MA 201: Lecture - 3
Solving quasilinear PDEs

Quasi-linear First-Order PDEs

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- This shows that the vector (a, b, c) and the gradient vector ∇F are orthogonal.
- In other words, the vector (a, b, c) lies in the tangent plane of the surface $\mathcal{S}$ at each point in the (x, y, u)-space where $\nabla F \neq 0$.

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$$x = x(t), \quad y = y(t), \quad \text{and} \quad u = u(t), \tag{3}$$

satisfying

$$\begin{aligned} \frac{dx}{dt} &= a(x(t), y(t), u(t)), \\ \frac{dy}{dt} &= b(x(t), y(t), u(t)), \\ \frac{du}{dt} &= c(x(t), y(t), u(t)), \end{aligned} \tag{4}$$

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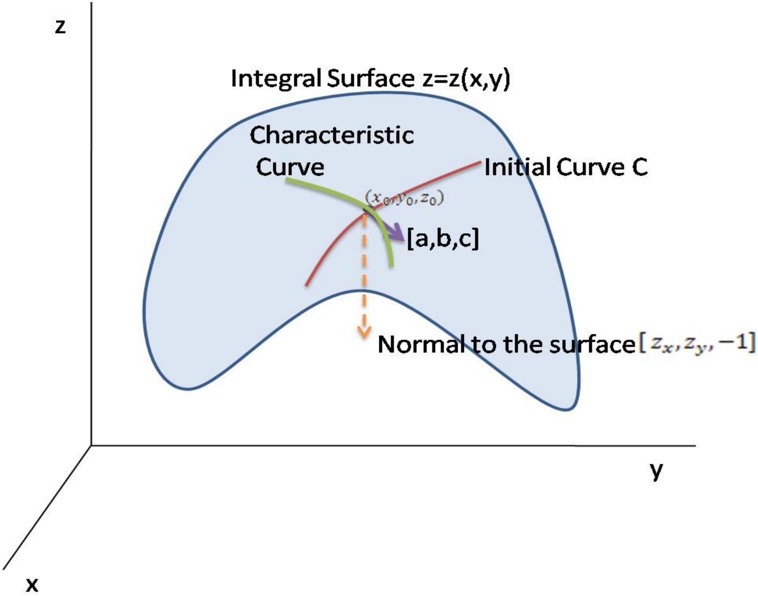
- The solutions of (4) are called the characteristic curves of the quasi-linear equation (1).


z
Integral Surface z=z(x,y)
Characteristic
Curve
Initial Curve C
(x_0, y_0, z_0)
[a,b,c]
Normal to the surface $[z_x, z_y, -1]$
y
x

Remarks.

- A smooth union of the characteristic curves given by (4) gives a surface in (x, y, u)-space having the property that this surface is tangent to the vector field (a, b, c) at every point. Thus it is an integral surface.

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- The characteristic equations (4) for x and y are not, in general, uncoupled from the equation for u and hence differ from those in the linear case.
- The characteristics equations (4) can be expressed in the nonparametric form as

$$\frac{dx}{a(x,y,u)} = \frac{dy}{b(x,y,u)} = \frac{du}{c(x,y,u)}. \tag{5}$$

Formation of quasi-linear First-Order PDEs

Theorem

If $\phi = \phi(x, y, u)$ and $\psi = \psi(x, y, u)$ are two given functions of x, y and u and if $G(\phi, \psi) = 0$, where G is an arbitrary function of ϕ and ψ, then $u = u(x, y)$ satisfies a first order PDE

$$\frac{\partial u}{\partial x}\frac{\partial(\phi,\psi)}{\partial(y,u)} + \frac{\partial u}{\partial y}\frac{\partial(\phi,\psi)}{\partial(u,x)} = \frac{\partial(\phi,\psi)}{\partial(x,y)} \tag{6}$$

where

$$\frac{\partial(\phi,\psi)}{\partial(x,y)} = \begin{vmatrix} \phi_x & \phi_y \\ \psi_x & \psi_y \end{vmatrix}.$$

Remarks.

- Since G is an arbitrary function that leads to equation (6), so G is called the general solution for the equation.
- This gives an idea to find the general solution for the quasi-linear equation
$$a(x, y, u)\frac{\partial u}{\partial x} + b(x, y, u)\frac{\partial u}{\partial y} = c(x, y, u).$$

How to prove it?

- Differentiate $G(\phi, \psi) = 0$ with respect to x and y respectively:

$$
\begin{aligned}
G_\phi\left(\phi_x + u_x\phi_u\right) + G_\psi\left(\psi_x + u_x\psi_u\right) &= 0, && (7)\\
G_\phi\left(\phi_y + u_y\phi_u\right) + G_\psi\left(\psi_y + u_y\psi_u\right) &= 0. && (8)
\end{aligned}
$$

- Nontrivial solutions for $\frac{\partial G}{\partial \phi} = G_\phi$ and $\frac{\partial G}{\partial \psi} = G_\psi$ can be found if

$$
\begin{vmatrix} \phi_x + u_x\phi_u & \psi_x + u_x\psi_u \\ \psi_y + u_y\psi_u & \psi_y + u_y\psi_u \end{vmatrix} = 0
$$

- Expanding this determinant gives first-order, quasi-linear equation (6).

Remarks (contd.)

- Consider following quasi-linear equation:

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u). \tag{9}$$

- Consider following equation with general solution $G(\phi,\psi) = 0$:

$$\frac{\partial u}{\partial x}\frac{\partial(\phi,\psi)}{\partial(y,u)} + \frac{\partial u}{\partial y}\frac{\partial(\phi,\psi)}{\partial(u,x)} = \frac{\partial(\phi,\psi)}{\partial(x,y)}. \tag{10}$$

- Both equation will be identical if we select $\phi = \phi(x,y,u)$ and $\psi = \psi(x,y,u)$ in such way that

$$a = \frac{\partial(\phi,\psi)}{\partial(y,u)}, \quad b = \frac{\partial(\phi,\psi)}{\partial(u,x)}, \quad c = \frac{\partial(\phi,\psi)}{\partial(x,y)}.$$

- What is (ϕ,ψ)?

Theorem (The method of Lagrange)

The general solution of the quasi-linear PDE (1) is

$$F(\phi, \psi) = 0, \tag{11}$$

where F is an arbitrary function, the functions $\phi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$ form a solution of the equations

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}. \tag{12}$$

Proof. If $\phi(x, y, u) = c_1$ and $\psi(x, y, u) = c_2$ satisfy equation (12) then the equations

$$\phi_x dx + \phi_y dy + \phi_u du = 0,$$
$$\psi_x dx + \psi_y dy + \psi_u du = 0$$

are compatible with (12).

Thus, we must have

$$
\begin{aligned}
a\phi_x + b\phi_y + c\phi_u &= 0, \\
a\psi_x + b\psi_y + c\psi_u &= 0.
\end{aligned}
$$

Solving these equations for a, b and c, we obtain

$$
\frac{a}{\frac{\partial(\phi,\psi)}{\partial(y,u)}} = \frac{b}{\frac{\partial(\phi,\psi)}{\partial(u,x)}} = \frac{c}{\frac{\partial(\phi,\psi)}{\partial(x,y)}}. \tag{13}
$$

Differentiate $F(\phi,\psi) = 0$ with respect to x and y, respectively, to have

$$
\frac{\partial F}{\partial \phi}\left\{\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x}\right\} + \frac{\partial F}{\partial \psi}\left\{\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial x}\right\} = 0
$$

$$
\frac{\partial F}{\partial \phi}\left\{\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y}\right\} + \frac{\partial F}{\partial \psi}\left\{\frac{\partial \psi}{\partial y} + \frac{\partial \psi}{\partial u}\frac{\partial u}{\partial y}\right\} = 0.
$$

Eliminating $\frac{\partial F}{\partial \phi}$ and $\frac{\partial F}{\partial \psi}$ from these equations, we obtain

$$\frac{\partial u}{\partial x}\frac{\partial(\phi,\psi)}{\partial(y,u)} + \frac{\partial u}{\partial y}\frac{\partial(\phi,\psi)}{\partial(u,x)} = \frac{\partial(\phi,\psi)}{\partial(x,y)}. \tag{14}$$

In view of (13), equation (14) yields

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = c.$$

Thus, we find that $F(\phi,\psi) = 0$ is a solution of equation (1). This completes the proof.

Example

Find the general integral of $xu_x + yu_y = u$.

Solution. The associated system of equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{du}{u}.$$

From the first two relations, we have

$$\frac{dx}{x} = \frac{dy}{y} \Longrightarrow \ln x = \ln y + \ln c_1 \Longrightarrow \frac{x}{y} = c_1.$$

Similarly,

$$\frac{du}{u} = \frac{dy}{y} \Longrightarrow \frac{u}{y} = c_2.$$

Take $\phi(x, y, u) = \frac{x}{y}$ and $\psi(x, y, u) = \frac{u}{y}$. The general integral is given by

$$F(\frac{x}{y}, \frac{u}{y}) = 0.$$

Example

Find the general integral of the equation

$$u(x+y)u_x + u(x-y)u_y = x^2 + y^2.$$

Solution. The characteristic equations are

$$\frac{dx}{u(x+y)} = \frac{dy}{u(x-y)} = \frac{du}{x^2+y^2}.$$

Each of these ratios is equivalent to

$$y\frac{dx}{dt} + x\frac{dy}{dt} - u\frac{du}{dt} = 0 = x\frac{dx}{dt} - y\frac{dy}{dt} - u\frac{du}{dt}.$$

Consequently, we have

$$d\{xy - \frac{u^2}{2}\} = 0 \quad \text{and} \quad d\{\frac{1}{2}(x^2 - y^2 - u^2)\} = 0.$$

Integrating we obtain two integrals

$$2xy - u^2 = c_1 \quad \text{and} \quad x^2 - y^2 - u^2 = c_2,$$

where c_1 and c_2 are arbitrary constants. Take $\phi(x,y,u) = 2xy - u^2$ and $\psi(x,y,u) = x^2 - y^2 - u^2$.
Hence the general solution is

$$F(2xy - u^2, x^2 - y^2 - u^2) = 0,$$

where F is an arbitrary function.

Theorem (Existence and uniqueness result)

Let $a(x, y, u)$, $b(x, y, u)$ and $c(x, y, u)$ in (1) have continuous partial derivatives with respect to x, y and u variables. Let the initial curve Γ be described parametrically as

$$x = x(s), \quad y = y(s), \quad \textit{and} \quad u(s) = u(x(s), y(s)).$$

The initial curve Γ has a continuous tangent vector and

$$J(s) = \frac{dy}{ds} a[x(s), y(s), u(s)] - \frac{dx}{ds} b[x(s), y(s), u(s)] \neq 0 \qquad (15)$$

on Γ. Then, there exists a unique solution $u = u(x, y)$, defined in some neighborhood of the initial curve Γ, satisfies (1) and the initial condition $u(x(s), y(s)) = u(s)$.

Example

$$\textbf{PDE}:\ u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0;\ \ \textbf{IC}:\ u(x,0) = f(x).$$

- **Step 1.** (Finding characteristic curves)
  To solve the IVP, we parameterize the initial curve as

  $$x = s, \quad y = 0, \quad u = f(s).$$

  The characteristic equations are

  $$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = 1, \quad \frac{du}{dt} = 0.$$

  Let the solutions be denoted as $x(t,s)$, $y(t,s)$, and $u(t,s)$. We immediately find that

  $$u(t,s) = c_1(s), \quad x(t,s) = c_1(s)t + c_2(s), \quad y(t,s) = t + c_3(s),$$

  where c_i, $i = 1,2,3$ are constants to be determined using IC.

- **Step 2.** (Applying IC) The initial conditions at $t = 0$ are given by

$$x(0,s) = s, \quad y(0,s) = 0, \quad u(0,s) = f(s).$$

Using these conditions, we obtain

$$u(t,s) = f(s), \quad x(t,s) = f(s)t + s, \quad y(t,s) = t.$$

- **Step 3.** (Writing the parametric form of the solution)
The solution is thus given by $u(t,s) = f(s)$ where

$$x(t,s) = f(s)t + s, \quad y(t,s) = t.$$

- **Step 4.** (Expressing $u(t,s)$ in terms of $u(x,y)$)
Applying the condition (15), we find that $J(s) = -1 \neq 0$, along the entire initial curve. We can immediately solve for $s(x,y)$ and $t(x,y)$ to obtain

$$s(x,y) = x - tf(s) = x - yu, \quad t(x,y) = y.$$

Thus the solution can also be given in implicit form as

$$u = f(x - yu).$$

Example

$$\textbf{PDE}:\ (y+u)\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x - y;\ \ \textbf{IC}:\ u(x,1) = 1 + x.$$

Observe that the Jacobian

$$J = \begin{vmatrix} 2+s & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0.$$

Therefore, we conclude that there exists an integral surface at least in the vicinity of Γ.

The characteristics equations and initial conditions are:

$$(i)\ \frac{dx}{dt} = y + u,\ \ (ii)\ \frac{dy}{dt} = y,\ \ (iii)\ \frac{du}{dt} = x - y.$$

$$x(0,s) = s, \qquad y(0,s) = 1, \qquad u(0,s) = 1 + s.$$

From (ii), we immediately have $y(t,s) = e^t$.
Adding (i) and (iii), we get

$$\frac{d}{dt}(u+x) = (u+x) \Rightarrow u(t,s) + x(t,s) = (1+2s)e^t.$$

From (i), we again obtain

$$\frac{dx}{dt} + x = (2+2s)e^t \Rightarrow x(s,t) = (1+s)e^t - e^{-t},$$

and

$$u(t,s) = se^t + e^{-t}.$$

Noting that

$$x - y = se^t - e^{-t},$$

we finally get

$$u(x,y) = 2/y + (x-y).$$

Note that the solution is not global (it becomes singular on the x-axis), but it is well defined near the initial curve.