Newton's observations on the Synopsis given in the Leipzig Acts of Jones's "Analysis per quantiatum Series" (London, 1711)

Author: Isaac Newton

Source: MS Add. 3968, ff. 460r-464v, Cambridge University Library, Cambridge, UK

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This Analysis is founded on three Rules the two first of which are equipollent to the solution of this Problem, Data æquatione fluentes duas quantitates involvente se invicem non multiplicantes fluxiones invenire; & contra. The third Rule directs the resolution of finite equations infinite ones when there is occasion. These Rules M$^{\rm r}$ Newton illustrates with various examples applies them to the quadrature of curves & then adds that all Problems concerning the length of curves & the contents & surfaces of solids & centers of gravity may be reduced to Quadratures after the following manner. Sit ABD Curva quævis — dato elicietur <460v> And after he had set down an example by computing the length of the arch of a circle from its moment he subjoyns. Sed notandum est quod unitas quæ pro momento ponitur est superficies cum de solidis & linea cum de superficiebus & punctum cum de lineis agitur. Nec vereor loqui de unitate in punctis, sive lineis infinite parvis, siquidem proportiones ibi jam contemplantur Geometræ dum utuntur methodis indivisibilem. So then by a point M$^{\rm r}$ Newton understands here an infinitely short line, & by a line an infinitely narrow surface & when he calls these moments & represents them by an unit it is to be understood that this unit is multiplied by an infinitely small quantity o, or moment of time, to make it infinitely little. The moment 1 is $1 \times \mathrm{o}$ & the moment y is $\mathrm{y} \times \mathrm{o}$, but the coefficient o for shortning the operation is not written down but understood. If o be not understood the lines 1 & y represent the fluxions of the areas BK & ABD, but if o be understood, these flusions multiplied by o become the moments of BK & AD. For fluxions are finite quantities but moments here are infinitely little. Thus you see his Notation when he wrote this Analysis is of the same kind with that which he uses at present.

So in demonstrating the first of his three Rules by this method, in the equation $\mathrm{c^n x^p = z^n}$ he supposes x & z to increase & be augmented by the moments o & ov & to become $\mathrm{x + o}$ & $\mathrm{z + ov}$, & by those moments understands the rectangles $\mathrm{o \times 1}$ & $\mathrm{o \times v}$ conteined under the fluxions 1 & v & the moment o. Then in the said æquation writing $\mathrm{x + o}$ for x & $\mathrm{z + ov}$ for z there arises $\mathrm{c^n}$ in $\mathrm{x^p + pox^{p-1} + \&c = z^n + novz^{n-1} + \&c}$. Where the first terms $\mathrm{c^n x^p}$ & $\mathrm{z^n}$ destroy one another & the next divided by o viz$^{\rm t}$ $\mathrm{c^n p x^{p-1}}$ & $\mathrm{nvz^{n-1}}$ become equal & determin the proportion of the fluxions 1 & v or of their moments o & ov. And its here observable that the series $\mathrm{x^p + pox^{p-1} + \&c}$ is the same with that in the beginning of M$^{\rm r}$ Newtons Letter of 13 Iun 1676, & being produced becomes $\mathrm{x^p + pox^{p-1} + \frac{pp-p}{2} oox^{p-2} + \frac{p^3-3pp+2p}{6} o^3 x^{p-3} + \&c}$, And the like is to be understood of the series $\mathrm{z^n + poz^{n-1} + \&c}$ in like manner to be produced. But M$^{\rm r}$ Newton having shewn before that all the terms after the second would vanish with the moment o, neglected them. He saw there fore in those days that the second terms of these series gave the fluxions & moments of the dignities of any fluent quantity x or $x+o$; & by this property of these series he Demonstrated the first of the three Rules upon which he founded his Analysis. And when he understood this he could not be long without seing the use of the third terms of these series & of the rest of the terms which follow the third. For that he knew the use of the third terms in those days is evident from his letter of 10 Dec. 1672, where he saith that by his method he determined the Curvature of Curves.

After this Demonstration M^r Newton subjoyns the following conclusion from it. Hinc in transitu notetur modus quo Curvæ quotcunque, quarum Areæ sunt cognitæ possunt inveniri; sumendo nempe quamlibet æquationem pro relatione inter aream z & basem [vel abscissam] x ut inde quæratur [ordinatim] applicata y. Vt si suppunas $\sqrt{aa+xx}=z$, ex calculo invenies $\frac{x}{\sqrt{aa+xx}}=y$. Et sic de reliquis. And this is the second Proposition of his book of Quadratures, & it is as much as to say that the method by which M^r Newton had now demonstrated the first of his three Rules was general, & extended to the determination of this Problem <u>Data Æquatione fluentes duas quantitates involvente fluxiones invenire</u>. And that he extended it also to more then two fluents appears by his letter of 10 December 1672 where he saith that his method stuck not at surds. For surds are in this method considered as fluents. So then the two first Propositions of the book of Quadratures were then known to M^r Newton

In this Tract of Analysis M^r Newton writes also that his Method extends to such Curve lines as were then called Mechanical. And instances in the Quadratrix by shewing how to find the Ordinate & Area of this Curve & adding that its length may <461r> be found by the same method. And then he subjoyns Nec quicquam hujusmodi scio ad quod hæc methodus, idque varijs modis sese non extendit Imo tangentes ad Curvas mechanicas (siquando id non alias fiat) hujus ope ducuntur. Et quicquid vulgaris Analysis per æquationes ex finito terminorum numero constantes (quando id sit possibile) perficit, hæc per æquationes infinitas semper perficit: Vt nil dubitaverium nomen Analysis etiam huic tribuere. Rationcinia nempe in hac non minus certa sunt quam in illa, nec æquationes minus exactæ. — Denique ad Analyticam merito pertinere censeatur, cujus beneficio Curvarum areæ & longitudines &c (id modo fiat () exacte et Geometrice determinentur. Sed ista narrandi non est locus. These words [id modo fiat] have respect to a sort of series which sometimes breake off & give the Quadrature in finite equations. One of these series is set down by M^r Newton in his Letter of 24 Octob 1676 as the first of certain Theoremes for Quadratures which he had formerly found by his method of fluxions This is the 5^th Proposition in his book of Quadratures & the first of those for squaring a given curve & the sixt is the second of the same kind, & these two depend on the 3^d & 4^th & those on of the 1^st & 2^d. So that the first six Propositions of that book were known to him when he wrote his Letter of 24 Octob 1676, & even when he wrote his Analysis as M^r Collins in his letter to M^r Strode dated 26 Iuly 1672 explains in these words: By the same method may be obteined the Quadrature or Area of the figure accurately. when it can be done, but always infinitely near.

Next after the Analysis M^r Iones has printed M^r Newton's Letter of 13 Iune 1676, wherein the Rule for reducing binomials into infinite series is set down at large & explained by Examples. And all this was known to M^r Newton when he wrote his Analysis, (the two first terms of this Rule being there set down,) as was also the reduction of finite æquations into infinite series by extraction of roots out of affected æquations, & the Quadratures of Curves by those series, which with the approximation of Quadratures makes up the body of the Epistle. And in the next place is an extract of M^r Newtons Letter of 24 Octob 1676, conteining a further explication of the method of extracting the roots of affected Equations. And next after is a fragment of a letter of M^r Newton written to D^r Wallis in the year 1692 wherein he represents that the sentence enigmatically exprest neare the end of his Letter of 24 Octob 1676, in which he setts down his double method for solving the Problem of determining Curves by the conditions of their Tangents & others more difficult, was this: Vna methodus consistit in Extractione fluentis quantitatis ex æquatione simul involvente fluxionem ejus Altera tantum in assumptione seriei pro quantitate qualibet incognita ex qua cætera commode derivari possunt; et in collatione terminorum homologorum æquationis resultantis ad eruendos terminos assumptæ seriei And then he sets down his method of Extracting a fluent out of an equation involving its fluxion representing it to be an operation of the same kind with the extraction of roots out of affected equations. This method was therefore known to him before he wrote his Letter of 24 Octob 1676.

Then follows a part of a letter writ by Newton to Mr Collins Nov. 8. 1676 wherein he represents that if any Curve be defined by an equation of no more than three terms expressing the relation between its abscissa & Ordinate, he could presently find the simplest figure with whose area its area might be compared. And this proves shews that he had then found out the 10th Proposition of his book of Quadratures & by consequence the 7th 8th & 9th upon which it depends. And this is further confirmed by the Catalogus of Quadratures in the Scholium of the 10th Proposition, mentioned in his Letter 24 Octob 1676 & there said to have been composed long before

By these things it appears that what has been printed about the method of fluxions since the year 1676 was known to Mr Newton when he wrote his two Letters of Iune 13th & Octob 24 1676, & even five years before: For he saith in the first of those Letters that he had been tired with these studies & left them off five years before, & in the second that he wrote a Tract of this method, five years before, designing to publish it with a Tract about Colours. But upon disputes raised against <461v> him about the nature of colours, he laid aside his designe for the sake of quiet before he had finished the Tract upon this Method, & when Mr L. rivalled him in this method he was further discouraged

Now as he understood this method in the year 1671 when he wrote the said Tract upon it, so the Analysis shews that he understood it when he wrote that Tract & in the Introduction to his Tract of Quadratures he tells us that he found it gradually in the years 1665 & 1666. The method of Series he found in the year 1665 & the second terms of the series gave him the moments of the first terms.

The next piece is Mr Newtons Tract de Quadratura Curvarum relates to the direct & inverse method of fluxions. It was published before at the end of his Opticks. The reason why it was published no sooner is given above. In the Scholium at the end thereof the word <u>ut</u> is accidentally omitted. It is expressed in the sentence: Hæ fluxiones sunt ut termini serierum. It should have been expressed afterwards in repeating the sense of this sentence & applying it to the particular terms of the series.

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But let us compare the symbols of Mr Newton & Mr Leibnitz & see which are the older & the better.

Mr Newton in his Analysis sometimes represents fluents by the areas of Curves & their fluxions by the Ordinates, & moments by the Ordinates drawn into the letter o. So where the Ordinate is $\frac{aa}{64x}$ he puts $\boxed{\frac{aa}{64x}}$ for the area. And so if the Ordinate be v or y the Area will be $\boxed{v}$ or $\boxed{y}$. And in this way of notation the moments will be $\frac{aao}{64x}$, vo, yo. Mr Leibnits instead of the Notes $\boxed{\frac{aa}{64x}}$, $\boxed{v}$, $\boxed{y}$ uses the notes $\int \frac{aa}{64x}$, $\int v$, $\int y$. Mr Newtons are much the older being used by him in the year 1669. When letters are put for fluents (as is commonly done) Mr Newton puts for the fluxions sometimes other letters, sometimes the same letters with a prick, sometimes the same letters in a different form or magnitude & still uses any of these notations without confining his method to any one of them. Mr Leibnits has no proper symbols for fluxions; these being finite quantities & the quantities being velocities of motion & differences dx dy &c being infinitely little ones. Mr Newton's symbols of fluxions are therefore the oldest. For moments Mr Newton puts the symbols of fluxions multiplied by the letter o which (as was said) represents an infinitly little quantity answering to a moment of time. Mr Leibnitz puts the symbols of the fluents with the letter d before them.

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In the Acta Leipsica of the month of February 1712, an Extract is given of a collection of Tracts published the year before by Mr Iones & Entituled Analysis per Quantitatum series, fluxiones ac differentias cum enumeratione linearum tertij Ordinis. And whereas in the beginning of this Account the author wishes that Mr Iones had given a fuller Account of the Commercium epistolicum of Mr Collins found in amongst his papers.

Something further has been since communicated by M^{r} Iones, & if M^{r} Leibnitz who had a correspondence with M^{r} Oldenburg & by his means with M^{r} Collins, would publish the Letters remaining in his custody relating to that correspondence or such extracts of them as may conduce to complete what is wanting in the collection of M^{r} Iones, he would equally oblige the world.

In the next place the Author of the Extract gives an account of the method used in the Analysis with the application thereof to the solving of Problemes & in the end of the extract subjoins: Cæterum quod Cl. Editor methodum rationum primarum & ultimarum methodo quantitatum infinite parvarum præfert; sciendum est, variari tantum in modo loquendi & pro rigorosa demonstratione utramque ad methodum Archimedeam revocari debere, ut error quovis dato minor ostendatur. Cumque in calculo præcedente adhibetur o et ov, quis non videt revera adhiberi in finite parvas nempe o pro dx et ov pro dz. By which words I perceive that the Author of the Abstract doth not yet understand the Method of the first & last ratios. For in this method quantities are never considered as infinitely little nor are right lines ever put for arches neither are any lines or quantities put by approximation for any other lines or quantities to which they are not exactly equal, but the whole operation is performed exactly in finite quantities by Euclides Geometry untill you come to an equation & then the equation is reduced by rejecting the terms which destroy one another & dividing the residue by the finite quantity o & making this quantity o not to become infinitely little but totally to vanish. For M^{r} Newtons words in explaining this method, are: Iam supponamus BB in infinitum diminui et evanescere, sive o esse nihil. Had BB or o been considered before as infinitely little, he would not have said Iam supponamus BB in infinitum diminui. Now By the vanishing of o there will remain an Equation which solves the Probleme. And this way of working being throughout as evident exact & demonstrative as any thing in Geometry is justly preferred by M^{r} Iones to the method of infinitely little quantities: which proceeding frequently by approximations is less Geometrical & more liable to errors, but yet may be usefull in some cases. And upon both these methods M^{r} Newton founded his method of fluxions as is manifest by this Analysis written in the year 1669, where he sometimes considers quantities as increasing or decreasing by continual motion or fluxion & gives the name of moments to their momentaneus increases or decreases. Fluxions or motions being finite quantities & the method of first & last ratios consisting in the consideration of nothing but finite quantities, & being exact & demonstrative & free from approximations: M^{r} Newton chose to calll this sort of Analysis the method of fluxions rather then the method of moments, or the method of Indivisibles or Infinitesimals. But yet he intended not thereby to exclude the working in <463v> moments & infinitely little figures, whenever it should be thought convenient this way of working being expedite . And this he has sufficiently explained in the Introduction to his Quadratura Curvarum.

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In Analysi per series numero terminorum infinitas quam Barrovius noster mense Iulio anni 1669 ad Collinium misit, dixi quod methodi ibi expositæ beneficio, Curvarum areæ & longitudines &c (id modo fiat) exacte & Geometrice determinantur: sed ista narrandi ibi non esse locum. Et Collinius in Epistola ad Thomam Strode, 26 Iulij anno 1672 data, ♁ < insertion from f 464r > ♁ scripsit quod haud multo post quam in publicum prodierat Mercatoris Logarithmotechnia exemplar ejus — Barrovio Cantabrigiam misit, qui quasdam Newtoni chartas — extemplo remisit: E quibus, et ex alijs, quæ olim ab Autore cum Barrovio communicata fuerant, patebat illam methodum a dicto Newtono aliquot annis antea [i. e. ante editam illam Logarithmotechniam] excogitatam et modo universali applicatam fuisse. < text from f 464r resumes > scripsit quod ex has Analysi et alijs quæ olim a me cum Barrovio communicata fuerant, pateret illam methodum a me aliquot annis antea (i.e. ante mensem Iulium anni 1669) excogitatam & modo universali applicatam fuisse: ita ut ejus ope in quavis figura Curvi linea proposita quæ una vel pluribus proprietatibus definitur, Quadratura vel Area dictæ figuræ, accurata si possibile sit, sin minus, infinite vero propinqua — obtineri queat. Et in Epistola mea ad Oldenburgum Octob. 24. 1676 data posui fundamentum harum operationum Propositione sequente [Data Æquatione quotcunque fluentes quantitates involvente fluxiones invenire; & vice versa.] deinde addidi Theorema primum inde derivatum quo Curvæ Geometrice quadrantur ubi fieri potest. Et hoc idem fit per Proposit. quintam hujus Libri. quæ Propositio pendet a Propositionibus quatuor primis: ideoque Methodus fluxionum quatenus exponitur in Propositionibus quinque primis hujus libri, mihi innotuit annis aliquot ante mensem Septembrem anni 1668 quo Mercatoris Logarithmotechnia prodijt.

Pag 45. In Analysi per series numero terminorum infinitas, pro fluxionibus posui literas quascunque ut z vel y; pro momentis literas easdam multiplicatas per literam o, & pro fluentibus vel literas alias quascunque ut vel et x, vel fluxiones in quadrato inclusas ut $\boxed{x}$ et $\boxed{y}$. Et sub finem Tractatus illius specimen dabam calculi demonstrando Propositionem primam illius Tractatus.

Pag. 46. Propositionis hujus solutionem cum exemplis in fluxionibus primis & secundis, Wallisius noster in lucem edidit anno 1693. Et hæc fuit Regula omnium prima quæ lucem vidit pro fluxionibus secundis tertijs, & alijs omnibus inveniendis.

Pag. 48 Extat hæc Propositio et ejus solutio in Analysi per Æquationes numero terminorum infinitas prope finem. Et ejus solutio eadem est cum solutione Propositionis primæ.

Pag 61. Prop. X. Corol. II Ad hoc Corollarium spectabat Epistola mea Novem 8. 1676 ad Collinium scripta his verbis. Nulla extat Curva cujus Æquatio — haud tamen adeo generaliter ‡ < insertion from f 464r > ‡ Vbi Librum hunc de Quadratura Curvilinearum ex chartis meis antiquioribus ad usque hoc Corrollarium composueram. Scripsi ad Collinium nostrum Epistolam sequentem Octob. 24 1676 datam. Nullam extat Curva – – – – haud tamen adeo generaliter. < text from f 464r resumes >

Pag 62. Hanc Tabulam diu ante annum 1676 proindeque compositam fuisse patet per ordinatas Curvarum in Epistola mea prædicta Octob. 24 1676 ad Oldenburgum data positas.

Pag 61. Prop X Corol 11.

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Ad Lectorem.

Interea dum componerem Philosophiæ naturalis Principia Mathematica, plura Problemata solvi per figurarum quadraturas quas Liber de Quadratura figurarum mihi suppeditavit Alia proposui solvenda concessis figurarum quadraturis. Et plurima demonstravi invertendo ordinem ivnentionis Analyticæ. Et propterea Librum de Quadratura Curvarum subjungere visum est quo figuras vel quadravi vel quadrandas proposui, et qui Analysin meam momentorum exhibet quo sæpissime usus sum. Et cum in exponenda Cometarum Theoria usus sim methodo mea differentiali, visum est etiam eandem methodum subjungere

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