30/10/2021 . Assignment 7-8 1

1(a) $f(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + x_3$ → ⟨Find local extremisers⟩

subjected to : $h(x) = x_1^2 + x_2^2 + x_3^2 = 16$

Solution :

$l(x, \lambda)$ or The lagrange function looks something like

$$l(x,\lambda) = x_1^2 + 3x_2^2 + x_3 + \lambda(x_1^2 + x_2^2 + x_3^2 - 16)$$

· According to Lagrange condition:

$\nabla l(x, \lambda) = 0$ ⟨FONC⟩

$$\therefore \frac{dl}{dx} + \frac{dL}{d\lambda} = 0$$

$$\therefore \begin{bmatrix} 2x_1 + 2\lambda x_1 \\ 6x_2 + 2\lambda x_2 \\ 1 + 2\lambda x_3 \\ x_1^2 + x_2^2 + x_3^2 - 16 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \therefore \quad \begin{aligned} 2x_1 + 2\lambda x_1 = 0 &\rightarrow (1) \\ 6x_2 + 2\lambda x_2 = 0 &\rightarrow (2) \\ 1 + 2\lambda x_3 = 0 &\rightarrow (3) \\ x_1^2 + x_2^2 + x_3^2 - 16 = 0 &\rightarrow (4) \end{aligned}$$

∴ For first value of λ, let $x_1 \neq 0$:

∴ $\lambda = -1$ from $2x_1 + 2\lambda x_1 = 0$

substituting $\lambda = -1$ and assumption $x_2 = 0$ in other equations we get :

$x_3 = \frac{1}{2}$ " from equation 3 "

and $x_1^2 = 16 - 0 - \frac{1}{4} \Rightarrow \pm\frac{\sqrt{63}}{2} = x_1$

$$\therefore x^{(1)} = \left[\frac{\sqrt{63}}{2}, 0, \frac{1}{2}\right]$$

$$x^{(2)} = \left[-\sqrt{63}/2, 0, \frac{1}{2}\right]$$

$\therefore$ Similarly assuming $x_2 \neq 0$ $\therefore$ $\lambda = -3$ from equation 2 $\therefore$ (2)

Substituting $\lambda = -3$ with assumption $x_1 = 0$ in other equation

$x_3 = 1/6$ from equation (3)

$0 + x_2^2 + \frac{1}{36} - 16 = 0$ $\quad x_2^2 = 16 - \frac{1}{36}$

$$\therefore \quad x^{(3)} = \left[0, \frac{\sqrt{575}}{6}, \frac{1}{6}\right]$$

$$x^{(4)} = \left[0, -\frac{\sqrt{575}}{6}, \frac{1}{6}\right]$$

Setting $x_1 = x_2 = 0$; we get

$x_3^2 = 16$ $\therefore x_3 = \pm 4$ Substituting this in equation (3)

we get $\lambda = \frac{1}{8}, -\frac{1}{8}$

$$\therefore \quad x^{(5)} = [0, 0, 4]$$

$$x^{(6)} = [0, 0, -4]$$

$\therefore$ The set of points we have as contenders for extremisers are as follows:

$x^{(1)} = \left[\frac{\sqrt{63}}{2}, 0, \frac{1}{2}\right]$ $\quad \lambda = -1$

$x^{(2)} = \left[-\frac{\sqrt{63}}{2}, 0, \frac{1}{2}\right]$ $\quad \lambda = -1$

$x^{(3)} = \left[0, \frac{\sqrt{575}}{6}, \frac{1}{6}\right]$ $\quad \lambda = -3$

$x^{(4)} = \left[0, -\frac{\sqrt{575}}{6}, \frac{1}{6}\right]$ $\quad \lambda = -3$

$x^{(5)} = [0, 0, +4]$ $\quad \lambda = -1/8$

$x^{(6)} = [0, 0, -4]$ $\quad \lambda = 1/8$

In order to check which of these points are regular:

$Dh(x^*) = [\nabla h(x^*)]^T = [2x_1 \quad 2x_2 \quad 2x_3]$

$\therefore$ Since the columns will always be independent, all the points qualify to be regular. We need to check at every point.

SONC:

$$H_f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

<Hessian of objective>

$$H_h(x) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 h}{\partial x_1^2} & \frac{\partial^2 h}{\partial x_1 \partial x_2} & \frac{\partial^2 h}{\partial x_1 \partial x_3} \\ \frac{\partial^2 h}{\partial x_2 \partial x_1} & \frac{\partial^2 h}{\partial x_2^2} & \frac{\partial^2 h}{\partial x_2 \partial x_3} \\ \frac{\partial^2 h}{\partial x_3 \partial x_1} & \frac{\partial^2 h}{\partial x_3 \partial x_2} & \frac{\partial^2 h}{\partial x_3^2} \end{bmatrix}$$

<Hessian of constraints>

- In order to find Tangent space $\quad Dh(x^*) = \nabla h(x^*)^T$

$\{y \mid Dh(x^*)y = 0\} \quad \therefore \{y \in \mathbb{R}^3 : [2x_1 \quad 2x_2 \quad 2x_3]\{y\} = 0\}$

$\rightarrow \quad 2x_1y_1 = 0 \; ; \; 2x_2y_2 = 0 \; ; \; 2x_3y_3 = 0 \qquad$ if $y = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix}$

Let: $y_2 = b$ & $y_1 = a$

$\therefore$ for $x^{(1)} \qquad x^{(1)} = \left[\frac{\sqrt{63}}{2}, 0, \frac{1}{2}\right]$

$2x_1y_1 + 2x_2y_2 + 2x_3y_3 = 0$

$\therefore \sqrt{63}\,y_1 + 0 + a = 0$

$\therefore y_1 = \frac{-a}{\sqrt{63}}$

$\therefore$ Tangent space of $x^{(1)} = \left\{\frac{-a}{\sqrt{63}}, b, a\right\}$

$\cdot$ Hessian of Lagrangian at $\{x^{(1)}, \lambda^{(1)}\} = \{\lambda^{(1)} = -1\}$ (4)

$$L(x^{(1)}, \lambda^{(1)}) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} + -1 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$L(x^{(1)}, \lambda^{(1)}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\therefore \quad y^T L(x^{(1)}, \lambda^{(1)}) y = \begin{bmatrix} -\frac{a}{\sqrt{63}} & b & a \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -a/\sqrt{63} \\ b \\ a \end{bmatrix}$$

$$= 4b^2 - 2a^2 \begin{cases} > 0 & \text{if } |a| < b\sqrt{2} \\ = 0 & \text{if } |a| = b\sqrt{2} \\ < 0 & |a| > b\sqrt{2} \end{cases}$$

$x^{(1)}$ does not satisfy the SONC, therefore can not be

$$L(x^{(2)}, \lambda^{(2)}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \quad \because \lambda = -1$$

$\therefore$ Along the same lines, $x^{(2)}$ can not be an extremiser either.

$\cdot$ For $(x^{(3)}, \lambda^{(3)})$ & $(x^{(4)}, \lambda^{(4)})$

$$L(x^3, \lambda^3) = L(x^4, \lambda^4) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} - 3 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -6 \end{bmatrix}$$

$\cdot$ In order to find the tangent space for $x^{(3)}$ & $x^{(4)}$ {which will be similar}

$\{y \mid Dh(x^*)y = 0\}$ $\therefore$ $\{y \in \mathbb{R}^3 : [2x_1 \quad 2x_2 \quad 2x_3]\{y\} = 0\}$

$\therefore \quad 2x_1y_1 + 2x_2y_2 + 2x_3y_3 = 0$

If $y_1 = a$, $y_2 = b$

$\rightarrow$ Then for $x^{(3)}$

$$2.0.a + \cancel{2}.\frac{\sqrt{575}}{\cancel{6}3}.b + \cancel{2}.\frac{1}{\cancel{6}3}.y_3 = 0$$

$$\therefore y_3 = -b\sqrt{575}$$

$\rightarrow$ Similarly for $x^{(4)}$ $y_3 = b\sqrt{575}$

$\therefore T(x^{(3)}) = \{a, b, -b\sqrt{575}\}$ $T(x^{(4)}) = \{a, b, b\sqrt{575}\} = y^{(3)}, y^{(4)}$

Tangent space for $x^{(3)}$ and $x^{(4)}$

$\therefore$ According to SONC:

$y^{(3)T}.L(x^{(3)}, \lambda^{(3)}).y^{(3)} = -4a^2 - 3450b^2 < 0$ {Always less than 0}

$y^{(4)T}.L(x^{(4)}, \lambda^{(4)}).y^{(4)} = -4a^2 - 3450b^2 < 0$

$\therefore$ $\boxed{x^{(3)} \text{ \& } x^{(4)} \text{ are strict local maximiser.}}$

- For $(x^{(5)}, \lambda^{(5)})$ $x^{(5)} = [0, 0, 4]^T$ $\lambda^{(5)} = -1/8$

$L(x^{(5)}, \lambda^{(5)})$
{Hessian of lagrange function at $x^{(5)}, \lambda^{(5)}$}

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} - 1/8 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{14}{8} & 0 & 0 \\ 0 & \frac{46}{8} & 0 \\ 0 & 0 & -1/4 \end{bmatrix}$$

- Similarly tangent space for $x^{(5)}$

$2x_1y_1 + 2x_2y_2 + 2x_3y_3 = 0$ $\{\Rightarrow \nabla h(x^*)^T\{y\} = 0\}$

$y_1 = a$, $y_2 = b$

$2.0.a + 2.0.b + 2.4.y_3 = 0$ $\boxed{\therefore y_3 = 0 \therefore y^{(3)} = [a, b, 0]^T = y^{(4)}}$

(6)

$\therefore$ SONC for $x^{(5)}$ & $x^{(6)}$ yields:

$$y^T \times L(x^5, \lambda^5) \times y = \frac{7}{4}a^2 + \frac{23}{4}b^2 > 0 \quad \{\text{Always}\}$$

$$y^T \times L(x^6, \lambda^6) \times y = \frac{9}{4}a^2 + \frac{25}{4}b^2 > 0 \quad \{\text{Always}\}$$

$\therefore$ $x^{(5)}$ & $x^{(6)}$ are strict local minimisers.

② Maximise $\underline{x}^T \begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix} \underline{x}$ subject to $\|x\|^2 = 1$ OR $x^T x = 1$ (7)

Solution

$$\underline{x}^T \begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix} \underline{x} = x^T \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} x$$ { Changing objective function $\frac{1}{2} \underline{x}^T \underline{Q} \underline{x}$ form }

$h(x) = \underline{x}^T \underline{x} - 1 = 0$

$\therefore f(x_1, x_2) = x^T \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} x$ with $\underline{\underline{Q}} = \underline{\underline{Q}}^T$

subjected to $h(x_1, x_2) = 1 - x^T x = 0$

Lagrangian $l(x, \lambda) = f + \lambda h$,

$$\nabla_x l = \nabla_x \left[x^T \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} x + \lambda \left(1 - x^T x \right) \right]$$

$$\Rightarrow \cancel{2} \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} x - \cancel{2} \lambda x = 0 \longrightarrow (A)$$

$\langle Ax = \lambda x$ form $\rangle$

- If we convert (A) in an equivalent form

$$\left(\lambda I_2 - \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} \right) x = 0$$

$\therefore$ Solving above eigen value problem:

$$\begin{bmatrix} \lambda - 3 & -2 \\ -2 & \lambda - 3 \end{bmatrix} x = 0$$

$\therefore$ Characteristic Polynomial $= \det \begin{bmatrix} \lambda - 3 & -2 \\ -2 & \lambda - 3 \end{bmatrix}$

$$\Rightarrow \lambda^2 - 6\lambda + 5 = 0 \Rightarrow (\lambda - 1)(\lambda - 5) = 0 \quad \lambda = (1, 5)$$

- The eigen values are 1 & 5. We need to find a maximiser, hence we choose λ_{max} for max^m function value ; $\lambda = 5$.

• Eigen vector corresponding to $\lambda = 5$, will give us the maximiser.

• $5I_2 - \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$\therefore$ Eigen vector can found as

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{matrix} 0 \\ 0 \end{matrix}$$

$\therefore 2x_1 \cancel{+} - 2x_2 = 0$

$\therefore$ If $x_1 = k \quad x_2 = +k$

$\therefore$ Maximiser ~~at~~ of form $\begin{bmatrix} k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\because$ Eigen vector can be found from non-zero column of adjoint matrix

$$\text{adj} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

$\therefore$ Maximiser can be found at $= \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Any scalar multiple of the above vector will be a maximiser of the given function in the problem.

3 Minimise : $x_1 x_2 - 2x_1$ $\quad x_1, x_2 \in \mathbb{R}$

Subject to : $x_1^2 - x_2^2 = 0$

Solution :

(a) Lagrange Function:

$$l(x, \lambda) = (x_1 x_2 - 2x_1) + \lambda^*(x_1^2 - x_2^2)$$

. Lagrange Condition $\quad (x_1^*, x_2^*)$ are extremiser.

$$\nabla f(x^*) + \lambda^* \nabla h(x^*) = 0$$

$$\therefore \begin{bmatrix} x_2^* - 2 \\ x_1^* \end{bmatrix} + \lambda^* \begin{bmatrix} 2x_1^* \\ 2x_2^* \end{bmatrix} = 0$$

$$\therefore \quad x_2^* - 2 + 2\lambda^* x_1^* = 0 \longrightarrow A$$

$$x_1^* + 2\lambda^* x_2^* = 0 \longrightarrow B$$

$$(x_1^*)^2 - (x_2^*)^2 = 0 \longrightarrow C$$

From equation A & B

$$\longrightarrow \quad \lambda^* = \left(\frac{2 - x_2^*}{2x_1^*}\right) = \left(\frac{x_1^*}{2x_2^*}\right)$$

$$\therefore \quad 2x_2^* - (x_2^*)^2 = (x_1^*)^2 \quad \therefore \quad (x_1)^{*2} + (x_2^*)^2 = 2(x_2^*) \longrightarrow D$$

$$\text{and } (x_1^2)^* = (x_2^*)^2 \longrightarrow C$$

$$\therefore \quad 2(x_2^*)^2 = 2(x_2^*)$$

$$\therefore \quad x_2^* = 1$$

$$\text{and} \quad x_1^* = 1, -1$$

$\therefore$ Possible combinations $= [1, 1]^T$ & $[-1, +1]^T$

Both these points are regular since $\nabla h(x^*)^T = [2x_1^* \; 2x_2^*]$ <Solution must exist at these two points>

$\therefore$ <Considering point : $[-1, 1]$>

From equation A $\quad \lambda = -1/2$

(10)

(b) $H_f(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ {Hessian of objective function}

$H_h(x) = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ {Hessian of constraints}

∴ Hessian of Lagrangian at $\lambda = -1/2$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

· Tangent space of $[-1, 1]$:

$\{ y \in \mathbb{R} \mid \nabla h(x^*)^T y = 0 \}$ $\qquad \nabla h(x)^T = [2x_1 \quad -2x_2]$

$\underset{[-1,1]}{\nabla h(x^*)} = [-2 \quad -2]$

∴ $[-2 \quad -2]\{y\} = 0 \Rightarrow -2y_1 - 2y_2 = 0$

If $y_1 = a$; $y_2 = -a$ $\quad$ ∴ $T(x^{(-1,1)}) = [a, -a]^T$

∴ $y^T L y = [a \quad -a]\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ -a \end{bmatrix}$

$= -2a^2 < 0$

∴ $[-1, 1]$ does not satisfy SONC and is not a strict local minimiser. It is a maximiser.

(c) $[1, 1]$

From equation (A) $\lambda = 1/2$

Hessian of Lagrangian

$L(x^{(2)}) \Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$y \in \mathbb{R}$: $[2 \quad -2]\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = 2y_1 - 2y_2 = 0$

If $y_1 = a$; $y_2 = a$ $\quad$ ∴ $T(x^{(2)}) = [a, a]^T$

∴ $y^T L(x^{(2)}, \lambda^2) . y = [a \quad a]\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a \\ a \end{bmatrix} = 2a^2 > 0$

∴ $[1, 1]$ is a strict local minimiser ; satisfies SOSC.