भारतीय विज्ञान संस्थान

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SEMESTER NOTES

Irish Debbarma

Department of Mathematics
Indian Institute of Science, Bangalore

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Contents

Part I.

Modular Forms

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3.1. Valence formula

Recall that $M_k(\Gamma_1)$ is the space of modular forms of weight k and level 1. It is also a vector space over $\mathbb{C}$.

Theorem 3.1.1.
$$\dim M_k(\Gamma_1) = \begin{cases} [k/12] + 1 & k \not\equiv \pmod{12} \\ [k/12] & k \equiv \pmod{12} \end{cases}$$

Proposition 3.1.2.
Let $f \in M_k(\Gamma_1)$. Then,

$$\sum_{p \in \Gamma_1 \backslash \mathbb{H}} \frac{1}{n_p} \mathrm{ord}_p(f) + \mathrm{ord}_\infty(f) = \frac{k}{12}$$

Proof. Let $\epsilon > 0$ be "small enough". Remove ϵ-balls around $\infty, i, \omega, \omega + 1$ in $\mathcal{F}_1$. ϵ is small enough so that the removed balls are disjoint. Truncate $\mathcal{F}_1$ at the line $y = \epsilon^{-1}$ and call the enclosed region D.
By Cauchy's theorem
$$\int_{\partial D} d(\log f(z)) = 0$$
This integral on the two vertical strips (just the straight lines not the semicircle part) is 0 since the contribution of left is same as right but orientation is different. On the segment joining $-1/2 + iY, 1/2 + iY$, the integral is $2\pi i \, \mathrm{ord}_\infty(f)$. Again,integral around each removed point in $\mathcal{F}_1$ is $\frac{1}{n_p}\mathrm{ord}_p(f)$. Next, divide the bottom arc into left and right parts and observe that

$$d(\log f(S \cdot z)) = d(\log f(z)) + k\frac{dz}{z}$$

$$\int_C d(\log f(z)) = \frac{k\pi i}{6}$$

□

Corollary 3.1.3.

$$\dim M_k(\Gamma_1) = \begin{cases} 0 & k < 0 \\ 0 & k \text{ is odd} \\ 1 & k = 0 \\ \begin{cases} [k/12]+1 & k \not\equiv \pmod{12} \\ [k/12] & k \equiv \pmod{12} \end{cases} \end{cases}$$

Proof.

- If $k < 0$, then f has poles but is holomorphic.
- If $k = 0$, then f is the constant function.
- We have seen
- For $m = [k/12] + 1$ let $f_1, \ldots, f_{m+1} \in M_k(\Gamma_1)$. Let $P_1, \ldots, P_m$ be any points on $\mathcal{F}_1$ not equal to $i, \omega, \omega + 1$ and consider $(f_i(P_j))_{i \in [m+1], j \in [m]}$.
  There exists a linear combination $f = \sum_{i=1}^{m+1} c_i f_i$ not all c_i being zero, such that $f(P_j) = 0$ for $1 \leq j \leq m$.
  From the previous theorem we get $f \equiv 0$ and this implies $\{f_i\}$ is linearly independent and thus $\dim_{\mathbb{C}} M_k(\Gamma_1) \leq m$.
  For $k \equiv 2 \pmod{12}$, the relation in previous theorem holds only if there is atleast a simple zero at $p = i$ and atleast a double zero at $p = \omega$. This gives

  $$\frac{k}{12} - \frac{7}{6} = m - 1$$

  Repeat the argument above.

∎

A slight notation. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we set $f|_\gamma(z) = (cz+d)^{-k} f(\gamma \cdot z)$.
Thus, $1|_\gamma(z) = (cz+d)^{-k}$. If $1|_\gamma(z) = 1 \Rightarrow c = 0$. Conversely, if $c = 0$, then $d^{-k} = 1$. So, $1|_\gamma(z) = 1 \Leftrightarrow c = 0$.
$\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\} = \mathrm{stab}(\infty)$

3.2. Eisenstein series

Definition 3.2.1.
The Eisenstein series $E_k(z)$ is defined to be

$$E_k(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_1} 1|_\gamma(z)$$

Proposition 3.2.2.

$$E_k(z) = \frac{1}{2} \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\},\gcd(c,d)=1} \frac{1}{(cz+d)^k}$$

Proof. □

Proposition 3.2.3.

$$\sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\},\gcd(c,d)=1} \frac{1}{(cz+d)^k}$$

converges absolutely for $k > 2$

Proof. □

Theorem 3.2.4.
$E_k(z) \in M_k(\Gamma_1)$ for $k > 2$.

Proof. □

Proposition 3.2.5.
$E_k(z) \not\equiv 0$ for $k > 2$, even.

Proof. Observe that

$$\frac{1}{(cz+d)^k} \to 0, \Im(z) \to \infty, c \neq 0$$

and if $c = 0$, then $c = \pm 1$. Hence, $E_k(z) = 1+$ bounded term as $\Im(z) \to \infty$. This implies $E_k(z) \not\equiv 0$ and

$$E_k(z) = 1 + \sum_{n=1}^{\infty} a_n e^{2\pi i z}$$

□

Another way of looking at Eisenstein series is a function on a lattice.
Consider $G_k(z) = G_k(\mathbb{Z}z + \mathbb{Z}) = \frac{1}{2}\sum'_{(c,d)\in\mathbb{Z}^2} \frac{1}{(cz+d)^k}$

Proposition 3.2.6.
$G_k(z)$ converges absolutely for $k > 2$.

Proposition 3.2.7.
$G_k(z) = \zeta(k)E_k(z)$

Proposition 3.2.8.
$\mathbb{G}_k(z) = \frac{(k-1)!}{(2\pi i)^k} G_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$ for $k > 2$, even.

4. Lecture-4 (12th January, 2023): Eisenstein series

4.1. Eisenstein series contd..

Recall that

$$M_*(\Gamma_1) = \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma_1)$$

is a graded ring.

Proposition 4.1.1.
The graded ring $M_*(\Gamma_1)$ is freely generated by E_4, E_6. This means that the map

$$\begin{aligned} f : C[X, Y] &\to M_*(\Gamma_1) \\ X &\mapsto E_4 \\ Y &\mapsto E_6 \end{aligned}$$

is an isomorphism of graded rings. Here, $\deg X = 4, \deg Y = 6$.

Proof. We want to show that E_4 and E_6 are algebraically independent. We start by showing that E_4^3 and E_6^2 are linearly independent over $\mathbb{C}$. Suppose $E_6(z)^2 = \lambda E_4(z)^3$. Consider $f(z) = E_6(z)/E_4(z)$. Now observe that $f(z)^2 = \lambda E_4(z)$. This means that f^2 is holomorphic and thus f is also holomorphic. But f is weakly modular of weight 2 which is a contradiction. So, our claim is proven.

Claim: Let f_1, f_2 be two nonzero modular forms of same weight. If f_1, f_2 are linearly independent, then they are algebraically independent as well.

Let $P(t_1, t_2) \in \mathbb{C}[t_1, t_2] \backslash \{0\}$ be such that $P(f_1, f_2) = 0$. Let $P_d(t_1, t_2)$ be the d degree parts of P. Using the fact that modular forms of different weights are linearly independent, we get that $P_d(f_1, f_2) = 0 \ \forall \ d \geq 0$. If $p_d(t_1/t_2) = P_d(t_1, t_2)/t_2^d$, then $p_d(f_1/f_2) = 0$. But this means that f_1/f_2 is a constant. But, f_1, f_2 are linearly independent which implies that they are algebraically independent as well.

All of this implies that E_4, E_6 are algebraically independent. Using □

Corollary 4.1.2.

$$\dim_{\mathbb{C}} M_k(\Gamma_1) = \begin{cases} [k/12]+1 & k \not\equiv \pmod{12} \\ [k/12] & k \equiv \pmod{12} \end{cases}$$

4.1.1. Fourier expansions of $E_k(z)$

Proposition 4.1.3.

$$\mathbb{G}_k(z) = \frac{(k-1)!}{(2\pi i)^k} G_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

for $k > 2$, even and B_k are Bernoulli numbers.

Proof. Use

$$\frac{\pi}{\tan \pi z} = \sum_{n\in\mathbb{Z}} \frac{1}{z+n} = \lim_{M,N\to\infty, N-M<\infty} \sum_{-M}^{N} \frac{1}{z+n}$$

and

$$\frac{\pi}{\tan \pi z} = \frac{\pi \cos \pi z}{\sin \pi z} = \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = -\pi i \frac{1+q}{1-q} = -2\pi i \left(\frac{1}{2} + \sum_{r=1}^{\infty} q^r \right)$$

This leads to the equality

$$\sum_{n\in\mathbb{Z}} \frac{1}{z+n} = -2\pi i \left(\frac{1}{2} + \sum_{r=1}^{\infty} q^r \right)$$

Differentiate both sides of equality $k-1$ times and divide by $(k-1)!$ to get

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} q^r$$

Next, if we look at

$$\begin{aligned} G_k(z) &= \frac{1}{2} {\sum}' \frac{1}{(mz+n)^k} \\ &= \frac{1}{2} \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{n^k} + \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2, m\neq 0} \frac{1}{(mz+n)^k} \\ &= \zeta(k) + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\ &= \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} r^{k-1} q^{mr} \\ &= \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \sigma_{k-1}(n) q^n \end{aligned}$$

The expression of $\mathbb{G}_k(z)$ is trivial after noting

$$\frac{(k-1)!}{(2\pi i)^k}\zeta(k) = B_k$$

□

Remark 4.1.4. 1. $\mathbb{G}_4(z) = \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + \cdots$

2. $\mathbb{G}_6(z) = -\frac{1}{504} + q + 33q^2 + 244q^3 + \cdots$

3. $\mathbb{G}_8(z) = \frac{1}{480} + q + 129q^2 + 2188q^3 + \cdots$

Proposition 4.1.5.

$$\sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) = \frac{\sigma_7(n) - \sigma_3(n)}{120}$$

Proof. □

4.1.2. Weight 2 Eisenstein series

Definition 4.1.6.

$$\begin{aligned}\mathbb{G}_2(z) &= -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)q^n \\ &= -\frac{1}{24} + q + 3q^2 + 4q^3 + 7q^4 + \cdots\end{aligned}$$

This converges rapidly on $\mathbb{H}$ and defines a holomorphic function.

Proposition 4.1.7.

$$G_2(z) = -4\pi^2\mathbb{G}_2(z)$$

Proof. Since we know that

$$G_2(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^2}$$

does not converge absolutely, we define

$$G_2(z) = \frac{1}{2}\sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{n^2} + \frac{1}{2}\sum_{m\neq 0}\sum_{n\in\mathbb{Z}} \frac{1}{(mz+n)^2}$$

This sum converges absolutely and we can show that this satisfies the functional equation as required. ☐

Proposition 4.1.8.
For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 G_2(z) - \pi i c(cz+d)$$

G_2 is called a quasi modular form.

Introduce (due to Hecke):

$$G_{2,s}(z) = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^2|mz+n|^{2s}}, \mathfrak{R}(s) > 0$$

4.2. Modular forms of higher level

Let $N \in \mathbb{Z}_{\geq 1}$

$$\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}/N\mathbb{Z}) \mid ad - bc \equiv 1 \pmod N\right\}$$

Lemma 4.2.1.
The map

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z}) &\to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \end{aligned}$$

is a surjective group homomorphism.

Proof. ☐

Definition 4.2.2.

$$\Gamma(N) = \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$$

is called the principal congruence subgroup.

Definition 4.2.3.
A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if there exists N such

that $\Gamma(N) \subseteq \Gamma$.

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}/N\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}/N\mathbb{Z}) \mid c \equiv d \equiv 1 \pmod{N} \right\}$$

5. Lecture-5 (17th January, 2023): Congruence subgroups and Δ function

5.1. Δ function

Consider
$$\Delta(z) = \frac{1}{1729}(E_4^3(z) - E_6^2(z)) = q + q^2() + \cdots$$
Clearly, $\Delta(z)$ is a normalised cusp form of weight 12 and level 1.

Theorem 5.1.1.
$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24}, q = e^{2\pi i z}$$

Proposition 5.1.2.
$\Delta(z)$ has no zero in $\mathbb{H}$.

Proof. From the valence formula we have
$$\sum_{p\in\mathbb{H}} \frac{1}{n_p}\mathrm{ord}_p(\Delta(z)) + \mathrm{ord}_\infty(\Delta(z)) = k/12 = 1$$
Moreover, $\mathrm{ord}_\infty(\Delta(z)) = 1$. Hence, we can conclude that $\mathrm{ord}_p(\Delta(z)) = 0\ \forall\ p \in \mathbb{H}$. □

Application: We use $\Delta(z)$ to write any modular form as a polynomial in E_4, E_6.

Take $f(z) \in M_k(\Gamma_1)$ with $4a + 6b, k \geq 4, a, b \geq 0$. The Fourier expansion of $f(z)$ can we written as
$$f(z) = a_0 + a_1 q + \cdots$$
Clearly, $f(z) - a_0 E_4^a(z)E_6^b(z) \in M_k(\Gamma_1) \subseteq S_k(\Gamma_1)$.
Next,
$$h(z) = \frac{f(z) - a_0 E_4^a(z)E_6^b(z)}{\Delta(z)} \in M_{k-12}(\Gamma_1)$$
Recursively, we can now find expression for $f(z)$.

Proposition 5.1.3.

$$j(z) = \frac{E_4^3}{\Delta(z)} = q^{-1} + \cdots$$

$$\begin{aligned} j : \bar{\mathbb{H}}/\Gamma_1 &\to \mathbb{P}^1(\mathbb{C}) \\ z &\mapsto j(z) \end{aligned}$$

is a bijection.

Proof. $E_4^3(z)$ and $\Delta(z)$ are linearly independent. For any $\lambda_1, \lambda_2 \in \mathbb{C}$ both not zero, $\lambda_1 E_4^3(z) + \lambda_2 \Delta(z)$ has an unique zero in $\bar{\mathbb{H}}/\Gamma_1$. □

Remark 5.1.4.
This j is called the j-invariant modular function. It attaches an elliptic curve in $\mathbb{P}^1(\mathbb{C})$ to any lattice in $\Lambda_z = \mathbb{Z}z + \mathbb{Z}$ and vice versa.

Next, the Fourier series of $\Delta(z)$ is of the form $\Delta(z) = \sum_{n\geq 1} \tau(n) q^n$ where $\tau(n)$ satisfies the following properties:

1. $\tau(pq) = \tau(p)\tau(q)$ if p, q are dinstinct primes.
2. $\tau(p^2) = \tau(p)^2 - p^{12-1}$.
3. $|\tau(p)| \leq 2p^{\frac{12-1}{2}}$.
4. $$\begin{aligned} \mathbb{G}_{12}(z) &= \Delta(z) + \frac{691}{156}\left(\frac{E_4^3(z)}{720} + \frac{E_6^2}{1008}\right) \\ \mathbb{G}_{12} &= -\frac{B_{12}}{24} + \sum_{n\geq 1} \sigma_{11}(n) q^n \\ &= \frac{691}{65520} + \sum_{n\geq 1} \sigma_{11}(n) q^n \\ \mathbb{G}_{12}(z) &\equiv \Delta(z) \pmod{691} \end{aligned}$$

   To conclude
   $$\tau(n) = \sigma_{11}(n) \pmod{691}$$
   (Related to the fact that $691 \mid \#\mathcal{C}\updownarrow(\mathbb{Q}(\gamma_{691}))$)

5.2. Congruence subgroup

Proposition 5.2.1.
Let $N = p_1^{a_1} \cdots p_r^{a_r}$ be the prime factorisation. Then,

$$\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) = \prod_{i=1}^{r} \mathrm{SL}_2(\mathbb{Z}/p^{a_i}\mathbb{Z})$$

Lemma 5.2.2.

$$\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right)$$

Definition 5.2.3.
A subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is called congruence subgroups if $\Gamma(N) \subseteq \Gamma$ for some $N \geq 1$.

Lemma 5.2.4.
A congruence subgroup has finite index in $\mathrm{SL}_2(\mathbb{Z})$.

Remark 5.2.5.
There are non-congruence subgroups of finite index in $\mathrm{SL}_2(\mathbb{Z})$.

Properties:

1. $\mathrm{PSL}_2(\mathbb{Z})$ is generated freely by an element of order 2 and an element of order 3.
2. S_7 is generated by an element of order 2 and an element of order 3. There is a surjection

$$\mathrm{PSL}_2(\mathbb{Z}) \xrightarrow{\pi} S_7$$
$$\pi^{-1}(\mathrm{Stab}_1) \subseteq \mathrm{PSL}_2(\mathbb{Z})$$

3. $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ is a simple group for $p \geq 5$.

Remark 5.2.6.
Γ is the smallest index subgroup that is non-congruence.

Definition 5.2.7.
A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k and level Γ if

1. $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$
2. f is holomorphic at all cusps.

Cusps of $X(\Gamma)$ are just elements of $\Gamma \backslash \mathbb{P}^1(\mathbb{Q})$.

Proposition 5.2.8.

$$\mathrm{orbit}(\infty) = \{\frac{a}{c} : p \mid c, p \nmid a\}, \mathrm{orbit}(1) = \{\frac{a}{c} : p \nmid c\}$$

6. Lecture-6 (19th January, 2023): Congruence subgroups and enhanced elliptic curves

6.1. Congruence subgroups and modular forms of higher levels

Suppose p is a prime.

Proposition 6.1.1.

$$\# \left(\Gamma_0(p) \backslash^{\mathbb{P}^1(\mathbb{Q})} \right) = 2$$

Proof. □

Proposition 6.1.2.
Let Γ be a congruence subgroup, then $\Gamma \backslash^{\mathbb{P}^1(\mathbb{Q})}$ is finite. (called the cusps of level Γ).

Proof. □

Exercise 6.1.3.

$$\mathrm{SL}_2(\mathbb{Z}) = \bigsqcup_{j=0}^{p-1} \alpha_j \Gamma_0(p) \bigsqcup \alpha_\infty \Gamma_0(p)$$

where $\alpha_j = \begin{pmatrix} 1 & 0 \\ j & 1 \end{pmatrix}$ $0lej \leq p-1, \alpha_\infty = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

Notation: Let $f : \mathbb{H} \to \mathbb{C}$ be a function. Let $k \in \mathbb{Z}$ and $\gamma \in SL_2(\mathbb{R})$. We define

$$f|_{[\gamma]_k} : \mathbb{H} \to \mathbb{C}$$
defined by $f|_{[\gamma]_k}(z) = (cz+d)^{-k} f(\gamma \cdot z)$

With this notation: $f \in M_k(\mathrm{SL}_2(\mathbb{Z}))$ if

- f is holomorphic on $\mathbb{H}$ and at ∞.
- $f|_{[\gamma]_k}(z) = f(z) \ \forall \ \gamma \in \mathrm{SL}_2(\mathbb{Z})$.

Definition 6.1.4.
Let Γ be a congruence subgroup and $k \in \mathbb{Z}$. A modular form of weight k and level Γ is a function $f : \mathbb{H} \to \mathbb{C}$ such that

1. f is holomorphic on $\mathbb{H}$.
2. $f|_{[\gamma]_k} = f \;\forall\, \gamma \in \Gamma$.
3. f is holomorphic at all cusps.
   Note that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \Gamma$ for some $h \in \mathbb{Z}_{>0}$, and if $\Gamma(N) \subseteq \Gamma$ then $h \mid N$. Thus,
   $$f(z+h) = f(z)$$
   and f admits the Fourier expansion
   $$f(z) = \sum_{n\in\mathbb{Z}} a_n \exp(2\pi i n z/h)$$
   f is said to be holomorphic at ∞ if $a_n = 0 \;\forall\, n \leq -1$. Suppose $\alpha \in \mathbb{P}^1(\mathbb{Q})$ is a cusp, then there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma \cdot \alpha = \infty$. f is holomorphic at α if $f|_{[\gamma]_k}$ is holomorphic at ∞.
4. $f|_{[\gamma]_k}$ is holomorphic at $\infty \;\forall\, \gamma \in \mathrm{SL}_2(\mathbb{Z})$

Example 6.1.5.
$\Gamma_1(p)$ has cusps $0, \infty$. We just need to check

- f is holomorphic at ∞.
- $f|_{[\gamma]_k}$ is holomorphic at ∞.

Notation:

$$M_k(\Gamma) = \text{ the space of modular forms of weight } k \text{ and level } \Gamma$$

Definition 6.1.6.
$f \in M_k(\Gamma)$ is said to be a cusp form if f vanishes at all cusps of level Γ, i.e., $f|_{[\gamma]_k}$ vanishes at $\infty \;\forall\, \gamma \in \mathrm{SL}_2(\mathbb{Z})$.

By $S_k(\Gamma)$ we denote the space of all cusp forms of weight k and level Γ.

$M(\Gamma) = \oplus_{k\in\mathbb{Z}} M_k(\Gamma)$ is the graded ring of modular forms of level Γ.

If $\Gamma_1 \subseteq \Gamma_2$ are two congruence subgroups, then $M_k(\Gamma_2) \subseteq M_k(\Gamma_1)$. This implies $M_k(\mathrm{SL}_2(\mathbb{Z})) \subseteq M_k(\Gamma)$ for any Γ.

Now, let Γ be a congruence subgroup. Define:

$$Y(\Gamma) = \Gamma \backslash \mathbb{H}$$
$$X(\Gamma) = \Gamma \backslash (\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}))$$

These are called modular curves.

We saw that $\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$ parametrises elliptic curves over $\mathbb{C}$ (upto isomorphism).

6.2. Enhanced elliptic curves

Let $N \in \mathbb{Z}_{\geq 1}$

Definition 6.2.1.

1. An enhanced elliptic curve of level $\Gamma_0(N)$ is a pair (E, C) where E is an elliptic curve and C is an order N cyclic subgroup of $E(\mathbb{C})$. Morphism between (E, C) and (E', C') is a homomorphism

$$\varphi : E \to E'$$

such that $\varphi(C) = C'$

2. An enhanced elliptic curve of level $\Gamma_1(N)$ is a pair (E, Q) such that E is an elliptic curve and Q is an order N point on $E(\mathbb{C})$. Morphism between (E, Q) and (E', Q') is a homomorphism

$$\varphi : E \to E'$$

such that $\varphi(Q) = Q'$

3. An enhanced elliptic curve of level $\Gamma(N)$ is a triplet (E, Q_1, Q_2) such that E is an elliptic curve and Q_1, Q_2 are points of order N and

$$\langle Q_1, Q_2 \rangle = E(\mathbb{C})[N] = \{x \in E(\mathbb{C}) \mid Nx = 0\}$$

Proposition 6.2.2.

1. $Y(\Gamma_0(N))$ parametrizes enhanced elliptic curves of level $\Gamma_0(N)$. The map $z \in \mathbb{H} \mapsto (\mathbb{C}/\Lambda_z, (1 + \Lambda_z)/\Lambda_z)$ gives a bijection between

$$Y(\Gamma_0(N)) \leftrightarrow \{\text{ isomorphism classes of enhanced elliptic curves of level } \Gamma_0(N)\}$$

2. $Y(\Gamma_1(N))$ parametrizes enhanced elliptic curves of level $\Gamma_0(N)$. The map $z \in \mathbb{H} \mapsto \left(\mathbb{C}/\Lambda_z, \frac{1}{N}\right)$ gives a bijection between

$$(\Gamma_1(N)) \leftrightarrow \{\text{ isomorphism classes of enhanced elliptic curves of level } \Gamma_1(N)\}$$

3. $Y(\Gamma(N))$ parametrizes enhanced elliptic curves of level $\Gamma_0(N)$. The map

$z \in \mathbb{H} \mapsto \left(\mathbb{C}/\Lambda_z, \frac{1}{N}, \frac{1}{N} \cdot z\right)$ gives a bijection between

$Y(\Gamma(N)) \leftrightarrow \{$ isomorphism classes of enhanced elliptic curves of level $\Gamma(N)\}$

Proof. □

Proposition 6.2.3.
The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ is properly discontinuous, i.e., for $z_1, z_2 \in \mathbb{H}$ there exists neighbourhoods U_i of z_i such that if $\gamma \in U_1 \cap U_2 \neq 0$, then $\gamma \cdot z_1 = z_2$.

Proof. □

7. Lecture-7 (24th January, 2023): Riemann surfaces

Corollary 7.0.1.
$Y(\Gamma) = \Gamma \backslash \mathbb{H}$ with the quotient topology is Hausdorff.

Proof. □

Proposition 7.0.2.
$X(\Gamma)$ is compact.

Proof. □

7.1. Riemann surfaces

Definition 7.1.1.
A Riemann surface consists of the following data:

1. X is a topological space (Hausdorff + second countable).
2. (U_i, V_i, ϕ_i) with V_i open in X, U_i open ball in $\mathbb{C}$ and $\phi : U_i \to V_i$ a homeomorphisms such that whenever $V_i \cap V_j \neq \emptyset$ we have

$$\phi_j^{-1} \circ \phi_i : U_i \cap \phi_i^{-1}(V_i \cap V_j) \to U_j \cap \phi_j^{-1}(V_i \cap V_j)$$

to be homeomorphisms.

GOAL: To make $X(\Gamma)$ a Riemann surface. That is we want to construct charts on $X(\Gamma)$.

Definition 7.1.2.
A point $P \in Y(\Gamma)$ is called an elliptic point if for any lift z of P in $\mathbb{H}$, we have $\mathrm{Stab}_\Gamma(z)/\left(\mathrm{Stab}_\Gamma(z) \cap \{\pm I_2\}\right)$ is nontrivial. That is to say $\mathrm{Stab}_{\bar{\Gamma}}(z)$ is nontrivial, where $\bar{\Gamma}$ is the image of Γ in $\mathrm{PSL}_2(\mathbb{Z})$.

P is an elliptic point in $Y(\Gamma)$ only if it lifts to a point equivalent to $i = \exp(2\pi i/4)$ or $\omega = 2\pi i/6$. If P is an elliptic point, then $\mathrm{Stab}_{\bar{\Gamma}}(z)$ has order 2 or 3.

7.1.1. Local charts on $Y(\Gamma)$

1. $P \in Y(\Gamma)$ is not an elliptic point.

   Let $z \in \mathbb{H}$ be a lift of P and U_1, U_2 be neighbourhoods of z. Put $U = U_1 \cup U_2$. If $\gamma U \cap U \neq \emptyset$, then the image set of Y in $\bar{\Gamma}$ is identity.

   $$\pi : \mathbb{H} \to Y(\Gamma)$$

   Put $\gamma = \pi(U)$. Then, $\pi|_U : U \to V$ is a homeomorphism.

2. Let $P \in Y(\Gamma)$ be an elliptic point. Let $z \in \mathbb{H}$ such that $\pi(z) = P$. Same as previous case get U_1, U_2 and define U as the the union of the two.
   Now, if $\gamma U \cap U \neq \emptyset$ then $\gamma \in \mathrm{Stab}_{\bar{\Gamma}}(z)$.
   Set $V = \pi(U)$. Notice that here $\pi|_U : U \to V$ need not be a homeomorphism. We instead have

   $$\begin{tikzcd} U \arrow[r] \arrow[d, "\pi|_U"'] & U' = U/\mathrm{Stab}_{\bar{\Gamma}}(z) \arrow[dl, "\simeq"] \\ V & \end{tikzcd}$$

3. We next want to extend this to cusps of $X(\Gamma)$.
   For $\mathrm{SL}_2(\mathbb{Z})$ we have already seen a local chart $z \mapsto \exp(2\pi i z)$. In general, take any cusp P of $X(\Gamma)$. Take $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma P = \infty$. Use the local charts at ∞.

Hence, $X(\Gamma)$ is a compact Riemann surface.

Next, we wish to compute genus of $X(\Gamma)$.
Genus $\mathfrak{g}$ of $X(\Gamma)$ is an integer such that

$$H^1(X(\Gamma), \mathbb{Z}) \cong \mathbb{Z}^{2g}$$
$$H_1(X(\Gamma), \mathbb{Z}) \cong \mathbb{Z}^{2g}$$

8. Lecture-8 (2nd February, 2023):

9. Lecture-9 (7th February, 2023):

Corollary 9.0.1.
Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Then,

$$g(X(\Gamma)) = 1 + \frac{N}{12} - \frac{\epsilon_2}{4} - \frac{\epsilon_3}{4} - \frac{\epsilon_\infty}{2}$$

where $N = [\mathrm{PSL}_2(\mathbb{Z}) : \bar{\Gamma}]$, ϵ_2 is the number of elliptic points of order 2, ϵ_3 is the number of elliptic points of order 3 and ϵ_∞ is the number of cusps of $\Gamma = \#(\Gamma\backslash\mathbb{P}^1(\mathbb{Q}))$.

Proof. Consider the map

$$X(\Gamma) \xrightarrow{f} X(\mathrm{SL}_2(\mathbb{Z}))$$

with $\deg f = n$. Then,

$$2g(X(\Gamma)) - 2 = N(-2) + \sum_{Q\in X(\Gamma)} (e_Q - 1)$$

If $f(Q)$ is not equivalent to i or ω, Q is unramified.

If $f(Q)$ is equivalent to i, then we have two cases:

1. Q is elliptic implies Q is unramified.
2. Q is not elliptic implies $e_Q = 2$.

$$\sum_{Q\in f^{-1}([i])} e_Q = N, \sum_{Q\in f^{-1}([i])} (e_Q - 1) = \frac{N - \epsilon_2}{2}$$

If $f(Q) \sim \omega$, then

$$\sum_{Q\in f^{-1}([i])} (e_Q - 1) = \frac{2(N - \epsilon_3)}{3}$$

Finally, let us talk about ramification at the cusps. Recall from the charts at cusps that Local coordinate at a cusp Q is $e^{2\pi i/h}$ for some integer $h \geq 1$. In this case $e_Q = h$ and

$$\sum_{Q\in f^{-1}([i])} (e_Q - 1) = \left(\sum_{Q} e_Q\right) - \epsilon_\infty = N - \epsilon_\infty$$

Therefore,

$$2g(X(\Gamma)) = -2N + \frac{N-\epsilon_2}{2} + 2\frac{N-\epsilon_3}{3} + N - \epsilon_\infty$$
$$2g(X(\Gamma)) = \frac{N}{12} - \frac{\epsilon_2}{4} - \frac{\epsilon_3}{6} - \frac{\epsilon_\infty}{2} + 1$$

□

Exercise 9.0.2. $g(X(\Gamma_0(p))) = 0$ *iff* $p \in \{2, 3, 5, 6, 7, 13\}$

Goal of the day

$$\dim_{\mathbb{C}} M_k(\Gamma) = (k-1)(g(X(\Gamma)) - 1) + \frac{k}{2}\epsilon_\infty + \left[\frac{k}{4}\right]\epsilon_2 + \left[\frac{k}{3}\right]\epsilon_3$$

and

$$\dim_{\mathbb{C}} S_k(\Gamma) = (k-1)(g(X(\Gamma)) - 1) - \frac{k}{2}\epsilon_\infty + \left[\frac{k}{4}\right]\epsilon_2 + \left[\frac{k}{3}\right]\epsilon_3$$

To proceed we will need some algebraic geometry language.

$X -$ is a smooth projective curve over $\mathbb{C}$
$\mathcal{O}_X -$ structure sheaf
$\mathcal{O}_X(U) - \{f : U \to \mathbb{C}$ regular for open $U\}$
$\mathcal{F}(U) - \mathcal{O}_X$- module sheaves

We are concerned with invertible $\mathcal{O}_X$-module sheaves, i.e., locally free of rank 1. These are called invertible sheaves.

Example 9.0.3.
For a number field F, take $\mathcal{O}_F$ as the structure sheaf. Any nonzero fractional ideal is invertible hence we can think of that as the invertible sheaves.

Invertible sheaves form a group under $\otimes_{\mathcal{O}_X}$.

$H^0(X, \mathcal{F}) =$ global sections.

Remark 9.0.4.
As X is a smooth projective curve, $H^0(X, \mathcal{F}) = \mathbb{C}$. This is like Liouville's theorem.

We can take a look at meromorphic sections of $\mathcal{F}$

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathrm{Frac}(\mathcal{O}_X(U))$$

where $\mathrm{Frac}(\mathcal{O}_X(U))$ are the meromorphic functions.

Theorem 9.0.5 (Riemann existence theorem)**.**
An invertible sheaf on a compact Riemann surface has a non-constant global meromorphic section.

Definition 9.0.6.
Let $\mathcal{F}$ be an invertible sheaf on X. Take a non-constant global section s of $\mathcal{F}$. We define $\deg(\mathcal{F})$ to be the sum of orders of zeros of s.

Remark 9.0.7.
Note that this definition does not depend on the choice of s. If we take a different s', then $s = fs'$ where f is a global meromorphic section of $\mathcal{O}_X$ ($\deg f = 0 =$ sum of orders of zeros)

Proposition 9.0.8.
Properties of the degree function:

1. $\deg(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) = \deg \mathcal{F} + \deg \mathcal{G}$
2. $\deg \mathcal{F}^{-1} = -\deg \mathcal{F}$

Theorem 9.0.9 (Riemann-Roch theorem)**.**

1. $H^0(X, \mathcal{F})$ is a finite dimensional $\mathbb{C}$-vector space.
2. Put $h^0(X, \mathcal{F}) = \dim_{\mathbb{C}} H^0(X, \mathcal{F})$. Then,

$$h^0(X, \mathcal{F}) - h^0(X, \Omega_X^1 \otimes \mathcal{F}^{-1}) = 1 - g(X) + \deg \mathcal{F}$$

where $\Omega = \Omega_X^1$ is the sheaf of holomorphic differentials on X.

Remark 9.0.10 (Facts)**.**

1. If $\deg \mathcal{F} < 0$, then $H^0(X, \mathcal{F}) = 0$
2. If $\deg \mathcal{F} >> 0$, then $H^0(X, \Omega \otimes \mathcal{F}^{-1}) = 0$

Lemma 9.0.11.

1. $h^0(X, \Omega) = g(X)$
2. $\deg \Omega = 2g - 2$

Proof.

1. Take $\mathcal{F} = \mathcal{O}_X$. Then

$$\begin{aligned} h^0(X, \mathcal{O}_X) - h^0(X, \Omega \otimes_{\mathcal{O}_X} \mathcal{O}_X^{-1}) &= 1 - g(X) + \deg \mathcal{O}_X \\ 1 + h^0(X, \Omega) &= 1 - g(X) + 0 \\ g(X) &= h^0(X, \Omega) \end{aligned}$$

2. Take $\mathcal{F} = \Omega$. Then,

$$\begin{aligned} h^0(X,\Omega) - h^0(X,\Omega_X) &= 1 - g(X) + \deg \Omega \\ g(X) - 1 &= 1 - g + \deg \Omega \end{aligned}$$

□

Lemma 9.0.12.

Definition 9.0.13 (Katz sheaf)**.**
Let $X(\Gamma), k \in \mathbb{Z}$ be as usual. For V open in $X(\Gamma)$, we define a sheaf ω_k as

$$\omega_k = \{f : \pi^{-1}(V) \subseteq \mathbb{H} \to \mathbb{C}\}$$

with f is holomorphic and $f(\gamma \cdot z) = (cz+d)^k f(z) \;\forall\; \gamma \in \Gamma, \forall\; z \in \pi^{-1}(V)$ and $\pi : \mathbb{H}^* = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) \to X(\Gamma)$

Remark 9.0.14.
$H^0(X(\Gamma), \omega_k)$ gives modular forms of weight k and level Γ.

Theorem 9.0.15. 1. ω_k is an invertible sheaf.

2. $\omega_2 = \Omega$ (cusps).

Let $\mathcal{L}$ be an invertible sheaf, $D = \sum_i n_i P_i$ divisor of X (P_i is a point of X). If x is a meromorphic section of $\mathcal{L}$, then

$$\mathrm{div}(x) := \sum_{P \in X} \mathrm{ord}_P(x)$$

and

$$\mathcal{L}(D) = \{x, \text{ a meromorphic section of } \mathcal{L} : \mathrm{div}(x) + D \geq 0\}$$

Example 9.0.16.
If P is a point of X. Then,

$$\begin{aligned} \mathcal{L}(-P) &= \text{ meromorphic section } x \text{ of } \mathcal{L} \text{ with atleast a simple zero of } P \\ \mathcal{L}(P) &= \text{meromorphic section } x \text{ of } \mathcal{L} \text{ with atmost a simple pole at } P \end{aligned}$$

Definition 9.0.17.

$$\text{cusps} = \sum_{P \in \Gamma \backslash \mathbb{P}^1(\mathbb{Q})} P$$

If Γ is a congruence subgroup. Then,

$$\Gamma_\infty = \{g \in \Gamma : g \cdot \infty = \infty\}$$

has one of the following forms $\{\pm I_2\}, \left\langle \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right\rangle, \left\langle - \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right\rangle$

Definition 9.0.18.
1. ∞ is called irregular cusp for Γ if $\Gamma_\infty = \left\langle - \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \right\rangle$
2. A cusp s is called irregular if ∞ is irregular for $\alpha\Gamma\alpha^{-1}$ where $s = \alpha \cdot \infty (\alpha \in \mathrm{SL}_2(\mathbb{Z}))$

Definition 9.0.19.
Let r be the least common multiple of integers in the following set

$$\{\mathrm{Stab}_\Gamma(P), P \in \mathbb{H}\} \cup \{2 \text{ if there exists an irregular cusp}\}$$

Exercise 9.0.20. $1 \leq r \leq 12$

Definition 9.0.21.
Γ is called neat if $r = 1$.

Exercise 9.0.22.
1. *$\Gamma_0(N)$ is neat for $N \geq 5$.*
2. *If Γ is neat, then it has no elliptic points and $-I \notin \Gamma$*

Theorem 9.0.23.
If Γ is neat, then for any $k \geq 0$, we have

$$\dim_{\mathbb{C}} M_k(\Gamma) = (k-1)(g(X(\Gamma)) - 1) + \frac{k}{2}\epsilon_\infty$$

Remark 9.0.24 (Fact)**.**
For an integer r as above, we have $\omega_{k+r} = \omega_k \otimes \omega_r$

In particular, if $r = 1$, then $\omega_r = \omega_1^{\otimes r} = \omega^{\otimes r}$

Proof of theorem.

$$\begin{aligned}\deg\omega &= \frac{1}{2}\deg\omega^{\otimes 2}\\ &= \frac{1}{2}\deg(\Omega(\text{cusps}))\\ &= \frac{1}{2}(\deg\Omega + \epsilon_\infty)\\ &= g - 1 + \frac{\epsilon_\infty}{2}\end{aligned}$$

$\therefore$

$$\begin{aligned}h^0(X(\Gamma), \omega_k) &= k(g - 1 + \frac{1}{2}\epsilon_\infty) - g + 1\\ &= (k-1)(g-1) + \frac{k}{2}\epsilon_\infty\end{aligned}$$

$\square$

Part II.

Elliptic Curves

10. Lecture-1 (3rd January): Introduction

11. Lecture-2 (5th January, 2023): Affine varieties

11.1. Affine Varieties

Suppose k is a perfect field (every extension is separable).
Let $G(\bar{k}/k)$ be the Galois group of the extension. It can also be viewed as $\varinjlim_{L/K \text{Galois, } L \text{ finite}} \mathrm{Gal}(L/K)$.

12. Lecture-3 (10 January, 2023): Projective varieties

12.1. Projective varieties

Definition 12.1.1.
A Projective n-space over k denoted by $\mathbb{P}^n$ or $\mathbb{P}^n(\bar{K})$ is the set $\mathbb{A}^{n+1} \backslash \{(0, \ldots, 0)\} / \sim$ with

$$(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$$

iff $\exists \lambda \in \bar{k}^\times$ such that $(y_0, \ldots, y_n) = (\lambda x_0, \ldots, \lambda x_n)$
The equivalence class $(x_0, \ldots, x_{n+1})$ is denoted by $[x_0, \ldots, x_n]$
The set of k-rational points of $\mathbb{P}^n$ is

$$\mathbb{P}^n = \{[x_0, \ldots, x_n] \mid x_i \in k\}$$

Caution: If $p = [x_0, \ldots, x_n] \in \mathbb{P}^n(k)$ and $x_i \neq 0$ for some i, then $x_j / x_i \in k \forall j$

Definition 12.1.2.
Let $p = [x_0, \ldots, x_n] \in \mathbb{P}^n(\bar{k})$. The minimal field of definition for p is the field

$$k(p) = k(x_0/x_i, \ldots, x_n/x_i) \text{ for any } i \text{ such that } x_i \neq 0$$

$k(p)\frac{x_i}{x_j} = k(x_0/x_j, \ldots, x_n/x_j)$ is the same as $k(p)$ as $x_i/x_j \in k(p)$

For $\sigma \in G(\bar{k}/k)$ and $p = [x_0, \ldots, x_n] \in \mathbb{P}^n$, we have the following action

$$\sigma(p) = [\sigma(x_0), \ldots, \sigma(x_n)]$$

This action is well defined as

$$\sigma(\lambda p) = [\sigma(\lambda)\sigma(x_0), \ldots, \sigma(\lambda)\sigma(x_n)] \sim [\sigma(x_0), \ldots, \sigma(x_n)]$$

Definition 12.1.3.
A polynomial $f \in \bar{k}[X_0, \ldots, X_n]$ is homogenous of degree d if

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n) \forall \lambda \in \bar{k}$$

Definition 12.1.4.
An ideal $I \subseteq \bar{k}[X_0, \dots, X_n]$ is called a homogenous ideal if it is generated by homogenous polynomial.

Definition 12.1.5.
Let $I \subseteq \bar{k}[X_0, \dots, X_n]$ be a homogenous ideal. Then,

$$V(I) = \{p \in \mathbb{P}^n(\bar{k}) \mid f(p) = 0 \;\forall\; f \in I\}$$

Definition 12.1.6.

- A projective algebraic set is any set of the form $V(I)$ for some homogenous ideal I.
- If V is a projective algebraic set, the homogenous ideal of V, denoted by $I(V)$ is the ideal of $\bar{k}[X_0 \dots, X_n]$ generated by $\{f \in \bar{k}[X_0 \dots, X_n] \mid f \text{ is homogenous and } f(p) = 0 \;\forall\; p \in V\}$
- Such a V is defined over k, denoted by V/k if its ideal $I(V)$ can be generated by homogenous polynomials $k[X_0 \dots, X_n]$.
- If V is defined over k, then the set of k-rational points of V is

$$V(k) = V \cap \mathbb{P}^n(k) = \{p \in V \mid \sigma(p) = p \;\forall\; \sigma \in G(\bar{k}/k)\}$$

Example 12.1.7.
A line in $\mathbb{P}^2$ is given by the equation $aX + bY + cZ = 0$ with $a, b, c \in \bar{k}$ and not all 0 simultaneously.
If $c \neq 0$, then such a line is defined over a field containing $a/c, b/c$.
More generally, a hyperplane in $\mathbb{P}^n$ is given by an equation $a_0X_0 + \cdots + a_nX_n = 0$ with all $a_i \neq 0$ simultaneously.

Example 12.1.8.
Let V be the projective algebraic set in $\mathbb{P}^2$ given by $X^2 + Y^2 = Z^2$.

$$\mathbb{P}^1 \overset{\sim}{\overbrace{\rightarrow}} V$$
$$[s, t] \mapsto [s^2 - t^2 : 2st : s^2 + t^2]$$

Remark 12.1.9.
For $p \in \mathbb{P}^n(\mathbb{Q})$ you can clear the denominators and then divide by common factor so that $x_i \in \mathbb{Z}$ and $\gcd(x_0, \dots, x_n) = 1$. So, $I = (f_1, \dots, f_m)$ and finding a rational point of V_I is same as finding coprime integer solutions to $f_i's$.

Example 12.1.10.
$V \subseteq \mathbb{P}^2$ such that $X^2 + Y^2 = 3Z^2$ over $\mathbb{Q}$. To find $V(\mathbb{Q})$, we just need to find integers a, b, c such that $a^2 + b^2 = 3c^2$

Example 12.1.11.
$V : 3X^3 + 4Y^3 + 5Z^3 = 0$. $V(\mathbb{Q}) = \emptyset$ but for all prime p we have $V(\mathbb{Q}_p) \neq \emptyset$

Definition 12.1.12.
A projective algebraic set is called a projective variety if its homogenous ideal $I(V)$ is prime $\bar{k}[X_0, \ldots, X_n]$

Relation between affine and projective varieties:

For $0 \leq i \leq n$

$$\begin{aligned} \phi_i : \mathbb{A}^n &\to \mathbb{P}^n \\ (Y_1, \ldots, Y_n) &\mapsto [Y_1, \ldots, Y_{i-1}, 1, Y_{i+1}, \ldots, Y_n] \end{aligned}$$

$\mathrm{Im}(\phi) = U_i = \{p \in \mathbb{P}^n \mid \ p = [x_0 : \ldots : x_n]$ with $x_i \neq 0\} = \mathbb{P}^n \backslash H_i$.
This process can also be reversed by the following map :

$$\begin{aligned} \phi_i^{-1} : U_i &\to \mathbb{A}^n \\ [x_0 : \ldots : x_n) &\mapsto [x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i] \end{aligned}$$

Let V be a projective algebraic set with homogenous ideal $I(V) \subseteq \bar{k}[X_0, \ldots, X_n]$. Then,

$$V \cap \mathbb{A}^n = \phi_i^{-1}(V \cap U_i) \text{ for fixed } i$$

is an affine algebraic set with $I(V \cap \mathbb{A}^n) \subset \bar{k}[X_0, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]$

Definition 12.1.13.
Let $V \subseteq \mathbb{A}^n$ be an affine algebraic set with ideal $I(V)$ and consider $V \subseteq \mathbb{P}^n$ and ϕ_i defined as before.
The projective closure of V is $\bar{V}$ is the projective algebraic set whose homogenous ideal $I(V)$ is generated by $\{f^* \mid f \in I(V)\}$.
Here, for $f \in k[X_0, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]$ we define

$$f^*(X_0, \ldots, X_n) = X_i^d(f(X_0/X_i, \ldots, X_{i-1}/X_i, X_{i+1}/X_i, \ldots, X_n/X_i))$$

with $d = \deg(f)$.

Definition 12.1.14.
Dehomogenization of $f(X_0, \ldots, X_n)$ with respect to i is $f(X_0, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n)$

Proposition 12.1.15.

1. Let V be an affine variety. Then $\bar{V}$ is a projective variety and $V = \bar{V} \cap \mathbb{A}^n$.
2. Let V be a projective variety. Then, $V \cap \mathbb{A}^n$ is an affine variety and either $V \cap \mathbb{A}^n = \emptyset$ or $V = \overline{V \cap \mathbb{A}^n}$.
3. If an affine (resp. projective) variety V is defined over k, then $\bar{V}$ (resp. $V \cap \mathbb{A}^n$) is also defined over k.

Proof.

1.
2.
3.

□

Example 12.1.16.
$V : Y^2 = X^3 + 17 \subseteq \mathbb{A}^2 \to \mathbb{P}^2$ with $(X, Y) \mapsto [X : Y : 1]$. Here, $\overline{V} : Y^2Z = X^3 + 17Z^3$ and $\overline{V} \backslash V = \{[0 : 1 : 0]\}$

13. Lecture-4 (12th January, 2023): Projective varieties and maps between varieties

13.1. Projective varieties contd..

Definition 13.1.1.
- Let Y/k be a projective variety and choose $\mathbb{A}^n \subseteq \mathbb{P}^n$ such that $V \cap \mathbb{A}^n \neq \emptyset$. The dimension of V is just dimension of $V \cap \mathbb{A}^n$.
- The function field of V, $\bar{k}(V) = \bar{k}(V \cap \mathbb{A}^n)$ is the function field for $V \cap \mathbb{A}^n$ over $\bar{k}$.
- Similarly, $k(V) = k(V \cap \mathbb{A}^n)$

$$\phi_i : \mathbb{A}^n \to \mathbb{P}^n \mathcal{I}(V \cap \mathbb{A}_i^n)$$
$$\phi_j : \mathbb{A}^n \to \mathbb{P}^n \mathcal{I}(V \cap \mathbb{A}_j^n)$$

For different ϕ_i we obtain $k(V)$s but they are canonically isomorphic to each other. This is because we can just switch x_i, x_j are dehomogenise accordingly.

Definition 13.1.2.
Let V be a projective variety and $p \in V$. Choose $\mathbb{A}^n \subseteq \mathbb{P}^n$ with $p \in \mathbb{A}^n$. Then, V is non-singular (or smooth) at p if $V \cap \mathbb{A}^n$ is non-singular at p.

The local ring of V at p, $\bar{k}[V]_p$ is just the local ring of $\bar{k}[V \cap \mathbb{A}^n]_p$

Remark 13.1.3.
Function field of a projective variety V is field of rational functions $f(X)/g(X)$ such that

1. f, g are homogenous of same degree.
2. $g \in \mathcal{I}(V)$.
3. $f_1/g_1 = f_2/g_2$ iff $f_1 g_2 - f_2 g_1 \in \mathcal{I}(V)$

Equivalently, take $f, g \in \bar{k}[X]/I(V)$ satisfying $1, 2$.

Here, X is just a short form for $(X_0, \dots, X_n)$

13.2. Maps between varieties

Definition 13.2.1.
Let $V_1, V_2 \in \mathbb{P}^n$ be projective varieties. A rational map

$$\phi : V_1 \to V_2$$

$\phi = [f_0 : \dots : f_n]$ where $f_i \in \bar{k}(V_1)$ such that $\forall p \in V_1$ at which f_i are defined, we have

$$\phi(p) = [f_0(p) : \dots : f_n(p)]$$

If V_1, V_2 are defined over k, we have a Galois action. For $\sigma \in G(\bar{k}/k)$ we have

$$\sigma(\phi)(p) = [\sigma(f_0) : \dots : \sigma(f_n)(p)]$$

We can check that $\sigma(\phi(p)) = \sigma(\phi)(\sigma(p))$.

Definition 13.2.2.
If $\exists \lambda \in \bar{k}^\times$ such that $\lambda f_i \in k(V_1)$, then ϕ is said to be defined over k.

Proposition 13.2.3.
ϕ is defined over k iff $\phi = \sigma(\phi)\ \forall\ \sigma \in G(\bar{k}/k)$.

Definition 13.2.4.
A rational map $\phi : V_1 \to V_2$ is said to be regular if there exists a function $g \in \bar{k}(V_1)$ such that

1. Each gf_i is regular at p.
2. There exists some i such that $(gf_i)(p) \neq 0$

If such a g exists, then we set

$$\phi(p) = [(gf_0)(p) : \dots : (gf_n)(p)]$$

Definition 13.2.5.
A rational map is called a morphism if it is regular everywhere.

Remark 13.2.6.
Let $V_1, V_2 \in \mathbb{P}^n$ be projective varieties.
$\bar{k}(V_1) =$ quotient of homogenous polynomials in $\bar{k}[X]$ of same degree.
A rational map $\phi = [f_0, \ldots, f_n]$ can be multiplied by a homogenous polynomial to clear denominators and get $\phi = [\phi_0, \ldots, \phi_n]$ such that

1. $\phi_i \in \bar{k}[X]$ homogenous polynomials not all in $\mathcal{I}(V_1)$ and have same degree.
2. For all $f \in \mathcal{I}(V_2)$ we have $f(\phi_0(X), \ldots, \phi_n(X)) \in \mathcal{I}(V_1)$.

Definition 13.2.7.
A rational map $\phi = [\phi_0, \ldots, \phi_n] : V_1 \to V_2$ as above is regular at $p \in V_1$ if there exists homogenous polynomials $\psi_0, \ldots, \psi_n \in \bar{k}[X]$ such that

1. ψ_is have the same degree
2. $\phi_i \psi_j \equiv \phi_j \psi_j \pmod{\mathcal{I}(V_1)}$ for all $0 \leq i, j \leq n$
3. $\psi_i(p) \neq 0$ for some i.

If this happens, we set
$$\phi(p) = [\psi_0(p), \ldots, \psi_n(p)]$$

Remark 13.2.8.
Let $\phi = [\phi_0, \ldots, \phi_n] : \mathbb{P}^m \to \mathbb{P}^n$ be a rational map. ϕ_is are homogenous polynomials having same degree. We can cancel common factors to assume $\gcd(\phi_0, \ldots, \phi_n) = 1$.

And, ϕ is regular at a point $p \in \mathbb{P}^n$ iff $\phi_i(p) \neq 0$ for some i.
So, ϕ is a morphism if ϕ_is have no common zeros in $\mathbb{P}^n$.

Definition 13.2.9.
Let V_1, V_2 be two projective varieties. We say that V_1, V_2 are isomorphic if there are morphisms
$$\phi : V_1 \to V_2, \psi : V_2 \to V_1$$
such that $\phi \circ \psi = \mathrm{id}_{V_2}, \psi \circ \phi = \mathrm{id}_{V_1}$.
V_1/k and V_2/k are isomorphic over k if both maps are defined over k.

Example 13.2.10.
$\mathrm{char}(k) \neq 2$, $V : X^2 + Y^2 = Z^2$.

$$\begin{aligned} \phi : V &\to \mathbb{P}^2 \\ [X : Y : Z] &\mapsto [X + Z : Y] \end{aligned}$$

ϕ is regular everywhere except $[1:0:1]$
Since $(X+Z)(X-Z) \equiv -Y^2 \equiv \pmod{\mathcal{I}(V)}$, we have $[X+Z:Y] = [X^2-Z^2 : Y(X-Z)] = [-Y^2 : Y(X-Z)] = [-Y : X-Z] = \psi$

$$\begin{aligned} \psi &: \mathbb{P}^1 \to V \\ [s:t] &\to [s^2-t^2 : 2st : s^2+t^2] \end{aligned}$$

$\psi \circ \phi$ and $\phi \circ \psi$ are both identity maps.

Example 13.2.11.

$$\begin{aligned} \phi &: \mathbb{P}^2 \to \mathbb{P}^2 \\ [X:Y:Z] &\mapsto [X^2 : YZ : Z^2] \end{aligned}$$

is regular everywhere but $[0:1:0]$ and this cannot be salvaged.

Example 13.2.12.
$V : Y^2Z = X^3 + X^2Z$

$$\begin{aligned} \psi &: \mathbb{P}^1 \to V \\ [s:t] &\mapsto [(s^2-t^2)t : (s^2-t^2)s : t^3] \phi : V && \to \mathbb{P}^1 \\ [X:Y:Z] &\mapsto [X:Y] \end{aligned}$$

ϕ is not regular at $[0:0:1]$. $[0:0:1]$ is a singular point of V which implies ϕ cannot be extended. So $\phi \circ \psi$ and $\psi \circ \phi$ are identities when they are defined.

Example 13.2.13.
$V_1 : X^2 + Y^2 = Z^2, V_2 : X^2 + Y^2 = 3Z^2$. $V_1 \ncong V_2$ over $\mathbb{Q}$ but $V_1 \cong V_2$ over $\mathbb{Q}(\sqrt{3})$.

14. Lecture-5 (17th January, 2023): Algebraic curves

14.1. Curves

Definition 14.1.1.
A curve is a projective variety of dimension 1.

Example 14.1.2.
Vanishing set of an irreducible polynomial in $\mathbb{P}^2$.

Proposition 14.1.3.
Let C be a curve and $p \in C$ be a smooth (non-singular) point. Then, $\bar{k}[C]_p$ is a discrete valuation ring.

Proof. $p \in C$ smooth implies M_p/M_p^2 is one dimensional over $\bar{k}[C]_p/M_p = \bar{k}$. Now, Nakayama will give us M_p is a principal ideal.

Claim: $\bigcap_n M_p^n = 0$.

Proof. If $\alpha \in \bigcap_n M_p^n$, then $\alpha = a_1 t = a_2 t^2 = a_3 t^3 = \cdots$. This implies $a_1 = a_2 t = a_3 t^2 = \cdots$. But this gives us a chain

$$\langle a_1 \rangle \subseteq \langle a_2 \rangle \subseteq \langle a_3 \rangle \subseteq \cdots$$

that must terminate at some point. This implies t is an unit which is a contradiction. Hence, we are done. □

□

Definition 14.1.4.
Let C be a curve and $p \in C$ a smooth point. The normalised valuation on $\bar{k}[C]_p$ is

$$\begin{aligned} \mathrm{ord}_p : \bar{k}[C]_p &\to \mathbb{N} \cup \{0, \infty\} \\ f &\mapsto \sup\{d \in \mathbb{Z} \mid f \in M_p^d\} \\ \mathrm{ord}_p(\frac{f}{g}) &= \mathrm{ord}_p(f) - \mathrm{ord}_p(g) \end{aligned}$$

Thus we can define

$$\mathrm{ord}_p : \bar{k}[C] \to \mathbb{Z} \cup \{\infty\}$$

Definition 14.1.5.
An uniformiser for C at p is any function $t \in \bar{k}(C)$ with $\mathrm{ord}_p(t) = 1$ that is the generator of M_p

Remark 14.1.6.
If C is defined over k, we can find a unit $t \in k(C)$.

Definition 14.1.7.
Let C be a curve and $p \in C$ a smooth point, $f \in \bar{k}(C)$, $\mathrm{ord}_p(f) =$ order of f at p.

1. If $\mathrm{ord}_p(f) > 0$, then f has a zero at p.
2. If $\mathrm{ord}_p(f) < 0$, then f has a pole at p.
3. If $\mathrm{ord}_p(f) \geq 0$, then f is regular at p.

Proposition 14.1.8.
Let C be a smooth curve and $0 \neq f \in \bar{k}(C)$. Then, there are only finitely many points in C at which f has a pole or 0. If f has no poles, then $f \in \bar{k}$.

Proof. A standard exercise in Riemann surfaces. □

Example 14.1.9.
Suppose $C_1 : Y^2 = X^3 + X, C_2 : Y^2 = X^3 + X^2$. C_1 is smooth everywhere but C_2 is smooth everywhere except $p = [0 : 0 : 1]$.

In $\bar{k}[C_1]_p$, $M_p = \langle X, Y \rangle$ and $X \in M_p^2$.

Proposition 14.1.10.
Let C/k be a curve and $p \in C$ be a smooth point, and $t \in k(C)$ an uniformiser at p. Then, $k(C)$ is a finite separable extension of $k(t)$.

Proof. $k(C)$ is a finite algebraic extension as it is finitely generated over k and has transcendence degree 0 over $k(t)$ as t is not algebraic over k (it is a local coordinate of C at p).

Now, take $x \in k(C)$ and let $\Phi(T,X) = \sum a_{ij}T^iX^j$ be the minimal polynomial at x over $k(t)$. Say $q = \mathrm{char}(k)$. If $\Phi(T,X)$ is not separable, then $\frac{\partial\Phi(T,X)}{\partial X} = 0$ as $\Phi(T,X)$ is irreducible.

$$\begin{aligned}
\Phi(T,X) &= \Psi(T,X^p)\\
&= \sum_{k=0}^{q-1}\left(\sum_{i,j} b_{ijk}T^{iq}X^{iq}\right)T^k\\
&= \sum_{k=0}^{q-1}\left(\Phi_k(T,X)\right)^q T^k \text{ since } k \text{ is perfect, every element is a } q\text{-th power}
\end{aligned}$$

$$\sum_{k=0}^{q-1}\left(\Phi_k(t,x)\right)^q t^k = 0$$

$$\mathrm{ord}_p(\Phi_k(t,x)^q t^k) \equiv k \pmod q$$

This implies that every term in the final sum has distinct order at p. And, hence

$$\Phi_0(t,x) = \Phi_1(t,x) = \cdots = \Phi_{q-1}(t,x) = 0$$

Atleast one of the Φ_is should have a nonzero power of X and $X-\deg\Phi_i < X-\deg\Phi$ and hence $\Phi_k(t,x) = 0$ which contradicts minimality of Φ. Hence, we are done. ☐

14.2. Morphism between curves

Proposition 14.2.1.
Let C be a curve, $V \subseteq \mathbb{P}^n$ be a variety, $p \in C$ a smooth point and

$$\phi : C \to V$$

a rational map. Then, ϕ is regular at p. In particular, if C is smooth, then ϕ is a morphism.

Proof. Suppose $\phi = [f_0 : \cdots : f_n]$ with $f_i\bar{k}(C)$ and $t \in \bar{k}(C)$ an uniformiser for C at p. Let

$$n = \min \mathrm{ord}_p f_i$$

Then, $\mathrm{ord}_p(t^{-n}f_i) \geq 0 \;\forall\, i$ and $\mathrm{ord}_p(t^{-n}f_j) = 0$ for some j. But then this means $t^{-n}f_i$ are regular at p, $t^{-n}f_j(p) \neq 0$ and thus ϕ is regular at p. ☐

Remark 14.2.2.
This proposition is not true if either $\dim(C) > 1$ or p is singular

1. $\phi : \mathbb{P}^n \to \mathbb{P}^n$ be $[X : Y : Z] \mapsto [X^2 : YZ : Z^2]$ is not regular at $p = [0 : 1 : 0]$.

2. Suppose $V : Y^2Z = X^3 + X^2Z$ and $V \to \mathbb{P}^1$ be given by $[X : Y : Z] \mapsto [Y : X]$ is not regular at $[0 : 0 : 1]$.

Example 14.2.3. 1. $V : X^2 + Y^2 = Z^2$

15. Lecture-6 (19th January, 2023):

16. Lecture-7 (24th January, 2023):

17. Lecture-8 (31st January, 2023): Group Law

17.1. Group Law and definition of Elliptic Curve

Proposition 17.1.1.
If E is a curve given by the Weierstrass cubic $f(x, y, z) = 0$ and L be a line. Then, number of points of $L \cap E$ counted with multiplicity is 3.

Proof. Suppose $P \in L \cap E$. Multiplicity of intersection of $L \cap E$ at P is given by $\dim_{\bar{k}} K[E]_P/L = \dim_{\bar{k}} \bar{k}[X, Y]_{\mathfrak{m}_P}/\langle \tilde{L}, \tilde{f}\rangle$ where $\tilde{}$ is the dehomogenisation depending on the affine chart.

Example 17.1.2.
$P = (0, 0), L : x = 0$. Multiplicity of $L \cap E$ at P equals $\dim_{\bar{k}}[X, Y]_{\mathfrak{m}_P}/\langle Y^2 - X^3 + X, X\rangle \simeq \dim_{\bar{k}} \bar{k}[Y]/Y^2 = 2$.

If $L' : Y - X = 0$, then $\dim_{\bar{k}}[X, Y]_{\mathfrak{m}_P}/\langle Y^2 - X^3 + X - Y, X\rangle \simeq \dim_{\bar{k}} \bar{k}[Y]/\langle Y(Y - Y^2 - 1) = \dim_{\bar{k}} \bar{k} = 1$ since $Y - Y^2 - 1$ is an unit.

All of this is a special case of Bezout's theorem which is stated after this proof.

Back to the proof.

Suppose $L : aX + bY + cZ = 0$.
Case-1: $b \neq 0$ so $O = [0 : 1 : 0] \notin L$. Dehomogenize with respect to Y to get $aX + bY + c = 0$. For $P \in L \cap E$,

$$\frac{\bar{k}[E]_P}{\langle L, f\rangle} = \frac{[X, \bar{Y}]_{\mathfrak{m}_P}}{\langle f, aX + bY + c\rangle} = \frac{\bar{k}[X, Y]}{\langle g(X)\rangle}(X - X(P))$$

where $g(X)$ is obtained by substituting $Y = -(aX+c)/b$ to f. Therefore $\dim_{\bar{k}} \bar{k}[E]_P/\langle L, f\rangle =$ multiplicity of $X(P)$ as a root of $g(X)$. This implies the number of points of $L \cap E$ with multiplicity is the number of roots of $g(X)$ with multiplicity which is 3.

Case-2: $b = 0$

1. Suppose $a \neq 0 \Rightarrow L : X - cZ = 0$.
   Multiplicity of $L \cap E$ at O is $\dim_{\bar{k}} \bar{k}[X, Y]_{\mathfrak{m}_O}/\langle \tilde{f}, X - cZ\rangle \simeq \dim_{\bar{k}} \bar{k} = 1$.
   For $P \in \mathbb{A}^2 \cap E$, multiplicity of $L \cap E$ at P is $\dim_{\bar{k}} \bar{k}[X, Y]_{\mathfrak{m}_P}\langle f, X - cZ\rangle =$

$\dim_{\bar{k}}(\bar{k}[Y]/h)(Y - Y(P))$ with h obtained from f by substituting $X = c$. Therefore, the total multiplicity is $1 + \#$ roots with multiplicity of $h = 1 + 2 = 3$

2. $L : z = 0$, then $L \cap E = \{0\}$. In the Weierstrass equation, after homogenization, this will mean that just X^3 survives. Therefore, $\dim_{\bar{k}} \bar{k}[X, Z]_{\mathfrak{m}_P}/\langle \tilde{f}, Z\rangle = \dim_{\bar{k}} \bar{k}[X]/\langle X^3\rangle = 3$

This concludes the proof. □

Theorem 17.1.3.
IF F, G are coprime homogenous polynomials in $k[X, Y, Z]$ and $V = \mathcal{V}(F), W = \mathcal{V}(G) \subseteq \mathbb{P}^2$. Then the number of points of $V \cap W$ with multiplicity equals mn where $m = \deg F, n = \deg G$.

17.1.1. Composition Law of E

Suppose $P, Q \in E$ and L be the line passing through P, Q (if $P = Q$, then L is the tangent line at P) and let R be the third point in the intersection of L with E. Let L' be the line joining R and O the point at infinity. L' intersects E at R, O and a third point. We denote this point by $P \oplus Q$

Proposition 17.1.4.
The composition law above makes E into an abelian group, i.e.,

1. If L intersects E in P, Q, R, then $(P \oplus Q) \oplus R = 0$
2. $P \oplus O = P \;\forall\; P$
3. $P \oplus Q = Q \oplus P \;\forall\; P, Q$
4. If $P \in E$, then there exists $-P \in E$ such that $P \oplus (-P) = 0$
5. If $P, Q, R \in E$, then $(P \oplus Q) \oplus R = P \oplus (Q \oplus R)$
6. If E is defined over k, then

$$E(k) = \{(x, y) \in k^2 : f(x, y) = 0\} \cup \{0\}$$

is a subgroup of E.

Proof. □

Notation: Suppose $P \in E$ $[m]P = \begin{cases} 0 & \\ P \oplus \cdots \oplus P & , m > 0 \\ -P \oplus \cdots \oplus -P & , m < 0 \end{cases}$

Now, we can explicitly compute the coordinates of the points. Suppose $P = (x_0, y_0) \in E$. Line passing through P and O: $X - x_0 Z$. Dehomogenize with respect to Z to get $X - x_0 = 0$

$$
\begin{aligned}
&f(X,Y) = Y^2 + a_1XY + a_3Y - (X^3 + a_2X^2 + a_4X - a_6)\\
&\Rightarrow X(-P) = x_0\\
&\Rightarrow Y(-P) \text{ is another root of } f(Y, x_0)\\
&\Rightarrow Y(-P) = -y_0 - a_1x_0 - a_3\\
&\quad \Rightarrow -P = (x_0, -y_0 - a_1x_0 - a_3)
\end{aligned}
$$

Suppose $P_i = (x_i, y_i), i = 1, 2$.

We want to find $P_1 \oplus P_2$'s coordinates.

If $x_1 = x_2$ and $y_1 + y_2 + ax_2 + a_3 = 0$, then $P_1 \oplus P_2 = 0$. Assume that this is not the case. This means that the line passing through P_1 and P_2 does not go through O and thus $L : Y = \lambda x + \nu$. By high school methods,

If $x_1 \neq x_2$, we get

$$
\begin{aligned}
\lambda &= \frac{y_1 - y_2}{x_1 - x_2}\\
\nu &= \frac{y_1x_2 - y_2x_1}{x_2 - x_1}
\end{aligned}
$$

If $x_1 = x_2$, we have

$$
\begin{aligned}
\lambda &= \frac{3x_1^2 + 2a_2x_1 + a_4 - a_1y_1}{2y_1 + a_1x_1 + a_3}\\
\nu &= \frac{-x_1^3 + a_4x_1 + 2a_6 - a_3y_1}{2y_1 + a_1x_1 + a_3}
\end{aligned}
$$

This implies $f(x, \lambda x + \nu)$ has 3 roots say x_1, x_2, x_3. Let $P_3 = (x_3, y_3 = \lambda x_3 + \nu)$

$$
\begin{aligned}
\therefore f(X, \lambda X + \nu - (X - x_1)(X - x_2)(X - x_3)) &= 0\\
\Rightarrow x_1 + x_2 + x_3 &= \lambda^2 + a_1\lambda - a_2\\
\Rightarrow X(P_1 \oplus P_2) &= \lambda^2 + a_1\lambda - a_2 - x_1 - x_2\\
Y(P_1 \oplus P_2) &= -(\lambda + a_1)x_3 - \nu - a_3
\end{aligned}
$$

Duplication formula: $X([2]P) = \dfrac{x^4 - b_4x^2 - 2b_6x - b_8}{4x^3 + b_2x^2 + 2b_4x + b_6}$

Example 17.1.5.
$E : Y^2 = X^3 + 17$. Some points on the curve are $P_1 = (-2, 3), P_2 = (-1, 4), P_3 = (2, 5), P_4 = (4, 9), P_5 = (8, 23), P_6 = (43, 282), P_7 = (52, 375), P_8 = (5234, 378661)$
$\therefore P_5 = [-2]P_1, P_4 = P_1 - P_3, P_7 = [3]P_1 - P_3.[2]P_1 = \left(\frac{137}{64}, \frac{-2651}{512}\right), P_2 + P_3 = \left(\frac{-8}{9}, \frac{-169}{27}\right)$

The point being that $E(\mathbb{Q}) = \mathbb{Z}P_1 \oplus \mathbb{Z}P_3$ (Mordell-Weil)
$E(\mathbb{Z}) = \{\pm P_1, \ldots, \pm P_8\}$ (Siegel's theorem)

Corollary 17.1.6.
Suppose $f \in \bar{k}(E) = \bar{k}(X, Y)$. Then

$$f \text{ is even } \Leftrightarrow f \in \bar{k}(X)$$

Proof. □

18. Lecture-9 (2nd February, 2023): Group Law and more algebraic geometry

18.1. Algebraic geometry

18.1.1. Divisors

18.1.2. Differentials

Part III.

Basic Algebraic Geometry

19. Lecture-1 (5th January): Introduction

20. Lecture-2 (10 January, 2023): Ideals and Zariski topology

20.1. Ideals

For I, J ideals

$$I + J = \{x + y \mid x \in I, y \in J\}$$

$$IJ = \{\sum x_i y_i \mid x_i \in I, y_i \in J\}$$

- $IJ \subseteq I \cap J$.
- If $I + J = R$, then $I^2 + J^2 = R$. This is because, say $I^2 + J^2 \neq R$, then there is a maximal ideal m such that $I^2 + J^2 \subseteq \mathfrak{m}$. This means $I^2, J^2 \subseteq \mathfrak{m}$. But $\mathfrak{m}$ is prime ideal, therefore $I, J \subseteq \mathfrak{m} \Rightarrow I + J \subseteq \mathfrak{m}$ which is a contradiction. Thus, we are done.
- If $\mathfrak{p}$ is a prime ideal and $IJ \subseteq \mathfrak{p}$. Then, $I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$. Suppose not, then there exists $x \in I \backslash \mathfrak{p}, y \in I \backslash \mathfrak{p}$. But then $xy \in IJ \subseteq \mathfrak{p}$.
- $\mathfrak{p} \supseteq I \cap J \Leftrightarrow IJ \subseteq \mathfrak{p}$.

20.2. Zariski topology

Definition 20.2.1.

- For an ideal I, let

$$V(I) = \{\mathfrak{p} \text{ prime ideal } \mid I \subseteq \mathfrak{p}\}$$

- $\mathrm{Spec}(R) = \{$ collection of all prime ideals of $R\}$

Definition 20.2.2 (Zariski Topology)**.**
It is the topology defined on $\mathrm{Spec}(R)$ such that the closed sets are $V(I)$.

Verification that this indeed is a topology.

1. $V(0) = \mathrm{Spec}(R), V(R) = \emptyset$.
2. $V(I) \cup V(J) = V(I \cap J) = V(IJ)$.
3. $\bigcap_{k \in k} V_k = V(\sum_{k \in K} I_k)$. This is because $\mathfrak{p} \supseteq I_k \Leftrightarrow \mathfrak{p} \supseteq \sum_{k \in K} I_k$

Let us now look at the open sets of this topology. The basis for the open sets is given by

$$D(f \in R) = \{ \text{ all prime ideals not containing } f\}$$

Clearly,

$$(V(I))^c = \bigcup_{f\in I} D(f)$$

and moreover, each $D(f)$ is open since $D(f) = (V(\langle f\rangle))^c$

Theorem 20.2.3.
$\mathrm{Spec}(R)$ is quasi-compact.

Proof. We wish to prove that every open cover has a finite subcover. This is equivalent to saying every cover by $D(f_i)$ has a finite subcover. Say

$$\mathrm{Spec}(R) = \bigcup_{i\in I} D(f_i)$$

Take J to be the ideal generated by $f_i's$. Either $J = R$ or $J \subseteq \mathfrak{m}$. Suppose $J \subseteq \mathfrak{m}$, then $f_i \in \mathfrak{m} \in \mathrm{Spec}(R) \Rightarrow \mathfrak{m} \notin D(f_i)\ \forall\ i \Rightarrow D(f_i)$ does not cover $\mathfrak{m}$. A contradiction. Therefore, $J = R$ and this implies $1 =$ some linear combination of f_i and notice that this sum is finite. So, just consider these finitely many $f_i's$ (say the indexing set is K). These cover J. Suppose that $\{D(f_k), k \in K\}$ do not cover $\mathrm{Spec}(R)$. Then, there is a prime ideal $\mathfrak{p} \notin \bigcup_{k\in K} D(f_k) \Rightarrow \mathfrak{p} \ni f_k\ \forall\ k \in K \Rightarrow R \subseteq \mathfrak{p} \Rightarrow\Leftarrow$. Hence, it covers all of $\mathrm{Spec}(R)$ as required.

Another proof:
Suppose $\mathrm{Spec}(R) = \bigcup_{j\in J} U_j = \bigcup_{j\in J} \mathrm{Spec}(R)\backslash\mathcal{V}(I_j) = \mathrm{Spec}(R)\backslash \bigcap_{j\in J} \mathcal{V}(I_j) = \mathrm{Spec}(R)\backslash\mathcal{V}(\sum_{j\in J} I_j)$. This is equivalent to saying that $\mathcal{V}(\sum_{j\in J} I_j) = \emptyset$. So, we conclude that $\sum_{j\in J} I_j = R \Rightarrow \sum_{k\in K} a_k = 1$ for some finite set K. We claim that $\{U_k : k \in K\}$ covers $\mathrm{Spec}(R)$. This is because

$$\begin{aligned}
\mathcal{V}(\sum_{k\in K} I_k) &= 0 \\
\Rightarrow \mathrm{Spec}(R) &= \mathrm{Spec}(R)\backslash\mathcal{V}(\sum_{k\in K} I_k) \\
&= \bigcup_{k\in K} \mathrm{Spec}(R)\backslash\mathcal{V}(I_k) \\
&= \bigcup_{k\in K} U_k
\end{aligned}$$

This completes the proof. □

Proposition 20.2.4.
Each $D(f)$ is quasi-compact.

Proof. Suppose
$$D(f) = \bigcup D(g_i)$$
and let J be the ideal generated by $g_i's$. Take $\mathfrak{p} \supseteq J$. Then, each $g_i \in J \subseteq \mathfrak{p} \Rightarrow \mathfrak{p} \notin D(g_i) \Rightarrow \mathfrak{p} \notin D(f) \Rightarrow f \in \mathfrak{p} \Rightarrow f \in \bigcap_{\mathfrak{p} \supseteq J} \mathfrak{p}$. Before completing this proof, we need to understand this intersection much better. Refer to following content on nilpotent elements and come back.
Now, we know that $f \in \mathrm{rad}(J)$ which implies $\exists n$ such that $f^n \in J$. We get

$$f^n = \sum_{\text{finite}} r_i g_i$$

Finally, we claim that these $D(g_i)$s cover $D(f)$. □

Definition 20.2.5.
$x \in R$ is nilpotent if $x^n = 0$ for some $n \in \mathbb{N}$.

Remark 20.2.6.
Any nilpotent element ($x^n = 0$ for some n) is clearly in every prime ideal ($0 \in \mathfrak{p}$) and thus in the intersection of all prime ideals. This can be recorded as

$$\bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p} \supseteq \mathrm{Nil}(R)$$

Proposition 20.2.7.

$$\bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p} \subseteq \mathrm{Nil}(R)$$

Proof. Take an element $x \in R \backslash \mathrm{Nil}(R)$ (not nilpotent) and consider the set

$$\Sigma = \{I \trianglelefteq R \mid x^n \notin I \ \forall\, n > 0\}$$

Notice that Σ is a poset with respect to inclusion. And every chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ has an upper bound (union of all the ideals). Thus, we can apply Zorn's lemma to get a maximal element $\mathfrak{p}$ which we claim is prime. Indeed, if $ab \in \mathfrak{p}$ but $a \notin \mathfrak{p}, b \notin \mathfrak{p}$ then $\mathfrak{p} + \langle a \rangle, \mathfrak{p} + \langle b \rangle$ are ideals strictly containing $\mathfrak{p}$ contradicting maximality of $\mathfrak{p}$. Therefore, we can conclude that $x \notin \mathfrak{p} \Rightarrow x \notin \bigcap_{\mathfrak{p} \supseteq J} \mathfrak{p}$ or rather not nilpotent implies not in intersection and hence we have proved the required inclusion. □

$$\mathrm{Nil}(R) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p} = \bigcap_{\mathfrak{p} \subseteq \{0\}} \mathfrak{p}$$
$$\{x \mid x^n \in J\} = \mathrm{rad}(J) = \bigcap_{\mathfrak{p} \supseteq J} \mathfrak{p}$$

21. Lecture-3 (12th January): Zariski topology

21.1. Zariski topology contd..

Definition 21.1.1.
If $J = \text{rad}(J)$, then J is called radical ideal.

Properties:

1. Every radical ideal is an intersection of prime ideals.
2. $\mathcal{V}(J) = \mathcal{V}(\text{rad}(J))$
3. $\mathcal{V}(J) = \mathcal{V}(J')$ implies $\text{rad}(J) = \text{rad}(J')$

Suppose $S \subseteq R$ such that

- $1 \in S, 0 \notin S$
- If $x, y \in S \Rightarrow xy \in S$

Proposition 21.1.2.
Take an ideal maximal wrt not intersecting S. Then, it is prime.

Proof. Suppose $\mathfrak{m}$ is the ideal in question. Next, suppose $\mathfrak{m}$ is not prime which implies $\exists a, b \in R$ such that $ab \in \mathfrak{m}$ but $a, b \notin \mathfrak{m}$. Then, $\mathfrak{m} + \langle a \rangle \supsetneq \mathfrak{m}, \mathfrak{m} + \langle b \rangle \supsetneq \mathfrak{m}$. But, this means $(\mathfrak{m} + \langle a \rangle) \cap S \neq \emptyset \Rightarrow m + ra \in S$ for some $m \in \mathfrak{m}, r \in R$. Similarly, $n + sb \in S$ for some $n \in \mathfrak{m}, s \in R$. But, S is multiplicative therefore $(m + ra)(n + sb) \in S \Rightarrow mn + ran + msb + rsab \in S \Rightarrow ((\langle ab \rangle + \mathfrak{m}) = \mathfrak{m}) \cap S \neq \emptyset$. This is a contradiction. Hence, we are done. $\square$

Proposition 21.1.3.
Say J is maximal wrt not being principal. Then, J is prime.

Proof. Suppose $\mathfrak{m}$ is the ideal in question. Next, suppose $\mathfrak{m}$ is not prime which implies $\exists a, b \in R$ such that $ab \in \mathfrak{m}$ but $a, b \notin \mathfrak{m}$. Next, we can consider the ideal $I = \mathfrak{m} + \langle a \rangle$.

By maximality of $\mathfrak{m}$, we have $I = \langle c \rangle$ for some $c \in R$. Now, consider $J = \{x \in R \mid xc \in \mathfrak{m}\}$. Clearly, $I \subseteq J$. Notice that $c = m + ar$ for some $m \in \mathfrak{m}, r \in R$.

$$\begin{aligned} bc &= b(m + ar) \\ &= bm + (ba)r \\ \Rightarrow bc &\in \mathfrak{m} \\ \Rightarrow b &\in J \end{aligned}$$

This means $b \in J \backslash \mathfrak{m}$. Therefore V is also principal and hence $V = \langle d \rangle$. Since $\mathfrak{m} \in I$, therefore $m = cr$ for some $r \in R$. But this means that $r \in V \Rightarrow r = r'd$ for some $r' \in R$. Hence, $m = cdr' \in \langle cd \rangle \Rightarrow \mathfrak{m} \subseteq \langle cd \rangle$. For the other direction, since $d \in V \Rightarrow cd \in U$. All of these tells us that $\mathfrak{m} = \langle cd \rangle$ a contradiction to our assumption. Therefore, $\mathfrak{m}$ must be prime. □

Proposition 21.1.4.
Say J is maximal wrt not being finitely generated. Then, J is prime.

Proof. Suppose $\mathfrak{m}$ is the ideal in question. Next, suppose $\mathfrak{m}$ is not prime which implies $\exists a, b \in R$ such that $ab \in \mathfrak{m}$ but $a, b \notin \mathfrak{m}$.

If we now look at $\mathfrak{m} + \langle a \rangle$, by our assumption, this ideal is finitely generated by say $u_1, \ldots, u_m$. □

Exercise 21.1.5. *Suppose J is maximal wrt not being generated by a cardinal number of generators. Then, J is prime.*

Definition 21.1.6.
A topological space X is said to be irreducible if it cannot be written as the union of proper closed subsets of X

21.2. Identify closed irreducible subsets of $\operatorname{Spec}(R)$

Proposition 21.2.1.
The sets $\mathcal{V}(\mathfrak{p})$ are exactly the irreducible components of $\operatorname{Spec}(R)$.

Lemma 21.2.2.
Let $I \subseteq R$ be a radical ideal. If $\mathcal{V}(I)$ is irreducible, then I is prime.

Proof. Suppose I is not prime. Then there exists a, b such that $ab \in I$ but $a \notin I$ and $b \notin I$. Consider a prime ideal $\mathfrak{p}$ that contains I, it will also contain ab and thus $\mathfrak{p}$ contains either a or b. This is summarised as

$$\mathcal{V}(I) = (\mathcal{V}(I) \cap \mathcal{V}(a)) \cup (\mathcal{V}(I) \cap \mathcal{V}(b))$$

Thus $\mathcal{V}(I)$ is union of closed sets. It remains to be shown that the sets are proper in order to conclude that $\mathcal{V}(I)$ is not irreducible. Since $\mathcal{V}(I) \cap \mathcal{V}(a) = \mathcal{V}(I + \langle a \rangle)$ and $a \notin I$ therefore $\mathcal{V}(I + \langle a \rangle)$ is a proper closed subset of I and same for b. This is a contradiction to our hypothesis. So, we are done. □

Lemma 21.2.3.
$\mathcal{V}(\mathfrak{p})$ is an irreducible closed subset for $\mathfrak{p}$ prime.

Proof. Suppose $\mathcal{V}(\mathfrak{p}) = V_1 \cup V_2$ with V_1, V_2 proper closed subsets of $V(\mathfrak{p})$. Then there exists ideals I, J such that $\mathcal{V}(\mathfrak{p}) = \mathcal{V}(I) \cup \mathcal{V}(J)$. Since $\mathfrak{p} \in \mathcal{V}(\mathfrak{p})$ this implies $\mathfrak{p} \in \mathcal{V}(I)$ or $\mathfrak{p} \in \mathcal{V}(J)$. Suppose $\mathfrak{p} \in \mathcal{V}(I)$, then $I \subseteq \mathfrak{p} \Rightarrow \mathcal{V}(\mathfrak{p}) \subseteq \mathcal{V}(I) \Rightarrow \mathcal{V}(\mathfrak{p}) = \mathcal{V}(I)$. This is a contradiction to our assumption and hence we are done. $\mathcal{V}(\mathfrak{p})$ is irreducible. □

Proposition 21.2.4.
Every irreducible closed subset of $\mathrm{Spec}(R)$ has an unique generic point.

Proof. Notice that any irreducible closed subset is of the form $\mathcal{V}(\mathfrak{p})$. Now, $\mathcal{V}(\mathfrak{p})$ is the closure of $\mathfrak{p}$. This is because $\mathrm{cl}(\mathfrak{p})$ is a closed set and hence of the form $\mathcal{V}(I)$ for some ideal I. Moreover $\mathfrak{p} \supseteq I$. The biggest ideal I such that $I \subseteq \mathfrak{p}$ is $\mathfrak{p}$ and this gives us what we want because $\mathcal{V}$ reverses inclusions. Therefore, $\mathrm{cl}(\mathfrak{p}) = \mathcal{V}(\mathfrak{p})$. And, such a generic point is unique for suppose $\mathcal{V}(\mathfrak{p}) = \mathcal{V}(\mathfrak{q})$ then clearly $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{q} \subseteq \mathfrak{p}$. So, we are done. □

To summarise, Zariski topology has the following properties:

1. $\mathrm{Spec}(R)$ is quasi-compact
2. $\mathrm{Spec}(R)$ has a basis of quasi-compact opens which is closed under intersection.
3. Every irreducible closed subset has a generic point.

Theorem 21.2.5 (Hochster)**.**
Any topological space with the 3 properties is the spectrum of some commutative ring.

Suppose X is spectral. Define a new space X^* with open sets as finite union of quasi-compact open sets in X. This new space is called the Hochster dual.

Theorem 21.2.6.
X^* is also spectral.

Proof. □

22. Lecture-4 (17th January, 2023): Noetherian spaces

22.1. Noetherian spaces

First, let us try to remember all the equivalent definitions of a ring being Noetherian.

Proposition 22.1.1.
The following are equivalent:

1. Every ideal is finitely generated.
2. Every ascending chain of ideals
$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$
stabilises.
3. Every non-empty family of ideals has a maximal element.

Nowhere do we use Zorn's lemma, so in some sense, these properties are essentially about some "finite-ness" property. Thus, Noetherian means strong finiteness in some sense.

Definition 22.1.2.

Definition 22.1.3.

Theorem 22.1.4.
A module M over R is Noetherian iff the module is finitely generated and finitely presented.

Proof. □

Proposition 22.1.5.
The direct sum of projective modules is projective.

Proposition 22.1.6.
The direct product of injective modules is injective.

A question we can ask is when is the direct sum of injective modules injective.

Proposition 22.1.7.
Direct sum of injective modules is injective iff the module is Noetherian.

23. Lecture-5 (19th January 2023):

Suppose A is a commutative ring and M an A-module.
Define $\mathrm{Sub}(M)$ = to be the set of all submodules of M. For any finite collection $m_1, \ldots, m_k \in M$, we next define

$$\begin{aligned}\mathbf{V}(m_1,\ldots,m_k) &= \text{ collection of submodules containing } m_1,\ldots,m_k \\ \mathbf{D}(m_1,\ldots,m_k) &= \mathrm{Sub}(M)\backslash\mathbf{V}(m_1,\ldots,m_k)\end{aligned}$$

Using these $\mathbf{D}(m_1, \ldots, m_k)$'s as open sets (sub-basis of open sets), we generate a topology.

Proposition 23.0.1 (read this here)**.**
The above mentioned topology is the same as Zariski topology OR the space is spectral.

Remark 23.0.2.
The takeaway point being this is also another way to get a spectral space.

Exercise 23.0.3. *Suppose X is spectral, $Y \subseteq X$ be a quasi-compact open subset. Then, Y is spectral.*

23.1. Localisation

Definition 23.1.1.
A multiplicatively closed set S is one that has the following properties:

1. $1 \in S, 0 \notin S$.
2. $x, y \in S \Rightarrow xy \in S$.

Example 23.1.2.
1. Invertible elements of a ring.
2.

Objective: We wish to construct a new ring in which each $s \in S$ is invertible.

If S was the collection of invertible elements, then localisation is just A.

Our objective can be summed up as follows:

$$\begin{tikzcd} A \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & B \\ & R \arrow[ur, dashed, "\exists ! \pi"'] & \end{tikzcd}$$

1. $\phi(s)$ is invertible in R for each $s \in S$
2. for any $\psi : A \to B$ such that each $\psi(s)$ is invertible, there is an unique map $\pi : R \to B$ that makes the diagram above commute.

Definition 23.1.3.
The localisation of A with respect to S, denoted by $S^{-1}A$ is the set of equivalence classes

$$\frac{a}{s} \quad , a \in A, s \in S$$

with

$$\frac{a}{s} \sim \frac{a'}{s'} \text{ if and only if } \exists\, t \in S \text{ such that } t(as' - sa') = 0$$

The ring addition and multiplication are the same as adding and multiplying fractions. Need to check it is well-defined!

Now, back to

$$A_f = \{ \text{ localisation of } A \text{ at } f\}$$

What we want to do is we essentially want to turn f into an unit. Take S to be all powers of f. Then, $S^{-1}A = A_f$.

This can also be realised as

$$\frac{A[X]}{\langle fX - 1\rangle}$$

Now, the question is why are the two notions equivalent.

$$\begin{tikzcd} A \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & B \\ & A_f \arrow[ur, dashed, "\pi"'] & \end{tikzcd}$$

with

$$\pi\left(\frac{a}{f^k}\right) = \frac{\psi(a)}{\psi(f)^k}$$

And,

$$\begin{tikzcd} A \arrow[rr, "\psi"] \arrow[dr, "\phi"'] & & B \\ & \frac{A[X]}{\langle fX-1\rangle} \arrow[ur, dashed, "\pi"'] & \end{tikzcd}$$

with $\pi(X) = \psi(f)^{-1}$

23.1.1. Prime ideals of A_f

Theorem 23.1.4.
The prime ideals of A_f are precisely $D(f)$ the set of primes not containing f.

Consider A_S and look at the ideals of A_S. They are precisely of the form

$$\left\{\frac{x}{s} \mid \ x \in I \trianglelefteq A, s \in S\right\}$$

There is a bijection between

$$\{\text{ prime ideals of } A_S\} \leftrightarrow \{\text{prime ideals of } A \text{ not intersecting S}\}$$

Say $\mathfrak{P}$ is a prime ideal of A_S. Then, $\mathfrak{P} = \frac{\mathfrak{p}}{s}$ with $\mathfrak{p}$ prime in A.

24. Lecture-6 (24th January, 2023): Localisation of modules, exact sequences

24.1. Localisation contd..

Suppose M is an A-module. And $S \subseteq A$ be a multiplicative set. Then, the localisation

$$M_S = \{\text{equivalence classes of all elements of the form } \frac{m}{s}\}$$

with $\frac{m}{s} \sim \frac{m'}{s'}$ if there exists $t \in S$ such that $t(s'm - m's) = 0$. This can be made into a module by standard operations.

Lemma 24.1.1.
M_S is an A_S-module.

Proof. $\square$

Some natural questions to ask are if $I \subseteq A$ is an ideal, whether

$$\left(\frac{A}{I}\right)_S \overset{?}{\cong} \frac{A_S}{I_S}$$

$$\text{More generally } \left(\frac{M}{M'}\right)_S \overset{?}{\cong} \frac{M_S}{M'_S}$$

We will need to introduce exact sequences to answer these questions.

24.2. Exact sequences

Suppose

$$f : M \to N$$

Then,

$$\begin{aligned} \ker(f) &= \{m \in M : f(m) = 0\} \\ \text{Coker} &= N/\text{Im}(f) \end{aligned}$$

This can be captured in the following diagram :

$$\begin{tikzcd}
\mathrm{Ker}(f) \arrow[r, "i"] & M \arrow[r, "f"] & N \arrow[r, "\pi"] \arrow[d, "h"] & \mathrm{Coker}(f) \arrow[dl, dashed] \\
& P \arrow[u, "g"] \arrow[ul, dashed, "\exists\,!"] & Q &
\end{tikzcd}$$

Here,

$$M/\ker(f) \cong \mathrm{Im}(f)$$

is equivalent to saying $\mathrm{Coker}(i) = \ker(\pi)$. This leads to the definition

Definition 24.2.1.

$$M' \xrightarrow{f} M \xrightarrow{g} M''$$

is exact at M if $\mathrm{Im}(f) = \ker(g)$.

Lemma 24.2.2. 1.

$$0 \to M' \xrightarrow{f} M$$

being exact means f is injective.

2.

$$M \xrightarrow{g} M'' \to 0$$

being exact means g is surjective.

Proof. 1. $\ker f = \mathrm{Im}(0 \to M')$

2. $\mathrm{Im}(g) = \ker(M'' \to 0) = M$

□

Definition 24.2.3.
A short exact sequence is a sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

is exact everywhere.

Proposition 24.2.4.
If

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is exact, then

$$0 \to M'_S \xrightarrow{f_S} M_S \xrightarrow{g_S} M''_S \to 0$$

is exact

Proof. **Claim**: $\ker(f)_S = \ker(f_S)$
$\subseteq$ is clear since $\frac{f(m)}{s} = 0$. Suppose $\frac{m}{s} \in \ker(f_S) \Rightarrow f(m)/s = 0 \in M_S$. This means that there is a $t \in S$ such that $tf(m) = 0 = f(tm) \Rightarrow tm \in \ker(f) \Rightarrow \frac{tm}{ts} = \frac{m}{s} \in \ker(f)_S$. This gives us " $\supseteq$ ".

Similarly, $\operatorname{Coker}(f)_S = \operatorname{Coker}(f_S)$. This completes the proof. □

Next, take $\mathfrak{p} \subseteq A$ be a prime ideal and $A \backslash \mathfrak{p}$ be the multiplicative set S. We denote M_S by $M_\mathfrak{p}$.

- If $M = 0$, then $M_\mathfrak{p} = 0$ for all prime ideals $\mathfrak{p}$. This implies $M_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m}$.
- If $M_\mathfrak{m} = 0 \; \forall \; \mathfrak{m} \Rightarrow M = 0$. Take an element $m \in M$ such that $\frac{m}{1} = 0 \in M_\mathfrak{m}$ for each maximal ideal $\mathfrak{m}$ in A. Suppose $\operatorname{Ann}(m) \neq A$, then $\operatorname{Ann}(A) \subsetneq \mathfrak{m}'$ for some maximal ideal $\mathfrak{m}'$. But then we will have $sm = 0$ for some $s \in A \backslash \operatorname{Ann}(m)$ which is a contradiction. Hence, $\operatorname{Ann}(m) = A$ and $m = 0$. This completes the claim.
- If $M \xrightarrow{f} N$ is an isomorphism iff $M_\mathfrak{m} \xrightarrow{f_\mathfrak{m}} N_\mathfrak{m}$ is an isomorphism for all maximal ideals $\mathfrak{m}$.

We can summarise in the following theorems

Theorem 24.2.5.
Let M be an A-module and $m \in M$. Then TFAE:

1. $m = 0$.
2. $\frac{m}{1} = 0$ in $M_\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of A.
3. $\frac{m}{1} = 0$ in $M_\mathfrak{m}$ for all maximal ideals $\mathfrak{m}$ of A.

Theorem 24.2.6.
Let M be an A-module. Then TFAE:

1. $M = 0$.
2. $M_\mathfrak{p} = 0$ for all prime ideals $\mathfrak{p}$ of A.
3. $M_\mathfrak{m} = 0$ for all maximal ideals $\mathfrak{m}$ of A.

Theorem 24.2.7.
Let $\phi : M \to N$ be an R-module homomorphism. Then, TFAE:

1. ϕ is injective.
2. $\phi_\mathfrak{p} : M_\mathfrak{p} \to N_\mathfrak{m}$ is injective for all prime ideals $\mathfrak{p}$.

3. $\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective for all maximal ideals $\mathfrak{m}$.

Proof. From the exactness of the sequence as per a proposition mentioned above we have $1 \Rightarrow 2, 1 \Rightarrow 3$. Moreover, $2 \Rightarrow 3$. We wish to show that $3 \Rightarrow 1$. Let $M' = \ker(\phi)$. Then we have the following exact sequence

$$0 \to M' \to M \to N$$

By the proposition above, we have

$$0 \to M'_{\mathfrak{m}} \to M_{\mathfrak{m}} \to N_{\mathfrak{m}}$$

exact. This implies $M'_{\mathfrak{m}} = \ker(\phi_{\mathfrak{m}}) = 0$ since $\phi_{\mathfrak{m}}$ is injective by hypothesis. Therefore, $M'_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$. Now, the result follows from previous theorem. $\square$

The same theorem can be repeated with injective replaced with surjective. This leads us thereby to the last conclusion in the points mentioned before these theorems.

Definition 24.2.8.
Suppose

$$M \xrightarrow{f} N$$

Then, f is a monomorphism means

$$T \underset{h}{\overset{g}{\rightrightarrows}} M \xrightarrow{f} N$$

such that $f \circ g = f \circ h \Rightarrow g = h$.

Epimorphism is the dual of this.

Remark 24.2.9.
For sets, these mean injection and surjection but monomorphism and epimorphism need not mean isomorphism in a random category.

25. Lecture-7 (7th February, 2023):

26. Lecture-8 (9th February, 2023):

Part IV.

Algebraic Geometry I

27. Lecture-1 (9th January, 2023): Topological properties and Zariski Topology

27.1. Topological properties

Consider a topological space X.

Definition 27.1.1.
1. We say X is quasi-compact if every open cover of X admits a finite subcover.
2. If $f : X \to Y$ is continuous, we call f quasi-compact if $f^{-1}(V)$ is quasi-compact for all quasi-compact open $V \subseteq Y$.

Exercise 27.1.2. *Composition of quasi-compact maps is quasi-compact.*

Consider the two maps $f : X \to Y$ and $g : Y \to Z$. Next, look at the composition $g \circ f : X \to Z$. For all quasi-compact open $V \subseteq Z$, $(g \circ f)^{-1}(V) = f^{-1} \circ g^{-1}(V)$. Since g is quasi-compact and continuous, $g^{-1}(V)$ is also quasi-compact and open. Similarly, f is also quasi-compact and continuous, therefore $f^{-1}(g^{-1}(V))$ is also quasi-compact and we are done.

Lemma 27.1.3.
X quasi-compact and $Y \subseteq X$ is closed implies Y is quasi-compact.

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of Y. Set $U = X - Y$. Since U_i is open in Y, we have $U_i = Y \cap V_i$ where V_i is open in X. Now we note that $\{V_i\}_{i\in I} \cup U$ covers X but X is quasi-compact and we obtain a finite subcover $\{V_i\}_{i\in J} \cup U$ where J is finite. The corresponding $U_i, i \in J$ must therefore cover Y and we are done. □

Proposition 27.1.4.
If X is quasi-compact and Hausdorff, then $E \subseteq X$ is quasi-compact iff E is closed.

Proof. $\Leftarrow$ direction is done.
$\Rightarrow$ direction is what we need to prove.
Take $x \in X \backslash E$. For each $y \in E$, due to Hausdorff-ness we have two disjoint open sets U_y and U_y containing x and y respectively. Do this for all $y \in E$. The collection

$\{U_y\}_{y \in E}$ covers E but it is quasi-compact thus we get a finite subcover $\{U_{y_i}\}_{i \in I}$ with I finite. Now, let

$$U = \bigcap_{i \in I} U_{y_i}$$

U is clearly open, contains x and is disjoint from E. Since x was chosen arbitrarily, $X \backslash E$ must be open. $\square$

> **Lemma 27.1.5.**
> Any finite union of quasi-compact spaces is quasi-compact.

Proof. Suppose $X_i, i = 1, 2, \ldots, n$ are the spaces in question. We want to show that

$$X = \bigcup_{i=1}^{n} X_i$$

is also quasi-compact. Take any cover $\{U_i\}_{i \in I}$ be an open cover of X. Then for each $i = 1, 2, \ldots, n$ it is clear that $\{U_i\}_{i \in I}$ also covers X_i. Using quasi-compactness of X_i we can get a finite subcollection $\{U_{i_j} : j = 1, \ldots, n_i\}$. This can be done for all i. Now, consider $\bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} U_{i_j}$. This union covers X and is finite. So, we are done. $\square$

> **Lemma 27.1.6.**
> Suppose $f : X \to Y$ is continuous, if X is quasi-compact then so is $f(X)$.

Proof. Let $\{U_i\}_{i \in I}$ be an open cover of $f(X)$. Now, $\{f^{-1}(U_i)\}_{i \in I}$ covers X and by continuity, each of them are open. Use quasi-compactness of X to get a finite subcover that covers X.

$$X = \bigcup_{i=1}^{n} f^{-1}(U_i)$$
$$\because f(f^{-1}(U_i)) \subseteq U_i$$
$$\therefore f(X) \subseteq \bigcup_{i=1}^{n} U_i$$

$\square$

Suppose Σ is a poset. Σ satisfies acc if every ascending chain

$$x_1 \leq x_2 \leq \cdots$$

is stationary.

> **Lemma 27.1.7.**
> The following are equivalent:
>
> 1. Σ satisfies acc.

2. Every non-empty subset of Σ has maximal element.

Proof. $1 \Rightarrow 2$. Suppose $S \subseteq \Sigma$ has no maximal element.
Then choose $x_0 \in S$ non-maximal, then we can find a x_1 such that $x_0 \lneq x_1$. By induction we can construct an infinite chain $x_0 \lneq x_1 \lneq \cdots \neq x_i \lneq \cdots$ which does not terminate which is a contradiction to our hypothesis. Thus, S must have a maximal element.
$2 \Rightarrow 1$. Suppose $x_1 \leq x_2 \leq \cdots \leq x_i \leq$ is an infinite ascending chain, then $S = \{x_i \mid i \geq 1\}$ has no maximal element. □

Definition 27.1.8.
A topological space is called Noetherian if set of all closed subsets of X satisfies dcc.

Lemma 27.1.9.
X Noetherian implies X is quasi-compact.

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open cover of X that does not have a finite subcover. Consider the collection $\mathcal{F}$ of union of finite number of elements of $\mathcal{U}$. Since being Noetherian is equivalent to saying any finite subset of open subsets has a maximal element, we know that $\mathcal{F}$ has a maximal element. Suppose that maximal element is $U_{i_1} \cup \ldots \cup U_{i_n}$. If this does not cover X, take an element x in the complement of the maximal element. Since $\mathcal{U}$ covers X, there is an $i \in I$ such that $x \in U_i$. Notice that now $U_{i_1} \cup \ldots \cup U_{i_n} \subsetneq U_{i_1} \cup \ldots \cup U_{i_n} \cup U_i$ which contradicts the maximality. Thus, we are done. □

Remark 27.1.10.
The converse need not be true. Consider $[0, 1]$ covered by $[1/2^n, 1]$.

Lemma 27.1.11.
If $X_1, \ldots, X_n$ are Noetherian subspaces of X, then so is $X = X_1 \cup X_2 \cup \ldots \cup X_n$

Proof. Let Y_is be closed in X that forms the chain

$$X \supseteq Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \cdots$$

For each i, we get a chain of closed sets in X_i by intersecting with X_i. This gives us

$$X_i \supseteq Y_1 \cap X_i \supseteq Y_2 \cap X_i \supseteq Y_3 \cap X_i \supseteq \cdots$$

Since X_i is Noetherian, this chain terminates at say r_i. Now, take $r = \max_i r_i$. The original chain will terminate after this point. Suppose $y \in Y_i$ with $i \leq r$, there is an j such that $y \in X_j$. This means $y \in X_j \cap Y_i = X_j \cap Y_r$. Hence, $y \in Y_r$ and we are done. □

Definition 27.1.12.
Locally Noetherian means every point $x \in X$ has a neighbourhood U which is Noetherian wrt subspace topology.

Lemma 27.1.13.
Quasi-compact and locally Noetherian implies Noetherian.

Proof. Since X is locally Noetherian, for each $x \in X$ we have a nbd. U_x that is Noetherian. $\{U_x\}_{x\in X}$ is an open cover of X. Quasi-compactness gives us a finite subcover $\{U_{x_i}\}_{i=1}^n$, i.e.,

$$X = \bigcup_{i=1}^{n} U_{x_i}$$

X is Noetherian from previous lemma. □

Exercise 27.1.14. *Give an example of a ring R such that $\operatorname{Spec}(R)$ is Noetherian but R is not.*

Consider the ring $R = k[X_1, X_2, \ldots,]$ and the ideal $I = \langle X_1^2, X_2^2, \ldots, \rangle$. Now, look at $R' = R/I$. $\operatorname{Spec}(R')$ is a singleton.

Definition 27.1.15.
A topological space X is called irreducible if it cannot be written as finite union of proper closed subsets.

A closed subset $Y \subseteq X$ is called irreducible component of X if it is a maximal irreducible closed subset of X.

Lemma 27.1.16.
If X is Noetherian and $Y \subseteq X$ is a subspace, then Y is Noetherian.

Proof. Let Y_is be closed in Y that forms the chain

$$Y \supseteq Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \cdots$$

For each i, we have a closed set in X such that $Y_i = Y \cap X_i$. This gives us

$$Y \supseteq X_1 \cap Y \supseteq X_2 \cap Y \supseteq X_3 \cap Y \supseteq \cdots$$

□

Lemma 27.1.17.
Let X be Noetherian. Then, X has finitely many irreducible components.

Proof. More generally, we will show that every closed subset for X has finitely many irreducible components.

Suppose that this is false. Let Σ be the collection of closed subsets of X that does not satisfy our condition. Order this as follows: $A \leq B$ if $A \supseteq B$. If $\{C_i\}$ is a chain in Σ, then it must eventually stabilise since X is Noetherian. This C_α is an upper bound for this chain. Therefore, by Zorn's lemma, there is a maximal element Y. Since $Y \in \Sigma$, therefore it is not irreducible. Suppose $Y = Y_1 \cup Y_2$ with Y_1, Y_2 proper closed subsets of Y. $Y \leq Y_1, Y \leq Y_2$. Since $Y \in \Sigma$, Y is not a finite union of irreducible components. Hence, either Y_1 or Y_2 is not irreducible. If Y_1 is not irreducible but $Y_1 \in \Sigma$, since Y is maximal in Σ and $Y \leq Y_1$, therefore $Y = Y_1$ a contradiction that Y_1 is a proper subset of Y. Thus, Σ must be empty and the claim is proven. □

Lemma 27.1.18.
X is Noetherian implies there exists an unique expression $X = X_1 \cup \cdots \cup X_n$ where $X_i's$ are irreducible components of X.

Proof. Suppose
$$X = X_1 \cup \cdots \cup X_n = X_1' \cup \cdots \cup X_m'$$
Clearly $X_1' \subseteq X$, this means $X_1' = \bigcup_{i=1}^n X_1' \cap X_i$. Since X_1' is irreducible, there must be a i_1 such that $X_1' = X_{i_1} \cap X_1'$. Thus, $X_1' \subseteq X_{i_1}$. We can choose i_1 to be 1 to get $X_1' \subseteq X_1$. Similarly, $X_1 \subseteq X_{j_1}'$. Since $X_1' \subseteq X_{j_1}'$ and our assumption that $X_i \not\subseteq X_j$ for $i \neq j$ we conclude that $j_1 = 1$. Finally, we conclude that $X_1 = X_1'$. Let Z be the closure of $X - X_1$, then $Z = X_2 \cup \cdots \cup X_n = X_2' \cup \cdots \cup X_m'$. We can argue inductively and conclude that $X_i = X_i'$ and $n = m$. □

Lemma 27.1.19.
Suppose X is Noetherian and $X_1 \subseteq X$ an irreducible component. Then, X_1 contains a non-empty open set in X.

Proof. Consider $U = X \backslash X_2 \cup \cdots \cup X_n$. Clearly, U is non-empty and open. Moreover, $U \subseteq X_1$ and we are done. □

Definition 27.1.20.
Let X be a topological space. We say that X is a spectral space if the following holds:

1. X is quasi-compact.

2. X is T_0.

3. X has a basis of quasi-compact open sets.

4. Every irreducible closed subset of X has a generic point ($\exists x \in Y$ such that $\overline{\{x\}} = X$)

27.2. Zariski Topology

Let A be a commutative ring with identity and $X = \mathrm{Spec}(A)$.

Zariski topology is the unique topology such that a subset $Y \subseteq X$ is closed iff $Y = \mathcal{V}(I)$ for some ideal $I \trianglelefteq A$. Here,

$$\mathcal{V}(I) = \{\mathfrak{p} \in X \mid \mathfrak{p} \supseteq I\}$$

Theorem 27.2.1.
$\mathrm{Spec}(A)$ is always spectral.

Proof.

1. X is T_0
   For all $f \neq 0$ in A, let $A_f = S^{-1}A$ be the localisation of A at f where $A_f = \{f^n \mid n \geq 0\}$. Next, let $V_f = X \backslash V(f) = \mathrm{Spec}(A_f)$. This forms a basis for the Zariski topology.
   Now, let $\mathfrak{p}, \mathfrak{P}$ be two distinct primes.
   - Suppose $\mathfrak{p} \not\subseteq \mathfrak{P}$.
     $Y = V(\mathfrak{p})$ is closed set and $\mathfrak{P} \notin V(\mathfrak{p})$. Take Y^c. Then $\mathfrak{P} \in Y^c$ and $\mathfrak{p} \notin Y^c$.
   - If $\mathfrak{p} \subseteq \mathfrak{P}$
     Then consider $\mathcal{V}(\mathfrak{P})$. Clearly, $\mathfrak{p} \notin \mathcal{V}(\mathfrak{P})$. Take $U = \mathcal{V}(\mathfrak{P})^c$, then $\mathfrak{p} \in U$ but $\mathfrak{P} \notin U$.
2. X is quasi-compact.
   Let $\{U_i\}$ be an open cover of X. WLOG, we can assume that $U_i = \mathrm{Spec}(A_{f_i}), f \neq 0$. Let I be the ideal generated by these $f_i s$.
   Case-1: Suppose that $I \neq A$. Then there exists a maximal ideal $\mathfrak{m} \supseteq I \Rightarrow \mathcal{V}(\mathfrak{m}) \subseteq \mathcal{V}(I) \Rightarrow X \backslash \mathcal{V}(\mathfrak{m}) \supseteq X \backslash \mathcal{V}(I) = X \backslash \bigcap_{i \in I} \mathcal{V}(f_i) = \bigcup U_i = X$ which is absurd. Hence, we conclude that $I = A$. Next,
   $$1 = \sum_{i=1}^{n} a_i f_i \qquad \text{for some } a_i \in A$$
   $$\Rightarrow \bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} X \backslash \mathcal{V}(f_i)$$
   And, we get the required refinement.
3. X has a basis of quasi-compact open sets follows from the above.
4. Let $Y \subseteq X$ be an irreducible closed subset. Then, $Y = \mathrm{Spec}(A/I)$. WLOG, we can assume X is irreducible. Next, observe that $\mathrm{Spec}(A) = \mathrm{Spec}(A_{\mathrm{red}}) = \mathrm{Spec}(A/\mathrm{Nil}(A))$. Since A is irreducible and reduced, we conclude that A is an integral domain. We are now done since 0 is a generic point in that case.

□

28. Lecture-2 (11th January, 2023): Zariski topology and affine schemes

28.1. Zariski topology contd..

Theorem 28.1.1 (Hochster)**.**
Every spectral space is homeomorphic to $\mathrm{Spec}(A)$ for some commutative ring A.

Notation: Ring be the category of commutative rings, **Top** be the category of topological spaces.

Theorem 28.1.2.
There is a contravariant functor

$$\begin{aligned} sp : \textbf{Ring} &\to \textbf{Top} \\ \mathrm{Spec}(B) &\mapsto \mathrm{Spec}(A) \end{aligned}$$

Proof. Consider $f : A \to B$. This induces a map

$$f_{\#} : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

such that $f_{\#}(\mathfrak{p}) = f^{-1}(\mathfrak{p})$.
Well-defined: Suppose $xy \in f^{-1}(\mathfrak{p}) \Rightarrow f(xy) = f(x)f(y) \in \mathfrak{p} \Rightarrow$ either x or y lies in $f^{-1}(\mathfrak{p})$ which completes our check.
We claim that $f_{\#}$ is continuous. This can be seen as follows:
Take a basic open set $D(a), a \in A$. Enough to show for these sets since $D(a)$ forms a basis for the topology on $\mathrm{Spec}(A)$. Now,

$$\mathfrak{p} \in f_{\#}^{-1}(D(a)) \Leftrightarrow f_{\#}(\mathfrak{p}) \in D(a) \Leftrightarrow a \notin f^{-1}(\mathfrak{p})$$

But this means

$$a \notin f^{-1}(\mathfrak{p}) \Leftrightarrow f(a) \notin \mathfrak{p} \Leftrightarrow \mathfrak{p} \in D(f(a))$$

□

28.2. Affine schemes

Definition 28.2.1.
$\text{Spec}(A)$ will be called an affine "scheme" (we will see this properly later on).

Definition 28.2.2.
Let $X = \text{Spec}(A), Y = \text{Spec}(B)$. Let $f : Y \to X$ be a continuous map. We call such a map f regular (holomorphic) if there is a ring homomorphism $g : A \to B$ such that $f = g_\#$

Example 28.2.3.
Take $\text{Spec}(\mathbb{Z})$ and consider the constant map. This cannot be regular because any ring homomorphism must take 1 to 1 and as a consequence fixes every element.

Proposition 28.2.4.
If $X = \text{Spec}(A)$. A regular function on X is a regular map from X to $\text{Spec}(\mathbb{Z}[t])$.

Proof. □

Remark 28.2.5.
On an affine scheme, the set of all regular maps is the ring A itself since, the map $\mathbb{Z}[t] \to A$ is determined by where t is sent to.

Lemma 28.2.6.
Every affine scheme has a closed point.

Proof. Every commutative ring has a maximal ideal. □

Definition 28.2.7.
Open in affine is called quasi-affine.

Example 28.2.8.
Take A a local integral domain with $\mathfrak{m}$ the maximal ideal. Suppose that all prime ideals of A are of the form

$$\langle 0 \rangle \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \{\mathfrak{m}\}$$

Consider $X = \text{Spec}(A) \backslash \mathfrak{m}$. X is open in affine scheme but has no closed point.

An example of such a ring is

$$\Gamma = \mathbb{Z}x_1 \oplus \mathbb{Z}x_2 \oplus \cdots$$

Give an ordering: $\sum a_i x_i \geq 0$ if the first nonzero term is > 0 or all $a_i = 0$
Γ is a totally ordered abelian group and hence there exists a valuation ring A with value group Γ and the prime ideals of Γ are in $1-1$ correspondence with prime ideals of A.

Exercise 28.2.9. *Let $A = k[X_1, X_2, \ldots], B = A_{\mathfrak{m}}, X = \operatorname{Spec}(B) \backslash \mathfrak{m}, \mathfrak{m} = \langle X_1, X_2, \ldots, \rangle$. Claim is that X has no closed point.*

28.2.1. Fiber products of affine schemes

Suppose A is a commutative ring, B, C are A-algebras. Let $X = \operatorname{Spec}(A), Y = \operatorname{Spec}(B), Z = \operatorname{Spec}(C)$. Next, suppose we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

Universal property of fiber products:

$$\begin{tikzcd} W' \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & Y \times_X Z \arrow[r] \arrow[d] & Z \arrow[d, "g_\#"] \\ & Y \arrow[r, "f_\#"'] & X \end{tikzcd}$$

Definition 28.2.10.
If a W exists such that the universal property is satisfied, then W is called the fiber product of Y, Z over X and we write $W = Y \times_X Z$

Theorem 28.2.11.
$\mathbf{Aff}_{\mathbb{Z}}$ = category of affine schemes admits fiber products.

Proof. Consider the following data:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

Let $D = B \otimes_A C$. We have the natural maps $f_1 : B \to B \otimes_A C$ sending $b \mapsto b \otimes 1$ and $f_2 : C \to B \otimes_A C$ sending $c \mapsto 1 \otimes c$. Both are ring homomorphisms and fit into the following diagram due to the nature of tensor product

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "f_1"] \\ C \arrow[r, "g_1"'] & B \otimes_A C \end{tikzcd}$$

Now, let $W = \operatorname{Spec}(B \otimes_A C)$ and we claim that this satisfies the universal property of fibre product. Apply $\operatorname{Spec}(-)$ functor to the diagram to get

$$\begin{tikzcd} A & B \arrow[l, "f_\#"'] \\ C \arrow[u, "g_\#"] & \operatorname{Spec}(B \otimes_A C) \arrow[l, "g_{1\#}"] \arrow[u, "f_{1\#}"'] \end{tikzcd}$$

From the universal property of tensor product we have the following diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "f_1"] \arrow[ddr, bend left] & \\ C \arrow[r, "g_1"'] \arrow[drr, bend right] & B \otimes_A C \arrow[dr, dashed, "\exists\,!"] & \\ & & U \end{tikzcd}$$

Again, apply the $\operatorname{Spec}(-)$ functor.

$$\begin{tikzcd} X & Y \arrow[l, "f_\#"'] & \\ Z \arrow[u, "g_\#"] & \operatorname{Spec}(B \otimes_A C) \arrow[l, "g_{1\#}"] \arrow[u, "f_{1\#}"'] & \\ & & \operatorname{Spec}(U) \arrow[ul, dashed, "\exists\,!"] \arrow[ull, bend left] \arrow[uul, bend right] \end{tikzcd}$$

This completes the proof. $\square$

29. Lecture-3 (16th January, 2023): Category theory brushup

Suppose we have a ring homomorphism $f : A \to B$ and $X = \operatorname{Spec}(A), Y = \operatorname{Spec}(B)$. This induces a map $f_\# : Y \to X$. From, the previous discussion, there is a fiber product $Y \times_X Y$ such that the following diagram makes sense

$$\begin{tikzcd}
X & Y \arrow[l, "f_\#"'] & \\
Y \arrow[u, "f_\#"] & Y \times_X Y \arrow[l, "p_1"] \arrow[u, "p_2"'] & \\
& & Y \arrow[ul, dashed, "\exists\, ! \, \Delta_Y"] \arrow[uul, bend right] \arrow[ull, bend left]
\end{tikzcd}$$

Here, $p_1 \circ \Delta_Y = p_2 \circ \Delta_Y = \mathrm{id}$ where

$$\Delta_Y : Y \to Y \times_X Y$$

is called the relative diagonal of Y/X.

Definition 29.0.1.
Say X_1, X_2 are affine schemes. $X_1 \to X_2$ is a closed immersion iff $A_1 \to A_2$ is a surjective. Here, $\operatorname{Spec}(A_i) = X_i, i = 1, 2$.

Lemma 29.0.2.
Δ_Y is a closed immersion.

Proof. $B \otimes_B B \to B$ is a surjection. □

Example 29.0.3.
Take $A = \mathbb{Z}, B = \mathbb{Z}[t]/\langle t^n \rangle$ for some $n \geq 2$. There is a canonical inclusion $f : A \to B$. This induces a map $Y = \operatorname{Spec}(B) \to X = \operatorname{Spec}(A)$ which is an identity map in terms of sets. Thus, it is a closed inclusion but not a closed immersion.

Remark 29.0.4.
We know that diagonal is closed iff the space is Hausdorff. This seems to contradict our assumptions! But we are fine because this claim is true only when the topology

is the product topology. Here, the topology we have is not the product topology.

Definition 29.0.5.
A regular map $f : X \to Y$ is called separated morphism if the relative diagonal of Y over X is closed in $Y \times_X Y$.

Lemma 29.0.6.
Let $X = \operatorname{Spec}(A)$. Suppose U_1, U_2 are two open affine subsets of X. Then, $U_1 \cap U_2$ is also affine.

Proof. We have two natural injections

$$U_1 \overset{j_1}{\hookrightarrow} X, U_2 \overset{j_2}{\hookrightarrow} X$$

then we naturally have the following

$$U_1 \times_Z U_2 \xrightarrow{j_1 \times j_2} X \times_Z X$$

where $Z = \operatorname{Spec}(\mathbb{Z})$ (if it is blank, just assume Z by default).
From previous discussion we get

$$\begin{tikzcd}
U_1 \times_Z U_2 \arrow[r, "j_1 \times j_2"] & X \times_Z X \\
& X \arrow[u, "\Delta_X"']
\end{tikzcd}$$

Since each term is affine, we can take the fiber product of $U_1 \times_Z U_2$ and X. Say the fiber product is W.

$$\begin{tikzcd}
& U_2 & & X & \\
U_1 & U_1 \times_Z U_2 \arrow[l, "q_2"'] \arrow[u, "q_1"'] \arrow[rr, "j_1 \times j_2"] & & X \times_Z X \arrow[u, "p_1"'] \arrow[r, "p_2"'] & X \\
& W \arrow[u, "\Delta'"] \arrow[rr, "j"'] \arrow[urr, phantom, "\square"] & & X \arrow[u, "\Delta_X"'] &
\end{tikzcd}$$

Then, we claim that

Claim: $W = U_1 \cap U_2$

Proof. Suppose $x \in W$, then $\square$

It now remains to show that W is affine but it is clear from the definition of fiber products.

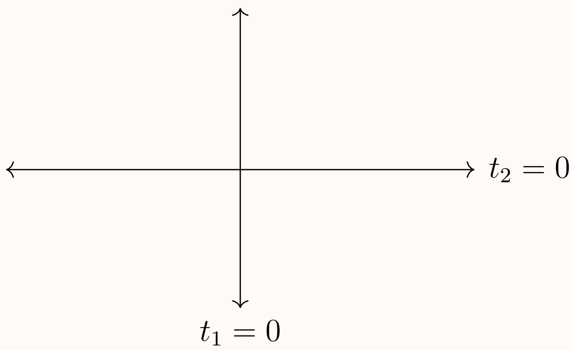
□

Remark 29.0.7.
If Δ_X is a closed immersion then so is Δ'. That is, closed immersions are preserved under fiber products. Follows from right exactness of tensor product.

Now, that we have discussed intersection, we naturally ask : What happens to $U_1 \cup U_2$. Is it still affine ?

The answer turns out to be NO. To see this,

Example 29.0.8 (NON-example)**.**
Consider k be an algebraically closed field. $A = k[t_1, t_2]$ and $X = \mathrm{Spec}(A)$. Let $U_i = \{x \mid t_i(x) \neq 0\} = X \backslash \mathcal{V}(t_i)$. Clearly, U_i is open and affine ($= \mathrm{Spec}(A_{t_i})$). But $U_1 \cup U_2$ is not affine.



U_1 is complement of the horizontal axis and U_2 of the vertical axis. But $U_1 \cup U_2$ is the complement of origin. The question is asking if the complement of origin is affine or not. A highly NON-TRIVIAL question to answer.

Exercise 29.0.9 (not trivial but do think about it)**.** *Suppose* $X = \mathrm{Spec}(A)$ *and* $U \hookrightarrow X$ *is affine open. Does this imply* $U = \mathrm{Spec}(S^{-1}A)$ *for some multiplicatively closed set* $S \subseteq A$*?*

Definition 29.0.10.
Suppose $S = \mathrm{Spec}(A)$ and $x \in X$. Let $K(A) = S^{-1}(A)$ where S is the set of all nonzero divisors in A. Here, we have $A \hookrightarrow S^{-1}(A) =$ the ring of all meromorphic functions on X. Then,

$$\mathcal{O}_{X,x} = \{f \in K(A) \mid f \text{ is regular in a nbd of } x\}$$

is called the germ of regular function.

Lemma 29.0.11.

$$\mathcal{O}_{X,x} = A_{\mathfrak{p}}$$

where $\mathfrak{p} = x$.

Proof. Suppose f is regular in a nbd of $\mathfrak{p}$ iff there exists $b \notin \mathfrak{p}$ such that $f \notin \mathcal{V}(b)$. But this means $f \notin A_b$ which in turn implies $f \in \bigcup_{b \notin \mathfrak{p}} A_b = A_{\mathfrak{p}}$. □

Definition 29.0.12.
The germs of analytic functions at x is the completion of $\mathcal{O}_{X,x}$, denoted by $\mathcal{O}^{\wedge}_{X,x}$ with respect to its maximal ideal.

Remark 29.0.13.
We have the natural map $\mathcal{O}_{X,x} \to \mathcal{O}^{\wedge}_{X,x}$ but if $\mathcal{O}_{X,x}$ is Noetherian then this map is also injective.

29.1. Categories and functors

A category $\mathcal{C}$ consists of a collection $\mathrm{ob}(\mathcal{C})$ and for all $X, Y \in \mathrm{ob}(\mathcal{C})$, there is a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and a map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

satisfying

1. $\forall\ X \in \mathrm{ob}(\mathcal{C})\ \exists\ 1_X \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ such that $f \circ 1_X = 1_X \circ f = f$
2. $f \circ (g \circ h) = (f \circ g) \circ h$

A functor (contravariant) $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ is a function $\mathcal{F} : \mathrm{ob}(\mathcal{C}_1) \to \mathrm{ob}(\mathcal{C}_2)$ and a map of sets $\mathcal{F} : \mathrm{Hom}_{\mathcal{C}_1}(X, Y) \to \mathrm{Hom}_{\mathcal{C}_2}(\mathcal{F}(X), \mathcal{F}(Y))$ such that

1. $f(1_X) = 1_{\mathcal{F}(X)}$
2. $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$

To each category $\mathcal{C}$, we associate a category $\mathcal{C}^{\mathrm{op}}$ such that

$$\mathrm{ob}(\mathcal{C}) = \mathrm{ob}(\mathcal{C}^{\mathrm{op}})$$

and

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X, Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

Suppose $\mathcal{F}, \mathcal{F}' : \mathcal{C} \to \mathcal{C}'$ be two functors. Then, a natural transformation is $T : \mathcal{F} \to \mathcal{F}'$ consisting of the following data:

1. $\forall\, X \in \mathcal{C}, \exists\, T_X : \mathcal{F}(X) \to \mathcal{F}'(X)$,i.e., $T_X \in \mathrm{Hom}_{\mathcal{C}'}(\mathcal{F}(X), \mathcal{F}'(X))$ such that for all $f : X \to Y$, the diagram commutes

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "T_X"] \arrow[d, "\mathcal{F}(f)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F}'(X) \arrow[d, "\mathcal{F}'(f)"] \\
\mathcal{F}(Y) \arrow[r, "T_Y"'] & \mathcal{F}'(Y)
\end{tikzcd}$$

Given, $\mathcal{C}, \mathcal{C}'$ then $F(\mathcal{C}, \mathcal{C}')$ = all functors from $\mathcal{C}$ to $\mathcal{C}'$ is a category and $\mathrm{Hom}_{F(\mathcal{C},\mathcal{C}')}(F_1, F_2) =$ all natural transformations from F_1 to F_2

30. Lecture-4 (20th January, 2023): Category theory

30.1. Category theory contd..

30.1.1. Equivalence of categories

Two categories $\mathcal{C}, \mathcal{C}'$ are equivalent if there exists functors

$$\mathcal{F} : \mathcal{C} \to \mathcal{C}' \text{ and } \mathcal{G} : \mathcal{C}' \to \mathcal{C}$$

and natural transformations

$$T : \mathrm{id}_{\mathcal{C}} \to \mathcal{G} \circ \mathcal{F} \text{ and } T' : \mathrm{id}_{\mathcal{C}'} \to \mathcal{F} \circ \mathcal{G}$$

which are isomorphisms.

Example 30.1.1.

1. The category of categories with all morphisms being identity is equivalent to the category of sets.
2. The category
3. Consider the category of A-modules and let $B = M_n(A)$. We claim that $\mathbf{Mod}_{\mathrm{A}}$ and $\mathbf{Mod}_{\mathrm{B}}$ are equivalent. This is also known as Morita equivalence.

30.1.2. Products and Co-products

In partially ordered sets, neither product nor co-product might exist.

30.2. Pre-sheaves and Yoneda lemma

Suppose $\mathcal{C}$ is a category. Then a presheaf on $\mathcal{C}$ is a contravariant functor

$$\mathcal{F} : \mathcal{C} \to \mathbf{Sets}(\text{ or } \mathbf{Ab})$$

The category of presheaves on $\mathcal{C}$ is denoted by $\mathbf{Presh}(\mathcal{C})$

Suppose $X \in \mathrm{ob}(\mathcal{C})$. Then we can construct $h_X \in \mathbf{Presh}(\mathcal{C})$ such that $h_X(Y) =$

$\mathrm{Hom}_{\mathcal{C}}(Y, X)$. Hence, we have a functor

$$h : \mathcal{C} \to \mathbf{Presh}(\mathcal{C})$$

that sends $X \mapsto h_X$. This h is called the Yoneda functor.

Lemma 30.2.1 (Yoneda Lemma)**.**
For every pre-sheaf F on $\mathcal{C}$ and for all $X \in \mathrm{ob}(\mathcal{C})$, there exists a natural bijection

$$\theta_X : \mathrm{Hom}_{\mathbf{Presh}(\mathcal{C})}(h_X, F) \to F(X)$$

Proof. Suppose we are given $f : h_X \to F$. This is a natural transformation and thus we obtain

$$f(X) : h_X(X) \to F(X)$$

but $h_X(X) = \mathrm{Hom}_{\mathcal{C}}(X, X)$ and $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$. This implies $f(X)(\mathrm{id}_X) \in F(X)$ and therefore $\theta_X(f) = f(X)(\mathrm{id}_X) \in F(X)$.
Now, let us construct the inverse. Construct

$$\psi_X : F(X) \to \mathrm{Hom}_{\mathbf{Presh}(\mathcal{C})}(h_X, F)$$

Let $\alpha \in F(X)$, we want to define

$$h_X(Y) \to F(Y) \;\forall\; Y \in \mathcal{C}$$

But then $f \in h_X(Y) = \mathrm{Hom}_{\mathcal{C}}(Y, X)$ implies $F(X) \xrightarrow{F(f)} F(Y) \Rightarrow F(f)(\alpha) \in F(Y)$.
We can easily check that these two maps are inverses which completes the proof. □

Suppose $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ is a functor. Then, $\mathcal{F}$ is called faithful if

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \hookrightarrow \mathrm{Hom}_{\mathcal{C}'}(\mathcal{F}(X), \mathcal{F}(Y)) \;\forall\; X, Y \in \mathrm{ob}(\mathcal{C})$$

We say that $\mathcal{F}$ is full if this map is an epimorphism and $\mathcal{F}$ is an embedding if $\mathcal{F}$ is fully faithful.

Lemma 30.2.2.
Yoneda functor is an embedding.

Proof. By Yoneda lemma we have

$$\mathrm{Hom}_{\mathbf{Presh}(\mathcal{C})}(h_X, h_Y) = h_Y(X)$$

But since $h_Y(X) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$, the proof is complete. □

30.2.1. Adjoint functors

Suppose we have two functors

$$\mathcal{F} : \mathcal{C} \to \mathcal{C}' \text{ and } \mathcal{G} : \mathcal{C}' \to \mathcal{C}$$

The pair $(\mathcal{F}, \mathcal{G})$ is an adjoint pair if for all $X \in \text{ob}(\mathcal{C})$ and $Y \in \text{ob}(\mathcal{C}')$, there exists a natural transformation

$$\text{Hom}_{\mathcal{C}}(X, \mathcal{G}(Y)) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}'}(\mathcal{F}(X), Y)$$

Example 30.2.3. 1. Take $\mathcal{C}, \mathcal{D}$ to be the category $\mathbf{Mod}_R$ of R-modules. Define the functors

$$\begin{aligned} F &: \mathcal{C} \to \mathcal{D} & G &: \mathcal{D} \to \mathcal{C} \\ F(A) &= A \otimes_R N & G(B) &= \text{Hom}_R(N, B) \end{aligned}$$

Consider $\text{Hom}_R(A \otimes_R N, B)$ and $\text{Hom}_R(A, \text{Hom}_R(N, B))$. These are both in bijective correspondence, in fact they are isomorphic as R-modules. Hence, (F, G) is an adjoint pair. This is also called the Hom-tensor adjunction.

2. (F, G) with F the free functor and G the forgetful functor is also an adjoint pair.

Proposition 30.2.4.
Left adjoint and right adjoint have to be unique (if they exist).

Suppose we have an adjoint pair $(\mathcal{F}, \mathcal{G})$. Then, for every $X \in \text{ob}(\mathcal{C})$ we have (follows from adjoint-ness)

$$\text{Hom}_{\mathcal{C}}(X, \mathcal{G} \circ \mathcal{F}(Y)) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}'}(\mathcal{F}(X), \mathcal{F}(Y))$$

This implies there is a canonical map

$$u_X : X \to \mathcal{G} \circ \mathcal{F}(X)$$

and this in turn implies the existence of a natural transformation

$$u : \text{id}_{\mathcal{C}} \to \mathcal{G} \circ \mathcal{F}$$

called the unit of adjunction.
Similarly, for all $Y \in \text{ob}(\mathcal{C}')$ we have

$$\text{Hom}_{\mathcal{C}}(\mathcal{F} \circ \mathcal{G}(X), Y) \xrightarrow{\sim} \text{Hom}_{\mathcal{C}'}(\mathcal{G}(Y), \mathcal{G}(Y))$$

This implies the existence of a natural transformation

$$\epsilon : \text{id}_{\mathcal{C}'} \to \mathcal{G} \circ \mathcal{F}$$

called the co-unit of adjunction.

Definition 30.2.5.
It is a category $\mathcal{C}$ such that

1. it admits finite coproduct.
2. it has a zero product (both final and initial object).
3. $\mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathbf{Ab}$ such that
$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$
is bilinear.

Definition 30.2.6.
It is an additive category such that every map $f : X \to Y$ has a kernel and a cokernel.

31. Lecture-5 (23rd January, 2023): Etale morphisms

Let A be a commutative ring and M be a A-module.

Definition 31.0.1.
M is flat if

$$N \hookrightarrow N' \Rightarrow N \times M \hookrightarrow N' \times M$$

Definition 31.0.2.
M is faithfully-flat if M is flat and

$$N = 0 \Leftrightarrow N \times_A M = 0$$

Definition 31.0.3.
M is projective if it is a direct summand of a free A-module. Or equivalently,

$$\begin{aligned} N \twoheadrightarrow N' &\Rightarrow \mathrm{Hom}(M, N) \twoheadrightarrow \mathrm{Hom}(M, N') \\ \Leftrightarrow \mathrm{Ext}^i_A(M, N) &= 0 \ \forall i > 0 \ \forall \ N \end{aligned}$$

Lemma 31.0.4.
Suppose A is Noetherian and M is a finitely generated A-module. TFAE:

1. M is projective.
2. M is flat.
3. $M_{\mathfrak{m}}$ is flat for all maximal ideals $\mathfrak{m}$.
4. $M_{\mathfrak{m}}$ is free for all maximal ideals $\mathfrak{m}$.

Proof. $1 \Rightarrow 2$ is obvious. $2 \Rightarrow 3$ is a local property. $3 \Rightarrow 4$ is done in commutative algebra. $4 \Rightarrow 1$: Note that $4 \Rightarrow 3 \Rightarrow 2$. So we just prove that $2 \Rightarrow 1$. Thus, enough to show that

$$\begin{aligned} \mathrm{Ext}^i_A(M, N) &= 0 && \forall \ N \forall \ i > 0 \\ \Leftrightarrow \mathrm{Ext}^i_A(M, N)_{\mathfrak{m}} &= 0 && \forall \mathfrak{m} \text{ maximal ideals} \\ \Leftrightarrow \mathrm{Ext}^i_{A_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) &= 0 \end{aligned}$$

This completes the proof. □

Let k be a field and A a k-algebra.

Definition 31.0.5.
A is called separable over k if $A \otimes_k k'$ is reduced for all field extensions k'/k.

Lemma 31.0.6.
A is separable over k iff every finitely generated subalgebra is separable over k.

Proposition 31.0.7.
Assume that A is finite dimensional over k. TFAE:

1. A is separable over k.
2. $\overline{A} := A \otimes_k \overline{k} = \prod_{i=1}^n \overline{k}$.
3. $A = \prod_{i=1}^n k_i$ where k_i/k is a finite separable field extension.
4. The trace form $A \times A \to k((w, w') \mapsto \mathrm{Tr}_{A/k}(ww'))$ is non-degenerate.

Proof. $(1 \Rightarrow 2)$

$$\bar{A} = \prod_{i=1}^n A_{\mathfrak{m}_i} = \prod_{i=1}^n \bar{k}$$

$(2 \Rightarrow 3)$

$$\frac{A}{\mathrm{Nil}(A)} = \prod_{i=1}^n k_i$$

where k_i is finite field extension of k.
$\bar{A}$ is reduced $\Rightarrow \mathrm{Nil}(A) = 0 \Rightarrow A \simeq \prod_{i=1}^n k_i$.
Now, say that A is a finite field extension of k. Say $A = k'$. We have the following inclusions

$$k \hookrightarrow k'' \hookrightarrow k'$$

where k' is the maximal purely inseparable extension of k inside k'. Need to show that $k'' = k$.
Can make

$$k'' = \frac{k[t]}{t^{p^n} - \alpha}, \alpha \in k$$

Therefore

$$k' \otimes_k \bar{k} = \frac{\bar{k}[t]}{t^{p^n} - \beta^{p^n}} = \frac{\bar{k}[t]}{(t - \beta)^{p^n}}$$

for some $\beta \in \bar{k}$. Since the last quotient is not reduced therefore $k'' = k$.

($3 \Rightarrow 4$) done in comm. alg.

($4 \Rightarrow 1$)

$$\begin{aligned} \varphi : A \times A &\to k \\ (w, w') &\mapsto \mathrm{Tr}_{A/k}(ww') \end{aligned}$$

is non-degenerate.
Let $\{w_1, \ldots, w_n\}$ be a k-basis of A.
Consider $B = (\mathrm{Tr}(w_i w_j))$ and $\mathrm{disc}_k(A) = \det(B) \neq 0 \Rightarrow \mathrm{disc}_{\bar{k}}(\bar{A}) \neq 0$.
Suppose that $\mathrm{Nil}(\bar{A}) \neq 0$. Suppose $\{w_1, \ldots, w_m\}$ be a $\bar{k}$-basis of $\bar{A}$. Extend this to a basis $\{w_1, \ldots, w_m, w_{m+1}, \ldots, w_n\}$ of A such that $w_i w_j$ is nilpotent for all i, j. This implies $\det(\mathrm{Tr}(w_i w_j)) = 0$ which is a contradiction. Therefore $\mathrm{Nil}(\bar{A}) = 0$. □

31.1. Kahler Differentials

Some reference materials.

- advanced1 advanced2 advanced3
- basic
- Matsumura book on Commutative algebra and Commutative ring theory
- Chapter 4 of T.A. Springer's Linear Algebraic Groups.

Let A be a commutative ring and M an A-module. A derivative $D : A \to M$ is an abelian group homomorphism such that

$$D(ab) = aD(b) + D(a)b$$

Let

$$\mathrm{Der}(A, M) = \text{ the set of derivations from } A \text{ to } M$$

If A is a k-algebra where k is a commutative ring, then we say that D is a k-derivation if $D(k) = 0$

More notation: Let $\mathrm{Der}_k(A, M)$ be the set of all k-derivations and $\mathrm{Der}_k(A) = \mathrm{Der}_k(A, A)$

We can make $\mathrm{Der}(A, M)$ is an A-module so that

$$(a \cdot D)(b) = aD(b)$$

Suppose

$$D : A \to M$$

then

$$\begin{aligned} D(\mathbb{Z}) &= 0 \\ \mathrm{Der}(A, M) &= \mathrm{Der}_{\mathbb{Z}}(A, M) \end{aligned}$$

Take $D, D' \in \mathrm{Der}_k(A)$, then we can define the bracket $[-,-]$ as

$$[D, D'] = DD' - D'D$$

This converts $\mathrm{Der}_k(A)$ into a Lie algebra

Remark 31.1.1.
1. $d(a^n) = na^{n-1}d(a)$
2. $d^n(ab) = \sum_{i=0}^{n} \binom{n}{i} d^i a d^{n-i} b$

In particular, if $\mathrm{char}(A) = p > 0$ then

1. $d^p(ab) = ad^p b + bd^p(a) \Rightarrow d^p$ is a k-derivation.
2. $d^p(a+b) = d^p a + d^p b$

Clearly, $\mathrm{Der}_k(A, -) : A - \mathrm{mod} \to A - \mathrm{mod}$ is a covariant functor.

Proposition 31.1.2.
$\mathrm{Der}_k(A, -)$ is a representable functor.

Proof. □

32. Lecture-6 (25th January, 2023):Kahler Differentials

32.1. Differentials and Derivations

Theorem 32.1.1.
There exists an unique A-module (upto isomorphism) $\Omega'_{A/k}$ with a k-derivation $d_{A/k} : A \to \Omega'_{A/k}$ such that for all A-modules M and a k-derivation $D : A \to M$, $\exists! A$-linear map $\varphi : \Omega'_{A/k} \to M$ such that $D = \varphi \circ d_{A/k}$

$$\mathrm{Der}_k(A, M) \xrightarrow{\sim} \mathrm{Hom}_A(\Omega'_{A/k}, M)$$

Proof. Take $\Omega^1_{A/k}$ to be the free A-module generated by symbols $\{da : a \in A\}$ modulo the relations $d(a+b) - d(a) - d(b) = 0$ and $d(ab) = ad(b) + bd(a)\ \forall\, a, b \in A$. $\square$

Definition 32.1.2.
A square zero extension of k-algebras is a surjection of k-algebras $g : B \twoheadrightarrow C$ such that $M^2 = 0$ where $M = \ker(g)$.
We can think of B as the manifold C plus some other tangent directions. B is some kind of thickening of C in the spec level.

Example 32.1.3.
Suppose $M \in \mathbf{Mod}_A$ and $B = A \oplus M$. Addition and multiplication are defined as follows:

$$(a, m) + (a', m') = (a + a', m + m')$$
$$(a, m) \cdot (a', m') = (aa', am' + a'm)$$

Here,

$$0 \longrightarrow M \longrightarrow B \xrightarrow{\varphi} A \longrightarrow 0$$

$$(a, m) \longmapsto a$$

is a square zero extension. We wish to ask when does the following lift exist.

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & B \arrow[r, "\varphi"] & C \arrow[r] & 0 \\ & & A \arrow[u, dashed, "?"] \arrow[ur, "g"'] & & \end{tikzcd}$$

Suppose we are given a lift $h : A \to B$ of g and h' is another lift of g. Then,

$$\begin{tikzcd} D := h - h' : A \arrow[r] \arrow[dr] & B \\ & M \arrow[u, hook] \end{tikzcd}$$

M is a C-module ($M = M/M^2 = M \otimes B/M = C$ is a C-module). So, M is also an A-module via the map g.

Claim: $D \in \mathrm{Der}_k(A, M)$

Proof. □

Conversely, if $D \in \mathrm{Der}_k(A, M)$, then $h' = h + D$ is also a lift.

Proof of main theorem. 1. Consider the map $A \otimes_k A \xrightarrow{\mu} A$ such that $a \otimes b \mapsto ab$. μ is a surjective k-algebra homomorphism. Let $I = \ker(\mu)$ and $B = A \otimes_k A/I^2$. We obtain the following square zero extension

$$0 \longrightarrow I/I^2 \longrightarrow B \xrightarrow{\varphi} A \longrightarrow 0$$

Let $\Omega'_{A/k} := I/I^2$ is the module of Kahler differentials.
$\Omega_{A/k}$ the canonical sheaf of diagonal embedding of $X \hookrightarrow X \times X$
Define

$$\alpha_1 : A \to B\alpha_1(a) = a \otimes 1 \pmod{I^2}$$
$$\alpha_2 : A \to B\alpha_2(a) = 1 \otimes a \pmod{I^2}$$

We obtain the following diagram that commutes

$$\begin{tikzcd} 0 \arrow[r] & \Omega'_{A/k} \arrow[r] & B \arrow[r, "\varphi"] & A \arrow[r] & 0 \\ & & A \arrow[u, "\alpha_i"] \arrow[ur, "\mathrm{id}"'] & & \end{tikzcd}$$

Next, define $d_{A/k} = \alpha_1 - \alpha_2$.
Let $M \in \mathbf{Mod}_A$ and $D : A \to M$ a k-derivation.

Now, define

$$\begin{aligned}\theta : A \otimes_k A &\to A * M (= A \oplus M) \text{ a square zero extension} \\ a \otimes b &\mapsto (ab, aDb)\end{aligned}$$

Claim: $\theta(I) \hookrightarrow M$

Proof. Suppose $\sum x_i \otimes y_i \in I \Rightarrow \sum x_i y_i = 0 \in A$. This implies $\theta(\sum x_i \otimes y_i) = (\sum x_i y_i = 0, \sum x_i D y_i) \in M$. Therefore,

$$\theta(I/I^2) \hookrightarrow M/M^2 = M$$

Thus θ descends to a map

$$\tilde{\theta} : I/I^2 \to M$$

or $\tilde{\theta} : \Omega'_{A/k} \to M$ □

Claim: $\tilde{\theta}$ is unique such that $\tilde{\theta} \circ d_{A/k} = D : A \to M$

Proof. Suffices to show that $\Omega'_{A/k}$ is generated by $\langle da : a \in A \rangle$ as an A-module.

$$\begin{aligned} a \otimes a' &= (a \otimes 1)(-a' \otimes 1 + 1 \otimes a') + aa' \otimes 1 \\ \alpha \in \Omega'_{A/k} \Rightarrow \alpha &= \sum x_i \otimes y_i \qquad \text{such that } \sum x_i y_i = 0 \\ \Rightarrow \alpha &= \sum x_i dy_i \end{aligned}$$

□

□

Corollary 32.1.4.

$$\Omega'_{A/k} = \frac{\text{free module on } da}{\text{additivity + Leibnitz rule}}$$

In particular, $\mathrm{Der}_k(A) = \Omega'^{*}_{A/k} = \mathrm{Hom}(\Omega'_{A/k}, A) = T_{A/k}$ (tangent space)

Definition 32.1.5.
We say that A is formally smooth over k if given any square zero extension

$$0 \longrightarrow M \longrightarrow B \xrightarrow{\varphi} A \longrightarrow 0$$

and a diagram of k-algebras, there exists a lifting $\tilde{g}$ of g

$$\begin{tikzcd} k \arrow[r, "f"] \arrow[d] & C \arrow[d, "g"] \arrow[dl, dashed, "\tilde{g}"] \\ B \arrow[r, "\varphi"'] & A \end{tikzcd}$$

Definition 32.1.6.
We say that A is formally unramified over k if g has atmost one lift.
We say that A is formally étale over k if A is formally smooth and formally unramified.

Definition 32.1.7.
We say that A is smooth (resp. unramified, étale) if it is formally smooth (unramified, étale) and finite type over k.

Exercise 32.1.8. $A = k[X_1, \dots, X_n], \Omega'_{A/k} = ?$
Claim: *There is a canonical isomorphism of A-modules*

$$\theta : \underbrace{AdX_1 \oplus \cdots \oplus AdX_n}_{F} \xrightarrow{\sim} \Omega'_{A/k}$$

Lemma 32.1.9.
Suppose $U \subseteq A$ is any set that generates A as k-algebra. Then, $\Omega'_{A/k}$ is generated by $\{da : a \in A\}$ as A-module.

Proof. □

This lemma implies the map is surjective.
Next, define $D_i : A \to A$ such that $f \mapsto \dfrac{\partial d}{\partial x_i}(f)$.
This gives an unique A-linear map $\psi_i : \Omega'_{A/k} \to A$. Define $\psi : \Omega'_{A/k} \to F$ such that $\psi = \sum_{i=1}^{n} \psi_i$. This implies $\psi \circ \theta = \mathrm{id}_F$. Hence, ψ is injective.

33. Lecture-7 (30th January, 2023): Module of differentials

Lemma 33.0.1.
A is formally unramified iff $\Omega^1_{A/k} = 0$.

Proof. Suppose that $\Omega^1_{A/k} = 0$. Let

$$0 \longrightarrow M \longrightarrow B \xrightarrow{\varphi} C \longrightarrow 0$$

be a square zero extension. We had seen that all liftings of $f : A \to C$ differ by $\mathrm{Der}_k(A, M) = \mathrm{Hom}_A(\Omega^1_{A/k}, M)$. This implies there is atmost one lifting of f and this concludes what we want.

For the other direction, suppose A is formally unramified over k. Recall

$$\mu : A \otimes_k A \to A$$

$$I = \ker(\mu)$$

and

$$0 \longrightarrow I/I^2 = \Omega^1_{A/k} \longrightarrow B = (A \otimes_k A)/I^2 \xrightarrow{\varphi} C \longrightarrow 0$$

We had two liftings from A to B, α_1, α_2 namely $\alpha_1(a) = a \otimes 1, \alpha_2(a) = 1 \otimes a$ and $d_{A/k} = \alpha_1 - \alpha_2$. Since A is formally unramified, $d_{A/k} = 0$ which implies $\Omega^1_{A/k} = 0$. $\square$

Exercise 33.0.2. *Suppose K/k be a finite separable extension. We claim that this extension is formally unramified.*

Lemma 33.0.3.
If $k \xrightarrow{f} A$ is of finite type (k is a ring), then $\Omega^1_{A/k}$ is a finitely generated A-module.

Proof. $\square$

Example 33.0.4.
If A is a commutative ring, S a multiplicatively closed subset of A, $B = S^{-1}A$. Then, $A \to B$ is formally etale.

Formally unramified: Enough to show that

$$d_{A/k}(fg^{-1}) = 0 \; \forall \; f \in A, g \in S$$

But this means

$$\begin{aligned} d_{B/A}(fg^{-1}) &= f d_{A/B}(g^{-1}) + g^{-1} d_{B/A}(f) \\ &= f d_{A/B}(g^{-1}) \\ g d_{A/B}(g^{-1}) + g^{-1} d_{A/B}(g) &= 0 \\ \Rightarrow g d_{A/B}(g^{-1}) &= 0 \\ \Rightarrow d_{B/A}(g^{-1}) &= 0 \end{aligned}$$

Next,
Formally unramified:

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & C \arrow[r, "\varphi"'] & C' \arrow[r] & 0 \\ & A \arrow[ur, "g"] \arrow[r, "\theta"'] & B \arrow[u, dashed, "\tilde{f}"'] \arrow[ur, "f"'] & & \end{tikzcd}$$

$$\begin{aligned} \tilde{f} \text{ exists } &\Leftrightarrow g(s) \in C^{\times} \; \forall \; g \in S \\ &\Leftrightarrow \varphi g(s) = f\theta(s) \in C'^{\times} \end{aligned}$$

Then use the lemma stated after this example.

Lemma 33.0.5.
If

$$0 \longrightarrow I \longrightarrow C \xrightarrow{\varphi} C' \longrightarrow 0$$

is an extension of rings such that I is nilpotent. Then $a \in C^{\times} \Leftrightarrow \varphi(a) \in C'^{\times}$.

Proof. □

Theorem 33.0.6 (First fundamental theorem for module of differentials)**.**
Let

$$k \xrightarrow[f]{} A \xrightarrow[g]{} B$$

be ring homomorphisms. Then,

$$\Omega^1_{A/k} \otimes_A B \xrightarrow{\alpha} \Omega^1_{B/k} \xrightarrow{\beta} \Omega^1_{B/A} \longrightarrow 0$$

is exact. Moreover, it is split exact if B is formally smooth over A. Here, $\alpha(ad_{A/k}a' \otimes b') = bad_{B/k}(a'), \beta(ad_{B/k}b) = ad_{B/A}(b)$.

Proof. We know that a sequence of B-modules

$$N' \longrightarrow N \longrightarrow N''$$

is exact iff

$$\mathrm{Hom}_B(N'', M) \longrightarrow \mathrm{Hom}_B(N, M) \longrightarrow \mathrm{Hom}_B(N', M)$$

is exact for all B-module M.

Thus, we just need to check that

$$\mathrm{Hom}_B(\Omega^1_{B/A}, M) \longrightarrow \mathrm{Hom}_B(\Omega^1_{B/k}, M) \longrightarrow \mathrm{Hom}_B(\Omega^1_{A/k} \otimes_A B, M) = \mathrm{Hom}_A(\Omega^1_{A/k}, M)$$

is exact. But this is equivalent to checking

$$\mathrm{Der}_A(B, M) \xrightarrow{\beta^*} \mathrm{Der}_k(B, M) \xrightarrow{\alpha^*} \mathrm{Der}_k(A, M)$$

is exact.

Next, assume that B is formally smooth over A. We need to show that α^* is surjective. Let $D \in \mathrm{Der}_k(A, M)$. We know that the diagram

$$\begin{tikzcd} B \arrow[r, "\mathrm{id}"] & B \\ A \arrow[u, "g"] \arrow[r, "\varphi"'] & B * M = B \oplus M \arrow[u, "p_1"'] \end{tikzcd}$$

commutes. But $B * M$ is a square zero extension. Thus, we get a map $B \to B * M$ such that diagram

$$\begin{tikzcd} B \arrow[r, "\mathrm{id}"] \arrow[dr, "\theta"] & B \\ A \arrow[u, "g"] \arrow[r, "\varphi"'] & B * M = B \oplus M \arrow[u, "p_1"'] \end{tikzcd}$$

commutes. Here, $\varphi(a) = (ga, Da)$. We write $\theta(b) = (b, D'b)$.

Claim: D' is a k-derivation from B to M.

It is clear that $D' \circ g = D$. This is equivalent to a B-linear map $\alpha' : \Omega^1_{B/k} \to M$. Define

$$D : A \to \Omega^1_{A/k} \otimes_A B$$
$$D(a) = d_{A/k}(a) \otimes 1$$

Check that $D \in \operatorname{Der}_k(A, \Omega^1_{A/k} \otimes_A B)$. This implies the existence of an extension $D' : B \to \Omega^1_{A/k} \otimes_A B$ such that $D' \circ g = D$ iff a B-linear map $\alpha' : \Omega^1_{B/k} \to \Omega^1_{A/k} \otimes_A B$ such that $\alpha' \circ g = \alpha$.

Claim: $\alpha' \circ \alpha = \mathrm{id}$

This concludes the proof. □

Suppose

$$k \xrightarrow{f} A \xrightarrow{g} B$$

From the previous theorem, we get

$$\Omega^1_{A/k} \otimes_A B \xrightarrow{\alpha} \Omega^1_{B/k} \xrightarrow{\beta} \Omega^1_{B/A} = 0 \longrightarrow 0$$

is exact. Or rather

$$\Omega^1_{A/k} \otimes_A B \xrightarrow{\alpha} \Omega^1_{B/k} \longrightarrow 0$$

is exact. What is the kernel of this map?

Theorem 33.0.7 (Second fundamental theorem of module of differentials)**.**
Let $I = \ker(A \twoheadrightarrow B)$. Then, there exists an exact sequence

$$I/I^2 \xrightarrow{\delta} \Omega^1_{A/k} \otimes_A B \twoheadrightarrow \Omega^1_{B/k} \longrightarrow 0$$

where $\delta(a) = d_{A/k}(a) \otimes 1$. Moreover, this sequence is split exact if B is formally smooth over k.

Example 33.0.8.
Let $B = k[X_1, X_2, \dots, X_n]/\langle f_1, \dots, f_n \rangle$. Then, what is $\Omega^1_{B/k}$.

If $A = k[X_1, X_2, \dots, X_n]$. Then,

$$\Omega^1_{A/k} = A dx_1 \oplus \cdots \oplus A dx_n$$

$X = \operatorname{Spec}(A), Y = \operatorname{Spec}(B = A/I)$

34. Lecture-8 (1st February, 2023):Differentials

contd..

Theorem 34.0.1 (Second fundamental theorem of module of differentials)**.**
Let $I = \ker(A \twoheadrightarrow B)$. Then, there exists an exact sequence

$$I/I^2 \xrightarrow{\delta} \Omega^1_{A/k} \otimes_A B \twoheadrightarrow \Omega^1_{B/k} \longrightarrow 0$$

where $\delta(a) = d_{A/k}(a) \otimes 1$. Moreover, this sequence is split exact if B is formally smooth over k.

Proof. Suffices to show that for any B-module M, the sequence

$$\mathrm{Hom}_B(\Omega^1_{B/k}, M) \xrightarrow{\alpha^*} \mathrm{Hom}_B(\Omega^1_{A/k} \otimes_A B, M) \xrightarrow{\delta^*} \mathrm{Hom}_B(I/I^2, M) \qquad (*)$$

is exact.
Well-definedness of δ:

$$\begin{aligned}
\delta(ab) &= d_{A/k}(ab) \otimes 1 \\
&= (ad_{A/k}(b) + bd_{A/k}a) \otimes 1 \\
&= ad_{A/k}b \otimes 1 + bd_{A/k} \otimes 1 \\
&= d_{A/k}b \otimes g(a) + d_{A/k} \otimes g(b) \text{ since} \\
&= 0 + 0 \\
&= 0
\end{aligned}$$

Now, we have

$$\mathrm{Der}_k(B, M) \xrightarrow{\alpha^*} \mathrm{Hom}_B(\mathrm{Der}_k(A, M) \xrightarrow{\delta^*} \mathrm{Hom}_B(I/I^2, M) \qquad = (*)$$

Take $D \in \mathrm{Der}_k(A, M)$, then $D(a) = 0 \;\forall\; a \in I$. Since,

$$\begin{tikzcd}
A \arrow[r, two heads] \arrow[d] & B \\
M &
\end{tikzcd}$$

Therefore there exists $D' \in \mathrm{Der}_k(B, M)$ such that $D' \circ g = D$. Thus, sequence is exact.

Now, assume that B is formally smooth over k. Look at the exact sequence of B-modules which is a square 0 extension of B.

$$\begin{tikzcd} 0 \arrow[r] & I/I^2 \arrow[r] & A/I^2 \arrow[r, "g'"] & B \arrow[r] & 0 \\ & & & B \arrow[u, "\mathrm{id}"'] \arrow[ul, dashed, "\exists\, h"] & \end{tikzcd}$$

This implies the existence of a k-algebra homomorphism $h : B \to A/I^2$ such that $g \circ h = \mathrm{id}_B$. Now, consider the map

$$h \circ g' : A/I^2 \to A/I^2$$

that kills I/I^2 so that $g'(1 - hg') = 0$.
Let $D' = 1 - hg' : A/I^2 \to A/I^2$. Check that this is a k-derivation of A/I^2.
Let $\psi \in \mathrm{Hom}_B(I/I^2, M)$ and consider the maps

$$D := A \longrightarrow A/I^2 \xrightarrow{D'} A/I^2 \xrightarrow{\psi} M$$

Since D' is a derivation, check that $D \in \mathrm{Der}_k(A, M)$. This means we get an A-linear map $\varphi : \Omega^1_{A/k} \to M$ which is equivalent to getting a map $\varphi : \Omega^1_{A/k} \otimes_A B \to M$ such that $\delta * (\phi) = \psi$.
Finally, take $M = I/I^2$ and $\psi = \mathrm{id}$.
$\Rightarrow$ $\exists$ a map from $\varphi : \Omega^1_{A/k} \to I/I^2 \Leftrightarrow \varphi : \Omega^1_{A/k} \otimes_A B \to I/I^2$ and $\varphi \circ \delta = \mathrm{id}_{I/I^2}$.
Finally,

$$\begin{tikzcd} 0 \arrow[r] & I/I^2 \arrow[r, "\delta"] & \Omega^1_{A/k} \otimes_A B \arrow[r] \arrow[l, bend left, "\varphi"] & \Omega^1_{B/k} \arrow[r] & 0 \end{tikzcd}$$

or the sequence splits. This concludes the proof. $\square$

Corollary 34.0.2.
Let $B = k[X_1, \ldots, X_n]/\langle f_1, \ldots, f_r \rangle$. Thus,

$$\Omega^1_{B/k} = \frac{Bd\bar{x}_1 + \cdots + Bd\bar{x}_n}{\langle df_1, \cdots, df_r \rangle}$$

with $df_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial x_j}$ modulo I^2.

Corollary 34.0.3.
Suppose $k \xrightarrow{f} A \xrightarrow{g} B$ are k-algebra homomorphisms such that A, B are

formally smooth over k. Then,

$$0 \longrightarrow I/I^2 \xrightarrow{\delta} \Omega^1_{A/k} \otimes_A B \longrightarrow \Omega^1_{B/k} \longrightarrow 0$$

Definition 34.0.4. 1. Let $k \xrightarrow{f} A$ be a ring homomorphism. We say that A is unramified over k if it is formally unramified and of finite type.

2. Let $q \in \mathrm{Spec}(A)$. Then, we say that A is unramified over k at q if there exists $g \in A \backslash k$ such that the map $f : K \to A_g$ is unramified.

3. We say that A is locally unramified over k if it is unramified at every $q \in \mathrm{Spec}(A)$.

The question is whether the $1, 3$ conditions are equivalent. $1 \Rightarrow 3$ is known. So we need to check if $3 \Rightarrow 1$. Suppose that for all $q \in \mathrm{Spec}(A)$ there exists $g \notin q$ such that $k \to A_g$ is unramified. Let $U_q = \mathrm{Spec}(A_g)$.
Since X is locally unramified, we must have $X = \bigcup\limits_q U_q$ but remember that X is spectral and thus quasi-compact. This implies that there is a finite subcover $X = \bigcup\limits_{i=1}^{n} U_{q_i} \Rightarrow A = \langle g_1, \ldots, g_n \rangle \Rightarrow A$ is of finite type over k.

Why is $\Omega^1_{A/k} = 0$?

Notice that $\Omega^1_{A_g/k} = 0$ and we have an exact sequence

$$\Omega^1_{A/k} \otimes_A A_g \longrightarrow \Omega^1_{A_g/k} \longrightarrow \Omega^1_{A_g/A} \longrightarrow 0$$

But $\Omega^1_{A_g/A}$ is formally étale and is therefore 0. This transforms the above sequence to

$$0 \longrightarrow \Omega^1_{A/k} \otimes_A A_g \longrightarrow \Omega^1_{A_g/k} \longrightarrow 0$$

Hence, $(\Omega^1_{A/k})_g = \Omega^1_{A_g/k} \, \forall \, g \Rightarrow \Omega^1_{A/k} = 0$. This finishes the proof.

Proposition 34.0.5. 1. Unramified maps are preserved under base change.

2. Unramified maps are preserved under composition.

3. Principal localisations are unramified.

4. Any surjection is unramified.

Proof. Let us prove 3. Look at

$$
\begin{tikzcd}
k \arrow[r, "f"] \arrow[d, "g"'] & A \arrow[d, "g'"] \\
B \arrow[r, "f'"'] & A \otimes_k B = C
\end{tikzcd}
$$

More generally, we have

Lemma 34.0.6.
$\Omega^1_{C/B} \simeq \Omega^1_{A/k} \otimes_k B$

Proof. Look at the maps as a consequence of first fundamental exact sequence

$$\Omega^1_{A/k} \longrightarrow \Omega^1_{C/k} \twoheadrightarrow \Omega^1_{C/B}$$

This gives the map $\Omega^1_{A/k} \otimes_k B \xrightarrow{\alpha} \Omega^1_{C/B}$. Now, we wish to construct an inverse map $\Omega_{C/B} \to \Omega^1_{A/k} \otimes_k B$. Look at the map $d_{A/k} : A \to \Omega^1_{A/k}$. This gives a map

$$
\begin{aligned}
d' : C = A \otimes_k B &\to \Omega^1_{A/k} \otimes_k B \\
a \otimes b &\mapsto d_{A/k} a \otimes b
\end{aligned}
$$

Check that d' is a B-derivation. This implies the existence of a map $\beta : \Omega^1_{C/B} \to \Omega^1_{A/k} \otimes_k B$. Check that $\alpha \circ \beta = \beta \circ \alpha = \mathrm{id}$. The proof is complete. □

□

Lemma 34.0.7.
Let $f \xrightarrow{f} A$ be a finite type morphism. Let $\mathfrak{p} \in \mathrm{Spec}(A)$. Then, A is unramified over k at $\mathfrak{p}$ if $\Omega^1_{A/k} \otimes_A k(\mathfrak{p}) = 0$ where $k(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{m}_\mathfrak{p}$.

Proof. It suffices to show that $(\Omega^1_{A/k})_\mathfrak{p} = 0$ since A is of finite type. We can deploy Nakayama lemma to conclude what we want. We just need to check that localisation of finitely generated implies finitely generated. □

35. Lecture-9 (6th February, 2023): Differentials

35.1. Differentials contd...

Recall that given a ring homomorphism $k \xrightarrow{A}$, A is unramified iff A is of finite type and $\Omega^1_{A/k} = 0$.

Also, for a finite type algebra A, if $\mathfrak{p} \in \operatorname{Spec}(A)$ such that $\Omega^1_{A/k} \otimes_A k(\mathfrak{p}) = 0$ then A is unramified at $\mathfrak{p}$.

In particular, if $\Omega^1_{A/k} \otimes_A k(\mathfrak{p}) = 0\ \forall \mathfrak{p} \in \operatorname{Spec}(A)$, then A is unramified.

Lemma 35.1.1.
1. Say A is a commutative ring and $I \subseteq A$ is a finitely generated ideal such that $I = I^2$. Then there exists an idempotent $e \in A$ such that $I = \langle e \rangle$.
2. $A/I = A_{e'}$ where $e'(1-e) = 0$.
3. $\mathcal{V}(I)$ is open in $\operatorname{Spec}(A)$.

Proof. Since $I = I^2$ and I is f.g. then Nakayama lemma implies the existence of an $a \in I$ such that $(1+a)I = 0$. Set $f = 1 + a$. Then $f^2 = ff = f(1+a) = f + af = f$.

Next, take $e' = f$ and $e = 1 - f$. Then $\forall\, b \in I$ we have $b = (1-f)b = eb \Rightarrow I = \langle e \rangle$.

In particular, the map

$$A \to \frac{A}{\langle e \rangle} \times \frac{A}{\langle e' \rangle}$$

is an isomorphism. This helps us conclude that $A/\langle e \rangle = A_{e'}$.

And, $V(I) = \operatorname{Spec}(A/I) = \operatorname{Spec}(A_{e'}) \hookrightarrow \operatorname{Spec}(A)$ is open. $\square$

Corollary 35.1.2.
If $k \xrightarrow{f} A$ is unramified, then the diagonal map $\Delta_X : X \hookrightarrow X \times_Y X$ is a closed and open immersion where $X = \operatorname{Spec}(A)$ and $Y = \operatorname{Spec}(k)$.

Proof. Recall that by definition $\Omega^1_{A/k} = I/I^2$ where I is the ideal of X inside $X \times_Y X$. The module of differential is of f.g. since A is of finite type.

Red Flag: I need not be f.g. as required in our previous lemma. So, WE HAVE TO change the definition of unramified by replacing finitely presented instead of finite type. So, work with this definition.

Working with the new definition, we know that $\Omega^1_{A/k} = 0 \Rightarrow I = I^2$ and hence from previous lemma, we know that this immersion is open. $\square$

Lemma 35.1.3.
Let $(A, \mathfrak{m})$ be a local ring which is a k-algebra for some field k such that $A/\mathfrak{m} = k$. Then, $\mathfrak{m}/\mathfrak{m}^2 \simeq \Omega^1_{A/k} \otimes_A k$

Proof. We have an exact sequence

$$0 \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \Omega^1_{A/k} \otimes_A k \longrightarrow \Omega^1_{k/k} \longrightarrow 0$$

Here, k sits inside A and acts on A as well, hence the sequence splits and is formally smooth. Now, using a result from before, we know that $\Omega^1_{k/k} = 0$ and hence the proof is complete. $\square$

Lemma 35.1.4.
Suppose A is as in prev. lemma such that A is essentially of finite type (localisation of a finite type) over k. Then, $\Omega^1_{A/k}$ is a free A-module of rank$=\dim(A)$ iff A is a regular ring.

Proof. ($\Rightarrow$) By previous lemma, $\mathfrak{m}/\mathfrak{m}^2$ is free k-module of rank $n = \dim(A)$. Now apply Nakayama to observe that $\mathfrak{m}$ is generated by n generators which is the minimal number required. Hence, A is regular.
Caution: Here, $\mathfrak{m}$ is f.g. requires Noetherian-ness which is guaranteed by the essentially finite-ness.

($\Leftarrow$) Suppose A is regular. Therefore $\mathfrak{m} = \langle x_1, x_2, \ldots, x_n \rangle$ where $n = \dim A$. By previous lemma, $\dim_k(\Omega^1_{A/k} \otimes_A k) = n$.

Let K be the quotient field of A (makes sense because a regular local ring is integral domain). This implies $\dim_K(\Omega^1_{K/k}) = \dim_K(\Omega^1_{A/k} \otimes_A K) = tr.\deg_k(K) = \dim(A) = n$.

PLEASE STOP: The third equality is because of Noether Normalisation theorem. The second equality is some other very non-trivial fact. Just take it for granted for the time being. We will justify it later.
Detour:

Lemma 35.1.5.
Let A be a local integral domain with fraction field K and residue field k. Let M be a f.g. A-module such that $\dim_k(M \otimes_A k) = \dim_K(M \otimes_A K) = n < \infty$. Then, M is a free module of rank n.

Proof. By Nakayama, can find a surjection $\phi : F = A^n \twoheadrightarrow M$. Let $N = \ker(\phi)$. This gives us the exact sequence

$$0 \longrightarrow N \longrightarrow F \xrightarrow{\phi} M \longrightarrow 0$$

which implies

$$0 \longrightarrow N_K \longrightarrow F_K \xrightarrow{\phi} M_K \longrightarrow 0$$

is exact.

We know that $F_k \to M_K$ is a surjection of vector spaces of same dimension (n). Hence, $N_K = 0$. But, N is torsion free therefore $N \hookrightarrow N_K = 0$. □

The proof is now complete using this lemma. □

Corollary 35.1.6.
If $\Omega^1_{A/k} = 0$, then $A = k$.

Lemma 35.1.7.
Let $k \hookrightarrow L$ be an algebraic extension of fields which is separable. Then, L/k is formally unramified.

Proof. Can assume that L/k is finite, and

Lemma 35.1.8.
If $A = \varinjlim_{i \in I} A_i \Rightarrow \Omega^1_{A/k} = \varinjlim_{i \in I} \Omega^1_{A_i/k}$

Proof. □

Can choose a primitive element $\alpha \in L$. Let $f(X)$ be the minimal polynomial of $\alpha \in k[X]$ such that $f(\alpha) = 0$ and $f'(\alpha) \in L^\times$. Now, write $A = k[X]$ and $I = \langle f(X) \rangle \in A \Rightarrow L = k[X]/\langle f(X) \rangle$. By the second fundamental exact sequence, we get that

$$I/I^2 \xrightarrow{\delta} \Omega^1_{A/k} \otimes_A L \xrightarrow{\phi} \Omega^1_{L/k} \longrightarrow 0$$

But $\Omega^1_{A/k} \otimes_A L = Ld\bar{X}$

And, $\delta(f) = \dfrac{\partial}{\partial X} f|_L d\bar{X} = f'(\alpha)d\bar{X}$. This implies δ is an isomorphism (takes basis to basis). $\Rightarrow \Omega^1_{L/k} = 0$

□

36. Lecture-10 (8th February, 2023):

Part V.

Topics in Analytic Number Theory

37. Lecture-1: Hardy-Littlewood proof of infinitely many zeros on the line $\Re(s) = 1/2$

38. Lecture-2:

39. Lecture-3 (10th January, 2023): Siegel's theorem

Theorem 39.0.1 (Siegel).
Let $\chi(q)$ be a real Dirichlet character modulo $q \geq 3$. Given any $\epsilon > 0$, we have

$$L(1,\chi) \geq \frac{C_\epsilon}{q^\epsilon}$$

A trivial lower bound: $L(1,\chi) \gg q^{-1/2}$

Goldfeld's proof. Consider

$$f(s) = \zeta(s)L(s,\chi_1)L(s,\chi_2)L(s,\chi_1\chi_2)$$

with $\chi_i, i = 1,2$ primitive quadratic characters. Notice that $f(s) = \sum_n b_n n^{-s}$ with $b_1 = 1, b_n \geq 0$. Let $\lambda = \mathrm{Res}_{s=1} f(s) = L(1,\chi_1)L(1,\chi_2)L(1,\chi_1\chi_2)$

Lemma 39.0.2.
Given any $\epsilon > 0$, one can find $\chi_1(q_1)$ and β with $1-\epsilon < \beta < 1$ such that $f(\beta) \leq 0$, independent of what $\chi_2(q_2)$ is.

Proof. **Case-1:** If there are no real zeros of $L(s,\psi)$ for any primitive quadratic character in $(1-\epsilon,1)$, then $f(\beta) < 0$ for any $\beta \in (1-\epsilon,1)$. This is because

$$f(\beta) = \underbrace{\zeta(\beta)}_{<0}\underbrace{L(s,\chi_1)L(s,\chi_2)L(s,\chi_1\chi_2)}_{>0}$$

as $L(1,\chi) > 0$ and L is continuous so any change of sign will lead to a zero which is a contradiction.
Case-2: If we cannot find such a ψ, then just set $\chi_1 = \chi$ and let β be the real zero. Then, $f(\beta) = 0$. We are done. $\square$

Next, consider the integral

$\square$

Corollary 39.0.3.

$$
\begin{aligned}
h(-d) &= \frac{L(1,\chi_d)\sqrt{|d|}\,\omega}{2\pi} \\
&= \frac{L(1,\chi_d)}{\log \epsilon_d}
\end{aligned}
$$

Theorem 39.0.4 (Y. Zhang)**.**

$$
L(1,\chi) \geq \frac{c}{(\log q)^{2022}}
$$

Theorem 39.0.5.
If $\chi(q)$ does not have a Siegel zero, then $L(1,\chi) \gg \frac{1}{\log q}$

40. Lecture-4 (12th January, 2023): PNT for Dirichlet characters and APs

Lemma 40.0.1.
If $\rho = \beta + i\gamma$ runs through nontrivial zeros of $L(s, \chi)$, then

$$\sum_{\rho} \frac{1}{1 + (T - \gamma)^2} = \mathcal{O}(\log q(|T| + 2)) \forall T \in \mathbb{R}$$

Lemma 40.0.2.

$$N(T + 1, \chi) - N(T, \chi) = \mathcal{O}(\log q(|T| + 2))$$

Lemma 40.0.3.

$$\sum_{\rho:|\gamma - t| \leq 1} \frac{1}{s - \rho} + \mathcal{O}(\log qt) = \frac{L'}{L}(s, \chi)$$

for $-1 \leq \sigma \leq 2, |t| \geq 2, L(s, \chi) \neq 0$

Lemma 40.0.4.
Let $\chi(q)$ be primitive, $q \geq 3, T \geq 2$. Then, there exists $T_1 \in [T, T + 1]$ such that $\frac{L'}{L}(\sigma \pm iT_1, \chi) \ll (\log qT)^2, -1 \leq \sigma \leq 2$.

Lemma 40.0.5.
Put $a = 1$ if χ is even and 0 otherwise.

$$\mathcal{A}(a) := \{s \in \mathbb{C} \mid \sigma \leq -1, |s + 2n - a| \geq \frac{1}{4} \; \forall \, n \geq 1\}$$

Then,

$$\frac{L'}{L}(s, \chi) \ll \log(q(|s| + 1))$$

on $\mathcal{A}(a)$

These are all the ingredients needed to prove the the explicit formula for $\psi_0(x, \chi)$.

Theorem 40.0.6.

$$\psi(s,\chi) = \sum_{n\leq x} \Lambda(n)\chi(n)$$

$$\psi_0(x,\chi) = \frac{1}{2}(\psi(x^+,\chi)+\psi(x^-,\chi)) = -\sum_{\rho:|\gamma|\leq t} \frac{x^\rho}{\rho} - \frac{1}{2}\log(x-1) - \frac{\chi(-1)}{2}\log(x+1) + C_\chi + R_\chi(T)$$

where $C_\chi = \frac{L'}{L}(1,\overline{\chi}) + \log\frac{q}{2\pi} - \gamma$ and $R_\chi(T) \ll (\log x)\min(1, x/T < x >) + \frac{x}{T}(\log(qxT))^2$. Letting $T \to \infty$ we see that $R_\chi(T) \to 0$.

Theorem 40.0.7 (Brun-Titsmarsh inequality)**.**
Let $x \geq 0, y \geq 2q$. Then,

$$\pi(x+y;q,a) - \pi(x;q,a) \leq \frac{2y}{\phi(q)\log(\frac{y}{q})}\left(1 + \mathcal{O}(\frac{1}{\log(\frac{y}{q})})\right)$$

Remind him to prove this later; uses Sieve theoretic methods

Theorem 40.0.8 (PNT for Dirichlet characters)**.**
There exists a $c_1 \geq 0$ such that for all $q \leq \exp(c_1\sqrt{\log x})$, we have

$$\psi(x,\chi) = \sum_{n\leq x} \Lambda(n)\chi(n) = \begin{cases} E_0(x) + \mathcal{O}(x\exp(-c_1\sqrt{\log x})) & \chi \text{ has no Siegel zero} \\ -\frac{x^{\beta_1}}{\beta_1} + \mathcal{O}(x\exp(-c_1\sqrt{\log x})) & \chi \text{ has Siegel zero} \end{cases}$$

Here, $E_0(\chi) = 1$ if $\chi = \chi_0$ and 0 otherwise.

Recall from MA317 that $L(x,\chi) \neq 0$ when $\sigma \geq 1 - \frac{c}{\log q\tau}$ for some constant $c > 0$ with the exception of atmost one real zero (β_1 the Siegel zero)

Proposition 40.0.9.
Let c be as above and assume that $\sigma \geq 1 - \frac{c}{2\log q\tau}$. Then,

1. If $L(s,\chi)$ has no Siegel zero or if β_1 is a Siegel zero (thus χ quadratic) but $|s-\beta_1| \geq \frac{1}{\log q}$, then
$$\frac{L'}{L}(s,\chi) \ll \log q\tau$$
$$|\log L(s,\chi)| \ll \log\log q\tau + \mathcal{O}(1)$$
$$\frac{1}{L(s,\chi)} \ll \log q\tau$$
2. If β_1 is a Siegel zero and $|s-\beta_1| \leq \frac{1}{\log q}$, then
$$\frac{L'}{L}(s,\chi) = \frac{1}{s-\beta_1} + \mathcal{O}(\log q)$$

$$|\arg L(s,\chi)| \leq \log\log q + \mathcal{O}(1)$$
$$|s-\beta_1| \ll |L(s,\chi)| \ll |s-\beta_1|(\log q)^2$$

Part VI.

Commutative Algebra

41. Ideals

42. Modules

43. Projective, Injective modules

44. Noetherian Rings

45. Artinian Rings

46. Localisation

Let R be a commutative ring, S a multiplicatively closed subset containing 1.

Theorem 46.0.1.
There exists a ring homomorphism $\theta : R \to S^{-1}R$ which has the following properties:

1. $\theta(s) \in (S^{-1}R)^{\times}\ \forall\ s \in S$
2. Given any ring homomorphism $g : R \to A$ such that $g(s) \in A^{\times}\ \forall\ s \in S\ \exists\,!$ ring homomorphism $\tilde{g} : S^{-1}R \to A$ such that $g = \tilde{g} \circ \theta$.

$$\begin{tikzcd} R \arrow[d, "\theta"'] \arrow[dr, "g"] & \\ S^{-1}R \arrow[r, dashed, "\exists\,!\,\tilde{g}"'] & A \end{tikzcd}$$

Proof. Existence: Define the relation on $R \times S$ given by $(a, s) \sim (a', s')$ if $t(as' - a's) = 0$ for some $t \in S$. It can be checked that this is an equivalence relation. Define □

47. Integral dependence

48. Completions

Part VII.

Algebraic Number Theory

49. Dedekind Domains

50. Splitting of primes

51. Finiteness of class number

52. Unit theorem

53. Cyclotomic Fields and Fermat's last theorem

54. Local Fields

55. Global Fields

56. Kronecker Weber

57. Adéles and Idéles

Part VIII.

Galois Theory

58. Finite Fields

59. Cyclotomic Fields

Part IX.

Representation theory

60. Introduction

61. Character theory

62. Wedderburn theorem

63. Induced characters

64. Brauer Induction theorem

Part X.

Miscellaneous

65. Galois representations

66. Artin L-functions

67. Riemann hypothesis for curves over finite fields