05/10/20

Assignment 1.

Q1> Find the Eigen values and Eigen vectors of $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Q2> Verify Cayley Hamilton's theorm and find A^{-1} & A^4 when $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Q3> Use Cayley Hamilton theorm to find the value of the matrix given by :-

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

if matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$

Ans 1>. $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

The ch. eq :- $|A - \lambda I| = 0$

$$\Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} = 0$$

Sum of diagonals of A $= 6+3+3 = 12$

" " minors of diagonals of A $= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$

$= (9-1) + (18-4) + (18-4)$

$= 8 + 14 + 14$

$= 36$

$|A| = 6\begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} - (-2)\begin{vmatrix} -2 & -1 \\ 2 & 3 \end{vmatrix} + 2\begin{vmatrix} -2 & 3 \\ 2 & -1 \end{vmatrix}$

$= 6[9-1] + 2[-6+2] + 2[2-6]$

$= 48 - 8 - 8 = 32$

$\therefore \lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

$$\begin{array}{c|cccc} & 1 & -12 & \cancel{36} & -32 \\ 1 & 0 & 1 & -11 & 25 \\ \hline & 1 & -11 & 25 & -7 \end{array}$$

By hit and trial method, the factors are 2,

Now $\because$ 2 is one root, $\therefore$ to find the other roots, we divide the eq. with $(\lambda-2)$ ie.

$$\begin{array}{r|l} & \lambda^2 - 10\lambda + 16 \\ \hline \lambda - 2 & \lambda^3 - 12\lambda^2 + 36\lambda - 32 \\ & \underline{-\lambda^3 \mp 2\lambda^2} \\ & \quad -10\lambda^2 + 36\lambda - 32 \\ & \quad \underline{-10\lambda^2 \pm 20\lambda} \\ & \qquad 16\lambda - 32 \\ & \qquad \underline{16\lambda \mp 32} \\ & \qquad \times \end{array}$$

$\therefore$ The roots are :-

& $(\lambda-2)(\lambda^2 - 10\lambda + 16)$

~~$= (\lambda-2)($~~

Now for $\lambda^2 - 10\lambda + 16$, we use :-

$$\frac{-(-10) \pm \sqrt{(-10)^2 - 4 \times 1 \times 16}}{2.1}$$

$$= \frac{10 \pm \sqrt{100-64}}{2} = \frac{10 \pm 6}{2} = 2, 8.$$

[Check :- $\lambda_1 \times \lambda_2 \times \lambda_3 = 32$ & $\det A = 32$]

Thus the ~~roots~~ Eigen values are 2, 2, 8.

To find the Eigen vectors:-

$|A - \lambda I| X = 0.$ where $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$\Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Case I :. when $\lambda = 2$,

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$4x_1 - 2x_2 + 2x_3 = 0 \quad -(i)$

$-2x_1 + x_2 - x_3 = 0 \quad -(ii)$

$2x_1 - x_2 + x_3 = 0 \quad -(iii)$

$$\begin{array}{cccc} -2 & 2 & 4 & -2 \\ 1 & -1 & -2 & 1 \end{array}$$

$$\frac{x_1}{2-2} = \frac{x_2}{-4+4} = \frac{x_3}{4-4} \quad \therefore \; X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ (not possible)}$$

$\Rightarrow$ putting $x_1 = 0$ in eq. (i) :-

$0 - 2x_2 + 2x_3 = 0$

$\Rightarrow -x_2 + x_3 = 0 \quad \therefore x_2 = -1 \;\&\; x_3 = 1$

i.e. $X_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

similarly putting $x_2 = 0$ in eq (ii) :-

$-2x_1 + 0 - x_3 = 0 \Rightarrow 2x_1 + x_3 = 0$

$\Rightarrow x_1 = \frac{1}{2} \;\&\; x_3 = 1$

i.e. $X_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$

Also putting $x_3 = 0$ in eq (iii) :-

$2x_1 - x_2 + 0 = 0. \Rightarrow 2x_1 - x_2 = 0.$

i.e. $x_1 = \frac{1}{2}$ and $x_2 = -1$

So $X_3 = \begin{bmatrix} 1/2 \\ -1 \\ 0 \end{bmatrix}$

Thus the Eigen vectors are :-

$X_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \; X_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}, \; X_3 = \begin{bmatrix} 1/2 \\ -1 \\ 0 \end{bmatrix}$ (Ans)

Ans 2) $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

The characteristic eq is :- $|A - \lambda I| = 0$

$\Rightarrow \begin{bmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{bmatrix} = 0$

Sum of diagonal elements of A $= 1 + 1 + (-1) = 1$

Sum of minors of diagonal elements A $= \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}$

$= (-1-3) + (-1+2) + (1-2)$

$= -4 + 1 - 1 = -4$

$|A| = \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} - \begin{vmatrix} 2 & -2 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix}$

$= (-1-3) - (-2+6) + (2+2)$

$= -4 - 4 + 4 = -4.$

so $\lambda^3 - \lambda^2 - 4\lambda + 4 = 0.$

By CH theorm, $A^3 - A^2 - 4A - 4I = 0$ —①

Now $A^2 = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1+2-2 & 2+2-6 & -2+2+2 \\ 1+1+1 & 2+1+3 & -2+1-1 \\ 1+3-1 & 2+3-3 & -2+3+1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1-2+2 & 2-2+6 & -2-2-2 \\ 3+6-2 & 6+6-6 & -6+6+2 \\ 3+2+2 & 6+2+6 & -6+2-2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix}$

Now $A^3 - A^2 - 4A - 4I$

$$= \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 8 & -8 \\ 4 & 4 & 4 \\ 4 & 12 & -4 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$= O$ (proved)

Now multiplying both sides of eq (i) with A.

$$A^4 - A^3 - 4A^2 - 4A = 0$$

$$\Rightarrow A^4 = A^3 + 4A^2 + 4A.$$

$$= \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix} + 4\begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} + 4\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix} + \begin{bmatrix} 4 & -8 & 8 \\ 12 & 24 & -8 \\ 12 & 8 & 8 \end{bmatrix} + \begin{bmatrix} 4 & 8 & -8 \\ 4 & 4 & 4 \\ 4 & 12 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 & -6 \\ 23 & 34 & -2 \\ 23 & 34 & -2 \end{bmatrix} \text{ (Ans)}$$

Also multiplying (i) with A^{-1},

$$A^2 - A - 4I - 4A^{-1} = 0$$

$$\therefore 4A^{-1} = A^2 - A - 4I.$$

$$= \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & -4 & 4 \\ 2 & 1 & -3 \\ 2 & -1 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & -1 & 1 \\ 1/2 & 1/4 & -3/4 \\ 1/2 & -1/4 & -1/4 \end{bmatrix} \quad \text{(Ans)}$$

Ans 37.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

The ch. eq : $|A - \lambda I| = 0$.

$$\Rightarrow \begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{bmatrix} = 0.$$

Sum of diagonals of A $= 2+1+2 = 5$

" " minors of diagonals of A $= \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}$

$= 2 + (4-1) + 2 = 7$

$$|A| = 2\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 2(2) + (-1) = 4-1 = 3.$$

So $\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$.

By Cayley Hamilton theorm,

$A^3 - 5A^2 + 7A - 3 = 0$ —①

After eq. (i), ~~[illegible]~~

given $f(x) = A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + 1$ —(ii)

Dividing (ii) by eq (i):-

$$
\begin{array}{r|l}
 & A^5 + A \\
\hline
A^3 - 5A^2 + 7A - 3 & \cancel{A^8} - \cancel{5A^7} + \cancel{7A^6} - \cancel{3A^5} + A^4 - 5A^3 + 8A^2 - 2A + I \\
 & A^8 \mp 5A^7 \pm 7A^6 \mp 3A^5 \\
\hline
 & \cancel{A^4} - \cancel{5A^3} + 8A^2 - 2A + I \\
 & A^4 \mp 5A^3 \pm 7A^2 \mp 3A \\
\hline
 & A^2 + A + I
\end{array}
$$

$\therefore f(x) = (A^3 - 5A^2 + 7A - 3)(A^5 + A) + A^2 + A + I.$

$= 0 + A^2 + A + 1$

Now. $A^2 = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0 & 1 & 0 \\ 2+0+2 & 1+1+2 & 1+0+4 \end{bmatrix}$

$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix}$

$\therefore f(x) = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$ (Ans)

5/10/20

Assignment 1 (Continued).

(4) Reduce the quadratic form $\phi = 3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2xz$ to canonical form and hence find its ~~nature~~ nature, rank, index and signature

Ans:- Given $\phi = 3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2xz$

$\therefore$ matrix $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

$\therefore$ Ch. eq :- $|A - \lambda I| = 0$.

$$\Rightarrow \begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

Now Sum of diagonal elements of $A = 3+5+3 = 11$

" " minors of " " " $A = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$

$= (15-1) + (9-1) + (15-1)$

$= 28 + 8 = 36$

$|A| = 3\begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} -1 & -1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} -1 & 5 \\ 1 & -1 \end{vmatrix}$

$= 3(15-1) + (-3+1) + (1-5)$

$= 42 - 2 - 4 = 36$

So $\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$ $\quad$ 2, 3, 4, 6,

$$\begin{array}{r|rrrr} 2 & 1 & -11 & 36 & -36 \\ & 0 & 2 & -18 & 36 \\ \hline & 1 & -9 & 18 & 0 \end{array}$$

$(\lambda-2)(\lambda^2 - 9\lambda + 18) = 0$

$\Rightarrow (\lambda-2)[\lambda^2 - 6\lambda - 3\lambda + 18] = 0.$

$\Rightarrow (\lambda-2)(\lambda-3)(\lambda-6) = 0$

So eigen values are 2, 3, 6

Case I: when $\lambda = 2$,

$[A - \lambda I]X = 0.$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow$ $x_1 - x_2 + x_3 = 0$ — (I)

$-x_1 + 3x_2 - x_3 = 0$ — (II)

$x_1 - x_2 + x_3 = 0$ — (III)

Applying cross multiplication rule,

$$\begin{array}{cccc} -1 & 1 & 1 & -1 \\ 3 & -1 & -1 & 3. \end{array}$$

$$\frac{x_1}{1-3} = \frac{x_2}{-1+1} = \frac{x_3}{3-1}$$

ie $\dfrac{x_1}{-2} = \dfrac{x_2}{0} = \dfrac{x_3}{2}$

$$\therefore X_1 = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Case II: when $\lambda = 3$.

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$0x_1 - x_2 + x_3 = 0$ —— (IV)

$-x_1 + 2x_2 - x_3 = 0$ —— (V)

$x_1 - x_2 + 0x_3 = 0$ —— (VI)

$\therefore$ Applying cross multiplication rule :-

$$\frac{x_1}{1-2} = \frac{x_2}{-1+0} = \frac{x_3}{0-1}$$

$$\Rightarrow \frac{x_1}{-1} = \frac{x_2}{-1} = \frac{x_3}{-1}$$

i.e. $X_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

When $\lambda = 6$,

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow -3x_1 - x_2 + x_3 = 0$ —— (VII)

$-x_1 - x_2 - x_3 = 0$ —— (VIII)

$x_1 - x_2 - 3x_3 = 0$ —— (IX).

Applying cross multiplication rule: $\begin{matrix} -1 & & 1 & & -3 & & -1 \\ & \times & & \times & & \times & \\ -1 & & -1 & & -1 & & -1 \end{matrix}$

$$\frac{x_1}{1+1} = \frac{x_2}{-1-3} = \frac{x_3}{3-1}$$

$$\Rightarrow \frac{x_1}{2} = \frac{x_2}{-4} = \frac{x_3}{2}$$

i.e. $X_3 = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

~~Now~~ $\cancel{\bar{X}_1 = \begin{bmatrix}-1 & 0 & 1\end{bmatrix}, \bar{X}_2 =}$

Now $\bar{X}_1 = \begin{bmatrix}\frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}}\end{bmatrix}$; $\bar{X}_2 = \begin{bmatrix}\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}}\end{bmatrix}$; $\bar{X}_3 = \begin{bmatrix}\frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}}\end{bmatrix}$

so $N = \begin{bmatrix}\frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{bmatrix}$

& $N^T = \begin{bmatrix}-1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6}\end{bmatrix}$

So $N^TAN = \begin{bmatrix}-1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6}\end{bmatrix}\begin{bmatrix}3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3\end{bmatrix} N$

$$= \begin{bmatrix}-3/\sqrt{2} + 0 + 1/\sqrt{2} & 1/\sqrt{2} + 0 - 1/\sqrt{2} & -1/\sqrt{2} + 0 + 3/\sqrt{2} \\ 3/\sqrt{3} - 1/\sqrt{3} + 1/\sqrt{3} & -1/\sqrt{3} + 5/\sqrt{3} - 1/\sqrt{3} & 1/\sqrt{3} - 1/\sqrt{3} + 3/\sqrt{3} \\ 3/\sqrt{6} + 2/\sqrt{6} + 1/\sqrt{6} & -1/\sqrt{6} - 10/\sqrt{6} - 1/\sqrt{6} & 1/\sqrt{6} + 2/\sqrt{6} + 3/\sqrt{6}\end{bmatrix} N$$

$$= \begin{bmatrix}-2/\sqrt{2} & 0 & 2/\sqrt{2} \\ 3/\sqrt{3} & 3/\sqrt{3} & 3/\sqrt{3} \\ 6/\sqrt{6} & -12/\sqrt{6} & 6/\sqrt{6}\end{bmatrix} N = \begin{bmatrix}-\sqrt{2} & 0 & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{3} \\ \sqrt{6} & -2\sqrt{6} & \sqrt{6}\end{bmatrix}\begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6}\end{bmatrix}$$

$$= \begin{bmatrix}\frac{2}{\sqrt{2}} + 0 + \frac{\sqrt{2}}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{3}} + \frac{\sqrt{3}}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{\sqrt{6}}{\sqrt{6}} + \frac{4\sqrt{6}}{\sqrt{6}} + \frac{\sqrt{6}}{\sqrt{6}}\end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Now $Q = Y^T (N^T A N) Y$ where $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 2y_1^2 + 3y_2^2 + 6y_3^2$$ (Ans) (canonical form)

~~Nature~~ rank :- $\because |A| = +ve$, so rank $(r) = 3$. (max order)

Index :- $p = 3$ [$\because$ no of +ve elements in canonical form]

Signature :- $(2p - r) = (2 \times 3) - 3 = 3$.

$\because r = p = n$ where n = no. of variables in

so nature is +ve definite. (Ans)

Q5. Reduce the quadratic form $Q = x_1^2 + 2x_2x_3$ to canonical form and hence find its nature, rank, index & signature.

Ans: The matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Now the Ch. eq :- $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$S_1 = 1 + 0 + 0 = 1.$

$S_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = -1$

$S_3 = |A| = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 \times (-1) = -1$

So $\lambda^3 - \lambda^2 - \lambda + 1 = 0$

$$\begin{array}{c|cccc} 1 & 1 & -1 & -1 & 1 \\ & 0 & 1 & 0 & -1 \\ \hline & 1 & 0 & -1 & \boxed{0} \end{array}$$

$(\lambda-1)(\lambda^2 + 0\lambda - 1) = 0$

$\Rightarrow (\lambda-1)(\lambda^2-1) = 0 \quad \Rightarrow (\lambda-1)(\lambda+1)(\lambda-1)$

so eigen values are :- $1, 1, -1.$

Now Case I :- when $\lambda = 1$

$|A - \lambda I| X = 0$ where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

~~From eq. (i) ; $x_1 = x_2 = x_3 = 0$~~

$\Rightarrow 0x_1 + 0x_2 + 0x_3 = 0$ —(i)

$0x_1 - x_2 + x_3 = 0$ —(ii)

$0x_1 + x_2 - x_3 = 0$ —(iii)

So from eq. (i), $x_1 = x_2 = x_3 = 0$

" eq. (ii), $x_2 = x_3 = 1.$

" eq (iii), $x_2 = x_3 = 1.$

so $X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Case II :- When $\lambda = -1$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$\Rightarrow \quad 2x_1 + 0x_2 + 0x_3 = 0 \quad \text{—(IV)}$

$0x_1 + x_2 + x_3 = 0 \quad \text{—(V)}$

$0x_1 + x_2 + x_3 = 0 \quad \text{—(VI)}$

Applying cross multiplication rule,

$$\begin{matrix} 0 & & 0 & & 2 & & 0 \\ & \times & & \times & & \times & \\ 1 & & 1 & & 0 & & 1 \end{matrix}$$

$$\frac{x_1}{0} = \frac{x_2}{0-2} = \frac{x_3}{2-0}$$

ie $\dfrac{x_1}{0} = \dfrac{x_2}{-2} = \dfrac{x_3}{2}$

so $X_2 = \begin{bmatrix} 0 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Now using orthogonal property,

$X_1^T \cdot X_3 = 0 \qquad$ where $X_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

$\Rightarrow \quad 0a + b + c = 0 \quad \text{—(VII)}$

& $X_2^T X_3 = 0$

$\Rightarrow \begin{bmatrix} 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

$\Rightarrow \quad 0 \cdot a - b + c = 0. \quad \text{—(VIII)}$