Homogeneous Coord

$[x \quad y \quad h]$

$$h \downarrow \quad h = 1$$

$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{bmatrix}$$

$[6 \quad 4 \quad 2]$

$\downarrow$ for homogeneous coord, this should be 1

$\Rightarrow [3 \quad 2 \quad 1]$

Translation Matrix

$$[x \quad y \quad 1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \underbrace{m \quad n}_{\text{rest same as identity}} & & 1 \end{bmatrix} = [x+m \quad y+n \quad 1]$$

Rotation Matrix :

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

third row & third column is identity

If say we have to shift origin from 0, 0 to (3, 4)

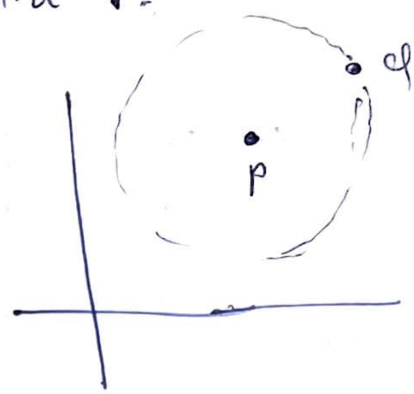
$$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 1 \end{bmatrix}$$

Reflection :

for $x = 0$ $\cancel{x=y}$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation about an arbitrary point P.



$P(m, n)$

We translate P to origin & Q is also correspondingly translated.

rotation is done anti-clockwise

After rotation, translate P back to its position & correspondingly Q as well.

Step 1 → perform translation such that pt P coincides w/ the origin.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m & -n & 1 \end{bmatrix}$$

Step 2 → Perform rotation.

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 3 → Inverse translation

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & n & 1 \end{bmatrix}$$

order is imp.

$$- \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m & -n & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & n & 1 \end{bmatrix}$$

Q Rotate the point Q(10, 15) about the point P(4,3) 90° in counter-clockwise sense.

$$\begin{bmatrix} 10 & 15 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix} \begin{bmatrix} \cancel{\cos 90}^{0} & \cancel{\sin 90}^{1} & 0 \\ -\cancel{\sin 90}^{-1} & \cancel{\cos 90}^{0} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$$

Repto

Reflection through an arbitrary line

step 1 → translate the line & the object so that the line passes through the origin.

step 2 → Rotate the line & the object about the origin until the line is co incident w/ one of the coordinate axes.

step 3 → Reflect through the coordinate axes

step 4 → Apply inverse rotation

step 5 → Translate back to original location.

$$[\text{Trans}][\text{Rot}][\text{Ref}][\text{Rot}]^{-1}[\text{Trans}]^{-1}$$

Q Reflect the ΔABC through the line L

$A(2,4)$ $B(4,6)$ $C(2,6)$

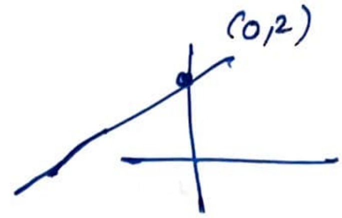
$L \rightarrow y = \frac{1}{2}(x+4)$

Sol^n $\rightarrow$

~~(coincide w/ origin or any of the axes)~~

Translation :

$$y = \frac{1}{2}(x+4)$$



$x = 0$, $y = 2$

coincide either this pt of ~~x~~y-intercept with the origin.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

~~Reflection:~~

Rotation :

$$\tan\theta = m$$

$$\theta = \tan^{-1}(m)$$

$$= \tan^{-1}\left(\frac{1}{2}\right) = 26.57$$

$\because$ coinciding w/ y-axis $\Rightarrow$ take $\underline{-m}$

i.e. cos(-m)

$$\begin{bmatrix} \cos(-26.57) & \sin(-26.57) & 0 \\ -\sin(-26.57) & \cos(-26.57) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Refl^n $\rightarrow$ $x = 0$

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotation inverse $\rightarrow$

$$\begin{bmatrix} \cos(26.57) & \sin(+26.57) & 0 \\ -\sin(26.57) & \cos(26.57) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translation inverse. (negation of prev. points)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Matrix for $\Delta \rightarrow$ $\begin{bmatrix} 2 & 4 & 1 \\ 4 & 6 & 1 \\ 2 & 6 & 1 \end{bmatrix}$

(write points row-wise)

$\downarrow$

$\because$ we are using homogenous coordinates these should always be 1

Final ans →

$$\begin{bmatrix} 2 & 4 & 1 \\ 4 & 6 & 1 \\ 2 & 6 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 3/5 & 4/5 & 0 \\ 4/5 & -3/5 & 0 \\ -8/5 & 16/5 & 1 \end{bmatrix}}_{\text{Final } T} = \underbrace{\begin{bmatrix} 14/5 & 12/5 & 1 \\ 28/5 & 14/5 & 1 \\ 22/5 & 6/5 & 1 \end{bmatrix}}_{\text{Final answer!}}$$

This cannot be 0.
If this is not 1,
divide the whole matrix by that no.

Scaling (enlargement or compression)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix}$$ → used for overall scaling by a factor of $\frac{1}{s}$

scaling matrix

(scaling is equal in x & y)

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix} = \begin{bmatrix} x & y & s \end{bmatrix}$$

⇓

divide w/ s

$$\begin{bmatrix} \frac{x}{s} & \frac{y}{s} & 1 \end{bmatrix}$$

If $s = 2$, the objected will shrink by a factor of 2

Q Write the overall scaling matrix to increase the size of the object to double ~~of~~ its size.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

If the scaling may or may not be equal for $x \neq y$

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Local scaling matrix}$$

x is scaled by a
y is scaled by b

$$\begin{bmatrix} x & y & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ax & by & 1 \end{bmatrix}$$

Q What will be the local scaling matrix to double the size of the object

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Points at ∞

$[x \quad y \quad 0]$

↓

pts x, y in the plane $h = 0$

If $h = 0 \Rightarrow$ lines are not intersecting & if $h \neq 0$ then yes

Q Consider a pair of intersecting lines $x + y = 1$, $2x - 3y = 0$.

Find the intersection pt.

Sol^n → $x + y - 1 = 0$

$2x - 3y = 0$

If the eq^n of line is given, write the coefficients columnwise.

$$\begin{array}{c} x \\ y \\ c \end{array} \begin{bmatrix} 1 & 2 \\ 1 & -3 \\ -1 & 0 \end{bmatrix}$$

Find inverse of this matrix.

$$[x \quad y \quad 1] \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 1 & -3 & 0 \\ -1 & 0 & 1 \end{bmatrix}}_{\text{convert into square matrix}} = [0 \quad 0 \quad 1]$$

→ m

$$[x \quad y \quad 1] = [0 \quad 0 \quad 1]\, m^{-1}$$

$$m^{-1} = \frac{1}{5}\begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 0 \\ 3 & 2 & 5 \end{bmatrix}$$

$$[x \quad y \quad 1] = \begin{bmatrix} \frac{3}{5} & \frac{2}{5} & 0 \\ \frac{1}{5} & \frac{-1}{5} & 0 \\ \frac{3}{5} & \frac{2}{5} & 1 \end{bmatrix} [0 \quad 0 \quad 1]$$

$$= [0 \quad 0 \quad 1]\begin{bmatrix} \frac{3}{5} & \frac{2}{5} & 0 \\ \frac{1}{5} & \frac{-1}{5} & 0 \\ \frac{3}{5} & \frac{2}{5} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{5} & \frac{2}{5} & 1 \end{bmatrix}$$

$\therefore h = 1 \Rightarrow$ intersecting lines

If $x + y = 1$

$x + y = 0$

$1 = 1$ (the third column to make a 3×3 sq. matrix)

$$[x \ y \ 1]\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = [0 \ 0 \ 1]$$

$\because$ 2 rows are equal, we cannot find the det. for inverse

Thus, we take the third column for $x = x$

$$[x \ y \ 1]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = [0 \ 0 \ x]$$

$$[x \ y \ 1] = [0 \ 0 \ x]\begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$= \{ [x \ \ -x \ \ 0] \}$$

$\therefore$ $h = 0 \Rightarrow$ parallel lines

Q Give a point at ∞ on +ve x-axis

Sol^n → $[\,3 \quad 0 \quad 0\,]$

3 ↓ +ve x

0 ↓ take y to be 0 (∵ pt is at x-axis)

0 ↓ ∵ at ∞

3-D

$[x \quad y \quad z \quad 1]_{1\times4}$

→ If nothing is mentioned in the ques, by default use homogeneous coordinates.

$$T = \begin{bmatrix} a & b & c & 0 \\ d & e & f & 0 \\ g & i & j & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Scaling

$$[x \quad y \quad z \quad 1] \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [ax \quad ey \quad jz \quad 1]$$

Local scaling

If $a = e = j$ ⇒ equal scaling otherwise unequal scaling.

Overall Scaling :

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix}$$

always equal

$$\begin{bmatrix} x & y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix} = \begin{bmatrix} x & y & z & s \end{bmatrix}$$

÷ by s

$$= \begin{bmatrix} \frac{x}{s} & \frac{y}{s} & \frac{z}{s} & 1 \end{bmatrix}$$

When $s > 1$ ⇒ uniform compression.

if $0 < s < 1$ ⇒ uniform expansion

$s = 1$ ⇒ no change

Q Given a unit cube at the origin, perform uniform scaling by a factor of 2. What will be the transformation matrix & the resultant cube

Coordinates →

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\text{Ans} = \begin{bmatrix} 0 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 0.5 \\ 0 & 1 & 0 & 0.5 \\ 0 & 1 & 1 & 0.5 \\ 1 & 0 & 0 & 0.5 \\ 1 & 0 & 1 & 0.5 \\ 1 & 1 & 0 & 0.5 \\ 1 & 1 & 1 & 0.5 \end{bmatrix} \implies$$

h should be 1

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 0 & 2 & 2 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 2 & 0 & 1 \\ 2 & 2 & 2 & 1 \end{bmatrix}$$

(i) Give the overall scaling matrix & local scaling matrix if I have to decrease the size ~~the~~ by a factor of 1/4

Overall $\rightarrow$ $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

Local $\rightarrow$ $\begin{bmatrix} 1/4 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3-D Shearing

$$[x \quad y \quad z \quad 1] \begin{bmatrix} 1 & b & c & 0 \\ d & 1 & e & 0 \\ f & g & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= [\, x + dy + fz \quad xb + y + gz \quad cx + ey + z \quad 1 \,]$$

$x + dy + fz$ $\downarrow$ Shearing in x wrt y & z

$xb + y + gz$ $\downarrow$ Shearing in y wrt x, z

$cx + ey + z$ $\downarrow$ Shearing in z wrt x & y.

Rotating will be performed wrt a line in 3-D

Rotation $\rightarrow$ (in anti-clockwise sense)

wrt x-axis :

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & & & & 0 \\ 0 & & & & 0 \\ & & & & 0 \\ 0 & 0 & 0 & & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

wrt y-axis :

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

wrt z-axis :

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Pg 109)

3D Rotation is non commutative i.e. order of rot^n affects the final result

(T.P., assume it is commutative, then prove by contradiction)

3-D Reflection →

wrt xy plane

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

wrt yz plane

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

xz plane

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(Pg 114, eg 3.6)

3-D Translation →
adding a constant value to x, y, z

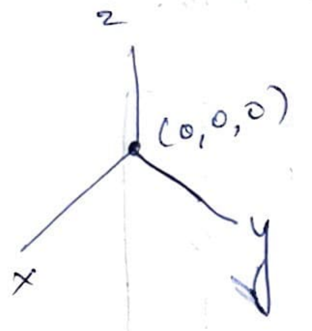
Translation is also k/a shifting

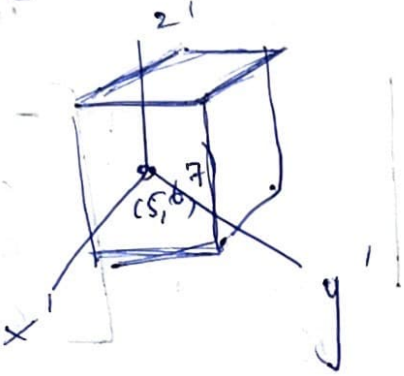
$$[x \; y \; z \; 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l & m & n & 1 \end{bmatrix} = [x+l \quad y+m \quad z+n \quad 1]$$

(Pg 116, eg 3.7)

Rotation about an axis parallel to coordinate axis

$$[\text{Trans}]\,[\text{Rot}(s)]\,[\text{Trans}]^{-1}$$





step 1 → Perform translation so that the local axis origin coincides w/ global axis origin.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & -6 & -7 & 1 \end{bmatrix}$$

Step 2 → Perform rotation as asked in the ques.

Step 3 → Translation inverse

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 6 & 7 & 1 \end{bmatrix}$$

(Pg 118, eg 3.8)

Multiple rotations

[Trans] [Rotations] [Trans]$^{-1}$

(Pg. 119, eg 3.9)

<u>Rotation</u>

$$[X] = \begin{array}{cc} \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} & \begin{matrix} A \\ B \\ C \\ D \\ E \\ F \\ G \\ H \end{matrix} \end{array}$$

Centroid - $\frac{12}{8}$ (x) $= \frac{12}{8}$ (z)

$= \frac{12}{8}$ (y)

$\left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right)$

$$[X^*] = [X][T_r][R][T_r^{-1}]$$

$$T_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3/2 & -3/2 & -3/2 & 1 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotⁿ about an arbitrary axis in space

$$[T]\underbrace{[R_x][R_y]}_{\text{extra}}[R_\delta][R_y]^{-1}[R_x]^{-1}[T]^{-1}$$

$\cancel{d = \sqrt{C_y^2 + C_z^2}}$ $\quad d = \sqrt{C_y^2 + C_z^2}$

$\cos\alpha = \dfrac{C_z}{d} \qquad \sin\alpha = \dfrac{C_y}{d}$

$\cos\beta = d \qquad \sin\beta = C_x$

Rotation matrix

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_z/d & C_y/d & 0 \\ 0 & -C_y/d & C_z/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[R_y] = \begin{bmatrix} d & 0 & C_x & 0 \\ 0 & 1 & 0 & 0 \\ -C_x & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[R_\delta] = \begin{bmatrix} \cos\delta & \sin\delta & 0 & 0 \\ -\sin\delta & \cos\delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Direction cosines

$$[C_x \quad C_y \quad C_z] = \frac{[(x_1-x_0) \quad (y_1-y_0) \quad (z_1-z_0)]}{[(x_1-x_0)^2 + (y_1-y_0)^2 + (z_1-z_0)^2]}$$

6/04/2023

Pg 125, example 3.9 Imp

When you inverse a rot^n matrix, just change the signs of $\sin\theta$ & not $\cos\theta$ as $\cos(-\theta) = \cos\theta$