Homeworks:-

Q1> Test the convergence of series :- $\frac{x}{1\cdot 2} + \frac{x^2}{3\cdot 4} + \frac{x^3}{5\cdot 6} + \cdots$

Ans:. $\Sigma u_n = \cancel{\frac{x^n}{n(n+1)}} \frac{x^n}{(2n-1)2n}$; $\Sigma u_{n+1} = \frac{x^{n+1}}{[2(n+1)-1]2(n+1)}$

$$\therefore \frac{\Sigma u_{n+1}}{\Sigma u_n} = \frac{x^{n+1}}{[2(n+1)-1][2(n+1)]} \times \frac{(2n-1)2n}{x^n}$$

$$= \frac{x^n \cdot x}{(2n+1)(2n+2)} \cdot \frac{(2n-1) \cdot 2n}{x^n}$$

$$= \frac{x}{4n^2\left(1+\frac{1}{2n}\right)\left(1+\frac{2}{2n}\right)} \cdot \frac{4n^2\left(1-\frac{1}{2n}\right)}{1}$$

$$= \frac{x \cdot \left(1-\frac{1}{2n}\right)}{\left(1+\frac{1}{2n}\right)\left(1+\frac{2}{2n}\right)}$$

Now $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x\left(1-\frac{1}{2n}\right)}{\left(1+\frac{1}{2n}\right)\left(1+\frac{2}{2n}\right)}$

$= x.$

By Rato test the series is convergent ~~divergent~~ if $x < 1$ and divergent if $x > 1$. The test fails when $x = 1$.

(PTO)

If $x = 1$, $\Sigma u_n = \dfrac{x^n}{(2n-1)2n} = \dfrac{1}{(2n-1)2n}$

$\Sigma v_n = \dfrac{1}{n^2}$ $\left[\because \Sigma v_n = \dfrac{\text{highest pow. of } n \text{ in numerator}}{\text{highest pow of } n \text{ in denominator}}\right.$

$\therefore$ On comparing Σv_n with $\dfrac{1}{n^p}$, $p = 2$ which is > 1

so Σv_n is convergent.

Again $\lim\limits_{n\to\infty} \dfrac{\Sigma u_n}{\Sigma v_n} = \lim\limits_{n\to\infty} \dfrac{1}{(2n-1)2n} \times n^2$

$= \lim\limits_{n\to\infty} \dfrac{1}{4n^2(1-\frac{1}{2n})} \cdot n^2$

$= \dfrac{1}{4} \lim\limits_{n\to\infty} \dfrac{1}{\left(1-\frac{1}{2n}\right)} = \dfrac{1}{4}$ which is finite & non zero

$\therefore$ The given series is converging for $x \leq 1$ and diverging for $x > 1$. (Ans)

Q2) Test the convergence :- $\sum \frac{n!}{n^n}$.

Ans:- $\sum u_n = \sum \frac{n!}{n^n}$; $\sum u_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}}$

Now $\sum \frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+1)^{n+1}} \times \frac{n^n}{n!}$

$= \frac{n!\,(n+1)}{(n+1)(n+1)^n} \times \frac{n^n}{n!}$

PTO.

$$\frac{\Sigma u_{n+1}}{\Sigma u_n} = \frac{n^n}{(n+1)^n} = \frac{n^n}{n^n\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n}$$

Now $\lim_{n\to\infty} \frac{\Sigma u_{n+1}}{\Sigma u_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e}$ which is < 1

So series is convergent. (Ans)

Q3> Test the convergence of series:-

$$\frac{x}{1} + \frac{1}{2} \cdot \frac{x^2}{3} + \frac{1.3}{2.4} \cdot \frac{x^3}{5} + \frac{1.3.5}{2.4.6} \cdot \frac{x^4}{7} + \cdots$$

Ans:- $\Sigma u_n = \dfrac{1.3.5.\ldots(2n-1)}{2.4.6\ldots 2n} \cdot \dfrac{x^n}{(2n+1)}$

$$\Sigma u_{n+1} = \frac{1.3.5.\ldots(2n-1)(2n+1)}{2.4.6.\ldots 2n.(2n+2)} \cdot \frac{x^{n+1}}{(2n+3)}$$

$$\therefore \frac{\Sigma u_{n+1}}{\Sigma u_n} = \frac{1.3.5.(2n-1)(2n+1)}{2.4.6.\cdots 2n(2n+2)} \cdot \frac{x^{n+1}}{(2n+3)} \cdot \times \frac{2.4.6.\cdots 2n}{1.3.5.\cdots(2n-1)} \cdot \frac{(2n+1)}{x^n}$$

$$= \frac{x.(2n+1)^2}{(2n+2)(2n+3)} = \frac{x.4n^2\left(1+\frac{1}{2n}\right)^2}{4n^2\left(1+\frac{2}{2n}\right)\left(1+\frac{3}{2n}\right)}$$

$$= \frac{x.\left(1+\frac{1}{2n}\right)^2}{\left(1+\frac{2}{2n}\right)\left(1+\frac{3}{2n}\right)}$$

Now $\lim_{n\to\infty} \frac{\Sigma u_{n+1}}{\Sigma u_n} = \lim_{n\to\infty} \frac{x.\left(1+\frac{1}{2n}\right)^2}{\left(1+\frac{2}{2n}\right)\left(1+\frac{3}{2n}\right)} = x.$

Now by Ratio test the series is convergent if $x < 1$ and divergent if $x > 1$

The test fails for $x = 1$.

If $x = 1$, $\frac{\Sigma u_n}{\Sigma u_{n+1}} = \frac{(2n+2)(2n+3)}{(2n+1)^2 . x} = \frac{(2n+2)(2n+3)}{(2n+1)^2}$ $[\because x = 1]$

$$\therefore \frac{\Sigma u_n}{\Sigma u_{n+1}} - 1 = \frac{(2n+2)(2n+3)}{(2n+1)^2} - 1$$

$$= \frac{\cancel{4n^2} + \cancel{4n} + 6n + 6 - \cancel{4n^2} - \cancel{4n} - 1}{(2n+1)^2}$$

$$= \frac{6n+5}{(2n+1)^2} = \frac{n\left(6+\frac{5}{n}\right)}{n^2\left(2+\frac{1}{n}\right)^2}$$

$$\Rightarrow n\left[\frac{\Sigma u_n}{\Sigma u_{n+1}} - 1\right] = \frac{n^2(6+5/n)}{n^2(2+1/n)^2} = \frac{(6+5/n)}{(2+1/n)^2}$$

PTO

Now $\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}} - 1\right] = \lim_{n\to\infty} \frac{(6+5/n)}{(2+\frac{1}{n})^2}$

$$= \frac{6}{4} \text{ which is } > 1$$

$\therefore$ u_n is convergent.

Thus the given series is convergent for $x \leq 1$ and divergent if $x > 1$. (Ans).