w| \leq \dfrac{|a_0|}{|z|^n} + \dfrac{|a_1|}{|z|^{n-1}} + \cdots + \dfrac{|a_{n-1}|}{|z|} < \dfrac{|a_n|}{2}$ for $|z| \geq R$.

$$|a_n + w| \geq \big||a_n| - |w|\big| > \frac{|a_n|}{2} \quad \text{for } |z| \geq R\,.$$

$$|P(z)| = |a_n + w||z|^n > \frac{|a_n|}{2} \quad \text{for } |z| \geq R\,.$$

Now

$$|P(z)| = |a_n + w||z|^n \leq |a_n||z|^n + |w||z|^n$$

$$< |a_n||z|^n + \frac{|a_n|}{2}|z|^n < \frac{3}{2}|a_n||z|^n \text{ for } |z| \geq R\,.$$

Exercise: By using the above inequality, show that $|P(z)| \to \infty$ (and hence $P(z) \to \infty$) as $z \to \infty$.

For $|z| \geq R$, we have $|P(z)| > \dfrac{1}{2}|a_n||z|^n = \dfrac{1}{2}|a_n|\, R^n$.
The above inequality is true for every $|z| = R^* > R$.
As $R^* \to \infty$, we have $|P(z)| \to \infty$.
Therefore $P(z) \to \infty$ as $z \to \infty$.

Fundamental Theorem of Algebra

Theorem

Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial of degree $n \geq 1$ with $a_n \neq 0$. Then $P(z)$ has at least one zero in $\mathbb{C}$.

Proof: Worked out on the board.

Corollary

Let $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ be a polynomial of degree $n \geq 1$ with $a_n \neq 0$. Then $P(z)$ has exactly n zeros counting multiplicities in $\mathbb{C}$.

Johann Carl Friedrich Gauss (1777 - 1855)

Gauss was a German mathematician who contributed significantly to many fields, including number theory, algebra, statistics, analysis, differential geometry, geodesy, geophysics, mechanics, electrostatics, astronomy, matrix theory, and optics.
Sometimes referred to as the Princeps mathematicorum (Latin, “the Prince of Mathematicians” or “the foremost of mathematicians”) and “greatest mathematician since antiquity”, Gauss had an exceptional influence in many fields of mathematics and science and is ranked as one of history’s most influential mathematicians.
Ref: https://en.wikipedia.org/wiki/Carl_Friedrich_Gauss

Gauss Mean Value Theorem

Theorem

Gauss Mean Value Theorem:

If f is analytic in a simply connected domain D that contains the circle $C : |z - z_0| = R$, then

$$f(z_0) \ = \ \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{i\theta}) \, d\theta \ .$$

That is, the value f at z_0 is the integral average of the values of $f(z)$ at points z on the circle C.

Proof: Worked out on the board.

Maximum-Modulus Theorem

Consider the function $f(x) = -x^2$ for $x \in [-1,\ 1]$. It attains the maximum value at $x = 0$ which is an interior point to the interval $[-1,\ 1]$. Whereas, the following theorem shows that for an analytic function $f(z)$, the maximum value of $|f(z)|$ cannot be attained in the interior point of a domain.

Theorem

Maximum-Modulus Theorem (or Maximum-Modulus Principle): If a function f is analytic and non-constant in a given domain D, then $|f(z)|$ has **no** maximum value in D. That is, there is **no** point z_0 in the domain D such that $|f(z)| \leq |f(z_0)|$ for all points z in D.

Another version of Maximum-Modulus theorem

Theorem

Maximum-Modulus Theorem (Stronger Version): Suppose that a function f is continuous in a closed bounded (that is, compact) region S and that $f(z)$ is analytic and non-constant in the interior of S. Then, the maximum value of $|f(z)|$ in S which is always reached, occurs somewhere on the boundary of S and never in the interior of S.

Minimum-Modulus Theorem

The following theorem is an immediate consequence of the maximum-modulus theorem and tells about the minimum-modulus of $f(z)$.

Theorem

Minimum-Modulus Theorem: Let a function f be continuous in a closed bounded region S and let f be analytic and non-constant throughout the interior of S. Further, assume that $f(z) \neq 0$ for every $z \in S$. Then, $|f(z)|$ has a minimum value in S which occurs on the boundary of S and never in the interior of S.

Hint: Apply Maximum Modulus Theorem to $1/f$.

Example

Find the maximum value and the minimum value of $|f(z)|$ in S if $f(z) = e^z$ and $S = \{z \in \mathbb{C} \ : \ |z| \leq R\}$ where $R > 0$.
Details worked out on the board.

Example

Find the maximum value of $|f(z)|$ in S if $f(z) = \sin(z)$ and $S = \{z = x + iy \in \mathbb{C} \ : \ 0 \leq x \leq \pi, \ 0 \leq y \leq 1\}$.

Hint:

$|f(z)| = |\sin(z)| = \sqrt{\sin^2(x) + \sinh^2(y)}$ for $z = x + iy \in \mathbb{C}$.

Apply the maximum modulus theorem and conclude that $|f|$ attains the maximum value in S at the point $z^* = (\pi/2) + i$ on the boundary of S and at no other point in S.

Maximum Values of $\Re(f)$ and $\Im(f)$, if f is analytic

- Let $f(z) = u(x, y) + iv(x, y)$ be a non-constant, analytic function in a closed and bounded set S.
- Consider the function $g(z) = \exp(f(z))$ and $h(z) = \exp(-i\ f(z))$. Observe that $g(z)$ and $h(z)$ are analytic in S.
- The maximum values of $|g(z)| = e^{u(x,y)}$ and $|h(z)| = e^{v(x,y)}$ are attained only on the boundary of S by maximum modulus theorem.
- Since $u(x, y)$ and $v(x, y)$ are real valued function and real exponential function e^t on $\mathbb{R}$ is strictly increasing on $\mathbb{R}$, it follows that the maximum values of $u(x, y)$ and $v(x, y)$ are attained only on the boundary of S and not at any interior point of S.

Hermann Schwarz (1843 - 1921)
German Mathematician

Schwarz originally studied chemistry in Berlin but Kummer and Weierstrass persuaded him to change to mathematics. Between 1867 and 1869 he worked in Halle, then in Zürich. From 1875 he worked at Gottingen University, dealing with the subjects of complex analysis, differential geometry and the calculus of variations. Schwarz is known for his work in Complex Analysis.

Schwarz Lemma

The following result is a consequence of maximum-modulus theorem.

Theorem

Schwarz Lemma:

Let $D = \{z \in \mathbb{C} \, : \, |z| < 1\}$ be the open unit disk. Let $f : D \to D$ be analytic in D with $f(0) = 0$. Then

- $|f(z)| \leq |z|$ for all $z \in D$,
- $|f'(0)| \leq 1$.

Moreover if $|f(z)| = |z|$ for some non-zero z in D or $|f'(0)| = 1$ then $f(z) = az$ for all $z \in D$ where a is a complex constant with $|a| = 1$.

Proof: Given in Tutorial Sheet to prove it.

Giacinto Morera (1856 - 1909)

Giacinto Morera was an Italian engineer and mathematician. He is known for Morera's theorem in the theory of functions of a complex variables and for his work in the theory of linear elasticity.
Ref: https://en.wikipedia.org/wiki/Giacinto_Morera

Morera's Theorem

The following theorem is some sort of converse to the Cauchy-Goursat theorem.

Theorem

Morera's Theorem: If a function f is continuous in a simply connected domain D and in $\int_C f(z)\, dz = 0$ for every simple closed contour C lying in D, then f is analytic throughout D.

Proof: Fix a point z_0 in D and define

$$F(z) = \int_{z_0}^{z} f(w)\, dw \qquad \text{for } z \in D\,.$$

$$\implies \quad F(z+\Delta z) - F(z) = \int_{z}^{z+\Delta z} f(w)\, dw\,.$$

Observe that $\int_{z}^{z+\Delta z} dw = \Delta z \quad \implies \quad f(z) = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\, dw\,.$

Continuation of Proof of Morera's Theorem

$$\frac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)=\frac{1}{\Delta z}\int_z^{z+\Delta z}(f(w)-f(z))\,dw$$

Since f is continuous at the point z, for any given $\epsilon > 0$, $\exists \delta > 0$ such that

$$|w-z|<\delta \qquad \Longrightarrow \qquad |f(w)-f(z)|<\epsilon\ .$$

Choose $|\Delta z| < \delta$. Then

$$\left|\frac{F(z+\Delta z)-F(z)}{\Delta z}-f(z)\right| \le \frac{1}{|\Delta z|}\int_z^{z+\Delta z}|f(w){-}f(z)|\,|dw| < \frac{\epsilon\,|\Delta z|}{|\Delta z|}=\epsilon$$

Thus, $F'(z)=f(z)$ for all $z \in D$. This implies that $F(z)$ is analytic in D. Since $F(z)$ is analytic in D, the derivatives $F^{(n)}(z)$ for all $n \in \mathbb{N}$ exist in D. Since $F^{(n)}(z)=f^{(n-1)}(z)$ for all $z \in D$ and for each $n \in \mathbb{N}$, it follows that f is analytic in D. This completes the proof.