School of Engineering and Computer Science

Mathematical tools applied to Computer Science

Ch5: Matrix calculus and analysis

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Week #**11** ✦ 21/NOV/2024 ✦

Solving Systems with Gauss-Jordan Elimination

Solving Systems with Gauss-Jordan Elimination

Definition

A matrix is a rectangular collection of numbers. Generally, matrices are denoted as bold capital letters.

Example

$$A = \begin{pmatrix} 1 & -5 & 4 \\ 2 & 5 & 3 \end{pmatrix}$$

A is a matrix with two rows and three columns. For that reason, it is called a 2 by 3 matrix. This is called the **dimension** of a matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$ is a matrix of size $(m \times n)$ over $\mathbb{R}$.

- $a_{ij} \in \mathbb{R}$ are called the entries of A (or the (i, j)-th entry of A) that describe the location of the elements of the matrix A
- $a_{ii} \in \mathbb{R}$ are called the diagonal entries of A, where $1 \leq i \leq m$ and $1 \leq j \leq n$.

Definition

A matrix A of size $(m \times n)$ is said to be

1. a **row matrix** if A has just one row (i.e $m = 1$).
2. a **column matrix** if A has just one column (i.e $n = 1$).
3. a **square matrix** of order n if A is a $(n \times n)$ matrix (i.e number of columns equal to that of rows $m = n$).
4. a **zero matrix** if the all entries equal to zero (i.e $a_{ij} = 0$ for any $1 \leq i \leq m$ and $1 \leq j \leq n$).

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Example

① $A = [1\ 2\ 3]$ is a row matrix of size (1×3).

② $B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a column matrix of size (3×1).

③ $C = \begin{pmatrix} 1 & 3 \\ 0 & 6 \end{pmatrix}$ is a square matrix of size (2×2).

④ $D = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 6 & 1 \end{pmatrix}$ is a matrix of size (2×3).

⋆ Transpose a Matrix ⋆

To take the **transpose** of a matrix, simply switch the rows and column of a matrix. The transpose of A can be denoted as A' or A^T.

Example

If $A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 0 \\ 2 & 1 & 2 \end{pmatrix}$ the $A^T = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 1 \\ -1 & 0 & 2 \end{pmatrix}$

If a matrix is its own transpose, then that matrix is said to be symmetric $A^T = A$. Symmetric matrices must be square matrices, with the same number of rows and columns.

Example

$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 0 & 2 \end{pmatrix}$ is symmetric because $A^T = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 3 & 0 \\ -1 & 0 & 2 \end{pmatrix} = A$

⋆ Matrix Addition ⋆

To perform matrix **addition**, two matrices must have the same dimensions. This means they must have the same number of rows and columns. In that case simply add each individual components, like below.

Example

$$A+B=\begin{pmatrix}1 & 0\\ 0 & 3\\ 4 & 1\end{pmatrix}+\begin{pmatrix}3 & 1\\ 3 & 4\\ 5 & 2\end{pmatrix}=\begin{pmatrix}1+3 & 0+1\\ 0+3 & 3+4\\ 4+5 & 1+2\end{pmatrix}=\begin{pmatrix}4 & 1\\ 3 & 7\\ 9 & 3\end{pmatrix}$$

Matrix addition does have many of the same properties as "normal" addition.

① $(A+B)+C=A+(B+C)$, (Associativity)

② $A+B=B+A$, (Commutativity)

③ $A+O=O+A=A$.

④ $A^T+B^T=(A+B)^T$.

⋆ Matrix Scalar Multiplication ⋆

To multiply a matrix by a **scalar**, also known as scalar multiplication, multiply every element in the matrix by the scalar

Example

If $A = \begin{pmatrix} 5 & -2 & 3 \\ 2 & -3 & 4 \end{pmatrix}$ and $\alpha \in \mathbb{R}$ then $\alpha A = \begin{pmatrix} 5\alpha & -2\alpha & 3\alpha \\ 2\alpha & -3\alpha & 4\alpha \end{pmatrix}$.

Matrix scalar multiplication has some properties, such as

① $\alpha(A + B) = \alpha A + \alpha B$,

② $(\alpha + \beta)A = \alpha A + \beta A$,

③ $\alpha(\beta A) = (\alpha\beta A)$.

④ $\alpha A^T = (\alpha A)^T$.

☆ Matrix Multiplication ☆

We start by special case when we have two vectors. To multiply two vectors with the same length together is to take the **dot product**, also called **inner product**. This is done by multiplying every entry in the two vectors together and then adding all the products up.

Example

$$x \cdot y = xy^T = (1 \ -5 \ 4)\begin{pmatrix} 4 \\ -2 \\ 5 \end{pmatrix} = 1 * 4 + (-5) * (-2) + 4 * 5 = 4 + 10 + 20 = 34$$

- To perform matrix multiplication, the first matrix must have the same number of columns as the second matrix has rows.

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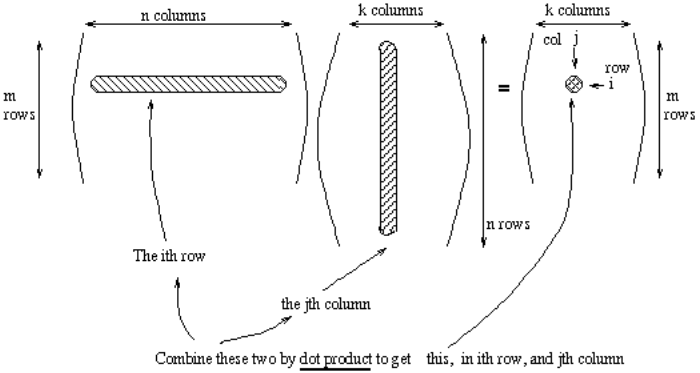
If A is an $m \times r$ matrix and B is an $r \times n$ matrix, then AB will be an $m \times n$ matrix.

$$\underset{m \times r}{A} \qquad \underset{r \times n}{B} = \underset{m \times n}{AB}$$

insides match

outsides give size of AB

- To perform matrix multiplication, the first matrix must have the same number of columns as the second matrix has rows.
- The number of rows of the resulting matrix equals the number of rows of the first matrix, and the number of columns of the resulting matrix equals the number of columns of the second matrix.
- To find the entries in the resulting matrix, simply take the dot product of the corresponding row of the first matrix and the corresponding column of the second matrix.



$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3} \qquad \times \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times 2}$$

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$$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \\ c_{31} & c_{32} \end{bmatrix}_{{\color{blue}3} \times {\color{blue}2}}$$

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Example

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 5 \\ 3 & 4 & 2 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 & 2 & 4 \\ 5 & 3 & 1 \\ 6 & 2 & 1 \end{bmatrix}_{3\times 3} = \begin{bmatrix} \square & \square & \boxed{12} \\ \square & \square & \square \\ \square & \boxed{22} & \square \end{bmatrix}_{3\times 3}$$

$(3)(2)+(4)(3)+(2)(2)=22$ $\quad$ $(2)(4)+(3)(1)+(1)(1)=12$

Example

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$(3)(2)+(4)(3)+(2)(2)=22$ $\quad$ $(2)(4)+(3)(1)+(1)(1)=12$

Matrix multiplication has some of the same properties as "normal" multiplication , such as

① $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$ (Distributivity)

② $AO = OA = O$ and $AI = IA = A$,

③ $(\alpha A)B = A(\alpha B) = \alpha(AB)$. (Product with a scalar)

④ $(AB)^T = B^T A^T$,

However matrix multiplication is not commutative $AB \neq BA$

Example

We consider the following real matrices

$$A = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$$

Calculate AB, BA. What do you conclude.

Example

We consider the following real matrices

$$A = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$$

Calculate AB, BA. What do you conclude.

Solution

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad BA = \begin{pmatrix} 6 & 3 \\ -12 & -6 \end{pmatrix}.$$

We conclude that $AB \neq BA$ and $AB = O_2$ even $A \neq O_2$ and $B \neq O_2$.

✎ **Wooclap 1.** Let A, B, C and D be four matrices such that:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 4 & -4 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 2 & 5 \\ 0 & 1 & 2 \\ -3 & 0 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 5 & -1 & 3 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\text{and} \quad D = \begin{pmatrix} 3 & 2 \\ 1 & -1 \end{pmatrix}.$$

Calculate (if possible) AB, BA, AD and DC.

✎ **Wooclap 2.** Find the real numbers x, y, z and t in the following cases:

① $$\begin{bmatrix} 3x+y & x-3y \\ 4z-2t & z+t \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$$

② $$2 \begin{bmatrix} x & y \\ z & t \end{bmatrix} - 5 \begin{bmatrix} x-2 & y+3 \\ z-2 & t+1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}$$

⋆ Matrix Properties ⋆

Identity matrix

An **identity matrix** is a square matrix where every diagonal entry is 1 and all the other entries are 0. The following two matrices are identity matrices and diagonal matrices.

Example

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

They are called identity matrices because any matrix multiplied with an identity matrix equals itself. The diagonal entries of a matrix are the entries where the column and row number are the same. $a_{2,2}$ is a diagonal entry but $a_{3,5}$ is not.

Definition

Let $A \in M_n(\mathbb{R})$. We say that

① A is **lower triangular** matrix if all the elements above the diagonal are zero (i.e. $a_{ij} = 0, \forall i < j$).

$$A \text{ is } \textbf{lower triangular} \iff A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

Example

$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 1 & 0 \end{pmatrix}$ is a lower triangular matrix.

Definition

Let $A \in M_n(\mathbb{R})$. We say that

② A is an **upper triangular** matrix if all the elements below the diagonal are zero (i.e. $a_{ij} = 0, \forall i > j$).

$$A \text{ is } \textbf{upper triangular} \iff A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}.$$

Example

$B = \begin{pmatrix} 1 & -1 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 5 \end{pmatrix}$ is an upper triangular matrix.

Definition

Let $A \in M_n(\mathbb{R})$. We say that

③ A is a **diagonal** matrix if all the elements above and below the diagonal are zero (i.e. $a_{ij} = 0, \forall i \neq j$).

$$A \text{ is } \textbf{diagonal matrix} \iff A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix}.$$

Example

$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$ is a diagonal matrix.

☆ Determinants of a Matrix ☆

Definition

Let A be a square, 2×2 matrix. The determmant of a A is denoted $|A|$ and is given by

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \,.$$

The notation $det(A)$ is also used for the determinant of A.

Example

The determinant of the matrix $A = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$ is
$|A| = (2 \times 1) - (4 \times (-3)) = 14$

Definition (Cofactor)

Let A be a square matrix. The **cofactor** of the element a_{ij} is denoted by C_{ij} and is given by

$$C_{ij} = (-1)^{i+j} M_{ij} \, ,$$

where M_{ij} is the determinant of the matrix A that remains after deleting row i and column j.

Example

Determine the cofactors of the elements a_{11} and a_{32} of $A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$.

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where M_{ij} is the determinant of the matrix A that remains after deleting row i and column j.

Example

Determine the cofactors of the elements a_{11} and a_{32} of $A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & -1 & 2 \\ 0 & -2 & 1 \end{bmatrix}$.

solution

- The cofactor of a_{11} : $C_{11} = (-1)^{1+1} M_{11} = \begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix} = 3$
- The cofactor of a_{32} : $C_{32} = (-1)^{3+2} M_{32} = -\begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 10\,.$

Definition

The determinant of a square matrix is the sum of the products of the elements of any row and their cofactors or any column and their coafactors. If A is $n \times n$, then

- $|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$, $\forall 1 \leq i \leq n$, with respect to the ith row
- $|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$, $\forall 1 \leq j \leq n$, with respect to the jth column

Example

Evaluate the determinant of matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ using the first row.

Definition

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Example

Evaluate the determinant of matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ using the first row.

solution

Using the elements of the first row and their corresponding cofactors we get

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= 1(-1)^2 \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} + (-1)(-1)^4 \begin{vmatrix} 3 & 0 \\ 4 & 2 \end{vmatrix} \\ &= -6\,. \end{aligned}$$

Example

Find the determinant of $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ using the second row. What do you conlclude?

Example

Find the determinant of $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ using the second row. What do you conlclude?

solution

$$|A| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} = -3\begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & -1 \\ 4 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = -6\,.$$

Note that we have already evaluated this determinant in terms of the first row. As is to be expected, we obtained the same value.

✎ **Wooclap 4.** Evaluate the determinant of $\begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 0 & 2 \\ 7 & -2 & 3 & 5 \\ 0 & 1 & 0 & -3 \end{bmatrix}$.

- If the entries of a row (resp. column) of A are zero, then $|A| = 0$.
- $|AB| = |A||B|$
- $|A^T| = |A|$
- $|A^n| = |A|^n$
- $|cA| = c^n|A|$

Example

If A is a 2×2 matrix with $|A| = 4$, use the properties of determinants to compute $|A^2|$ and $|A^T 3A|$.

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Example

If A is a 2×2 matrix with $|A| = 4$, use the properties of determinants to compute $|A^2|$ and $|A^T 3A|$.

Solution

- $|A^2| = |A|^2 = 4 \times 4 = 16.$
- $|A^T 3A| = |A^T||3A| = 3^2|A^T||A| = 144.$

Theorem

The determinant of a triangular matrix is the product of its diagonal elements. Thus

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix}$$

$$= a_{11} \times a_{22} \times \cdots \times a_{nn}.$$

Example

The determinant of the matrix $A = \begin{bmatrix} 2 & 0 & -7 & 4 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & -5 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is $|A| = 2 \times 3 \times (-5) \times 1 = -30$

☆ Matrix Inverse ☆

Definition

A squared matrix A is said to be **invertible** (or **non-singular**) if there exists a matrix B of same dimension, such that $AB = BA = I$. B is called the inverse of A and denoted by A^{-1}

- The notion of an inverse matrix only applies to square matrices.
- Not all square matrices are invertible.

Example

Show that the following pairs of matrices are inverses

a) $\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix}$ and $\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix}$

b) $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix}$ and $\begin{pmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$.

solution

We can easly calculate the product of two matrices and find I

a) $\begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 7 & -4 \\ -5 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 4 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

b) $\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \times \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

c) $\begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix} \times \begin{pmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and
$\begin{pmatrix} 7 & -3 & -3 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 3 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

✎ **Exercise 1.** Let matrix A be given as:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

And let its inverse, A^{-1}, be given as:

$$A^{-1} = \begin{bmatrix} a & -\frac{3}{5} \\ b & +\frac{2}{5} \end{bmatrix}$$

Find the values of a and b.

Let A and B be an invertible matrices and α a nonzero scalar. Then

① $(A^{-1})^{-1} = A$,

② $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$

③ AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

④ A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

⑤ $|A^{-1}| = \frac{1}{|A|}$.

⑥ $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

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Example

If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then $A^{-1} = \dfrac{1}{4-6} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = -\dfrac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$.

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For finding the matrix inverse in general, you can use **Gauss-Jordan Algorithm**. However, this is a rather complicated algorithm, so usually one relies upon the computer or calculator to find the matrix inverse.

⋆ Calculating A^{-1} by Adjoint Matrix Formula ⋆

Definition

Let A be $n \times n$ matrix with $n \geq 2$. We define the **adjoint matrix** of A to be the $n \times n$ matrix over $\mathbb{R}$

$$\operatorname{adj}(A) = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T$$

where for all $i, j \in \{1, \cdots, n\}$, $C_{ij} = (-1)^{i+j} M_{ij}$ is the (i,j)-th cofactor of A, and M_{ij} is the determinant of the matrix A that remains.

Example

Calculate the adjoint of $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$.

Solution

We know that

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

where $C_{ij} = (-1)^{i+j} M_{ij}$ and M_{ij} is the determinant of the matrix A that remains after deleting row i and column j of A.

$$\begin{array}{lll} C_{11} = +\begin{vmatrix} 4 & 0 \\ 0 & 5 \end{vmatrix} = 20 & C_{21} = -\begin{vmatrix} 0 & 3 \\ 0 & 5 \end{vmatrix} = 0 & C_{31} = +\begin{vmatrix} 0 & 3 \\ 4 & 0 \end{vmatrix} = -12 \\ C_{12} = -\begin{vmatrix} 2 & 0 \\ 1 & 5 \end{vmatrix} = -10 & C_{22} = +\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = 2 & C_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 0 \end{vmatrix} = 6 \\ C_{13} = +\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} = -4 & C_{23} = -\begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} = 0 & C_{33} = +\begin{vmatrix} 1 & 0 \\ 2 & 4 \end{vmatrix} = 4 \end{array}$$

hence

$$\operatorname{adj}(A) = \begin{bmatrix} 20 & 0 & -12 \\ -10 & 2 & 6 \\ -4 & 0 & 4 \end{bmatrix}$$

✎ **Exercise 2.** Calculate the adjoint of $B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 4 \\ 0 & 2 & 5 \end{bmatrix}$.

Proposition

Let A be a square matrix of order n. Then

1. $A \cdot \mathrm{adj}(A) = \mathrm{adj}(A) \cdot A = |A| \cdot I_n$.
2. A is invertible if and only if $|A| \neq 0$.
3. If $n \geq 2$ and A is invertible, then $A^{-1} = \dfrac{1}{|A|}\mathrm{adj}(A)$.

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Example

Show that $B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 4 \\ 0 & 2 & 5 \end{bmatrix}$ is invertible then find B^{-1} by calculating $\mathrm{adj}(B)$.

solution

We have

$$|B| = \begin{vmatrix} 1 & 2 & 3 \\ -1 & 1 & 4 \\ 0 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} - 2\begin{vmatrix} -1 & 4 \\ 0 & 5 \end{vmatrix} + 3\begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} = 1 \neq 0$$

Therefore B is invertible. We have already calculate $\mathbf{adj}(B) = \begin{bmatrix} -3 & -4 & 5 \\ 5 & 5 & -7 \\ -2 & -2 & 3 \end{bmatrix}$, then

$$B^{-1} = \frac{1}{|B|}\mathbf{adj}(B) = \begin{bmatrix} -3 & -4 & 5 \\ 5 & 5 & -7 \\ -2 & -2 & 3 \end{bmatrix}$$

✎ **Exercise 3.** Show that $C = \begin{bmatrix} 0 & -1 & 3 \\ 2 & 1 & 5 \\ -4 & 3 & -1 \end{bmatrix}$ is invertible then find C^{-1} by calculating $\mathbf{adj}(C)$.

⋆ Elementary row operations ⋆

- **Purpose of Row Operations:**
  - Simplify matrices to solve systems of linear equations.
  - Find ranks, inverses, and determinants of matrices.

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  - Simplify matrices to solve systems of linear equations.
  - Find ranks, inverses, and determinants of matrices.

- **Applications:** Used in Gaussian elimination for solving linear equations.

- **Overview of Elementary Row Operations**
  - **Row Swapping:** Interchanging two rows. Notation: $R_i \leftrightarrow R_j$
  - **Row Multiplication:** Multiplying all entries in a row by a non-zero scalar. Notation: kR_i
  - **Row Addition:** Adding or subtracting a multiple of one row to another. Notation: $R_i + kR_j \to R_i$

Example (Row Swapping)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \overset{R_1 \leftrightarrow R_3}{\longrightarrow}$$

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Example (Row Replacement)

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \overset{\longrightarrow}{R_3 \to R_3 - 7R_1}$$

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Example (Combination of Operations)

$$D = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix} \underset{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1}}{\rightarrow}$$

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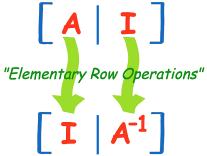
☆ Calculating A^{-1} by Gauss-Jordan Elimination ☆

- The elimination steps create the inverse matrix while changing $\boldsymbol{A}$ to $\boldsymbol{I}$

Play around with the rows (adding, multiplying or swapping) until we make Matrix **A** into the Identity Matrix **I**



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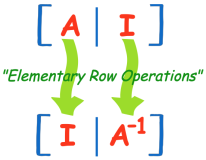

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- Let $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, so to start with Gauss-Jardan on $\boldsymbol{A}$, we first write the augmented matrix with $\boldsymbol{I_3}$.

$$\left[\boldsymbol{A} \vdots \boldsymbol{I_3}\right] = \left[\begin{array}{ccc:ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

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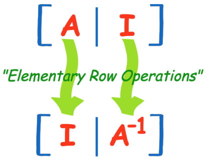

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- Now we do our best to turn $\boldsymbol{A}$ into an **Identity** Matrix by using the **elementary Row operations**. The goal is to make Matrix $\boldsymbol{A}$ have 1s on the diagonal and 0s elsewhere, and the right hand side comes along for the ride, with every operation being done on it as well.

Our row operations procedure is as follows:

1. We keep the first diagonal entry a_{11} and we make the rest of the **first column** zero 0.

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

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2. Then we keep the second diagonal entry a_{22} and we make all the other entries in the **second column** zero 0.

$$\left[\begin{array}{ccc:ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

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1. We keep the first diagonal entry a_{11} and we make the rest of the **first column** zero 0.

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❸ Then we keep the third diagonal entry a_{33} and we make all the other entries in the **third column** zero 0.

$$\begin{bmatrix} 2 & 0 & -\frac{2}{3} & \vdots & \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & \frac{3}{2} & -1 & \vdots & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \vdots & \frac{1}{3} & \frac{2}{3} & 1 \end{bmatrix}$$

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$$\left[\begin{array}{ccc|ccc} 2 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + \frac{1}{2}R_3 \\ R_2 \to R_2 + \frac{3}{4}R_3 \end{array}$$

❸ Then we keep the third diagonal entry a_{33} and we make all the other entries in the **third column** zero 0.

$$\left[\begin{array}{ccc:ccc} 2 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 + \frac{1}{2}R_3 \\ R_2 \to R_2 + \frac{3}{4}R_3 \end{array} \to \left[\begin{array}{ccc:ccc} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

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❹ The last Gauss-Jordan step is to divide each row by its pivot. The new pivots are 1.

$$\left[\begin{array}{ccc:ccc} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

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$$\left[\begin{array}{ccc:ccc} 2 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 + \frac{1}{2}R_3 \\ R_2 \to R_2 + \frac{3}{4}R_3 \end{array} \to \left[\begin{array}{ccc:ccc} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

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$$\left[\begin{array}{ccc|ccc} 2 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 + \frac{1}{2}R_3 \\ R_2 \to R_2 + \frac{3}{4}R_3 \end{array} \to \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

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$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right] \begin{array}{l} R_1 \to \frac{1}{2}R_1 \\ R_2 \to \frac{3}{2}R_2 \\ R_3 \to \frac{3}{4}R_3 \end{array} \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{array}\right] = [I_3 \vdots A^{-1}]$$

❸ Then we keep the third diagonal entry a_{33} and we make all the other entries in the **third column** zero 0.

$$\left[\begin{array}{ccc|ccc} 2 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & \frac{3}{2} & -1 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right] \begin{array}{l} R_1 \to R_1 + \frac{1}{2}R_3 \\ R_2 \to R_2 + \frac{3}{4}R_3 \end{array} \to \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{3}{2} & 1 & \frac{1}{2} \\ 0 & \frac{3}{2} & 0 & \frac{3}{4} & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & 1 \end{array}\right]$$

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We have reached I_3 in the first half of the matrix, because A is invertible. The three columns of A^{-1} are in the second half of $[I_3 \vdots A^{-1}]$.

✎ **Exercise 4.** Show that $C = \begin{bmatrix} \mathbf{0} & \mathbf{-1} & \mathbf{3} \\ \mathbf{2} & \mathbf{1} & \mathbf{5} \\ \mathbf{-4} & \mathbf{3} & \mathbf{-1} \end{bmatrix}$ is invertible by finding its inverse C^{-1} using the Gauss-Jordan elimination method.

We will use Gauss-Jordan elimination to compute C^{-1}.

- **Step 1: Write the augmented matrix** We start with the augmented matrix $[C \mid I]$, where I is the 3×3 identity matrix:

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 3 & 1 & 0 & 0 \\ 2 & 1 & 5 & 0 & 1 & 0 \\ -4 & 3 & -1 & 0 & 0 & 1 \end{array}\right].$$

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- **Step 2: Perform row operations to reduce the left-hand side to the identity matrix**
  - **Swap R_1 and R_2**: This is because the pivot in R_1 is zero. After the swap:

$$\left[\begin{array}{ccc|ccc} 0 & -1 & 3 & 1 & 0 & 0 \\ 2 & 1 & 5 & 0 & 1 & 0 \\ -4 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\mathrm{R}_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 2 & 1 & 5 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 0 \\ -4 & 3 & -1 & 0 & 0 & 1 \end{array}\right].$$

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  - **Scale R_1 by $\frac{1}{2}$** to make the pivot 1:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 5 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & 0 & 0 \\ -4 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\mathrm{R}_1 \leftrightarrow \frac{1}{2} R_1} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 3 & 1 & 0 & 0 \\ -4 & 3 & -1 & 0 & 0 & 1 \end{array}\right].$$

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  - **Eliminate the first column below the pivot:** Add $4R_1$ to R_3:

$$\left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 3 & 1 & 0 & 0 \\ -4 & 3 & -1 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \rightarrow \\ \mathrm{R}_3 \leftrightarrow R_3 + 4R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & \frac{1}{2} & \frac{5}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -1 & 3 & 1 & 0 & 0 \\ 0 & 5 & 9 & 0 & 2 & 1 \end{array}\right].$$

- **Step 3: Continue row reduction** After further row operations, we reduce the left-hand side to the identity matrix. The resulting matrix is:

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{6} & -\frac{1}{6} \\ 0 & 1 & 0 & -\frac{3}{8} & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 1 & \frac{5}{24} & \frac{1}{12} & \frac{1}{24} \end{bmatrix}.$$

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- **Step 4: Extract the inverse matrix** The right-hand side of the augmented matrix is C^{-1}:

$$C^{-1} = \begin{bmatrix} -\frac{1}{3} & \frac{1}{6} & -\frac{1}{6} \\ -\frac{3}{8} & \frac{1}{4} & \frac{1}{8} \\ \frac{5}{24} & \frac{1}{12} & \frac{1}{24} \end{bmatrix}.$$

Definition

A system of m linear equations in n variables $x_1, x_2, x_3, \ldots, x_n$ is one that can be written in the form

$$(S)\quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n & = & b_2 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & b_m \end{cases}$$

where the coefficients a_{ij} and b_j are constants for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

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where the coefficients a_{ij} and b_j are constants for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Example

The following is an example of a system of three linear equations.

$$\begin{cases} x_1 + x_2 + x_3 & = & 2 \\ 2x_1 + 3x_2 + x_3 & = & 3 \\ x_1 - x_2 - 2x_3 & = & -6 \end{cases}$$

It can be seen on substitution that $x_1 = -1, x_2 = 1, x_3 = 2$ is a solution to this system.

The system (S), can be written in the form $AX = B$, where

$$A = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}}_{\text{coefficient matrix}}, B = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\text{variable matrix}} \text{ and } X = \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}}_{\text{constant matrix}}.$$

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How to write A

- Write the coefficients of the x-terms as the numbers down the first column.
- Write the coefficients of the y-terms as the numbers down the second column.
- If there are z-terms, write the coefficients as the numbers down the third column.

$$\begin{cases} x + 2y - z = 3 \\ 2x - y + 2z = 6 \\ x - 3y + 3z = 4 \end{cases} \Longrightarrow A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 2 \\ 1 & -3 & 3 \end{pmatrix}$$

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How to write B

- Write the constants raw by raw.

$$\begin{cases} x + 2y - z &= 3 \\ 2x - y + 2z &= 6 \\ x - 3y + 3z &= 4 \end{cases} \Longrightarrow B = \begin{pmatrix} 3 \\ 6 \\ 4 \end{pmatrix}$$

Definition

We call augmented matrix of the system (S), is one that can be written in the form

$$(A|B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

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Example

The augmented matrix of $\left\{\begin{array}{rcr} x_1 + x_2 + x_3 & = & 2 \\ 2x_1 + 3x_2 + x_3 & = & 3 \\ x_1 - x_2 - 2x_3 & = & -6 \end{array}\right.$ is $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{array}\right)$.

- To solve a system of linear equations using an inverse matrix, let A be the coefficient matrix, let X be the variable matrix, and let B be the constant matrix. Thus, we want to solve a system

$$AX = B\,.$$

For example, look at the following system of equations.

$$\begin{cases} x + y + z & = & 1 \\ x + z & = & 1 \\ x + 2y + 4z & = & 1 \end{cases}$$

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For example, look at the following system of equations.

$$\begin{cases} x+y+z & = & 1 \\ x+z & = & 1 \\ x+2y+4z & = & 1 \end{cases}$$

1. From this system, the coefficient matrix is

$$\underbrace{A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 4 \end{pmatrix},}_{\text{coefficient matrix}} \quad \underbrace{X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}}_{\text{variable matrix}} \quad \text{and} \quad \underbrace{B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{\text{constant matrix}}.$$

Then $AX = B$ looks like

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

- To solve the system $AX = B$ for X, we would simply multiply both sides by the multiplicative inverse (reciprocal) of A. Thus,

$$\begin{aligned} (A^{-1})AX &= (A^{-1})B \\ \big((A^{-1})A\big)X &= (A^{-1})B \\ IX &= (A^{-1})B \\ X &= (A^{-1})B \end{aligned}$$

- To solve the system $AX = B$ for X, we would simply multiply both sides by the multiplicative inverse (reciprocal) of A. Thus,

$$\begin{aligned}(A^{-1})AX &= (A^{-1})B\\ \left((A^{-1})A\right)X &= (A^{-1})B\\ IX &= (A^{-1})B\\ X &= (A^{-1})B\end{aligned}$$

2. So first we need to prove that A is invertible matrix. Using row echelon form we have

$$|A| = \begin{vmatrix} 1 & 1 & 1\\ 1 & 0 & 1\\ 1 & 2 & 4\end{vmatrix} \begin{matrix} \\ R_2 \to R_2 - R_1\\ R_3 \to R_3 - R_1\end{matrix} = \begin{vmatrix} 1 & 1 & 1\\ 0 & -1 & 0\\ 0 & 1 & 3\end{vmatrix} \begin{matrix} \\ \\ R_3 \to R_3 + R_2\end{matrix}$$

$$= \begin{vmatrix} 1 & 1 & 1\\ 0 & -1 & 0\\ 0 & 0 & 3\end{vmatrix} = -3\,.$$

Since $|A| \neq 0$, then A is invertible.

❸ Now, we need to calculate A^{-1}. Using any method to calculate the inverse of a 3 by 3 matrix, we have:

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

where $C_{ij} = (-1)^{i+j} M_{ij}$ and M_{ij} is the determinant of the matrix A that remains after deleting row i and column j of A.

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$C_{11} = +$	$\begin{matrix} 0 & 1 \\ 2 & 4 \end{matrix}$	$= -2$	$C_{12} = -$	$\begin{matrix} 1 & 1 \\ 1 & 4 \end{matrix}$	$= -3$	$C_{13} = +$	$\begin{matrix} 1 & 0 \\ 1 & 2 \end{matrix}$	$= 2$
$C_{21} = -$	$\begin{matrix} 1 & 1 \\ 2 & 4 \end{matrix}$	$= -2$	$C_{22} = +$	$\begin{matrix} 1 & 1 \\ 1 & 4 \end{matrix}$	$= 3$	$C_{23} = -$	$\begin{matrix} 1 & 1 \\ 1 & 2 \end{matrix}$	$= -1$
$C_{31} = +$	$\begin{matrix} 1 & 1 \\ 0 & 1 \end{matrix}$	$= 1$	$C_{32} = -$	$\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$	$= 0$	$C_{33} = +$	$\begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}$	$= -1$

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Hence $\operatorname{adj}(A)=\begin{bmatrix}-2&-2&1\\-3&3&0\\2&-1&-1\end{bmatrix}$ and so

$$A^{-1}=\frac{1}{|A|}\operatorname{adj}(A)=\begin{bmatrix}2/3&2/3&-1/3\\1&-1&0\\-2/3&1/3&1/3\end{bmatrix}.$$

❹ Now we are ready to solve. Multiply both sides of the equation by A^{-1}. We have

$$AX = B \iff X = A^{-1}B \iff X = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ 1 & -1 & 0 \\ -2/3 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

✎ **Exercise 5.** Solve the following system using the inverse of a matrix

$$\begin{cases} 5x + 15y + 56z & = & 35 \\ -4x - 11y - 41z & = & -26 \\ -x - 3y - 11z & = & -7 \end{cases}$$

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- The first step of the Gaussian strategy includes obtaining a 1 as the first entry, so that row 1 may be used to alter the rows below.

1. The first equation should have a leading coefficient of 1. Interchange rows or multiply by a constant, if necessary.
2. Use row operations to obtain zeros down the first column below the first entry of 1
3. Use row operations to obtain a 1 in row 2, column 2.
4. Use row operations to obtain zeros down column 2, below the entry of 1.
5. Use row operations to obtain a 1 in row 3, column 3.
6. Continue this process for all rows until there is a 1 in every entry down the main diagonal and there are only zeros below.
7. If any rows contain all zeros, place them at the bottom.

✎ Wooclap 5.

Consider the following linear system .

$$(\mathcal{S}) := \begin{cases} 6x + y + z & = & 12 \\ 2x + 4y & = & 0 \\ x + 2y + 6z & = & 6 \end{cases}$$

(a) Find the matrices A, B and X such that $(\mathcal{S}) \Leftrightarrow AX = B$.

(b) Calculate the determinant of A. Deduce that A is invertible.

(c) Calculate the adjoint of A and find A^{-1}.

(d) Deduce the solution of the linear system $(\mathcal{S})$.

(e) Refind the solution of the system $(\mathcal{S})$ by an another method.

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Note: When doing step 2, row operations can be performed in any order. Try to choose row operations so that as few fractions as possible are carried through the computation. This makes calculation easier when working by hand.

Example

Use the Gauses-Jordan elimimnation method to solve the system:

$$\begin{cases} x + 2y - z &= 4 \\ 2x - y + 5z &= 5 \\ 3x + 4y + z &= 9 \end{cases} .$$

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$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 4 \\ 2 & -1 & 5 & 5 \\ 3 & 4 & 1 & 9 \end{array}\right)$$

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Example

Use the Gauses-Jordan elimimnation method to solve the system:

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Thus $x = 11/2,\ y = -3/2,\ z = -3/2$.

✎ **Exercise 6.** Use the Gauses-Jordan elimimnation method to solve the system:

$$\begin{cases} x + y - 2z & = & 9 \\ 2x + 4y - 3z & = & 1 \\ 3x + 6y - 5z & = & 0 \end{cases}.$$