Sparse Portfolio Optimization For Index Tracking In Finance

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***Abstract*—In the investment industry, fund managers follow two types of strategies, active and passive portfolio management. We explore one of the commonly used passive portfolio optimization strategy with financial index tracking. Historical data suggests that financial index has always outperformed the market in a stable manner with low risk factor. While tracking a financial index, the tracking error can be minimized with convex optimization methods. The best method surely is replicating the benchmark weights of a financial index but this results in small and positions. Also this is a Single Period Optimization algorithm, so the weights need to be recalculated frequently and the portfolio should be rebalanced accordingly. This results in selling and buying of stocks which can increase the commission of the exchange. For this reason we use three sparse portfolio management algorithms on NIFTY 50 data for asset allocation and asset selection. We have explored Single Period Optimization (SPO) given in [1]. The three algorithms which have been implemented by us is LAIT, LAITH, SLAIT.**

***Index Terms*—financial index,tracking,sparse portfolio,NIFTY 50, LAIT, SLAIT, LAITH**

I. Introduction

There are two methods for managing a portfolio in finance, active portfolio management and passive portfolio management. In the active portfolio management method, the fund manager builds a portfolio in the hope of outperforming the market, whereas in the passive portfolio management method, the fund manager builds a portfolio to track the market index. The passive portfolio management technique is a safer way to manage the portfolio because it has been observed over time that the actively managed portfolio rarely outperforms the market[4]. In this work, we investigated the signal processing aspect of portfolio optimization in finance to build a portfolio that can track the market index.

Our focus will be on the passive portfolio management strategies. The passive portfolio management strategy involves two approaches, static and dynamic. The portfolio is built at the start of the investment process and held throughout the investment period in the static approach. This method is relatively easy to implement and has lower transaction costs. However, in the static approach, the tracking error, which measures the difference between the performance of the tracking portfolio and the benchmark index, may be higher. The portfolio is adjusted over time in the dynamic approach based on a trading strategy that aims to reduce tracking error. This approach may produce lower tracking error than the static approach, but it incurs higher transaction costs and may necessitate more complex trading strategies.

The simplest way to construct a portfolio is to include all assets in the same proportion as the index (full replication strategy), but this is not a good strategy because it includes a lot of small and illiquid assets that are difficult to sell and may result in selling these illiquid assets at a price lower than market price. Another downside of this strategy is the high transaction cost associated with the portfolio rebalancing process. The best method for managing a portfolio is to choose a portion of the assets linked to the original index, thereby optimizing a sparse portfolio. Index tracking is accomplished by minimizing index tracking error and introducing sparsity into the portfolio using norm regularization techniques.

In this study, we looked into Single Period Optimization (SPO) for the optimization of sparse portfolios from the paper [1]. Single Period Optimization uses a static approach, holding the constructed portfolio for the duration of the relevant period.

Maintaining the relative amounts of the holding portfolio, which can fluctuate due to price changes in the underlying assets, is one of the major challenges. The tracking error may rise as a result of the relative quantities in the portfolio gradually deviating from their intended value. This problem can be mitigated by rebalancing the constructed portfolio frequently.Another important consideration is the amount of data required to track the market index. Because market dynamics change at such a rapid pace, data from a long time ago is no longer relevant, as we verified during our experimentation.So, for the final tracking, we select data up to a relevant time period. It is recommended to rebalance the portfolio frequently according to market changes to reduce tracking error and track the market index appropriately, but this incurs a high transaction cost due to the large number of assets involved. The solution is to construct a sparse portfolio by selecting only a few assets from the index's entire set of constituent assets that is constructing a sparse portfolio.

Our entire work will be based on the following basic concepts. Consider an index made of N assets. Then the return of the market for index T days is given by the vector $\mathbf{r}^b = \left[r_1^b, ..., r_T^b\right] \epsilon \mathbb{R}^T$. The return of the N assets in the past T days is given by $\mathbf{X} = \left[\mathbf{r}_1, ..., \mathbf{r}_T\right]^T \epsilon\ \mathbb{R}^{T\times N}$ where $\mathbf{r}_t\ \epsilon\ \mathbb{R}^N$

denotes the return of the N assets at the t-th day. $\mathbf{b} \epsilon \mathbb{R}_+^N$ is the benchmark index weights of the original market index satisfying $\mathbf{b} > 0$, $\mathbf{b}^T\mathbf{1} = 1$ and $\mathbf{Xb} = \mathbf{r}^b$. $\mathbf{w}$ denotes the weight vector of the designed portfolio which satisfies the constraints $\mathbf{w} \geq 0$ signifying no short-selling and $\mathbf{w}^T\mathbf{1} = 1$. The term $\mathbf{Xw}$ denotes the return of the constructed portfolio.

This paper will be divided into the sections listed below. In the following section, we will define the implementation and experimentation, which will be divided into the fundamentals of index error tracking, followed by the details of the implementation of the three algorithms. Then we will present the results and plots associated with them, followed by a section defining the mathematical notations used in this paper, and finally we will mention the references.

II. Implementation and Experimentation

The basic idea used from the papers [5],[6] for tracking the index error is to find the l_2-norm distance between the return of the constructed portfolio and the return of the market index and minimize this distance. Mathematically this empirical tracking error is given by

$$ETE(\mathbf{w}) = \frac{\left\|\mathbf{Xw} - \mathbf{r^b}\right\|_2^2}{T} \tag{1}$$

In the following sections, we will use this objective and modify it by incorporating various types of regularization. We used NIFTY 50 data for this work, which has never been done before.

A. Algorithm-1: Linear Approximation For Index Tracking Problem (LAIT)

The basic optimization problem behind this is algorithm given below [7]

$$\min_{w} \frac{\left\|\mathbf{Xw} - \mathbf{r^b}\right\|_2^2}{T} + \lambda \left\|\mathbf{w}\right\|_0$$

$$\text{subject to} \quad \mathrm{w}^T 1 = 1$$
$$\mathrm{w} \geq 0$$

The authors have replaced the l_0-norm by a continuous and differentiable function. The problem still remains a non-convex optimization problem. Then the majorization-minimization approach has been used by the authors coupled with First order Taylor's series expansion making the regularizer a linear function of $\mathbf{w}$.

A good approximation for l_0-norm is done by using l_1-norm but according to the authors of the original paper this will not be a good approximation as the weights need to non-negative and they should sum up to one and because of which l_1-norm becomes a constant. So the l_0-norm has been approximated by the following function :

$$\rho_p(w) = \frac{\log(1 + |w|/p)}{\log(1 + 1/p)}$$

The parameter p has value in the range $0 < p << 1$. and the function behaves like a Indicator function of the form $\rho_p(w) \to I_{w \neq 0}$ as $p \to 0$.

The above function is a good approximation for l_0-norm when $|w| \, \epsilon [0, 1]$. The authors have mentioned a more general form of the above function for the cases when $|w| \, \epsilon [0, u]$ where $u \leq 1$ is the upper limit on the weights or $|w| \, \epsilon [0, l]$ where l is the lower bound on the weights. The more general form of the above function is given by:

$$\rho_{p,\gamma}(w) = \frac{\log(1 + |w|/p)}{\log(1 + \gamma/p)}$$

where $\gamma > 0$. The above function approximates the indicator function in the interval [0,γ] satisfactorily. This can be seen from the figure 1.

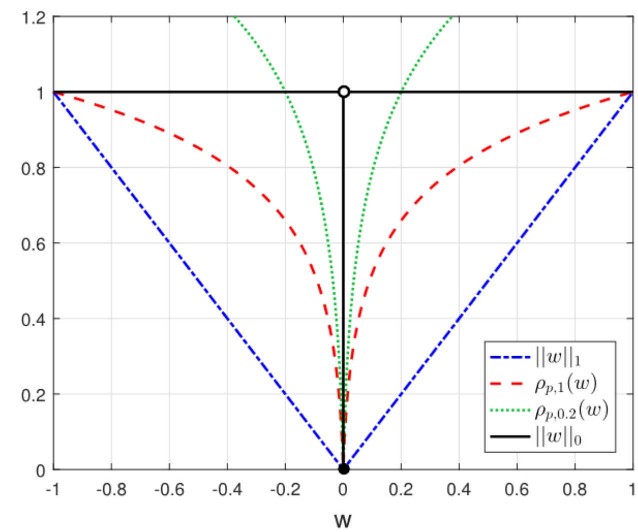

Fig. 1. Image showing the approximation function mentioned above.The image has been taken directly from the paper [1]

The optimization challenge (with the aforementioned function) is presented below:

$$\min_{w} \frac{\left\|\mathbf{Xw} - \mathbf{r^b}\right\|_2^2}{T} + \lambda \mathbf{1}^T \rho_{p,u}(\mathbf{w})$$

$$\text{subject to} \quad \mathrm{w} \epsilon W$$

Here $W = \left\{\mathbf{w} | \mathbf{w}^T\mathbf{1} = 1, 0 < \mathbf{w} < u\mathbf{1}^T\right\}$ and γ has been replaced by u.

This is a non-convex optimization problem because the objective is a sum of l_2-norm which is convex and $\rho_{p,u}(\mathbf{w})$ which is a concave function for $w > 0$. So the authors of the paper[1] have taken a majorization-minimization approach using a surrogate function. Then the first order Taylor series expansion is applied on the concave part of the objective and a linear approximation for the majorization function of $\rho_{p,u}(\mathbf{w})$ is obtained in the form given by $g_{p,\gamma}(w, w^{(k)}) = d_{p,\gamma}(w^{(k)})w + c_{p,\gamma}(w^{(k)})$. This function majorizes $\rho_{p,u}(\mathbf{w})$ at $w^{(k)}$ where the function $g_{p,\gamma}(w, w^{(k)})$ sets an upper bound for $\rho_{p,u}(\mathbf{w})$ at $w^{(k)}$. Here $d_{p,\gamma}(w^{(k)}) = \frac{1}{\kappa_1(p+w^{(k)})}$, $c_{p,\gamma}(w^{(k)}) = \frac{\log(1+w^{(k)}/p)}{\kappa_1} - \frac{w^{(k)}}{\kappa_1(p+w^{(k)})}$ and $\kappa_1 = \log(1 + \gamma/p)$.

We have implemented the above optimization problem with all the details given by the authors. We have the following hyperparameters: $\gamma = 0.4$, $p = 1e-3$, $\lambda = 1e-6$, Testing period = 240 days and training period = 120 days. The algorithm converged within 0.2 seconds. We are attaching the plots below: The figure 2 is the convergence plot for algorithm 1,figure 3 depicts training plot for algorithm 1 and figure 4 is the testing plot for algorithm 1.

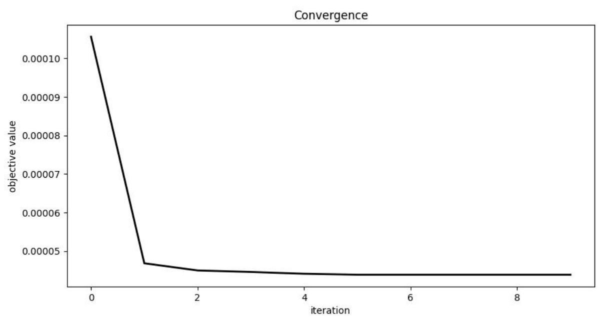

Fig. 2. The plot shows the objective value convergence with iterations for algorithm 1

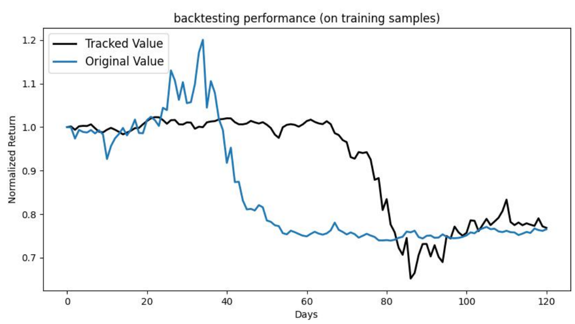

Fig. 3. The plot shows the training of the algorithm1

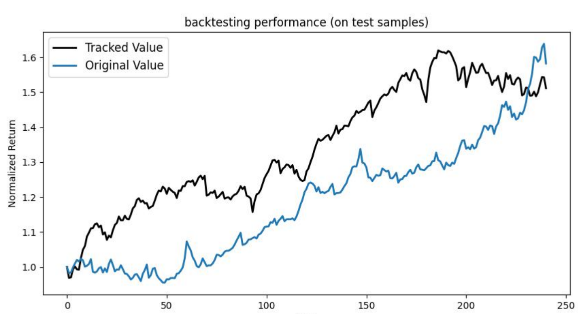

Fig. 4. The plot shows the testing of the algorithm1

B. Algorithm-2: Linear Approximation for Index Tracking problem with Holding Constraints(LAITH)

This algorithm is an improvement in the LAIT algorithm (algorithm 1) enforcing holding constraints on the weights. Holding constraints are non-convex in nature. Here also we approximate l_0-norm with the help of the function $\rho_{p,\gamma}$. This problem with holding constraints can be written like this in below.

$$\min_{\mathbf{w}} \frac{\left\|\mathbf{X}\mathbf{w} - \mathbf{r}^b\right\|_2^2}{T} + \lambda \mathbf{1}^T \rho_{p,u}(\mathbf{w})$$

$$\text{subject to} \quad \mathrm{w}\epsilon W$$

$$1 \odot \mathbf{I_{w>0}} \leq \mathbf{w} \leq \mathbf{u} \odot \mathbf{I_{w>0}}$$

As we can see, the lower bound constraint is non-convex in nature,this is not dealt with directly.Rather an additional term is added in the objective function which penalizes for all non-zero weights less than the lower holding bound,l. A suitable function for penalization can be

$$f_l(w) = max((I_{0<w<l}.l - w), 0) \tag{2}$$

Above function is applied element wise on a vector. Here also we have approximated the indicator function with the help of $\rho_{p,\gamma}$.

$$\tilde{f}_{p,l}(w) = max((\rho_{p,l}(w).l - w), 0) \tag{3}$$

The problem now takes the following form where ν is the hyper-parameter vector that penalizes weights which are below the lower bound.

$$\min_{\mathbf{w}} \frac{\left\|\mathbf{X}\mathbf{w} - \mathbf{r}^b\right\|_2^2}{T} + \lambda \mathbf{1}^T \rho_{p,u}(\mathbf{w}) + \nu^{\mathbf{T}} \mathbf{f}_{p,l}\tilde{(\mathbf{w})}$$

$$\text{subject to} \quad \mathrm{w}\epsilon W$$

The above problem is not convex as $\tilde{f}_{p,l}(\mathbf{w})$ is neither concave nor convex.$\tilde{f}_{p,l}(\mathbf{w})$ is separable and the uni-variate case can be majorized at $w^{(k)}$ as

$$h_{p,l}(w, w^{(k)}) = max((d_{p,l}(w, w^{(k)}).l - 1)w + c_{p,l}(w^{(k)}), 0) \tag{4}$$

where $d_{p,l}(w^{(k)})$ and $c_{p,l}(w^{(k)})$ are calculated in Algorithm 1.This is a convex function as it is maximum of two convex(affine) functions.After this the problem takes the following form

$$\min_{\mathbf{w}} \frac{\left\|\mathbf{X}\mathbf{w} - \mathbf{r}^b\right\|_2^2}{T} + \lambda \mathbf{d}_{p,u}^{(k)} \mathbf{w} + \nu^T max(Diag(\mathbf{d}_{p,u}^{(k)} \odot \mathbf{l} - \mathbf{1})\mathbf{w} + \mathbf{c}_{p,l}^{(k)} \odot \mathbf{l}, 0)$$

$$\text{subject to} \quad \mathrm{w}\epsilon W$$

The following are the hyperparameters associated with the algorithm 2: $u = 0.15$, $l = 0.08$, $p = 1e-3$, $\lambda = 1e-5$, $\nu = 1e-3$. Testing period = 240 days and training period = 120 days. The algorithm converged within 0.47 seconds. We are attaching the plots below: The figure 5 is the convergence plot for algorithm 2,figure 6 depicts training plot for algorithm 2 and figure 7 is the testing plot for algorithm 2.

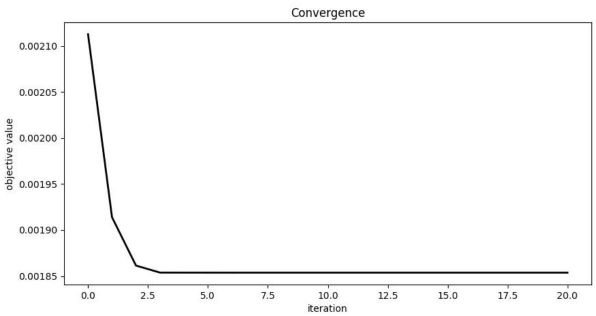

Fig. 5. The plot shows the objective value convergence with iterations for algorithm 2

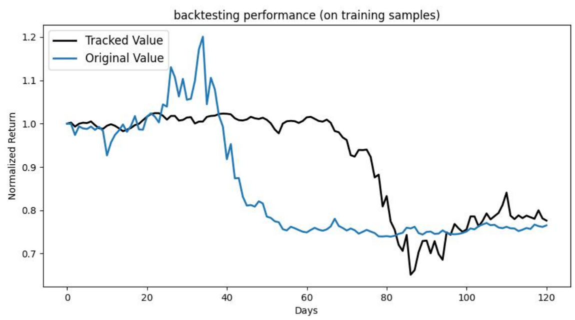

Fig. 6. The plot shows the training of the algorithm2

C. *Algorithm-3: Specialized Linear Approximation For Index Tracking Problem (SLAIT)*

While implementing the LAIT(algorithm 1) we had to use the solver from python library cvxpy. Because of the closed form nature of the constraint $W_u = \left\{\mathbf{w}|\mathbf{w}^T\mathbf{1} = 1, 0 \leq \mathbf{w} \leq u\mathbf{1}\right\}$ this problem can be solved without a solver. Consider the optimization problem:

$$\min_{\mathbf{w}} \mathbf{w}^T\mathbf{w} + \mathbf{q}^T\mathbf{w}$$

$$\text{subject to} \quad \mathbf{w}\epsilon W_u$$

where $\mathbf{q}\epsilon\mathbb{R}^N$. This problem is solved using dual formulation and using the KKT conditions. The solution has been divided into two parts. First it is solved for $u = 1$ and then the solution is used to initialize the closed form solution for the case $u \leq 1$. This algorithm doesn't use any solver thus proving to be the fastest algorithm amongst all the three algorithms. The closed form solution for the $u = 1$ case is given by:

$$\mathbf{w}^* = \max(-\frac{(\mu\mathbf{1} + \mathbf{q})}{2}, 0)$$

where

$$\mu = \frac{-(\sum_{i\epsilon A} q_i + 2)}{Card(A)}$$

and

$$A = \{i|\mu + q_i < 0\}$$

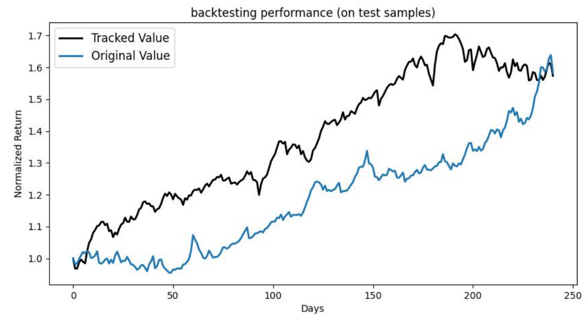

Fig. 7. The plot shows the testing of the algorithm2

For $u \leq 1$ the solutions obtained by solving the KKT conditions are given below:

$$w^* = max(min(\frac{-(\mu\mathbf{1} + \mathbf{q})}{2}, u\mathbf{1}), 0)$$

where

$$\mu = \frac{-(\sum_{i\epsilon B_2} q_i + 2 - card(B_1)2u)}{card(B_2)}$$

and

$$B_1 = \{i|\mu + q_i < -2u\}$$

$$B_2 = \{i| - 2u < \mu + q_i < 0\}$$

While implementing the $u \leq 1$ case using the solution of $u = 1$ case we were getting oscillations in the solution space. So we used the initialization $\mu = 0$ and the solution converged.

The optimization problem defined for the algorithm has been expanded using the properties of norm and is written in the form:

$$\min_{\mathbf{w}} \frac{\mathbf{w}^T\mathbf{X}^T\mathbf{X}\mathbf{w}}{T} + (\lambda\mathbf{d}_{p,u}^{(k)} - \frac{2\mathbf{X^T r^b}}{T})^T\mathbf{w}$$

subject to $\quad \mathbf{w}\epsilon W_u$ This problem has been reduced by the authors to the form:

$$\min_{\mathbf{w}} \mathbf{w}^T\mathbf{w} + \mathbf{q_1}^T\mathbf{w}$$

$$\text{subject to} \quad \mathbf{w}\epsilon W_u$$

where,

$$q_1 = \frac{(2(\mathbf{L}_1 - \mathbf{M}_1)\mathbf{w}^{(k)} + \lambda\mathbf{d}_{p,u}^{(k)} - \frac{2\mathbf{X}^T\mathbf{r}^b}{T})}{\lambda_{max}^{\mathbf{L}^{(1)}}}$$

This algorithm 2 uses the hyperparameters: $\mu = 0$,$u = 1$,$p = 1e-3$, $\lambda = 1e-6$, Testing period = 240 days and training period = 120 days. The algorithm converged within 0.13 seconds. We are attaching the plots below. The fig 8 shows the convergence of the objective function with iterations of algorithm 3, fig 9 shows training of algorithm 3 and fig 10 shows testing of the algorithm 3.

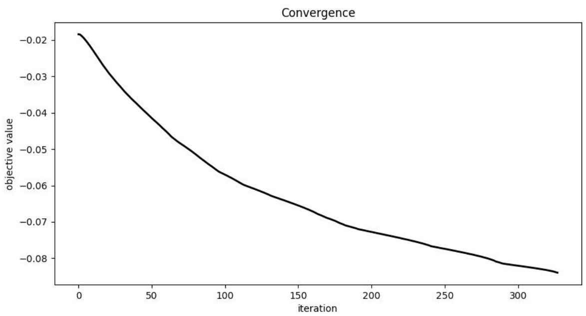

Fig. 8. The plot shows the objective value convergence with iterations for algorithm 3

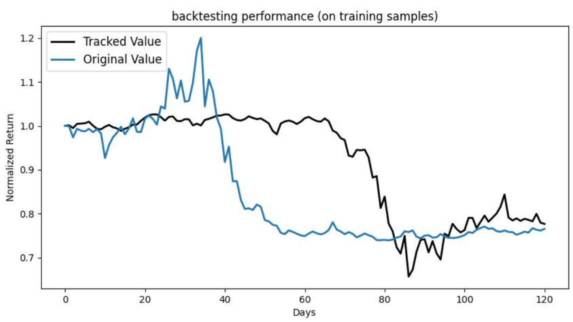

Fig. 9. The plot shows the training of the algorithm 3

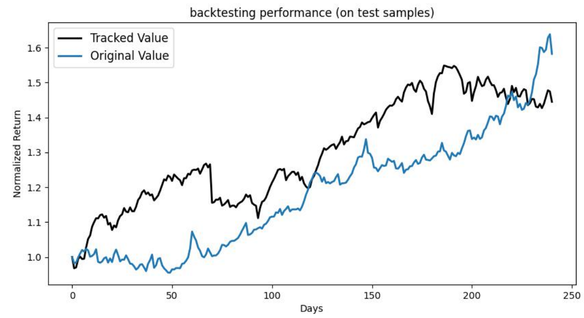

Fig. 10. The plot shows the testing of the algorithm 3

III. Conclusion and Performance Comparison of the Three Algorithms

In conclusion, we can state that all three algorithms have, for a brief period of time, been able to outperform the market by a respectable margin. Due to the holding constraints being included in LAITH, algorithm 2 (LAITH), which performs better than algorithm 1 (LAIT), does so at the cost of a longer convergence time. Since no solver has been used to solve algorithm 3, which is SLAIT, it is the fastest of the three. Utilizing KKT conditions on the objective and exploiting the constraint's closed form property, this was accomplished. Over all the algorithms, the same amount of training and testing time was used in all experiments.

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