School of Engineering and Computer Science

Mathematics Applied to Digital Engineering

(NUMERATION SYSTEM)

Kamel ATTAR

kamel.attar@epita.fr

attar.kamel@gmail.com

Week **#1** ✦ Friday 27/OCT/2023 ✦

Week **#2** ✦ Friday 3/NOV/2023 ✦

★ Numeration system ★

Definition

- A **numeration system** consists of a set of symbols (numerals) to represent numbers, and a set of rules for combining those symbols.
- A **number** is a concept, or an idea, used to represent some quantity. It answers the question "How many?"

A **numeral**, on the other hand, is a symbol such as $\cap$, **10**, ***X*** used to represent a number.

System of numeration

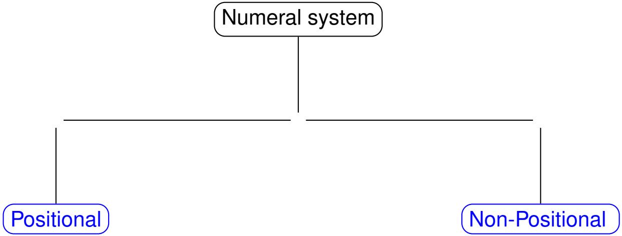
The numeral system can be classified into two types namely:

- Positional Numeral System

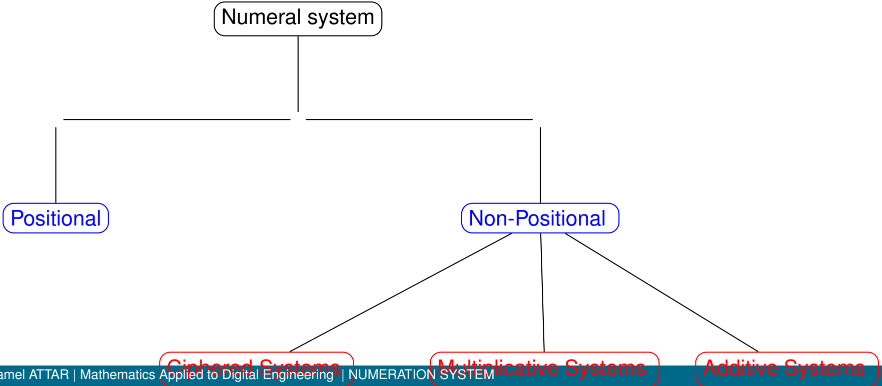
The Non-Positional numeral system is also known as the Non-Weighted Numeral System. In the olden days, people used this type of numeral system for simple calculations like additions and subtractions. In this system, the digit value is independent of its position.

- Non-Positional Numeral System

The Positional numeral system is also known as the Weighted Numeral System. As the name implies, there will be a weight associated with each digit, also known as the digit's Place Value.



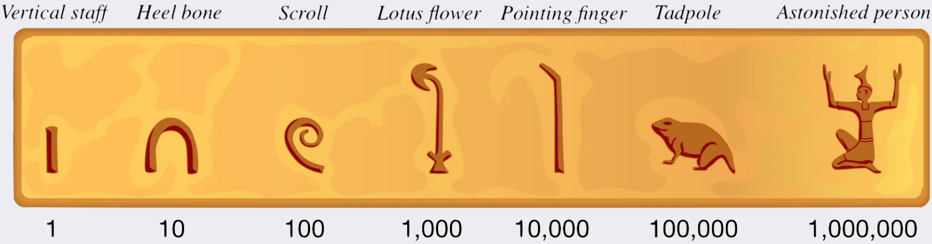
- An **additive system** is one in which the number represented by a set of numerals is simply the sum of the values of the numerals.
- A **Multiplicative Systems** are more similar to our Hindu-Arabic system which we use today than are additive systems.
- In the **Ciphered Systems**, after a base $\boldsymbol{b}$ has been selected, sets of symbols are adopted for $\mathbf{1, 2, \cdots, b-1; b, 2b, \cdots, (b-1)b; b^2, 2b^2, \cdots, (b-1)b^2}$.



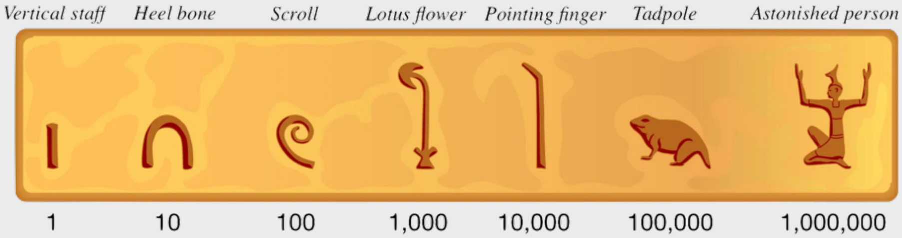
- Numeral system
  - Positional
    - *Hindu-Arabic*
    - *Babylonian*
    - *Mayan*
  - Non-Positional
    - Ciphered Systems
      - *Ionic Greek*
    - Multiplicative Systems
      - *Traditional Chinese*
    - Additive Systems
      - *Ancient Egyptian*
      - *Roman*

Definition (Ancient Egyptian hieroglyphic numeral system)

One of the earliest formal numeration systems was developed by the Egyptians sometime prior to **3000** BCE. It used a system of hieroglyphics using pictures to represent numbers.

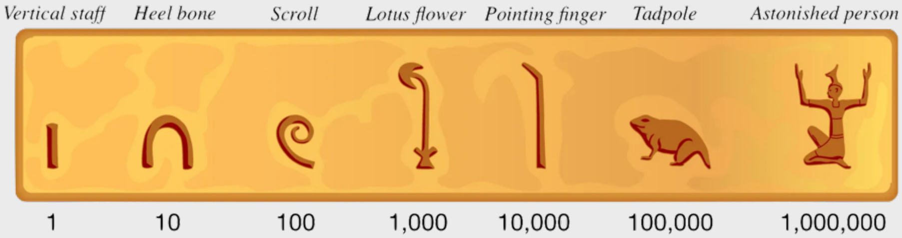


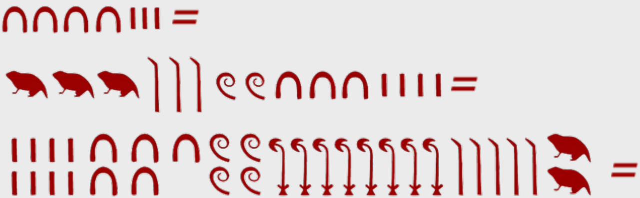
Example


Vertical staff
Heel bone
Scroll
Lotus flower
Pointing finger
Tadpole
Astonished person
1
10
100
1,000
10,000
100,000
1,000,000


=
=
=

Example





8(1) + 5(10) + 4(100) + 8(1,000) + 5(10,000) + 2(100,000) = 258,458

Definition (Roman Numeration System)

Roman numerals are devised by the ancient Romans to count and perform other day-to-day transactions. Several letters from the Latin alphabet are used for the representation of roman numerals.

Roman Symbols	***I***	***V***	***X***	***L***	***C***	***D***	***M***
Numbers	**1**	**5**	**10**	**50**	**100**	**500**	**1000**

The Romans also used the concept of subtraction.

Example

For example, **8** is written as ***VIII***, but **9** is written as ***IX***, meaning that **1** is subtracted from **10** to get **9**.

There are four rules for writing Roman numerals in numbers :

R.❶ When a letter is repeated in sequence, its numerical value is added.

- For example, ***XXX*** represents $\mathbf{10} + \mathbf{10} + \mathbf{10}$ or **30**.
- ***I***, ***X*** and ***C*** can be repeated only up to three times. ***V***, ***L*** and ***D*** are never repeated.

R.❷ When larger-value letters follows smaller-value, the numerical values of each are added.

- For example, ***LXVI*** represents $\mathbf{50} + \mathbf{10} + \mathbf{5} + \mathbf{1}$ or **66**.

R.❸ When a smaller-value letter precedes a larger-value letter, the smaller value is subtracted from the larger value.

- For example, ***IV*** $= \mathbf{5} - \mathbf{1} = \mathbf{4}$, ***XC*** $= \mathbf{100} - \mathbf{10} = \mathbf{90}$ and ***CM*** $= \mathbf{1000} - \mathbf{100} = \mathbf{900}$.
- ***I*** can only precede ***V*** or ***X***, ***X*** can only precede ***L*** or ***C***, ***C*** can only precede ***D*** or ***M***.

R.❹ A symbol of smaller value, put between two symbols of greater value is subtracted from the symbol on its right.

- For example, ***XIV*** $= \mathbf{10} + \mathbf{5} - \mathbf{1} = \mathbf{14}$, ***LIX*** $= \mathbf{50} + \mathbf{10} - \mathbf{1} = \mathbf{59}$, ***XCIX*** $= \mathbf{90} + \mathbf{10} - \mathbf{1} = \mathbf{99}$ and ***XLIX*** $= \mathbf{40} + \mathbf{10} - \mathbf{1} = \mathbf{49}$.

Example

Find the value of each Roman Numeral

(a) ***LXVII*** (b) ***XCIV*** (c) ***MCML*** (d) ***CCCXLVI*** (e) ***DCCCLV***

Solution

Example

Find the value of each Roman Numeral

(a) $\boldsymbol{LXVII}$ (b) $\boldsymbol{XCIV}$ (c) $\boldsymbol{MCML}$ (d) $\boldsymbol{CCCXLVI}$ (e) $\boldsymbol{DCCCLV}$

Solution

(a) $\boldsymbol{L = 50}$, $\boldsymbol{X = 10}$, $\boldsymbol{V = 5}$ and $\boldsymbol{III = 3}$; so $\boldsymbol{LXVIII = 68}$.

(b) $\boldsymbol{XC = 90}$ and $\boldsymbol{IV = 4}$; so $\boldsymbol{XCIV = 94}$

(c) $\boldsymbol{M = 1000}$, $\boldsymbol{CM = 900}$, $\boldsymbol{L = 50}$; so $\boldsymbol{MCML = 1950}$

(d) $\boldsymbol{CCC = 300}$, $\boldsymbol{XL = 40}$, $\boldsymbol{V = 5}$ and $\boldsymbol{I = 1}$; so $\boldsymbol{CCCXLVI = 346}$

(e) $\boldsymbol{D = 500}$, $\boldsymbol{CCC = 300}$, $\boldsymbol{L = 50}$, $\boldsymbol{V = 5}$; so $\boldsymbol{DCCCLV = 855}$.

★ Roman Numeration System ★

Convert our numbers to Roman numbers

Roman Symbols	$\overline{V}$	$\overline{X}$	$\overline{L}$	$\overline{C}$	$\overline{D}$	$\overline{M}$
Numbers	**5 000**	**10 000**	**50 000**	**100 000**	**500 000**	**1000 000**

Example

$$\mathbf{579 = 500 + 70 + 9 = D + LXX + IX = DLXXIX}$$

$$\mathbf{5605 = 5000 + 600 + 5 = \overline{V} + DC + V = \overline{V}DCV}$$

★ Roman Numeration System ★

Conversion of Roman Numeration

Roman Symbols	$\overline{V}$	$\overline{X}$	$\overline{L}$	$\overline{C}$	$\overline{D}$	$\overline{M}$
Numbers	**5 000**	**10 000**	**50 000**	**100 000**	**500 000**	**1000 000**

Example

Convert **1347** and **25493** in roman numerals.

Solution

★ Roman Numeration System ★

Conversion of Roman Numeration

Roman Symbols	$\overline{V}$	$\overline{X}$	$\overline{L}$	$\overline{C}$	$\overline{D}$	$\overline{M}$
Numbers	**5 000**	**10 000**	**50 000**	**100 000**	**500 000**	**1000 000**

Example

Convert **1347** and **25493** in roman numerals.

Solution

$$\begin{aligned} 1347 &= 1000 + 300 + 40 + 7 \\ &= M + CCC + LX + VII \\ &= MCCCXLVII \end{aligned}$$

$$\begin{aligned} 25493 &= 25000 + 400 + 90 + 3 \\ &= \overline{XXV} + CD + XC + III \\ &= \overline{XXV}CDXCIII \end{aligned}$$

★ Roman Numeration System ★

Operations on Roman Numerals

Example (Addition of Roman Numerals:)

Find the sum of $\boldsymbol{LXXX}$ and $\boldsymbol{VI}$. Give the answer is Roman Numerals.

solution

$\boldsymbol{LXXX} = \mathbf{50} + \mathbf{10} + \mathbf{10} + \mathbf{10} = \mathbf{80}$ and $\boldsymbol{VI} = \mathbf{5} + \mathbf{1} = \mathbf{6}$.

Now,

$$\boldsymbol{LXXX} + \boldsymbol{VI} = \mathbf{80} + \mathbf{6} = \mathbf{86}$$

We write **86** in Roman numerals as $\boldsymbol{LXXXVI}$.

Therefore, $\boldsymbol{LXXX} + \boldsymbol{VI} = \boldsymbol{LXXXVI}$

Example (Subtraction of Roman Numerals:)

Subtract $\boldsymbol{LI}$ from $\boldsymbol{XCV}$. Write the answer in Roman numerals.

solution

$\boldsymbol{LI} = \mathbf{50} + \mathbf{1} = \mathbf{51}$ and $\boldsymbol{XCV} = \mathbf{100} - \mathbf{10} + \mathbf{5} = \mathbf{95}$

Now,

$$\boldsymbol{XCV} - \boldsymbol{LI} = \mathbf{95} - \mathbf{51} = \mathbf{44}$$

We write **44** in Roman numerals as $\boldsymbol{XLIV}$.

Therefore, $\boldsymbol{XCV} - \boldsymbol{LI} = \boldsymbol{XLIV}$

Example (Multiplication of Roman Numerals:)

Find the product of the Roman numerals ***IX*** and ***XC***.

solution

$\boldsymbol{IX} = \mathbf{10} - \mathbf{1} = \mathbf{9}$ and $\boldsymbol{XC} = \mathbf{100} - \mathbf{10} = \mathbf{90}$

Now,

$$\boldsymbol{IX} \times \boldsymbol{XC} = \mathbf{9} \times \mathbf{90} = \mathbf{810} = \boldsymbol{DCCCX}$$

Therefore, $\boldsymbol{IX} \times \boldsymbol{XC} = \boldsymbol{DCCCX}$

Example (Division of Roman Numerals:)

Divide ***CXXV*** by ***XXV***

solution

$\boldsymbol{CXXV} = \mathbf{100} + \mathbf{10} + \mathbf{10} + \mathbf{5} = \mathbf{125}$ and $\boldsymbol{XXV} = \mathbf{10} + \mathbf{10} + \mathbf{5} = \mathbf{25}$ Now,

$$\boldsymbol{CXXV} \div \boldsymbol{XXV} = \mathbf{125} \div \mathbf{25} = \mathbf{5} = \boldsymbol{V}$$

✎ **Exercise 1.**

1. Add ***DCIX*** + ***MCII***. Give the answer is Roman Numerals.
2. Subtract ***LXIII*** from ***CLVII***. Write the answer in Roman numerals.
3. Find the product of the Roman numerals ***LIX*** and ***XIV***.

Definition

In a multiplicative grouping system some integer $\boldsymbol{b} > \mathbf{1}$ is selected for the base, a first set of symbols are chosen for the values $\mathbf{1}, \mathbf{2}, \cdots, \boldsymbol{b} - \mathbf{1}$, and a second set of symbols are chosen for $\boldsymbol{b}, \boldsymbol{b}^{\mathbf{2}}, \boldsymbol{b}^{\mathbf{3}}, \cdots$. Then a number is expressed using the symbols multiplicatively, with a symbol from the first set giving the number of units from the second set (followed by an understood summation).

The traditional Chinese numeral system is a multiplicative grouping system to base **10**. Writing vertically, the symbols of the two basic groups are shown below

0	零		
1	一		
2	二		
3	三		
4	四		
5	五		
6	六	100	一百
7	七	1000	一千
8	八	10000	一万
9	九		

Example

5625	9327	10327
五千六百二十五	九千三百二十七	一万零三百二十七

- Like many ancient languages, **Greek** employed a type of numbering system now called a “ciphered numeral system.” Using base **10**, the Greeks represented numbers with letters from their alphabet.
- The so-called **Ionic**, or alphabetic, **Greek numeral system** can be traced as far back as about 450 BC.
- It is a system that is based on **10** and employs twenty-seven characters–the twenty-four letters of the Greek alphabet together with the symbols for the obsolete digamma, koppa, and sampi.
- Although the capital letters were used (the small letters were substituted much later), we shall now illustrate the system with the small letters. The following equivalents had to be memorized.

Ionic Number Systems

Numeral	Value	Name	Numeral	Value	Name	Numeral	Value	Name
α	**1**	*alpha*	ι	**10**	*iota*	ρ	**100**	*rho*
β	**2**	*beta*	κ	**20**	*kappa*	σ	**200**	*sigma*
γ	**3**	*gamma*	λ	**30**	*lambda*	τ	**300**	*tau*
δ	**4**	*delta*	μ	**40**	*mu*	υ	**400**	*upsilon*
ϵ	**5**	*epsilon*	ν	**50**	*nu*	ϕ	**500**	*phi*
Ϝ	**6**	*digamma*	ξ	**60**	*xi*	χ	**600**	*chi*
ζ	**7**	*zeta*	o	**70**	*omicron*	ψ	**700**	*psi*
η	**8**	*eta*	π	**80**	*pi*	ω	**800**	*omega*
θ	**9**	*theta*	ϙ	**90**	*koppa*	ϡ	**900**	*sampi*

Accompanying bars or accents were used for larger numbers. Thus the numbers **1000**, **2000**, and **3000** would appear as α', β', and γ', respectively.

Example

$$\mathbf{12} = \iota\beta\,, \qquad \mathbf{21} = \kappa\alpha\,, \qquad \mathbf{247} = \sigma\mu\zeta\,, \qquad \mathbf{4537} = \delta'\phi\lambda\zeta\,.$$

✎ Exercise 2.

❶ How would these numbers appear in Hindu-Arabic systems?

a) $\boldsymbol{\pi\eta}$ b) $\boldsymbol{\tau o \delta}$ c) $\boldsymbol{\zeta'}$ϡ$\boldsymbol{\eta}$

❷ How would these numbers be written in Greek?

a) **46** b) **838** c) **9697**

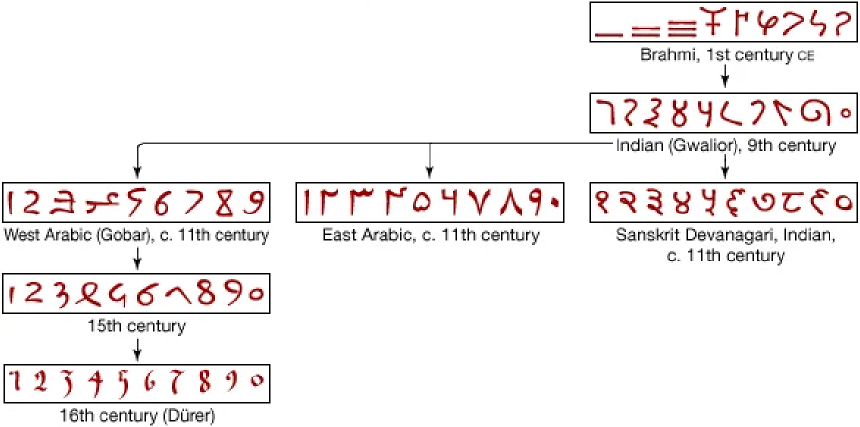
Definition (Positional systems)

- In a **positional system** no multiplier is needed. The value of the symbol is understood by its position in the number.
- In a **positional number system**, the value of each (symbol) digit determined by which place it appears in the full number.
- A true **positional number system** requires a base b and a set of symbols, including a symbol for zero and one for each counting number less than the base. $0, 1, 2, \cdots, b-1$.
- Any number N can be written uniquely in the form

$$N = a_n b^n + a_{n-1} b^{n-1} + \cdots + a_1 b + a_0 \Longleftrightarrow N = (a_n a_{n-1} \cdots a_1 a_0)_b \,,$$

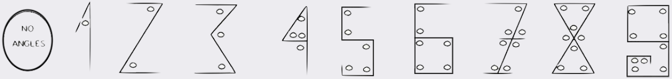
where $0 \leqslant a_i < b$, $i = 0, 1, \cdots, n$.

The most common place-value system is the base **10** system. It is called the **decimal number system**.



Definition (Hindu-Arabic Numeration System)

The numeration system we use today is called the **Hindu-Arabic** system. It uses **10** symbols called **digits**:



This is a positional system since the position of each digit indicates a specific value. The lowest place value is the **rightmost** position, and each successive position to the left has a **higher place value**.

billion	hundred million	ten million	million	hundred thousand	ten thousand	thousand	hundred	ten	one
10^9	10^8	10^7	10^6	10^5	10^4	10^3	10^2	$10^1 = 10$	$10^0 = 1$

Example

The number **82, 653** means there are **8** ten thousands, **2** thousands, **6** hundreds, **5** tens, and **3** ones. We say that the place value of the **6** in this numeral is hundreds.

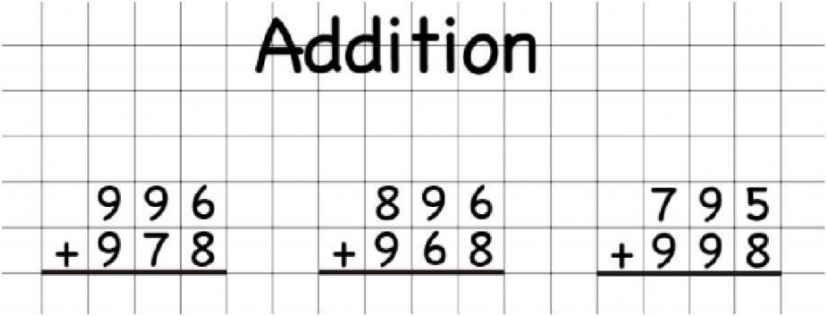
- To clarify the place values, Hindu-Arabic numbers are sometimes written in **expanded notation**. To evaluate a numeral in this system,
  - Multiply the first digit on the right by $\mathbf{10^0}$ or **1**.
  - Multiply the second digit from the right by base **10**.
  - Multiply the third digit from the right by base $\mathbf{10^2}$ or **100**, and so on.
  - In general, we multiply the digit $\boldsymbol{n}$ places from the right by $\mathbf{10^{n-1}}$ to show a number in expanded form.

Example (Expanded notation)

$$\mathbf{32,569 = 3 \times 10^4 + 2 \times 10^3 + 5 \times 10^2 + 6 \times 10^1 + 9 \times 10^0 .}$$

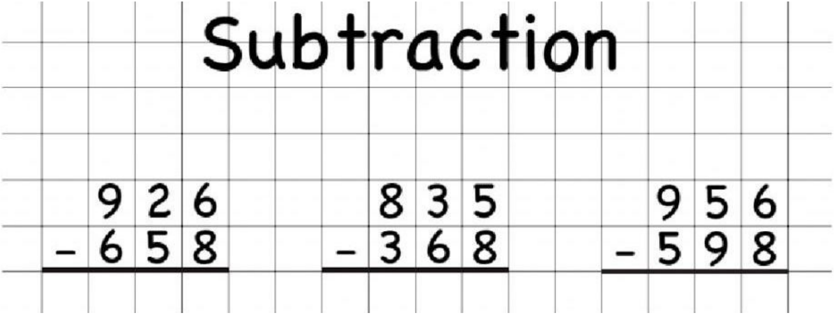
★ Hindu-Arabic Numeration System ★

Operations on Hindu-Arabic Numerals



★ Hindu-Arabic Numeration System ★

Operations on Hindu-Arabic Numerals



★ Hindu-Arabic Numeration System ★

Operations on Hindu-Arabic Numerals

Multiplication

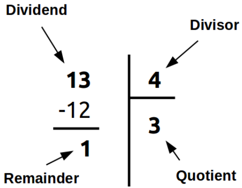
$$\begin{array}{r} 21 \\ \times 32 \\ \hline \end{array} \qquad \begin{array}{r} 42 \\ \times 12 \\ \hline \end{array} \qquad \begin{array}{r} 53 \\ \times 52 \\ \hline \end{array} \qquad \begin{array}{r} 84 \\ \times \ \ 2 \\ \hline \end{array}$$

★ Hindu-Arabic Numeration System ★

Euclidean Division on Hindu-Arabic Numerals

Each part of a division has a name:

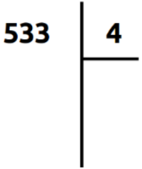
- Dividend: The dividend is the number you are dividing up.
- Divisor: The divisor is the number you are dividing by.
- Quotient: The quotient is the main result.
- Remainder: The left over.



★ Hindu-Arabic Numeration System ★

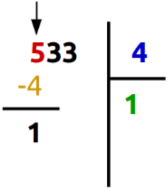
Euclidean Division on Hindu-Arabic Numerals

Starting point



We start the process using the most left digit of the dividend.

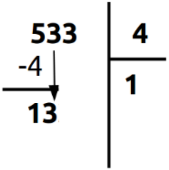
A. How many times do we have 4 (blue) in 5 (red)?



In five, we may count to four only 1 time (green).

We put this number in the quotient box (green) and **perform the subtraction (yellow):**
5 - (4 x 1) = 1

B. Bring down the next digit of the dividend.

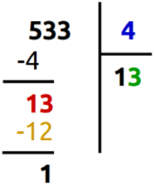


All we have to do now is to bring down the next digits of the dividend and repeat from A.

★ Hindu-Arabic Numeration System ★

Euclidean Division on Hindu-Arabic Numerals

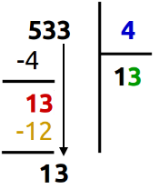
A. How many times do we have 4 (blue) in 13 (red)?



In thirteen, we may count to four only 3 times (green).

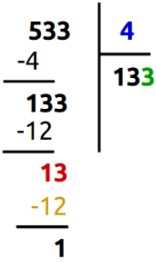
We put this number in the quotient box (green) and **perform the subtraction (yellow):**
13 - (4 x 3) = 1

B. Bring down the next digit of the dividend.



Just like before (step B), we bring down the next digits of the new dividend and repeat from A.

A. Finished!



Here we perform the same operations as before...

Once the new dividend is less than the divisor, our process is over!

★ Hindu-Arabic Numeration System ★

Euclidean Division on Hindu-Arabic Numerals

Example

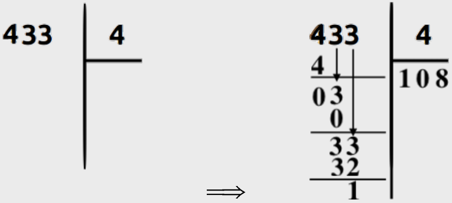
$$\begin{array}{r|l} 433 & 4 \\ \hline & \end{array}$$

$\Longrightarrow$

★ Hindu-Arabic Numeration System ★

Euclidean Division on Hindu-Arabic Numerals

Example



★ Positional numeral systems ★

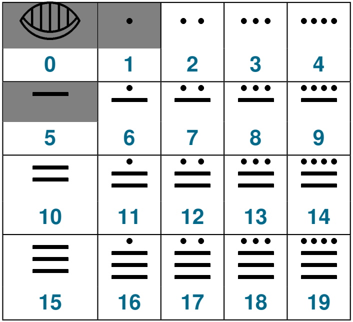
Mayan Numeration System

Definition (Mayan Numeration System)

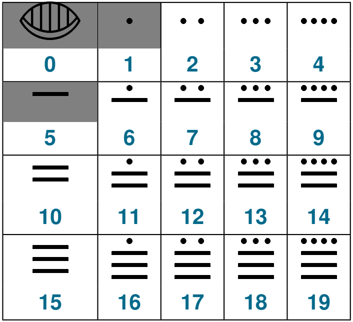
- The Mayans have lived in the Yucatan peninsula of Central America from about **400** AD until the present time.
- They were advanced mathematicians and the one of the only ancient civilizations to use a zero.
- They used a base **20** (vigesimal) number system.
- Their number system has three symbols:
  - shell to represent **0**
  - dot to represent **1**
  - line (or bar) to represent **5**

zero one five

- These twenty numbers (**0** through **19**) can be written with bars and dots
  - **1** to **4** are represented by dots.
  - Multiples of five are represented by lines, with extra dots being added to complete the numbers as shown.
- To write numbers higher than **19**, a system of PLACE NOTATION, based on units of TWENTY, is used.



- These twenty numbers (**0** through **19**) can be written with bars and dots
  - **1** to **4** are represented by dots.
  - Multiples of five are represented by lines, with extra dots being added to complete the numbers as shown.
- To write numbers higher than **19**, a system of PLACE NOTATION, based on units of TWENTY, is used.



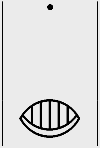
Can you predict what the numeral is for **20**?

Base **10**	Base **20**
▸ Our own number system is base **10**, which means that we have **9** 'numerals' (**1, 2, 3, 4, 5, 6, 7, 8, 9**) plus a zero. ▸ When writing numbers, once we get to '**9**' we then have to move across to the next column. We write a 'one' followed by a 'zero' to show that we have moved across – zero is a 'place-holder'.	▸ The Maya number system was base **20**, which means that they had **19** numerals plus a shell for zero. ▸ The Maya used a similar system using their **19** numerals and then moving to the next section and putting a zero (represented by the shell) as a placeholder.

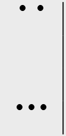
- In the Maya civilization, the writing of numbers is generally vertical (the units are placed under the tens/twenties, under the (four-)hundreds etc.) each row has the value of a power of **20** (**1**, **20**, **400**, **8000**, etc.).
- The number of units are MULTIPLIED by the value of the row, and the results are summed for the entire column.

The following are examples of how to write different numbers using this system.

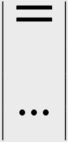
Example



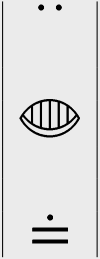
$$\left.\begin{array}{l} = \mathbf{1} \times \mathbf{20} = \mathbf{20} \\ = \mathbf{0} \times \mathbf{1} = \mathbf{0} \end{array}\right\} = \mathbf{20}$$



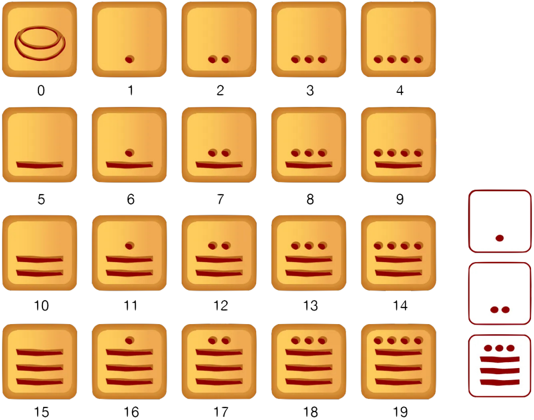
$$\left.\begin{array}{l} = \mathbf{2} \times \mathbf{20} = \mathbf{40} \\ = \mathbf{3} \times \mathbf{1} = \mathbf{3} \end{array}\right\} = \mathbf{43}$$



$$\left.\begin{array}{l} = \mathbf{10} \times \mathbf{20} = \mathbf{200} \\ = \mathbf{3} \times \mathbf{1} = \mathbf{3} \end{array}\right\} = \mathbf{203}$$



$$\left.\begin{array}{l} = \mathbf{2} \times \mathbf{400} = \mathbf{800} \\ = \mathbf{0} \times \mathbf{20} = \mathbf{0} \\ = \mathbf{11} \times \mathbf{1} = \mathbf{11} \end{array}\right\} = \mathbf{811}$$


0
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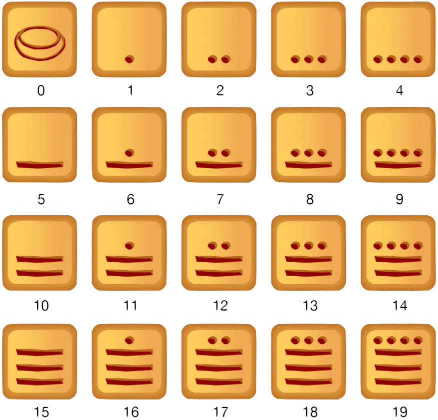
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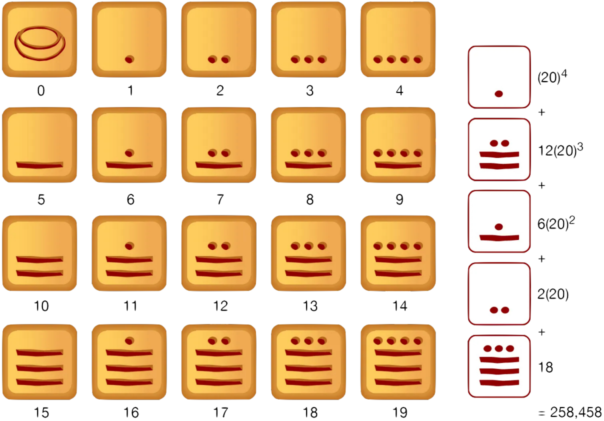

0 1 2 3 4
5 6 7 8 9
10 11 12 13 14
15 16 17 18 19
$(20)^4$
+
$12(20)^3$
+
$6(20)^2$
+
$2(20)$
+
18
= 258,458

★ Mayan Numeration system ★

Hindu-Arabic system to Mayan

To convert **Hindu-Arabic** numerals to **Mayan** numerals, we Successively dividing quotients by **20**.

Example

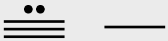
Write **745** and **4025** as a Mayan numeral.

solution

$$\begin{aligned} 745 &= 20 \cdot 37 + 5 \\ 37 &= 20 \cdot 1 + 17 \\ 1 &= 20 \cdot 0 + 1 \end{aligned}$$

The successive remainders that we have found, **5**, **17** and **1**, are the digits from the right to the left of **745** in base **20**. Hence,

$$745 = (1, 17, 5)_{20} = 1 \times 20^2 + 17 \times 20 + 5 \times 1 = \bullet$$



$$4025 = \cdots\cdots\cdots$$

★ Mayan Numeration system ★

Hindu-Arabic system to Mayan

To convert **Hindu-Arabic** numerals to **Mayan** numerals, we Successively dividing quotients by **20**.

Example

Write **745** and **4025** as a Mayan numeral.

solution

$$
\begin{aligned}
\mathbf{745} &= \mathbf{20 \cdot 37 + 5} \\
\mathbf{37} &= \mathbf{20 \cdot 1 + 17} \\
\mathbf{1} &= \mathbf{20 \cdot 0 + 1}
\end{aligned}
$$

The successive remainders that we have found, **5**, **17** and **1**, are the digits from the right to the left of **745** in base **20**. Hence,

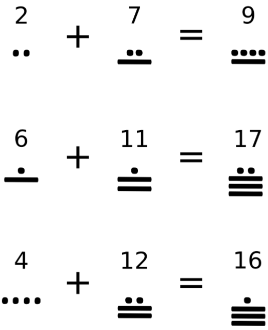
$$\mathbf{745 = (1, 17, 5)_{20} = 1 \times 20^2 + 17 \times 20 + 5 \times 1 =}$$

$$\mathbf{4025 = (11, 3, 5)_{20} =}$$

★ Mayan Numeration system ★

Addition

- Adding and subtracting in the Mayan system is simply a matter of juggling the dots and bars.
- We use the same algorithms for calculating with Hindu-Arabic numbers work with Maya numbers as well.
- Addition is performed by combining the numeric symbols at each level and we replace **5** dots by one bar.

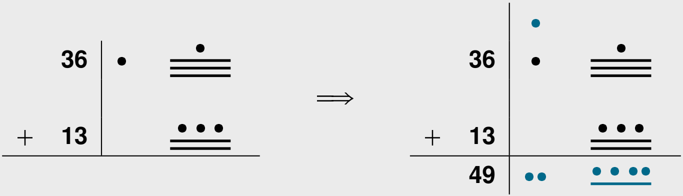


★ Mayan Numeration system ★

Addition

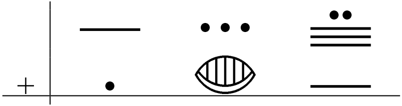
Example

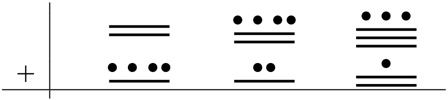
To calculate **36** + **13**, for example, you start by adding the units (i.e., **16** + **13**). This gives you **29**, so you leave **9** in the ones column and carry the **20** up, giving you a grand total of **2** twenties and **9** ones = **49**



★ Mayan Numeration system ★

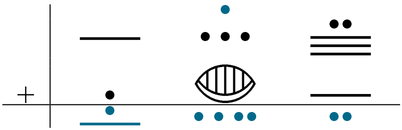
Addition

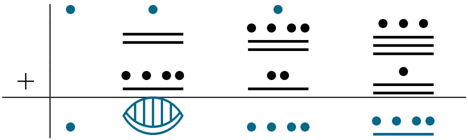




★ Mayan Numeration system ★

Addition

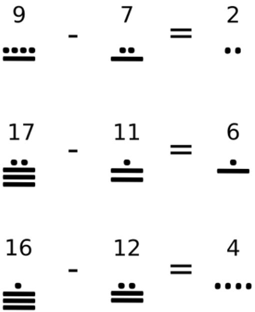




★ Mayan Numeration system ★

Subtraction

- Adding and subtracting in the Mayan system is simply a matter of juggling the dots and bars.
- We use the same algorithms for calculating with Hindu-Arabic numbers work with Maya numbers as well.
- Subtraction is performed by removing the numeric symbols.

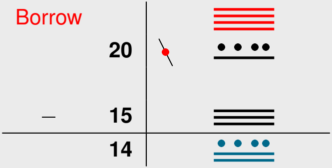


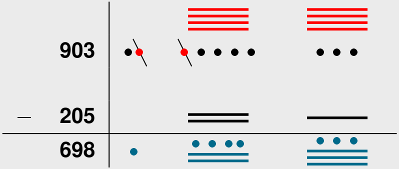
★ Mayan Numeration system ★

Subtraction

Example

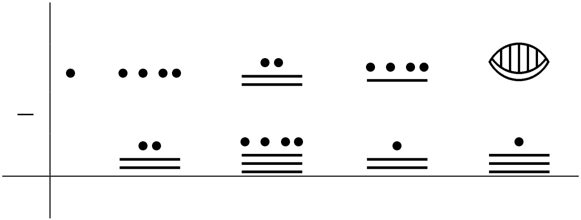
We begin with the units. Since the units of the minuend (i.e. the number from which we are subtracting) are less than the units of the subtrahend (the number being subtracted), we have to borrow from the twenties, just as is the case of HA numbers. After borrowing, the problems looks like this:





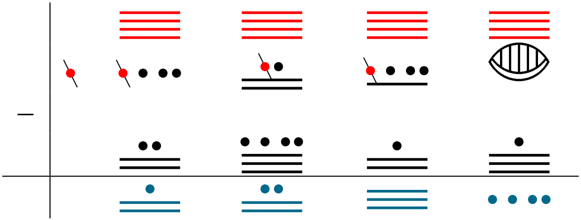
★ Mayan Numeration system ★

Subtraction



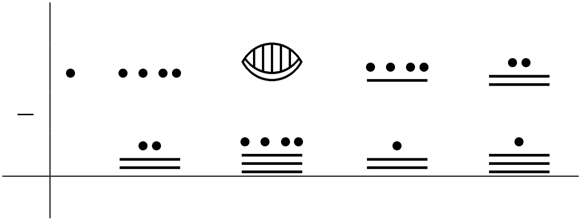
★ Mayan Numeration system ★

Subtraction



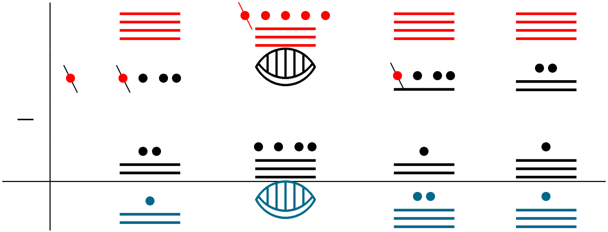
★ Mayan Numeration system ★

Subtraction

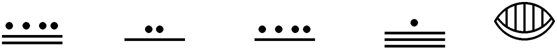


★ Mayan Numeration system ★

Subtraction



Exercise 3. Write the following Mayan numerals in decimal notation (base 10).

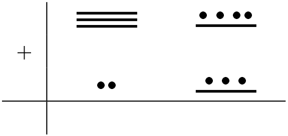
1. 
2. 
3. 

Exercise 4. Write the following in Mayan numerals.

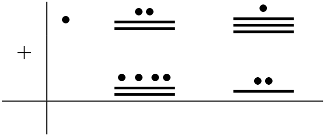
1. **238**
2. **1280**
3. **239836**

✎ **Exercise 5.** Add the following (without converting to decimal form).

❶



❷



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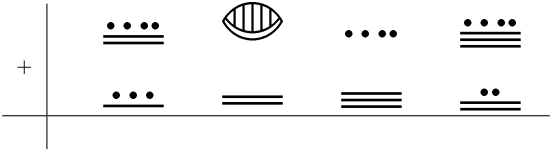
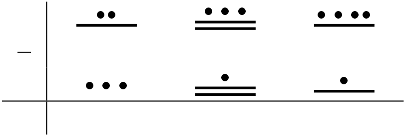
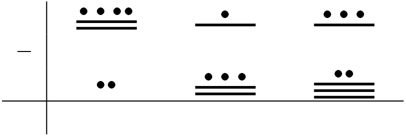


✎ **Exercise 6.** Subtract the following (without converting to decimal form).

❶



❷



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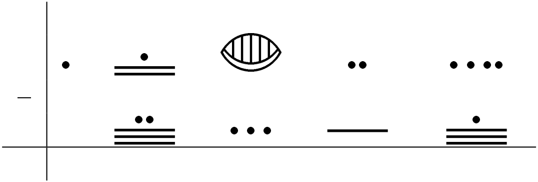
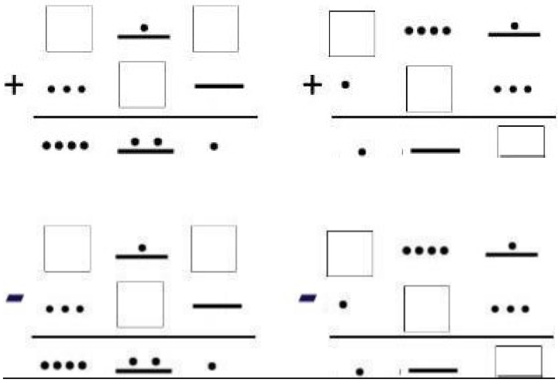


✎ Exercise 7.

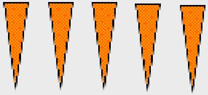
Find the sum and the subtraction of the Mayan numerals:



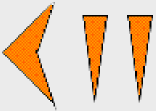
Definition (Babylonian Numeration System)

The **Babylonian numeration system** was developed between **3000** and **2000** BCE. It uses three numerals or symbols, a **one**, **Zero** and a **ten** to represent numbers and they looked this these

Example



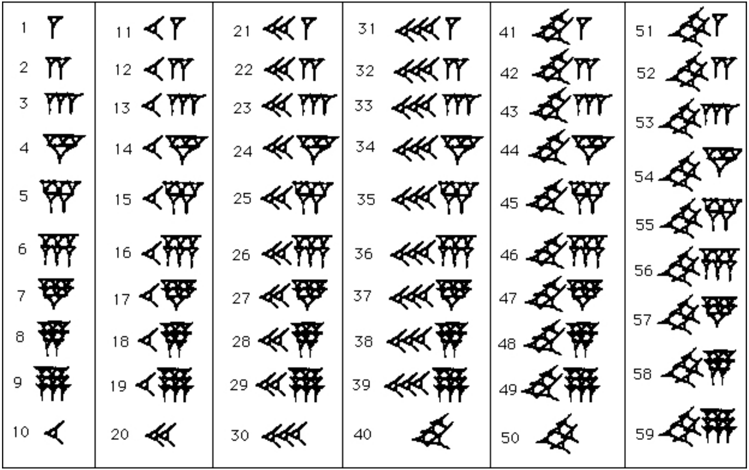
5



12

45

To represent numbers from **2** to **59**, the system was simply additive

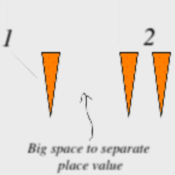


★ Babylonian Numeration System ★

Convert Babylonian Numbers to our numbers

- The ancient Babylonian system is sort of a cross between a multiplier system and a positional system.
- You might think it would be cumbersome to write large numbers in this system; the Babylonian system was also positional in base **60**.
- Numbers from **1** to **59** were written using the two symbols, but after the number **60** to represent numbers, use several groups of symbols, separated by spaces, and multiply the value of these groups by increasing powers of **60**.

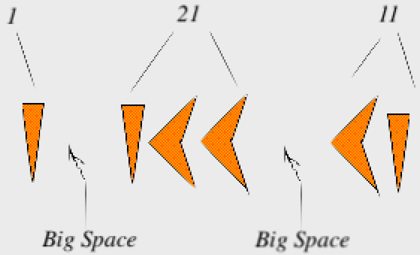
Example



$$= \mathbf{1} \times \mathbf{60}^{\mathbf{1}} + \mathbf{2} \times \mathbf{60}^{\mathbf{0}} = \mathbf{62} \Longrightarrow (\mathbf{1}, \mathbf{2})_{\mathbf{60}} = \mathbf{62}$$

Example

Find the value:



$$= 1 \times 60^2 + 21 \times 60^1 + 11 \times 60^0$$
$$= 3600 + 1260 + 11 = 4871$$

Thus $(1, 21, 11)_{60} = 4871$

Example

Convert to Hindu-Arabic notation.

Solution

Example

Convert to Hindu-Arabic notation.

Solution

$$(2, 31, 23)_{60} = 2 \times 60^2 + 31 \times 60^1 + 23 \times 60^0 = 9083\,.$$

★ Babylonian Numeration system ★

Convert our numbers to Babylonian numbers

To convert **Hindu-Arabic** numerals to **Babylonian** numerals, divide by powers of **60**, similar to the way seconds are converted to hours and minutes.

Example

Write **745** as a Babylonian numeral.

solution

Successively dividing quotients by **60** gives

$$\begin{aligned} \mathbf{745} &= \mathbf{60} \cdot \mathbf{12} + \mathbf{25} \\ \mathbf{12} &= \mathbf{60} \cdot \mathbf{0} + \mathbf{12} \end{aligned}$$

The successive remainders that we have found, **25** and **12**, are the digits from the right to the left of **745** in base **60**. Hence, $\mathbf{745} = \mathbf{12} \times \mathbf{60} + \mathbf{25} \times \mathbf{1} = (\mathbf{12}, \mathbf{25})_{\mathbf{60}} =$ 𒌋𒁹𒁹 𒌋𒌋𒁹𒁹𒁹𒁹𒁹.

Example

Write **12, 221** as a Babylonian numeral.

solution

Example

Write $\mathbf{12,221}$ as a Babylonian numeral.

solution

Successively dividing quotients by $\mathbf{60}$ gives

$$\begin{aligned}
\mathbf{12221} &= \mathbf{60 \cdot 203 + 41} \\
\mathbf{203} &= \mathbf{60 \cdot 3 + 23} \\
\mathbf{3} &= \mathbf{60 \cdot 0 + 3}
\end{aligned}$$

The successive remainders that we have found, $\mathbf{41}$, $\mathbf{23}$, and $\mathbf{3}$, are the digits from the right to the left of $\mathbf{12221}$ in base $\mathbf{60}$. Hence,

$$\mathbf{12,221 = 3 \times 60^2 + 23 \times 60 + 41 = (3,23,41)_{60}} =$$


What about $\mathbf{7223}$?

Example

Write **12, 221** as a Babylonian numeral.

solution

Successively dividing quotients by **60** gives

$$\begin{aligned}
12221 &= 60 \cdot 203 + 41 \\
203 &= 60 \cdot 3 + 23 \\
3 &= 60 \cdot 0 + 3
\end{aligned}$$

The successive remainders that we have found, **41**, **23**, and **3**, are the digits from the right to the left of **12221** in base **60**. Hence,

$$12,221 = 3 \times 60^2 + 23 \times 60 + 41 = (3, 23, 41)_{60} =$$ 

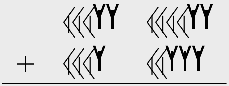
$7223 =$ 

★ Babylonian Numeration system ★

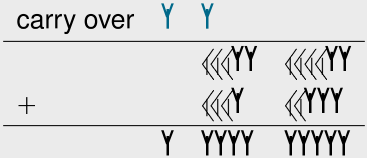
Addition

Example

Find the sum of the Babylonian numeral. Write the answer as a Babylonian numeral.



solution

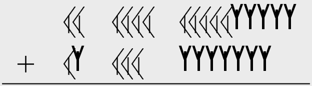


★ Babylonian Numeration system ★

Addition

Example

Find the sum of the Babylonian numeral. Write the answer as a Babylonian numeral.



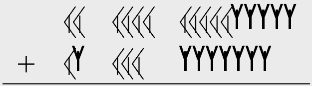
solution

★ Babylonian Numeration system ★

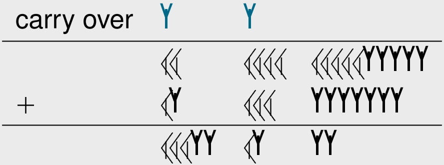
Addition

Example

Find the sum of the Babylonian numeral. Write the answer as a Babylonian numeral.



solution

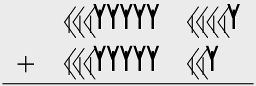


★ Babylonian Numeration system ★

Addition

Example

Find the sum of the Babylonian numeral. Write the answer as a Babylonian numeral.



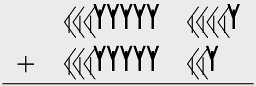
solution

★ Babylonian Numeration system ★

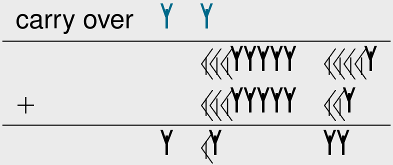
Addition

Example

Find the sum of the Babylonian numeral. Write the answer as a Babylonian numeral.



solution



★ Babylonian Numeration system ★

Subtraction

The Babylonians used the symbol  for subtraction.

For example, the numeral  represents **20** − **3** = **17**.

Example

write the Babylonian numeral as a Hindu-Arabic numeral

solution

★ Babylonian Numeration system ★

Subtraction

The Babylonians used the symbol  for subtraction.

For example, the numeral  represents **20** $-$ **3** $=$ **17**.

Example

write the Babylonian numeral as a Hindu-Arabic numeral

solution

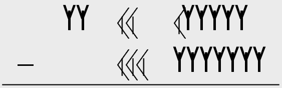
$$\mathbf{10} - \mathbf{2} = \mathbf{8} \qquad \mathbf{60} - \mathbf{3} = \mathbf{57} \qquad \mathbf{50} - \mathbf{4} = \mathbf{46}\,.$$

★ Babylonian Numeration system ★

Subtraction

Example

Find the subtraction of the Babylonian numeral. Write the answer as a Babylonian numeral.



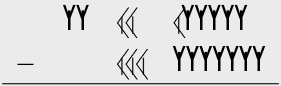
solution

★ Babylonian Numeration system ★

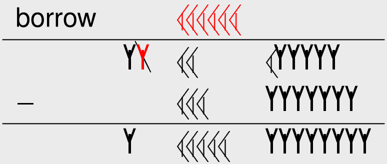
Subtraction

Example

Find the subtraction of the Babylonian numeral. Write the answer as a Babylonian numeral.



solution



✎ Exercise 8.

Convert the numbers expressed in Babylonian numeral to our Hindu-Arabic number system (Standard form).

Babylonian Numerals	Written in Srandard Form
𒌋𒌋𒌋𒌋𒌋𒁹𒁹𒁹𒁹𒁹𒁹𒁹𒁹𒁹 𒁹𒁹𒁹𒁹	
𒁹𒁹𒁹𒁹𒁹𒁹𒁹 𒌋𒌋𒌋𒌋𒁹𒁹𒁹𒁹 𒌋𒌋	
𒌋𒌋𒌋 𒌋𒁹𒁹 𒌋𒌋𒌋𒁹𒁹𒁹	

✎ Exercise 9.

Find the subtraction of the Babylonian numeral:

$$\begin{array}{rcccc} & 𒁹𒁹𒁹 & 𒁹𒁹𒁹 & 𒌋𒁹𒁹𒁹 & 𒌋 \\ - & 𒁹𒁹 & 𒌋𒌋𒌋𒁹 & 𒌋 & 𒌋𒌋𒌋𒁹 \\ \hline \end{array}$$

Thank you! Questions?