Method of Curves and Infinite Series, and application to the Geometry of Curves (Part 2)

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Published online: August 2013

<57>

Problema 5.
Curvæ alicujus ad datum punctum
curvaturam invenire.

Problema cum primis elegans videtur et ad curvarum scientiam utile. In ejus autem constructionem generalia quædam præmittere convenit.

1**{.}** Ejusdem circuli eadem est undique curvatura et Inæqualium circulorum curvaturæ sunt reciprocè proportionales diametris. Si alicujus diameter diametro alterius duplo minor est, ejus periferiæ curvatura erit duplo major, si diameter triplo minor est curvatura erit triplo major, &c.

2. Si Circulus Curvam aliquam ad partem concavam in dato puncto tangat, sitque talis magnitudinis ut alius contingens circulus in angulis contactûs proximè punctum istud interscribi nequeat, circulus ille ejusdem est curvitatis ac Curva in isto puncto contactûs. Nam circulus, qui inter curvam et alium circulum juxta punctum contactus interjacet, minus deflectit a curva ejusque curvaturam magis appropinquat quam ille alius circulus; et proinde curvaturam ejus maximè appropinquat inter quem et Curvam non alius quisquam potes intercedere.

3. Itaque centrum curvaminis ad aliquod Curvæ punctum est centrum tangentis circuli æqualiter incurvatæ; et sic radius vel semidiameter curvaminis est pars perpendiculi ad istud centrum terminata.

4**{.}** Et proportio curvaminis ad diversa ejus puncta e proportione cui curvaminis circulorum æque curvorum sive e reciproca proportione radiorum curvaminis innotescit.

Problema itaque ad hunc locum redijt ut radius vel centrum curvaminis inveniatur**{.}**

Concipe ergo quod ad tria curvæ puncta δ, D, ac d ducantur perpendicula quorum quæ sunt ad D et δ conveniant in H; et quæ ad D et d, conveniant in h. Et puncto D existente medio si major est curvitas a parte Dδ quam Dd, erit $\delta H < dh$. Sed quo perpendicula δH ac dh propiora sunt intermedio perpendiculo, eò minùs distabunt puncta H et h. Et convenientibus <58> tandem perpendiculis, coalescent. Coalescant autem in puncto C et erit illud C centrum curvaminis ad curvæ punctum D cui perpendicula insistunt. Id quod per se manifestum est.

Hujus autem C varia sunt symptomata quæ ad ejus determinationem inservire possunt: Quemadmodum 1**{.}** Quod sit concursus perpendiculorum hinc et inde a DC infinitè parùm distantium.

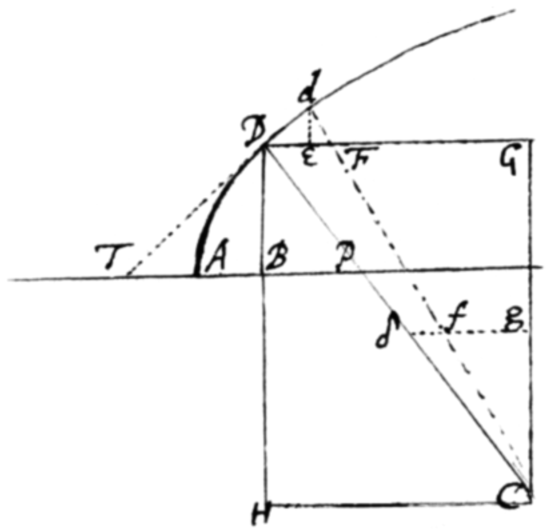
2**{.}** Quod perpendiculorum finitè parùm distantium intersectiones hinc et inde dirimit ac disterminat. Ita ut quæ sunt a parte curviori Dδ citiùs ad H conveniant, et quæ sunt ex alterâ minùs curvâ parte Dδ remotiùs conveniant ad h.

3. Si DC dum curvæ perpendiculariter insistat moveri concipiatur, illud ejus punctum C (si demas motum accedendi vel recedendi a puncto insistentiæ C) minimè movebitur sed centri motionis rationem habebit.

4**{.}** Si centro C intervallo DC circulus describatur, non potest alius describi circulus qui juxta contactum interjacebit.

5. Denique si alterius alicujus tangentis circuli centrum ut H vel h paulatim ad hujus centrum C accedat donec tandem conveniat, tunc aliquod e punctis in quibus circulus ille curvam secavit simul conveniet punctum contactûs D.

Et unumquodque horum Symptomatum ansam præbet diversimodè resolvendi Problema. Nos autem primum tanquam simplicissimum eligemus.



Ad quodlibet Curvæ punctum D esto DT tangens, DC perpendiculum et C centrum curvaminis ut ante. Sitque AB basis ad quam DB in angulo recto applicatur, et cui DC occurrit in P. Age DG parallelam AB, Et CG perpendiculum, inque eo cape Cg cujuslibet datæ magnitudinis, et age gδ perpendiculum quod occurrat DC in δ: eritque $\mathrm{Cg}\ .\ \mathrm{g}\delta\ \ (::\ \ \mathrm{TB}\ .\ \mathrm{BD})\ \ ::$ fluxio Basis ad fluxionem Applicatæ. Concipe præterea punctum D per infinitè parvum intervallum Dd in curva promoveri et actis dE ad DG et Cd ad curvam normalibus quarum Cd occurrit DG in F et δg in f; erit DE momentum Basis <59> dE momentum Applicatæ, ac δf contemporaneum momentum rectæ gδ. Estque $\mathrm{DF} = \mathrm{DE} + \frac{\mathrm{dE}\times\mathrm{dE}}{\mathrm{DE}}$. Habitis itaque horum momentorum sive quod perinde est fluxionum generantium rationibus, habebitur ratio GC ad datam gC (quippe quæ est DF ad δf,) et inde punctum C determinabitur.

Sit ergo $\mathrm{AB} = \mathrm{x}$, $\mathrm{BD} = \mathrm{y}$, $\mathrm{Cg} = 1$, et $\mathrm{g}\delta = \mathrm{z}$ et erit $1\ .\ \mathrm{z}\ \ ::\ \ \mathrm{m}\ .\ \mathrm{n}$ seu $\mathrm{z} = \frac{\mathrm{n}}{\mathrm{m}}$, hujus autem z momentum δf dic $\mathrm{r}\times\mathrm{o}$ (factum nempe ex velocitate et infinite parva quantitate,) eritque momentum $\mathrm{DE} = \mathrm{m}\times\mathrm{o}$, $\mathrm{dE} = \mathrm{n}\times\mathrm{o}$, et inde $\mathrm{DF} = \mathrm{mo} + \frac{\mathrm{nno}}{\mathrm{m}}$. Est ergo $\mathrm{Cg}(1)\ .\ \mathrm{CG}\ \ ::\ \ (\delta\mathrm{f}\ .\ \mathrm{DF}\ \ ::)\ \ \mathrm{ro}\ .\ \mathrm{mo} + \frac{\mathrm{nno}}{\mathrm{m}}$. Adeoque $\mathrm{CG} = \frac{\mathrm{mm}+\mathrm{nn}}{\mathrm{mr}}$.

Cùm insuper Basis fluxioni m (ad quam tanquam uniformem fluxionem cæteras referre convenit) liberum sit quancunque velocitatem tribuere; dic esse 1, et erit $\mathrm{n} = \mathrm{z}$, et $\mathrm{CG} = \frac{1+\mathrm{zz}}{\mathrm{r}}$. Et inde $\mathrm{DG} = \frac{\mathrm{z}+\mathrm{z}^3}{\mathrm{r}}$, ac $\mathrm{DC} = \frac{\overline{1+\mathrm{zz}}\sqrt{\overline{1+\mathrm{zz}}}}{\mathrm{r}}$**{.}**

Expositâ itaque quâvis æquatione quâ relatio BD ad AB pro curvâ definiendâ designetur, imprimis quære relationem inter m et n per Problema 1, et interea substitue 1 pro m et z pro n. Dein ex æquatione resultante per idem Problema 1 quære relationem inter m, n, et r et interea substitue 1 pro m et z pro n ut ante. Atque ita per priorem operationem obtinebis valorem z, et per posteriorem obtinebis valorem r; quibus habitis, produc DB ad H versus concavam partem curvæ ut sit $DH = \frac{1+zz}{r}$, et age HC parallelam AB et perpendiculo DC occurrentem in C, eritque C centrum curvaturæ ad curvæ punctum D. Vel cùm sit $1+zz = \frac{PT}{BP}$, fac $DH = \frac{PT}{r \times BP}$, vel $DC = \frac{DP^3}{r \times DB^3}$.

Exemplum 1. Sic exposita $ax + bxx - yy = 0$, æquatione ad Hyperbolam cujus latus rectum est a ac transversum $\frac{a}{b}$; emerget (per Problema 1) $a + 2bx - 2zy = 0$ (scriptis nempe 1 pro m et z pro n in æquatione resultante, quæ secus foret $am + 2bmx - 2ny = 0$) et hinc denuò prodit $2b - 2zz - 2ry = 0$ scriptis iterum 1 pro m et z pro n. Per priorem est $z = \frac{a+2bx}{2y}$, et per posteriorem $r = \frac{b-zz}{y}$. Dato itaque quovis curvæ puncto D et per consequentiam x et y, ex his dabuntur z et r, quibus cognitis fac $\frac{1+zz}{r} =$ GC vel DH, et age HC Quemadmodum si definitè sit $a = 3$, & $b = 1$, adeoque $3x + xx = yy$ Hyperbolæ conditio: et si <60> assumatur $x = 1$, erit $y = 2$, $z = \frac{5}{4}$, $r = -\frac{9}{32}$ & $DH = -9\frac{1}{9}$. Invento H, erige HC occurrente perpendiculo DC priùs ducto. Vel quod perinde est fac $HD \,.\, HC \quad (:: \ 1 \,.\, z) \ :: \ 1 \,.\, \frac{5}{4}$ et age DC curvedinis Radium.

Siquando computationem non admodum perplexam fore censeas, possis indefinitos valores ipsorum r et z in $\frac{1+zz}{r}$ valore CG substituere. Et sic in hoc exemplo per debitam reductionem obtinebis $DH = y + \frac{4y^3+4by^3}{aa}$. Cujus tamen DH valor per calculum negativus prodit sicut in exemplo numerali videre est. At hoc tantùm arguit DH ad partes versus B capiendam esse**{.}** Nam si fuisset affirmativus ad contrarias partes duxisse oporter**{et}**.

Corollarium. Hinc si signum symbolo $+b$ præfixum mutetur, ut fiat $ax - bxx - yy = 0$ æquatio ad Ellipsin; erit $DH = y + \frac{4y^3-4by^3}{aa}$ **{.}**

At posito $b = 0$ ut æquatio fiat $ax - yy = 0$ ad Parabolam; erit $DH = y + \frac{4y^3}{aa}$. Indéque $DG = \frac{1}{2}a + 2x$. Ex his facilè colligitur radium curvaturæ cujusvis conicæ sectionis valere $\frac{4DP^{cubum}}{aa}$.

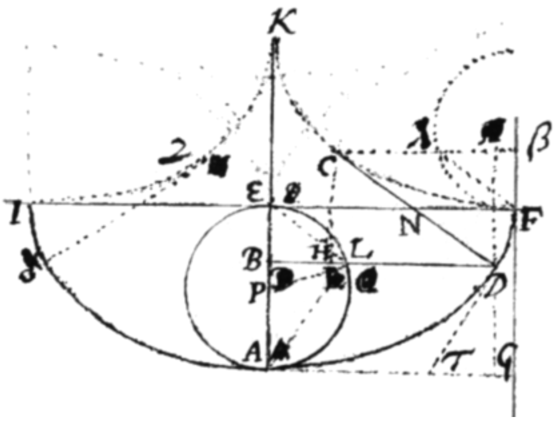
Exemplum 2. Si $x^3 = ayy - xyy$ (æquatio ad Cissoidem Dioclis) exponatur; Per Problema 1 imprimis obtinebitur $3xx = 2azy - 2xzy - yy$; ac deinde $6x = 2ary - 2azz - 2zy - 2xry - 2xzz - 2zy$. Adeoque est $z = \frac{3xx+yy}{2ay-2xy}$. Et $r = \frac{3x-azz+2zy+xzz}{ay-xy}$. Dato itaque quolibet Cissoidis puncto et inde x et y, dabuntur z et r: Quibus cognitis fac $\frac{1+zz}{r} = CG$.

Exemplum 3**{.}** Si detur $\overline{b+y}\sqrt{cc-yy} = xy$ æquatio ad Conchoidem, ut supra; Finge $\sqrt{cc-yy} = v$, et emerget $bv + yv = xy$. Jam harum prior (viz $cc - yy = vv$) per Problema 1 dat $-2yz = 2vl$ (scripto nempe z pro n,) et posterior dat $bl + yl + zv = y + xz$. Et ex his æquationibus rite dispositis determinantur l et z. Ut autem r præterea determinetur, e novissimâ æquatione extermina fluxionem l substituendo $\frac{-yz}{v}$ et emerget $-\frac{byz}{v} - \frac{yyz}{v} + zv = y + xz$, æquatio quæ fluentes quantitates sine aliquibus earum fluxionibus (prout exigit resolutio Problematis primi) complectitur. Hinc itaque per Problema 1 elicies $-\frac{bzz}{v} - \frac{byr}{v} + \frac{byzl}{vv} - \frac{2yzz}{v} - \frac{yyr}{v} + \frac{yyzl}{vv} + rv + zl = 2z + xr$. Qua æquatione in ordinem redactâ et concinnatâ, dabitur r. <61> Inventis autem z et r fac $\frac{1+zz}{r} = CG$.

Si penultimam æquationem per z divisisses, exinde postmodum per Problema 1 obtinuisses $-\frac{bz}{v} + \frac{byl}{vv} - \frac{2yz}{v} + \frac{yyl}{vv} + l = 2 - \frac{yr}{zz}$, æquationem priori simpliciorem pro determinando r.

Dedi quidem hoc exemplum ut modus operandi in surdis æquationibus constaret. At Conchoidis curvatura sic breviùs inveniri potuit. Æquationis $\overline{b+y}\sqrt{cc-yy} = xy$ partibus quadratis et per yy divisis, exurgit $\frac{bbcc}{yy} + \frac{2bcc}{y} \begin{matrix} +cc \\ -bb \end{matrix} - 2by - yy = xx$ **{.}** Et inde per Problema 1 exoritur $-\frac{2bbccz}{y^3} - \frac{2bccz}{yy} - 2bz - 2yz = 2x$. sive $-\frac{bbcc}{y^3} - \frac{bcc}{yy} - b - y = \frac{x}{z}$. Et hinc denuo per Problema 1 exoritur $\frac{3bbccz}{y^4} + \frac{2bccz}{y^3} - z = \frac{1}{z} - \frac{xr}{zz}$. Per priorem exitum determinatur z, et per posteriorem r.

Exemplum 4. Sit IADF Trochois ad circulum ALE (cujus diameter est AE) accommodata; et ordinatâ BD secante circulum in L, dic $AE = a$, $AB = x$, $BD = y$, $BL = v$, et arcus $AL = t$ ejusque arcûs fluxionem dic k. Et imprimis (ducto PL semidiametro) erit Fluxio Basis AB ad fluxionem arcus AL ut BL ad PL; hoc est, m sive $1 \,.\, k \ :: \ v \,.\, \frac{1}{2}a$. Atque adeo $\frac{a}{2v} = k$.



Porrò ex natura circuli est $ax - xx = vv$. Et inde per Problema 1 $a - 2x = 2lv$, sive $\frac{a-2x}{2v} = l$.

Adhæc ex natura Trochoidis est $LD =$ arcus AL; adeoque $v + t = y$. Et inde per Problema 1, $l + k = z$.

Denique pro fluxionibus l et k valores hic substituantur et emerget $\frac{a-x}{v} = z$. Unde per Problema 1 deducitur $-\frac{al}{vv} + \frac{xl}{vv} - \frac{1}{v} = r$ **{.}** Et his inventis fac $\frac{1+zz}{r} = -DH$ et erige HC.

Corollarium. Cæterum ex his consectatur, 1**{.}** Quod sit $DH = 2BL$ et $CH = 2BE$, sive quod EF in N bisecat CD radium curvaminis. Et hoc patebit substituendo valores r et z jam inventos in æquatione $\frac{1+zz}{r} = DH$ et exitum probè reducendo.

2. Hinc Curva FCK in qua centrum curvaminis indefinite versatur est alia <62> huic æqualis Trochois cujus vertices ad I et F adjacent hujus cuspidibus. Nam circulus Fλ æqualis ALE et similiter positus describatur et agatur Cβ parallela EF circuloque occurrens in λ; et erit arcus Fλ (= arcus EL = NF)= Cλ**{.}**

3. CD quæ recta est ad Trochoidem IAF, contingit Trochoidem IKF in C.

4. Hinc (inversis Trochoidibus) si superioris Trochoidis cuspidi K pondus ad distantiam KA sive 2EA filo appensum innitatur, et undulante pondere filum se applicet ad Trochoidis partes KF et KI hinc inde obsistentes ne in rectum distendatur, et cogentes ut ad earum normam dum digreditur a perpendiculo paulatim desuper inflectatur, parte CD sub infimo contactûs puncto manente rectâ: pondus in inferioris Trochoidis perimetro movebitur, utpote cui filum CD semper perpendiculare est.

5. Est itaque tota fili longitudo KA æqualis perimetro Trochoidis KCF, ejusque pars CD æqualis parti perimetri CF.

6. Cum filum circa mobile punctum C tanquam centrum undulando convolvatur; superficies per quam tota CD continuò trajicitur erit ad superficiem per quam pars CN supra rectam IF simul trajicitur ut CD^q ad CN^q hoc est ut 4 ad 1. Est itaque area CFN quarta pars areæ CFD, et area KCNE quarta pars areæ KCDA.

7. Quinimò cùm subtensa EL sit æqualis et parallela CN, et circa immobile centrum E perinde ac CN circa mobile centrum C circumagitur, æquales erunt superficies per quas simul trajiciuntur; nempe area CFN et circuli segmentum EL. Et inde area NFD tripla erit segmenti istius, ac tota EADF tripla semicirculi**{.}**

8. Denique cùm pondus D attingit punctum F, totum filum circum Trochoidis perimetrum KCF flectetur, radio curvaminis CD manente nullo. Et proinde Trochois IAF ad ejus cuspidem F curvior est quàm quilibet circulus, et cum tangente BF productâ constituit angulum contactus infinitè majorem quàm circulus cum rectâ potest constituere.

Sunt etiam anguli contactûs Trochoidalibus infinitè majores <63> et illis deinceps alij infinite majores et sic in infinitum, et tamen maximi sunt infinitè minores rectilineis. Sic $xx = ay$. $x^3 = byy$. $x^4 = cy^3$. $x^5 = dy^4$ &c denotant seriem curvarum quarum quælibet posterior cum Basi constituit angulum contactus infinitè majorem quàm prior cùm eadem Basi potest constituere. Estque angulus contactus quem prima $xx = ay$ constituit, ejusdem generis cum circularibus, et ille quem secunda $x^3 = byy$ constituit, ejusdem generis cum Trochoidalibus. Et quamvis subsequentium anguli angulos præcedentium perpetim infinitè superant, tamen anguli rectilinei magnitudinem nunquam possunt assequi.

Ad eundem modum $x = y$. $xx = ay$. $x^3 = bby$. $x^4 = c^3y$ &c denotant seriem linearum quarum subsequentium anguli ad vertices cum basibus confecti sunt angulis præcedentium perpetim infinitè minores. Quinetiam inter angulos contactus duorum quorumlibet ex his generibus possunt alia angulorum se infinite superantium intercedentia genera in infinitum excogitari.

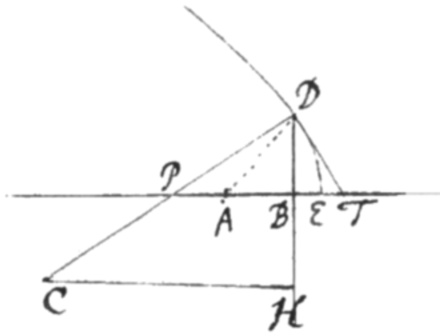
Angulorum verò contactus unum genus esse infinitè majus alio constat cùm unius generis curva utcunque magna inter rectam tangentem et alterius generis curvam quantumvis parvam juxta punctum contactus non potest interjacere: Sive cujus angulus contactus necessariò continet alterius angulum contactûs ut partem totius. Sic curva $x^4 = cy^3$ angulus contactûs quem cum basi constituit, necessario continet angulum contactus curvæ $x^3 = byy$. Qui verò se mutuò superare possunt anguli sunt ejusdem generis, uti de præfatis angulis Trochoidis et hujus curvæ $x^3 = byy$ contigit.

Ex his patet curvas in quibusdam punctis posse infinitè rectiores esse vel infinitè curviores quolibet circulo et tamen formam curvarum non ideo amittere. Sed hæc in transitu. ***

< insertion from p 65 >

***[1]

Exemplum 5. Esto ED Quadratrix ad circulum centro A descriptum pertinens, ac DB ad AE normaliter demissâ dic $AB = x$. $BD = y$ et $AE = 1$. Eritque $nx - nyy - nxx = my$ ut supra. quæ æquatio, scriptis 1 pro m et z pro n, fit $zx - zyy - zxx = y$; Et inde per Problema 1 elicitur $rx - ryy - rxx + zm - 2zmx - 2zny = n$. Factâque reductione et scriptis iterum 1 pro m et z pro n, exit $r = \frac{2zzy + 2zx}{x - xx - yy}$. Inventis autem z et r fac $\frac{1+zz}{r} = DH$, et age HC ut supra.



Si constructionem concinnare placet, perbrevem invenies; nempe ad DT duc normalem DP occurrentem AT in P, et fac esse $2AP \,.\, AE \;::\; PT \,.\, CH$.

Scilicet est $z = \frac{y}{x - xx - yy} = \frac{BD}{-BT}$, et $zy = \frac{BD^q}{-BT} = -BP$. et $zy + x = -AP$, et $\frac{2z}{x - xx - yy}$ in $zy + x = \frac{2BD}{AE \times BT^q}$ in $-AP = r$ Præterea est $1 + zz = \frac{PT}{BT}$, (utpote $= 1 + \frac{BD^q}{BT^q} = \frac{DT^q}{BT^q}$,) adeóque $\frac{1+zz}{r} = \frac{PT \times AE \times BT}{-2BD \times AP} = DH$. Denique est $BT \,.\, BD \;::\; DH \,.\; CH = \frac{PT \times AE}{-2AP}$. Ubi valor negativus tantum arguit CH capiendam esse ad partes DH versus AB.

Eadem methodo Spiralium et aliarum quarumvis Curvarum curvatura calculo brevissimo determinari potest.

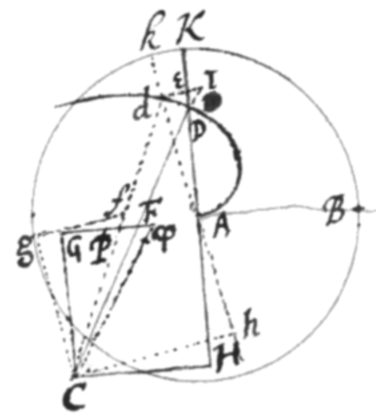
Ad curvaturam praæterea, cum curvæ alijs modis ad rectas referuntur, sine prævia reductione determinandam, jam potuit hæc methodus applicari perinde ut in determinando Tangentes factum est. Sed cùm omnes Geometricæ curvæ ut et Mechanicæ (præsertim ubi definientes conditiones ad infinitas æquationes uti post ostendam reducantur) ad rectangulas ordinatas referri possent videor satis præstitisse.

[2]

< text from p 63 resumes >

<64> Qui plura desiderat haud difficulter proprio Marte supplebit Præsertim si in ejus rei illustrationem ex abundanti methodum pro Spiralibus adjecero.

Esto BK circulus, A centrum ejus, B punctum in circumferentia datum, ADd spiralis, DC perpendiculum ejus, et C centrum curvitatis ad punctum D. Ductâque ADK recta, et ei parallela et æquali CG, ut et normali GF occurrente CD in F; dic AB vel $AK = 1 = CG$, $BK = x$, $AD = y$, & $GF = z$. Præterea concipe punctum D per infinitè parvum spatium Dd in spirali moveri, et perinde per d agi semidiatrum Ak, eique parallelam et æqualem Cg, et normalem gf occurentem Cd in f, cui etiam GF occurrit in P; Produc GF ad φ ut sit $G\varphi = gf$, et ad AK demitte normalem dE et produc donec cum CD conveniat ad I: Et ipsarum BK, AD, ac Gφ contemporanea momenta erunt Kk, DE, et Fφ, quæ proinde dicentur $m \times o$, $n \times o$, et $r \times o$.



Jam est $AK \,.\, AE\,(AD) \;::\; kK \,.\, dE = oy$ ubi assumo $m = 1$ ut supra. Item
$CG \,.\, GF \;::\; dE \,.\, ED = oyz$ adeóque $yz = n$. Præterea
$CG \,.\, CF \;::\; dE \,.\, dD = oy \times CF \;::\; dD \,.\, dI = oy \times CF$. Ad hæc propter
$\text{ang } PC\varphi (= \text{ang } GCg) = \text{ang } DAd$, $\text{ang } CP\varphi (= \text{ang } CdI = \text{ang } EdD + \text{rect:}) = \text{ang } ADd$,
triangula CPφ et ADd sunt similia, et inde $AD \,.\, Dd \;::\; CP\,(CF) \,.\, P\varphi = o \times CF^q$, unde aufer Fφ et restabit $PF = o \times CF^q - o \times r$. Denique demissa CH normali ad AD est
$PF \,.\, dI \;::\; CG \,.\, EH$ vel $DH = \frac{y \times CF^q}{CF^q - r}$. Vel substituto $1 + zz$ pro CF^q, erit $DH = \frac{1+yzz}{1+zz-r}$ **{.}**

Et nota quod in hujusmodi computationibus quantitates (ut AD et AE) pro ǽqualibus habeo quarum ratio a ratione aequa <67> litatis non nisi infinitè parùm differt.

Ex his autem prodit hujusmodi Regula: Relatione inter x et y per quamlibet æquationem definitâ, quære relationem fluxionum m et n ope Problematis 1, et substitue 1 pro m et yz pro n. Deinde ex æquatione prodeunte quære denuò per Problema 1 relationem inter m n et r et iterum substitue 1 pro m. Prior exitus per debitam reductionem dabit n et z et posterior dabit r; quibus cognitis fac $\frac{y+yzz}{1+zz-r} = DH$, et erige normalem HC spiralis perpendiculo DC priùs ducto occurrentem in C, et erit C centrum curvaminis. Vel quod eodem recidit cape $CH \,.\, HD \;::\; z \,.\, 1$, et age CD.

Exemplum 1. Si detur $ax = y$ æquatio ad Spiralem Archimedeam; erit per Problema 1 $am = n$ sive (scripto 1 pro m et yz pro n) $a = yz$. Et hinc denuò per Problema 1 exit $0 = nz + yr$. Quare ex dato quolibet spiralis puncto D et inde longitudine AD sive y, dabuntur z $\left(= \frac{a}{y}\right)$ et r $\left(= -\frac{nz}{y} \text{ sive } = \frac{-az}{y}\right)$: Quibus cognitis fac $1 + zz - r \,.\, 1 + zz \;::\; DA\,(y) \,.\, DH$. et $1 \,.\, z \;::\; DH \,.\, CH$.

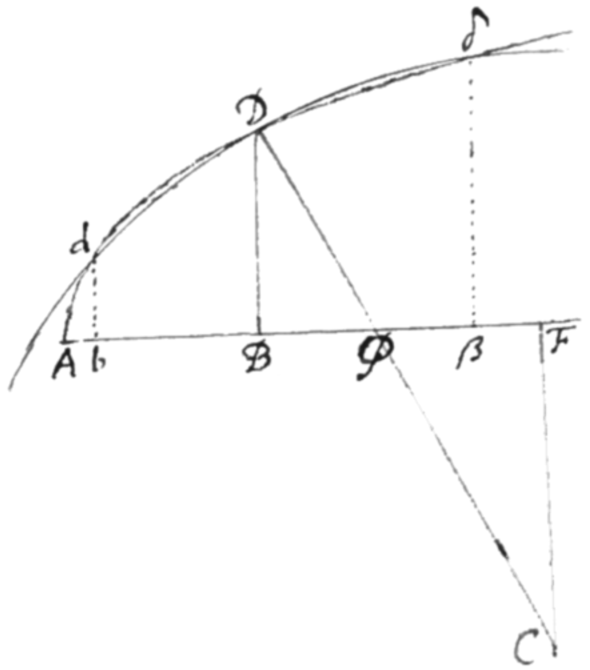
Et hinc facilè deducitur hujusmodi constructio. Produc AB ad Q ut sit $AB \,.\, \text{arc } BK \;::\; \text{arc } BK \,.\, BQ$, et fac $AB + AQ \,.\, AQ \;::\; DA \,.\, DH \;::\; a \,.\, HC$.[3]

Exemplum 2. Si $axx = y^3$ definit relationem inter BK et AD: obtinebis (per Problema 1) $2amx = 3ny^2$, sive $2ax = 3zy^3$, et inde rursus $2am = 3ry^3 + 9znyy$. Est itaque $z = \frac{2ax}{3y^3}$ et $r = \frac{2a - 9zzy^3}{3y^3}$. Quibus cognitis fac $1 + zz - r \,.\, 1 + zz \;::\; DA \,.\, DH$. Vel opere concinnato, fac $9xx + 6 \,.\, 9xx + 4 \;::\; DA \,.\, DH$.

Exemplum 3. Ad eundem modum si $axx - bxy = y^3$ determinat relationem BK ad AD, orietur $\frac{2ax - by}{bxy + 3y^3} = z$, et $\frac{2a - 2bzy - bzzxy - 9zzy^3}{bxy + 3y^3} = r$. Ex quibus DH, et inde punctum C determinatur ut ante.

Et sic aliarum quarumvis spiralium curvaturam nullo negotio determinabis. Imo et ad horum exemplar Regulas pro <68> quibuslibet curvarum generibus excogitare.

Absolvi tandem Problema sed cum methodum adhibueri**{m}** a vulgaribus operandi modis satis diversam, et ipsum Problema non sit ex eorum numero quorum contemplatio apud Geometras increbuit: in ablatæ solutionis illustrationem et confirmationem non gravabor aliam solutionem attingere, magis obviam et usitatis in ducendo tangentes methodis affinem. Utpote si centro et intervallo quovis circulus describi concipiatur, qui curvam quamlibet in pluribus punctis secet, et circulus ille contrahetur vel dilatetur donec duo intersectionum puncta conveniant, is curvam ibidem tanget. Et præterea si centrum ejus accedere vel recedere a puncto contactûs fingatur, donec tertium intersectionis punctum cum prioribus in puncto contactûs conveniat, is æque curvus ac Curva in illo puncto contactûs evadet. Quemadmodum in ultimo quinque symptomatum centri curvaminis supra monui, e quorum singulis dixi Problema diversimodè confici potuisse.



Centro itaque C et radio CD describatur circulus secans curvam in punctis d, D, ac δ. Et demissis db, DB, δβ, et CF ad Basin AB normalibus: dic $AB=x$, $BD=y$, $AF=v$, $FC=t$, ac $DC=s$; et erit $BF=v-x$, ac $DB+FC=y+t$; Quorum quadratorum aggregatum æquatur quadrato DC. Hoc est $vv-2vx+xx+yy+2ty+tt=ss$. Quam < insertion from p 201 > **{**æquationem si pl**}**acet abbreviare possis fingendo $vv+tt-ss=$ symbolo cuivis qq, et evadet $xx-2vx+yy+2ty+qq=0$. Postquam verò t, v, et qq inveneris si s desideres fac $=\sqrt{vv+tt-qq}$. < text from p 68 resumes >

Proponatur jam quælibet æquatio pro Curva definienda cujus flexuræ quantitatem invenire oportet et ejus ope alterutram quantitatem x vel y externima et emerget æquatio cujus radices (db, DB, δβ &c si extermines x, vel Ab, AB, Aβ &c si extermines y) sunt ad intersectionum puncta (d, D, δ &c). Et proinde cùm <69> ex istis tres evadent æquales, circulus et curvam continget et erit ejusdem curvitatis ac curva in puncto contactus**{.}** Æquales autem evadent conferendo æquationem cum alia totidem dimensionum æquatione fictitia cujus tres sunt æquales radices ut docuit Cartesius; vel expeditiùs multiplicando terminos ejus bis per Arithmeticam progressionem.

Exemplum. Sit $ax=yy$ æquatio ad Parabolam, et exterminato x (substituendo nempe in æquatione superiori valorem ejus $\frac{yy}{a}$) prodibit

$$\begin{array}{cccccccc} \frac{y^4}{a} & - & \frac{2v}{a}yy & + & 2ty & + & qq & = 0 \\ & + & yy & & & & & \\ 4. & & 2. & & 1. & & 0. & \\ 3. & & 1. & & 0. & & -1. & \end{array}$$

cujus e radicibus y tres debent fieri æquales. Et in hunc finem terminos per Arithmeticam progressionem bis multiplico ut hic videre est, et exit $\frac{12y^4}{aa}-\frac{4v}{a}yy+2yy=0$ sive $v=\frac{3yy}{a}+\frac{1}{2}a$. Unde facilè colligitur esse $BF=2x+\frac{1}{2}a$ ut supra.

Quamobrem dato quovis Parabolæ puncto D, duc perpendiculum DP et in axe cape $PF=2AB$ et erige normalem FC occurentem DP in C et erit C desideratum centrum curvitatis.

Idem in Ellipsi et Hyperbola præstare possis sed calculo satis molesto, et in alijs curvis utplurimùm fastidiosissimo.

De Quæstionibus quibusdam cognatis.

Ex hujus Problematis resolutione consectantur aliorum nonnullorum confectiones. Cujusmodi sunt

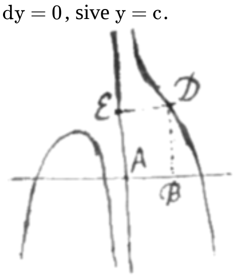
1. Invenire punctum ubi linea datam habet curvaturam.

Sic in Parabola $ax=yy$ si punctum quæratur ad quod radius curvaturæ sit datæ longitudinis f: e centro curvaturæ ut prius invento radium determinabis esse $\frac{a+4x}{2a}\sqrt{aa+4ax}$, quem pone æqualem f. Et factâ reductione emerget $x=-\frac{1}{4}a+\sqrt{c:\ \frac{1}{16}aff}$.

2. Invenire punctum rectitudinis.

Punctum rectitudinis voco ad quod radius flexionis infinitus evadit, sive centrum infinitè distans; quale est ad verticem Parabolæ $ax^3=y^4$. Et hoc idem plerumque limes est flexionis contrariæ cujus <70> determinationem supra posui. Sed et alia haud inelegans ex hoc Problemate scaturit. [4]Nempe quo longior est radius flexionis eo minor evadit angulus DCd, et pariter momentum δf adeóque fluxio quantitatis z unà diminuitur, ita ut per ejus radij infinitatem prorsus evanescant. Quære ergo fluxionem r et suppone nullam esse**{.}**

Quemadmodum si limitem flexûs contrarij in Parabola secundi generis cujus ope Cartesius construxit æquationes sex dimensionum determinare oportet. Ad illam Curvam æquatio est $x^3-bxx-cdx+bcd+dxy=0$. Et hinc per Problema 1 exit $3mxx-2bmx-cdm+dmy+dxn=0$; Quæ, scripto 1 pro m et z pro n, fit $3xx-2bx-cd+dy+dxz=0$: Unde rursus per Problema 1 exit $6mx-2bm+dn+dmz+dxr=0$, Et hæc, scripto iterum 1 pro m, z pro n, et 0 pro r, fit $6x-2b+2dz=0$. Jam extermina z scribendo pro dz valorem $b-3x$, in æquatione $3xx-2bx-cd+dy+dxz=0$, et proveniet $-cd+dy=0$, sive $y=c$. Quamobrem ad punctum A erige perpendiculum $AE=c$, et per E duc ED parallelam AB, et punctum D ubi Parabolæ partem convexo-concavam secuerit erit in confinio flexionis contrariæ.



Similique methodo alia rectitudinis puncta quæ non interjacent partibus contrariè flexis determinari possunt. Veluti si $x^4-4ax^3+6aaxx-b^3y=0$ Curvam definiat, Exinde per Problema 1 imprimis producetur $4x^3-12axx+12aax-b^3z=0$ et hinc denuò $12xx-24ax+12aa-b^3r=0$, Ubi suppone $r=0$ et factâ reductione prodibit $x=a$. Quamobrem sume $AB=a$ et BD normaliter erecta curvæ in desiderato rectitudinis puncto D occurret**{.}**

<71>

3. Invenire punctum flexûs infiniti

Quære radium curvaminis et suppone nullum esse. Sic ad Parabolam secundi generis æquatione $x^3=ayy$ definitam, erit radius ille $CD=\frac{4a+9x}{6a}\sqrt{4ax+9xx}$; qui nullus evadit cùm sit $x=0$.

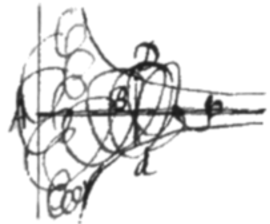
4{.} Flexûs maximi minimive punctum determinare.

Ad hujusmodi puncta radius curvaturæ aut maximus aut minimus evadit. Quare centrum curvaturæ ad id temporis momentum nec versus punctum contactus neque ad contrarias partes movetur sed penitus quiescit. Quæratur itaque fluxio Radij CD; vel expeditiùs, quæratur fluxio alterutrius rectæ BH vel AK, et supponatur nulla.

Quemadmodum si de Parabola secundi generis $x^3=aay$ quæstio proponatur: imprimis ad curvaturæ centrum determinandum invenies $DH=\frac{aa+9xy}{6x}$, adeoque est $BH=\frac{aa+15xy}{6x}$, dic autem $BH=v$, et erit $\frac{aa}{6x}+\frac{5}{2}y=v$, unde juxta Problema 1 educitur $-\frac{aam}{6xx}+\frac{5n}{2}=l$. Jam vero [5]l ipsius BH fluxionem suppone nullam esse, et insuper cùm ex hypothesi sit $x^3 = aay$, et inde per Problema 1 $3mxx = aan$, posito $m = 1$ substitue $\frac{3xx}{aa}$ pro n, et emerget $45x^4 = a^4$. Cape ergo $AB = \sqrt{}\,4{:}\ \frac{x^4}{45}$. Et BD normaliter erecta occurret curvæ in puncto maximæ curvaturæ. Vel, quod perinde est fac $AB \,.\, BD \;::\; 3\sqrt{5} \,.\, 1$.

Ad eundem modum Hyperbola secundi generis per æquationem $xxy = a^3$ designata maximè flectitur in [6] punctis D, d, quæ determinabis sumendo $AQ = 1$ in Basi, et erigendo $QP = \sqrt{5}$. eique æqualem Qp ex altera parte et agendo AP et Ap, quæ curvæ occurrent in desideratis punctis D ac d.

5. Locum centri curvaminis determinare; sive Curvam describere in quâ centrum istud perpetuo versatur.



Trochoidis centrum curvaminis in alia Trochoide <72> versari ostensum est. Et sic Parabolæ centrum istud in alia secundi generis (quam æquatio $axx = y^3$ definit) Parabola versatur, ut inito calculo facilè constabit.

6. Luce in quamlibet curvam incidente, invenire focum sive concursum radiorum circa quodpiam ejus punctum refractorum.

Curvaturam ad istud Curvæ punctum quære, et centro radioque curvaturæ Circulum describe; Dein quære concursum radiorum a Circulo circa istud punctum refractorum. Nam idem erit concursus refractorum a propositâ Curvâ.

7. His addi potest particularis inventio curvaturæ ad vertices curvarum ubi normaliter secant Bases. Nempe punctum in quo Curvæ perpendiculum cum Basi conveniens ipsam ultimò secuerit, est centrum curvaturæ ejus. Quamobrem habitâ relatione inter Basin x et rectangulum applicatum y et inde (per Problema 1) relationem inter fluxiones m et n; valor mn; si in eo scribas 1 pro m et fingas $y = 0$, erit radius curvaturæ.

Sic in Ellipsi $ax - \frac{a}{b}xx = yy$, est $\frac{am}{2} - \frac{amx}{b} = ny$, qui valor ny si supponas $y = 0$ et consequenter $x = 0$ et scribas 1 pro m evadet $\frac{1}{2}a$ radius curvaturæ. Et sic ad vertices Hyperbolæ et Parabolæ radius curvaturæ erit etiam dimidium lateris recti.

Atque ita ad Conchoiden æquatione $\frac{bbcc}{xx} + \frac{2bcc}{x} \begin{matrix} +cc \\ -bb \end{matrix} - 2bx - xx\,yy$ definitam valor ny ope Problematis 1 invenietur $-\frac{bbcc}{x^3} - \frac{bcc}{xx} - b - x$. Qui supponendo $y = 0$, et inde $x = c$ vel $-c$ evadet $-\frac{bb}{c} - 2b - c$, vel $\frac{bb}{c} - 2b + c$ radius curvaturæ. [7]Fac ergo $AE \,.\, EG \;::\; EG \,.\, EC$, et $Ae \,.\, eG \;::\; eG \,.\, ec$, et habes curvaturæ centra C et c ad vertices conjugatarum Conchoidum E et e.

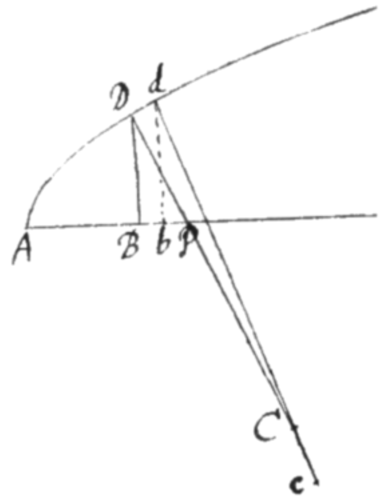
<73>

Problema 6.
Curvaturæ ad datum Curvæ alicujus punctum qualitatem determinare.

Per qualitatem Curvaturæ intelligo formam ejus quatenus est plus vel minùs inæquabilis, sive quatenus plus vel minùs variatur in processu per diversas partes Curvæ. Sic interroganti qualis sit circuli curvatura, responderi potest quod sit uniformis, sive invariata; [8]et interroganti qualis sit curvatura Spiralis quæ describitur per motum puncti D cum accelerata celeritate AD in recta AK uniformitèr circa centrum A gyrante progredientis ab A, adeo ut recta AD ad arcum BK dato puncto K descriptum rationem habeat numeri ad Logarithmum ejus, responderi potest quod sit uniformiter variata sive quod sit æquabiliter inæquabilis. Et sic aliæ curvæ in singulis earum punctis aliquales pro curvaturæ variatione denominari possunt.

Quæritur itaque Curvaturæ circa aliquod Curvæ punctum inæquabilitas sive variatio. Qua de causa animadvertendum est 1 Quod ad puncta in similibus curvis similiter posita similis est inæquabilitas sive variatio curvaturæ. 2 Et quod momenta radiorum curvaturæ ad illa puncta sunt proportionalia contemporaneis momentis curvarum, et fluxiones fluxionibus. 3 Atque adeò quod ubi fluxiones illæ non sunt proportionales dissimilis erit inæquabilitas curvaturæ. Utpote major erit inæquabilitas ubi major est ratio fluxionis radij curvaturæ ad fluxionem Curvæ, Adue fluxionum ratio illa non immeritò dici potest index inæquabilitatis sive variationis curvaturæ.

Ad Curvæ alicujus AD puncta D ac d infinitè parùm distantia sunto radij curvaturæ DC ac dc, et existente Dd momento Curvæ erit Cc contemporaneum momentum radij curvaturæ, et $\frac{Cc}{Dd}$ index inæquabilitatis curvaturæ. Nempe tanta dicetur inæquabilitas illa, quantam esse indicat rationis illius $\frac{Cc}{Dd}$ quantitas. Sive curvatura dicetur tanto dissimilior curvaturæ circuli.



Demissis jam ad quamlibet AB occurrentem DC in P, <74> rectangulis applicatis DB ac db dic $AB = x$, $BD = y$, $DP = t$, $DC = v$, et inde $Bb = m \times o$, eritque $Cc = l \times o$, et $BD \,.\, DP \;::\; Bb \,.\, Dd = \frac{tmo}{y}$. ac $\frac{Cc}{Dd} = \frac{ly}{tm}$ sive $= \frac{ly}{t}$ supposito $m = 1$. Quamobrem relatione inter x et y per quamlibet æquationem definitâ, et inde juxta Problema 4 & 5 invento perpendiculo DP sive t et radio curvaturæ v, ejusque radij fluxione l per Problema 1; dabitur index inæquabilitatis curvaturæ $\frac{ly}{t}$.

Exemplum 1. Sit $2ax = yy$ (æquatio ad Parabolam**()**} et per Problema 4) erit $BP = a$, adeoque $DP = \sqrt{aa + yy} = t$. Item per Problema 5 $BF = a + 2x$ *[9] et $BP \,.\, DP \;::\; BF \,.\, DC = \frac{at+2tx}{a} = v$. Jam æquationes $2ax = yy$ et $aa + yy = tt$, et $\frac{at+2tx}{a} = v$ per Problema 1 dant $2am = 2ny$, et $2ny = 2kt$, et $\frac{ak+2kx+2tm}{a} = l$. Quibus ordinatis et posito $m = 1$, orientur $n = \frac{a}{y}$, $k = \frac{ny}{t}$ vel $= \frac{a}{t}$ et $l = \frac{ak+2kx+2t}{a}$. Et sic inventis n, k, et l habebitur $\frac{ly}{t}$ index inæquabilitatis curvaturæ.

Quemadmodum si in numeris definiatur $a = 1$, sive $2x = yy$, et $x = \frac{1}{2}$, erit $y\left(\sqrt{2x}\right) = 1$, $n\left(\frac{a}{y}\right) = 1$, $t\left(\sqrt{aa + yy}\right) = \sqrt{2}$, $k\left(\frac{a}{t}\right) = \sqrt{\frac{1}{2}}$, et $l\left(\frac{ak+2kx+2t}{a}\right) = 3\sqrt{2}$. Adeoque $\frac{ly}{t} = 3$ indici inæquabilitatis**{.}**

Sin autem definiatur $x = 2$, erit $y = 2$, $n = \frac{1}{2}$, $t = \sqrt{5}$, $k = \sqrt{\frac{1}{5}}$ et $l = 3\sqrt{5}$, Adeoque $\frac{ly}{t} = 6$ index inæquabilitatis. Quamobrem inæquabilitas Curvaturæ ad punctum a quo ad axin demissa ordinatim applicata æquatur lateri recto Parabolæ dupla est ejus ad punctum a quo demissa ordinatim applicata æquatur dimidio ejusdem lateris recti. Hoc est curvatura in priori casu duplo dissimilior est curvaturæ circuli, quàm in posteriori.

Exemplum 2. Sit $2ax - bxx = yy$, et per Problema 4 erit $a - bx =$BP et inde $aa - 2abx + bbxx + yy = tt$, sive $aa - byy + yy = tt$. Item per Problema 5 erit $DH = y + \frac{y^3 - by^3}{aa}$ ubi si $yy - byy$ substituas $tt - aa$ evadet $DH = \frac{tty}{aa}$. Et est $BP \,.\, DP \;::\; DH \,.\, DC = \frac{t^3}{aa} = v$. Jam per Problema 1 æquationes $2ax - bxx = yy$ et $aa - byy + yy = tt$ et $\frac{t^3}{aa} = v$ dant $a - bx = ny$ et $ny - bny = tk$, et $\frac{3ttk}{aa} = l$. Et sic invento l, dabitur $\frac{ly}{t}$ index inæquabilitatis curvaturæ.

Sic ad Ellipsin $2x - 3xx = yy$, ubi est $a = 1$ et $b = 3$ si supponatur $x = \frac{1}{2}$, erit $y = \frac{1}{2}$, $n = -1$, $t = \sqrt{\frac{1}{2}}$, $k = \sqrt{2}$, $l = 3\sqrt{\frac{1}{2}}$ et $\frac{ly}{t} = \frac{3}{2}$ indici inæquabilitatis curvaturæ. Unde patet curvaturam <75> hujus Ellipsis ad hic definitum punctum D, esse duplo minus inæquabilem (sive duplo similiorem curvaturæ circuli,) quàm curvatura Parabolæ ad illud ejus punctum a quo ad axin demissa ordinatim applicata æquatur dimidio ejus lateris recti.

Si conclusiones in his exemplis concinnare placet, ad Parabolam $2ax = yy$ exibit $\frac{ly}{t} = \frac{3y}{a}$ index inæquabilitatis et ad Ellipsin $2ax - bxx = yy$ exibit index $\frac{ly}{t} = \frac{3y - 3by}{aa} \times BP$, et sic ad Hyperbolam $2ax + bxx = yy$, observata analogiâ, erit index $\frac{ly}{t} = \frac{3y + 3by}{aa} \times BP$. Unde patet quod ad diversa puncta cujusvis Conicæ sectionis seorsim spectatæ curvaminis inæquabilitas est ut rectangulum $BD \times BP$. Et quod ad diversa puncta Parabolæ est ut ordinatim applicata BD.

Cæterùm cum Parabola sit simplicissima linearum inæquabili curvaturâ flexarum, ejusque curvaturæ inæquabilitas tam levi negotio determinatur (utpote cujus index sit $\frac{6 \times \text{ordin: applic.}}{\text{lat: rect:}}$;) aliarum curvarum curvaturæ ad curvaturam hujus non incommodè referri possunt. Quemadmodum si quæratur qualis sit Ellipsis $2x - 3xx = yy$ curvatura ad illud ejus punctum quod definitur assumendo $x = \frac{1}{2}$: Quoniam index ejus (ut supra) sit $\frac{3}{2}$, responderi potest esse similem curvaturæ Parabolæ $6x = yy$ ad illud ejus punctum inter quod et axin recta $= \frac{3}{2}$ ordinatim applicatur.

Sic cum lineæ Spiralis ADE jam ante descriptæ[10] fluxio sit ad fluxionem subtensæ AD in data quadam ratione, puta d ad e: versus partes concavas ejus erige ad AD normalem $AP = \frac{e}{\sqrt{dd - ee}} \times AD$, et erit P centrum curvaturæ, et $\frac{AP}{AD}$ sive $\frac{e}{\sqrt{dd - ee}}$ index inæquabilitatis ejus. Quare Spiralis hæcce curvaturam habet ubique similiter inæquabilem ac Parabola $6x = yy$ habet in illo ejus puncto a quo demittitur ad axin ordinatim applicata $= \frac{\sqrt{dd - ee}}{e}$.

Et sic index inæquabilitatis ad quodvis Trochoidis punctum D (fig   ) invenietur esse $\frac{AB}{BL}$. Quare curvatura ejus ad idem D tam inæquabilis est sive tam dissimilis curvaturæ <76> circuli, quàm curvatura Parabolæ cujusvis $ax = yy$ ad illud ejus punctum ubi ordinatim applicata æquatur $\frac{1}{6}a \times \frac{AB}{BL}$.

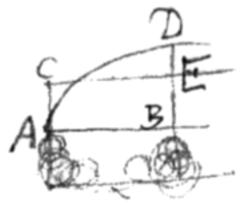
Ex his credo sensus Problematis satis elucescet, quo benè perspecto non difficile erit animadvertenti seriem rerum supra traditarum plura exempla de proprio suppeditare et hujusmodi complures alias operandi methodos, prout res exiget, concinnare. Quinetiàm cognata Problemata (ubi perplexa computatione non conteritur et fatigatur,) haud majori difficultate transiget: Cujusmodi sunt, 1**{.}** Invenire punctum curvæ alicujus ubi vel nullam, vel infinitam, vel maximam aut minimam, vel datam quamvis habeat inæquabilitatem curvaturæ. Sic ad vertices Conicarum sectionum nulla est inæquabilitas curvaturæ, ad cuspidem Trochoidis infinita est, et ad puncta Ellipseos maxima est ubi rectangulum $BD \times BP$ fit maximum, hoc est ubi lineæ diagonales rectanguli Parallelogrammi circumscripti Ellipsin secant cujus latera tangunt illam in principalibus verticibus**{.}**

2. Curvam alicujus definitæ speciei, puta Conicam Sectionem, determinare, cujus curvaturæ ad aliquod punctum & æqualis sit et similis curvaturæ alterius alicujus curve ad datum punctum ejus.

3. Conicam Sectionem determinare ad cujus punctum aliquod curvatura & lineæ tangentis (respectu axis) positio sit similis curvaturæ ac tangentis positioni alterius alicujus Curvæ ad assignatum punctum ejus. Et hujus problematis usus est ut vice Ellipsium secundi generis quarum refringendi proprietates Cartesius in Geometria demonstravit, Conicæ sectiones idem in refractionibus quàm proximè præstantes subrogari possint. Atque idem de alijs curvis intellige.

<77>

Problema 7.
Curvas pro arbitrio multas invenire quarum areæ per finitas æquationes designari possunt.



Sit AB basis curvæ, ad cujus initium A erigatur normalis $AC = 1$ et agatur CE parallela AB, sit etiam DB rectangula applicata occurrens rectæ DE in E et curvæ AD in D. Et concipe has areas ACEB et ADB a rectis BE et BD per AB delatis generari. Et earum incrementa sive fluxiones perpetim erunt ut lineæ describentes BE et BD. Quare parallelogrammum ACEB sive $AB \times 1$ dic x, et curvæ aream ADB dic z: et fluxiones m et r erunt ut BE et BD, adeoque posito $m = 1 = BE$ erit $r = BD$.

Si jam ad arbitrium assumatur æquatio quævis pro definienda relatione z ad x, exinde per problema 1 elicietur r. Atque ita duæ habebuntur æquationes quarum posterior Curvam definiet et prior aream ejus.

Exempla. Assumatur $xx = z$ et inde per Problema 1 elicietur $2mx = r$, sive $2x = r$ siquidem est $m = 1$.

Assumatur $\frac{x^3}{a} = z$ et inde prodibit $\frac{3xx}{a} = r$, æquatio ad Parabolam.

Assumatur $ax^3 = zz$, sive $a^{\frac{1}{2}}x^{\frac{3}{2}} = z$, et emerget $\frac{3}{2}a^{\frac{1}{2}}x^{\frac{1}{2}} = r$, sive $\frac{9}{4}ax = rr$ æquatio iterum ad Parabolam.

Assumatur præterea $a^3x = zz$, sive sive $a^{\frac{3}{2}}x^{\frac{1}{2}} = z$ et elicietur $\frac{1}{2}a^{\frac{3}{2}}x^{-\frac{1}{2}} = r$ sive $a^3 = 4xrr$**{.}**

Item assumatur $\frac{a^3}{x} = z$ sive $a^3x^{-1} = z$ et elicietur $-a^3x^{-2} = r$ sive $a^3 + rxx = 0$ Ubi negativus valor ipsius r tantùm denotat BD capiendam esse ad partes contra BE.

Adhæc si assumas $ccaa + ccxx = zz$, elicies $2ccx = 2zr$ et exterminato z proveniet $\frac{cx}{\sqrt{aa+xx}} = r$.

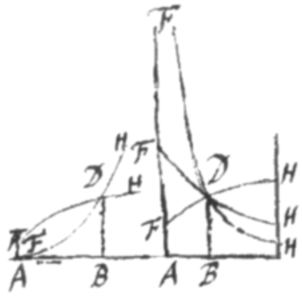
Vel si assumas $\frac{aa+xx}{b}\sqrt{aa + xx} = z$, dic $\sqrt{aa + xx} = v$ et erit $\frac{v^3}{b} = x$, et inde per Problema 1 $\frac{3lvv}{b} = r$. Item æquatio $aa + xx = vv$ per Problema 1 dat $2x = 2vl$ cujus ope si extermines l fiet $\frac{3vx}{b} = r = \frac{3x}{b}\sqrt{aa + xx}$.

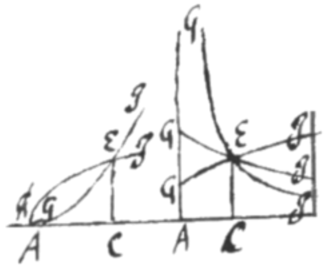
Si denique assumas $8 - 3xz + \frac{2}{5}z = zz$, elicies $-3z - 3xr + \frac{2}{5}r = 2rz$. Quare per assumptam æquationem imprimis quære aream z, ac deinde applicatam r per elicitam**{.}**

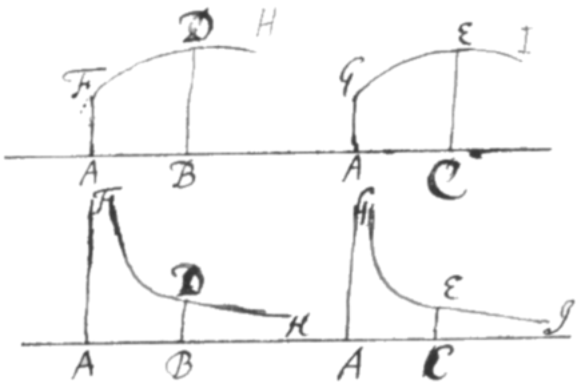
Atque ita ex areis qualescunque effingas semper possis applicatas determinare.

<78>

Problema 8.
Curvas pro arbitrio multas invenire quarum areæ ad aream datæ alicujus Curvæ relationem habent per finitas æquationes designabilem.







Sit FDH data curva, ac GEI quæsita et earum applicatas DB et EC concipe super Basibus AB et AC erectas incedere: Et arearum quas ita transigunt incrementa sive fluxiones erunt ut applicatæ illæ ductæ in earum velocitates incedendi, hoc est in fluxiones basium. Sit ergo $AB = x$, $BD = v$, $AC = z$ ac $CE = y$, area $AFDB = s$, & area $AGEC = t$, ac arearum fluxiones sint p, et q, nempe p ipsius s, et q ipsius t: Eritque $m \times v \;.\; r \times y \;::\; p \;.\; q$. Quare si supponatur $m = 1$, et $v = p$, ut supra; erit $ry = q$ et inde $\frac{q}{r} = y$.

Assumantur itaque duæ quævis æquationes quarum una definiat relationem arearum s ac t, et altera relationem basium x et z et inde per Problema 1 quærantur fluxiones q et r, et statuatur $\frac{q}{r} = y$.

Exemplum 1. Data curva AFD sit circulus æquatione $ax - xx = vv$ designatus, et quærantur aliæ curvæ quarum areæ adæquant aream ejus. Ex hypothesi ergo est $s = t$ et inde $p = q = v$. et $y = \left(\frac{q}{r} =\right) \frac{v}{r}$. Superest ut r determinetur assumendo relationem aliquam inter bases x et z.

Veluti si fingas $ax = zz$ erit per Problema 1 $a = 2rz$. Quare substitue $\frac{a}{zz}$ pro r et fiet $y = \left(\frac{v}{r} =\right) \frac{2vz}{a}$. Est autem $v = \left(\sqrt{ax - xx} =\right) \frac{z}{a}\sqrt{aa - zz}$, adeoque $\frac{2zz}{aa}\sqrt{aa - zz} = y$, æquatio ad curvam cujus area æquatur areæ circuli{.}

Ad eundem modum si fingas $xx = z$, proveniet $2x = r$, et inde $y = \left(\frac{v}{r} =\right) \frac{v}{2x}$ et exterminato v et x fiet $y = \frac{\sqrt{az^{\frac{1}{2}} - z}}{2z^{\frac{1}{2}}}$.

<79>

Vel si fingas $cc = xz$, proveniet $0 = z + xr$; et inde $\frac{-vx}{z} = y = -\frac{c^3}{z^3}\sqrt{az - cc}$.

Atque ita si fingas $ax + \frac{s}{1} = z$, ope Problema 1 obtinebitur $a + p = r$ et inde $\frac{v}{a+p} = y = \frac{v}{a+v}$ quæ Curvam Mechanicam designat.

Exemplum 2. Detur iterum Circulus $ax - xx = vv$ et quærantur Curvæ quarum areae ad aream ejus habeant aliam quamlibet assumptam relationem. Veluti si assumes $cx + s = t$, et præterea fingas $ax = zz$, mediante Problema 1 elicies $c + p = q$ et $a = 2rz$. Quare est $y = \left(\frac{q}{r} =\right) \frac{2cz + 2pz}{a}$, et substituto $\sqrt{ax - xx}$ pro p, et $\frac{zz}{a}$ pro x fit $y = \frac{2cz}{a} + \frac{2zz}{aa}\sqrt{aa - zz}${.}

Quod si assumas $s - \frac{2v^3}{3a} = t$, et $x = z$, invenies ope Problema 1 $p - \frac{2lvv}{a} = q$ et $1 = r$. Adeoque $y = \left(\frac{q}{r} =\right) p - \frac{2lvv}{a}$ sive $= v - \frac{2lvv}{a}$. Jam vero pro exterminando l, æquatio $ax - xx = vv$ per Problema 1 dat $a - 2x = 2vl$ et proinde est $y = \frac{2vx}{a}$ ubi si supprimas v et x substituendo valores $\sqrt{ax - xx}$ et z, emerget $y = \frac{2z}{a}\sqrt{az - zz}$.

Sin assumas $ss = t$, et $x = tt$ emerget $2ps = q$, et $1 = 2rz$ atque adeò $y = \left(\frac{q}{r} =\right) 4psz$, et pro p et x substitutis $\sqrt{ax - xx}$ et zz fiet $y = 4szz\sqrt{a - zz}$ æquatio ad Curvam Mechanicam.

Exemplum 3. Ad eundem modum figuræ assumptam relationem ad aliam quamvis datam figuram habentes inveniuntur. Sic datâ Hyperbolâ $cc + xx = vv$, si assumas $s = t$ et $xx = cz$ elicies per Problema 1 $p = q$ et $2x = cr$ et inde $y \left(= \frac{q}{r}\right) = \frac{cp}{2x}$, et substitutis $\sqrt{cc + xx}$ pro p et $c^{\frac{1}{2}} z^{\frac{1}{2}}$ pro x, proveniet $y = \frac{c}{2z}\sqrt{cz + zz}$ {.}

Atque ita si assumas $xv - s = t$, et $xx = cz$, elicies $v + lx - p = q$, et $2x = cr$. Est autem $v = p$ et inde $lx = q$. Quare $y \left(= \frac{q}{r}\right) = \frac{cl}{2}$. Jam vero $cc + xx = vv$ ope Problema 1 dat $x = lv$. Adeóque est $y = \frac{cx}{2v}$ et substitutis $\sqrt{cc + xx}$ pro v et $c^{\frac{1}{2}} z^{\frac{1}{2}}$ pro x, fit $y = \frac{cz}{2\sqrt{cz + zz}}$.

Exemplum 4. Ad hæc si detur Cissoides $\frac{xx}{\sqrt{ax - xx}} = v$ ad quam relatæ aliæ figuræ sunt inveniendæ, et ea de causa assumatur $\frac{x}{3}\sqrt{ax - xx} + \frac{2}{3}s = t$, finge $\frac{x}{3}\sqrt{ax - xx} = h$ ejusque fluxionem k et erit $h + \frac{2}{3}s = t$ et inde per Problema 1 $k + \frac{2}{3}p = q$. Æquatio autem $\frac{ax^3 - x^4}{9} = hh$ per Problema 1 dat $\frac{3axx - 4x^3}{9} = 2kh$ ubi si extermines h fiet $k = \frac{3ax - 4xx}{6\sqrt{ax - xx}}$. Quare cùm præterea sit $\frac{2}{3}p = \left(\frac{2}{3}v =\right) \frac{4xx}{6\sqrt{ax - xx}}$ erit $\frac{ax}{2\sqrt{ax - xx}} = q$. Porro ad determinandum <80> z et r assumatur $\sqrt{aa - ax} = z$ et ope

Problema 1 emerget $-\mathrm{a} = 2\mathrm{rz}$ sive $\mathrm{r} = \frac{-\mathrm{a}}{2\mathrm{z}}$. Quare est $\mathrm{y} \left(= \frac{\mathrm{q}}{\mathrm{r}} = \frac{-\mathrm{zx}}{\sqrt{\mathrm{ax}-\mathrm{xx}}} = \sqrt{\frac{\mathrm{zzx}}{\mathrm{a}-\mathrm{x}}} = \sqrt{\mathrm{ax}}\right) = \sqrt{\mathrm{aa}-\mathrm{zz}}$. Quæ æquatio cùm sit ad circulum, habebitur relatio arearum circuli et Cissoidis.

Atque ita si assumpsisses $\frac{2\mathrm{x}}{3}\sqrt{\mathrm{ax}-\mathrm{xx}} + \frac{1}{3}\mathrm{s} = \mathrm{t}$ et $\mathrm{x} = \mathrm{z}$ prodijsset $\mathrm{y} = \sqrt{\mathrm{az}-\mathrm{zz}}$ æquatio denuò ad circulum.

Haud secus si detur curva aliqua Mechanica, possunt aliæ ad eam relatæ curvæ Mechanicæ inveniri, sed ad eliciendum Geometricas convenit ut e rectis ab invicem Geometricè dependentibus aliqua pro Basi adhibeatur, et ut area ad parallelogrammum complementalis quæratur supponendo fluxionem ejus valere Basin ductam in fluxionem ordinatim applicatæ.

Exemplum 5. [11]Sic Trochoide ADF propositâ, refero ad Basin AB et completo parallelogrammo ABDG quæro complementalem superficiem ADG concipiendo descriptam esse per motum rectæ GD, et proinde fluxionem ejus valere illam GB in celeritatem progrediendi ductam, hoc est $\mathrm{x} \times \mathrm{l}$. Jam cùm AL sit parallela tangenti DT, erit AB ad BL ut fluxio ejusdem AB ad fluxionem applicatæ BD hoc est ut 1 ad l. Quare est $\mathrm{l} = \frac{\mathrm{BL}}{\mathrm{AB}}$, adeóque $\mathrm{x} \times \mathrm{l} = \mathrm{BL}$, Et proinde area ADG describitur fluxione BL; Atque adeo cùm area circularis ALB eadem fluxione describátur æquales erunt.

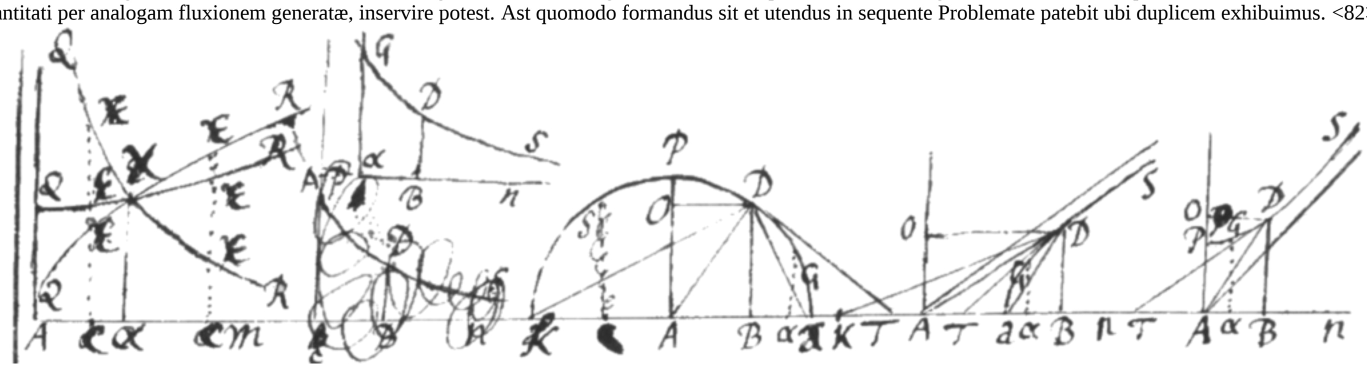
[12]Pari ratione si concipias ADF esse figuram arcuum sive sinuum versorum, hoc est cujus applicata BD æquatur arcui AL: cùm fluxio arcus AL sit ad fluxionem Basis AB ut PL ad BL, hoc est $\mathrm{l} \,.\, 1 :: \frac{1}{2}\mathrm{a} \,.\, \sqrt{\mathrm{ax}-\mathrm{xx}}$ erit $\mathrm{l} = \frac{\mathrm{a}}{2\sqrt{\mathrm{ax}-\mathrm{xx}}}$. Adeoque $\mathrm{l} \times \mathrm{x}$ fluxio areæ ADG erit $\frac{\mathrm{ax}}{2\sqrt{\mathrm{ax}-\mathrm{xx}}}$. Quare si ad ipsius AB punctum B recta æqualis $\frac{\mathrm{ax}}{2\sqrt{\mathrm{ax}-\mathrm{xx}}}$ in angulo recto applicari concipiatur, illa ad curvam quandam Geometricam terminabitur cujus area Basi AB adjacens æquatur areæ ADG.

Et sic alijs figuris per arcuum circuli, Hyperbolæ vel cujusvis Curvæ ad arcuum istorum sinus rectos vel versos aut alias quasvis geometricè determinabiles rectas lineas in datis angulis applicationem constitutis, æquales Geometric**{æ}** <81> figuræ inveniri possunt.

[13]Circa Spiralium areas levissimum est negotium. Utpote centro convolutionis A radio quovis AG descripto arcu DG occurrente AF in G et spirali in D; cùm arcus ille ad instar lineæ super Basi AG incedentis describat Spiralis Aream AHDG, ita ut ejus areæ fluxio sit ad fluxionem rectanguli $1 \times \mathrm{AG}$, ut arcus GD ad 1; si rectam GL arcui isti æqualem erigas illa similiter incedendo super eadem AG describet aream ALG æqualem areæ Spiralis AHDG; curvâ EIL existente Geometricâ. Et præterea si subtensa AL ducatur, erit triangulum $\mathrm{ALG} \left(= \frac{1}{2}\mathrm{AG} \times \mathrm{GL} = \frac{1}{2}\mathrm{AG} \times \mathrm{GD}\right) = \text{sectori AGD}$, adeoque complementalia segmenta ALI et ADH erunt etiam æqualia. Et hæc non tantum Spirali Archimedeæ (ubi AIL evadit Parabola Apolloniana) , sed et alijs quibuscunque conveniunt, adeo ut omnes eodem negotio in æquales Geometricas converti possint.

Possem plura hujus construendi Problematis specimina afferre, sed hæc sufficiant cùm sint adeò generalia ut quicquid hactenus circa curvarum areas inventum fuerit, vel ni fallor inveniri possit, aliquo saltem modo complectantur, et utplurimùm leviori curâ sine solitis ambagibus determinent.

Præcipuus autem hujus & præcedentis Problematis usus est, ut assumptis conicis sectionibus vel quibuslibet notæ magnitudinis curvis, aliæ curvæ quæ cum his conferri possunt, investigentur, et earum definientes æquationes in Catalogum ordinatim disponantur. Et constructo ejusmodi Catalogo, cum curvæ alicujus area quæritur, si æquatio ejus definiens vel immediatè in Catalogo reperiatur, vel in aliam quam Catalogus complectitur transformari potest, exinde cognosces aream ejus. Quinetiam Catalogus ille determinandis Curvarum longitudinibus, centris gravitatum, solidis per convolutionem generatis, solidorum superficiebus, et cuilibet fluenti quantitati per analogam fluxionem generatæ, inservire potest. Ast quomodo formandus sit et utendus in sequente Problemate patebit ubi duplicem exhibuimus. <82>



<83>

Problema 9.
Propositæ alicujus Curvæ aream determinare{.}

Problematis resolutio in eo fundatur ut quantitatum fluentium relatio ex relatione fluxionum (per Problema 2) eliciatur. Et imprimis si recta BD cujus motu quæsita area AFDB describitur, super basi AB positione datâ erectè incedat, concipe ut supra parallelogrammum ABEC a parte ejus BE unitatem æquante interea describi. Et posita BE fluxione parallelogrammi erit BD fluxio areæ quæsitæ.

Dic ergo $\mathrm{AB} = \mathrm{x}$, et erit etiam $\mathrm{ABEC}\,(= 1 \times \mathrm{x}) = \mathrm{x}$ et $\mathrm{BE} = \mathrm{m}$ dic insuper aream $\mathrm{AFDB} = \mathrm{z}$, et erit $\mathrm{BD} = \mathrm{r}$ ut et $= \frac{\mathrm{r}}{\mathrm{m}}$, eo quod sit $\mathrm{m} = 1$. Et proin per æquationem definientem BD simul definitur fluxionum ratio $\frac{\mathrm{r}}{\mathrm{m}}$, et exinde per Problema 2, Casum 1, elicietur relatio fluentium quantitatum x et z.

Exempla prima. Ubi BD sive r valet simplicem
aliquam quantitatem{.}

Detur $\frac{\mathrm{xx}}{\mathrm{a}} = \mathrm{r}$ vel $= \frac{\mathrm{r}}{\mathrm{m}}$ æquatio nempe ad Parabolam, et (per Problema 2) emerget $\frac{\mathrm{x}^3}{3\mathrm{a}} = \mathrm{z}$. Est ergo $\frac{\mathrm{x}^3}{3\mathrm{a}}$ sive $\frac{1}{3}\mathrm{AB} \times \mathrm{BD} =$ areæ Parabolicæ AFDB**{.}**

Detur $\frac{\mathrm{x}^3}{\mathrm{aa}} = \mathrm{r}$ æquatio ad Parabolam secundi generis et (per Problema 2) emerget $\frac{\mathrm{x}^4}{4\mathrm{aa}} = \mathrm{z}$, hoc est $\frac{1}{4}\mathrm{AB} \times \mathrm{BD} = \text{areæ AFDB}$.

Detur $\frac{\mathrm{a}^3}{\mathrm{xx}} = \mathrm{r}$ sive $\mathrm{a}^3\mathrm{x}^{-2} = \mathrm{r}$ æquatio ad Hyperbolam secundi generis, et emerget $-\mathrm{a}^3\mathrm{x}^{-1} = \mathrm{z}$ sive $-\frac{\mathrm{a}^3}{\mathrm{x}} = \mathrm{z}$: hoc est $\mathrm{AB} \times \mathrm{BD} =$ areæ infinite longæ HDBH ex altera parte applicatæ BD jacentis, ut innuit valor negativus.

Atque ita si detur $\frac{\mathrm{a}^4}{\mathrm{x}^3} = \mathrm{r}$, emerget $-\frac{\mathrm{a}^4}{2\mathrm{xx}} = \mathrm{z}$**{.}**

<84>

Præterea sit $\mathrm{ax} = \mathrm{rr}$. sive $\mathrm{a}^{\frac{1}{2}}\mathrm{x}^{\frac{1}{2}} = \mathrm{r}$, æquatio iterum ad Parabolam, et proveniet $\frac{2}{3}\mathrm{a}^{\frac{1}{2}}\mathrm{x}^{\frac{3}{2}} = \mathrm{z}$, hoc est $\frac{2}{3}\mathrm{AB} \times \mathrm{BD} = \text{areæ AFDB}$**{.}**

Sit $\frac{\mathrm{a}^3}{\mathrm{x}} = \mathrm{rr}$, et fiet $2\mathrm{a}^{\frac{3}{2}}\mathrm{x}^{\frac{1}{2}} = \mathrm{z}$, sive $2\mathrm{AB} \times \mathrm{BD} = \mathrm{AFDB}$.

Sit $\frac{a^5}{x^3} = rr$, et fiet $-\frac{2a^{\frac{5}{2}}}{x^{\frac{1}{2}}} = z$, sive $2AB \times BD = HDBH$.

Sit $axx = r^3$, et fiet $\frac{3}{5}a^{\frac{1}{3}}x^{\frac{5}{3}} = z$, sive $\frac{3}{5}AB \times BD = AFDB$.

Et sic in alijs.

Exempla secunda. Ubi r valet plures ejusmodi connexas quantitates.

Sit $x + \frac{xx}{a} = r$, et fiet $\frac{xx}{2} + \frac{x^3}{3a} = z$ **{.}**

Sit $a + \frac{a^3}{xx} = r$, et fiet $ax - \frac{a^3}{x} = z$.

Sit $3x^{\frac{1}{2}} - \frac{5}{xx} - \frac{2}{x^{\frac{1}{2}}} = r$ et fiet $2x^{\frac{3}{2}} + \frac{5}{x} - 4x^{\frac{1}{2}} = z$.

Exempla 3{.} ubi prævia reductio per divisionem requiritur.

Detur $\frac{aa}{b+x} = r$, æquatio ad Hyperbolam Apollonianam et factâ in infinitum divisione, evadet $r = \frac{aa}{b} - \frac{aax}{bb} + \frac{aaxx}{b^3} - \frac{aax^3}{b^4}$ &c . Et inde per Problema 2 (ut in secundis exemplis) obtinebitur $z = \frac{aax}{b} - \frac{aaxx}{2bb} + \frac{aax^3}{3b^3} - \frac{aax^4}{4b^4}$ &c **{.}**

Detur $\frac{1}{1+xx} = r$ et per divisionem elicietur $r = 1 - xx + x^4 - x^6$ &c vel etiam $r = \frac{1}{xx} - \frac{1}{x^4} + \frac{1}{x^6}$ &c **{.}** Indeque per problema 2, $z = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7$ &c $= AFDB$ vel $z = -\frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5}$ &c $= HDBH$.

Detur $\frac{2x^{\frac{1}{2}} - x^{\frac{3}{2}}}{1 + x^{\frac{1}{2}} - 3x} = r$, et per divisionem evadet $r = 2x^{\frac{1}{2}} - 2x + 7x^{\frac{3}{2}} - 13x^2 + 34x^{\frac{5}{2}}$ &c et inde per Problema 2, $z = \frac{4}{3}x^{\frac{3}{2}} - xx + \frac{14}{5}x^{\frac{5}{2}} - \frac{13}{4}x^3 + \frac{68}{7}x^{\frac{7}{2}}$ &c.

<85>

Exempla 4. Ubi prævia reductio per extractionem radicum requiritur.

Detur $r = \sqrt{aa + xx}$ æquatio nempe ad Hyperbolam et radice ad usque terminos infinitè multos extractâ, evadet $r = a + \frac{xx}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{112a^7}$ &c Atque inde ut in præcedentibus $z = ax + \frac{x^3}{6a} - \frac{x^5}{40a^3} + \frac{x^7}{112a^5} - \frac{5x^9}{1008a^7}$ &c &c**{.}**

Ad eundem modum si detur $r = \sqrt{aa - xx}$ æquatio scilicet ad circulum, obtinebitur $z = ax - \frac{x^3}{6a} - \frac{x^5}{40a^3} - \frac{x^7}{112a^5} - \frac{5x^9}{1008a^7}$ &c **{.}**

Atque ita si detur $r = \sqrt{x - xx}$ æquatio iterum ad circulum proveniet extrahendo radicem $r = x^{\frac{1}{2}} - \frac{1}{2}x^{\frac{3}{2}} - \frac{1}{8}x^{\frac{5}{2}} - \frac{1}{16}x^{\frac{7}{2}}$ &c adeoque est $z = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{5}x^{\frac{5}{2}} - \frac{1}{28}x^{\frac{7}{2}} - \frac{1}{72}x^{\frac{9}{2}}$ &c**{.}**

Sic $r = \sqrt{aa + bx - xx}$ æquatio denuò ad circulum per extractionem radicis dat $r = a + \frac{bx}{2a} - \frac{xx}{2a} - \frac{bbxx}{8a^3}$ &c unde per Problema 2 elicitur $z = ax + \frac{bxx}{4a} - \frac{x^3}{6a} - \frac{bbx^3}{24a^3}$ &c .

Et sic $\sqrt{\frac{1+axx}{1-bxx}} = r$, per debitam reductionem dat $\begin{array}{lllll} r = 1 & +\frac{1}{2}b & x^2 & +\frac{3}{8}bb & x^4 \quad \&c. \\ & +\frac{1}{2}a & & +\frac{1}{4}ab & \\ & & & -\frac{1}{8}aa & \end{array}$. Unde per Problema 2 fit $\begin{array}{lllll} z = x & +\frac{1}{6}b & x^3 & +\frac{3}{40}bb & x^5 \quad \&c. \\ & +\frac{1}{6}a & & +\frac{1}{20}ab & \\ & & & -\frac{1}{40}aa & \end{array}$ **{.}**

Sic denique $r = \sqrt{3: \overline{a^3 + x^3}}$ per extractionem radicis cubicæ dat $r = a + \frac{x^3}{3aa} - \frac{x^6}{9a^5} + \frac{5x^9}{81a^8}$ &c . Indeque $z = ax + \frac{x^4}{12aa} - \frac{x^7}{63a^5} + \frac{x^{10}}{162a^8}$ &c $= AFDB$. vel etiam $r = x + \frac{a^3}{3xx} - \frac{a^6}{9x^5} + \frac{5a^9}{81x^8}$ &c . Indeque $z = \frac{xx}{2} - \frac{a^3}{3x} + \frac{a^6}{36x^4} - \frac{5a^9}{567x^7}$ &c $= HDBH$.

Exempla 5. Ubi prævia reductio per æquationis affectæ resolutionem requiritur.

Si curva per æquationem $r^3 + aar + axr - 2a^3 - x^3 = 0$ definiatur, extrahe radicem et proveniet $r = a - \frac{x}{4} + \frac{xx}{64a} + \frac{131x^3}{512aa}$ &c. Unde ut in prioribus obtinebitur $z = ax - \frac{xx}{8} + \frac{x^3}{192a} + \frac{131x^4}{2048aa}$ &c**{.}**

Sin $r^3 - crr - 2xxr - ccr + 2x^3 + c^3 = 0$ sit æquatio ad curvam resolutio dabit triplicem radicem nempe $r = c + x - \frac{xx}{4c} + \frac{x^3}{32cc}$ &c et $r = c - x + \frac{3xx}{4c} - \frac{15x^3}{32cc}$ &c , et $r = -c - \frac{xx}{2c} - \frac{x^3}{2cc} + \frac{x^5}{4c^4}$ &c et inde trium correspondentium arearum valores $z = cx + \frac{1}{2}xx - \frac{x^3}{12c} + \frac{x^4}{128cc}$ &c . $z = cx - \frac{1}{2}x + \frac{x^3}{4c} - \frac{15x^4}{128cc}$ &c , ac $z = -cx - \frac{x^3}{6c} - \frac{x^4}{6cc} + \frac{x^6}{24c^4}$ &c .

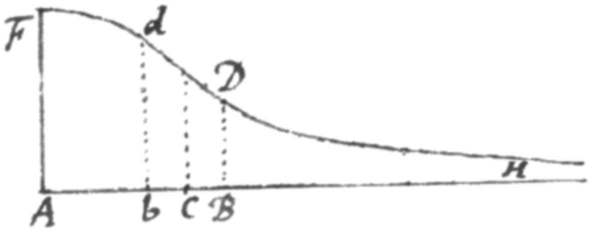
<86>

De Curvis Mechanicis hic nihil adjicio, siquidem reductio ad formam Geometricarum post ostenditur.

Cæterum cum sic inventi valores z areis quandoque ad Basis finitam partem AB, quandoque ad partem BH infinitè versus H productam, et quandoque ad utramque partem sitis secundum diversos eorum terminos competant: quò debitus areæ ad quamlibet Basis portionem sitæ valor assignetur, Area illa semper ponenda est æqualis differentiæ valorum z partibus Basis ad initium et finem istius areæ terminatis competentium.

Exempli Gratia. Ad curvam quam æquatio $\frac{1}{1+xx} = r$ definit inventum est $z = x - \frac{1}{3}x^3 + \frac{1}{5}x^5$ &c . Jam ut quantitatem areæ bdDB adjacentis parti Basis bB determinem, a valore z qui fit ponendo $AB = x$ subduco valorem z qui fit ponendo $Ab = x$, et (distinctionis gratia scriptâ X majuscula pro AB et x minusculâ pro Ab) restat $X - \frac{1}{3}X^3 + \frac{1}{5}X^5$ &c $- x - \frac{1}{3}x^3 + \frac{1}{5}x^5$ &c valor areæ illius bdDB. Unde si Ab seu x ponatur nullum habebitur tota area $AFDB = X - \frac{1}{3}X^3 + \frac{1}{5}X^5$ &c .

Ad eandem Curvam inventum est etiam $z=-\frac{1}{x}+\frac{1}{3x^3}-\frac{1}{5x^5}$ &c unde rursus juxta præcedentia erit area illa bdDB$=\frac{1}{x}-\frac{1}{3x^3}+\frac{1}{5x^5}$ &c $-\frac{1}{X}+\frac{1}{3X^3}-\frac{1}{5X^5}$ &c . Adeoque si AB seu x statuatur infinitum, area adjacens bdH a parte H similiter infinite longa valebit $\frac{1}{x}-\frac{1}{3x^3}+\frac{1}{5x^5}$ &c. Siquidem posterior series $-\frac{1}{X}+\frac{1}{3X^3}-\frac{1}{5X^5}$ &c propter infinitatem denominatorum evanescat.



Ad Curvam æquatione $a+\frac{a^3}{xx}=r$ designatam, inventum est $ax-\frac{a^3}{x}=z$. Unde fit $aX-\frac{a^3}{X}-ax+\frac{a^3}{x}=$ areæ bdDB. Haec autem evadit infinita sive x fingatur nulla sive X infinita et proinde utraque area AFDB et bdH infinitè magna est, ac solæ partes intermediæ (qualis bdDB) exhiberi possunt. Id quod semper evenit ubi basis x cum in numeratoribus aliquorum tum in denominatoribus aliorum terminorum valoris z reperitur. Ubi vero x < insertion from p 183 > in numeratoribus solummodo, ut in primo exemplo, reperitur; valor z competit areæ sitæ ad AB cis parallelè incedentem. Et ubi in denominatoribus tantùm, ut in secundo exemplo; valor ille mutatis omnium terminorum signis, competit areæ omni ultra parallelè incedentem infinitè productæ. < text from p 86 resumes >

Siquando Curva linea secat Basin inter puncta b et B puta in E, vice areæ habebitur arearum ad diversas Basis partes differentia $bdE-BDE$, cui si addatur rectangulum BDGb obtinebitur area dEDG

<87>

Præcipuè autem notandum est quod ubi in valore r terminus aliquis per x unius tantùm dimensionis dividitur, area illi termino correspondens pertinet ad Hyperbolam conicam et proinde per infinitam seriem seorsim exhibenda est; quemadmodum in sequentibus factum.

Sit $\frac{a^3-aax}{ax+xx}=r$ æquatio ad Curvam et per divisionem fiet $r=\frac{aa}{x}-2a+2x-\frac{2xx}{a}+\frac{2x^3}{aa}$ &c . Indeque $z=\boxed{\frac{aa}{x}}-2ax+xx-\frac{2x^3}{3a}+\frac{x^4}{2aa}$ &c**{.}** Et area bdDB $=\boxed{\frac{aa}{X}}-2aX+X^2-\frac{2X^3}{3a}$ &c $-\boxed{\frac{aa}{x}}+2ax-xx+\frac{2x^3}{3a}$ &c . Ubi per notas $\boxed{\frac{aa}{X}}$ et $\boxed{\frac{aa}{x}}$ designo areolas terminis $\frac{aa}{X}$ et $\frac{aa}{x}$. Jam ut $\boxed{\frac{aa}{X}}-\boxed{\frac{aa}{x}}$ investigetur, fingo Ab seu x definitam esse et bB indefinitam seu fluentem lineam, quam itaque si dicam y, erit $\boxed{\frac{aa}{x+y}}=$ areæ isti Hyperbolicæ adjacenti bB, nempe $\boxed{\frac{aa}{X}}-\boxed{\frac{aa}{x}}$. Est autem, factâ divisione, $\frac{aa}{x+y}=\frac{aa}{x}-\frac{aay}{xx}+\frac{aayy}{x^3}-\frac{aay^3}{x^4}$ &c**{.}** Adeoque $\boxed{\frac{aa}{x+y}}$ seu $\boxed{\frac{aa}{X}}-\boxed{\frac{aa}{x}}=\frac{aay}{x}-\frac{aayy}{2xx}+\frac{aay^3}{3x^3}-\frac{aay^4}{4x^4}$ &c . Et proinde tota area quæsita bdDB $=\frac{aay}{x}-\frac{aayy}{2xx}+\frac{aay^3}{3x^3}$ &c $-2aX+XX-\frac{2X^3}{3a}$ &c $+2ax-xx+\frac{2x^3}{3a}$ &c .

Ad eundem modum AB seu X pro definita linea adhiberi potuit et sic prodijsset $\boxed{\frac{aa}{X}}-\boxed{\frac{aa}{x}}=\frac{aay}{X}-\frac{aayy}{2XX}+\frac{aay^3}{3X^3}-\frac{aay^4}{4X^4}$ &c **{.}**

Quinetiam si bisecetur bB in C et assumatur AC esse definitæ longitudinis et Cb ac CB indefinitæ. Tum dicto $AC=e$ et Cb vel $CB=y$, erit $bd=\frac{aa}{e-y}=\frac{aa}{e}+\frac{aay}{ee}+\frac{aayy}{e^3}+\frac{aay^3}{e^4}+\frac{aay^4}{e^5}$ &c, indeque area Hyperbolica parti Basis bC adjacens $\frac{aay}{e}+\frac{aayy}{2ee}+\frac{aay^3}{3e^3}+\frac{aay^4}{4e^4}+\frac{aay^5}{4e^5}$ &c. Erit etiam $CB=\frac{aa}{e+y}=\frac{aa}{e}-\frac{aay}{ee}+\frac{aayy}{e^3}-\frac{aay^3}{e^4}+\frac{aay^4}{e^5}$ &c et inde area alteri basis parti CB adjacens $\frac{aay}{e}-\frac{aayy}{2ee}+\frac{aay^3}{3e^3}-\frac{aay^4}{4e^4}+\frac{aay^5}{4e^5}$ &c **{.}**

Et harum arearum summa $\frac{2aay}{e}+\frac{2aay^3}{3e^3}+\frac{2aay^5}{3e^5}$ &c valebit $\boxed{\frac{aa}{X}}-\boxed{\frac{aa}{x}}$.

Sic æquatione $r^3+rr+r-x^3=0$ ad Curvam existente, ejus radix erit $r=x-\frac{1}{3}-\frac{2}{9x}+\frac{7}{81xx}+\frac{5}{81x^3}$ &c . Unde fit <88> $z=\frac{1}{2}xx-\frac{1}{3}x-\boxed{\frac{2}{9x}}-\frac{7}{81x}-\frac{5}{162xx}$ &c , et area bdDB $=\frac{1}{2}XX-\frac{1}{3}X-\boxed{\frac{2}{9X}}-\frac{7}{81X}$ &c $-\frac{1}{2}xx+\frac{1}{3}x+\boxed{\frac{2}{9x}}+\frac{7}{81x}$ &c , hoc est $=\frac{1}{2}XX-\frac{1}{3}X-\frac{7}{81X}$ &c $-\frac{1}{2}xx+\frac{1}{3}x+\frac{7}{81x}$ &c $-\frac{4y}{9e}-\frac{4y^3}{27e^3}-\frac{4y^5}{45e^5}$ &c.

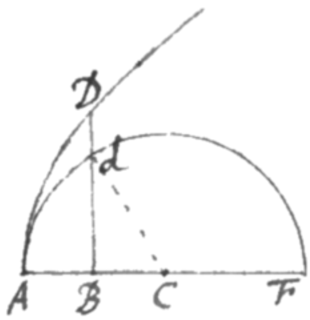
Potest autem terminus iste Hyperbolicus utplurimùm com
modè devitari mutando initium Basis, id est, augendo vel minuendo eam per datam aliquam quantitatem. Quemadmodum in exemplo priori ubi $\frac{a^3-aax}{ax+xx}=r$ erat æquatio ad Curvam, si faciam b esse initium Basis, et fingens Ab cujuslibet esse determinatæ longitudinis puta $\frac{1}{2}a$, pro Basis residuo bB jam scribam x: Hoc est si diminuam Basem per $\frac{1}{2}a$ scribendo $x+\frac{1}{2}a$ pro x: evadet $\frac{\frac{1}{2}a^3-aax}{\frac{3}{4}aa+2ax+xx}=r$, et per divisionem $r=\frac{2}{3}a-\frac{28}{9}x+\frac{200xx}{27a}$ &c. Unde fit $z=\frac{3}{2}ax-\frac{14}{9}xx+\frac{200x^3}{81a}$ &c $=$ areæ bdDB.

Et sic pro initio Basis adhibendo aliud atque aliud ejus punctum, potest area cujusvis curvæ modis infinitis exprimi.

Potuit etiam æquatio $\frac{a^3-aax}{ax+xx}=r$ in duas series infinitas resolvi prodeunte $r=\frac{a^3}{xx}-\frac{a^4}{x^3}+\frac{a^5}{x^4}$ &c $-a+x-\frac{xx}{a}+\frac{x^3}{aa}$ &c ubi terminus per x unius tantùm dimensionis divisus non reperitur. Sed hujusmodi series, ubi dimensiones x in unius numeratoribus et alterius denominatoribus infinitè ascendunt, minùs aptæ sunt ex quibus z per computum Arithmeticum obtineri possit, cùm in ejus valore numeri pro speciebus substituuntur.

Instituenti computum hujusmodi numerosum, postquam valor areæ in speciebus habetur, haud aliquid difficile occurret. Tamen in præcedentem doctrinam penitiùs illustrandam exemplum unum et alterum subjungere placuit**{.}**

Proponatur Hyperbola AD quam æquatio $\sqrt{x+xx}=r$ designat, utpote cujus vertex est ad A, et uterque Axis æquatur unitati. Et e præcedentibus Area ejus ADB erit $\frac{2}{3}x^{\frac{3}{2}}+\frac{1}{5}x^{\frac{5}{2}}-\frac{1}{28}x^{\frac{7}{2}}+\frac{1}{72}x^{\frac{9}{2}}-\frac{5}{704}x^{\frac{11}{2}}$ &c hoc est $x^{\frac{1}{2}}$ in $\frac{2}{3}x+\frac{1}{5}xx-\frac{1}{28}x^3+\frac{1}{72}x^4$ <89> $-\frac{5}{704}x^5$ &c. Quæ series infinitè producitur multiplicando ultimum terminum continuò per succedaneos terminos hujus progressionis $\frac{1,3}{2,5}x.\ \frac{-1,5}{4,7}x.\ \frac{-3,7}{6,9}x.\ \frac{-5,9}{8,11}x.\ \frac{-7,11}{10,13}x$ &c. Nempe primus terminus $\frac{2}{3}x^{\frac{3}{2}}$ in $\frac{1,3}{2,5}x$ facit $\frac{1}{5}x^{\frac{5}{2}}$ secundum terminum. Hic in $\frac{-1,5}{4,7}x$ facit $-\frac{1}{28}x^{\frac{7}{2}}$ tertium terminum. Hic in $\frac{-3,7}{6,9}x$ facit $\frac{1}{72}x^{\frac{9}{2}}$ quartum terminum. Et sic in infinitum. Sumatur jam AB cujuslibet longitudinis puta $\frac{1}{4}$, et hunc numerum scribe pro x ejusque radicem $\frac{1}{2}$ pro $x^{\frac{1}{2}}$, et primus terminus $\frac{2}{3}x^{\frac{3}{2}}$ sive $\frac{2}{3}\times\frac{1}{8}$ in decimalem fractionem reductus evadit 0,08333333 &c. Hic in $\frac{1,3}{2,5,4}$ facit 0,00625 secundum terminum. Hic in $\frac{-1,5}{4,7,4}$ facit $-0,0002790178$ &c tertium terminum. Et sic in infinitum. Terminos autem quos sic gradatim elicio dispono in duas Tabulas affirmativos nempe in unam et negativos in aliam, et addo, ut hic vides.



+0,08333,33333,333333	−0,00027,90178,571429
625,00000,000000	34679,066051
2,71267,361111	834,465027
5135,169396	26,285354
144,628917	961296
4,954581	38676
190948	1663
7963	75
352	4
16	−0,00028,25719,389575
1	
+0,08961,09885,646618	

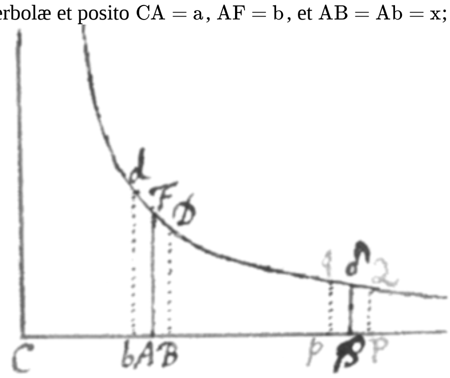
.
Dein a summa affirmativorum aufero summam negativorum et restat 0,0893284166257043 quantitas areæ Hyperbolicæ ADB quam quærere oportuit.

Proponatur jam circulus AdF quem æquatio $\sqrt{\mathrm{x}-\mathrm{xx}}=\mathrm{r}$ designat, hoc est cujus diameter AF sit unitas, et e præcedentibus area ejus AdB erit $\frac{2}{3}\mathrm{x}^{\frac{3}{2}}-\frac{1}{5}\mathrm{x}^{\frac{5}{2}}-\frac{1}{28}\mathrm{x}^{\frac{7}{2}}-\frac{1}{72}\mathrm{x}^{\frac{9}{2}}$ &c **{.}** In qua serie cùm termini non differant a terminis seriei supra exprimentis aream Hyperbolicam nisi in signis + et −, nihil aliud agendum restat quam ut eosdem numerales terminos cum alijs signis nectamus, subducendo nempe connexas ambarum præfatarum Tabularum summas 0,0898935605036193 <90> a primo termino duplicato 0,1666666666666666 et residuum 0,0767731061630473 erit areæ circularis portio AdB, posito scilicet AB quadrante diametri. Atque ita videre est quod etsi areæ circuli et Hyperbolæ non conferantur ratione geometrica, tamen utraque eodem computo arithmetico prodit.

Inventa circuli portione AdB, exinde tota area facilè eruitur. Nempe radio dC acto, duc Bd seu $\frac{1}{4}\sqrt{3}$ in BC seu $\frac{1}{4}$ et facti dimidium $\frac{1}{32}\sqrt{3}$ seu 0,0541265877365274 valebit triangulum CdB, quod adde areæ AdB et habebitur Sector ACd= 0,1308996938995747 cujus sextuplum 0,7853981633974482 est area tota.

Et hinc obiter exit peripheriæ longitudo 3,1415926535897928, dividendo nempe aream per quadrantem diametri.

Hisce calculum areæ inter Hyperbolam dFD et ejus Asymptoton CA interjectæ subnectimus. Sit C centrum Hyperbolæ et posito $\mathrm{CA}=\mathrm{a}$, $\mathrm{AF}=\mathrm{b}$, et $\mathrm{AB}=\mathrm{Ab}=\mathrm{x}$; erit $\frac{\mathrm{ab}}{\mathrm{a}+\mathrm{x}}=\mathrm{BD}$, et $\frac{\mathrm{ab}}{\mathrm{a}-\mathrm{x}}=\mathrm{bd}$ et inde area $\mathrm{AFDB}=\mathrm{bx}-\frac{\mathrm{bxx}}{2\mathrm{a}}+\frac{\mathrm{bx}^3}{3\mathrm{aa}}-\frac{\mathrm{bx}^4}{4\mathrm{a}^3}$ &c, et area $\mathrm{AFdb}=\mathrm{bx}+\frac{\mathrm{bxx}}{2\mathrm{a}}+\frac{\mathrm{bx}^3}{3\mathrm{aa}}+\frac{\mathrm{bx}^4}{4\mathrm{a}^3}$ &c ac earum summa $\mathrm{bdDB}=2\mathrm{bx}+\frac{2\mathrm{bx}^3}{3\mathrm{aa}}+\frac{2\mathrm{bx}^5}{5\mathrm{aa}}+\frac{2\mathrm{bx}^7}{7\mathrm{aa}}$ &c. Ponamus jam $\mathrm{CA}=\mathrm{AF}=1$, et Ab vel $\mathrm{AB}=\frac{1}{10}$, existente Cb = 0,9 et CB = 1,1: et substituendo hos numeros pro a b et x, primus seriei terminus evadet 0,2, secundus 0,00066666 &c, tertius 0,000004, et sic deinceps ut vides in hac Tabula



	0,20000,00000,000000	
	66,66666,666666	
	40000,000000	
	285,714286	
	2,222222	
	18182	
	154	
	1	
Summa	0,20067,06954,621511	= areæ bdDB

.

Quod si areæ hujus partes Ad et AD seorsim desiderentur subduc minorem AD e majori Ad et restabit $\frac{\mathrm{bxx}}{\mathrm{a}}+\frac{\mathrm{bx}^4}{2\mathrm{a}^3}+\frac{\mathrm{bx}^6}{3\mathrm{a}^5}+\frac{\mathrm{bx}^8}{4\mathrm{a}^7}$ &c. <91> Ubi si 1 scribatur pro a et b, ac $\frac{1}{10}$ pro x, termini in decimales redacti conficient sequentem Tabulam

	0,01000,00000,000000	
	5,00000,000000	
	3333,333333	
	25,000000	
	200000	
	1667	
	1	
Summa	0,01005,03358,535014	= Ad − AD

.

Jam si hæc arearum differentia addatur et auferatur summæ earum priùs inventæ, aggregati dimidium 0,10536,05156,578263 erit major area Ad, et residui dimidium 0,09531,01798,043248 minor AD.

Per easdem Tabulas obtinentur etiam areæ illæ AD et Ad ubi AB et Ab ponuntur $\frac{1}{100}$ sive CB = 1,01 & Cb = 0,99 si modo numeri in depressiora loca debitè transferantur ut hic videre est

	0,02000,00000,000000	
	6666,666667	
	400000	
	28	
Sum	0,02000,06667,066695	= bD.

	0,00010,00000,000000	
	50,000000	
	3333	
Sum	0,00010,00050,003333	= Ad − AD.

$\frac{1}{2}$ Aggreg $0{,}01005{,}03358{,}535014=\mathrm{Ad}$. $\frac{1}{2}$ Resid $0{,}00995{,}03308{,}531681=\mathrm{AD}$.

Et sic positis AB & Ab $\frac{1}{1000}$ seu CB = 1,001 et Cb = 0,999, obtinebitur Ad = 0,00100,05003,335835 et AD = 0,00099,95003,330835.

Ad eundem modum si stantibus CA et $\mathrm{AF} = 1$, ponantur AB et $\mathrm{Ab} = 0{,}2$ vel $= 0{,}02$ vel $= 0{,}002$ elicientur areæ illæ,

$$\begin{array}{rll} & \mathrm{Ad} = 0{,}22314{,}35513{,}142097\,, & \text{et}\quad \mathrm{AD} = 0{,}18232{,}15567{,}939546 \\ \text{vel} & \mathrm{Ad} = 0{,}02020{,}27073{,}175194\,, & \text{et}\quad \mathrm{AD} = 0{,}01980{,}26272{,}961797 \\ \text{vel} & \mathrm{Ad} = 0{,}002002 & \text{et}\quad \mathrm{AD} = 0{,}001 \end{array}$$

Ex inventis hisce areis jam facile est alias per solam additionem et subductionem derivare. Utpote cum sit $\frac{1{,}2}{0{,}8}$ in $\frac{1{,}2}{0{,}9} = 2$, arearum pertinentium ad rationes $\frac{1{,}2}{0{,}8}$ & $\frac{1{,}2}{0{,}9}$ (hoc est, insistentium partibus Basis $1{,}2 - 0{,}8$ et $1{,}2 - 0{,}9$) summa 0,6931471805599453 erit area AFδβ, existente $\mathrm{C}\beta = 2$, ut notum est. Dein cum sit $\frac{1{,}2}{0{,}8}$ in $2 = 3$, arearum pertinentium ad $\frac{1{,}2}{0{,}8}$ et 2 summa 1,0986122886681097 erit area AFδβ, existente $\mathrm{C}\beta = 3$. Pariter cùm sit $\frac{2 \text{ in } 2}{0{,}8} = 5$, et 2 in $5 = 10$, per debitam arearum additionem obtinebitur $1{,}6093379124341004 = \mathrm{AF}\delta\beta$, existente <92> $\mathrm{C}\beta = 5$, et $2{,}3025850929940457 = \mathrm{AF}\delta\beta$ existente $\mathrm{C}\beta = 10$. Atque ita cùm sit 10 in $10 = 100$, et 10 in $100 = 1000$, et $\sqrt{5 \text{ in } 10 \text{ in } 0{,}98} = 7$, et 10 in $1{,}1 = 11$, et $\frac{1000 \text{ in } 1{,}001}{7 \text{ in } 11} = 13$ $\frac{100 \text{ in } 1{,}02}{2 \text{ in } 3} = 17$, et $\frac{1000 \text{ in } 0{,}999}{3 \text{ in } 3 \text{ in } 3} = 37$, et 100 in $1{,}01 = 101$, et $\frac{1000 \text{ in } 1{,}002}{2 \text{ in } 3} = 167$, et $\frac{1000 \text{ in } 0{,}998}{2} = 499$. patet aream AFδβ per arearum supra inventarum compositionem inveniri posse, existente $\mathrm{C}\beta = 100$; 1000; 7; aut alio quolibet e recensitis numeris, et stante $\mathrm{CA} = \mathrm{AF} = 1$.Id quod significare volui ut Methodus construendo Logarithmorum Canoni aptissima pateret quæ areas Hyperbolicas (ex quibus Logarithmi facilè deducuntur) tot numeris primis correspondentes, quasi per binas tantum haud molestas operationes determinat. Cæterùm cùm Canon iste ex hoc fonte præ cæteris feliciter depromi videatur, quid si Constructionem ejus coronidis loco perstringam.

Imprimis itaque assumpto 0 pro Logarithmo numeri 1, et 1 pro Logarithmo numeri 10 ut solet, investigandi sunt Logarithmi primorum numerorum 2, 3, 5, 7, 11, 13, 17, 37, dividendo inventas areas Hyperbolicas per 2,3025850929940457 aream nempe correspondentem numero 10, vel quod eodem recidit, multiplicando per ejus reciprocum 0,4342944819032518. Sic enim e.g. Si 0,69314718&c area correspondens numero 2 multiplicetur per 0,43429**{&c}** facit 0,3010299956639812 Logarithmum numeri 2.

Deinde Logarithmi numerorum omnium in Canone qui ex horum multiplicatione fiunt indagandi sunt per additionem eorum Logarithmorum, ut solet, et loca vacua postmodum interpolanda ope hujus Theorematis. Sit n numerus Logarithmo donandus, x differentia inter illum et proximos numeros hinc inde æqualiter distantes quorum logarithmi habentur ac d semissis differentiæ logarithmorum, et quæsitus Logarithmus numeri n obtinebitur addendo $\mathrm{d} + \frac{\mathrm{dx}}{2\mathrm{n}} + \frac{\mathrm{dx}^3}{12\mathrm{n}^3}$ logarithmo minoris numeri. Nam si numeri exponantur per Cp, Cβ et CP. Et existente rectangulo CBD vel $\mathrm{C}\beta\delta = 1$ ut supra, ac erectis parallelè incedentibus pq et PQ, si n scribatur pro Cβ et x pro βp vel BP erit area pqQP sive $\frac{2\mathrm{x}}{\mathrm{n}} + \frac{2\mathrm{x}^3}{3\mathrm{n}^3} + \frac{2\mathrm{x}^5}{5\mathrm{n}^5}$ &c ad aream pqδβ <93> sive $\frac{\mathrm{x}}{\mathrm{n}} + \frac{\mathrm{xx}}{2\mathrm{nn}} + \frac{\mathrm{x}^3}{3\mathrm{n}^3}$ &c, ut differentia inter logarithmos extremorum numerorum sive $2\mathrm{d}$, ad differentiam inter logarithmos minoris et medij, quæ proinde erit $\frac{\frac{\mathrm{dx}}{\mathrm{n}} + \frac{\mathrm{dxx}}{2\mathrm{nn}} + \frac{\mathrm{dx}^3}{3\mathrm{n}^3} \ \&\mathrm{c}}{\frac{\mathrm{x}}{\mathrm{n}} + \frac{\mathrm{x}^3}{3\mathrm{n}} + \frac{\mathrm{x}^5}{5\mathrm{n}} \ \&\mathrm{c}}$, hoc est facta divisione $\mathrm{d} + \frac{\mathrm{dx}}{2\mathrm{n}} + \frac{\mathrm{dx}^3}{12\mathrm{n}^3}$ &c.

Hujus autem seriei duos primos terminos $\mathrm{d} + \frac{\mathrm{dx}}{2\mathrm{n}}$ pro Canone construendo sat accuratos existimo etiamsi ad usque quatuordecim vel forte quindecim figurarum loca logarithmi producerentur, si modò numerus logarithmo donandus non sit minor quam 1000. Quod sane calculum haud difficilem præbere potest siquidem x utplurimùm erit unitas vel numerus binarius. Non opus est tamen omnia loca beneficio hujus regular interpolare. Nam logarithmi numerorum qui prodeunt e multiplicatione vel divisione numeri novissimè transacti per numeros quorum logarithmi prius habebantur obtineri possunt per additionem vel subductionem eorum logarithmorum. Quinetiam per differentias logarithmorum et illarum differentiarum secundas differentias tertiasque si opus est, loca vacua expeditiùs impleri possunt, adhibitâ tantùm prædictâ regulâ ubi ad obtinendum illas differentias continuatio aliquot locorum plenorum desideratur.

Eadem methodo Regulæ pro intercalatione Logarithmorum inveniri possunt ubi e tribus numeris dantur logarithmi minoris et medij, vel medij et majoris, idque licet numeri non sint in Arithmetica progressione.

Imò et hujus methodi vestigijs insistendo Regulæ pro construendis artificialium sinuum et Tangentium Tabulis sine adminiculo naturalium haud difficulter depromi possunt. Sed hæc in transitu.

<94>

Hactenus Curvarum quæ per æquationes minùs simplices definiuntur Quadraturam mediante reductione in æquationes ex infinite multis terminis simplicibus constantes ostendimus. Cum verò ejusmodi curvæ per finitas etiam æquationes nonnunquam quadrari possint vel saltem comparari cum alijs curvis quarum areæ quodammodo pro cognitis habeantur, quales sunt sectiones conicæ: eapropter sequentes duos Theorematum catalogos in illum usum ope Problematis septimi & octavi ut promisimus constructos, jam visum est adjungere. Horum prior exhibet areas curvarum quæ quadrari possunt, et posterior complectitur curvas quarum areas cum areis conicarum sectionum conferre liceat. In utrisque literæ latinæ <95> d, e, f, g, et h datas quasvis quantitates, x et z bases curvarum, v et y parallelè incedentes , et s ac t areas ut supra denotant. Graecæ autem η et θ quantitati z suffixæ denotant ejusdem z dimensionum numerum sive sit integer vel fractus, sive affirmativus aut negativus. Veluti si sit $\eta = 3$ erit $\mathrm{z}^\eta = \mathrm{z}^3$, $\mathrm{z}^{2\eta} = \mathrm{z}^6$, $\mathrm{z}^{-\eta} = \mathrm{z}^{-3}$ sive $\frac{1}{\mathrm{z}^3}$, $\mathrm{z}^{\eta+1}$ vel $\mathrm{z}^{\overset{\eta}{+1}} = \mathrm{z}^4$, & $\mathrm{z}^{\eta-1}$ vel $\mathrm{z}^{\overset{\eta}{-1}} = \mathrm{zz}$. Insuper in valoribus arearum abbreviandi causâ scribitur R vice radicalis illius $\sqrt{\mathrm{e} + \mathrm{fz}^\eta}$ vel $\sqrt{\mathrm{e} + \mathrm{fz}^\eta + \mathrm{gz}^{2\eta}}$ quâ valor incedentis y afficitur.

Catalogus Curvarum aliquot ad rectilineas figuras relatarum, ope Problematis 7 constructus.

Curvarum	
Ordo primus.	Arearum valores.
$dz^{\eta-1} = y.$	$\frac{d}{\eta} z^{\eta} = t.$
Ordo secundus.	
$\frac{dz^{\eta-1}}{ee+2efz^{\eta}+ffz^{2\eta}} = y.$	$\frac{dz^{\eta}}{\eta ee+\eta efz^{\eta}} = t,$ vel $\frac{-d}{\eta ef+\eta ffz^{\eta}} = t.$
Ordo tertius.	
1. $dz^{\eta-1}\sqrt{e+fz^{\eta}} = y.$	$\frac{2d}{3\eta f} R^3 = t.$
2. $dz^{2\eta-1}\sqrt{e+fz^{\eta}} = y.$	$\frac{-4e+6fz^{\eta}}{15\eta ff} dR^3 = t.$
3. $dz^{3\eta-1}\sqrt{e+fz^{\eta}} = y.$	$\frac{16ee-24efz^{\eta}+30ffz^{2\eta}}{105\eta f^3} dR^3 = t.$
4. $dz^{4\eta-1}\sqrt{e+fz^{\eta}} = y.$	$\frac{-96e^3-144eefz^{\eta}-180effz^{2\eta}+210f^3z^{3\eta}}{945\eta f^4} dR^3 = t.$
Ordo quartus.	
1. $\frac{dz^{\eta-1}}{\sqrt{e+fz^{\eta}}} = y.$	$\frac{2d}{\eta f} R = t.$
2. $\frac{dz^{2\eta-1}}{\sqrt{e+fz^{\eta}}} = y.$	$\frac{-4e+2fz^{\eta}}{3\eta ff} dR = t.$

<96>

3. $\frac{dz^{3\eta-1}}{\sqrt{e+fz^{\eta}}} = y.$	$\frac{16ee+8efz^{\eta}+6ffz^{2\eta}}{15\eta f^3} dR = t.$
4. $\frac{dz^{4\eta-1}}{\sqrt{e+fz^{\eta}}} = y.$	$\frac{-96e^3+48eefz^{\eta}-36effz^{2\eta}+30f^3z^{3\eta}}{105\eta f^4} dR = t.$

His adjiciantur sequentia magis generalia Theoremata quibus via ad altiora sternitur.

Ordo quintus. Arearum valores.

1. $2\theta e z^{\theta-1} \begin{matrix}+2\theta\\+3\eta\end{matrix} f z^{\theta+\eta-1}$ in $\frac{1}{2}\sqrt{e+fz^{\eta}}=y.$ $\qquad z^{\theta}R^{3}=t.$

2. $2\theta e z^{\theta-1} \begin{matrix}+2\theta\\+3\eta\end{matrix} f z^{\theta+\eta-1} \begin{matrix}+2\theta\\+6\eta\end{matrix} g z^{\theta+2\eta-1}$ in $\frac{1}{2}\sqrt{e+fz^{\eta}+gz^{2\eta}}=y.$ $\qquad z^{\theta}R^{3}=t.$

Ordo sextus.

1. $\dfrac{2\theta e z^{\theta-1} \begin{matrix}+2\theta\\+\eta\end{matrix} f z^{\theta+\eta-1}}{2\sqrt{e+fz^{\eta}}}=y.$ $\qquad z^{\theta}R=t.$

2. $\dfrac{2\theta e z^{\theta-1} \begin{matrix}+2\theta\\+\eta\end{matrix} f z^{\theta+\eta-1} \begin{matrix}+2\theta\\+2\eta\end{matrix} g z^{\theta+2\eta-1}}{2\sqrt{e+fz^{\eta}+gz^{2\eta}}}=y.$ $\qquad z^{\theta}R=t.$

Ordo septimus.

1. $\dfrac{2\theta e z^{\theta-1} \begin{matrix}+2\theta\\-\eta\end{matrix} f z^{\theta+\eta-1}}{e+fz^{\eta} \text{ in } 2\sqrt{e+fz^{\eta}}}=y.$ $\qquad \frac{z^{\theta}}{R}=t.$

2. $\dfrac{2\theta e z^{\theta-1} \begin{matrix}+2\theta\\-\eta\end{matrix} f z^{\theta+\eta-1} \begin{matrix}+2\theta\\-2\eta\end{matrix} g z^{\theta+2\eta-1}}{e+fz^{\eta}+gz^{2\eta} \text{ in } 2\sqrt{e+fz^{\eta}+gz^{2\eta}}}=y.$ $\qquad \frac{z^{\theta}}{R}=t.$

Ordo octavus.

1. $\dfrac{2\theta e z^{\theta-1} \begin{matrix}+2\theta\\-2\eta\end{matrix} f z^{\theta+\eta-1}}{ee+2efz^{\eta}+ffz^{2\eta}}=2y.$ $\qquad \frac{z^{\theta}}{RR}\left(\text{sive } \frac{z^{\theta}}{e+fz^{\eta}}\right)=t.$

2. $\dfrac{2\theta e z^{\theta-1} \begin{matrix}+2\theta\\-2\eta\end{matrix} f z^{\theta+\eta-1} \begin{matrix}+2\theta\\-4\eta\end{matrix} g z^{\theta+2\eta-1}}{ee+2efz^{\eta} \begin{matrix}+ff\\+2eg\end{matrix} z^{2\eta}+2fgz^{3\eta}+ggz^{4\eta}}=2y.$ $\qquad \frac{z^{\theta}}{RR}\left(\text{sive } \frac{z^{\theta}}{e+fz^{\eta}+gz^{2\eta}}\right)=t.$

Ordo nonus, ubi (ut et in decimo) pro radicali $\sqrt{h+iz^{\eta}}$ in arearum valoribus substituitur P.

$2\theta ehz^{\theta-1} \begin{matrix}+2\theta\\+3\eta\\+2\theta\\+\eta\end{matrix}\begin{matrix}fh\\ \\ ei\\ \end{matrix} z^{\theta+\eta-1} \begin{matrix}+2\theta\\+4\eta\end{matrix} fi\, z^{\theta+2\eta-1}$ in $\dfrac{\sqrt{e+fz^{\eta}}}{2\sqrt{h+iz^{\eta}}}=y.$ $\qquad z^{\theta}R^{3}p=t.$

<97>

Ordo decimus.

$2\theta ehz^{\theta-1} \begin{matrix}+2\theta\\+3\eta\\+2\theta\\-\eta\end{matrix}\begin{matrix}fh\\ \\ ei\\ \end{matrix} z^{\theta+\eta-1} \begin{matrix}+2\theta\\+2\eta\end{matrix} fi\, z^{\theta+2\eta-1}$ in $\dfrac{\sqrt{e+fz^{\eta}}}{h+iz^{\eta} \text{ in } 2\sqrt{h+iz^{\eta}}}=y.$ $\qquad \frac{z^{\theta}R^{3}}{p}=t.$

Possint et hujusmodi alia adjici, sed ad alterius generis curvas quæ cum Conicis sectionibus conferri possunt jam transeo. Et in hoc Catalogo expositam Curvam linea QEχR (fig) designatam habes, cujus basis principium sit A, basis AC, parallelè incedens CE areæ principium αχ, et area descripta αχEC. Ejus autem areæ principium sive terminus initialis (quod utplurimùm vel basis principio A insistit, vel ad infinitam distantiam recedit) invenitur quærendo basis longitudinem Aα cùm areæ valor nullus est, et erigendo normalem αχ.

Ad eundem modum Conicam sectionem (fig) habes designatam linePDG, cujus centrum sit A, vertex a, rectangulæ semidiametri Aa & AP, basis principium A vel a, vel α, basis AB vel aB, vel αB**{,}** ordinatim applicata BD tangens DT occurrens AB in T, subtensa aD et inscriptum vel ascriptum rectangulum ABDO.

Itaque retentis jam ante definitis literis, erit $AC=z$, $CE=y$, $\alpha\chi EC=t$, AB vel $aB=x$, $BD=v$, et ABDP, vel $aGDB=s$. et præterea siquando ad alicujus areæ determinationem duæ Conicæ Sectiones requiruntur, posterioris area dicetur σ, basis ξ, et parallelè incedens Y.

Catalogus Curvarum aliquot ad Conicas Sectiones relatarum ope Problematis 8 constructus.

		Sectionis Conicæ			
	Curvarum	Basis.	parall. Incedens.	Arearum valores.	
	Ordo primus.				
1.	$\frac{dz^{\eta-1}}{e+fz^{\eta}}=y.$	$z^{\eta}=x.$	$\frac{d}{e+fx}=v.$	$\frac{1}{\eta}s=t=\frac{\alpha GDB}{\eta}.$	Fig
2.	$\frac{dz^{2\eta-1}}{e+fz^{\eta}}=y.$	$z^{\eta}=x.$	$\frac{d}{e+fx}=v.$	$\frac{d}{\eta f}z^{\eta}-\frac{e}{\eta f}s=t.$	
3.	$\frac{dz^{3\eta-1}}{e+fz^{\eta}}=y.$	$z^{\eta}=x.$	$\frac{d}{e+fx}=v.$	$\frac{d}{2\eta f}z^{2\eta}-\frac{de}{\eta ff}z^{\eta}+\frac{ee}{\eta ff}s=t.$	
	Ordo secundus.				
1.	$\frac{dz^{\frac{1}{2}\eta-1}}{e+fz^{\eta}}=y.$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x.$	$\sqrt{\frac{d}{f}-\frac{e}{f}xx}=v.$	$\frac{2xv\ \div\ 4s}{\eta}=t=\frac{4}{\eta}ADGa.$	

<98>

2.	$\frac{dz^{\frac{3}{2}\eta-1}}{e+fz^{\eta}}=y.$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x.$	$\sqrt{\frac{d}{f}-\frac{e}{f}xx}=v.$	$\frac{2de}{\eta f}z^{\frac{\eta}{2}}\frac{+4es-2exv}{\eta f}=t.$	
3.	$\frac{dz^{\frac{5}{2}\eta-1}}{e+fz^{\eta}}=y.$	$\sqrt{\frac{d}{e+fz^{\eta}}}=x.$	$\sqrt{\frac{d}{f}-\frac{e}{f}xx}=v.$	$\frac{2de}{3\eta f}z^{\frac{3\eta}{2}}-\frac{2dee}{\eta ff}z^{\frac{\eta}{2}}\frac{+2eexv-4ees}{\eta ff}=t.$	

< insertion from p 171 >

Ordo tertius

$\frac{dz^{\eta-1}}{e+fz^{\eta}+gz^{2\eta}}=y.$		$\sqrt{\frac{d}{e+fz^{\eta}+gz^{2\eta}}}=x.$	$\sqrt{\frac{d}{g}\quad\frac{+ff-4eg}{4gg}xx}=v.$	$\frac{xv-2s}{\eta}=t.$
	Vel sic,	$\sqrt{\frac{dz^{2\eta}}{e+fz^{\eta}+gz^{2\eta}}}=x.$	$\sqrt{\frac{d}{e}\quad\frac{+ff-4eg}{4ee}xx}=v.$	$\frac{2s-xv}{\eta}=t.$
$\frac{dz^{2\eta-1}}{e+fz^{\eta}+gz^{2\eta}}=y.$		$\left\{\begin{array}{l}\sqrt{\frac{d}{e+fz^{\eta}+gz^{2\eta}}}=x.\\ fz^{\eta}+gz^{2\eta}=\xi.\end{array}\right.$	$\left.\begin{array}{c}\sqrt{\frac{d}{g}\quad\frac{+ff-4eg}{4gg}xx}=v.\\ \frac{1}{e+\xi}=\Upsilon.\end{array}\right\}$	$\frac{d\sigma+2fs-fxv}{2\eta g}=t.$

< text from p 98 resumes >

Ordo quartus, ubi abbreviandi causâ scribitur p pro $\sqrt{ff-4eg}.$

1.	$\frac{dz^{\frac{1}{2}\eta-1}}{e+fz^{\eta}+gz^{2\eta}}=y.$	$\left\{\begin{array}{l}\sqrt{\frac{2dg}{f-p+2gz^{\eta}}}=x.\\ \sqrt{\frac{2dg}{f+p+2gz^{\eta}}}=\xi.\end{array}\right.$	$\left.\begin{array}{l}\sqrt{d\quad\frac{-f+p}{2g}xx}=v.\\ \sqrt{d\quad\frac{-f-p}{2g}\xi\xi}=\Upsilon.\end{array}\right\}$	$\frac{2xv-4s-2\xi r+4\sigma}{\eta p}=t.$
2.	$\frac{dz^{\frac{3}{2}\eta-1}}{e+fz^{\eta}+gz^{2\eta}}=y.$	$\left\{\begin{array}{l}\sqrt{\frac{2dez^{\eta}}{fz^{\eta}-pz^{\eta}+2e}}=x.\\ \sqrt{\frac{2dez^{\eta}}{fz^{\eta}-pz^{\eta}+2e}}=\xi.\end{array}\right.$	$\left.\begin{array}{l}\sqrt{d\quad\frac{-f+p}{2e}xx}=v.\\ \sqrt{d\quad\frac{-f-p}{2e}xx}=\Upsilon.\end{array}\right\}$	$\frac{4s-2xv-4\sigma+2\xi r}{\eta p}=t.$

<99>

Ordo quintus.

1. $\frac{d}{z}\sqrt{e+fz^\eta}=y.$	$\frac{1}{z^\eta}=xx.$	$\sqrt{f+exx}=v.$	$\frac{4de}{\eta f}$ in $\frac{v^3}{2ex}-s=t=\frac{4de}{\eta f}$ in aGDT, vel in APDB
Vel sic,	$\frac{1}{z^\eta}=x.$	$\sqrt{fx+exx}=v.$	$\frac{8dee}{\eta ff}$ in: $\overline{s-\frac{1}{2}xv-\frac{fv}{4e}+\frac{ffv}{4eex}}=t=\frac{8dee}{\eta ff}$ in aGDA +
2. $\frac{d}{z^{\eta+1}}\sqrt{e+fz^\eta}=y.$	$\frac{1}{z^\eta}=xx.$	$\sqrt{f+exx}=v.$	$\frac{-2d}{\eta}s=t=\frac{2d}{\eta}\times$ APDB, ceu $\frac{2d}{\eta}\times$ aGDB.
Vel sic,	$\frac{1}{z^\eta}=x.$	$\sqrt{fx+exx}=v.$	$\frac{4de}{\eta f}$ in $s-\frac{1}{2}xv-\frac{fv}{2e}=t=\frac{4de}{\eta f}\times$ aGDK.
3. $\frac{d}{z^{2\eta+1}}\sqrt{e+fz^\eta}=y.$	$\frac{1}{z^\eta}=x.$	$\sqrt{f+exx}=v.$	$\frac{-d}{\eta}s=t=\frac{d}{\eta}\times-$aGDB, vel BDPK.
4. $\frac{d}{z^{3\eta+1}}\sqrt{e+fz^\eta}=y.$	$\frac{1}{z^\eta}=x.$	$\sqrt{f+exx}=v.$	$\frac{3dfs-2dv^3}{6\eta e}=t.$

Ordo sextus.

1. $\frac{d}{z}\sqrt{e+fz^\eta+gz^{2\eta}}=y.$	$\left\{\begin{matrix} z^\eta=x. \\ \frac{1}{z^\eta}=\xi. \end{matrix}\right.$	$\left.\begin{matrix}\sqrt{e+fx+gxx}=v. \\ \sqrt{g+f\xi+e\xi\xi}=\Upsilon.\end{matrix}\right\}$	$\frac{4dee\xi\Upsilon+2def\Upsilon-2dffv-8dee\sigma+4dfgs}{4\eta eg-\eta ff}=t.$
2. $\frac{d}{z^{\eta-1}}\sqrt{e+fz^\eta+gz^{2\eta}}=y.$	$z^\eta=x.$	$\sqrt{e+fx+gxx}=v.$	$\frac{d}{\eta}s=t=\frac{d}{\eta}$ in αGDB.
3. $\frac{d}{z^{2\eta-1}}\sqrt{e+fz^\eta+gz^{2\eta}}=y.$	$z^\eta=x.$	$\sqrt{e+fx+gxx}=v.$	$\frac{d}{3\eta g}v^3-\frac{df}{2\eta g}s=t.$
4. $\frac{d}{z^{3\eta-1}}\sqrt{e+fz^\eta+gz^{2\eta}}=y.$	$z^\eta=x.$	$\sqrt{e+fx+gxx}=v.$	$\frac{6dgx-5df}{24\eta gg}v^3\ \ \frac{+5dff-4deg}{16\eta gg}v^3=t.$

Ordo septimus.

1. $\frac{d}{z\sqrt{e+fz^\eta}}=y.$	$\frac{1}{z^\eta}=xx.$	$\sqrt{f+exx}=v.$	$\frac{4d}{\eta f}$ in $\frac{1}{2}xv\div s=t=\frac{4d}{\eta f}$ in PAD vel in aGDA.
Vel sic	$\frac{1}{z^\eta}=x.$	$\sqrt{fx+exx}=v.$	$\frac{8de}{\eta ff}$ in $s-\frac{1}{2}xv-\frac{fv}{4e}=t=\frac{8de}{\eta ff}$ in aGDA.
2. $\frac{d}{z^{\eta+1}\sqrt{e+fz^\eta}}=y.$	$\frac{1}{z^\eta}=xx.$	$\sqrt{f+exx}=v.$	$\frac{2d}{\eta e}$ in $s-xv=t=\frac{2d}{\eta e}$ in POD, vel in AODGa.
Vel sic	$\frac{1}{z^\eta}=x.$	$\sqrt{fx+exx}=v.$	$\frac{4d}{\eta f}$ in $\frac{1}{2}xv\div s=t=\frac{4d}{\eta f}$ in aDGA.
3. $\frac{d}{z^{2\eta+1}\sqrt{e+fz^\eta}}=y.$	$\frac{1}{z^\eta}=xx.$	$\sqrt{fx+exx}=v.$	$\frac{d}{\eta e}$ in $3s\div 2xv=t=\frac{d}{\eta e}$ in 3aDGA $\div\ \Delta$aDB.
4. $\frac{d}{z^{3\eta+1}\sqrt{e+fz^\eta}}=y.$	$\frac{1}{z^\eta}=xx.$	$\sqrt{fx+exx}=v.$	$\frac{10dfxv-15dfs-2dexxv}{6\eta ee}=t.$

<100>

Ordo octavus.

1. $\frac{dz^{\eta-1}}{\sqrt{e+fz^\eta+gz^{2\eta}}}=y.$	$z^\eta=x.$	$\sqrt{e+fx+gxx}=v.$	$\frac{8dgs-4dgxv-2dfv}{4\eta eg-\eta ff}=t=\frac{8dg}{4\eta eg-\eta ff}$ in αGDB $\pm\ \Delta$DBA.
2. $\frac{dz^{2\eta-1}}{\sqrt{e+fz^\eta+gz^{2\eta}}}=y.$	$z^\eta=x.$	$\sqrt{e+fx+gxx}=v.$	$\frac{-4dfs+2dfxv+4dev}{4\eta eg-\eta ff}=t.$
3. $\frac{dz^{3\eta-1}}{\sqrt{e+fz^\eta+gz^{2\eta}}}=y.$	$z^\eta=x.$	$\sqrt{e+fx+gxx}=v.$	$\frac{\begin{matrix}3dff\\-4deg\end{matrix}s\ \ \begin{matrix}-2dff\\+4deg\end{matrix}xv\ \ -2defv}{4\eta egg-\eta ffg}=t.$
4. $\frac{dz^{4\eta-1}}{\sqrt{e+fz^\eta+gz^{2\eta}}}=y.$	$z^\eta=x.$	$\sqrt{e+fx+gxx}=v.$	$\frac{\begin{matrix}-36defg\\-15dfff\end{matrix}s\ \ \begin{matrix}+8degg\\-2dffg\end{matrix}xxv\ \ \begin{matrix}-28defg\\+10dfff\end{matrix}xv\ \ \begin{matrix}+10defg\\-16deeg\end{matrix}v}{24\eta eg^3-6\eta ffgg}=t.$

Ordo nonus.

1. $\frac{dz^{\eta-1}\sqrt{e+fz^\eta}}{g+hz^\eta}=y.$	$\sqrt{\frac{d}{g+hz^\eta}}=x.$	$\sqrt{\frac{df}{h}\ \ \frac{+eh-fg}{h}xx}=v.$	$\frac{\begin{matrix}4fg\\-4eh\end{matrix}s\ \ \begin{matrix}-2fg\\+2eh\end{matrix}xv\ \ +2df\frac{v}{x}}{\eta fh}=t.$
2. $\frac{dz^{2\eta-1}\sqrt{e+fz^\eta}}{g+hz^\eta}=y.$	$\sqrt{\frac{d}{g+hz^\eta}}=x.$	$\sqrt{\frac{df}{h}\ \ \frac{+eh-fg}{h}xx}=v.$	$\frac{\begin{matrix}4egh\\-4fgg\end{matrix}s\ \ \begin{matrix}-2egh\\+2fgg\end{matrix}xv\ \ +\frac{2}{3}dh\frac{v^3}{x^3}-2dfg\frac{v}{x}}{\eta fhh}=t.$

Ordo decimus.

1. $\frac{dz^{\eta-1}}{g+hz^\eta\sqrt{e+fz^\eta}}=y.$	$\sqrt{\frac{d}{g+hz^\eta}}=x.$	$\sqrt{\frac{df}{h}\ \ \frac{+eh-fg}{h}xx}=v.$	$\frac{2xv-4s}{\eta f}=t=\frac{4}{\eta f}$ADGa.
2. $\frac{dz^{2\eta-1}}{g+hz^\eta\sqrt{e+fz^\eta}}=y.$	$\sqrt{\frac{d}{g+hz^\eta}}=x.$	$\sqrt{\frac{df}{h}\ \ \frac{+eh-fg}{h}xx}=v.$	$\frac{4gs-2gxv+2d\frac{v}{x}}{\eta fh}=t.$

Ordo undecimus.

2. $dz^{\eta-1}\sqrt{\frac{e+fz^\eta}{g+hz^\eta}}=y.$	$\sqrt{g+hz^\eta}=x.$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}xx}=v.$	$\frac{2d}{\eta h}s=t.$
3. $dz^{2\eta-1}\sqrt{\frac{e+fz^\eta}{g+hz^\eta}}=y.$	$\sqrt{g+hz^\eta}=x.$	$\sqrt{\frac{eh-fg}{h}+\frac{f}{h}xx}=v.$	$\frac{dhxv^3\ \ \begin{matrix}-3dfg\\-deh\end{matrix}s}{2\eta fhh}=t.$
1. $dz^{-1}\sqrt{\frac{e+fz^\eta}{g+hz^\eta}}=y.$	$\left\{\begin{matrix}\sqrt{g+hz^\eta}=x. \\ \sqrt{h+gz^{-\eta}}=\xi.\end{matrix}\right.$	$\left.\begin{matrix}\sqrt{\frac{eh-fg}{h}+\frac{f}{h}xx}=v. \\ \sqrt{\frac{fg-eh}{h}+\frac{e}{g}\xi\xi}=v.\end{matrix}\right\}$	$\frac{dxv^3z^{-\eta}-4dfs-4de\sigma}{\eta fg-\eta eh}s=t.$

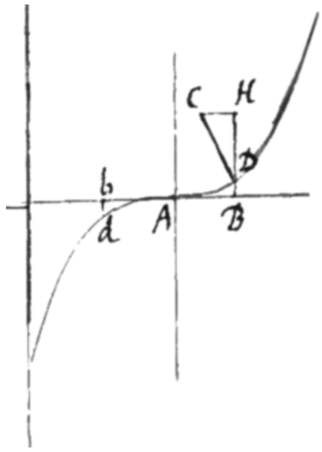
[1] This pag: must bee inserted at the end of pag 59.

[2] pag 60, lin. 6

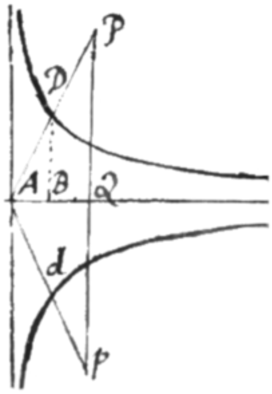
[3] here a particular figure is required

[4] Fig

[5]



[6]



[7] Fig

[8] Fig.

[9] is. fig. for. 5.

[10] pag & fig

[11] Fig

[12] Fig

[13] Fig