भारतीय विज्ञान संस्थान

TATE'S THESIS – FOURIER ANALYSIS ON NUMBER FIELDS

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Abstract

Tate's PhD thesis (attributed to Iwasawa as well) develops Fourier analysis on Global Fields, proving in particular the Poisson summation formula which enables us to study analytic continuation and functional equation of Hecke L-functions attached to Größencharactere. The content of this thesis is also known as the GL_1 theory of automomorphic forms in modern literature. An exposition of this viewpoint can be found in [Kud04][Bum97].

1 Overview

The Riemann-zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

defined for $\text{Re}(s) > 1$, can be extended meromorphically to other values of s by analytic continuation and follows the functional equation [Lan94] [Dav80]

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s) \tag{2}$$

Riemann's proof crucially depends on the Poisson summation formula

$$\sum_{n\in\mathbb{Z}} f(at) = \frac{1}{|t|}\sum_{n\in\mathbb{Z}} \widehat{f}(a/t) \tag{3}$$

where $\widehat{f}$ is the Fourier transform on the reals $\widehat{f}(\xi) := \int_{\mathbb{R}} \exp(-2\pi i\xi x) f(x) dx$ where $\exp(x) = 2\pi i x$ is the standard character from $\mathbb{R}$ to S^1. Applying this formula to the Gaussian function $\exp(\pi|x|^2)$ which is its own (additive) Fourier transform, and then applying Mellin transform (multiplicative Fourier transform), one obtains the functional equation (2). An important thing to notice in the proof is how the additive and multiplicative Fourier transform combine to give the functional equation.

In his PhD thesis [Cas+76] (independently found by Iwasawa [Iwa]), Tate was able to extend this idea to the adéle ring of a number field. He was able to develop Poisson summation formula in this setting 8, apply this formula to the adélic Gauss function which is its own Fourier transform and then apply adélic Mellin transform to obtain a functional equation 13 of the type we saw before.

The significance of this thesis lies in the fact that the theory developed in the process allows one to seamlessly generalise the functional equation to Dedekind zeta functions and Hecke L-functions, thereby giving us a large class of functions whose analytic continuation and functional equation is readily available.

This article is just a synopsis of [Cas+76], but there are other expositions as well, as can be seen in [Kud04][RV99][Bum97][Lan94]. These notes [Poo] are also an excellent resource. [1]

2 Local theory

Let $K_\mathfrak{p}$ be a local field (completion of a global field 3 K at $\mathfrak{p}$). Depending on whether $\mathfrak{p}$ gives rise to Archimedean valuation or a discrete valuation, we can choose a normalised absolute value $|\cdot|$.

2.1 Additive local theory

Let $K_\mathfrak{p}^+$ be the additive group of the local field $K_\mathfrak{p}$. Each $\eta \in K_\mathfrak{p}^+$ can be identified with the character χ_η where $\chi_\eta(\xi) = \exp(2\pi i\Lambda(\eta\xi))$ giving an isomorphism (algebraic and topological) between $K_\mathfrak{p}^+$ and the character group $\widehat{K_\mathfrak{p}^+}$.
For a measure, let μ be the measure for $K_\mathfrak{p}^+$. Then, multiplication by any $0 \neq \alpha \in K_\mathfrak{p}$ is an automorphism and hence $\mu_1(M) = \mu(\alpha M)$ is also a Haar measure, moreover $\mu(\alpha M) = |\alpha|\mu(M)$. This allows us to construct measures that are self dual. As a consequence, we have the following

Theorem 1. *[Cas+76, Theorem 2.2.2] For $f \in L_1(K_\mathfrak{p}^+)$, if we define the Fourier transform $\widehat{f}$ by $\widehat{f}(\eta) = \int_{K_\mathfrak{p}^+} f(\xi)\exp(-2\pi i\Lambda(\eta\xi))d\xi$. Then with our choice of measure we have the inversion formula $f(\xi) = \int_{\widehat{K_\mathfrak{p}^+}} \widehat{f}(\eta)\exp(2\pi i\Lambda(\xi\eta))d\eta = \widehat{\widehat{f}}(-\xi)$*

[1] I am writing a more detailed exposition here.

2.2 Multiplicative local theory

We repeat the same process and construct the multiplicative characters. The map $\alpha \mapsto |\alpha|$ is a continuous homomorphism from the group $K_\mathfrak{p}^\times$ to the group of positive reals. The kernel of this map (denoted by U) is called the unit group. The quasi-characters that are trivial on U are said to be unramified. It can be shown that any unramified character is of the form $c(\alpha) = |\alpha|^s \equiv e^{s \log |\alpha|}$ where $s \in \mathbb{C}$ is determined by c if $\mathfrak{p}$ is Archimedean and s is determined only mod $2\pi i / \log(\mathrm{Norm}(\mathfrak{p}))$ if $\mathfrak{p}$ is discrete. As a consequence, every quasi-character $c(\alpha)$ of $K_\mathfrak{p}^\times$ is of the form $c(\alpha) = \tilde{c}(\tilde{\alpha})|\alpha|^s$ where $\tilde{c}$ is a character of U (uniquely determined by c) and s is determined as in the case of unramified quasi-characters.
A measure on $K_\mathfrak{p}^\times$ is constructed using the additive measure $d\xi$ on $K_\mathfrak{p}^+$. For every $g(\alpha) \in L_1(K_\mathfrak{p}^\times)$ we have $g(\xi)|\xi|^{-1} \in L_1(K_\mathfrak{p}^+ - 0)$. Thus, we can define e linear functional on $L_1(K_\mathfrak{p}^\times)$ by

$$\Phi(g) = \int_{K_\mathfrak{p}^+ - 0} g(\xi)|\xi|^{-1} d\xi$$

Then, by Riesz representation theorem, we have a Haar measure $d^\times \alpha$ such that

$$\int_{K_\mathfrak{p}^\times} g(\alpha) d^\times \alpha = \int_{K_\mathfrak{p}^+ - 0} g(\xi)|\xi|^{-1} d\xi \tag{4}$$

For our computations, we need to choose $d^\times \alpha$ such that it gives the subgroup U measure 1.

2.3 Local zeta function's analytic continuation and functional equation

Let $f(\xi)$ be the complex valued function defined on $K_\mathfrak{p}^+$ and $f(\alpha)$ its restriction to $K_\mathfrak{p}^\times$. By $\mathfrak{z}$ denote the class of functions such that: $f(\xi), \widehat{f}(\xi)$ are continuous and belong to $L_1(K_\mathfrak{p}^+)$; $f(\alpha)|\alpha|^\sigma$ and $\widehat{f}(\alpha)|\alpha|^\sigma \in L_1(K_\mathfrak{p}^\times)$ for $\sigma > 0$.

Definition 2. *[Cas+76, Definition 2.4.1] For each $f \in \mathfrak{z}$, we define a function of quasi-characters c, defined for all quasi-characters of exponent greater than 0 by*

$$\zeta(f, c) = \int_{K_\mathfrak{p}^\times} f(\alpha) c(\alpha) d^\times \alpha$$

It can be shown that $f \in \mathfrak{z}$ is regular in the domain of quasi-characters of exponent greater than 0. This lets us get to the fundamental result in the local theory, namely

Theorem 3 (Analytic Continuation and Functional Equation)**.** *[Cas+76, Theorem 2.4.1] A ζ-function has an analytic continuation to the domain of all quasi-characters and satisfies the functional equation*

$$\zeta(f, c) = \rho(c)\zeta(\widehat{f}, c^\vee) \tag{5}$$

where $c^\vee(\alpha) = |\alpha| c^{-1}(\alpha)$. Moreover, the factor $\rho(c)$ is independent of the function f, is meromorphic function of quasi-characters with exponent in the region $(0, 1)$ by the functional equation itself and for all quasi-characters by analytic continuation.

3 Global theory

In this section, let K be a global field (finite extension of $\mathbb{Q}$ or $\mathbb{F}_q(t)$), $\mathcal{O}_K$ be the associated ring of integers.

3.1 Restricted product topology

Definition 4. *[Cas+76, §§3.1] Let $\{\mathfrak{p}\}$ be a set of indices. Suppose for each $\mathfrak{p}$ we have a locally compact abelian group G_p and for almost all $\mathfrak{p}$, a fixed subgroup $H_\mathfrak{p} \subseteq G_\mathfrak{p}$ which is compact and open. Then, we can define a new group G as the restricted product topology of $G_\mathfrak{p}$ with respect to $H_\mathfrak{p}$. Or, $G := \prod_\mathfrak{p} (G_\mathfrak{p}, H_\mathfrak{p})$*

3.1.1 Characters

Let $c(\mathfrak{a})$ be a quasi-character of G and $c_\mathfrak{p}$ be the restriction of c to $G_\mathfrak{p}$. We have a very important characterisation of such quasi-characters: Any a quasi-character of G, $c(\mathfrak{a})$ is of the form $c(\mathfrak{a}) = \prod_\mathfrak{p} c_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})$ with

$c_\mathfrak{p}$ being trivial on $H_\mathfrak{p}$ for almost all $\mathfrak{p}$. Moreover,if $H_\mathfrak{p}^* \subseteq \widehat{G}_\mathfrak{p}$ is the subgroup of all quasi-characters that are trivial on $H_\mathfrak{p}$, then $\widehat{G} \simeq \prod_\mathfrak{p}(\widehat{G}_\mathfrak{p}, H_\mathfrak{p}^*)$ (the isomorphism being algebraic and topological).

3.1.2 Measures

For each $\mathfrak{p}$, we choose a measure $d\mathfrak{a}_\mathfrak{p}$ on each $G_\mathfrak{p}$ such that $\int_{H_\mathfrak{p}} d\mathfrak{a}_\mathfrak{p} = 1$ for almost all $\mathfrak{p}$. Then, we can construct a measure on G which can be written symbolically as $d\mathfrak{a} = \prod_\mathfrak{p} d\mathfrak{a}_\mathfrak{p}$. For each choice of measure $d\mathfrak{a}_\mathfrak{p}$ we can choose a dual measure $dc_\mathfrak{p}$ of $\widehat{G}_\mathfrak{p}$. If $f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})$ and $\widehat{f}_\mathfrak{p}(c_\mathfrak{p})$ is a function and Fourier transform of the function as in the local setting, then letting $f_\mathfrak{p}(\mathfrak{a}_\mathfrak{p})$ be the characteristic function of $H_\mathfrak{p}$, then it can be easily shown that $\int_{H_\mathfrak{p}^*} dc_\mathfrak{p} = 1$ for almost all $\mathfrak{p}$. This means we can put $dc = \prod_\mathfrak{p} dc_\mathfrak{p}$. This measure dc is the dual of $d\mathfrak{a}$.

3.2 Poisson summation formula

Definition 5. *[Cas+76, Definition 4.1.1] Let $\{\mathfrak{p}\}$ be the places (Archimedean and non-Archimedean) of the global field K. If $K_\mathfrak{p}$ is the completion of K at $\mathfrak{p}$ and $\mathcal{O}_\mathfrak{p}$ the ring of integers of $K_\mathfrak{p}$. Then, the adele ring of K, $\mathbb{A}_K := \prod(K_\mathfrak{p}, \mathcal{O}_\mathfrak{p})$ is the restricted product topology of $K_\mathfrak{p}$ with respect to $\mathcal{O}_\mathfrak{p}$.*

Theorem 6. *[Cas+76, Theorem 4.1.1] $\mathbb{A}_K$ is naturally its own character group $\widehat{\mathbb{A}_K}$ if we identify each element $\eta \in \mathbb{A}_K$ with the character $\chi_\eta : \mathfrak{X} \mapsto \exp(2\pi i \Lambda(\eta\mathfrak{X}))$ of $\mathbb{A}_K$.*

As described in the previous section, $\mathbb{A}_K$ has the measure $d\mathfrak{X} = \prod_\mathfrak{p} d\mathfrak{X}_\mathfrak{p}$ with $d\mathfrak{X}_\mathfrak{p}$ being the local self-dual additive measure. Since, the local measures are self-dual, so is $d\mathfrak{X}$. Thus, we have the following

Theorem 7. *[Cas+76, Theorem 4.1.2] If for a function $f \in L_1(\mathbb{A}_K)$, we define the Fourier transform $\widehat{f}(\mathfrak{n}) = \int_{\mathbb{A}_K} f(\mathfrak{X}) e^{-2\pi i \Lambda(\mathfrak{X})} d\mathfrak{X}$ Then, the inversion formula is*

$$f(\mathfrak{X}) = \int_{\widehat{\mathbb{A}_K}} \widehat{f}(\mathfrak{n}) e^{2\pi i \Lambda(\mathfrak{X})} d\mathfrak{n} \tag{6}$$

Next, we define the additive fundamental domain $D \subseteq \mathbb{A}_K$ such that $\mathbb{A}_K = \bigsqcup_{\xi \in K} (\xi + D)$. A consequence of this is that K is a discrete subgroup of $\mathbb{A}_K$ and the factor group $\mathbb{A}_K/K$ is compact.

A function $\varphi(\mathfrak{X})$ is called periodic if $\varphi(\mathfrak{X} + \xi) = \varphi(\mathfrak{X})$ for all $\xi \in K$. It can be shown that if $\varphi(\mathfrak{X})$ is continuous and periodic, then $\int_D \varphi(\mathfrak{X}) d\mathfrak{X} = \int_{\mathbb{A}_K/K} \Phi(\mathfrak{y}) d\mathfrak{y}$ such that $\int_{\mathbb{A}_K/K} d\mathfrak{y} = 1$.
Noting that $K \simeq \widehat{\mathbb{A}_K/K}$, the Fourier transform $\widehat{\varphi}(\xi)$ of the continuous function on $\mathbb{A}_K/K$ represented by $\varphi(\mathfrak{X})$ is defined as

$$\widehat{\varphi}(\xi) = \int_D \varphi(\mathfrak{X}) e^{-2\pi i \Lambda(\xi\mathfrak{X})} d\mathfrak{X} \tag{7}$$

Moreover, we are also able to obtain the inverse Fourier transform enabling us to obtain the following

Lemma 8 (Poisson summation Formula). *[Cas+76, Lemma 4.2.4] If $f(\mathfrak{X})$ satisfies the conditions: $f(\mathfrak{X})$ is continuous, and is in $L_1(\mathbb{A}_K)$; $\sum_{\xi \in K} f(\mathfrak{X} + \xi)$ is uniformly convergent for $\mathfrak{X} \in D$; $\sum_{\xi \in K} |\widehat{f}(\xi)|$ is convergent. Then,*

$$\sum_{\xi \in K} \widehat{f}(\xi) = \sum_{\xi \in K} f(\xi) \tag{8}$$

3.3 Riemann-Roch theorem (Harmonic Analysis version)

Recall from the local theory, multiplication by $a \in K^\times$ is an automorphism of $K_\mathfrak{p}^+$. We want to find such an a in the case of adeles as well. Turns out that such elements are exactly the ideles. Multiplication by idele $\mathfrak{a}$, *i.e.*, $(\mathfrak{X} \mapsto \mathfrak{a}\mathfrak{X})$ is an automorphism and $|\mathfrak{a}| = \prod_\mathfrak{p} |\mathfrak{a}_\mathfrak{p}|_\mathfrak{p}$. This and 8 allows us to deduce the

Theorem 9 (Riemann-Roch theorem). *[Cas+76, Theorem 4.2.1] If $f(\mathfrak{X})$ satisfies the conditions: $f(\mathfrak{X})$ is continuous, and is in $L_1(\mathbb{A}_K)$; $\sum_{\xi\in K} f(\mathfrak{a}(\mathfrak{X}+\xi))$ is convergent for all idéle s $\mathfrak{a}$ and uniformly convergent for $\mathfrak{X}\in D$; $\sum_{\xi\in K} |\widehat{f}(\mathfrak{a}\xi)|$ is convergent for all idéle s $\mathfrak{a}$. Then,*

$$\frac{1}{|\mathfrak{a}|}\sum_{\xi\in K}\widehat{f}(\xi/\mathfrak{a}) = \sum_{\xi\in K} f(\mathfrak{a}\xi) \tag{9}$$

3.4 Global functional equation and analytic continuation

Definition 10. *[Cas+76, Definition 4.3.1] Let K be a global field with $K_{\mathfrak{p}}$ the completion of K at $\mathfrak{p}$. Let $U_{\mathfrak{p}}$ be the unit group, then the multiplicative group $\mathbb{I}_K$ of idéle s is defined as $\mathbb{I}_K := \prod(K_{\mathfrak{p}}^{\times}, U_{\mathfrak{p}})$.*

Embedding $K^{\times}$ in idéle s by identifying $\alpha \in K^{\times}$ with $\mathfrak{a} = (\alpha, \alpha, \dots)$ and using the product formula $\prod_{\mathfrak{p}} |\alpha|_{\mathfrak{p}} = 1$ for $\alpha \in K^{\times}$, allows us to use the full potential of these objects. Just like the additive global theory (the adelic setting), we would like to repeat the process and obtain a fundamental domain which will come in handy while computing integrals. This will also allow us to obtain the global zeta function and talk of its analytic continuation.

Definition 11. *[Cas+76, Definition 4.3.2] Let h be the class number of K and select idéle s $\mathfrak{b}^{(1)}, \mathfrak{b}^{(h)} \in J$ such that $\varphi(\mathfrak{b}^{(1)}), \dots, \varphi(\mathfrak{b}^{(h)})$ represent different ideal classes. Let w be the number of roots of unity in K. Let E_0 be the subset of all $\mathfrak{b} \in \ell(P)$ such that $0 \le \arg \mathfrak{b}_{\mathfrak{p}_0} < 2\pi/w$. We define the multiplicative fundamental domain E for $J/K^{\times}$ to be*

$$E = E_0\mathfrak{b}^{(1)} \cup \dots \cup E_0\mathfrak{b}^{(h)}$$

where $\ell\colon J_{S_\infty} \to \mathbb{R}^r$ sends $\mathfrak{b} \in J_{S_\infty}$ to $\ell(\mathfrak{b}) = (\dots, \log|\mathfrak{b}|_{\mathfrak{p}}, \dots)_{\mathfrak{p}\in S_\infty'}$, S_∞' is the set of all Archimedean primes except $\mathfrak{p}_0$ and $P := \left\{\sum_{i=1}^{r} x_i\ell(\epsilon_i) : 0 \le x_i < 1 \;\forall\, 1 \le i \le r\right\}$, $\{\epsilon_i\}_{1\le i\le r}$ is a basis for groups of units modulo the roots of unity.

It can be shown that $J = \bigsqcup_{\alpha\in K^{\times}} \alpha E$ and the volume of E is $\kappa = (2^{r_1}(2\pi)^{r_2}hR)/(\sqrt{|d|}w)$. As a consequence $K^{\times}$ is discrete subgroup of J and $J/K^{\times}$ is compact.

Let $f(\mathfrak{X})$ be a function on $\mathbb{A}_K$ and $f(\mathfrak{a})$ its restriction to idéle s. We let $\mathfrak{z}$ be the class of all functions satisfying the conditions: $f(\mathfrak{X}), \widehat{f}(\mathfrak{X})$ are continuous, $\in L_1(\mathbb{A}_K)$; $\sum_{\xi\in K} f(\mathfrak{a}(\mathfrak{X}+\xi))$ and $\sum_{\xi\in K}\widehat{f}(\mathfrak{a}(\mathfrak{X}+\xi))$ are both convergent for each idéle $\mathfrak{a}$ and adéle $\mathfrak{X}$, the convergence being uniform in the pair $(\mathfrak{a}, \mathfrak{X})$ for $\mathfrak{X} \in D$ and $\mathfrak{a}$ ranging over any fixed compact subset $\mathbb{I}_K$; $f(\mathfrak{a})|\mathfrak{a}|^{\sigma}$ and $\widehat{f}(\mathfrak{a})|\mathfrak{a}|^{\sigma}$ belong to $L_1(\mathbb{I}_K)$ for $\sigma > 1$. Just as in the case of local theory, we define the "global" ζ-function.

Definition 12. *[Cas+76, Definition 4.4.1] For each $f \in \mathfrak{z}$, we define a function $\zeta(f, c)$ of quasi-characters c for all quasi-characters with exponent greater than 1 by*

$$\zeta(f, c) = \int_{\mathbb{I}_K} f(\mathfrak{a})c(\mathfrak{a})d\mathfrak{a}$$

We call such a function a ζ-function of K.

A simple application of the Poisson Summation Formula 8 and the Riemann-Roch theorem 9 gives us:

Theorem 13 (Global Analytic Continuation and Functional Equation). *[Cas+76, Theorem 4.4.1] By analytic continuation we may extend the domain of any $\zeta(f, c)$ to the domain of all quasi-characters. The extended function is single-valued and regular except at $c(\mathfrak{a}) = 1$ and $c(\mathfrak{a}) = |\mathfrak{a}|$ where it has simple poles with residues $-\kappa f(0)$ and $\kappa\widehat{f}(0)$ respectively. Moreover, $\zeta(f, c)$ satisfies the functional equation*

$$\zeta(f, c) = \zeta(\widehat{f}, c^{\vee})$$

This theorem is the required generalisation that allows us to talk about the analytic continuation and functional equation of Dedekind zeta functions and Hecke L-functions in full generality.

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