Mathematics Assignment

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Q1) Find the radius of curvature at the point $(3a/2, 3a/2)$ on the curve $x^3 + y^3 = 3axy$.

Ans:- given equation of curve:

$$x^3 + y^3 = 3axy$$

Diff. with respect to x :-

$$3x^2 + 3y^2 dy/dx = 3a(xy_1 + y)$$

$$\Rightarrow 3(y^2 - ax)y_1 = 3(ay - x^2)$$

$$\Rightarrow y_1 = \frac{(ay - x^2)}{(y^2 - ax)} \quad \text{—} ①$$

$$\therefore y_1(3a/2, 3a/2) = \frac{a \cdot 3a/2 - 9a^2/4}{9a^2/4 - a \cdot 3a/2}$$

$$= \frac{-3a^2/4}{+3a^2/4} = -1$$

Again diff. eq. ①, we get :-

$$y_2 = \frac{(y^2 - ax).(ay_1 - 2x) - (ay - x^2).(2yy_1 - a)}{(y^2 - ax)^2}$$

$$\therefore y_2(3a/2, 3a/2) = \frac{[(3a/2)^2 - a \cdot 3a/2][a(-1) - 2 \cdot 3a/2] - [a \cdot 3a/2 - (3a/2)^2][2 \cdot 3a/2 \cdot (-1) - a]}{[(3a/2)^2 - a \cdot 3a/2]^2}$$

$$\Rightarrow \quad y_2 = \frac{\left[\frac{9a^2}{4} - \frac{3a^2}{2}\right]\left[-a-3a\right] - \left[\frac{3a^2}{2} - \frac{9a^2}{4}\right]\left[-3a-a\right]}{\left[\frac{9a^2}{4} - \frac{3a^2}{2}\right]^2}$$

$$= \frac{(3a^2/4)(-4a) - (-3a^2/4)(-4a)}{9a^4/16}$$

$$= -6a^3 / 9a^4/16 = -\frac{32}{3a}$$

So radius of curvature $(\rho)_{(3a/2,\ 3a/2)} = \frac{\left[1+(y_1)^2\right]^{3/2}}{y_2}$

$$= \frac{\left[1+(-1)^2\right]^{3/2}}{-32/3a}$$

$$= -\frac{2\sqrt{2} \cdot 3a}{32}$$

$$= -3a/8\sqrt{2}$$

Thus Radius of curvature of given curve at $(3a/2,\ 3a/2)$ point is $-3a/8\sqrt{2}$ (Ans)

Q2> Show that circle of curvature of $\sqrt{x} + \sqrt{y} = \sqrt{a}$ at $(a/4,\ a/4)$ is $(x - 3a/4)^2 + (y - 3a/4)^2 = a^2/2$

Ans:- The given curve:

$$\sqrt{x} + \sqrt{y} = \sqrt{a}$$

Now diff. w.r.to x:-

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow y_1 = \frac{dy}{dx} = -\sqrt{y}/\sqrt{x} \quad — (1)$$

Again diff. eq. (1),

$$y_2 = -\left[\frac{\sqrt{x} \cdot \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} - \sqrt{y} \cdot \frac{1}{\sqrt{x}} \cdot \frac{1}{2}}{x}\right]$$

$$= -\left[\frac{\frac{1}{2} - \sqrt{y}/\sqrt{x} \cdot \frac{1}{2}}{x}\right]$$

$$= \frac{-1}{2x} \cdot (1 + y_1)$$

$$= \frac{1}{2x} \cdot (y_1 - 1) \quad — (11)$$

Now $y_1(a/4, a/4) = -\frac{a/4}{a/4} = -1$

& $y_2(a/4, a/4) = \frac{1}{2 \cdot a/4} \cdot (-1 - 1) = 4/a$.

Now radius of curvature $(\rho)_{(a/4, a/4)} = \frac{(1 + y_1^2)^{3/2}}{y_2}$

$$= \frac{[1 + (-1)^2]^{3/2}}{4/a} = \frac{2\sqrt{2}}{4/a} = a/\sqrt{2}$$

Now $(\bar{x}, \bar{y})$ is the centre of curvature.

$\therefore \bar{x} = x - y_1(1+y_1^2)/y_2$

$= a/4 + (1+1)/4/a = 3a/4$

$\bar{y} = y + (1+y_1^2)/y_2$

$= a/4 + (1+1)/4/a = 3a/4$

So equation of circle of curvature :-

$(x-\bar{x})^2 + (y-\bar{y})^2 = \rho^2$

$\Rightarrow (x - 3a/4)^2 + (y - 3a/4)^2 = a^2/2$ (Proved)

Q3> Find the radius of curvature at the origin for
$x^3 + y^3 - 2x^2 + 6y = 0$

Ans:. Given eq :- $x^3 + y^3 - 2x^2 + 6y = 0$

Diff. w.r.to x : $3x^2 + 3y^2 \cdot \frac{dy}{dx} - 2 \cdot 2x + 6 \cdot \frac{dy}{dx} = 0$

$\Rightarrow 3x^2 + 3y^2 \cdot y_1 - 4x + 6y_1 = 0$

$\Rightarrow (3y^2 + 6)y_1 = 4x - 3x^2$

$\Rightarrow y_1 = \frac{4x - 3x^2}{3y^2 + 6}$ —(1)

$\therefore y_1(0,0) = \frac{0}{6} = 0$

Diff. eq. (1) w.r.to x :-

$$y_2 = \frac{(3y^2+6).(4-6x) - (4x-3x^2)(6y.y_1)}{(3y^2+6)^2}$$

$$\therefore y_2(0,0) = \frac{(6 \cdot 4) - 0}{36} = 24/36 = 2/3$$

$$\therefore \text{Radius of curvature } (\rho)_{(0,0)} = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+0)^{3/2}}{2/3}$$

$$= 3/2 \text{ (Ans)}$$

Q4) Find the equation of evolute of ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Ans:. Given eq. of ellipse :- $x^2/a^2 + y^2/b^2 = 1$

Diff. w.r.to x :- $\frac{2x}{a^2} + \frac{2y.y_1}{b^2} = 0$

$$\Rightarrow \frac{2yy_1}{b^2} = -\frac{2x}{a^2}$$

$$\Rightarrow y_1 = -\frac{x}{y}.\left(\frac{b}{a}\right)^2 \quad \text{(1)}$$

Again diff. eq. (1) :- $y_2 = -\left[\frac{y - xy_1}{y^2}\right]\frac{b^2}{a^2}$

$$= -\left[\frac{y + x.\frac{x}{y}.\left(\frac{b}{a}\right)^2}{y^2}\right]\left(\frac{b}{a}\right)^2$$

$$\Rightarrow y_2 = -\left[\frac{y^2 \cdot a^2 + x^2 \cdot b^2}{a^2 \cdot y^3}\right] \times \frac{b^2}{a^2}$$

We know, parametric eqns of ellipse are :- $x = a\cos\theta$;

$y = b\sin\theta$

putting them in eq. ①, $y_1 = -\frac{a\cos\theta}{b\sin\theta} \cdot \left(\frac{b}{a}\right)^2$

$$= -\frac{b}{a}\cot\theta$$

" " " eq. ⑪, $y_2 = -\left[\frac{(b\sin\theta)^2 \cdot a^2 + (a\cos\theta)^2 \cdot b^2}{a^2 \cdot (b\sin\theta)^3}\right] \times \frac{b^2}{a^2}$

$$= -\left[\frac{a^2b^2\sin^2\theta + a^2b^2\cos^2\theta}{a^2b^3\sin^3\theta}\right] \times \frac{b^2}{a^2}$$

$$= -\frac{b}{a^2}\operatorname{cosec}^3\theta$$

Now $\bar{x} = x - \dfrac{y_1(1+y_1^2)}{y_2}$

$$= \frac{a\cos\theta - \left[(-b/a)\cot\theta\right]\left(1 + \frac{b^2}{a^2}\cot^2\theta\right)}{\left(-b/a^2 \operatorname{cosec}^3\theta\right)}$$

$$= a\cos\theta - \left[a \cdot \frac{\cos\theta}{\sin\theta} \cdot \sin^3\theta\right]\left(1 + \frac{b^2}{a^2}\cot^2\theta\right)$$

$$= a\cos\theta - a\cos\theta \cdot \sin^2\theta - \frac{b^2}{a} \cdot \cos^3\theta.$$

$$= a\cos\theta(1 - \sin^2\theta) - \frac{b^2}{a^2}\cos^3\theta$$

$$\Rightarrow \bar{x} = (a - b^2/a)\cos^3\theta$$

$$\bar{y} = y_0 + \frac{(1+y_1^2)}{y_2}$$

$$= b\sin\theta + \frac{(1 + b^2/a^2\cos^2\theta)}{-(b/a^2)\operatorname{cosec}^3\theta}$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin^3\theta \cdot \frac{\cos^2\theta}{\sin^2\theta}$$

$$= b\sin\theta(1 - \cos^2\theta) - a^2/b\sin^3\theta$$

$$= b\sin^3\theta - a^2/b\sin^3\theta$$

$$\Rightarrow \bar{y} = \sin^3\theta(b - a^2/b)$$

PTO

Eliminating the parameter:-

$$\bar{x} = (a - b^2/a) \text{ cosine}^3\theta$$

$$\Rightarrow a\bar{x} = (a^2 - b^2)\cos^3\theta$$

$$\Rightarrow (a\bar{x})^{2/3} = (a^2-b^2)^{2/3}\cos^2\theta$$

$$\Rightarrow \cos^2\theta = \frac{(a\bar{x})^{2/3}}{(a^2-b^2)^{2/3}} \quad \text{— (I)}$$

$$\bar{y} = \sin^3\theta\,(b^2-a^2)/b$$

$$\Rightarrow b\bar{y} = \sin^3\theta\,(b^2-a^2)$$

$$\Rightarrow (b\bar{y})^{2/3} = \sin^2\theta\,(b^2-a^2)^{2/3}$$

$$\Rightarrow \sin^2\theta = \frac{(b\bar{y})^{2/3}}{(b^2-a^2)^{2/3}} \quad \text{— (II)}$$

Now adding eq. (I) & (II) :-

$$\left(\frac{a\bar{x}}{a^2-b^2}\right)^{2/3} + \left(\frac{b\bar{y}}{b^2-a^2}\right)^{2/3} = 1$$

$$\Rightarrow (a\bar{x})^{2/3} + (b\bar{y})^{2/3} = (a^2-b^2)^{2/3} \quad \text{(Ans)}$$

Q5> Find the envelope of family of st. lines :

$$y\cos\alpha - x\sin\alpha = a\cos 2\alpha$$

Ans:- Given eq: $y\cos\alpha - x\sin\alpha = a\cos 2\alpha$ — (I)

Diff. w.r.to α:

$$-y\sin\alpha - x\cos\alpha = -a\cdot 2\sin 2\alpha$$

$$\Rightarrow y\sin\alpha + x\cos\alpha = 2a\sin 2\alpha \quad \text{— (II)}$$

Doing : (I) × $\sin\alpha$ & (II) × $\cos\alpha$:-

$$y\cos\alpha\sin\alpha - x\sin^2\alpha = a\sin\alpha\cos 2\alpha$$

$$\underset{-}{y\sin\alpha\cos\alpha} \;\underset{-}{+}\; x\cos^2\alpha = \underset{-}{2a\cos\alpha\sin 2\alpha}$$

$$x(\sin^2\alpha + \cos^2\alpha) = (2a\cos\alpha\sin 2\alpha - a\sin\alpha\cos 2\alpha)$$

$$\Rightarrow x = 2a\cdot 2\sin\alpha\cos^2\alpha - a\cos 2\alpha\sin\alpha$$

$$= 2a\cdot 2\sin\alpha\cos^2\alpha - a\sin\alpha(\cos^2\alpha - \sin^2\alpha)$$

$$= 4a\sin\alpha\cos^2\alpha - a\sin\alpha\cos^2\alpha + a\sin^3\alpha$$

$$\Rightarrow x = a\sin^3\alpha + 3a\sin\alpha\cos^2\alpha.$$

Now $(i) \times \cos\alpha$ & $(ii) \times \sin\alpha$

$$y\cos^2\alpha - x\sin\alpha\cos\alpha = a\cos 2\alpha\cos\alpha$$
$$y\sin^2\alpha + x\cos\alpha\sin\alpha = 2a\sin 2\alpha\sin\alpha$$

$$y(\sin^2\alpha + \cos^2\alpha) = 2a\sin 2\alpha\sin\alpha + a\cos 2\alpha\cos\alpha$$

$$\Rightarrow y = 2a\sin\alpha(2\sin\alpha\cos\alpha) + a\cos\alpha(\cos^2\alpha - \sin^2\alpha)$$

$$= 4a\sin^2\alpha\cos\alpha + a\cos^3\alpha - a\sin^2\alpha\cos\alpha$$

$$= a\cos^3\alpha + 3a\sin^2\alpha\cos\alpha.$$

$$\therefore x + y = a\sin^3\alpha + 3a\sin\alpha\cos^2\alpha + a\cos^3\alpha + 3a\sin^2\alpha\cos\alpha$$

$$= a(\sin\alpha + \cos\alpha)^3$$

$$\Rightarrow \frac{x+y}{a} = (\sin\alpha + \cos\alpha)^3$$

$$\Rightarrow \left(\frac{x+y}{a}\right)^{1/3} = \sin\alpha + \cos\alpha$$

Similarly:

$$x - y = a\sin^3\alpha + 3a\sin\alpha\cos^2\alpha - a\cos^3\alpha - 3a\sin^2\alpha\cos\alpha$$

$\Rightarrow x - y = a(\sin\alpha - \cos\alpha)^3$

$\Rightarrow \dfrac{x-y}{a} = (\sin\alpha - \cos\alpha)^3$

$\Rightarrow \left(\dfrac{x-y}{a}\right)^{1/3} = (\sin\alpha - \cos\alpha)$

Now $\left(\dfrac{x+y}{a}\right)^{2/3} = \sin^2\alpha + 2\sin\alpha\cos\alpha + \cos^2\alpha$ — (iii)

& $\left(\dfrac{x-y}{a}\right)^{2/3} = \sin^2\alpha - 2\sin\alpha\cos\alpha + \cos^2\alpha$ — (iv)

Doing (iii) + (iv) :-

$\left(\dfrac{x+y}{a}\right)^{2/3} + \left(\dfrac{x-y}{a}\right)^{2/3} = 2$ which is the eq. of envelope. (Ans)