Copy of a letter from Newton to Henry Oldenburg, dated 13 June 1676

Author: Isaac Newton

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Dignissime Domine

Quanquam D. Leibnitij modestia in excerptis quæ ex Epistola ejus ad me nuper misisti, nostratibus multùm tribuat circa speculationem quandam infinitarum Serierum de qua jam cœpit esse rumor: nullus dubito tamen quin ille, non tantùm (quod asserit) methodum reducendi quantitates quascunque in ejusmodi series, sed et varia compendia, fortè nostris similia, si non et meliora, adinvenerit. Quoniam tamen ea scire pervelit quæ ab Anglis hâc in re inventa sunt, et ipse ante annos aliquot in hanc speculationem inciderim: ut votis ejus aliqua saltern ex parte satisfacerem nonnulla eorum quæ mihi occurrerunt, ad te transmisi.

Fractiones in infinitas Series reducuntur per divisionem et quantitates radicales per extractionem radicum, perindè instituendo operationes istas in speciebus ac institui solent in decimalibus numeris. Hæc sunt fundamenta harum reductionum; sed extractiones radicum multùm abbreviantur per hoc Theorema.

$$\overline{\mathrm{P+PQ}}\Big\}^{\frac{m}{n}} = \mathrm{P}^{\frac{m}{n}} + \frac{m}{n}\mathrm{AQ} + \frac{m-n}{2n}\mathrm{BQ} + \frac{m-2n}{3n}\mathrm{CQ} + \frac{m-3n}{4n}\mathrm{DQ} + \&c\ .$$

Ubi $\mathrm{P+PQ}$ significat quantitatem cujus radix, vel etiam dimensio, quævis vel radix dimensionis investiganda est, P primum terminum quantitatis ejus, Q reliquos terminos divisos per primum, & $\frac{m}{n}$ numeralem indicem dimensionis ipsius $\mathrm{P+PQ}$ sive dimensio illa integra sit, sive (ut ita loquar) fracta, sive affermativa sive negativa. Nam sicut Analystæ pro aa, aaa &c scribere solent a^2, a^3, sic ego pro $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{c.\ a^5}$ &c scribo $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, $a^{\frac{5}{3}}$, & pro $\frac{1}{a}$, $\frac{1}{aa}$, $\frac{1}{a^3}$ scribo a^{-1}, a^{-2}, a^{-3}. et sic pro $\frac{aa}{\sqrt{c:\ \overline{a^3+bbx}}}$ scribo $aa \times \overline{a^3+bbx}\Big\}^{-\frac{1}{3}}$, & pro $\frac{aab}{\sqrt{c:\ \overline{a^3+bbx} \times \overline{a^3+bbx}}}$ scribo $aab \times \overline{a^3+bbx}\Big\}^{-\frac{2}{3}}$ in quo ultimo casu si $\overline{a^3+bbx}\Big\}^{-\frac{2}{3}}$ conciapiatur esse $\overline{\mathrm{P+PQ}}\Big\}^{\frac{m}{n}}$ in Regula; erit $\mathrm{P} = a^3$, $\mathrm{Q} = \frac{bbx}{a^3}$, $m = -2$, & $n = 3$. Denique pro terminis inter operandum inventis in quoto, usurpo A, B, C, D &c nempe A pro primo termino $\mathrm{P}^{\frac{m}{n}}$, B pro secundo $\frac{m}{n}\mathrm{AQ}$, & sic deinceps. Cæterùm usus Regulæ patebit exemplis.

Exempl: 1. est $\sqrt{\overline{cc+xx}}$ $\left(\text{seu } \overline{cc+xx}\Big\}^{\frac{1}{2}}\right) = c + \frac{xx}{2c} - \frac{x^4}{8c^3} + \frac{x^6}{16c^5} - \frac{5x^8}{128c^7} + \frac{7x^{10}}{256a^9} + \&c.$. Nam in hoc casu est $\mathrm{P} = cc$, $\mathrm{Q} = \frac{xx}{cc}$, $m = 1$, $n = 2$, $\mathrm{A}\left(= \mathrm{P}^{\frac{m}{n}} = \overline{cc}\Big\}^{\frac{1}{2}}\right) = c$. $\mathrm{B}\left(= \frac{m}{n}\mathrm{AQ}\right) = \frac{xx}{2c}$. $\mathrm{C}\left(= \frac{m-n}{2n}\mathrm{BQ}\right) = \frac{-x^4}{8c^3}$, & sic deinceps.

Exempl: 2. est $\sqrt{⑤}\ \overline{c^5+c^4x-x^5}$ $\left(\text{i.e. } \overline{c^5+c^4x-x^5}\Big\}^{\frac{1}{5}}\right) = c + \frac{c^4x-x^5}{5c^4} - \frac{2c^8xx+4c^4x^6-2x^{10}}{25c^9} + \&c$ ut patebit substituendo in allatam Regulam, 1 pro m, 5 pro n, c^5 pro P, & $\frac{c^4x-x^5}{c^5}$ pro Q. Potest etiam $-x^5$ substitui pro P, & $\frac{c^4x+c^5}{-x^5}$ pro Q, et tunc evadet $\sqrt{⑤}\ \overline{c^5+c^4x-x^5} = -x + \frac{c^4x+c^5}{5x^4} + \frac{2c^8xx+4c^9x+c^{10}}{25x^9} + \&c.$ Prior modus eligendus est si x valde parvum sit, posterior si valde magnum.

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Exempl 3. Est $\frac{N}{\sqrt{③}\ \overline{y^3-aay}}$ $\left(\text{hoc est } N \times \overline{y^3-aay}\Big\}^{-\frac{1}{3}}\right) = N \times \overline{\frac{1}{y} + \frac{aa}{3y^3} + \frac{2a^4}{9y^5} + \frac{7a^6}{81y^7}} +$ Nam $\mathrm{P} = y^3$. $\mathrm{Q} = \frac{-aa}{yy}$. $m = -1$. $n = 3$. A $\left(= \mathrm{P}^{\frac{m}{n}} = y^{3 \times \frac{-1}{3}}\right) = y^{-1}$. hoc est $\frac{1}{y}$. B $\left(= \frac{m}{n}\mathrm{AQ} = \frac{-1}{3} \times \frac{1}{y} \times \frac{-aa}{yy}\right) = \frac{aa}{3y^3}$. &c

Exempl. 4. Radix cubica ex quadrato-quadrato ipsius $d+e$ (hoc est $\overline{d+e}\Big\}^{\frac{4}{3}}$) est $d^{\frac{4}{3}} + \frac{4ed^{\frac{1}{3}}}{3} + \frac{2ee}{9d^{\frac{2}{3}}} - \frac{4e^3}{81d^{\frac{5}{3}}} + \&c$. Nam $\mathrm{P} = d$. $\mathrm{Q} = \frac{e}{d}$. $m = 4$. $n = 3$. A $\left(= \mathrm{P}^{\frac{m}{n}}\right) = d^{\frac{4}{3}}$ &c.

Eodem modo simplices etiam potestates eliciuntur. Ut si quadrato-cubus ipsius $\mathrm{d}+\mathrm{e}$ $\left(\text{hoc est } \overline{\mathrm{d}+\mathrm{e}}\right\}^5, \text{ seu } \overline{\mathrm{d}+\mathrm{e}}\}^{\frac{5}{1}}\Big)$ desideretur: erit juxta Regulam $\mathrm{P}=\mathrm{d}$. $\mathrm{Q}=\frac{\mathrm{e}}{\mathrm{d}}$. $\mathrm{m}=5$ & $\mathrm{n}=1$; adeoque A $\left(=\mathrm{P}^{\frac{\mathrm{m}}{\mathrm{n}}}\right)=\mathrm{d}^5$, B $\left(=\frac{\mathrm{m}}{\mathrm{n}}\mathrm{AQ}\right)=5\mathrm{d}^4\mathrm{e}$, & sic $\mathrm{C}=10\mathrm{d}^3\mathrm{ee}$, $\mathrm{D}=10\mathrm{dde}^3$, $\mathrm{E}=5\mathrm{de}^4$, $\mathrm{F}=\mathrm{e}^5$, & G $\left(=\frac{\mathrm{m}-5\mathrm{n}}{6\mathrm{n}}\mathrm{FQ}\right)=0$. Hoc est $\overline{\mathrm{d}+\mathrm{e}}\}^5=\mathrm{d}^5+5\mathrm{d}^4\mathrm{e}+10\mathrm{d}^3\mathrm{ee}+10\mathrm{dde}^3+5\mathrm{de}^4+\mathrm{e}^5$.

Quinetiam Divisio, sive simplex sit, sive repetita, per eandem Regulam perficitur. Ut si $\frac{1}{\mathrm{d}+\mathrm{e}}$, $\left(\text{hoc est } \overline{\mathrm{d}+\mathrm{e}}\}^{-1} \text{ sive } \overline{\mathrm{d}+\mathrm{e}}\}^{\frac{-1}{1}}\right)$ in seriem simplicium terminorum resolvendum sit: erit juxta Regulam $\mathrm{P}=\mathrm{d}$. $\mathrm{Q}=\frac{\mathrm{e}}{\mathrm{d}}$. $\mathrm{m}=-1$. $\mathrm{n}=1$. & A $\left(=\mathrm{P}^{\frac{\mathrm{m}}{\mathrm{n}}}=\mathrm{D}^{\frac{-1}{1}}\right)=\mathrm{d}^{-1}$ seu $\frac{1}{\mathrm{d}}$. B $\left(=\frac{\mathrm{m}}{\mathrm{n}}\mathrm{AQ}=-1\times\frac{1}{\mathrm{d}}\times\frac{\mathrm{e}}{\mathrm{d}}\right)=-\frac{\mathrm{e}}{\mathrm{dd}}$, & sic $\mathrm{C}=\frac{\mathrm{ee}}{\mathrm{d}^3}$, $\mathrm{D}=\frac{-\mathrm{e}^3}{\mathrm{d}^4}$ &c Hoc est $\frac{1}{\mathrm{d}+\mathrm{e}}=\frac{1}{\mathrm{d}}-\frac{\mathrm{e}}{\mathrm{dd}}+\frac{\mathrm{ee}}{\mathrm{d}^3}-\frac{\mathrm{e}^3}{\mathrm{d}^4}+$ &c .

Sic et $\overline{\mathrm{d}+\mathrm{e}}\}^{-3}$ (hoc est unitas ter divisa per $\mathrm{d}+\mathrm{e}$ vel semel per cubum ejus,) evadit $\frac{1}{\mathrm{d}^3}-\frac{3\mathrm{e}}{\mathrm{d}^4}+\frac{6\mathrm{ee}}{\mathrm{d}^5}-\frac{10\mathrm{e}^3}{\mathrm{d}^6}+$ &c .

Et $\mathrm{N}\times\overline{\mathrm{d}+\mathrm{e}}\}^{-\frac{1}{3}}$ hoc est N divisum per radicem cubicam ipsius $\mathrm{d}+\mathrm{e}$ evadit $\mathrm{N}\times\frac{1}{\mathrm{d}^{\frac{1}{3}}}-\frac{\mathrm{e}}{3\mathrm{d}^{\frac{4}{3}}}+\frac{2\mathrm{ee}}{9\mathrm{d}^{\frac{7}{3}}}-\frac{14\mathrm{e}^3}{81\mathrm{d}^{\frac{10}{8}}}+$ &c

Et $\mathrm{N}\times\overline{\mathrm{d}+\mathrm{e}}\}^{-\frac{3}{5}}$ (hoc est N divisum per radicem quadrato-cubicam ex cubo ipsius $\mathrm{d}+\mathrm{e}$, sive $\frac{\mathrm{N}}{\sqrt{⑤}\ \overline{\mathrm{d}^3+3\mathrm{dde}+3\mathrm{dee}+\mathrm{e}^3}}$) evadit $\mathrm{N}\times\frac{1}{\mathrm{d}^{\frac{3}{5}}}-\frac{3\mathrm{e}}{5\mathrm{d}^{\frac{9}{5}}}+\frac{12\mathrm{ee}}{25\mathrm{d}^{\frac{13}{5}}}-\frac{52\mathrm{e}^3}{125\mathrm{d}^{\frac{18}{5}}}$ &c .

Per eandem Regulam Genesses Potestatum, Divisiones per potestates aut per quantitates radicales, & extractiones radicum altiorum in numeris etiam commodè instituuntur.

Extractiones Radicum affectarum in speciebus imitantur earum extractiones in numeris, sed methodus Vietæ et Oughtredi nostri huic negotio minùs idonea est, Quapropter aliam excogitare adactus sum cujus specimen exhibent sequentia Diagrammata ubi dextra columna prodit substituendo in media columnâ Valores ipsorum y, p, q, r &c in sinistra columna expressos. Prius Diagramma exhibet resolutionem hujus numeralis æquationis $\mathrm{y}^3-2\mathrm{y}-5=0$; et hic in supremis numeris pars negativa radicis subducta de parte affirmativa relinquit absolutam Radicem $2\,|\,\underline{09455148}$ et posterius Diagramma exhibet resolutionem hujus literariæ æquationis $\mathrm{y}^3+\mathrm{axy}+\mathrm{aay}-\mathrm{x}^3-2\mathrm{a}^3=0$.

		$\left(\begin{array}{l}+2{,}10000000\\ -0{,}00544852\end{array}\right.$
		$+2{,}09455148$
$2+\mathrm{p}=\mathrm{y}$	y^3	$+\,8+12\mathrm{p}+6\mathrm{pp}+\mathrm{p}^3$
	$-2\mathrm{y}$	$-\,4-\;\;2\mathrm{p}$
	-5	$-\,5$
	summa	$-\,1+10\mathrm{p}+6\mathrm{pp}+\mathrm{p}^3$
$+0{,}1+\mathrm{q}=\mathrm{p}$	$+\mathrm{p}^3$	$+\,0{,}001+\;\;0{,}03\mathrm{q}+0{,}3\mathrm{qq}+\mathrm{q}^3$
	$+6\mathrm{pp}$	$+\,0{,}06\;\;+\;\;1{,}2\;\;\;+\,6$
	$+10\mathrm{p}$	$+\,1\;\;\;\;\;\;+\,10,$
	-1	$-\,1$
	summa	$0{,}061+11{,}23\mathrm{q}+6{,}3\mathrm{qq}+\mathrm{q}^3$
$-0{,}0054+\mathrm{r}=\mathrm{q}$	$+\mathrm{q}^3$	$-\,0{,}0000001+\;\;0{,}000\mathrm{r}$ &c
	$+6{,}3\mathrm{qq}$	$+\,0{,}0001837-\;\;0{,}068$
	$+11{,}23\mathrm{q}$	$-\,0{,}060642\;\;+\,11{,}23$
	$+0{,}061$	$+\,0{,}061$
	summa	$-\,0{,}0005416+11{,}162\mathrm{r}$
$-0{,}00004852+\mathrm{s}=\mathrm{r}$		

		$\Big(a-\frac{x}{4}+\frac{xx}{64a}+\frac{131x^3}{512aa}+\frac{509x^4}{16384a^3}$ &c
$a+p=y$	y^3 $+axy$ $+aay$ $-x^3$ $-2a^3$	$a^3+3aap+3app+p^3$ $+aax+axp$ $+a^3+aap$ $-x^3$ $-2a^3$
$-\frac{1}{4}x+q=p$	p^3 $+3app$ $+axp$ $+4aap$ $+aax$ $-x^3$	$-\frac{1}{64}x^3+\frac{3}{16}xxq$ &c $+\frac{3}{16}axx-\frac{3}{2}axq+3aqq$ $-\frac{1}{4}axx+axq$ $-axx+4aaq$ $+aax$ $-x^3$
$+\frac{xx}{64a}+r=q$	$3aqq$ $+\frac{3}{16}xxq$ $-\frac{1}{2}axq$ $+4aaq$ $-x^3$ $-\frac{65}{64}a^3$ $-\frac{1}{16}aax$	$+\frac{3x^4}{4096a}$ &c $+\frac{3x^4}{1024a}$ &c $-\frac{1}{128}x^3-\frac{1}{2}axr$ $+\frac{1}{16}axx+4aar$ $-x^3$ $-\frac{65}{64}a^3$ $-\frac{1}{16}aax$

$$+4aa-\frac{1}{2}ax\Big)+\frac{131}{128}x^3-\frac{15x^4}{4096a}\Big(+\frac{131x^3}{512aa}+\frac{509x^4}{16384a^3}.$$

In priori diagrammate primus terminus valoris ipsorum p, q, r, in prima columna invenitur dividendo primum terminum summæ proximè superioris per coefficientem secundi termini ejusdem summæ: et idem terminus eodem ferè modo invenitur in secundo diagrammate. Sed hic præcipua difficultas est in inventione primi termini radicis: id quod methodo generali perficitur, sed hoc brevitatis gratia jam prætereo, ut et alia quædam quæ ad concinnandam operationem spectant. Neque hic compendia tradere vacat, sed dicam tantum in genere, quod radix cujusvis æquationis semel extracta pro regula resolvendi consimiles æquationes asservari possit; & quod ex pluribus ejusmodi regulis, regulam generaliorem plerumque efformare liceat; quodque radices omnes, sive simplices sint sive affectæ, modis infinitis extrahi possint, de quorum simplicioribus itaque semper consulendum est.

<2v>

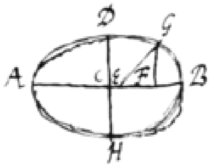
Quomodo ex æquationibus, sic ad infinitas series reductis, areæ & longitudines curvarum, contenta et superficies solidorum, vel quorumlibet segmentorum figurarum quarumvis eorumque centra gravitatis determinantur, et quomodo etiam curvæ omnes Mechanicæ ad ejusmodi æquationes infinitarum serierum reduci possint, indeque Problemata circa illas resolvi perinde ac si geometricæ essent, nimis longum foret describere. Sufficiat specimina quædam talium Problematum recensuisse: inque iis brevitatis gratia literas A, B, C, D &c pro terminis seriei, sicut sub initio, nonnunquam usurpabo.

1. Si ex dato sinu recto vel sinu verso arcus desideretur: sit radius r et sinus rectus x eritque arcus $=x+\frac{x^3}{6rr}+\frac{3x^5}{40r^4}+\frac{5x^7}{112r^6}+\&c$. hoc est $=x+\frac{1\times1\times xx}{2\times3\times rr}A+\frac{3\times3xx}{4\times5rr}B+\frac{5\times5xx}{6\times7rr}C+\frac{7\times7xx}{8\times9rr}D+\&c$. Vel sit d diameter ac x sinus versus, et erit arcus $=d^{\frac{1}{2}}x^{\frac{1}{2}}+\frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}}+\frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}}+\frac{5x^{\frac{7}{2}}}{112d^{\frac{5}{2}}}+\&c$ hoc est $=\sqrt{dx}$ in $1+\frac{x}{6}+\frac{3xx}{40d}+\frac{5x^3}{112dd}+\&c$.

2. Si vicissim ex dato arcu desiderentur sinus: sit radius r et arcus z, eritque sinus rectus $= z - \frac{z^3}{6rr} + \frac{z^5}{120r^4} - \frac{z^7}{5040r^6} + \frac{z^9}{36288r^8} - \&c$, hoc est $= z - \frac{zz}{2\times 3rr}A - \frac{zz}{4\times 5rr}B - \frac{zz}{6\times 7rr}C - \&c$; Et sinus versus $= \frac{zz}{2r} - \frac{z^4}{24r^3} + \frac{z^6}{720r^5} - \frac{z^8}{4032r^7} + \&c$, hoc est $\frac{zz}{1\times 2r} - \frac{zz}{3\times 4rr}A - \frac{zz}{5\times 6rr}B - \frac{zz}{7\times 8}C$.

3. Si arcus capiendus sit in ratione data ad alium arcum: esto diameter $= d$, chorda arcûs dati $= x$, & arcus quæsitus ad arcum illum datum ut n ad 1; eritque arcus quæsiti chorda $= nx + \frac{1-nn}{2\times 3dd}xxA + \frac{9-nn}{4\times 5dd}xxB + \frac{25-nn}{6\times 7dd}xxC + \frac{36-nn}{8\times 9dd}xxD + \frac{49-nn}{10\times 11dd}xxE + \&c$ Ubi nota quod cùm n est numerus impar, series desinet esse infinita, & evadet eadem quæ prodit per vulgarem Algebram ad multiplicandum datum angulum per istum numerum n.

4. Si in axe alterutro AB ellipseos ADB (cujus centrum C & axis alter DH) detur punctum aliquod E circa quod recta EG occurrens Ellipsi in G motu angulari feratur, et ex data area sectoris Ellipticæ BEG quæratur recta GF quæ a puncto G ad axem AB normaliter demittitur: esto $BC = q$, $DC = r$, $EB = t$, ac duplum areæ $BEG = z$; et erit $GF = \frac{z}{t} - \frac{qz^3}{6rrt^4} + \frac{10qq - qqt}{120r^4t^7}z^5 - \frac{280q^3 + 504qqt - 225qtt}{5040r^6t^{10}}z^7 + \&c$.
Sic itaque Astronomicum illud Kepleri Problema resolvi potest.



5. In eâdem Ellipsi si statuatur $CD = r$, $\frac{CB^q}{CD} = c$, & $CF = x$, erit arcus Ellipticus

$$\begin{array}{llllll}
DG = x + \frac{1}{6cc}x^3 & + \frac{1}{10rc^3}x^5 & + \frac{1}{14rrc^4}x^7 & + \frac{1}{18r^3c^5}x^9 & + \frac{1}{22r^4c^6}x^{11} & + \&c \\
 & - \frac{1}{40c^4} & - \frac{1}{28rc^5} & - \frac{1}{24rrc^6} & - \frac{1}{22r^3c^7} & \\
 & & + \frac{1}{112c^6} & + \frac{1}{48rc^7} & + \frac{3}{88rrc^8} & \\
 & & & - \frac{5}{1152c^8} & - \frac{5}{352rc^9} & \\
 & & & & + \frac{7}{2816c^{10}} &
\end{array}$$

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Hic numerales coefficientes supremorum terminorum $\left(\frac{1}{6} \cdot \frac{1}{10} \cdot \frac{1}{14}\ \&c\right)$ sunt in musica progressione, & numerales coefficientes omnium inferiorum in una quaque columna prodeunt multiplicando continuò numeralem coefficientem supremi termini per terminos hujus progressionis $\frac{\frac{1}{2}n-1}{2} \cdot \frac{\frac{3}{3}n-3}{4} \cdot \frac{\frac{5}{4}n-5}{6} \cdot \frac{\frac{7}{5}n-7}{8} \cdot \frac{\frac{9}{6}n-9}{10}$ &c: ubi n significat numerum dimensionum ipsius c in denominatore istius supremi termini. E.g. ut terminorum infra $\frac{1}{22r^4c^6}$, numerales coefficientes inveniantur, pono $n = 6$, ducoque $\frac{1}{22}$ (numeralem coefficientem ipsius $\frac{1}{22r^4c^6}$) in $\frac{\frac{1}{2}n-1}{2}$ hoc est in 1; et prodit $\frac{1}{22}$ numeralis coefficiens termini proximè inferioris; dein duco hunc $\frac{1}{22}$ in $\frac{\frac{3}{3}n-3}{4}$ sive in $\frac{n-3}{4}$ hoc est in $\frac{3}{4}$ & prodit $\frac{3}{88}$ numeralis coefficiens tertij termini in ista columna. Atque ita $\frac{3}{88} \times \frac{\frac{5}{4}n-5}{6}$ facit $\frac{5}{352}$ num: coeff: quarti termini & $\frac{5}{352} \times \frac{\frac{7}{5}n-7}{8}$ facit $\frac{7}{2816}$ numeralem coefficientem infimi termini Idem in alijs ad infinitum columnis præstari potest, adeoque valor ipsius DG per hanc regulam pro lubitu produci.

Ad hæc si BF dicatur x, sitque r latus rectum Ellipseos & $e = \frac{r}{AB}$; erit arcus Ellipticus

$$BG = \sqrt{rx} \ \text{in} \ \overline{\frac{\left.\begin{array}{c} 1+2 \\ -\frac{3}{2}e \end{array}\right\}}{3r} x \quad \frac{\left.\begin{array}{c} -2 \\ +3e \\ -\frac{5}{8}ee \end{array}\right\}}{5rr} xx \quad \frac{\left.\begin{array}{c} +4 \\ -9e \\ +\frac{23}{4}ee \\ -\frac{7}{16}e^3 \end{array}\right\}}{7r^3} x^3 \quad \frac{\left.\begin{array}{c} -10 \\ +30e \\ -\frac{123}{4}ee \\ +\frac{91}{8}e^3 \\ -\frac{45}{128}e^4 \end{array}\right\}}{9r^4} x^4} \ +\&c.$$

Quare si ambitus totius Ellipseos desideretur: biseca CB in F, & quære arcum DG per prius Theorema & arcum GB per posterius.

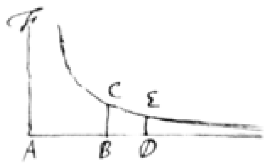
6 Si vice versa ex dato arcu Elliptico DG quæratur sinus ejus CF, tum dicto $CD = r$, $\frac{CB^q}{CD} = c$, & arcu illo $DG = z$ erit

$$\begin{array}{llll}
CF = z - \frac{1}{6cc}z^3 & - \frac{1}{10rc^3}z^5 & - \frac{1}{14rrc^4}z^7 & - \&c. \\
 & + \frac{13}{120c^4} & + \frac{71}{420rc^5} & \\
 & & - \frac{493}{5040c^6} &
\end{array}$$

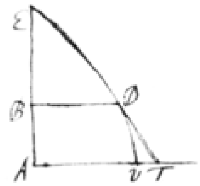
Quæ autem de Ellipsi dicta sunt, omnia facilè accommodantur ad Hyperbolam: mutatis tantum signis ipsorum c & e ubi sunt imparium dimentionum.

7. Præterea si sit CE Hyperbola cujus Asymptoti AD, AF rectum angulum FAD constituant et ad AD erigantur utcunque perpendicula BC DE occurrentia Hyperbolæ in C & E, & AB dicatur a, BC b, & area BCED z, erit $BD = \frac{z}{b} + \frac{zz}{2abb}$

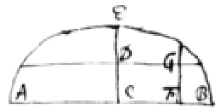
$+\frac{z^3}{6aab^3}+\frac{z^4}{24a^3b^4}+\frac{z^5}{120a^4b^5}$ &c Ubi coefficientes denominatorum prodeunt multiplicando terminos hujus arithmeticæ progressionis, 1 , 2 , 3 , 4 , 5 &c in se continuò. Et hinc ex Logarithmo dato potest numerus ei competens inveniri.



8. Esto VDE Quadratrix cujus vertex V, existente A centro et AE diametro circuli ad quem aptatur, et angulo, VAE recto. Demissoque ad AE perpendiculo quovis DB et acta quadratricis tangente DT occurrente axi ejus AV in T: dic $AV = a$, & $AB = x$, eritque <3v> $BD = a-\frac{xx}{3a}-\frac{x^4}{45a^3}-\frac{2x^6}{945a^5}-$&c. Et $VT=\frac{xx}{3a}+\frac{x^4}{15a^3}+\frac{2x^6}{189a^5}+$&c et area $AVDB = ax-\frac{x^3}{9a}-\frac{x^5}{225a^3}-\frac{2x^7}{6615a^5}-$&c Et arcus $VD = x+\frac{2x^3}{27aa}+\frac{14x^5}{2025a^4}+\frac{604x^7}{893025a^6}+$&c. Unde vicissim ex dato BD, vel VT, aut areâ AVDB arcuve VD, per resolutionem affectarum æquationum erui potest x seu AB.



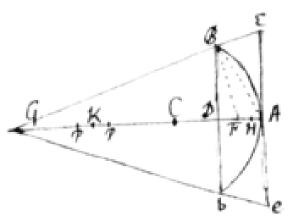
9 Esto denique AEB sphæroides, revolutione Ellipseos AEB circa axem AB genita, & secta planis quatuor, AB per axem transeunte, DG parallelo AB, CDE perpendiculariter bisecante axem, et FC parallelo CE: sitque recta $CB = a$. $CE = c$. $CF = x$. & $FG = y$; et sphæroideos segmentum CDFG, dictis quatuor planis comprehensum erit.



$$\begin{array}{llllll}
+2cx\,y & -\frac{x}{3c}\,y^3 & -\frac{x}{20c^3}\,y^5 & -\frac{x}{56c^5}\,y^7 & -\frac{5x}{576c^7}\,y^9 & -\&c \\
-\frac{cx^3}{3aa} & -\frac{x^3}{18caa} & -\frac{x^3}{40c^3aa} & -\frac{5x^3}{336c^5aa} & -\&c. & \\
-\frac{cx^5}{20a^4} & -\frac{x^5}{40ca^4} & -\frac{3x^5}{160c^3a^4} & -\&c. & & \\
-\frac{cx^7}{56a^6} & -\frac{5x^7}{336ca^6} & -\&c & & & \\
-\frac{5cx^9}{576a^7} & -\&c. & & & & \\
-\&c. & & & & &
\end{array}$$

Ubi numerales coefficientes supremorum terminorum $\left[2,\ \frac{-1}{3},\ \frac{-1}{20},\ \frac{-1}{56},\ \frac{-5}{576}\ \&c\right]$ in infinitum producuntur multiplicando primum coefficientem 2 continuò per terminos hujus progressionis $\frac{-1\times1}{2\times3}$. $\frac{1\times3}{4\times5}$. $\frac{3\times5}{6\times7}$ $\frac{5\times7}{8\times9}$ $\frac{7\times9}{10\times11}$. &c Et numerales coefficientes terminorum in unaquaque coluna descendentium in infinitum producuntur multiplicando continuò coefficientem supremi termini in prima columna per eandem progressionem, in secunda autem per terminos hujus $\frac{1\times1}{2\times3}$. $\frac{3\times3}{4\times5}$. $\frac{5\times5}{6\times7}$ $\frac{7\times7}{8\times9}$. $\frac{9\times9}{10\times11}$&c; in tertia per terminos hujus $\frac{3\times1}{2\times3}$. $\frac{5\times3}{4\times5}$. $\frac{7\times5}{6\times7}$. $\frac{9\times7}{8\times9}$. &c , in quarta per terminos huius $\frac{5\times1}{2\times3}$. $\frac{7\times3}{4\times5}$. $\frac{9\times5}{6\times7}$. &c , in quinta per terminos huius $\frac{7\times1}{2\times3}$. $\frac{9\times3}{4\times5}$ $\frac{11\times5}{6\times7}$. &c {Ac} sic in infinitum, et eodem modo segmenta aliorum solidorum designari, et valores eorum aliquando commodè per series quasdem numerales in infinitum produci possunt.

Ex his videre est quantum fines Analyseos per hujusmodi infinitas æquationes ampliantur: quippe quæ earum beneficio, ad omnia, pene dixerim, problemata (si numeralia Diophanti et similia excipias) sese extendit non tamen omninò universalis evadit, nisi per ulteriores quasdem methodos eliciendi series infinitas. Sunt enim quædam Problemata in quibus non liceat ad series infinitas per divisionem vel extractionem radicum simplicium affectarumve pervenire: Sed quomodo in istis casibus procedendum sit jam non vacat dicere; ut neque alia quædam tradere quæ circa reductionem infinitarum serierum in finitas, ubi rei natura tulerit, excogitavi. Nam parcius scribo quòd hæ speculationes diu mihi fastidio esse cœperunt, adeò ut ab ijsdem jam per quinque ferè annos abstinuerim. Unum tamen addam: quod postquam Problema aliquod ad infinitam æquationem deducitur, possint inde variæ approximationes in usum Mechanicæ nullo ferè negotio formari, quæ per alias methodos quæsitæ, multo labore temporisque dispendio constare solent Cujus rei exemplo esse possunt Tractatus Hugenij aliorumque de quadratura circuli. Nam ut ex data Arcûs chorda A & dimidij arcus chorda B arcum illum proxime assequaris, finge arcum illum esse Z, et circuli radium r; juxtaque superiora erit A (nempe duplum sinûs dimidij z) $= z-\frac{z^3}{4\times6rr}+\frac{z^5}{4\times4\times120r^4}-$&c. Et $B=\frac{1}{2}z-\frac{z^3}{2\times16\times6rr}$ <4r> $+\frac{z^5}{2\times16\times16\times120r^4}-$&c. Duc jam B in numerum fictitium n & a producto aufer A, et residui secundum terminum (nempe $-\frac{nz^3}{2\times16\times6rr}+\frac{z^3}{4\times6rr},$) eo ut evanescat, pone $=0$, indeque emerget $n=8$, & erit $8B-A=3z*-\frac{3z^5}{64\times120r^4}+$&c : hoc est $\frac{8B-A}{3}=z$ errore tantum existente $\frac{z^5}{7680r^4}-$&c in excessu. Quod est Theorema Hugenianum.



Insuper si in arcûs Bb sagittâ AD indefinitè productâ quæratur punctum G à quo actæ rectæ GB, Gb abscindant tangentem Ee quamproximè æqualem arcui isti: esto circuli centrum C diameter $AK = d$, et sagitta $AD = x$ et erit $DB\left(=\sqrt{dx-xx}\right)=d^{\frac{1}{2}}x^{\frac{1}{2}}-\frac{x^{\frac{3}{2}}}{2d^{\frac{1}{2}}}-\frac{x^{\frac{5}{2}}}{8d^{\frac{3}{2}}}-\frac{x^{\frac{7}{2}}}{16d^{\frac{5}{2}}}-$&c . Et $AE(=AB)=d^{\frac{1}{2}}x^{\frac{1}{2}}+\frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}}+\frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}}+\frac{5x^{\frac{7}{2}}}{112d^{\frac{5}{2}}}+$&c . Et $AE-DB$. $AD :: AE$. AG . Quare $AG=\frac{3}{2}d-\frac{1}{5}x-\frac{12xx}{175d}-vel+$&c . Finge ergo $AG=\frac{3}{2}d-\frac{1}{5}x$, et vicissim erit $DG\ \left(\frac{3}{2}d-\frac{6}{5}x\right)$. $DB :: DA$. $AE-DB$. Quare $AE-DB=\frac{2x^{\frac{3}{2}}}{3d^{\frac{1}{2}}}+\frac{x^{\frac{5}{2}}}{5d^{\frac{3}{2}}}+\frac{23x^{\frac{7}{2}}}{300d^{\frac{5}{2}}}+$&c . Adde AB et prodit $AE=d^{\frac{1}{2}}x^{\frac{1}{2}}+\frac{x^{\frac{3}{2}}}{6d^{\frac{1}{2}}}+\frac{3x^{\frac{5}{2}}}{40d^{\frac{3}{2}}}+\frac{17x^{\frac{7}{2}}}{1200d^{\frac{5}{2}}}+$&c . Hoc aufer de valore ipsius AE supra habito et restabit error $\frac{16x^{\frac{7}{2}}}{525d^{\frac{5}{2}}}+vel-$&c. Quare in AG cape AH quintam partem DH, et $KG = HC$; & actæ GBE, Gbe abscindent tangentem Ee quamproximè æqualem arcui Bab errore tantum existente $\frac{32x^3}{525d^3}\sqrt{dx}+\text{vel}-\&c$; multò minore scilicet quam in Theoremate Hugenij. Quod si fiat $7AK\ .\ 3AH :: DH\ .\ n$, & capiatur $KG = CH - n$ erit error adhuc multò minor.

Atque ita si circuli segmentum aliquod BAb per Mechanicam designandum esset: primo reducerem aream istam in infinitam seriem; puta hanc $BbA = \frac{4}{3}d^{\frac{1}{2}}x^{\frac{3}{2}} - \frac{2x^{\frac{5}{2}}}{5d^{\frac{1}{2}}} - \frac{x^{\frac{7}{2}}}{14d^{\frac{3}{2}}} - \frac{x^{\frac{9}{2}}}{36d^{\frac{5}{2}}} - \&c$; dein quærerem constructiones mechanicas quibus hanc seriem proximè assequerer; cujusmodi sunt hæc. Age rectam AB, & erit Segmentum $BbA = \overline{\frac{2}{3}AB + BD} \times \frac{4}{5}AD$ proximè, existente scilicet errore tantum $\frac{x^3}{70dd}\sqrt{dx}+\&c$, in defectu: vel proximiùs erit segmentum illud, (bisecto AD in F et acta recta BF,) $= \frac{4BF+AB}{15} \times 4AD$, existente errore solummodo $\frac{x^3}{560dd}\sqrt{dx}+\&c$. qui semper minor est quàm $\frac{1}{1500}$ totius segmenti, etiamsi segmentum illud ad usque semicirculum augeatur.

Sic in Ellipsi BAb cujus vertex A, axis alteruter AK, et latus rectum AP, cape $PG = \frac{1}{2}AP + \frac{19AK-21AP}{10AK} \times AP$; in Hyperbola verò cape $PG = \frac{1}{2}AP + \frac{19AK+21AP}{10AK} \times AP$: et acta recta GBE abscindet tangentem AE quamproximè æqualem arcui Elliptico vel Hyperbolico AB, dummodo ar**{cus}** ille non sit nimis magnus. Et pro area segmenti Hyperbolici BbA, dic latus rectum d, latus transversum e, et AD x; et cape $m = \sqrt{dx}$ et $n = \sqrt{dx + \frac{3d}{4e}xx}$ eritque $\frac{4n+m}{15} \times 4AD = BbA$; vel forte melius cape $n = \sqrt{dx + \frac{5d}{7e}xx}$, et erit $\frac{21n+4m}{75} \times 4AD = BbA$.

Et ejusdem methodi vestigijs insistendo.
