भारतीय विज्ञान संस्थान

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SEMESTER NOTES

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Contents

Part I.

Modular Forms

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3.1. Valence formula

Recall that $M_k(\Gamma_1)$ is the space of modular forms of weight k and level 1. It is also a vector space over $\mathbb{C}$.

Theorem 3.1.1.
$$\dim M_k(\Gamma_1) = \begin{cases} [k/12]+1 & k \not\equiv \pmod{12} \\ [k/12] & k \equiv \pmod{12} \end{cases}$$

Proposition 3.1.2.
Let $f \in M_k(\Gamma_1)$. Then,

$$\sum_{p \in \Gamma_1 \backslash \mathbb{H}} \frac{1}{n_p} \mathrm{ord}_p(f) + \mathrm{ord}_\infty(f) = \frac{k}{12}$$

Proof. Let $\epsilon > 0$ be "small enough". Remove ϵ-balls around $\infty, i, \omega, \omega+1$ in $\mathcal{F}_1$. ϵ is small enough so that the removed balls are disjoint. Truncate $\mathcal{F}_1$ at the line $y = \epsilon^{-1}$ and call the enclosed region D.
By Cauchy's theorem
$$\int_{\partial D} d(\log f(z)) = 0$$
This integral on the two vertical strips (just the straight lines not the semicircle part) is 0 since the contribution of left is same as right but orientation is different. On the segment joining $-1/2+iY, 1/2+iY$, the integral is $2\pi i\, \mathrm{ord}_\infty(f)$. Again,integral around each removed point in $\mathcal{F}_1$ is $\frac{1}{n_p}\mathrm{ord}_p(f)$. Next, divide the bottom arc into left and right parts and observe that

$$d(\log f(S \cdot z)) = d(\log f(z)) + k\frac{dz}{z}$$

$$\int_C d(\log f(z)) = \frac{k\pi i}{6}$$

□

Corollary 3.1.3.

$$\dim M_k(\Gamma_1) = \begin{cases} 0 & k < 0 \\ 0 & k \text{ is odd} \\ 1 & k = 0 \\ \begin{cases} [k/12]+1 & k \not\equiv \pmod{12} \\ [k/12] & k \equiv \pmod{12} \end{cases} \end{cases}$$

Proof.

- If $k < 0$, then f has poles but is holomorphic.
- If $k = 0$, then f is the constant function.
- We have seen
- For $m = [k/12] + 1$ let $f_1, \dots, f_{m+1} \in M_k(\Gamma_1)$. Let $P_1, \dots, P_m$ be any points on $\mathcal{F}_1$ not equal to $i, \omega, \omega + 1$ and consider $(f_i(P_j))_{i\in[m+1], j\in[m]}$.
  There exists a linear combination $f = \sum_{i=1}^{m+1} c_i f_i$ not all c_i being zero, such that $f(P_j) = 0$ for $1 \leq j \leq m$.
  From the previous theorem we get $f \equiv 0$ and this implies $\{f_i\}$ is linearly independent and thus $\dim_{\mathbb{C}} M_k(\Gamma_1) \leq m$.
  For $k \equiv 2 \pmod{12}$, the relation in previous theorem holds only if there is atleast a simple zero at $p = i$ and atleast a double zero at $p = \omega$. This gives

  $$\frac{k}{12} - \frac{7}{6} = m - 1$$

  Repeat the argument above.

$\square$

A slight notation. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we set $f|_\gamma(z) = (cz+d)^{-k} f(\gamma \cdot z)$.
Thus, $1|_\gamma(z) = (cz+d)^{-k}$. If $1|_\gamma(z) = 1 \Rightarrow c = 0$. Conversely, if $c = 0$, then $d^{-k} = 1$. So, $1|_\gamma(z) = 1 \Leftrightarrow c = 0$.
$\Gamma_\infty = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\} = \mathrm{stab}(\infty)$

3.2. Eisenstein series

Definition 3.2.1.
The Eisenstein series $E_k(z)$ is defined to be

$$E_k(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_1} 1|_\gamma(z)$$

Proposition 3.2.2.

$$E_k(z) = \frac{1}{2} \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\},\gcd(c,d)=1} \frac{1}{(cz+d)^k}$$

Proof. □

Proposition 3.2.3.

$$\sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\},\gcd(c,d)=1} \frac{1}{(cz+d)^k}$$

converges absolutely for $k > 2$

Proof. □

Theorem 3.2.4.
$E_k(z) \in M_k(\Gamma_1)$ for $k > 2$.

Proof. □

Proposition 3.2.5.
$E_k(z) \not\equiv 0$ for $k > 2$, even.

Proof. Observe that

$$\frac{1}{(cz+d)^k} \to 0, \Im(z) \to \infty, c \neq 0$$

and if $c = 0$, then $c = \pm 1$. Hence, $E_k(z) = 1+$ bounded term as $\Im(z) \to \infty$. This implies $E_k(z) \not\equiv 0$ and

$$E_k(z) = 1 + \sum_{n=1}^{\infty} a_n e^{2\pi i z}$$

□

Another way of looking at Eisenstein series is a function on a lattice.
Consider $G_k(z) = G_k(\mathbb{Z}z + \mathbb{Z}) = \frac{1}{2}\sum'_{(c,d)\in\mathbb{Z}^2} \frac{1}{(cz+d)^k}$

Proposition 3.2.6.
$G_k(z)$ converges absolutely for $k > 2$.

Proposition 3.2.7.
$G_k(z) = \zeta(k)E_k(z)$

Proposition 3.2.8.
$\mathbb{G}_k(z) = \frac{(k-1)!}{(2\pi i)^k} G_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$ for $k > 2$, even.

4. Lecture-4 (12th January, 2023): Eisenstein series

4.1. Eisenstein series contd..

Recall that

$$M_*(\Gamma_1) = \bigoplus_{k \in \mathbb{Z}} M_k(\Gamma_1)$$

is a graded ring.

Proposition 4.1.1.
The graded ring $M_*(\Gamma_1)$ is freely generated by E_4, E_6. This means that the map

$$\begin{aligned} f : C[X,Y] &\to M_*(\Gamma_1) \\ X &\mapsto E_4 \\ Y &\mapsto E_6 \end{aligned}$$

is an isomorphism of graded rings. Here, $\deg X = 4, \deg Y = 6$.

Proof. We want to show that E_4 and E_6 are algebraically independent. We start by showing that E_4^3 and E_6^2 are linearly independent over $\mathbb{C}$. Suppose $E_6(z)^2 = \lambda E_4(z)^3$. Consider $f(z) = E_6(z)/E_4(z)$. Now observe that $f(z)^2 = \lambda E_4(z)$. This means that f^2 is holomorphic and thus f is also holomorphic. But f is weakly modular of weight 2 which is a contradiction. So, our claim is proven.

Claim: Let f_1, f_2 be two nonzero modular forms of same weight. If f_1, f_2 are linearly independent, then they are algebraically independent as well.

Let $P(t_1, t_2) \in \mathbb{C}[t_1, t_2] \backslash \{0\}$ be such that $P(f_1, f_2) = 0$. Let $P_d(t_1, t_2)$ be the d degree parts of P. Using the fact that modular forms of different weights are linearly independent, we get that $P_d(f_1, f_2) = 0 \ \forall \ d \geq 0$. If $p_d(t_1/t_2) = P_d(t_1, t_2)/t_2^d$, then $p_d(f_1/f_2) = 0$. But this means that f_1/f_2 is a constant. But, f_1, f_2 are linearly independent which implies that they are algebraically independent as well.

All of this implies that E_4, E_6 are algebraically independent. Using $\square$

Corollary 4.1.2.

$$\dim_{\mathbb{C}} M_k(\Gamma_1) = \begin{cases} [k/12]+1 & k \not\equiv \pmod{12} \\ [k/12] & k \equiv \pmod{12} \end{cases}$$

4.1.1. Fourier expansions of $E_k(z)$

Proposition 4.1.3.

$$\mathbb{G}_k(z) = \frac{(k-1)!}{(2\pi i)^k} G_k(z) = -\frac{B_k}{2k} + \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$$

for $k > 2$, even and B_k are Bernoulli numbers.

Proof. Use

$$\frac{\pi}{\tan \pi z} = \sum_{n\in\mathbb{Z}} \frac{1}{z+n} = \lim_{M,N\to\infty, N-M<\infty} \sum_{-M}^{N} \frac{1}{z+n}$$

and

$$\frac{\pi}{\tan \pi z} = \frac{\pi \cos \pi z}{\sin \pi z} = \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}} = -\pi i \frac{1+q}{1-q} = -2\pi i \left(\frac{1}{2} + \sum_{r=1}^{\infty} q^r \right)$$

This leads to the equality

$$\sum_{n\in\mathbb{Z}} \frac{1}{z+n} = -2\pi i \left(\frac{1}{2} + \sum_{r=1}^{\infty} q^r \right)$$

Differentiate both sides of equality $k-1$ times and divide by $(k-1)!$ to get

$$\sum_{n\in\mathbb{Z}} \frac{1}{(z+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{r=1}^{\infty} r^{k-1} q^r$$

Next, if we look at

$$\begin{aligned} G_k(z) &= \frac{1}{2} \sum{}' \frac{1}{(mz+n)^k} \\ &= \frac{1}{2} \sum_{n\in\mathbb{Z}, n\neq 0} \frac{1}{n^k} + \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2, m\neq 0} \frac{1}{(mz+n)^k} \\ &= \zeta(k) + \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{1}{(mz+n)^k} \\ &= \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} r^{k-1} q^{mr} \\ &= \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \sigma_{k-1}(n) q^n \end{aligned}$$

The expression of $\mathbb{G}_k(z)$ is trivial after noting

$$\frac{(k-1)!}{(2\pi i)^k}\zeta(k) = B_k$$

□

Remark 4.1.4.
1. $\mathbb{G}_4(z) = \frac{1}{240} + q + 9q^2 + 28q^3 + 73q^4 + \cdots$
2. $\mathbb{G}_6(z) = -\frac{1}{504} + q + 33q^2 + 244q^3 + \cdots$
3. $\mathbb{G}_8(z) = \frac{1}{480} + q + 129q^2 + 2188q^3 + \cdots$

Proposition 4.1.5.

$$\sum_{m=1}^{n-1}\sigma_3(m)\sigma_3(n-m) = \frac{\sigma_7(n)-\sigma_3(n)}{120}$$

Proof. □

4.1.2. Weight 2 Eisenstein series

Definition 4.1.6.

$$\begin{aligned}\mathbb{G}_2(z) &= -\frac{1}{24} + \sum_{n=1}^{\infty}\sigma_1(n)q^n \\ &= -\frac{1}{24} + q + 3q^2 + 4q^3 + 7q^4 + \cdots\end{aligned}$$

This converges rapidly on $\mathbb{H}$ and defines a holomorphic function.

Proposition 4.1.7.

$$G_2(z) = -4\pi^2\mathbb{G}_2(z)$$

Proof. Since we know that

$$G_2(z) = \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}}\frac{1}{(mz+n)^2}$$

does not converge absolutely, we define

$$G_2(z) = \frac{1}{2}\sum_{n\in\mathbb{Z},n\neq 0}\frac{1}{n^2} + \frac{1}{2}\sum_{m\neq 0}\sum_{n\in\mathbb{Z}}\frac{1}{(mz+n)^2}$$

This sum converges absolutely and we can show that this satisfies the functional equation as required. □

Proposition 4.1.8.
For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 G_2(z) - \pi i c(cz+d)$$

G_2 is called a quasi modular form.

Introduce (due to Hecke):

$$G_{2,s}(z) = \frac{1}{2} \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{1}{(mz+n)^2|mz+n|^{2s}}, \mathfrak{R}(s) > 0$$

4.2. Modular forms of higher level

Let $N \in \mathbb{Z}_{\geq 1}$

$$\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}/N\mathbb{Z}) \mid ad - bc \equiv 1 \pmod N \right\}$$

Lemma 4.2.1.
The map

$$\begin{aligned} \mathrm{SL}_2(\mathbb{Z}) &\to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \end{aligned}$$

is a surjective group homomorphism.

Proof. □

Definition 4.2.2.

$$\Gamma(N) = \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}))$$

is called the principal congruence subgroup.

Definition 4.2.3.
A subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ is called a congruence subgroup if there exists N such

that $\Gamma(N) \subseteq \Gamma$.

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}/N\mathbb{Z}) \mid c \equiv 0 \pmod{N} \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}/N\mathbb{Z}) \mid c \equiv d \equiv 1 \pmod{N} \right\}$$

5. Lecture-5 (17th January, 2023): Congruence subgroups and Δ function

5.1. Δ function

Consider
$$\Delta(z) = \frac{1}{1729}(E_4^3(z) - E_6^2(z)) = q + q^2() + \cdots$$
Clearly, $\Delta(z)$ is a normalised cusp form of weight 12 and level 1.

Theorem 5.1.1.

$$\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24}, q = e^{2\pi i z}$$

Proposition 5.1.2.
$\Delta(z)$ has no zero in $\mathbb{H}$.

Proof. From the valence formula we have

$$\sum_{p\in\mathbb{H}} \frac{1}{n_p}\text{ord}_p(\Delta(z)) + \text{ord}_\infty(\Delta(z)) = k/12 = 1$$

Moreover, $\text{ord}_\infty(\Delta(z)) = 1$. Hence, we can conclude that $\text{ord}_p(\Delta(z)) = 0 \;\forall\; p \in \mathbb{H}$. $\square$

Application: We use $\Delta(z)$ to write any modular form as a polynomial in E_4, E_6.

Take $f(z) \in M_k(\Gamma_1)$ with $4a + 6b, k \geq 4, a, b \geq 0$. The Fourier expansion of $f(z)$ can we written as
$$f(z) = a_0 + a_1 q + \cdots$$
Clearly, $f(z) - a_0 E_4^a(z)E_6^b(z) \in M_k(\Gamma_1) \subseteq S_k(\Gamma_1)$.
Next,
$$h(z) = \frac{f(z) - a_0 E_4^a(z)E_6^b(z)}{\Delta(z)} \in M_{k-12}(\Gamma_1)$$
Recursively, we can now find expression for $f(z)$.

Proposition 5.1.3.

$$j(z) = \frac{E_4^3}{\Delta(z)} = q^{-1} + \cdots$$

$$\begin{aligned} j : \bar{\mathbb{H}}/\Gamma_1 &\to \mathbb{P}^1(\mathbb{C}) \\ z &\mapsto j(z) \end{aligned}$$

is a bijection.

Proof. $E_4^3(z)$ and $\Delta(z)$ are linearly independent. For any $\lambda_1, \lambda_2 \in \mathbb{C}$ both not zero, $\lambda_1 E_4^3(z) + \lambda_2 \Delta(z)$ has an unique zero in $\bar{\mathbb{H}}/\Gamma_1$. □

Remark 5.1.4.
This j is called the j-invariant modular function. It attaches an elliptic curve in $\mathbb{P}^1(\mathbb{C})$ to any lattice in $\Lambda_z = \mathbb{Z}z + \mathbb{Z}$ and vice versa.

Next, the Fourier series of $\Delta(z)$ is of the form $\Delta(z) = \sum_{n\geq 1} \tau(n) q^n$ where $\tau(n)$ satisfies the following properties:

1. $\tau(pq) = \tau(p)\tau(q)$ if p, q are dinstinct primes.
2. $\tau(p^2) = \tau(p)^2 - p^{12-1}$.
3. $|\tau(p)| \leq 2p^{\frac{12-1}{2}}$.
4. $$\begin{aligned} \mathbb{G}_{12}(z) &= \Delta(z) + \frac{691}{156}\left(\frac{E_4^3(z)}{720} + \frac{E_6^2}{1008}\right) \\ \mathbb{G}_{12} &= -\frac{B_{12}}{24} + \sum_{n\geq 1} \sigma_{11}(n) q^n \\ &= \frac{691}{65520} + \sum_{n\geq 1} \sigma_{11}(n) q^n \\ \mathbb{G}_{12}(z) &\equiv \Delta(z) \pmod{691} \end{aligned}$$

   To conclude
   $$\tau(n) = \sigma_{11}(n) \pmod{691}$$
   (Related to the fact that $691 \mid \#\mathcal{C}\updownarrow(\mathbb{Q}(\gamma_{691}))$)

5.2. Congruence subgroup

Proposition 5.2.1.
Let $N = p_1^{a_1} \cdots p_r^{a_r}$ be the prime factorisation. Then,

$$\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) = \prod_{i=1}^{r} \mathrm{SL}_2(\mathbb{Z}/p^{a_i}\mathbb{Z})$$

Lemma 5.2.2.

$$\#\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z}) = N^3 \prod_{p|N} \left(1 - \frac{1}{p^2}\right)$$

Definition 5.2.3.
A subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is called congruence subgroups if $\Gamma(N) \subseteq \Gamma$ for some $N \geq 1$.

Lemma 5.2.4.
A congruence subgroup has finite index in $\mathrm{SL}_2(\mathbb{Z})$.

Remark 5.2.5.
There are non-congruence subgroups of finite index in $\mathrm{SL}_2(\mathbb{Z})$.

Properties:

1. $\mathrm{PSL}_2(\mathbb{Z})$ is generated freely by an element of order 2 and an element of order 3.
2. S_7 is generated by an element of order 2 and an element of order 3. There is a surjection

$$\begin{aligned} \mathrm{PSL}_2(\mathbb{Z}) &\xrightarrow{\pi} S_7 \\ \pi^{-1}(\mathrm{Stab}_1) &\subseteq \mathrm{PSL}_2(\mathbb{Z}) \end{aligned}$$

3. $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ is a simple group for $p \geq 5$.

Remark 5.2.6.
Γ is the smallest index subgroup that is non-congruence.

Definition 5.2.7.
A holomorphic function $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight k and level Γ if

1. $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$
2. f is holomorphic at all cusps.

Cusps of $X(\Gamma)$ are just elements of $\Gamma\backslash\mathbb{P}^1(\mathbb{Q})$.

Proposition 5.2.8.

$$\mathrm{orbit}(\infty) = \{\frac{a}{c} : p \mid c, p \nmid a\}, \mathrm{orbit}(1) = \{\frac{a}{c} : p \nmid c\}$$

Part II.

Elliptic Curves

6. Lecture-1 (3rd January): Introduction

7. Lecture-2 (5th January, 2023): Affine varieties

7.1. Affine Varieties

Suppose k is a perfect field (every extension is separable).
Let $G(\bar{k}/k)$ be the Galois group of the extension. It can also be viewed as $\varinjlim_{L/K\text{Galois, } L \text{ finite}} \mathrm{Gal}(L/K)$.

8. Lecture-3 (10 January, 2023): Projective varieties

8.1. Projective varieties

Definition 8.1.1.
A Projective n-space over k denoted by $\mathbb{P}^n$ or $\mathbb{P}^n(\bar{K})$ is the set $\mathbb{A}^{n+1} \backslash \{(0, \ldots, 0)\} / \sim$ with

$$(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$$

iff $\exists \lambda \in \bar{k}^\times$ such that $(y_0, \ldots, y_n) = (\lambda x_0, \ldots, \lambda x_n)$
The equivalence class $(x_0, \ldots, x_{n+1})$ is denoted by $[x_0, \ldots, x_n]$
The set of k-rational points of $\mathbb{P}^n$ is

$$\mathbb{P}^n = \{[x_0, \ldots, x_n] \mid x_i \in k\}$$

Caution: If $p = [x_0, \ldots, x_n] \in \mathbb{P}^n(k)$ and $x_i \neq 0$ for some i, then $x_j / x_i \in k \forall j$

Definition 8.1.2.
Let $p = [x_0, \ldots, x_n] \in \mathbb{P}^n(\bar{k})$. The minimal field of definition for p is the field

$$k(p) = k(x_0/x_i, \ldots, x_n/x_i) \text{ for any } i \text{ such that } x_i \neq 0$$

$k(p)\frac{x_i}{x_j} = k(x_0/x_j, \ldots, x_n/x_j)$ is the same as $k(p)$ as $x_i/x_j \in k(p)$

For $\sigma \in G(\bar{k}/k)$ and $p = [x_0, \ldots, x_n] \in \mathbb{P}^n$, we have the following action

$$\sigma(p) = [\sigma(x_0), \ldots, \sigma(x_n)]$$

This action is well defined as

$$\sigma(\lambda p) = [\sigma(\lambda)\sigma(x_0), \ldots, \sigma(\lambda)\sigma(x_n)] \sim [\sigma(x_0), \ldots, \sigma(x_n)]$$

Definition 8.1.3.
A polynomial $f \in \bar{k}[X_0, \ldots, X_n]$ is homogenous of degree d if

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^d f(x_0, \ldots, x_n) \forall \lambda \in \bar{k}$$

Definition 8.1.4.
An ideal $I \subseteq \bar{k}[X_0, \ldots, X_n]$ is called a homogenous ideal if it is generated by homogenous polynomial.

Definition 8.1.5.
Let $I \subseteq \bar{k}[X_0, \ldots, X_n]$ be a homogenous ideal. Then,

$$V(I) = \{p \in \mathbb{P}^n(\bar{k}) \mid f(p) = 0 \;\forall\, f \in I\}$$

Definition 8.1.6.

- A projective algebraic set is any set of the form $V(I)$ for some homogenous ideal I.
- If V is a projective algebraic set, the homogenous ideal of V, denoted by $I(V)$ is the ideal of $\bar{k}[X_0 \ldots, X_n]$ generated by $\{f \in \bar{k}[X_0 \ldots, X_n] \mid f \text{ is homogenous and } f(p) = 0 \;\forall\, p \in V\}$
- Such a V is defined over k, denoted by V/k if its ideal $I(V)$ can be generated by homogenous polynomials $k[X_0 \ldots, X_n]$.
- If V is defined over k, then the set of k-rational points of V is

$$V(k) = V \cap \mathbb{P}^n(k) = \{p \in V \mid \sigma(p) = p \;\forall\, \sigma \in G(\bar{k}/k)\}$$

Example 8.1.7.
A line in $\mathbb{P}^2$ is given by the equation $aX + bY + cZ = 0$ with $a, b, c \in \bar{k}$ and not all 0 simultaneously.
If $c \neq 0$, then such a line is defined over a field containing $a/c, b/c$.
More generally, a hyperplane in $\mathbb{P}^n$ is given by an equation $a_0X_0 + \cdots + a_nX_n = 0$ with all $a_i \neq 0$ simultaneously.

Example 8.1.8.
Let V be the projective algebraic set in $\mathbb{P}^2$ given by $X^2 + Y^2 = Z^2$.

$$\mathbb{P}^1 \overbrace{\rightarrow}^{\sim} V$$
$$[s, t] \mapsto [s^2 - t^2 : 2st : s^2 + t^2]$$

Remark 8.1.9.
For $p \in \mathbb{P}^n(\mathbb{Q})$ you can clear the denominators and then divide by common factor so that $x_i \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_n) = 1$. So, $I = (f_1, \ldots, f_m)$ and finding a rational point of V_I is same as finding coprime integer solutions to $f_i's$.

Example 8.1.10.
$V \subseteq \mathbb{P}^2$ such that $X^2 + Y^2 = 3Z^2$ over $\mathbb{Q}$. To find $V(\mathbb{Q})$, we just need to find integers a, b, c such that $a^2 + b^2 = 3c^2$

Example 8.1.11.
$V : 3X^3 + 4Y^3 + 5Z^3 = 0$. $V(\mathbb{Q}) = \emptyset$ but for all prime p we have $V(\mathbb{Q}_p) \neq \emptyset$

Definition 8.1.12.
A projective algebraic set is called a projective variety if its homogenous ideal $I(V)$ is prime $\bar{k}[X_0, \ldots, X_n]$

Relation between affine and projective varieties:

For $0 \leq i \leq n$

$$\begin{aligned} \phi_i : \mathbb{A}^n &\to \mathbb{P}^n \\ (Y_1, \ldots, Y_n) &\mapsto [Y_1, \ldots, Y_{i-1}, 1, Y_{i+1}, \ldots, Y_n] \end{aligned}$$

$\mathrm{Im}(\phi) = U_i = \{p \in \mathbb{P}^n \mid \ p = [x_0 : \ldots : x_n] \text{ with } x_i \neq 0\} = \mathbb{P}^n \backslash H_i$.
This process can also be reversed by the following map :

$$\begin{aligned} \phi_i^{-1} : U_i &\to \mathbb{A}^n \\ [x_0 : \ldots : x_n) &\mapsto [x_0/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i] \end{aligned}$$

Let V be a projective algebraic set with homogenous ideal $I(V) \subseteq \bar{k}[X_0, \ldots, X_n]$. Then,

$$V \cap \mathbb{A}^n = \phi_i^{-1}(V \cap U_i) \text{ for fixed } i$$

is an affine algebraic set with $I(V \cap \mathbb{A}^n) \subset \bar{k}[X_0, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]$

Definition 8.1.13.
Let $V \subseteq \mathbb{A}^n$ be an affine algebraic set with ideal $I(V)$ and consider $V \subseteq \mathbb{P}^n$ and ϕ_i defined as before.
The projective closure of V is $\bar{V}$ is the projective algebraic set whose homogenous ideal $I(V)$ is generated by $\{f^* \mid f \in I(V)\}$.
Here, for $f \in k[X_0, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n]$ we define

$$f^*(X_0, \ldots, X_n) = X_i^d(f(X_0/X_i, \ldots, X_{i-1}/X_i, X_{i+1}/X_i, \ldots, X_n/X_i))$$

with $d = \deg(f)$.

Definition 8.1.14.
Dehomogenization of $f(X_0, \ldots, X_n)$ with respect to i is $f(X_0, \ldots, X_{i-1}, 1, X_{i+1}, \ldots, X_n)$

Proposition 8.1.15.

1. Let V be an affine variety. Then $\bar{V}$ is a projective variety and $V = \bar{V} \cap \mathbb{A}^n$.
2. Let V be a projective variety. Then, $V \cap \mathbb{A}^n$ is an affine variety and either $V \cap \mathbb{A}^n = \emptyset$ or $V = \overline{V \cap \mathbb{A}^n}$.
3. If an affine (resp. projective) variety V is defined over k, then $\bar{V}$ (resp. $V \cap \mathbb{A}^n$) is also defined over k.

Proof.

1.
2.
3.

□

Example 8.1.16.
$V : Y^2 = X^3 + 17 \subseteq \mathbb{A}^2 \to \mathbb{P}^2$ with $(X, Y) \mapsto [X : Y : 1]$. Here, $\overline{V} : Y^2Z = X^3 + 17Z^3$ and $\overline{V} \backslash V = \{[0 : 1 : 0]\}$

9. Lecture-4 (12th January, 2023): Projective varieties and maps between varieties

9.1. Projective varieties contd..

Definition 9.1.1.

- Let Y/k be a projective variety and choose $\mathbb{A}^n \subseteq \mathbb{P}^n$ such that $V \cap \mathbb{A}^n \neq \emptyset$. The dimension of V is just dimension of $V \cap \mathbb{A}^n$.
- The function field of V, $\bar{k}(V) = \bar{k}(V \cap \mathbb{A}^n)$ is the function field for $V \cap \mathbb{A}^n$ over $\bar{k}$.
- Similarly, $k(V) = k(V \cap \mathbb{A}^n)$

$$\phi_i : \mathbb{A}^n \to \mathbb{P}^n \mathcal{I}(V \cap \mathbb{A}^n_i)$$
$$\phi_j : \mathbb{A}^n \to \mathbb{P}^n \mathcal{I}(V \cap \mathbb{A}^n_j)$$

For different ϕ_i we obtain $k(V)$s but they are canonically isomorphic to each other. This is because we can just switch x_i, x_j are dehomogenise accordingly.

Definition 9.1.2.
Let V be a projective variety and $p \in V$. Choose $\mathbb{A}^n \subseteq \mathbb{P}^n$ with $p \in \mathbb{A}^n$. Then, V is non-singular (or smooth) at p if $V \cap \mathbb{A}^n$ is non-singular at p.

The local ring of V at p, $\bar{k}[V]_p$ is just the local ring of $\bar{k}[V \cap \mathbb{A}^n]_p$

Remark 9.1.3.
Function field of a projective variety V is field of rational functions $f(X)/g(X)$ such that

1. f, g are homogenous of same degree.
2. $g \in \mathcal{I}(V)$.
3. $f_1/g_1 = f_2/g_2$ iff $f_1 g_2 - f_2 g_1 \in \mathcal{I}(V)$

Equivalently, take $f, g \in \bar{k}[X]/I(V)$ satisfying $1, 2$.

Here, X is just a short form for $(X_0, \dots, X_n)$

9.2. Maps between varieties

Definition 9.2.1.
Let $V_1, V_2 \in \mathbb{P}^n$ be projective varieties. A rational map

$$\phi : V_1 \to V_2$$

$\phi = [f_0 : \dots : f_n]$ where $f_i \in \bar{k}(V_1)$ such that $\forall p \in V_1$ at which f_i are defined, we have

$$\phi(p) = [f_0(p) : \dots : f_n(p)]$$

If V_1, V_2 are defined over k, we have a Galois action. For $\sigma \in G(\bar{k}/k)$ we have

$$\sigma(\phi)(p) = [\sigma(f_0) : \dots : \sigma(f_n)(p)]$$

We can check that $\sigma(\phi(p)) = \sigma(\phi)(\sigma(p))$.

Definition 9.2.2.
If $\exists \lambda \in \bar{k}^{\times}$ such that $\lambda f_i \in k(V_1)$, then ϕ is said to be defined over k.

Proposition 9.2.3.
ϕ is defined over k iff $\phi = \sigma(\phi)\ \forall\ \sigma \in G(\bar{k}/k)$.

Definition 9.2.4.
A rational map $\phi : V_1 \to V_2$ is said to be regular if there exists a function $g \in \bar{k}(V_1)$ such that

1. Each gf_i is regular at p.
2. There exists some i such that $(gf_i)(p) \neq 0$

If such a g exists, then we set

$$\phi(p) = [(gf_0)(p) : \dots : (gf_n)(p)]$$

Definition 9.2.5.
A rational map is called a morphism if it is regular everywhere.

Remark 9.2.6.
Let $V_1, V_2 \in \mathbb{P}^n$ be projective varieties.
$\bar{k}(V_1) =$ quotient of homogenous polynomials in $\bar{k}[X]$ of same degree.
A rational map $\phi = [f_0, \ldots, f_n]$ can be multiplied by a homogenous polynomial to clear denominators and get $\phi = [\phi_0, \ldots, \phi_n]$ such that

1. $\phi_i \in \bar{k}[X]$ homogenous polynomials not all in $\mathcal{I}(V_1)$ and have same degree.
2. For all $f \in \mathcal{I}(V_2)$ we have $f(\phi_0(X), \ldots, \phi_n(X)) \in \mathcal{I}(V_1)$.

Definition 9.2.7.
A rational map $\phi = [\phi_0, \ldots, \phi_n] : V_1 \to V_2$ as above is regular at $p \in V_1$ if there exists homogenous polynomials $\psi_0, \ldots, \psi_n \in \bar{k}[X]$ such that

1. ψ_is have the same degree
2. $\phi_i \psi_j \equiv \phi_j \psi_j \pmod{\mathcal{I}(V_1)}$ for all $0 \leq i, j \leq n$
3. $\psi_i(p) \neq 0$ for some i.

If this happens, we set
$$\phi(p) = [\psi_0(p), \ldots, \psi_n(p)]$$

Remark 9.2.8.
Let $\phi = [\phi_0, \ldots, \phi_n] : \mathbb{P}^m \to \mathbb{P}^n$ be a rational map. ϕ_is are homogenous polynomials having same degree. We can cancel common factors to assume $\gcd(\phi_0, \ldots, \phi_n) = 1$.

And, ϕ is regular at a point $p \in \mathbb{P}^n$ iff $\phi_i(p) \neq 0$ for some i.
So, ϕ is a morphism if ϕ_is have no common zeros in $\mathbb{P}^n$.

Definition 9.2.9.
Let V_1, V_2 be two projective varieties. We say that V_1, V_2 are isomorphic if there are morphisms
$$\phi : V_1 \to V_2, \psi : V_2 \to V_1$$
such that $\phi \circ \psi = \mathrm{id}_{V_2}, \psi \circ \phi = \mathrm{id}_{V_1}$.
V_1/k and V_2/k are isomorphic over k if both maps are defined over k.

Example 9.2.10.
$\mathrm{char}(k) \neq 2$, $V : X^2 + Y^2 = Z^2$.

$$\begin{gathered} \phi : V \to \mathbb{P}^2 \\ [X : Y : Z] \mapsto [X + Z : Y] \end{gathered}$$

ϕ is regular everywhere except $[1:0:1]$
Since $(X+Z)(X-Z) \equiv -Y^2 \equiv \pmod{\mathcal{I}(V)}$, we have $[X+Z:Y] = [X^2-Z^2 : Y(X-Z)] = [-Y^2 : Y(X-Z)] = [-Y : X-Z] = \psi$

$$\begin{aligned}\psi : \mathbb{P}^1 &\to V\\ [s:t] &\to [s^2-t^2 : 2st : s^2+t^2]\end{aligned}$$

$\psi \circ \phi$ and $\phi \circ \psi$ are both identity maps.

Example 9.2.11.

$$\begin{aligned}\phi : \mathbb{P}^2 &\to \mathbb{P}^2\\ [X:Y:Z] &\mapsto [X^2 : YZ : Z^2]\end{aligned}$$

is regular everywhere but $[0:1:0]$ and this cannot be salvaged.

Example 9.2.12.
$V : Y^2Z = X^3 + X^2Z$

$$\begin{aligned}\psi : \mathbb{P}^1 &\to V\\ [s:t] &\mapsto [(s^2-t^2)t : (s^2-t^2)s : t^3] \phi : V &\to \mathbb{P}^1\\ [X:Y:Z] &\mapsto [X:Y]\end{aligned}$$

ϕ is not regular at $[0:0:1]$. $[0:0:1]$ is a singular point of V which implies ϕ cannot be extended. So $\phi \circ \psi$ and $\psi \circ \phi$ are identities when they are defined.

Example 9.2.13.
$V_1 : X^2+Y^2 = Z^2, V_2 : X^2+Y^2 = 3Z^2$. $V_1 \ncong V_2$ over $\mathbb{Q}$ but $V_1 \cong V_2$ over $\mathbb{Q}(\sqrt{3})$.

10. Lecture-5 (17th January, 2023):

10.1. Curves

Definition 10.1.1.
A curve is a projective variety of dimension 1.

Example 10.1.2.
Vanishing set of an irreducible polynomial in $\mathbb{P}^2$.

Proposition 10.1.3.
Let C be a curve and $p \in C$ be a smooth (non-singular) point. Then, $\bar{k}[C]_p$ is a discrete valuation ring.

Proof. $p \in C$ smooth implies M_p/M_p^2 is one dimensional over $\bar{k}[C]_p/M_p = \bar{k}$. Now, Nakayama will give us M_p is a principal ideal.

Claim: $\bigcap_n M_p^n = 0$.

Proof. If $\alpha \in \bigcap_n M_p^n$, then $\alpha = a_1 t = a_2 t^2 = a_3 t^3 = \cdots$. This implies $a_1 = a_2 t = a_3 t^2 = \cdots$. But this gives us a chain

$$\langle a_1 \rangle \subseteq \langle a_2 \rangle \subseteq \langle a_3 \rangle \subseteq \cdots$$

that must terminate at some point. This implies t is an unit which is a contradiction. Hence, we are done. □

□

Definition 10.1.4.
Let C be a curve and $p \in C$ a smooth point. The normalised valuation on $\bar{k}[C]_p$ is

$$\begin{aligned}
\mathrm{ord}_p : \bar{k}[C]_p &\to \mathbb{N} \cup \{0, \infty\} \\
f &\mapsto \sup\{d \in \mathbb{Z} \mid f \in M_p^d\} \\
\mathrm{ord}_p(\frac{f}{g}) &= \mathrm{ord}_p(f) - \mathrm{ord}_p(g)
\end{aligned}$$

Thus we can define

$$\operatorname{ord}_p : \bar{k}[C] \to \mathbb{Z} \cup \{\infty\}$$

Definition 10.1.5.
An uniformiser for C at p is any function $t \in \bar{k}(C)$ with $\operatorname{ord}_p(t) = 1$ that is the generator of M_p

Remark 10.1.6.
If C is defined over k, we can find a unit $t \in k(C)$.

Definition 10.1.7.
Let C be a curve and $p \in C$ a smooth point, $f \in \bar{k}(C)$, $\operatorname{ord}_p(f) =$ order of f at p.

1. If $\operatorname{ord}_p(f) > 0$, then f has a zero at p.
2. If $\operatorname{ord}_p(f) < 0$, then f has a pole at p.
3. If $\operatorname{ord}_p(f) \geq 0$, then f is regular at p.

Proposition 10.1.8.
Let C be a smooth curve and $0 \neq f \in \bar{k}(C)$. Then, there are only finitely many points in C at which f has a pole or 0. If f has no poles, then $f \in \bar{k}$.

Proof. A standard exercise in Riemann surfaces. □

Example 10.1.9.
Suppose $C_1 : Y^2 = X^3 + X, C_2 : Y^2 = X^3 + X^2$. C_1 is smooth everywhere but C_2 is smooth everywhere except $p = [0 : 0 : 1]$.

In $\bar{k}[C_1]_p$, $M_p = \langle X, Y \rangle$ and $X \in M_p^2$.

Proposition 10.1.10.
Let C/k be a curve and $p \in C$ be a smooth point, and $t \in k(C)$ an uniformiser at p. Then, $k(C)$ is a finite separable extension of $k(t)$.

Proof. $k(C)$ is a finite algebraic extension as it is finitely generated over k and has transcendence degree 0 over $k(t)$ as t is not algebraic over k (it is a local coordinate of C at p).

Now, take $x \in k(C)$ and let $\Phi(T,X) = \sum a_{ij}T^iX^j$ be the minimal polynomial at x over $k(t)$. Say $q = \mathrm{char}(k)$. If $\Phi(T,X)$ is not separable, then $\frac{\partial \Phi(T,X)}{\partial X} = 0$ as $\Phi(T,X)$ is irreducible.

$$
\begin{aligned}
\Phi(T,X) &= \Psi(T,X^p) \\
&= \sum_{k=0}^{q-1}\left(\sum_{i,j} b_{ijk}T^{iq}X^{iq}\right)T^k \\
&= \sum_{k=0}^{q-1}\left(\Phi_k(T,X)\right)^q T^k \text{ since } k \text{ is perfect, every element is a } q\text{-th power}
\end{aligned}
$$

$$\sum_{k=0}^{q-1}\left(\Phi_k(t,x)\right)^q t^k = 0$$

$$\mathrm{ord}_p(\Phi_k(t,x)^q t^k) \equiv k \pmod q$$

This implies that every term in the final sum has distinct order at p. And, hence

$$\Phi_0(t,x) = \Phi_1(t,x) = \cdots = \Phi_{q-1}(t,x) = 0$$

Atleast one of the Φ_is should have a nonzero power of X and $X - \deg \Phi_i < X - \deg \Phi$ and hence $\Phi_k(t,x) = 0$ which contradicts minimality of Φ. Hence, we are done. □

10.2. Morphism between curves

Proposition 10.2.1.
Let C be a curve, $V \subseteq \mathbb{P}^n$ be a variety, $p \in C$ a smooth point and

$$\phi : C \to V$$

a rational map. Then, ϕ is regular at p. In particular, if C is smooth, then ϕ is a morphism.

Proof. Suppose $\phi = [f_0 : \cdots : f_n]$ with $f_i\bar{k}(C)$ and $t \in \bar{k}(C)$ an uniformiser for C at p. Let

$$n = \min \mathrm{ord}_p f_i$$

Then, $\mathrm{ord}_p(t^{-n}f_i) \geq 0 \;\forall\; i$ and $\mathrm{ord}_p(t^{-n}f_j) = 0$ for some j. But then this means $t^{-n}f_i$ are regular at p, $t^{-n}f_j(p) \neq 0$ and thus ϕ is regular at p. □

Remark 10.2.2.
This proposition is not true if either $\dim(C) > 1$ or p is singular

1. $\phi : \mathbb{P}^n \to \mathbb{P}^n$ be $[X : Y : Z] \mapsto [X^2 : YZ : Z^2]$ is not regular at $p = [0 : 1 : 0]$.

2. Suppose $V : Y^2Z = X^3 + X^2Z$ and $V \to \mathbb{P}^1$ be given by $[X : Y : Z] \mapsto [Y : X]$ is not regular at $[0 : 0 : 1]$.

Example 10.2.3. 1. $V : X^2 + Y^2 = Z^2$

Part III.

Basic Algebraic Geometry

11. Lecture-1 (5th January): Introduction

12. Lecture-2 (10 January, 2023): Ideals and Zariski topology

12.1. Ideals

For I, J ideals

$$I + J = \{x + y \mid x \in I, y \in J\}$$
$$IJ = \{\sum x_i y_i \mid x_i \in I, y_i \in J\}$$

- $IJ \subseteq I \cap J$.
- If $I + J = R$, then $I^2 + J^2 = R$. This is because, say $I^2 + J^2 \neq R$, then there is a maximal ideal m such that $I^2 + J^2 \subseteq \mathfrak{m}$. This means $I^2, J^2 \subseteq \mathfrak{m}$. But $\mathfrak{m}$ is prime ideal, therefore $I, J \subseteq \mathfrak{m} \Rightarrow I + J \subseteq \mathfrak{m}$ which is a contradiction. Thus, we are done.
- If $\mathfrak{p}$ is a prime ideal and $IJ \subseteq \mathfrak{p}$. Then, $I \subseteq \mathfrak{p}$ or $J \subseteq \mathfrak{p}$. Suppose not, then there exists $x \in I \backslash \mathfrak{p}, y \in I \backslash \mathfrak{p}$. But then $xy \in IJ \subseteq \mathfrak{p}$.
- $\mathfrak{p} \supseteq I \cap J \Leftrightarrow IJ \subseteq \mathfrak{p}$.

12.2. Zariski topology

Definition 12.2.1.
- For an ideal I, let

$$V(I) = \{\mathfrak{p} \text{ prime ideal } \mid I \subseteq \mathfrak{p}\}$$

- $\mathrm{Spec}(R) = \{$ collection of all prime ideals of $R\}$

Definition 12.2.2 (Zariski Topology)**.**
It is the topology defined on $\mathrm{Spec}(R)$ such that the closed sets are $V(I)$.

Verification that this indeed is a topology.

1. $V(0) = \mathrm{Spec}(R), V(R) = \emptyset$.
2. $V(I) \cup V(J) = V(I \cap J) = V(IJ)$.
3. $\bigcap_{k \in k} V_k = V(\sum_{k \in K} I_k)$. This is because $\mathfrak{p} \supseteq I_k \Leftrightarrow \mathfrak{p} \supseteq \sum_{k \in K} I_k$

Let us now look at the open sets of this topology. The basis for the open sets is given by

$$D(f \in R) = \{ \text{ all prime ideals not containing } f\}$$

Clearly,

$$(V(I))^c = \bigcup_{f \in I} D(f)$$

and moreover, each $D(f)$ is open since $D(f) = (V(\langle f \rangle))^c$

Theorem 12.2.3.
$\mathrm{Spec}(R)$ is quasi-compact.

Proof. We wish to prove that every open cover has a finite subcover. This is equivalent to saying every cover by $D(f_i)$ has a finite subcover. Say

$$\mathrm{Spec}(R) = \bigcup_{i \in I} D(f_i)$$

Take J to be the ideal generated by $f_i's$. Either $J = R$ or $J \subseteq \mathfrak{m}$. Suppose $J \subseteq \mathfrak{m}$, then $f_i \in \mathfrak{m} \in \mathrm{Spec}(R) \Rightarrow \mathfrak{m} \notin D(f_i)\ \forall\ i \Rightarrow D(f_i)$ does not cover $\mathfrak{m}$. A contradiction. Therefore, $J = R$ and this implies $1 =$ some linear combination of f_i and notice that this sum is finite. So, just consider these finitely many $f_i's$ (say the indexing set is K). These cover J. Suppose that $\{D(f_k), k \in K\}$ do not cover $\mathrm{Spec}(R)$. Then, there is a prime ideal $\mathfrak{p} \notin \bigcup_{k \in K} D(f_k) \Rightarrow \mathfrak{p} \ni f_k\ \forall\ k \in K \Rightarrow R \subseteq \mathfrak{p} \Rightarrow\Leftarrow$. Hence, it covers all of $\mathrm{Spec}(R)$ as required.

Another proof:
Suppose $\mathrm{Spec}(R) = \bigcup_{j \in J} U_j = \bigcup_{j \in J} \mathrm{Spec}(R) \backslash \mathcal{V}(I_j) = \mathrm{Spec}(R) \backslash \bigcap_{j \in J} \mathcal{V}(I_j) = \mathrm{Spec}(R) \backslash \mathcal{V}(\sum_{j \in J} I_j)$. This is equivalent to saying that $\mathcal{V}(\sum_{j \in J} I_j) = \emptyset$. So, we conclude that $\sum_{j \in J} I_j = R \Rightarrow \sum_{k \in K} a_k = 1$ for some finite set K. We claim that $\{U_k : k \in K\}$ covers $\mathrm{Spec}(R)$. This is because

$$\begin{aligned}
&\mathcal{V}(\sum_{k \in K} I_k) = 0 \\
&\Rightarrow \mathrm{Spec}(R) = \mathrm{Spec}(R) \backslash \mathcal{V}(\sum_{k \in K} I_k) \\
&\qquad\qquad\quad = \bigcup_{k \in K} \mathrm{Spec}(R) \backslash \mathcal{V}(I_k) \\
&\qquad\qquad\quad = \bigcup_{k \in K} U_k
\end{aligned}$$

This completes the proof. □

Proposition 12.2.4.
Each $D(f)$ is quasi-compact.

Proof. Suppose
$$D(f) = \bigcup D(g_i)$$
and let J be the ideal generated by $g_i's$. Take $\mathfrak{p} \supseteq J$. Then, each $g_i \in J \subseteq \mathfrak{p} \Rightarrow \mathfrak{p} \notin D(g_i) \Rightarrow \mathfrak{p} \notin D(f) \Rightarrow f \in \mathfrak{p} \Rightarrow f \in \bigcap_{\mathfrak{p} \supseteq J} \mathfrak{p}$. Before completing this proof, we need to understand this intersection much better. Refer to following content on nilpotent elements and come back.

Now, we know that $f \in \mathrm{rad}(J)$ which implies $\exists n$ such that $f^n \in J$. We get

$$f^n = \sum_{\text{finite}} r_i g_i$$

Finally, we claim that these $D(g_i)$s cover $D(f)$. □

Definition 12.2.5.
$x \in R$ is nilpotent if $x^n = 0$ for some $n \in \mathbb{N}$.

Remark 12.2.6.
Any nilpotent element ($x^n = 0$ for some n) is clearly in every prime ideal ($0 \in \mathfrak{p}$) and thus in the intersection of all prime ideals. This can be recorded as

$$\bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p} \supseteq \mathrm{Nil}(R)$$

Proposition 12.2.7.

$$\bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p} \subseteq \mathrm{Nil}(R)$$

Proof. Take an element $x \in R \backslash \mathrm{Nil}(R)$ (not nilpotent) and consider the set

$$\Sigma = \{I \trianglelefteq R \mid x^n \notin I \;\forall\, n > 0\}$$

Notice that Σ is a poset with respect to inclusion. And every chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ has an upper bound (union of all the ideals). Thus, we can apply Zorn's lemma to get a maximal element $\mathfrak{p}$ which we claim is prime. Indeed, if $ab \in \mathfrak{p}$ but $a \notin \mathfrak{p}, b \notin \mathfrak{p}$ then $\mathfrak{p} + \langle a \rangle, \mathfrak{p} + \langle b \rangle$ are ideals strictly containing $\mathfrak{p}$ contradicting maximality of $\mathfrak{p}$. Therefore, we can conclude that $x \notin \mathfrak{p} \Rightarrow x \notin \bigcap_{\mathfrak{p} \supseteq J} \mathfrak{p}$ or rather not nilpotent implies not in intersection and hence we have proved the required inclusion. □

$$\mathrm{Nil}(R) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p} = \bigcap_{\mathfrak{p} \subseteq \{0\}} \mathfrak{p}$$

$$\{x \mid x^n \in J\} = \mathrm{rad}(J) = \bigcap_{\mathfrak{p} \supseteq J} \mathfrak{p}$$

13. Lecture-3 (12th January): Zariski topology

13.1. Zariski topology contd..

Definition 13.1.1.
If $J = \mathrm{rad}(J)$, then J is called radical ideal.

Properties:

1. Every radical ideal is an intersection of prime ideals.
2. $\mathcal{V}(J) = \mathcal{V}(\mathrm{rad}(J))$
3. $\mathcal{V}(J) = \mathcal{V}(J')$ implies $\mathrm{rad}(J) = \mathrm{rad}(J')$

Suppose $S \subseteq R$ such that

- $1 \in S, 0 \notin S$
- If $x, y \in S \Rightarrow xy \in S$

Proposition 13.1.2.
Take an ideal maximal wrt not intersecting S. Then, it is prime.

Proof. Suppose $\mathfrak{m}$ is the ideal in question. Next, suppose $\mathfrak{m}$ is not prime which implies $\exists a, b \in R$ such that $ab \in \mathfrak{m}$ but $a, b \notin \mathfrak{m}$. Then, $\mathfrak{m} + \langle a \rangle \supsetneq \mathfrak{m}, \mathfrak{m} + \langle b \rangle \supsetneq \mathfrak{m}$. But, this means $(\mathfrak{m} + \langle a \rangle) \cap S \neq \emptyset \Rightarrow m + ra \in S$ for some $m \in \mathfrak{m}, r \in R$. Similarly, $n + sb \in S$ for some $n \in \mathfrak{m}, s \in R$. But, S is multiplicative therefore $(m + ra)(n + sb) \in S \Rightarrow mn + ran + msb + rsab \in S \Rightarrow ((\langle ab \rangle + \mathfrak{m}) = \mathfrak{m}) \cap S \neq \emptyset$. This is a contradiction. Hence, we are done. $\square$

Proposition 13.1.3.
Say J is maximal wrt not being principal. Then, J is prime.

Proof. Suppose $\mathfrak{m}$ is the ideal in question. Next, suppose $\mathfrak{m}$ is not prime which implies $\exists a, b \in R$ such that $ab \in \mathfrak{m}$ but $a, b \notin \mathfrak{m}$. Next, we can consider the ideal $I = \mathfrak{m} + \langle a \rangle$.

By maximality of $\mathfrak{m}$, we have $I = \langle c \rangle$ for some $c \in R$. Now, consider $J = \{x \in R \mid xc \in \mathfrak{m}\}$. Clearly, $I \subseteq J$. Notice that $c = m + ar$ for some $m \in \mathfrak{m}, r \in R$.

$$\begin{aligned} bc &= b(m + ar) \\ &= bm + (ba)r \\ \Rightarrow bc &\in \mathfrak{m} \\ \Rightarrow b &\in J \end{aligned}$$

This means $b \in J \backslash \mathfrak{m}$. Therefore V is also principal and hence $V = \langle d \rangle$. Since $\mathfrak{m} \in I$, therefore $m = cr$ for some $r \in R$. But this means that $r \in V \Rightarrow r = r'd$ for some $r' \in R$. Hence, $m = cdr' \in \langle cd \rangle \Rightarrow \mathfrak{m} \subseteq \langle cd \rangle$. For the other direction, since $d \in V \Rightarrow cd \in U$. All of these tells us that $\mathfrak{m} = \langle cd \rangle$ a contradiction to our assumption. Therefore, $\mathfrak{m}$ must be prime. □

Proposition 13.1.4.
Say J is maximal wrt not being finitely generated. Then, J is prime.

Proof. Suppose $\mathfrak{m}$ is the ideal in question. Next, suppose $\mathfrak{m}$ is not prime which implies $\exists a, b \in R$ such that $ab \in \mathfrak{m}$ but $a, b \notin \mathfrak{m}$.

If we now look at $\mathfrak{m} + \langle a \rangle$, by our assumption, this ideal is finitely generated by say $u_1, \ldots, u_m$. □

Exercise 13.1.5. *Suppose J is maximal wrt not being generated by a cardinal number of generators. Then, J is prime.*

Definition 13.1.6.
A topological space X is said to be irreducible if it cannot be written as the union of proper closed subsets of X

13.2. Identify closed irreducible subsets of $\operatorname{Spec}(R)$

Proposition 13.2.1.
The sets $\mathcal{V}(\mathfrak{p})$ are exactly the irreducible components of $\operatorname{Spec}(R)$.

Lemma 13.2.2.
Let $I \subseteq R$ be a radical ideal. If $\mathcal{V}(I)$ is irreducible, then I is prime.

Proof. Suppose I is not prime. Then there exists a, b such that $ab \in I$ but $a \notin I$ and $b \notin I$. Consider a prime ideal $\mathfrak{p}$ that contains I, it will also contain ab and thus $\mathfrak{p}$ contains either a or b. This is summarised as

$$\mathcal{V}(I) = (\mathcal{V}(I) \cap \mathcal{V}(a)) \cup (\mathcal{V}(I) \cap \mathcal{V}(b))$$

Thus $\mathcal{V}(I)$ is union of closed sets. It remains to be shown that the sets are proper in order to conclude that $\mathcal{V}(I)$ is not irreducible. Since $\mathcal{V}(I) \cap \mathcal{V}(a) = \mathcal{V}(I + \langle a \rangle)$ and $a \notin I$ therefore $\mathcal{V}(I + \langle a \rangle)$ is a proper closed subset of I and same for b. This is a contradiction to our hypothesis. So, we are done. $\square$

Lemma 13.2.3.
$\mathcal{V}(\mathfrak{p})$ is an irreducible closed subset for $\mathfrak{p}$ prime.

Proof. Suppose $\mathcal{V}(\mathfrak{p}) = V_1 \cup V_2$ with V_1, V_2 proper closed subsets of $V(\mathfrak{p})$. Then there exists ideals I, J such that $\mathcal{V}(\mathfrak{p}) = \mathcal{V}(I) \cup \mathcal{V}(J)$. Since $\mathfrak{p} \in \mathcal{V}(\mathfrak{p})$ this implies $\mathfrak{p} \in \mathcal{V}(I)$ or $\mathfrak{p} \in \mathcal{V}(J)$. Suppose $\mathfrak{p} \in \mathcal{V}(I)$, then $I \subseteq \mathfrak{p} \Rightarrow \mathcal{V}(\mathfrak{p}) \subseteq \mathcal{V}(I) \Rightarrow \mathcal{V}(\mathfrak{p}) = \mathcal{V}(I)$. This is a contradiction to our assumption and hence we are done. $\mathcal{V}(\mathfrak{p})$ is irreducible. $\square$

Proposition 13.2.4.
Every irreducible closed subset of $\operatorname{Spec}(R)$ has an unique generic point.

Proof. Notice that any irreducible closed subset is of the form $\mathcal{V}(\mathfrak{p})$. Now, $\mathcal{V}(\mathfrak{p})$ is the closure of $\mathfrak{p}$. This is because $\operatorname{cl}(\mathfrak{p})$ is a closed set and hence of the form $\mathcal{V}(I)$ for some ideal I. Moreover $\mathfrak{p} \supseteq I$. The biggest ideal I such that $I \subseteq \mathfrak{p}$ is $\mathfrak{p}$ and this gives us what we want because $\mathcal{V}$ reverses inclusions. Therefore, $\operatorname{cl}(\mathfrak{p}) = \mathcal{V}(\mathfrak{p})$. And, such a generic point is unique for suppose $\mathcal{V}(\mathfrak{p}) = \mathcal{V}(\mathfrak{q})$ then clearly $\mathfrak{p} \subseteq \mathfrak{q}$ and $\mathfrak{q} \subseteq \mathfrak{p}$. So, we are done. $\square$

To summarise, Zariski topology has the following properties:

1. $\operatorname{Spec}(R)$ is quasi-compact
2. $\operatorname{Spec}(R)$ has a basis of quasi-compact opens which is closed under intersection.
3. Every irreducible closed subset has a generic point.

Theorem 13.2.5 (Hochster)**.**
Any topological space with the 3 properties is the spectrum of some commutative ring.

Suppose X is spectral. Define a new space X^* with open sets as finite union of quasi-compact open sets in X. This new space is called the Hochster dual.

Theorem 13.2.6.
X^* is also spectral.

Proof. $\square$

14. Lecture-5 (17th January, 2023): Noetherian spaces

14.1. Noetherian spaces

First, let us try to remember all the equivalent definitions of a ring being Noetherian.

Proposition 14.1.1.
The following are equivalent:

1. Every ideal is finitely generated.
2. Every ascending chain of ideals
$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$
stabilises.
3. Every non-empty family of ideals has a maximal element.

Nowhere do we use Zorn's lemma, so in some sense, these properties are essentially about some "finite-ness" property. Thus, Noetherian means strong finiteness in some sense.

Definition 14.1.2.

Definition 14.1.3.

Theorem 14.1.4.
A module M over R is Noetherian iff the module is finitely generated and finitely presented.

Proof. □

Proposition 14.1.5.
The direct sum of projective modules is projective.

Proposition 14.1.6.
The direct product of injective modules is injective.

A question we can ask is when is the direct sum of injective modules injective.

Proposition 14.1.7.
Direct sum of injective modules is injective iff the module is Noetherian.

Part IV.

Algebraic Geometry I

15. Lecture-1 (9th January, 2023): Topological properties and Zariski Topology

15.1. Topological properties

Consider a topological space X.

Definition 15.1.1. 1. We say X is quasi-compact if every open cover of X admits a finite subcover.

2. If $f : X \to Y$ is continuous, we call f quasi-compact if $f^{-1}(V)$ is quasi-compact for all quasi-compact open $V \subseteq Y$.

Exercise 15.1.2. *Composition of quasi-compact maps is quasi-compact.*

Consider the two maps $f : X \to Y$ and $g : Y \to Z$. Next, look at the composition $g \circ f : X \to Z$. For all quasi-compact open $V \subseteq Z$, $(g \circ f)^{-1}(V) = f^{-1} \circ g^{-1}(V)$. Since g is quasi-compact and continuous, $g^{-1}(V)$ is also quasi-compact and open. Similarly, f is also quasi-compact and continuous, therefore $f^{-1}(g^{-1}(V))$ is also quasi-compact and we are done.

Lemma 15.1.3.
X quasi-compact and $Y \subseteq X$ is closed implies Y is quasi-compact.

Proof. Let $\{U_i\}_{i \in I}$ be an open cover of Y. Set $U = X - Y$. Since U_i is open in Y, we have $U_i = Y \cap V_i$ where V_i is open in X. Now we note that $\{V_i\}_{i \in I} \cup U$ covers X but X is quasi-compact and we obtain a finite subcover $\{V_i\}_{i \in J} \cup U$ where J is finite. The corresponding $U_i, i \in J$ must therefore cover Y and we are done. □

Proposition 15.1.4.
If X is quasi-compact and Hausdorff, then $E \subseteq X$ is quasi-compact iff E is closed.

Proof. $\Leftarrow$ direction is done.
$\Rightarrow$ direction is what we need to prove.
Take $x \in X \backslash E$. For each $y \in E$, due to Hausdorff-ness we have two disjoint open sets U_y and U_y containing x and y respectively. Do this for all $y \in E$. The collection

$\{U_y\}_{y\in E}$ covers E but it is quasi-compact thus we get a finite subcover $\{U_{y_i}\}_{i\in I}$ with I finite. Now, let

$$U = \bigcap_{i\in I} U_{y_i}$$

U is clearly open, contains x and is disjoint from E. Since x was chosen arbitrarily, $X\backslash E$ must be open. □

Lemma 15.1.5.
Any finite union of quasi-compact spaces is quasi-compact.

Proof. Suppose $X_i, i = 1, 2, \ldots, n$ are the spaces in question. We want to show that

$$X = \bigcup_{i=1}^{n} X_i$$

is also quasi-compact. Take any cover $\{U_i\}_{i\in I}$ be an open cover of X. Then for each $i = 1, 2, \ldots, n$ it is clear that $\{U_i\}_{i\in I}$ also covers X_i. Using quasi-compactness of X_i we can get a finite subcollection $\{U_{i_j} : j = 1, \ldots, n_i\}$. This can be done for all i. Now, consider $\bigcup_{i=1}^{n}\bigcup_{j=1}^{n_i} U_{i_j}$. This union covers X and is finite. So, we are done. □

Lemma 15.1.6.
Suppose $f : X \to Y$ is continuous, if X is quasi-compact then so is $f(X)$.

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of $f(X)$. Now, $\{f^{-1}(U_i)\}_{i\in I}$ covers X and by continuity, each of them are open. Use quasi-compactness of X to get a finite subcover that covers X.

$$X = \bigcup_{i=1}^{n} f^{-1}(U_i)$$
$$\because f(f^{-1}(U_i)) \subseteq U_i$$
$$\therefore f(X) \subseteq \bigcup_{i=1}^{n} U_i$$

□

Suppose Σ is a poset. Σ satisfies acc if every ascending chain

$$x_1 \leq x_2 \leq \cdots$$

is stationary.

Lemma 15.1.7.
The following are equivalent:

1. Σ satisfies acc.

2. Every non-empty subset of Σ has maximal element.

Proof. $1 \Rightarrow 2$. Suppose $S \subseteq \Sigma$ has no maximal element.
Then choose $x_0 \in S$ non-maximal, then we can find a x_1 such that $x_0 \lneq x_1$. By induction we can construct an infinite chain $x_0 \lneq x_1 \lneq \cdots \neq x_i \lneq \cdots$ which does not terminate which is a contradiction to our hypothesis. Thus, S must have a maximal element.
$2 \Rightarrow 1$. Suppose $x_1 \leq x_2 \leq \cdots \leq x_i \leq$ is an infinite ascending chain, then $S = \{x_i \mid i \geq 1\}$ has no maximal element. $\square$

Definition 15.1.8.
A topological space is called Noetherian if set of all closed subsets of X satisfies dcc.

Lemma 15.1.9.
X Noetherian implies X is quasi-compact.

Proof. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open cover of X that does not have a finite subcover. Consider the collection $\mathcal{F}$ of union of finite number of elements of $\mathcal{U}$. Since being Noetherian is equivalent to saying any finite subset of open subsets has a maximal element, we know that $\mathcal{F}$ has a maximal element. Suppose that maximal element is $U_{i_1} \cup \ldots \cup U_{i_n}$. If this does not cover X, take an element x in the complement of the maximal element. Since $\mathcal{U}$ covers X, there is an $i \in I$ such that $x \in U_i$. Notice that now $U_{i_1} \cup \ldots \cup U_{i_n} \subsetneq U_{i_1} \cup \ldots \cup U_{i_n} \cup U_i$ which contradicts the maximality. Thus, we are done. $\square$

Remark 15.1.10.
The converse need not be true. Consider $[0, 1]$ covered by $[1/2^n, 1]$.

Lemma 15.1.11.
If $X_1, \ldots, X_n$ are Noetherian subspaces of X, then so is $X = X_1 \cup X_2 \cup \ldots \cup X_n$

Proof. Let Y_is be closed in X that forms the chain

$$X \supseteq Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \cdots$$

For each i, we get a chain of closed sets in X_i by intersecting with X_i. This gives us

$$X_i \supseteq Y_1 \cap X_i \supseteq Y_2 \cap X_i \supseteq Y_3 \cap X_i \supseteq \cdots$$

Since X_i is Noetherian, this chain terminates at say r_i. Now, take $r = \max_i r_i$. The original chain will terminate after this point. Suppose $y \in Y_i$ with $i \leq r$, there is an j such that $y \in X_j$. This means $y \in X_j \cap Y_i = X_j \cap Y_r$. Hence, $y \in Y_r$ and we are done. $\square$

Definition 15.1.12.
Locally Noetherian means every point $x \in X$ has a neighbourhood U which is Noetherian wrt subspace topology.

Lemma 15.1.13.
Quasi-compact and locally Noetherian implies Noetherian.

Proof. Since X is locally Noetherian, for each $x \in X$ we have a nbd. U_x that is Noetherian. $\{U_x\}_{x\in X}$ is an open cover of X. Quasi-compactness gives us a finite subcover $\{U_{x_i}\}_{i=1}^n$, i.e.,

$$X = \bigcup_{i=1}^{n} U_{x_i}$$

X is Noetherian from previous lemma. □

Exercise 15.1.14. *Give an example of a ring R such that $\operatorname{Spec}(R)$ is Noetherian but R is not.*

Consider the ring $R = k[X_1, X_2, \ldots,]$ and the ideal $I = \langle X_1^2, X_2^2, \ldots, \rangle$. Now, look at $R' = R/I$. $\operatorname{Spec}(R')$ is a singleton.

Definition 15.1.15.
A topological space X is called irreducible if it cannot be written as finite union of proper closed subsets.

A closed subset $Y \subseteq X$ is called irreducible component of X if it is a maximal irreducible closed subset of X.

Lemma 15.1.16.
If X is Noetherian and $Y \subseteq X$ is a subspace, then Y is Noetherian.

Proof. Let Y_is be closed in Y that forms the chain

$$Y \supseteq Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq \cdots$$

For each i, we have a closed set in X such that $Y_i = Y \cap X_i$. This gives us

$$Y \supseteq X_1 \cap Y \supseteq X_2 \cap Y \supseteq X_3 \cap Y \supseteq \cdots$$

□

Lemma 15.1.17.
Let X be Noetherian. Then, X has finitely many irreducible components.

Proof. More generally, we will show that every closed subset for X has finitely many irreducible components.

Suppose that this is false. Let Σ be the collection of closed subsets of X that does not satisfy our condition. Order this as follows: $A \leq B$ if $A \supseteq B$. If $\{C_i\}$ is a chain in Σ, then it must eventually stabilise since X is Noetherian. This C_α is an upper bound for this chain. Therefore, by Zorn's lemma, there is a maximal element Y. Since $Y \in \Sigma$, therefore it is not irreducible. Suppose $Y = Y_1 \cup Y_2$ with Y_1, Y_2 proper closed subsets of Y. $Y \leq Y_1, Y \leq Y_2$. Since $Y \in \Sigma$, Y is not a finite union of irreducible components. Hence, either Y_1 or Y_2 is not irreducible. If Y_1 is not irreducible but $Y_1 \in \Sigma$, since Y is maximal in Σ and $Y \leq Y_1$, therefore $Y = Y_1$ a contradiction that Y_1 is a proper subset of Y. Thus, Σ must be empty and the claim is proven. □

Lemma 15.1.18.
X is Noetherian implies there exists an unique expression $X = X_1 \cup \cdots \cup X_n$ where $X_i's$ are irreducible components of X.

Proof. Suppose
$$X = X_1 \cup \cdots \cup X_n = X_1' \cup \cdots \cup X_m'$$
Clearly $X_1' \subseteq X$, this means $X_1' = \bigcup_{i=1}^n X_1' \cap X_i$. Since X_1' is irreducible, there must be a i_1 such that $X_1' = X_{i_1} \cap X_1'$. Thus, $X_1' \subseteq X_{i_1}$. We can choose i_1 to be 1 to get $X_1' \subseteq X_1$. Similarly, $X_1 \subseteq X_{j_1}'$. Since $X_1' \subseteq X_{j_1}'$ and our assumption that $X_i \notin X_j$ for $i \neq j$ we conclude that $j_1 = 1$. Finally, we conclude that $X_1 = X_1'$. Let Z be the closure of $X - X_1$, then $Z = X_2 \cup \cdots \cup X_n = X_2' \cup \cdots \cup X_m'$. We can argue inductively and conclude that $X_i = X_i'$ and $n = m$. □

Lemma 15.1.19.
Suppose X is Noetherian and $X_1 \subseteq X$ an irreducible component. Then, X_1 contains a non-empty open set in X.

Proof. Consider $U = X \backslash X_2 \cup \cdots \cup X_n$. Clearly, U is non-empty and open. Moreover, $U \subseteq X_1$ and we are done. □

Definition 15.1.20.
Let X be a topological space. We say that X is a spectral space if the following holds:

1. X is quasi-compact.
2. X is T_0.
3. X has a basis of quasi-compact open sets.

4. Every irreducible closed subset of X has a generic point ($\exists x \in Y$ such that $\overline{\{x\}} = X$)

15.2. Zariski Topology

Let A be a commutative ring with identity and $X = \operatorname{Spec}(A)$.

Zariski topology is the unique topology such that a subset $Y \subseteq X$ is closed iff $Y = \mathcal{V}(I)$ for some ideal $I \trianglelefteq A$. Here,

$$\mathcal{V}(I) = \{\mathfrak{p} \in X \mid \mathfrak{p} \supseteq I\}$$

Theorem 15.2.1.
$\operatorname{Spec}(A)$ is always spectral.

Proof.

1. X is T_0
   For all $f \neq 0$ in A, let $A_f = S^{-1}A$ be the localisation of A at f where $A_f = \{f^n \mid n \geq 0\}$. Next, let $V_f = X \backslash V(f) = \operatorname{Spec}(A_f)$. This forms a basis for the Zariski topology.
   Now, let $\mathfrak{p}, \mathfrak{P}$ be two distinct primes.
   - Suppose $\mathfrak{p} \not\subseteq \mathfrak{P}$.
     $Y = V(\mathfrak{p})$ is closed set and $\mathfrak{P} \notin V(\mathfrak{p})$. Take Y^c. Then $\mathfrak{P} \in Y^c$ and $\mathfrak{p} \notin Y^c$.
   - If $\mathfrak{p} \subseteq \mathfrak{P}$
     Then consider $\mathcal{V}(\mathfrak{P})$. Clearly, $\mathfrak{p} \notin \mathcal{V}(\mathfrak{P})$. Take $U = \mathcal{V}(\mathfrak{P})^c$, then $\mathfrak{p} \in U$ but $\mathfrak{P} \notin U$.
2. X is quasi-compact.
   Let $\{U_i\}$ be an open cover of X. WLOG, we can assume that $U_i = \operatorname{Spec}(A_{f_i}), f \neq 0$. Let I be the ideal generated by these $f_i s$.
   Case-1: Suppose that $I \neq A$. Then there exists a maximal ideal $\mathfrak{m} \supseteq I \Rightarrow \mathcal{V}(\mathfrak{m}) \subseteq \mathcal{V}(I) \Rightarrow X \backslash \mathcal{V}(\mathfrak{m}) \supseteq X \backslash \mathcal{V}(I) = X \backslash \bigcap_{i \in I} \mathcal{V}(f_i) = \bigcup U_i = X$ which is absurd. Hence, we conclude that $I = A$. Next,
   $$1 = \sum_{i=1}^{n} a_i f_i \qquad \text{for some } a_i \in A$$
   $$\Rightarrow \bigcup_{i=1}^{n} U_i = \bigcup_{i=1}^{n} X \backslash \mathcal{V}(f_i)$$
   And, we get the required refinement.
3. X has a basis of quasi-compact open sets follows from the above.
4. Let $Y \subseteq X$ be an irreducible closed subset. Then, $Y = \operatorname{Spec}(A/I)$. WLOG, we can assume X is irreducible. Next, observe that $\operatorname{Spec}(A) = \operatorname{Spec}(A_{\text{red}}) = \operatorname{Spec}(A/\operatorname{Nil}(A))$. Since A is irreducible and reduced, we conclude that A is an integral domain. We are now done since 0 is a generic point in that case.

□

16. Lecture-2 (11th January, 2023): Zariski topology and affine schemes

16.1. Zariski topology contd..

Theorem 16.1.1 (Hochster)**.**
Every spectral space is homeomorphic to $\mathrm{Spec}(A)$ for some commutative ring A.

Notation: **Ring** be the category of commutative rings, **Top** be the category of topological spaces.

Theorem 16.1.2.
There is a contravariant functor

$$\begin{aligned} sp : \textbf{Ring} &\to \textbf{Top} \\ \mathrm{Spec}(B) &\mapsto \mathrm{Spec}(A) \end{aligned}$$

Proof. Consider $f : A \to B$. This induces a map

$$f_\# : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

such that $f_\#(\mathfrak{p}) = f^{-1}(\mathfrak{p})$.
Well-defined: Suppose $xy \in f^{-1}(\mathfrak{p}) \Rightarrow f(xy) = f(x)f(y) \in \mathfrak{p} \Rightarrow$ either x or y lies in $f^{-1}(\mathfrak{p})$ which completes our check.
We claim that $f_\#$ is continuous. This can be seen as follows:
Take a basic open set $D(a), a \in A$. Enough to show for these sets since $D(a)$ forms a basis for the topology on $\mathrm{Spec}(A)$. Now,

$$\mathfrak{p} \in f_\#^{-1}(D(a)) \Leftrightarrow f_\#(\mathfrak{p}) \in D(a) \Leftrightarrow a \notin f^{-1}(\mathfrak{p})$$

But this means

$$a \notin f^{-1}(\mathfrak{p}) \Leftrightarrow f(a) \notin \mathfrak{p} \Leftrightarrow \mathfrak{p} \in D(f(a))$$

□

16.2. Affine schemes

Definition 16.2.1.
$\text{Spec}(A)$ will be called an affine "scheme" (we will see this properly later on).

Definition 16.2.2.
Let $X = \text{Spec}(A), Y = \text{Spec}(B)$. Let $f : Y \to X$ be a continuous map. We call such a map f regular (holomorphic) if there is a ring homomorphism $g : A \to B$ such that $f = g_{\#}$

Example 16.2.3.
Take $\text{Spec}(\mathbb{Z})$ and consider the constant map. This cannot be regular because any ring homomorphism must take 1 to 1 and as a consequence fixes every element.

Proposition 16.2.4.
If $X = \text{Spec}(A)$. A regular function on X is a regular map from X to $\text{Spec}(\mathbb{Z}[t])$.

Proof. □

Remark 16.2.5.
On an affine scheme, the set of all regular maps is the ring A itself since, the map $\mathbb{Z}[t] \to A$ is determined by where t is sent to.

Lemma 16.2.6.
Every affine scheme has a closed point.

Proof. Every commutative ring has a maximal ideal. □

Definition 16.2.7.
Open in affine is called quasi-affine.

Example 16.2.8.
Take A a local integral domain with $\mathfrak{m}$ the maximal ideal. Suppose that all prime ideals of A are of the form

$$\langle 0 \rangle \subset \mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \cdots \subset \{\mathfrak{m}\}$$

Consider $X = \text{Spec}(A) \backslash \mathfrak{m}$. X is open in affine scheme but has no closed point.

An example of such a ring is

$$\Gamma = \mathbb{Z}x_1 \oplus \mathbb{Z}x_2 \oplus \cdots$$

Give an ordering: $\sum a_i x_i \geq 0$ if the first nonzero term is > 0 or all $a_i = 0$
Γ is a totally ordered abelian group and hence there exists a valuation ring A with value group Γ and the prime ideals of Γ are in $1-1$ correspondence with prime ideals of A.

Exercise 16.2.9. *Let* $A = k[X_1, X_2, \ldots], B = A_{\mathfrak{m}}, X = \operatorname{Spec}(B) \backslash \mathfrak{m}, \mathfrak{m} = \langle X_1, X_2, \ldots, \rangle$. *Claim is that* X *has no closed point.*

16.2.1. Fiber products of affine schemes

Suppose A is a commutative ring, B, C are A-algebras. Let $X = \operatorname{Spec}(A), Y = \operatorname{Spec}(B), Z = \operatorname{Spec}(C)$. Next, suppose we have

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

Universal property of fiber products:

$$\begin{tikzcd} W' \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & Y \times_X Z \arrow[r] \arrow[d] & Z \arrow[d, "g_\#"] \\ & Y \arrow[r, "f_\#"'] & X \end{tikzcd}$$

Definition 16.2.10.
If a W exists such that the universal property is satisfied, then W is called the fiber product of Y, Z over X and we write $W = Y \times_X Z$

Theorem 16.2.11.
$\mathbf{Aff}_{\mathbb{Z}}$ = category of affine schemes admits fiber products.

Proof. Consider the following data:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \\ C & \end{tikzcd}$$

Let $D = B \otimes_A C$. We have the natural maps $f_1 : B \to B \otimes_A C$ sending $b \mapsto b \otimes 1$ and $f_2 : C \to B \otimes_A C$ sending $c \mapsto 1 \otimes c$. Both are ring homomorphisms and fit into the following diagram due to the nature of tensor product

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "f_1"] \\ C \arrow[r, "g_1"'] & B \otimes_A C \end{tikzcd}$$

Now, let $W = \mathrm{Spec}(B \otimes_A C)$ and we claim that this satisfies the universal property of fibre product. Apply $\mathrm{Spec}(-)$ functor to the diagram to get

$$\begin{tikzcd} A & B \arrow[l, "f_{\#}"'] \\ C \arrow[u, "g_{\#}"] & \mathrm{Spec}(B \otimes_A C) \arrow[l, "g_{1\#}"] \arrow[u, "f_{1\#}"'] \end{tikzcd}$$

From the universal property of tensor product we have the following diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "f_1"] \arrow[ddr, bend left] & \\ C \arrow[r, "g_1"'] \arrow[drr, bend right] & B \otimes_A C \arrow[dr, dashed, "\exists\,!"] & \\ & & U \end{tikzcd}$$

Again, apply the $\mathrm{Spec}(-)$ functor.

$$\begin{tikzcd} X & Y \arrow[l, "f_{\#}"'] & \\ Z \arrow[u, "g_{\#}"] & \mathrm{Spec}(B \otimes_A C) \arrow[l, "g_{1\#}"] \arrow[u, "f_{1\#}"'] & \\ & & \mathrm{Spec}(U) \arrow[ul, dashed, "\exists\,!"] \arrow[ull, bend left] \arrow[uul, bend right] \end{tikzcd}$$

This completes the proof. $\square$

17. Lecture-3 (16th January, 2023): Category theory brushup

Suppose we have a ring homomorphism $f : A \to B$ and $X = \operatorname{Spec}(A), Y = \operatorname{Spec}(B)$. This induces a map $f_\# : Y \to X$. From, the previous discussion, there is a fiber product $Y \times_X Y$ such that the following diagram makes sense

$$\begin{tikzcd}
X & Y \arrow[l, "f_\#"'] & \\
Y \arrow[u, "f_\#"] & Y \times_X Y \arrow[l, "p_1"] \arrow[u, "p_2"'] & \\
& & Y \arrow[ul, dashed, "\exists\,!\,\Delta_Y"'] \arrow[uul, bend right] \arrow[ull, bend left]
\end{tikzcd}$$

Here, $p_1 \circ \Delta_Y = p_2 \circ \Delta_Y = \mathrm{id}$ where

$$\Delta_Y : Y \to Y \times_X Y$$

is called the relative diagonal of Y/X.

Definition 17.0.1.
Say X_1, X_2 are affine schemes. $X_1 \to X_2$ is a closed immersion iff $A_1 \to A_2$ is a surjective. Here, $\operatorname{Spec}(A_i) = X_i, i = 1, 2$.

Lemma 17.0.2.
Δ_Y is a closed immersion.

Proof. $B \otimes_B B \to B$ is a surjection. □

Example 17.0.3.
Take $A = \mathbb{Z}, B = \mathbb{Z}[t]/\langle t^n \rangle$ for some $n \geq 2$. There is a canonical inclusion $f : A \to B$. This induces a map $Y = \operatorname{Spec}(B) \to X = \operatorname{Spec}(A)$ which is an identity map in terms of sets. Thus, it is a closed inclusion but not a closed immersion.

Remark 17.0.4.
We know that diagonal is closed iff the space is Hausdorff. This seems to contradict our assumptions! But we are fine because this claim is true only when the topology

is the product topology. Here, the topology we have is not the product topology.

Definition 17.0.5.
A regular map $f : X \to Y$ is called separated morphism if the relative diagonal of Y over X is closed in $Y \times_X Y$.

Lemma 17.0.6.
Let $X = \operatorname{Spec}(A)$. Suppose U_1, U_2 are two open affine subsets of X. Then, $U_1 \cap U_2$ is also affine.

Proof. We have two natural injections

$$U_1 \overset{j_1}{\hookrightarrow} X, U_2 \overset{j_2}{\hookrightarrow} X$$

then we naturally have the following

$$U_1 \times_Z U_2 \xrightarrow{j_1 \times j_2} X \times_Z X$$

where $Z = \operatorname{Spec}(\mathbb{Z})$ (if it is blank, just assume Z by default).
From previous discussion we get

$$\begin{tikzcd}
U_1 \times_Z U_2 \arrow[r, "j_1 \times j_2"] & X \times_Z X \\
& X \arrow[u, "\Delta_X"']
\end{tikzcd}$$

Since each term is affine, we can take the fiber product of $U_1 \times_Z U_2$ and X. Say the fiber product is W.

$$\begin{tikzcd}
& U_2 & & X & \\
U_1 & U_1 \times_Z U_2 \arrow[l, "q_2"'] \arrow[u, "q_1"'] \arrow[rr, "j_1 \times j_2"] & & X \times_Z X \arrow[u, "p_1"'] \arrow[r, "p_2"'] & X \\
& W \arrow[u, "\Delta'"] \arrow[rr, "j"'] \arrow[rru, phantom, "\square"] & & X \arrow[u, "\Delta_X"'] &
\end{tikzcd}$$

Then, we claim that

Claim: $W = U_1 \cap U_2$

Proof. Suppose $x \in W$, then $\square$

It now remains to show that W is affine but it is clear from the definition of fiber products.

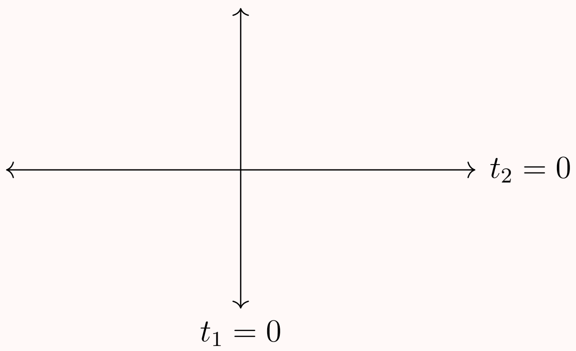
□

Remark 17.0.7.
If Δ_X is a closed immersion then so is Δ'. That is, closed immersions are preserved under fiber products. Follows from right exactness of tensor product.

Now, that we have discussed intersection, we naturally ask : What happens to $U_1 \cup U_2$. Is it still affine ?

The answer turns out to be NO. To see this,

Example 17.0.8 (NON-example)**.**
Consider k be an algebraically closed field. $A = k[t_1, t_2]$ and $X = \mathrm{Spec}(A)$. Let $U_i = \{x \mid t_i(x) \neq 0\} = X \backslash \mathcal{V}(t_i)$. Clearly, U_i is open and affine ($= \mathrm{Spec}(A_{t_i})$). But $U_1 \cup U_2$ is not affine.



U_1 is complement of the horizontal axis and U_2 of the vertical axis. But $U_1 \cup U_2$ is the complement of origin. The question is asking if the complement of origin is affine or not. A highly NON-TRIVIAL question to answer.

Exercise 17.0.9 (not trivial but do think about it)**.** *Suppose $X = \mathrm{Spec}(A)$ and $U \hookrightarrow X$ is affine open. Does this imply $U = \mathrm{Spec}(S^{-1}A)$ for some multiplicatively closed set $S \subseteq A$?*

Definition 17.0.10.
Suppose $S = \mathrm{Spec}(A)$ and $x \in X$. Let $K(A) = S^{-1}(A)$ where S is the set of all nonzero divisors in A. Here, we have $A \hookrightarrow S^{-1}(A) =$ the ring of all meromorphic functions on X. Then,

$$\mathcal{O}_{X,x} = \{f \in K(A) \mid f \text{ is regular in a nbd of } x\}$$

is called the germ of regular function.

Lemma 17.0.11.

$$\mathcal{O}_{X,x} = A_{\mathfrak{p}}$$

where $\mathfrak{p} = x$.

Proof. Suppose f is regular in a nbd of $\mathfrak{p}$ iff there exists $b \notin \mathfrak{p}$ such that $f \notin \mathcal{V}(b)$. But this means $f \notin A_b$ which in turn implies $f \in \bigcup_{b \notin \mathfrak{p}} A_b = A_{\mathfrak{p}}$. $\square$

Definition 17.0.12.
The germs of analytic functions at x is the completion of $\mathcal{O}_{X,x}$, denoted by $\mathcal{O}^{\wedge}_{X,x}$ with respect to its maximal ideal.

Remark 17.0.13.
We have the natural map $\mathcal{O}_{X,x} \to \mathcal{O}^{\wedge}_{X,x}$ but if $\mathcal{O}_{X,x}$ is Noetherian then this map is also injective.

17.1. Categories and functors

A category $\mathcal{C}$ consists of a collection $\mathrm{ob}(\mathcal{C})$ and for all $X, Y \in \mathrm{ob}(\mathcal{C})$, there is a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ and a map

$$\mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

satisfying

1. $\forall\ X \in \mathrm{ob}(\mathcal{C})\ \exists\ 1_X \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$ such that $f \circ 1_X = 1_X \circ f = f$
2. $f \circ (g \circ h) = (f \circ g) \circ h$

A functor (contravariant) $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ is a function $\mathcal{F} : \mathrm{ob}(\mathcal{C}_1) \to \mathrm{ob}(\mathcal{C}_2)$ and a map of sets $\mathcal{F} : \mathrm{Hom}_{\mathcal{C}_1}(X,Y) \to \mathrm{Hom}_{\mathcal{C}_2}(\mathcal{F}(X), \mathcal{F}(Y))$ such that

1. $f(1_X) = 1_{\mathcal{F}(X)}$
2. $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$

To each category $\mathcal{C}$, we associate a category $\mathcal{C}^{\mathrm{op}}$ such that

$$\mathrm{ob}(\mathcal{C}) = \mathrm{ob}(\mathcal{C}^{\mathrm{op}})$$

and

$$\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$$

Suppose $\mathcal{F}, \mathcal{F}' : \mathcal{C} \to \mathcal{C}'$ be two functors. Then, a natural transformation is $T : \mathcal{F} \to \mathcal{F}'$ consisting of the following data:

1. $\forall\, X \in \mathcal{C}, \exists\, T_X : \mathcal{F}(X) \to \mathcal{F}'(X)$,i.e., $T_X \in \mathrm{Hom}_{\mathcal{C}'}(\mathcal{F}(X), \mathcal{F}'(X))$ such that for all $f : X \to Y$, the diagram commutes

$$\begin{tikzcd}
\mathcal{F}(X) \arrow[r, "T_X"] \arrow[d, "\mathcal{F}(f)"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F}'(X) \arrow[d, "\mathcal{F}'(f)"] \\
\mathcal{F}(Y) \arrow[r, "T_Y"'] & \mathcal{F}'(Y)
\end{tikzcd}$$

Given, $\mathcal{C}, \mathcal{C}'$ then $F(\mathcal{C}, \mathcal{C}')$ = all functors from $\mathcal{C}$ to $\mathcal{C}'$ is a category and $\mathrm{Hom}_{F(\mathcal{C},\mathcal{C}')}(F_1, F_2) =$ all natural transformations from F_1 to F_2

Part V.

Topics in Analytic Number Theory

18. Lecture-1: Hardy-Littlewood proof of infinitely many zeros on the line $\mathfrak{R}(s) = 1/2$

19. Lecture-2:

20. Lecture-3 (10th January, 2023): Siegel's theorem

Theorem 20.0.1 (Siegel)**.**
Let $\chi(q)$ be a real Dirichlet character modulo $q \geq 3$. Given any $\epsilon > 0$, we have

$$L(1,\chi) \geq \frac{C_\epsilon}{q^\epsilon}$$

A trivial lower bound: $L(1,\chi) \gg q^{-1/2}$

Goldfeld's proof. Consider

$$f(s) = \zeta(s)L(s,\chi_1)L(s,\chi_2)L(s,\chi_1\chi_2)$$

with $\chi_i, i = 1,2$ primitive quadratic characters. Notice that $f(s) = \sum_n b_n n^{-s}$ with $b_1 = 1, b_n \geq 0$. Let $\lambda = \mathrm{Res}_{s=1} f(s) = L(1,\chi_1)L(1,\chi_2)L(1,\chi_1\chi_2)$

Lemma 20.0.2.
Given any $\epsilon > 0$, one can find $\chi_1(q_1)$ and β with $1-\epsilon < \beta < 1$ such that $f(\beta) \leq 0$, independent of what $\chi_2(q_2)$ is.

Proof. **Case-1:** If there are no real zeros of $L(s,\psi)$ for any primitive quadratic character in $(1-\epsilon, 1)$, then $f(\beta) < 0$ for any $\beta \in (1-\epsilon, 1)$. This is because

$$f(\beta) = \underbrace{\zeta(\beta)}_{<0}\underbrace{L(s,\chi_1)L(s,\chi_2)L(s,\chi_1\chi_2)}_{>0}$$

as $L(1,\chi) > 0$ and L is continuous so any change of sign will lead to a zero which is a contradiction.
Case-2: If we cannot find such a ψ, then just set $\chi_1 = \chi$ and let β be the real zero. Then, $f(\beta) = 0$. We are done. □

Next, consider the integral

□

Corollary 20.0.3.

$$
\begin{aligned}
h(-d) &= \frac{L(1,\chi_d)\sqrt{|d|}\,\omega}{2\pi} \\
&= \frac{L(1,\chi_d)}{\log \epsilon_d}
\end{aligned}
$$

Theorem 20.0.4 (Y. Zhang)**.**

$$L(1,\chi) \geq \frac{c}{(\log q)^{2022}}$$

Theorem 20.0.5.
If $\chi(q)$ does not have a Siegel zero, then $L(1,\chi) \gg \frac{1}{\log q}$

21. Lecture-4 (12th January, 2023): PNT for Dirichlet characters and APs

Lemma 21.0.1.
If $\rho = \beta + i\gamma$ runs through nontrivial zeros of $L(s, \chi)$, then

$$\sum_{\rho} \frac{1}{1 + (T - \gamma)^2} = \mathcal{O}(\log q(|T| + 2)) \forall T \in \mathbb{R}$$

Lemma 21.0.2.

$$N(T + 1, \chi) - N(T, \chi) = \mathcal{O}(\log q(|T| + 2))$$

Lemma 21.0.3.

$$\sum_{\rho : |\gamma - t| \leq 1} \frac{1}{s - \rho} + \mathcal{O}(\log qt) = \frac{L'}{L}(s, \chi)$$

for $-1 \leq \sigma \leq 2, |t| \geq 2, L(s, \chi) \neq 0$

Lemma 21.0.4.
Let $\chi(q)$ be primitive, $q \geq 3, T \geq 2$. Then, there exists $T_1 \in [T, T + 1]$ such that $\frac{L'}{L}(\sigma \pm iT_1, \chi) \ll (\log qT)^2, -1 \leq \sigma \leq 2$.

Lemma 21.0.5.
Put $a = 1$ if χ is even and 0 otherwise.

$$\mathcal{A}(a) := \{s \in \mathbb{C} \mid \sigma \leq -1, |s + 2n - a| \geq \frac{1}{4} \, \forall \, n \geq 1\}$$

Then,

$$\frac{L'}{L}(s, \chi) \ll \log(q(|s| + 1))$$

on $\mathcal{A}(a)$

These are all the ingredients needed to prove the the explicit formula for $\psi_0(x, \chi)$.

Theorem 21.0.6.

$$\psi(s,\chi) = \sum_{n \le x} \Lambda(n)\chi(n)$$

$$\psi_0(x,\chi) = \frac{1}{2}(\psi(x^+,\chi)+\psi(x^-,\chi)) = -\sum_{\rho:|\gamma|\le t} \frac{x^\rho}{\rho} - \frac{1}{2}\log(x-1) - \frac{\chi(-1)}{2}\log(x+1) + C_\chi + R_\chi(T)$$

where $C_\chi = \frac{L'}{L}(1,\overline{\chi}) + \log\frac{q}{2\pi} - \gamma$ and $R_\chi(T) \ll (\log x)\min(1, x/T < x >) + \frac{x}{T}(\log(qxT))^2$. Letting $T \to \infty$ we see that $R_\chi(T) \to 0$.

Theorem 21.0.7 (Brun-Titsmarsh inequality)**.**
Let $x \ge 0, y \ge 2q$. Then,

$$\pi(x+y;q,a) - \pi(x;q,a) \le \frac{2y}{\phi(q)\log(\frac{y}{q})}\left(1 + \mathcal{O}(\frac{1}{\log(\frac{y}{q})})\right)$$

Remind him to prove this later; uses Sieve theoretic methods

Theorem 21.0.8 (PNT for Dirichlet characters)**.**
There exists a $c_1 \ge 0$ such that for all $q \le \exp(c_1\sqrt{\log x})$, we have

$$\psi(x,\chi) = \sum_{n \le x} \Lambda(n)\chi(n) = \begin{cases} E_0(x) + \mathcal{O}(x\exp(-c_1\sqrt{\log x})) & \chi \text{ has no Siegel zero} \\ -\frac{x^{\beta_1}}{\beta_1} + \mathcal{O}(x\exp(-c_1\sqrt{\log x})) & \chi \text{ has Siegel zero} \end{cases}$$

Here, $E_0(\chi) = 1$ if $\chi = \chi_0$ and 0 otherwise.

Recall from MA317 that $L(x,\chi) \neq 0$ when $\sigma \ge 1 - \frac{c}{\log q\tau}$ for some constant $c > 0$ with the exception of atmost one real zero (β_1 the Siegel zero)

Proposition 21.0.9.
Let c be as above and assume that $\sigma \ge 1 - \frac{c}{2\log q\tau}$. Then,

1. If $L(s,\chi)$ has no Siegel zero or if β_1 is a Siegel zero (thus χ quadratic) but $|s-\beta_1| \ge \frac{1}{\log q}$, then
$$\frac{L'}{L}(s,\chi) \ll \log q\tau$$
$$|\log L(s,\chi)| \ll \log\log q\tau + \mathcal{O}(1)$$
$$\frac{1}{L(s,\chi)} \ll \log q\tau$$
2. If β_1 is a Siegel zero and $|s-\beta_1| \le \frac{1}{\log q}$, then
$$\frac{L'}{L}(s,\chi) = \frac{1}{s-\beta_1} + \mathcal{O}(\log q)$$

$$|\arg L(s,\chi)| \leq \log\log q + \mathcal{O}(1)$$

$$|s-\beta_1| \ll |L(s,\chi)| \ll |s-\beta_1|(\log q)^2$$

Part VI.

Commutative Algebra