27/09/2021 Assignment - 4B. <AML-771> Debjit Hore (1)

{2020AMY7556}

1. Given: $f : \mathbb{R}^n \rightarrow \mathbb{R}$ quadratic function

- $f(x) = \frac{1}{2} \underline{x}^T \underline{\underline{Q}} \underline{x} - \underline{x}^T \underline{b}$
- $g^{(k+1)^T} d^{(i)} = 0 \quad \forall\ k = 0, 1, \dots n-1 \ \& \ i = 0, 1, \dots k$

  $\{ g^{(k+1)} = \nabla f(x^{(k+1)}) \}$
- Show, if $g^{(k)^T} d^k \neq 0$ for $\forall\ k = 0, 1, \dots (n-1)$ then $d^0, d^1, \dots d^{n-1}$ are Q conjugates.

Solution:

$\phi_k(\alpha) = f(x^{(k)} + \alpha d^{(k)})$ {Single-variate function of α, that can be minimised by any line search method}

$\therefore \phi'_k(\alpha) = \nabla f(x^{(k)} + \alpha d^{(k)})^T d^{(k)} = g^{(k+1)^T} d^{(k)}$ {By chain rule}

Since $g^{(k+1)^T} d^{(k)} = 0 \qquad \phi'_k(\alpha) = 0$

$\because \phi_k(\alpha) = f(x^{(k)} + \alpha d^{(k)}) = \frac{1}{2}(x^{(k)} + \alpha d^{(k)})^T \underline{\underline{Q}} (x^{(k)} + \alpha d^{(k)}) - (x^{(k)} + \alpha d^{(k)})^T d$

{substituting $x \rightarrow x^{(k)} + \alpha d^{(k)}$ in expression of $f(x)$}

$\Rightarrow \frac{1}{2} \underline{\underline{Q}} (x^{(k)} + \alpha d^{(k)})^2 - (x^{(k)} + \alpha d^{(k)})^T b$

$\Rightarrow \frac{1}{2} \underline{\underline{Q}} \{ x^{(2k)} + \alpha^2 d^{(2k)} + 2x^{(k)} \alpha d^{(k)} \} - x^{(k)^T} b - \alpha d^{(k)^T} b$

$\Rightarrow \frac{1}{2} \alpha^2 (d^{(k)T} \underline{\underline{Q}} d^{(k)}) + Q x^{(k)} d^{(k)} \alpha - \alpha d^{(k)^T} b$ + {Terms not containing 'α'}

$\Rightarrow \frac{1}{2}\alpha^2 \left(d^{(k)T} \underline{\underline{Q}} d^{(k)} \right) + \alpha d^{(k)} \left\{ \left(Q x^{(k)} - b \right) \right\}$ + Constant terms not containing 'α'.

$\Rightarrow \frac{1}{2}\alpha^2 \left(\underline{d}^{(k)T} \underline{\underline{Q}} \underline{d}^{(k)} \right) + \alpha g^{(k)T} d^{(k)}$ + Constant Terms.

Now, since $g^{(k)T} d^{(k)} \neq 0$ is co-efficient of linear term in ϕ_k, we have $\alpha_k \neq 0$.

» Therefore, for $i \in \{0, 1 \ldots k-1\}$ we have :

$$\frac{x^{(k+1)} - x^{(k)}}{\alpha^{(k)}} = d^{(k)}$$

$$\therefore \quad \underline{d}^{(k)T} \underline{\underline{Q}} \underline{d}^{(i)} = \frac{1}{\alpha^{(k)}} \left(\underline{x}^{(k+1)} - \underline{x}^{(k)} \right)^T \underline{\underline{Q}} \underline{d}^{(i)}$$

$$= \frac{1}{\alpha^{(k)}} \left(g^{(k+1)} - g^{(k)} \right)^T d^{(i)}$$

$$= \frac{1}{\alpha^{(k)}} \left(g^{(k+1)T} d^{(i)} - g^{(k)T} d^{(i)} \right)$$

$$= 0$$

$$\therefore \quad \underline{d}^{(k)T} \underline{\underline{Q}} \underline{d}^{(i)} = 0 \quad \text{for } \forall \; k = 0, 1, \ldots (n-1) \text{ and } i = 0, 1 \ldots k$$

$\therefore$ $d^{(0)}, d^{(1)} \ldots d^{(n-1)}$ are Q-conjugates, since they satify the requisite condition.

(3)

(2) $\underline{x}^{(k+1)} = \underline{x}^{(k)} + \alpha_k \underline{d}^{(k)}$ where $g^{(k)} = \nabla f(x^{(k)})$

$d^{(k+1)} = \gamma_k \underline{g}^{(k+1)} + \underline{d}^{(k)}$ → We choose considering f is quadratic, such that $\underline{d}^{(k)}$ and $\underline{d}^{(k+1)}$ are Q-conjugate.

- Find γ_k in terms of $d^{(k)}$, $g^{(k+1)}$ and $\underline{\underline{Q}}$

Solution:

If $d^{(k)}$ and $d^{(k+1)}$ are Q-conjugate.

$$\underline{d}^{(k)T} \underline{\underline{Q}}\, \underline{d}^{(k+1)} = 0$$

Substituting $\underline{d}^{(k+1)} = \gamma_k \underline{g}^{(k+1)} + \underline{d}^{(k)}$

$$\underline{d}^{(k)T} \underline{\underline{Q}} \left\{ \gamma_k \underline{g}^{(k+1)} + \underline{d}^{(k)} \right\} = 0$$

$$\therefore \gamma_k \underline{d}^{(k)T} \underline{\underline{Q}}\, \underline{g}^{(k+1)} + \underline{d}^{(k)T} \underline{\underline{Q}}\, \underline{d}^{(k)} = 0$$

$$\therefore \gamma_k = -\left\{ \frac{\underline{d}^{(k)T} \underline{\underline{Q}}\, \underline{d}^{(k)}}{\underline{d}^{(k)T} \underline{\underline{Q}}\, \underline{g}^{(k+1)}} \right\}$$

<Expression of γ_k in terms of $\underline{d}^{(k)}$, $\underline{\underline{Q}}$ and $\underline{g}^{(k+1)}$>

(4)

(iii) $f(x) = \frac{5}{2}x_1^2 + \frac{1}{2}x_2^2 + 2x_1x_2 - 3x_1 - x_2$

(a) Express $f(x)$ as $\frac{1}{2}\underline{x}^T \underline{\underline{Q}}\, \underline{x} - \underline{x}^T \underline{b}$

(b) Find minimiser using CGA with $x^0 = [0, 0]^T$

(c) Calculate analytically from $\underline{Q}$ & $\underline{b}$ and check.

Solution:

$$\frac{1}{2} Q = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = f(x)$$

$$-b_1x_1 - b_2x_2 = -3x_1 - x_2$$

$$\therefore b_1 = 3 \quad b_2 = 1 \qquad \therefore b = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \rightarrow (A)$$

$$ex_1^2 + fx_1x_2 + gx_2x_1 + hx_2^2 = \frac{5}{2}x_1^2 + \frac{1}{2}x_2^2 + 2x_1x_2$$

{ $\underline{\underline{Q}}$ is symmetric so g & f are same }

$$e = 5/2 \; ; \; f = g = 1 \; ; \; h = \frac{1}{2}$$

$$\therefore \begin{bmatrix} 5/2 & 1 \\ 1 & 1/2 \end{bmatrix} = \frac{1}{2}\underline{\underline{Q}}$$

$$\therefore \underline{\underline{Q}} \begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix}$$

- Since f is quadratic function, in conjugate gradient algorithm, the number of iterations will be 2. to reach minimiser.

⑤

Iteration 1

$$d^{(0)} = -g^{(0)} = -\left[\underline{\underline{Q}}\,\underline{x} - \underline{b}\right] = -\left\{\begin{bmatrix}5 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix}\right\} = \begin{bmatrix}3\\1\end{bmatrix}$$

$$\left\{x^0 = \begin{bmatrix}0 & 0\end{bmatrix}^T\right\}$$

Also $$\alpha_0 = -\frac{g^{(0)T} d^0}{\underline{d}^{(0)T}\,\underline{\underline{Q}}\,\underline{d}^{(0)}} = \frac{5}{29} = 0.1724.$$

$$\therefore \quad \underline{x}^{(1)} = \underline{x}^{(0)} + \alpha_0 \underline{d}^{(0)}$$

$$= \begin{bmatrix}0\\0\end{bmatrix} + 0.1724\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}0.51724\\0.17241\end{bmatrix}$$

$$g^{(1)} = \begin{bmatrix}5 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}0.51724\\0.17241\end{bmatrix} - \begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}-0.069\\0.2069\end{bmatrix}$$

Iteration 2

$$\beta^{(0)} = \frac{g^{(1)T} Q\, d^{(0)}}{d^{(0)T} Q\, d^{(0)}} = 0.00475$$

$$d^{k+1} = d^{(k)} \cdot \beta_k - g^{k+1}$$

$$\therefore d^{(1)} = \begin{bmatrix}+0.069\\-0.2069\end{bmatrix} + 0.00475\begin{bmatrix}3\\1\end{bmatrix}$$

$$d^{(1)} = \begin{bmatrix}0.0832\\-0.2021\end{bmatrix}$$

$$\alpha_1 = -\frac{\underline{g}^{(1)T}\underline{d}^{(1)}}{\underline{d}^{(1)T}\,\underline{\underline{Q}}\,\underline{d}^{(1)}} = 5.8000.$$

$$x^{(2)} = x^{(1)} + \alpha_1 d^{(1)} = \begin{bmatrix}1.000\\-1.000\end{bmatrix}$$

⑥

Analytical Minimiser:

$\nabla f(x) = 0$

$\therefore$ we need to solve $\underline{\underline{Q}}\,\underline{x} = \underline{b}$

$$\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\therefore \quad 5x_1^* + 2x_2^* = 3 \longrightarrow (1)$

$\quad\;\; 2x_1^* + x_2^* = 1 \longrightarrow (2)$

$\rightarrow \quad (1) - 2 \times (2) \quad \Rightarrow$

$$\begin{array}{r} 5x_1^* + \cancel{2x_2^*} = 3 \\ 4x_1^* + \cancel{2x_2^*} = 2 \\ \hline x_1^* = 1 \end{array}$$

Substituting $\quad x_2^* = -1$

$\therefore \quad x^* = [1, -1]^T$. {Matches with minimum found through CGA}.

4. OUTPUT OF MATLAB CODE. KINDLY RUN THE ‘conjugateGradient.m’ file. That is the main file.

```
conjugateGradient
 Choose 1 for Fletcher Reeves updation of Beta, and 2 for Hestenes Stiefel Formula
Enter 1 or 2 ONLY  1
The final interval after 27.0000 iterations is      1.0000E+00      1.0000E+00
>> conjugateGradient
 Choose 1 for Fletcher Reeves updation of Beta, and 2 for Hestenes Stiefel Formula
Enter 1 or 2 ONLY  2
The final interval after 11.0000 iterations is      1.0000E+00      9.9999E-0
```