Sound wave equation:-

$$\frac{\partial P}{\partial t} + \sigma' P = -\rho c^2 (\nabla \cdot v) \quad \text{— (i)}$$

$$\beta \frac{\partial v}{\partial t} + \sigma' v = -\beta^2 \frac{\nabla P}{\rho} + \sigma' v_b \quad \text{— (ii)}$$

Discretization

Solving equation -(i)

$$\nabla \cdot v = \left(\frac{\partial u_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) = C_x u + C_y v$$

$$\frac{\partial u_x}{\partial x} = C_x u = \frac{u_x(i)|_t - u_x(i-1)|_t}{\Delta x}$$

$$\frac{\partial u_y}{\partial y} = C_y v = \frac{v_y(j)|_t - v_y(j-1)|_t}{\Delta y}$$

$$\Rightarrow \frac{\partial P}{\partial t} + \sigma' P = -\rho c^2 \left(C_x u + C_y v \right)$$

$$\Rightarrow \left(\frac{P|_{t+1/2} - P|_{t-1/2}}{\Delta t} \right) + \sigma' \left(\frac{P_{t+1/2} + P_{t-1/2}}{2} \right) = -\rho c^2 (C_x u + C_y v)$$

$$\Rightarrow P_{t+1/2} \left[\frac{1}{\Delta t} + \frac{\sigma'}{2} \right] - P_{t-1/2} \left[\frac{1}{\Delta t} - \frac{\sigma'}{2} \right] = -\rho c^2 (C_x u + C_y v)$$

$$\Rightarrow P|_{t+1/2} = P_{t-1/2}\left(\frac{1/\Delta t - \sigma'/2}{1/\Delta t + \sigma'/2}\right) + \frac{(-\rho c^2)}{\left(1/\Delta t + \sigma'/2\right)}\left(C_x u + C_y v\right)$$

Solving eqn - (ii)

$$\beta \frac{\partial v}{\partial t} + \sigma' v = -\beta^2 \frac{\nabla \rho}{\rho} + \sigma' v_b$$

$$\Rightarrow \beta\left(\frac{u_x|_{t+\Delta t} - u_x|_t}{\Delta t}\right) + \sigma'\left(\frac{u_x|_{t+\Delta t} + u_x|_t}{2}\right) =$$

$$-\frac{\beta^2}{\rho} C_x P + \sigma' v_b$$

$$\Rightarrow u_x|_{t+\Delta t}\left(\frac{\beta}{\Delta t} + \frac{\sigma'}{2}\right) - u_x|_t\left(\frac{\beta}{\Delta t} - \frac{\sigma'}{2}\right) = \frac{-\beta^2}{\rho} C_x P + \sigma' v_b$$

$$\Rightarrow u_x|_{t+\Delta t} = \left(\frac{\beta/\Delta t - \sigma'/2}{\beta/\Delta t + \sigma'/2}\right) u_x|_t + \frac{\left(\frac{-\beta^2}{\rho}\right)}{\left(\frac{\beta}{\Delta t} + \sigma'/2\right)} C_x P +$$

$$\left(\frac{\sigma'}{\beta/\Delta t + \sigma'/2}\right) v_b.$$

Same could be proved for v_y