Exercise 5.12 [**] *(Taken from the 2019-2020 final exam)*

A pair of trousers is held together by a pair of braces and a belt.

The lifetimes of the braces and the belt are given by continuous random variables T_1 and T_2 following exponential distributions with parameters λ_1 and λ_2.

It is assumed that the trousers are held in place as long as at least one of the two accessories, braces or belt, is functional. Let T be the continuous random variable “duration of the trousers been held”

a) Find the cumulative distribution functions F_1 and F_2 of T_1 and T_2.

b) Compute $P(T_1 > 2t)$ for $t > 0$.

c) Compute $P(t < T_2 < 2t)$ for $t > 0$.

d) Find the cumulative distribution function of T.

e) Deduce the probability density function of T.

f) Compute the expected values $E(T_1), E(T_2)$ and $E(T)$.

Exercise 5.13 [***] **Memory-less random variables** *(Proof of proposition 5.10)*

A random variable T is *memory-less* if it satisfies

$$\forall s \geq 0, P(T \geq t+s \big| T \geq t) = P(T \geq s).$$

a) Show that an exponential random variable is memory-less.

b) Conversely, let us show that if T is a memory-less continuous random variable, then it must be exponential.

  i) Prove that the function $G_T(t) = P(T \geq t)$ satisfies

$$\forall t, s \geq 0, G_T(t+s) = G_T(t) G_T(s).$$

  ii) Deduce from this the expression of G_T and then those of the cumulative distribution function F_T and the probability density function of T.

♦ **Exercise** 5.14 [**] **Minimum et maximum de deux lois exponentielles**

This exercise is taken from M. Ghassany’s probability course at EFREI in 2022/2023.

he goal of this exercise is to solve the following problem: two counters are open at a bank. The service time T_1 at the first counter (respectively, T_2 at the second counter) follows an exponential distribution with mean 5 minutes (respectively, 8 minutes). The two counters are assumed to be independent. Two customers enter simultaneously, one choosing counter 1 and the other counter 2. We want to find the mean time δT after which the first customer exits and the mean time ΔT after which the last customer exits.

a) Introduce the variable $Z = min(T_1, T_2)$. Express the cumulative distribution function of Z in terms of those of T_1 and T_2.

b) Deduce from this that Z is an exponential random variable of parameter $\lambda = 40/13$.

c) Find δT and ΔT.

Hint: for ΔT, use the fact $T_1 + T_2 = min(T_1, T_2) + max(T_1, T_2)$.

Exercise 5.15 [**] **Exponential and uniform random variables**

Let X be a uniform continuous random variable over $[0, 1]$. Show that the random variable $Y = -ln(X)$ is exponential.

♦ **Exercise 5.16** [*]

Let X be a normal random variable of parameters $\mu = 10$ and $\sigma = 2$.

a) Compute $P(X \leq 15)$.

b) Compute $P(|X| \leq 8)$.

c) Compute $P(8 \leq X \leq 15)$.

♦ **Exercise 5.17** [*]

Suppose that the distribution of cholesterol levels in a population of children (in cg) follows a normal distribution with mean μ and standard deviation σ.

Studies show that 57,9% of these children have cholesterol levels below 165cg and that 4,5% have cholesterol levels above 180cg.

a) Compute μ and σ.

b) Children whose cholesterol levels are 183cg must follow a treatment. hat percentage of the population should be treated?

♦ **Exercise 5.18** [*] **Score de Maddrey**

In patients with severe damage, the Maddrey score is A random variable X which is distributed according to a normal distribution $\mathcal{N}(\mu = 54, \sigma^2 = 14^2)$, whereas in patients with minimal damage, the Maddrey score is A random variable Y which is distributed according to a normal distribution $\mathcal{N}(\mu = 20, \sigma^2 = 6^2)$.

a) For a patient with severe damage, compute the probability that his or her Maddrey score is below 68.

b) For a patient with minimal damage, compute the probability that his or her Maddrey score is above 8.