Complex Analysis
Lecture 03

MA201 Mathematics III

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Learning Outcome of this Lecture

We learn

- Complex Functions and its visualization
- Limits of Functions
- Point at Infinity (∞), Extended Complex Plane and Riemann Sphere
- Limits involving ∞
- Continuity
- Properties of Continuous Functions
- Differentiation
- Properties of Differentiable Functions

Complex Functions

Definition

A complex valued function f of a complex variable is a rule that assigns to each complex number z in a set $D \subseteq \mathbb{C}$ one and only complex value w. We write $w = f(z)$ and call w the image of z under f. The set D is called the domain of the definition of f and the set of all images $R = \{w = f(z) \ : \ z \in D\}$ is called the range of f.

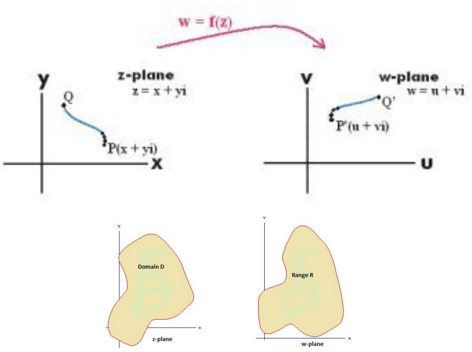
Usually, the real and imaginary parts of z are denoted by x and y, and those of the image point w are denoted by u and v respectively, so that $w = f(z) = u + i\,v$, where $u \equiv u(z) = u(x,\ y)$ and $v \equiv v(z) = v(x,\ y)$ are real valued functions of $z = x + iy$.

Example: Consider the function $f(z) = z^2$ for $z \in \mathbb{C}$. This function assigns to each complex number z in $\mathbb{C}$ one and only complex value $w = z^2$. The real and imaginary parts of $f(z)$ are given by

$$\Re(f(z)) = u(x,\ y) = x^2 - y^2 \qquad \Im(f(z)) = v(x,\ y) = 2xy\ .$$

Visualizing Complex Functions

In order to investigate a complex function $w = f(z)$, it is necessary to visualize it.
We view z and its image w as points in the complex plane, so that f becomes a transformation or mapping from D in the z-plane (xy-plane) on to the range R in the w-plane (uv-plane).



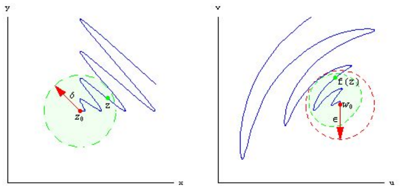
Limits of functions

Definition

Let $w = f(z)$ be a complex function of a complex variable z that is defined for all values of z in some neighborhood of z_0, except perhaps at the point z_0. We say that f has the limit w_0 as z approaches z_0 if **for each** positive number $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(z) - w_0| < \epsilon \qquad \text{whenever} \qquad 0 < |z - z_0| < \delta \,.$$

We write it as $\lim\limits_{z \to z_0} f(z) = w_0$.



- Geometrically, this says that for each ϵ-neighborhood $B_\epsilon(w_0) = \{w \in \mathbb{C} \,:\, |w - w_0| < \epsilon\}$ of the point w_0 in the w-plane, there exists a deleted or punctured δ-neighborhood $B_\delta^*(z_0) = \{z \in \mathbb{C} \,:\, 0 < |z - z_0| < \delta\}$ of z_0 in the z-plane such that $f(B_\delta^*(z_0)) \subset B_\epsilon(w_0)$.
- In case of functions $f : \mathbb{R} \to \mathbb{R}$, the variable x approaches the point x_0 in only two directions, either right or left. But, in the complex case, z can approach z_0 from any direction. That is, for the limit $\lim\limits_{z \to z_0} f(z)$ to exist, it is required that $f(z)$ *must* approach the same value no matter how z approaches z_0.

Example 1: If $f(z) = 2i/z$ then examine the existence of $\lim\limits_{z \to i} f(z)$.

Example 2: If $f(z) = \overline{z}$ then examine the existence of $\lim\limits_{z \to (1+2i)} f(z)$.

Example 3: If $f(z) = \Re(z)/|z|$ then examine the existence of $\lim\limits_{z \to 0} f(z)$.

Example 4: If $f(z) = z/\overline{z}$ then examine the existence of $\lim\limits_{z \to 0} f(z)$.

Limit of $f(z)$ and Limit of $\Re(f(z))$ and $\Im(f(z))$

Theorem

Let $f(z) = u(x,\ y) + i\,v(x,\ y)$ *be a complex function that is defined in some neighborhood of* z_0, *except perhaps at* $z_0 = x_0 + i\,y_0$. *Then*

$$\lim_{z \to z_0} f(z) = w_0 = u_0 + i\,v_0$$

if and only if

$$\lim_{(x,\,y)\to(x_0,\,y_0)} u(x,\ y) = u_0 \qquad \text{and} \qquad \lim_{(x,\,y)\to(x_0,\,y_0)} v(x,\ y) = v_0\ .$$

Example: Let $f(z) = z^2$. Then, $f(z) = u(x,\ y) + i\,v(x,\ y)$ where $u(x,\ y) = x^2 - y^2$ and $v(x,\ y) = 2xy$. Using above theorem, show that

$$\lim_{z \to (1+2i)} z^2 = -3 + 4\,i\ .$$

Limit of Functions and Algebraic Operations

Theorem

If $\lim_{z \to z_0} f(z) = A$ *and* $\lim_{z \to z_0} g(z) = B$ *then*

$$\lim_{z \to z_0} k\, f(z) \;=\; k\, A\,, \quad \textit{where } k \textit{ is a complex constant}\,,$$

$$\lim_{z \to z_0} (f(z) + g(z)) = A + B\,,$$

$$\lim_{z \to z_0} (f(z) - g(z)) = A - B\,,$$

$$\lim_{z \to z_0} f(z)g(z) = AB\,,$$

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{A}{B} \quad \textit{provided } B \neq 0\,.$$

Point at Infinity ∞ and the Extended Complex Plane

It is convenient to include with the complex number system $\mathbb{C}$ one ideal element, called point at infinity, denoted by the symbol ∞. Then the set $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called the extended complex plane and satisfies the following properties.

- For $z \in \mathbb{C}$,

$$z + \infty = \infty + z = z - \infty = \infty, \qquad \text{and} \qquad \frac{z}{\infty} = 0\ .$$

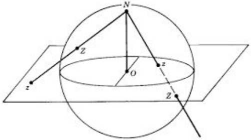
- For $z \in \mathbb{C} \setminus \{0\}$,

$$z \cdot \infty = \infty \cdot z = \infty, \qquad \text{and} \qquad \frac{z}{0} = \infty\ .$$

- $\infty \cdot \infty = \infty$.

Expressions such as $\infty + \infty$, $\infty - \infty$, $0 \cdot \infty$, ∞/∞ are not defined since they do not lead to meaningful results.

Riemann Sphere and Stereographic Projection



- Join the North Pole $N = (0, 0, 1)$ with the complex number $z = x + iy$ by a straight line L which pierce the sphere at Z.
- The mapping $z \mapsto Z$ gives one-to-one correspondence between $S \setminus \{N\}$ and $\mathbb{C}$.
- As $|z|$ approaches ∞ (along any direction in the plane), the corresponding point Z on S approaches N.
- Associate the North Pole N with the point at infinity ∞.
- $|z| > 1 \mapsto$ Upper hemisphere of S. $|z| < 1 \mapsto$ Lower hemisphere of S. $|z| = 1 \mapsto$ Equator of S.
- S is called the Riemann sphere. This bijection between S and $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is called the Stereographic Projection.

Limits involving infinity

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$. *Let* z_0 *be a limit point of* D. *Then,* $\lim_{z \to z_0} f(z) = \infty$ *if for each* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that*

$$0 < |z - z_0| < \delta \qquad \Longrightarrow \qquad |f(z)| > 1/\epsilon\ .$$

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$. *Let* ∞ *be a limit point of* D. *Then,* $\lim_{z \to \infty} f(z) = w_0$ *if for each* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that*

$$|z| > 1/\delta \qquad \Longrightarrow \qquad |f(z) - w_0| < \epsilon\ .$$

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$. *Let* ∞ *be a limit point of* D. *Then,* $\lim_{z \to \infty} f(z) = \infty$ *if for each* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that*

$$|z| > 1/\delta \qquad \Longrightarrow \qquad |f(z)| > 1/\epsilon\ .$$

Results related Limits involving Infinity

1. $$\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0 \, .$$

2. $$\lim_{z \to \infty} f(z) = w_0 \iff \lim_{z \to 0} f\left(1/z\right) = w_0 \, .$$

3. $$\lim_{z \to \infty} f(z) = \infty \iff \lim_{z \to 0} \frac{1}{f(1/z)} = 0 \, .$$

Exercises: Find (i) $\displaystyle\lim_{z \to \infty} \frac{4z^2}{(z-1)^2}$, (ii) $\displaystyle\lim_{z \to 1} \frac{1}{(z-1)^3}$,
(iii) $\displaystyle\lim_{z \to \infty} \frac{z^2+1}{z-1}$.

Continuous functions

Definition

Let $f(z)$ be a complex function of a complex variable z that is defined for all values of z in some neighborhood of z_0. We say that f is continuous at z_0 if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|z - z_0| < \delta \qquad \Longrightarrow \qquad |f(z) - f(z_0)| < \epsilon \, .$$

Equivalently, $f(z)$ is continuous at the point z_0 if $\lim_{z \to z_0} f(z)$ exists and is equal to $f(z_0)$.

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$. We say that f is continuous in the set D if f is continuous at each point of D.

Geometrical Interpretation of Continuity

To be continuous at z_0, the function f should map Near by points of z_0 in to Near by points of $f(z_0)$.
Near by concept is written in terms of neighborhood.

The continuity of $f(z)$ at a point z_0 can be interpreted geometrically as for each ϵ-neighborhood $B_\epsilon(f(z_0)) = \{w \in \mathbb{C} \, : \, |w - f(z_0)| < \epsilon\}$ of the point $f(z_0)$ in the w-plane, there exists a δ-neighborhood $B_\delta(z_0) = \{z \in \mathbb{C} \, : \, |z - z_0| < \delta\}$ of z_0 in the z-plane such that the function $f(z)$ maps $B_\delta(z_0)$ inside $B_\epsilon(f(z_0))$.

Example: *Let* $f(z) = z^2$*. Then,*

$$\lim_{z \to (1+2i)} f(z) = \lim_{z \to (1+2i)} z^2 = (1+2i)^2 = -3 + 4\,i = f(1+2i)\;.$$

Therefore, the function $f(z)$ *is continuous at the point* $(1+2i)$*.*

Example: *Let* $f(z) = \Re(z)/|z|$ *for* $z \neq 0$ *and* $f(0) = 1$*. The function* $f(z)$ *is* **not** *continuous at* 0*, since* $\lim_{z \to 0} \dfrac{\Re(z)}{|z|}$ *does not exist.*

Example: *Let* $f(z) = \Re(z)/|1+z|$ *for* $z \neq 0$ *and* $f(0) = 1$*. The function* $f(z)$ *is* **not** *continuous at* 0*, since* $\lim_{z \to 0} \dfrac{\Re(z)}{|1+z|} = 0$ *which is not equal to* $f(0) = 1$*.*

Results on Continuity

Theorem

Let $f(z) = u(x,\ y) + i\ v(x,\ y)$ *be defined in some neighborhood of* $z_0 = x_0 + i\ y_0$*. Then,* f *is continuous at* z_0 *if and only if* $u(x, y)$ *and* $v(x, y)$ *are continuous at* $(x_0,\ y_0)$*.*

Theorem

Suppose that the functions f *and* g *are continuous at* z_0*. Then, the following functions are continuous at* z_0*: (i)* $f(z) + g(z)$*, (ii)* $f(z) - g(z)$*, (iii)* $f(z)g(z)$ *and (iv)* $\frac{f(z)}{g(z)}$ *provided that* $g(z_0) \neq 0$*.*

Theorem

Suppose that f *is continuous at* z_0 *and* $g(z)$ *is continuous at* $f(z_0)$*. Then, the composition function* $h = g \circ f = g(f(z))$ *is continuous at* z_0*.*

Results on Continuity (continuation...)

Theorem

Suppose that $f(z)$ is continuous at z_0. Then, $|f(z)|$ and $\overline{f(z)}$ are continuous at z_0.

Theorem

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$. If D is a connected set and f is continuous in D then the set $f(D)$ is a connected set. That is, Continuous image of connected set is connected.

Theorem

Let $f : D \subset \mathbb{C} \to \mathbb{C}$. If D is a compact set and f is continuous in D then the set $f(D)$ is a compact set. That is, Continuous image of compact set is compact. Further $|f|$ attains its maximum and minimum values in D.

DIFFERENTIABLE FUNCTIONS

Differentiability

Definition

Let $f : D \subseteq \mathbb{C} \to \mathbb{C}$ where D is an open set. Let $z_0 \in D$. If

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, then f is said to be differentiable at the point z_0, and the number

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

is called the derivative of f at z_0.

If we write $\Delta z = z - z_0$, then the above definition can be expressed in the form

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z}$$

We can also use the Leibnitz notation for the derivative, $\frac{df}{dz}(z_0)$, or $\frac{df}{dz}|_{z=z_0}$.

Note: In the complex variable case there are infinitely many directions in which a variable can approach a point z_0. But, in the real case, there are only two directions, namely, left and right to approach. So the statement that a function of a complex variable has a derivative is **stronger** than the same statement about a function of a real variable. For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is differentiable on $\mathbb{R} \setminus \{0\}$. But, if we consider the same function as a function of complex variable, that is, $f : \mathbb{C} \to \mathbb{R}$ given by $f(z) = |z|$, then it is nowhere differentiable in $\mathbb{C}$ (which will be proved later). Therefore, it has lost the differentiability on the set $\mathbb{R} \setminus \{0\}$ in the complex case.

Examples

Example 1: By using the definition of derivative, let us compute $f'(z)$ at an arbitrary point $z_0 \in \mathbb{C}$ for the function $f(z) = z$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z - z_0}{z - z_0} = \lim_{z \to z_0} 1 = 1 .$$

Therefore, the derivative of $f(z) = z$ is $f'(z) = 1$ for any $z \in \mathbb{C}$.

Example 2: By using the definition of derivative, let us compute $f'(z)$ at an arbitrary point $z_0 \in \mathbb{C}$ for the function $f(z) = z^2$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} (z + z_0) = 2z_0 .$$

Therefore, the derivative of $f(z) = z^2$ is $f'(z) = 2z$ for any $z \in \mathbb{C}$.

Examples

Example 3:
Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = \overline{z}$ for $z \in \mathbb{C}$.
Examine the differentiability of $f(z)$ at each point z of $\mathbb{C}$.
Answer: $f(z) = \overline{z}$ is not differentiable at any point of $\mathbb{C}$ (No where differentiable in $\mathbb{C}$).
Details are worked out on the board.

Example 4:
Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (x, -y)$ for $(x, y) \in \mathbb{R}^2$.
Examine the Frechet differentiability of f on $\mathbb{R}^2$ (Differentiability in $\mathbb{R}^2$ from multivariable calculus)
Compare with the above example (Example 3).
Details are worked out on the board.

Examples

Example 5:
On $\mathbb{C}$, examine the differentiability of $f(z) = |z|^2$ for $z \in \mathbb{C}$.
Answer: The $f(z) = |z|^2$ is not differentiable in $\mathbb{C} \setminus \{0\}$. Further it is differentiable at $z = 0$.
That is, $|z|^2$ is differentiable only at the point 0 in $\mathbb{C}$.
Comparison: The function $f(x) = |x|^2$ for $x \in \mathbb{R}$ is (real) differentiable at each point of $\mathbb{R}$.
Details are worked out on the board.

Example 6:
On $\mathbb{C}$, examine the differentiability of $f(z) = |z|$ for $z \in \mathbb{C}$.
Answer:The $f(z) = |z|$ is not differentiable at any point of $\mathbb{C}$.
That is, $|z|$ is nowhere differentiable in $\mathbb{C}$.
Comparison: The function $f(x) = |x|$ for $x \in \mathbb{R}$ is (real) differentiable at each point of $\mathbb{R} \setminus \{0\}$.
Details are worked out on the board.

Results and Properties

1. If $f(z)$ is differentiable at z_0, then f is continuous at z_0.
2. If $f(z) \equiv c$ is a constant function, then $f'(z) = 0$.

Theorem

Let $f(z)$ and $g(z)$ be two differentiable functions. Then,

1. *Sum:* $\dfrac{d}{dz}\left[f(z) + g(z)\right] = f'(z) + g'(z)$
2. *Product:* $\dfrac{d}{dz}\left[f(z)g(z)\right] = f'(z)g(z) + f(z)g'(z)$
3. *Quotient:* $\dfrac{d}{dz}\left[\dfrac{f(z)}{g(z)}\right] = \dfrac{f'(z)g(z) - f(z)g'(z)}{(g(z))^2}$ *provided* $g(z) \neq 0$
4. *Composition:* $\dfrac{d}{dz}\left[f(g(z))\right] = f'(g(z))\, g'(z)$