Declaration

I hereby declare that the project titled "Estimation of Scale Parameter for Cauchy Distribution - A Moment based approach " has been carried out by us as part of our summer internship under the guidance of Prof. Sabyasachi Bhattacharya at the Indian Statistical Institute, Kolkata.

This work is the result of our joint efforts, and it has not been submitted elsewhere for any academic or professional purpose. All simulations, analyses, and results presented in this report were conducted using R programming, and appropriate references have been cited wherever required.

I have maintained academic integrity and followed ethical research practices throughout the course of this project.

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Team Members:

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Introduction

I undertook my internship at the Indian Statistical Institute (ISI), Kolkata. The internship was conducted in offline mode during the summer of 2025, spanning a period of six weeks under the guidance of Prof. Sabyasachi Bhattacharya.

In statistics, Cauchy distribution is a well known probability distribution. It is symmetric in shape, lies over whole real line. Also, it has heavy tail on both sides ie. contains significant extreme values. An exceptional property of this distribution is that no integer order moment exists. In such case, for parameter estimation procedure, usually quantile based estimators are used as method of moments should not be suitable here.One can note that, though any integer order moment does not exists, but fractional order moments of order (-1,1) are finite.With view from this angle, we use a modified method of moments estimation tool where we equate population fractional moments to sample generalized mean, for scale parameter of that distribution.From here, we derive a much simple scale estimator, depending on order of fractional moments.We also run a simulation experiment for comparison, and watch that there exists a range of order for which our estimator performs much better than other existing estimators (quartile deviation, maximum likelihood estimator) in respect to absolute bias, mean square error(mse).

Objectives

- To understand the statistical challenges in estimating the parameters of the Cauchy distribution, particularly the scale parameterσ.
- To explore and implement fractional moment-based estimators for the scale parameter and study about the limiting behavior of these estimators.
- To assess estimator performance through simulation studies in R using metrics like bias and mean squared error (MSE).

Work Done

Cauchy Distribution

The Cauchy distribution is a continuous probability distribution with location parameter μ and scale parameter σ , defined by the probability density function:

$$f(x; \mu, \sigma) = \frac{1}{\pi\sigma} \cdot \frac{1}{1 + \left(\frac{x-\mu}{\sigma}\right)^2}, \quad x \in \mathbb{R}$$

The scale parameter of a continuous probability distribution measures the dispersion of the distribution. In other words, a variation in the scale parameter stretches or shrinks the distribution horizontally and vertically without altering its basic shape.For this reason we want a estimator for σ.

Our approach: Fractional moment

We calculate,

$$\begin{aligned}
\mathbb{E}|X-\mu|^\alpha &= \frac{1}{\pi}\int_{-\infty}^{\infty} |x-\mu|^\alpha \cdot \frac{\sigma}{\sigma^2 + (x-\mu)^2} dx \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} |\sigma z|^\alpha \cdot \frac{\sigma}{\sigma^2 + (\sigma z)^2} \cdot \sigma dz \quad \left(\frac{x-\mu}{\sigma} = z\right) \\
&= \frac{\sigma^\alpha}{\pi}\int_{-\infty}^{\infty} \frac{|z|^\alpha}{1+z^2} dz \\
&= \frac{\sigma^\alpha}{\pi}\left[\int_{-\infty}^{0} \frac{(-z)^\alpha}{1+z^2} dz + \int_{0}^{\infty} \frac{z^\alpha}{1+z^2} dz\right] \\
&= \frac{2\sigma^\alpha}{\pi}\int_{0}^{\infty} \frac{z^\alpha}{1+z^2} dz \\
&= \frac{2\sigma^\alpha}{\pi}\int_{0}^{\infty} \frac{t^{\alpha/2} t^{-1/2}}{1+t} dt \quad (z^2 = t) \\
&= \frac{\sigma^\alpha}{\pi}\int_{0}^{\infty} \frac{t^{\frac{\alpha-1}{2}}}{1+t} dt \\
&= \frac{\sigma^\alpha}{\pi}\int_{0}^{\infty} \frac{t^{\frac{\alpha+1}{2}-1}}{(1+t)^{\frac{\alpha+1}{2}+1-\frac{\alpha+1}{2}}} dt \\
&= \frac{\sigma^\alpha}{\pi} \cdot \mathrm{Beta}\left(\frac{1+\alpha}{2}, \frac{1-\alpha}{2}\right) \\
&= \frac{\sigma^\alpha}{\pi} \cdot \frac{\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)}{\Gamma(1)} \\
&= \sigma^\alpha g(\alpha),
\end{aligned}$$

As we can see, the value of $\Gamma\left(\frac{1+\alpha}{2}\right)$ exists only when α takes values in the range $-1 < \alpha < 0$, and the value of $\Gamma\left(\frac{1-\alpha}{2}\right)$ exists only when α takes values in the range $0 < \alpha < 1$.Therefore, the range of α for which both $\Gamma\left(\frac{1+\alpha}{2}\right)$ and $\Gamma\left(\frac{1-\alpha}{2}\right)$ are defined is

$$\alpha \in (-1, 0) \cup (0, 1).$$

where,

$$g(\alpha) = \frac{1}{\pi}\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right) = \frac{1}{\cos(\alpha\pi/2)}.$$

Variance Estimate

So, from method of moment estimation procedure [Mukhopadhyay et al. (2021)], we get,

$$\frac{1}{n}\sum_{i=1}^{n}|x_i-\mu|^\alpha = \sigma^\alpha g(\alpha),$$

i.e.

$$\hat{\sigma}_\alpha = \left[\frac{1}{g(\alpha)n}\sum_{i=1}^{n}|x_i-\mu|^\alpha\right]^{\frac{1}{\alpha}}.$$

See that, our estimator is not defined for $\alpha = 0$. Here, we should use limiting case. First taking log,

$$\ln\hat{\sigma}_\alpha = \frac{\ln\sum_{i=1}^{n}|x_i-\mu|^\alpha - \ln ng(\alpha)}{\alpha} \quad (0/0) - \text{form};$$

and then taking limit,

$$\begin{aligned}\lim_{\alpha\to 0}\ln\hat{\sigma}_\alpha &= \lim_{\alpha\to 0}\frac{\frac{d}{d\alpha}\left(\sum_{i=1}^{n}|x_i-\mu_c|^\alpha\right)}{\sum_{i=1}^{n}|x_i-\mu_c|^\alpha} - \lim_{\alpha\to 0}\frac{g'(\alpha)}{g(\alpha)} = \lim_{\alpha\to 0}\frac{\sum_{i=1}^{n}\left[|x_i-\mu_e|^\alpha\ln|x_i-\mu_e|\right]}{\sum_{i=1}^{n}|x_i-\mu_e|^\alpha}\\ &= \frac{1}{n}\sum_{i=1}^{n}\ln|x_i-\mu_e|,\end{aligned}$$

$$\left(\lim_{\alpha\to 0}g(\alpha) = 1, \quad g'(\alpha) = \left(\frac{\pi}{2}\right)\sec(\alpha\pi/2)\tan(\alpha\pi/2), \quad \lim_{\alpha\to 0}g'(\alpha) = 0\right);$$

we get,

$$\ln\left[\lim_{\alpha\to 0}\hat{\sigma}_\alpha\right] = \ln\left(\prod_{i=1}^{n}|x_i-\mu_e|\right)^{1/n} \quad \text{i.e.,} \quad \lim_{\alpha\to 0}\sigma_\alpha = \left(\prod_{i=1}^{n}|x_i-\mu_e|\right)^{1/n}.$$

So, our *proposed estimator* is given by,

$$\hat{\sigma}_\alpha = \begin{cases}\left(\frac{1}{g(\alpha)n}\sum_{i=1}^{n}|x_i-\mu_e|^\alpha\right)^{1/\alpha}, & \text{for } \alpha\in(-1,0)\cup(0,1),\\ \left(\prod_{i=1}^{n}|x_i-\mu_e|\right)^{1/n}, & \text{for } \alpha\to 0,\end{cases}$$

$$g(\alpha) = \frac{1}{\pi}\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right) = \frac{1}{\cos(\alpha\pi/2)}.$$

Limitation with existing estimate:

As we can see, there is no direct limitation of the estimator, so we will compare it with:-

Quartile Deviation: Qd considers only central half of data, ignoring remaining half (ie, not amenable to algebraic treatment).

Maximum Likelihood Estimate: Mle estimate can be heavily biased for small samples. Also, computation may not always intensive.

Note 1. Let $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} C(\mu, \sigma^2)$, where μ is known, say μ_0. Then, for $-1 < \alpha < 1$,

$$\mathbb{E}\left[\frac{1}{g(\alpha)n}\sum_{i=1}^{n}|x_i - \mu_0|^\alpha\right] = \sigma^\alpha.$$

Note 2. Let $X_1, X_2, \ldots, X_n \overset{\text{iid}}{\sim} C(\mu, \sigma^2)$, where μ is unknown but we have a pilot estimate μ_p of it. Then, for $-1 < \alpha < 1$,

$$\hat{\sigma}_\alpha = \left(\frac{1}{g(\alpha)n}\sum_{i=1}^{n}|x_i - \mu_p|^\alpha\right)^{1/\alpha}$$

serves as an estimator of σ.

Note: The most popular choice of μ_p is sample median (me)

Implementation in R

This project investigates and compares multiple scale estimators for the Cauchy distribution, which is known for having undefined mean and variance. The data is simulated from a Cauchy distribution with location parameter $\mu = 5$ and scale parameter $\sigma = 2$

Estimators Used

- **Proposed Estimator $\hat{\sigma}_\alpha$:**

$$\hat{\sigma}_\alpha = \left(\frac{1}{g(\alpha)} \cdot \frac{1}{n}\sum_{i=1}^{n}|x_i - \text{Median}(x)|^\alpha\right)^{1/\alpha}, \quad \alpha \neq 0$$

  where

$$g(\alpha) = \frac{\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)}{\pi}$$

- **Geometric Mean Estimator $\hat{\sigma}_0$:** Limiting case of the above estimator as $\alpha \to 0$, implemented using `geometric.mean()` from the `psych` package.

- **MLE Estimator**: Estimated using `cauchy.mle()` from the `cauchypca` package.

- **Quartile Deviation (QD)**:

$$\text{QD} = \frac{Q_3 - Q_1}{2}$$

Simulation Details

- **Sample size:** $n = 14$
- **Number of simulations:** 1000
- **Alpha values:** $\alpha \in (-0.99, -0.01] \cup [0.01, 0.99)$, chosen to avoid singularities at $\alpha = 0$.

Additional analysis was also conducted near $\alpha = 0$. For each replication, Cauchy-distributed samples were generated, and all estimators were computed and stored for analysis.

Evaluation Metrics

For each estimator, the following statistical properties were computed:

- **Bias:**
$$\text{Bias} = \mathbb{E}[\hat{\sigma}] - \sigma$$
- **Mean Squared Error (MSE):**
$$\text{MSE} = \mathbb{E}\left[(\hat{\sigma} - \sigma)^2\right]$$

These metrics were used to evaluate the accuracy and consistency of each estimator under repeated sampling from the Cauchy distribution.

Visualisation of Results

To evaluate and compare the performance of the proposed scale estimator $\hat{\sigma}_\alpha$ with classical estimators like the sample QD and MLE, several visual tools were used. These include line plots of bias and MSE as functions of α, as well as histograms of the simulated estimates.

Absolute Bias vs α

A line plot of the absolute bias of $\hat{\sigma}_\alpha$ was generated over a range of α values from -0.5 to 0.5. Horizontal reference lines were also added to indicate the absolute bias of the QD and MLE estimators.

- The minimum absolute bias of the proposed estimator was observed near $\alpha = 0.06$.
- This result outperformed both QD and MLE estimators.
- The plot showed a clear U-shape, suggesting that careful selection of α improves estimator performance.

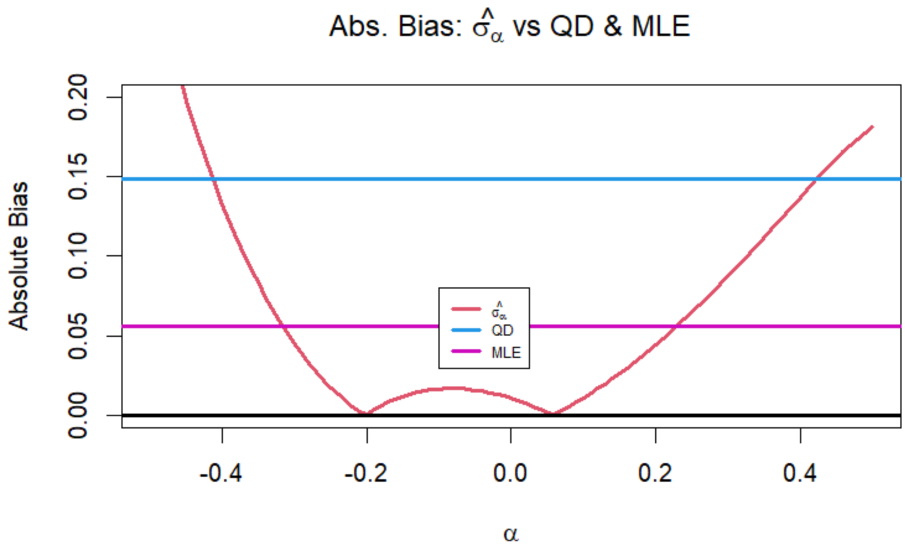


Figure 1: Bias of Estimator vs α

Mean Squared Error (MSE) vs α

Another line plot shows how the MSE of $\hat{\sigma}_\alpha$ changes with α. Horizontal lines were again drawn to mark the MSE values of QD and MLE.

- The minimum absolute bias of the proposed estimator was observed near $\alpha = 0$.
- This result outperformed both QD and MLE estimators.
- The plot showed a clear U-shape, suggesting that careful selection of α improves estimator performance.

These visual comparisons confirm that the proposed estimator is more reliable and accurate when the tuning parameter α is appropriately chosen.

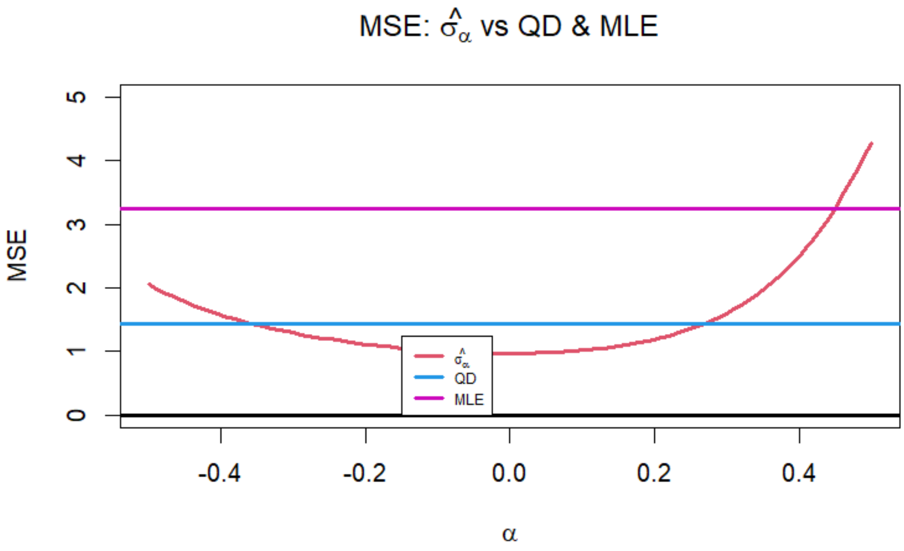


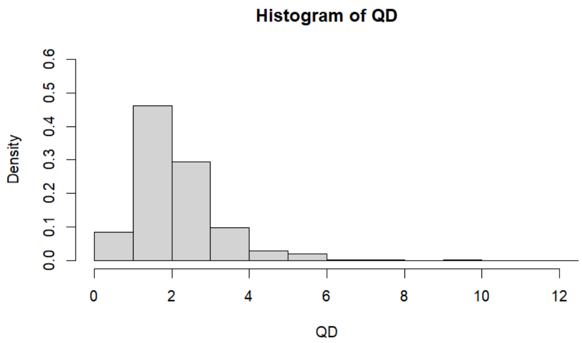
Figure 2: Bias of Estimator vs α

Histograms of Estimators

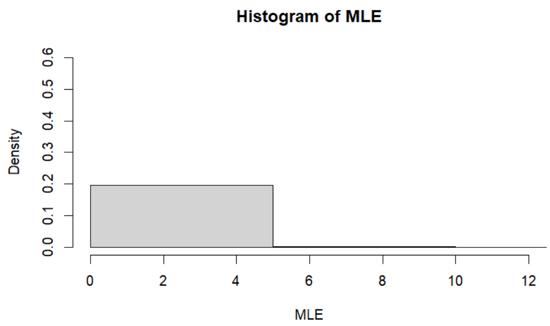
Histograms were plotted to visualise the distribution of the estimators across 1000 simulations:

- The QD estimator exhibited a moderate concentration near the true value but with a noticeable right skew and moderate spread, reflecting its sensitivity to sample variation. It ignores half the data (central 50%), making it unstable in small samples
- The MLE estimator showed a strong peak around the true scale but had a very heavy tail, indicating occasional extreme outliers that increase its variance.
- The proposed estimator $\hat{\sigma}_\alpha$ (for α minimizing bias and MSE) showed:
  - Strong concentration around σ=2 with little skew.
  - A more balanced and narrower distribution compared to both QD and MLE.

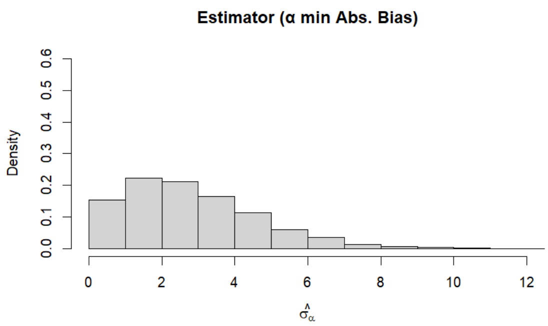
These visual comparisons confirm that the proposed estimator is more reliable and accurate when the tuning parameter α is appropriately chosen.



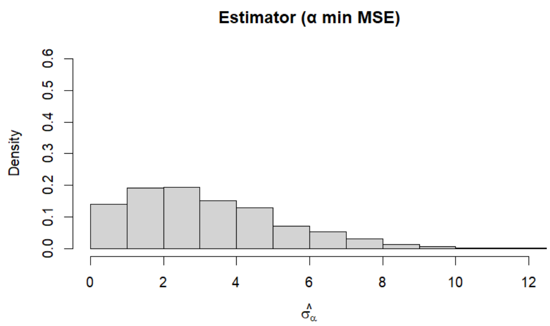
(a) **Histogram of QD Estimator**



(b) **Histogram of MLE Estimator**



(c) **Estimator with Minimum Absolute Bias**



(d) **Estimator with Minimum MSE**

Figure 3: **Histograms and Performance Plots of Estimators**

Resources and Tools

- **Books:** *Continuous Univariate Distributions, Volume 1* by Johnson et al. (1995), Casella, G., and Berger, R. L. (2002),

  Statistical Inference (2nd ed.). Duxbury,
- **Software:** R (v4.3.2), Overleaf
- **R Packages:** `psych`, `cauchypca`, `graphics`

R Code

```
set.seed(2025)

# g() function as per theory
g = function(a) {
  return(gamma((1 + a)/2) * gamma((1 - a)/2) / pi)
}

# Our proposed scale estimator using median as pilot location
f1 = function(x, a) {
  y = abs(x - median(x))
  return((mean(y^a) / g(a))^(1 / a))
}
# Geometric mean-based scale estimator ( = 0 limit)
library(psych)
f3 = function(x) {
  y = abs(x - median(x))
  return(geometric.mean(y))
}

# MLE scale estimator
library(cauchypca)
m1 = function(x) {
  mle = cauchy.mle(x)
  return(as.vector(mle$param)[2])
}

# Simulation parameters
n = 14
runs = 1000
alpha = c(seq(-0.99, -0.01, 0.01), seq(0.01, 0.99, 0.01))
q = length(alpha)
mu = 5
sigma = 2

# Storage
x = matrix(0, runs, n)
sig_hat1 = matrix(0, runs, q)
sig_est1 = rep(0, q)
med = rep(0, runs)
qd = rep(0, runs)
mle_sc = rep(0, runs)
est0_1 = rep(0, runs)
bias1 = rep(0, q)
MSE1 = rep(0, q)
v1 = rep(0, q)
```

```
# Main simulation loop
for (k in 1:runs) {
  x[k, ] = rcauchy(n, mu, sigma)

  med[k] = median(x[k, ])
  qd[k] = (quantile(x[k, ], 0.75) - quantile(x[k, ], 0.25)) / 2
  mle_sc[k] = m1(x[k, ])
  est0_1[k] = f3(x[k, ])

  for (i in 1:q) {
    sig_hat1[k, i] = f1(x[k, ], alpha[i])
  }
}

# Bias, variance, MSE calculations
for (i in 1:q) {
  sig_est1[i] = mean(sig_hat1[, i])
  bias1[i] = sig_est1[i] - sigma
  v1[i] = var(sig_hat1[, i])
  MSE1[i] = v1[i] + bias1[i]^2
}

# Special case for  = 0 (geometric mean)
bias0_1 = mean(est0_1 - sigma)
MSE0_1 = mean((est0_1 - sigma)^2)

# Absolute Bias and MSE of sample QD and MLE
abs(mean(qd - sigma))
abs(mean(mle_sc - sigma))
mean((qd - sigma)^2)
mean((mle_sc - sigma)^2)

# Construct  vector including  = 0, and extended bias/MSE
alpha_new = seq(-0.25, 0.25, 0.01)
bias1_new = c(bias1[75:99], bias0_1, bias1[100:124])
MSE1_new = c(MSE1[75:99], MSE0_1, MSE1[100:124])

# Identify  with minimum absolute bias and MSE
alpha_new[which.min(abs(bias1_new))]
abs(bias1_new)[which.min(abs(bias1_new))]
MSE1_new[which.min(abs(bias1_new))]

alpha_new[which.min(MSE1_new)]
abs(bias1_new)[which.min(MSE1_new)]
MSE1_new[which.min(MSE1_new)]

# Compare to QD and MLE scale
alpha_new[which.min(abs(abs(bias1_new) - abs(mean(qd - sigma))))]
```

```
abs(bias1_new)[which.min(abs(abs(bias1_new) - abs(mean(qd - sigma))))]
MSE1_new[which.min(abs(abs(bias1_new) - abs(mean(qd - sigma))))]

alpha_new[which.min(abs(MSE1_new - mean((qd - sigma)^2)))]
abs(bias1_new)[which.min(abs(MSE1_new - mean((qd - sigma)^2)))]
MSE1_new[which.min(abs(MSE1_new - mean((qd - sigma)^2)))]

alpha_new[which.min(abs(MSE1_new - mean((mle_sc - sigma)^2)))]
abs(bias1_new)[which.min(abs(MSE1_new - mean((mle_sc - sigma)^2)))]
MSE1_new[which.min(abs(MSE1_new - mean((mle_sc - sigma)^2)))]

# Plot: Absolute Bias vs alpha
alpha_0 = seq(-0.5, 0.5, 0.01)
bias1_0 = c(bias1[50:99], bias0_1, bias1[100:149])
plot(alpha_0, abs(bias1_0), col = 2, type = "l", lwd = 3,
xlim = c(-0.5, 0.5), ylim = c(0, 0.20),
     xlab = expression(alpha), ylab = "Absolute Bias",
     main = expression(paste("Abs. Bias: ", hat(sigma[alpha]), " vs QD & MLE")))
abline(h = c(0, abs(mean(qd - sigma)), abs(mean(mle_sc - sigma))), col = c(1, 4, 6),
lwd = 3)
legend(-0.1, 0.08, legend = c(expression(hat(sigma[alpha])), "QD", "MLE"),
col = c(2, 4, 6),
lty = 1, lwd = 3, cex = 0.6)

# Plot: MSE vs alpha
MSE1_0 = c(MSE1[50:99], MSE0_1, MSE1[100:149])
plot(alpha_0, MSE1_0, col = 2, type = "l",  lwd = 3, xlim = c(-0.5, 0.5),
ylim = c(0, 5),
     xlab = expression(alpha), ylab = "MSE",
     main = expression(paste("MSE: ", hat(sigma[alpha]), " vs QD & MLE")))
abline(h = c(0, mean((qd - sigma)^2), mean((mle_sc - sigma)^2)), col = c(1, 4, 6),
lwd = 3)
legend(-0.15, 1.25, legend = c(expression(hat(sigma[alpha])), "QD", "MLE"),
       col = c(2, 4, 6), lty = 1, lwd = 3, cex = 0.6)

# Histograms
hist(qd, prob = TRUE, xlab = "QD", main = "Histogram of QD",
xlim = c(0, 12), ylim = c(0,0.6))
hist(mle_sc, prob = TRUE, xlab = "MLE", main = "Histogram of MLE",
xlim = c(0, 12), ylim = c(0,0.6))


hist(sig_hat1[, which.min(abs(bias1_new))], prob = TRUE,
     xlab = expression(hat(sigma[alpha])), main = "Estimator ( min Abs. Bias)",
     xlim = c(0, 12), ylim = c(0,0.6))
hist(sig_hat1[, which.min(MSE1_new)], prob = TRUE,
     xlab = expression(hat(sigma[alpha])), main = "Estimator ( min MSE)",
     xlim = c(0, 12), ylim = c(0,0.6))
```

Key learnings

- the range of α for which moments exist

$$\alpha \in (-1,0) \cup (0,1).$$

- here the fractional momemt based estimator of scale parameter σ can be written as

$$\hat{\sigma_\alpha} = \begin{cases} \left(\frac{1}{g(\alpha)n} \sum_{i=1}^{n} |x_i - \mu_e|^\alpha\right)^{1/\alpha}, & \text{for } \alpha \in (-1,0) \cup (0,1), \\ \left(\prod_{i=1}^{n} |x_i - \mu_e|\right)^{1/n}, & \text{for } \alpha \to 0, \end{cases}$$

- The QD estimator showed a wider spread, consistent with its higher MSE.
- The MLE estimator had a tighter concentration but remained sensitive to outliers.
- The proposed estimator $\hat{\sigma}_\alpha$ (for α minimizing bias and MSE)
- The minimum absolute bias of the proposed estimator was observed near $\alpha = -0.07$.
- The QD estimator showed a wider spread, consistent with its higher MSE.
- The MLE estimator had a tighter concentration but remained sensitive to outliers.
- The proposed estimator $\hat{\sigma}_\alpha$ (for α minimizing bias and MSE) showed:
  - Concentration around the true value $\sigma = 2$
  - Narrower spread compared to both QD and MLE

Future Work

- **Generalization to Other Heavy-Tailed Distributions:** Adapt and test the fractional moment estimation strategy on other heavy-tailed distributions (e.g., Lévy, stable distributions) where integer moments do not exist.

- **Multivariate Extensions:** Formulate analogous estimators for the multivariate Cauchy distribution, which could benefit robust covariance estimation.

- **Application in Real Data Problems:** Apply the estimator to real datasets in signal processing, finance, or physics where Cauchy noise or outlier contamination is common, and compare its performance against existing robust estimators.

Conclusion

This internship has been a highly enriching experience that bridged the gap between classroom knowledge and real-world research. Working on scale estimation in the Cauchy distribution not only deepened my theoretical understanding but also trained me in robust statistical methods and computational thinking. The experience of deriving estimators, implementing them, and evaluating their performance through simulation has been intellectually rewarding. It has reaffirmed my interest in statistical inference and robust methods, and inspired me to pursue further studies and research in theoretical statistics and data science.