Assignment - II :

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① Identify the saddle points & extreme points of
$$f(x,y) = x^4 - y^4 - 2x^2 + 2y^2$$

Ans:- Given $f(x,y) = x^4 - y^4 - 2x^2 + 2y^2$

$$\therefore f_x = 4x^3 - 4x$$

$$\& \; f_y = -4y^3 + 4y$$

Now $f_x = 0$,

$$\therefore 4x^3 - 4x = 0$$

$$\Rightarrow 4x(x^2 - 1) = 0$$

$$\therefore x = 0 \quad \& \quad x = \pm 1$$

& $f_y = 0$

$$\therefore -4y^3 + 4y = 0$$

$$\Rightarrow 4y(1 - y^2) = 0$$

$$\therefore y = 0, \quad \& \quad y = \pm 1$$

So the pts are:- $(0,0); (0,1); (0,-1); (1,0); (1,1); (1,-1);$
$(-1,0); (-1,1); (-1,-1)$

Pg-1

Now $f_{xx} = 12x^2 - 4$ — A

$f_{yy} = -12y^2 + 4$ — C

$f_{xy} = 0$ — B.

	(0,0)	(0,1)	(0,-1)	(1,0)	(1,1)	(1,-1)	(-1,0)	(-1,1)	(-1,-1)
A	-4	-4	-4	8	8	8	8	8	8
B	0	0	0	0	0	0	0	0	0
C	+4	-8	-8	4	-8	-8	4	-8	-8
$AC-B^2$	-16 <0	32 >0	32 >0	32 >0	-64 <0	-64 <0	32 >0	-64 <0	-64 <0

At pts (0,1) & (0,-1), $AC-B^2 > 0$ & A is -ve i.e. <0,

so (0,1) & (0,-1) are maximum. (Ans)

At pts (1,0) & (-1,0), $AC-B^2 > 0$ & A is +ve i.e. >0,

so (1,0) & (-1,0) are minimum. (Ans)

& at pts (0,0), (1,1), (1,-1), (-1,1) & (-1,-1), $AC-B^2 < 0$

so they are saddle pts. (Ans)

Q2> A rectangular box open at the top is to have a ~~max~~ volume of 32 cubic feet. Find its dimensions if total surface area is minimum.

Ans:- $f =$ surface area $= xy + 2yz + 2zx$

$g =$ volume $= xyz = 32$ (given) i.e. $g : xyz - 32 = 0$

Now $F = f + \lambda g$

$$= (xy + 2yz + 2zx) + \lambda(xyz - 32)$$

Now $F_x = (y + 2z) + \lambda(yz)$

$$F_y = (x + 2z) + \lambda(xz)$$

$$F_z = (2y + 2x) + \lambda(xy)$$

Now $\cancel{F_x = 0}$

$$\Rightarrow \cancel{(y+2z) + \lambda(xyz - 32) = 0}$$

$$\Rightarrow \cancel{-\lambda = \frac{y + 2z}{xyz - 32}}$$

Now $F_x = 0$

$$\Rightarrow (y + 2z) + \lambda(yz) = 0$$

$\boxed{Pg-3}$

$$\Rightarrow -\lambda = \frac{y + 2z}{yz} = \left(\frac{1}{z} + \frac{2}{y}\right) \quad \text{— (I)}$$

$F_y = 0$

$$\Rightarrow (x + 2z) + \lambda(xz) = 0$$

$$\Rightarrow -\lambda = \frac{x + 2z}{xz} = \left(\frac{1}{z} + \frac{2}{x}\right) \quad \text{— (II)}$$

& $F_z = 0$

$\Rightarrow (2y + 2x) + \lambda(xy) = 0$

$\Rightarrow -\lambda = \dfrac{2y + 2x}{xy} = \left(\dfrac{2}{x} + \dfrac{2}{y}\right)$ — (iii)

From (i) & (ii) :-

$$\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$$

$$\therefore \quad x = y.$$

From (ii) & (iii) :-

$$\frac{1}{z} + \frac{2}{x} = \frac{2}{x} + \frac{2}{y}$$

$$\Rightarrow \quad 2z = y.$$

From (i) & (iii) :-

$$\frac{1}{z} + \frac{2}{y} = \frac{2}{x} + \frac{2}{y}$$

$$\therefore \quad 2z = x.$$

i.e. $x = y = 2z$.

Now $g(x, y, z) \Rightarrow xyz = 32.$

On substitution; i.e. $\cancel{x \times x \times}\ 2z \times 2z \times z = 32$ $\boxed{\text{Pg-4}}$

$$\Rightarrow 4z^3 = 32$$

$$\therefore z = 2.$$

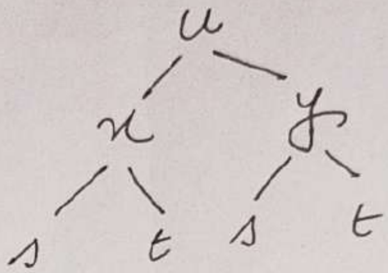
So $x = 2z = 2 \times 2 = 4$

& $y = x = 4$

So point (dimensions) :- $(4, 4, 2)$ (Ans)

③ ⓐ If $u = x^2 + y^2$ where $x = s + 3t$ & $y = 2s - t$

Find :- $\dfrac{\partial u}{\partial s}$ & $\dfrac{\partial u}{\partial t}$.



Ans: $\dfrac{\partial u}{\partial s} = \left(\dfrac{\partial u}{\partial x} \times \dfrac{\partial x}{\partial s}\right) + \left(\dfrac{\partial u}{\partial y} \times \dfrac{\partial y}{\partial s}\right)$

Now $\dfrac{\partial u}{\partial x} = 2x$; $\dfrac{\partial u}{\partial y} = 2y$;

$\dfrac{\partial x}{\partial s} = 1$; $\dfrac{\partial y}{\partial s} = 2$;

so $\dfrac{\partial u}{\partial s} = (2x \times 1) + (2y \times 2)$

$\Rightarrow \boxed{\dfrac{\partial u}{\partial s} = 2x + 4y}$ (Ans)

Again: $\dfrac{\partial u}{\partial t} = \left(\dfrac{\partial u}{\partial x} \times \dfrac{\partial x}{\partial t}\right) + \left(\dfrac{\partial u}{\partial y} \times \dfrac{\partial y}{\partial t}\right)$

Now: $\dfrac{\partial u}{\partial x} = 2x$; $\dfrac{\partial u}{\partial y} = 2y$

$\dfrac{\partial x}{\partial t} = 3$; $\dfrac{\partial y}{\partial t} = -1$.

so $\dfrac{\partial u}{\partial t} = (2x \times 3) + [2y \times (-1)]$

$\Rightarrow \boxed{\dfrac{\partial u}{\partial t} = 6x - 2y}$ (Ans)

(36) If $z = \sin\left(\frac{x}{y}\right)$, $x = e^t$; $y = t^2$; find $\frac{dz}{dt}$.

Ans: $\frac{dz}{dt} = \left(\frac{\partial z}{\partial x} \cdot \frac{dx}{dt}\right) + \left(\frac{\partial z}{\partial y} \cdot \frac{dy}{dt}\right)$

$$\begin{tikzcd}[column sep=small] & z \arrow[dl, no head] \arrow[dr, no head] & \\ x \arrow[d, no head] & & y \arrow[d, no head] \\ t & & t \end{tikzcd}$$

Now: $\frac{\partial z}{\partial x} = \cancel{\cos} \frac{\partial}{\partial x}\left[\sin\left(\frac{x}{y}\right)\right]$

$$= \cos\left(\frac{x}{y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{y}\right)$$

$$= \frac{1}{y}\cos\left(\frac{x}{y}\right)$$

$$\frac{dx}{dt} = \frac{d}{dt}(e^t) = e^t$$

& $\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left[\sin\left(\frac{x}{y}\right)\right] =$

$$= \cos\left(\frac{x}{y}\right) \cdot \frac{\partial}{\partial y} \cdot \left(\frac{x}{y}\right)$$

$$= \cos\left(\frac{x}{y}\right) \cdot \left(-\frac{x}{y^2}\right) = -\frac{x}{y^2}\cos(x/y)$$

$$\frac{dy}{dt} = 2t$$

so $\frac{dz}{dt} = \left[\frac{1}{y}\cos(x/y)\right]e^t + \left[-\frac{x}{y^2}\cos(x/y)\right] \cdot 2t$

$$\Rightarrow \boxed{\frac{dz}{dt} = \frac{e^t}{y}\cos(x/y) - 2t\frac{x}{y^2}\cos(x/y)}$$ (Ans)

④ If $u = \frac{yz}{x}$; $v = \frac{zx}{y}$; $w = \frac{xy}{z}$

show that : $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$

Ans:- $\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

$$= \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= -\frac{yz}{x^2}\begin{vmatrix} -\frac{xz}{y^2} & \frac{x}{y} \\ \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix} - \frac{z}{x}\begin{vmatrix} \frac{z}{y} & \frac{x}{y} \\ \frac{y}{z} & -\frac{xy}{z^2} \end{vmatrix} + \frac{y}{x}\begin{vmatrix} \frac{z}{y} & -\frac{xz}{y^2} \\ \frac{y}{z} & \frac{x}{z} \end{vmatrix}$$

$$= -\frac{yz}{x^2}\left[\frac{x^2yz}{y^2z^2} - \frac{x^2}{yz}\right] - \frac{z}{x}\left[-\frac{x\cancel{y}}{z} - \frac{x}{z}\right] + \frac{y}{x}\left[\frac{x}{y} + \frac{x\cancel{y}}{y}\right]$$

$$= -\frac{yz}{x^2}\left\{\frac{x^2}{yz}\left(\frac{yz}{yz} - 1\right)\right\} + \frac{z}{x}\left\{\frac{2x}{z}\cancel{(y-1)}\right\} + \frac{y}{x}\left\{\frac{2x}{y}\right\}$$

$\cancel{= 0 + 2y - 1} + 2 = 2 + 2$

$\cancel{= y + 1}$ $= 4$ (proved)

⑤ expand $e^{2x}\cos 2y$ in powers of x & y at $(0, \pi/2)$.
upto second degree.

Ans:- Given: $f(x,y) = e^{2x}\cos 2y$

$\therefore f(x,y) = e^{2x}\cos 2y \quad ; \quad f(0,\pi/2) = \cancel{\cos} e^0 \cos\pi = -1$

$f_x(x,y) = 2e^{2x}\cos 2y \quad ; \quad f_x(0,\pi/2) = 2\times e^0\cos\pi = -2$

$f_{xx}(x,y) = 4e^{2x}\cdot\cos 2y \quad ; \quad f_{xx}(0,\pi/2) = 4e^0\cos\pi = -4$

$\cancel{f_{xxx}(x,y) = 8e^{2x}\cos 2y}$

$f_y(x,y) = -2e^{2x}\sin 2y \quad ; \quad f_y(0,\pi/2) = -2\cdot e^0\cdot\sin\pi = 0$

$f_{yy}(x,y) = -4e^{2x}\cos 2y \quad ; \quad f_{yy}(0,\pi/2) = -4e^0\cdot\cos\pi = 4$

$\cancel{f_{yyy}(x,y) = 8e^{2x}\sin 2y}$

$f_{xy}(x,y) = -4e^{2x}\cdot\sin 2y \quad ; \quad f_{xy}(0,\pi/2) = -4e^0\cdot\sin\pi = 0$

$$\therefore f(x,y) = f(0,\pi/2) + \frac{1}{1!}\left[(x-0)f_x(0,\pi/2) + (y-\pi/2)f_y(0,\pi/2)\right]$$

$$+ \frac{1}{2!}\left[(x-0)^2 f_{xx}(0,\pi/2) + (y-\pi/2)^2 f_{yy}(0,\pi/2) \cancel{+ 2\times(x-0)f_x(0,\pi/2)\times(y-\pi/2)f_y(0,\pi/2)]}\right.$$
$$\left. + 2(x-0)(y-\pi/2)\cdot f_{xy}(0,\pi/2)\right]$$

$\boxed{Pg-8}$

$$= -1 + \left[x\cdot(-2) + (y-\pi/2)\cdot 0\right] + \frac{1}{2}\left[x^2\cdot(-4) + (y-\pi/2)^2(4) \cancel{+2} + 2x(y-\pi/2)\cdot 0\right]$$

$$\boxed{f(x,y) = -1 - 2x - 2x^2 + 2(y-\pi/2)^2} \text{ (Ans)}$$