School of Engineering and Computer Science

Mathematical tools applied to Computer Science

Ch1 : Elements of Logic

Kamel ATTAR
kamel.attar@epita.fr

Week #**1** ✦ 10/SEP/2024 ✦
Week #**2** ✦ 19/SEP/2024 ✦

- In developing a mathematical theory, **statements** or **assertions** are made. *These statements are made in the form of sentences using words and mathematical symbols to convey ideas..*

- In developing a mathematical theory, **statements** or **assertions** are made. *These statements are made in the form of sentences using words and mathematical symbols to convey ideas..*

Example (Mathematical Statements and Assertions)

"For all real numbers x, if $x > 2$, then $x^2 > 4$."

This statement uses both words ("for all real numbers") and mathematical symbols ($x > 2$, $x^2 > 4$) to express a mathematical assertion.

- In developing a mathematical theory, **statements** or **assertions** are made. *These statements are made in the form of sentences using words and mathematical symbols to convey ideas..*

- When proving a theory, mathematicians rely on a system of logic.
  - This logical approach is similarly applied in computer science when designing algorithms or systems of programs.

- In developing a mathematical theory, **statements** or **assertions** are made. *These statements are made in the form of sentences using words and mathematical symbols to convey ideas..*

- When proving a theory, mathematicians rely on a system of logic.
  - This logical approach is similarly applied in computer science when designing algorithms or systems of programs.

Example (Use of Logic in Proving Theorems or Developing Algorithms)

- **Mathematics:** To prove the Pythagorean theorem, a mathematician uses logical steps, such as defining a right triangle, applying the properties of similar triangles, and deducing that $a^2 + b^2 = c^2$.
- **Computer Science:** When developing an algorithm to sort a list of numbers, a programmer uses logical reasoning to decide how to compare elements, swap them if needed, and repeat the process until the list is sorted.

- In developing a mathematical theory, **statements** or **assertions** are made. *These statements are made in the form of sentences using words and mathematical symbols to convey ideas..*

- When proving a theory, mathematicians rely on a system of logic.
  - This logical approach is similarly applied in computer science when designing algorithms or systems of programs.
  - This logical framework is used to determine whether a statement logically follows from, or is a consequence of, one or more other statements (premises).

- In developing a mathematical theory, **statements** or **assertions** are made. *These statements are made in the form of sentences using words and mathematical symbols to convey ideas..*

- When proving a theory, mathematicians rely on a system of logic.
  - This logical approach is similarly applied in computer science when designing algorithms or systems of programs.
  - This logical framework is used to determine whether a statement logically follows from, or is a consequence of, one or more other statements (premises).

Example (Applying a Logical System to Determine Consequences)

Given the statements "If it rains, the ground will be wet" and "It is raining," you can logically conclude that "The ground is wet."

In **mathematical logic**, this is an example of the **modus ponens** rule, where if "A implies B" and "A is true," then "B must also be true."

- In developing a mathematical theory, **statements** or **assertions** are made. *These statements are made in the form of sentences using words and mathematical symbols to convey ideas..*

- When proving a theory, mathematicians rely on a system of logic.
  - This logical approach is similarly applied in computer science when designing algorithms or systems of programs.
  - This logical framework is used to determine whether a statement logically follows from, or is a consequence of, one or more other statements (premises).

- You are already familiar with using numbers in arithmetic, symbols in algebra, and the rules governing both of these areas. In arithmetic an operation is a rule for producing new numbers from a pair of given numbers, like addition $(+)$ or multiplication $(\times)$.

- In developing a mathematical theory, **statements** or **assertions** are made. *These statements are made in the form of sentences using words and mathematical symbols to convey ideas..*

- When proving a theory, mathematicians rely on a system of logic.
  - This logical approach is similarly applied in computer science when designing algorithms or systems of programs.
  - This logical framework is used to determine whether a statement logically follows from, or is a consequence of, one or more other statements (premises).

- You are already familiar with using numbers in arithmetic, symbols in algebra, and the rules governing both of these areas. In arithmetic an operation is a rule for producing new numbers from a pair of given numbers, like addition $(+)$ or multiplication $(\times)$.

Example (Familiarity with Arithmetic and Algebra Rules)

- **Arithmetic**: When adding two numbers, such as $3 + 5$, you use the rule that the sum is 8. This follows the commutative property of addition, which states that $a + b = b + a$.
- **Algebra**: In solving the equation $2x + 3 = 7$, you apply algebraic rules to isolate x (subtract 3 from both sides and then divide by 2) to find that $x = 2$.

- In developing a mathematical theory, **statements** or **assertions** are made. *These statements are made in the form of sentences using words and mathematical symbols to convey ideas..*

- When proving a theory, mathematicians rely on a system of logic.
  - This logical approach is similarly applied in computer science when designing algorithms or systems of programs.
  - This logical framework is used to determine whether a statement logically follows from, or is a consequence of, one or more other statements (premises).

- You are already familiar with using numbers in arithmetic, symbols in algebra, and the rules governing both of these areas. In arithmetic an operation is a rule for producing new numbers from a pair of given numbers, like addition $(+)$ or multiplication $(\times)$.

- In a similar way, Logic deals with **statements** or **sentences** by defining symbols and establishing 'rules'. In formal logic, we can use symbols like p and q to represent statements, and symbols like $\wedge$ (and), $\vee$ (or), and $\rightarrow$ (implies) to form new statements by combining short statements using these connectives, thereby forming complex logical expressions.

- In developing a mathematical theory, **statements** or **assertions** are made. *These statements are made in the form of sentences using words and mathematical symbols to convey ideas..*

- When proving a theory, mathematicians rely on a system of logic.
    - This logical approach is similarly applied in computer science when designing algorithms or systems of programs.
    - This logical framework is used to determine whether a statement logically follows from, or is a consequence of, one or more other statements (premises).

- You are already familiar with using numbers in arithmetic, symbols in algebra, and the rules governing both of these areas. In arithmetic an operation is a rule for producing new numbers from a pair of given numbers, like addition $(+)$ or multiplication $(\times)$.

- In a similar way, Logic deals with **statements** or **sentences** by defining symbols and establishing 'rules'. In formal logic, we can use symbols like p and q to represent statements, and symbols like $\wedge$ (and), $\vee$ (or), and $\rightarrow$ (implies) to form new statements by combining short statements using these connectives, thereby forming complex logical expressions.

Example (Logic Using Symbols and Rules)

For instance, if p represents "It is raining," and q represents "The ground is wet," then the expression $p \rightarrow q$ can be read as "If it is raining, then the ground is wet."

⋆ Statement ⋆

Definition

A **statement** is a declarative sentence that is either true or false, but not both.

Example

1. Paris is the capital of France. THIS IS A **TRUE** STATEMENT
2. $\mathbf{2+2=4.}$ THIS IS ALSO A STATEMENT BECAUSE IT CAN BE VERIFIED TO BE **TRUE**.
3. The number $\mathbf{4}$ is positive and the number $\mathbf{3}$ is negative. THIS IS A **FALSE** STATEMENT

Definition

A **statement** is a declarative sentence that is either true or false, but not both.

Example

1. Paris is the capital of France. THIS IS A **TRUE** STATEMENT
2. $\mathbf{2+2=4.}$ THIS IS ALSO A STATEMENT BECAUSE IT CAN BE VERIFIED TO BE **TRUE**.
3. The number $\mathbf{4}$ is positive and the number $\mathbf{3}$ is negative. THIS IS A **FALSE** STATEMENT

❢ In logic and mathematics, statements are used to express propositions, assertions, or claims that can be evaluated for their truth value.

Example

The following sentences are not statements:

1. Your place or mine?
2. What's your name?
3. Knock before entering!
4. $x - y = y - x$. IT IS NOT A STATEMENT BECAUSE IT IS NEITHER **true** NOR **false**.

Note that the last sentences can be turned into a statement if we specify the symbols.

$$x - y = y - x \text{ for all } x, y \in \mathbb{R}\,,$$

then this is a false proposition. If the intention is

$$x - y = y - x \text{ for all } x, y \in \{0\}\,,$$

then this is a true statement.

Definition (Simple Statement)

A *simple* or *primitive* statement is a statement which cannot be broken down into anything simpler.

Definition (Simple Statement)

A *simple* or *primitive* statement is a statement which cannot be broken down into anything simpler.

A simple statement is denoted by use of letters $p, q, r, \cdots$

Definition (Simple Statement)

A *simple* or *primitive* statement is a statement which cannot be broken down into anything simpler.

A simple statement is denoted by use of letters $p, q, r, \cdots$

Example

- p: There are seven days in a week
  p is a simple statement.
- $q: 2+3=6$
  q is a simple statement

Definition (Compound Statements)

A *compound* or *composite* statement is a statement which is comprised of simple statements and logical operations.

Definition (Compound Statements)

A *compound* or *composite* statement is a statement which is comprised of simple statements and logical operations.

A compound statement is denoted by use of letters P, Q, $R \cdots$

Definition (Compound Statements)

A *compound* or *composite* statement is a statement which is comprised of simple statements and logical operations.

A compound statement is denoted by use of letters P, Q, $R\cdots$

Example

- P: There are seven days in a week and twelve months in a year.

  Is a compound statement.

  p: There are seven days in a week

  q: There twelve months in a year

  Operation: and
- P: $2+3=6$ or $5-(3-2)=(5-3)-2$. Is a compound statement.

Definition (Compound Statements)

A *compound* or *composite* statement is a statement which is comprised of simple statements and logical operations.

A compound statement is denoted by use of letters P, Q, $R \cdots$

Example

- P: If it is not raining then I will go outside and eat my lunch.

  Is a compound statement

  p: It is raining

  q: I will go outside

  r: I will eat my lunch

  Negation of p

  Operations: If $\cdots$ then, and

✯ Truth and Falsity in Logic: Demonstrations and Proofs ✯

The truth value of a mathematical statement can be determined by application of known rules, axioms and laws of mathematics.

⋆ Truth and Falsity in Logic: Demonstrations and Proofs ⋆

The truth value of a mathematical statement can be determined by application of known rules, axioms and laws of mathematics.

- A statement which is true requires a **proof**.

⋆ Truth and Falsity in Logic: Demonstrations and Proofs ⋆

The truth value of a mathematical statement can be determined by application of known rules, axioms and laws of mathematics.

- A statement which is true requires a **proof**.

❢ *For a statement that is true, a proof is required to rigorously show that the statement holds in all cases under the given conditions. Proofs can involve direct reasoning, mathematical induction, or other methods of logical inference.*

⋆ Truth and Falsity in Logic: Demonstrations and Proofs ⋆

The truth value of a mathematical statement can be determined by application of known rules, axioms and laws of mathematics.

- A statement which is true requires a **proof**.

Example

Is the following statement True or False?

For a real number x, if $x^2 = 1$, then $x = 1$ or $x = -1$.

⭐ Truth and Falsity in Logic: Demonstrations and Proofs ⭐

The truth value of a mathematical statement can be determined by application of known rules, axioms and laws of mathematics.

- A statement which is true requires a **proof**.

Example

Is the following statement True or False?

For a real number x, if $x^2 = 1$, then $x = 1$ or $x = -1$.

Solution

The statement is TRUE. Therefore, we must prove it.

Consider $x^2 = 1$.

Adding -1 to both sides gives $x^2 - 1 = 0$.

Factorising this equation, we have $(x - 1)(x + 1) = 0$.

Therefore, $x - 1 = 0$ or $x + 1 = 0$.

- Case 1: $x - 1 = 0$. Add 1 to both sides and we have $x = 1$.
- Case 2: $x + 1 = 0$. Add -1 to both sides and we have $x = -1$.

- A statement which is false requires a **demonstration**.

- A statement which is false requires a **demonstration**.

> *A demonstration for a false statement typically involves showing a counterexample or providing evidence that contradicts the statement. Demonstrating a false statement means finding a specific case or logical argument that clearly proves the statement cannot be true in all cases.*

- A statement which is false requires a **demonstration**.

❢ *A demonstration for a false statement typically involves showing a counterexample or providing evidence that contradicts the statement. Demonstrating a false statement means finding a specific case or logical argument that clearly proves the statement cannot be true in all cases.*

Example

Is the following statement True or False?

$$\mathbf{5 - (3 - 2) = (5 - 3) - 2}$$

- A statement which is false requires a **demonstration**.

❢ *A demonstration for a false statement typically involves showing a counterexample or providing evidence that contradicts the statement. Demonstrating a false statement means finding a specific case or logical argument that clearly proves the statement cannot be true in all cases.*

Example

Is the following statement True or False?

$$\mathbf{5 - (3 - 2) = (5 - 3) - 2}$$

Solution

The statement is FALSE. Therefore, we must demonstrate it.

$\mathbf{5 - (3 - 2) = 5 - 1 = 4}$

$\mathbf{(5 - 3) - 2 = 2 - 2 = 0}$

Therefore $\mathbf{5 - (3 - 2) \neq (5 - 3) - 2}$

✎ **Wooclap 1.** Determine which of the following sentences are statements. For those which are statements, determine their truth value.

1	$2+3=5$		
2	It is hot and sunny outside.		
3	$2+3=6$		
4	Is it raining?		
5	Go away!		
6	There exists an even prime number.		
7	There are six people in this room.		
8	For some real number x, $x<2$		
9	$x<2$		
10	$x+y=y+x$		

- **Initial Problem**:

"It is raining today"

- **Initial Problem**:

"It is raining today"

- If it is raining, THE STATEMENT IS **TRUE**.
- If it is not raining, THE STATEMENT IS **FALSE**.

- **Initial Problem**:
  - We encountered a difficulty in determining the truth of the statement THAT GUY IS GOING TO THE STORE. The issue is that we cannot decide if it is true or false because the statement does not specify who THAT GUY is. Some people are going to the store, while others are not.
  - Similar ambiguities also appear in mathematics. For example, the statement X/2 IS AN INTEGER is true for some values of x but false for others. This is not a proposition in the strict sense because its truth depends on the value of x.

- **Initial Problem**:
  - We encountered a difficulty in determining the truth of the statement THAT GUY IS GOING TO THE STORE. The issue is that we cannot decide if it is true or false because the statement does not specify who THAT GUY is. Some people are going to the store, while others are not.
  - Similar ambiguities also appear in mathematics. For example, the statement X/2 IS AN INTEGER is true for some values of x but false for others. This is not a proposition in the strict sense because its truth depends on the value of x.

Definition (Predicate)

A **predicate** is a sentence that contains one or more variables and becomes a statement when specific values are substituted for the variables.

- **Initial Problem**:
  - We encountered a difficulty in determining the truth of the statement THAT GUY IS GOING TO THE STORE. The issue is that we cannot decide if it is true or false because the statement does not specify who THAT GUY is. Some people are going to the store, while others are not.
  - Similar ambiguities also appear in mathematics. For example, the statement X/2 IS AN INTEGER is true for some values of x but false for others. This is not a proposition in the strict sense because its truth depends on the value of x.

Definition (Predicate)

A **predicate** is a sentence that contains one or more variables and becomes a statement when specific values are substituted for the variables.

- **Using Predicates**: The earlier statements can be expressed using predicates:
  - Let $P(x)$ be true if x is going to the store.
  - Let $Q(x)$ be true if $x/2$ is an integer.

- **Initial Problem**:
  - We encountered a difficulty in determining the truth of the statement THAT GUY IS GOING TO THE STORE. The issue is that we cannot decide if it is true or false because the statement does not specify who THAT GUY is. Some people are going to the store, while others are not.
  - Similar ambiguities also appear in mathematics. For example, the statement X/2 IS AN INTEGER is true for some values of x but false for others. This is not a proposition in the strict sense because its truth depends on the value of x.

Definition (Predicate)

A **predicate** is a sentence that contains one or more variables and becomes a statement when specific values are substituted for the variables.

- **Using Predicates**: The earlier statements can be expressed using predicates:
  - Let $P(x)$ be true if x is going to the store.
  - Let $Q(x)$ be true if $x/2$ is an integer.
- Once the variable has a fixed value, the predicate becomes a proposition with a determined truth value. For example, $Q(8)$ is a true proposition, while $Q(9.3)$ is a false proposition.

- **Initial Problem**:
  - We encountered a difficulty in determining the truth of the statement THAT GUY IS GOING TO THE STORE. The issue is that we cannot decide if it is true or false because the statement does not specify who THAT GUY is. Some people are going to the store, while others are not.
  - Similar ambiguities also appear in mathematics. For example, the statement X/2 IS AN INTEGER is true for some values of x but false for others. This is not a proposition in the strict sense because its truth depends on the value of x.

Definition (Predicate)

A **predicate** is a sentence that contains one or more variables and becomes a statement when specific values are substituted for the variables.

- **Using Predicates**: The earlier statements can be expressed using predicates:
  - Let $P(x)$ be true if x is going to the store.
  - Let $Q(x)$ be true if $x/2$ is an integer.
- Once the variable has a fixed value, the predicate becomes a proposition with a determined truth value. For example, $Q(8)$ is a true proposition, while $Q(9.3)$ is a false proposition.
- **Conclusion**: Predicates allow us to create logical expressions whose truth can be evaluated once the variables are specified. They are essential for formalizing reasoning in mathematics and logic.

Example

Consider the predicate "x is greater than 5," where "x" is a variable:

- When we assign a specific value to "x," such as 8, the predicate becomes a statement: "8 is greater than 5," which is true.
- If we assign the value 3 to "x," the predicate becomes "3 is greater than 5," which is false.

Example

Consider the predicate "x is greater than 5," where "x" is a variable:

- When we assign a specific value to "x," such as 8, the predicate becomes a statement: "8 is greater than 5," which is true.
- If we assign the value 3 to "x," the predicate becomes "3 is greater than 5," which is false.

❢ Predicates are frequently used in mathematics, logic, and computer science to define conditions or relationships that can be evaluated for specific values.
It is foundational in fields like artificial intelligence, database querying, and formal verification of software.

Consider the previous example. Does it make sense to assign to x the value "?

Consider the previous example. Does it make sense to assign to x the value "?

Intuitively, the universe of discourse is the set of all things we wish to talk about; that is, the set of all objects that we can sensibly assign to a variable in a propositional function.

Definition (Domain)

The **domain** of a predicate variable consists of all values that may be substituted in place of the variable

Consider the previous example. Does it make sense to assign to x the value "?

Intuitively, the universe of discourse is the set of all things we wish to talk about; that is, the set of all objects that we can sensibly assign to a variable in a propositional function.

Definition (Domain)

The **domain** of a predicate variable consists of all values that may be substituted in place of the variable

Example

If we have a **predicate** "x is an even number," the domain for the variable "x" would consist of all integers since we want to check if any integer value can satisfy the predicate.

In logic and mathematics, defining the domain of a predicate variable is crucial. It establishes the set of elements under consideration, allowing us to determine the truth or falsity of a statement based on possible substitutions for the variable within that domain.

In logic and mathematics, defining the domain of a predicate variable is crucial. It establishes the set of elements under consideration, allowing us to determine the truth or falsity of a statement based on possible substitutions for the variable within that domain.

Definition (Truth Set)

If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements in D that make $P(x)$ true. The truth set is denoted by $\{x \in D : P(x)\}$ and is read "the set of all x in D such that $P(x)$."

In logic and mathematics, defining the domain of a predicate variable is crucial. It establishes the set of elements under consideration, allowing us to determine the truth or falsity of a statement based on possible substitutions for the variable within that domain.

Definition (Truth Set)

If $P(x)$ is a predicate and x has domain D, the **truth set** of $P(x)$ is the set of all elements in D that make $P(x)$ true. The truth set is denoted by $\{x \in D : P(x)\}$ and is read "the set of all x in D such that $P(x)$."

Example

Let $P(x)$ be the predicate "$x^2 > x$" with $x \in \mathbb{R}$, meaning the domain is the set of all real numbers $\mathbb{R}$.

- Write down $P(2)$, $P(1)$, $P(-2)$ and indicate which are true and which are false.
- Determine the truth set of $P(x)$

Solution

- Evaluate the predicate for specific values:

$$\begin{aligned} &P(2): 2^2 > 2 \Rightarrow 4 > 2 \quad \text{(True)} \\ &P(1): 1^2 > 1 \Rightarrow 1 > 1 \quad \text{(False)} \\ &P(-2): (-2)^2 > -2 \Rightarrow 4 > -2 \quad \text{(True)} \end{aligned}$$

- Determine the truth set of $P(x)$:
  The truth set consists of all real numbers $x \in \mathbb{R}$ for which $x^2 > x$. To find this, solve the inequality for x.

Example

Let $Q(n)$ be the predicate "n is factor of 8". Then the truth set of $Q(n)$ if $n \in \mathbb{Z}^+$ is

$$\{n \in \mathbb{Z}^+ : \text{"}n \text{ is a factor of } 8\text{"}\} = \{1, 2, 4, 8\}.$$

✎ **Wooclap 2. Predicate Analysis with Integers** Let $P(x)$ be the predicate “ $x^3 > x$” with $x \in \mathbb{Z}$ i.e. domain the set of integers, $\mathbb{Z}$.

a. Write down $P(2)$, $P(0)$, $P(-2)$ and indicate which are true and which are false.

b. Determine the truth set of $P(x)$

✎ **Wooclap 3. Factors and Divisibility** Let $Q(n)$ be the predicate “n is factor of 6”. Determine the truth set of $Q(n)$ if $n \in \mathbb{Z}$

✎ **Wooclap 4. Creating New Predicates** Create new predicates and define appropriate domains.

a. Define a predicate $S(x)$.

b. Specify the domain of x .

c. Determine the truth set for their predicate.

✎ **Wooclap 5. Real-World Scenario Predicate** Create a predicate $R(y)$, such as "y is a person who is eligible to vote," and define a suitable domain (e.g., $y \in$ set of all people).

a. Determine the conditions for $R(y)$ to be true.

b. Describe the truth set of $R(y)$.

- A predicate becomes a statement when we assign it fixed values.

- However, another way to make a predicate into a statement is to quantify it. That is, the predicate is true (or false) for all possible values in the universe of discourse or for some value(s) in the universe of discourse.

- Such quantification can be done with two quantifiers: the universal quantifier and the existential quantifier.

- Quantifiers are words that refer to quantities such as “all”, “every”, "any", "there is" or “some” and tell for how many elements a given predicate is true.

The symbol ∀ denotes “for all” and is called the *universal quantifier*.

Definition (Universal Statement)

Let $P(x)$ be a predicate and D the domain of x. The *universal quantification* of $P(x)$ is the statement

" $P(x)$ IS TRUE FOR ALL VALUES OF x IN THE UNIVERSE OF DISCOURSE"

We use the notation

$$\forall x \in D,\ P(x)\,.$$

- It is defined to be true if, and only if, $P(x)$ is true for every $x \in D$.
- It is defined to be false if, and only if, $P(x)$ is false for at least one x in D.

A value of x for which $P(x)$ is false is called a *counterexample* to the universal statement.

The symbol $\forall$ denotes “for all” and is called the *universal quantifier*.

Definition (Universal Statement)

Let $P(x)$ be a predicate and D the domain of x. The *universal quantification* of $P(x)$ is the statement

" $P(x)$ IS TRUE FOR ALL VALUES OF x IN THE UNIVERSE OF DISCOURSE"

We use the notation

$$\forall x \in D,\ P(x)\,.$$

- It is defined to be true if, and only if, $P(x)$ is true for every $x \in D$.
- It is defined to be false if, and only if, $P(x)$ is false for at least one x in D.

A value of x for which $P(x)$ is false is called a *counterexample* to the universal statement.

Example

Let $P(x)$ be "x is an even number." The universal statement $\forall x \in \mathbb{N}, P(x)$ reads "every natural number is even," which is **false**.

- *Counter example:* $x = 1$ (since 1 is not an even number).

The symbol ∀ denotes “for all” and is called the *universal quantifier*.

Definition (Universal Statement)

Let $P(x)$ be a predicate and D the domain of x. The *universal quantification* of $P(x)$ is the statement

" $P(x)$ IS TRUE FOR ALL VALUES OF x IN THE UNIVERSE OF DISCOURSE"

We use the notation

$$\forall x \in D,\ P(x)\,.$$

- It is defined to be true if, and only if, $P(x)$ is true for every $x \in D$.
- It is defined to be false if, and only if, $P(x)$ is false for at least one x in D.

A value of x for which $P(x)$ is false is called a *counterexample* to the universal statement.

Significance:

- Universal quantifiers are essential in mathematics, logic, and computer science for formulating theorems, writing proofs, and defining algorithmic constraints.
- The universal quantifier (∀) is often contrasted with the *existential quantifier* (∃), which denotes "there exists at least one."

✎ **Wooclap 6. Expressing Statements with the Universal Quantifier** Write the following statements using the universal quantifier. Determine whether each statement is true or false.

a. "All human beings are mortal" .
b. "All dogs are animals".
c. "The square of any real number is positive".
d. "Every integer is a rational number".

✎ **Wooclap 7. Translating Quantified Statements into Words** Translate the following mathematical statements into words and determine whether each statement is true or false:

a. $\forall x \in \mathbb{Z},\ x + 1 > x$.
b. $\forall x \in \mathbb{Q},\ x^2 \geq 0$.
c. $\forall x \in \mathbb{N},\ 2x \geq x$.
d. $\forall x \in \mathbb{R},\ x + (-x) = 0$.

✎ **Wooclap 8. Identifying Counterexamples** For each of the following universal statements, determine if the statement is true or false. If false, provide a counterexample:

a. "For all $x \in \mathbb{N},\ x^2 - x + 2 > 0$."
b. "For all $x \in \mathbb{R},\ x^2 \geq x$."
c. "For all $n \in \mathbb{Z},\ n + n = 2n$."
d. "For all $x \in \mathbb{R},\ x + 1 \geq x$."

✎ **Wooclap 9. Constructing Your Own Universal Statements** Create your own universal statement using the quantifier $\forall$:

a. Choose a domain for your variable (e.g., integers, real numbers, natural numbers).
b. Write a predicate involving your variable.
c. Determine whether your universal statement is true or false. If false, provide a counterexample.

⋆ Existential Quantifier ⋆

Definition (Existential Statement)

Let $P(x)$ be a predicate and D the domain of x. The *existential quantification* of $P(x)$ is the statement

"THERE EXISTS AN X IN THE UNIVERSE OF DISCOURSE SUCH THAT P(X) IS TRUE."

We use the notation

$$\exists x \in D,\ P(x)$$

- It is defined to be true if, and only if, $P(x)$ is true for at least one x in D.
- It is defined to be false if, and only if, $P(x)$ is false for all x in D.

✎ **Wooclap 10.** Write the following statements using the existential quantifier. Determine whether each statement is true or false.

a. "Some people are vegetarians".
b. "Some cats are black".
c. " There is a real number whose square is negative ".
d. "Some programs are structured".

✎ **Wooclap 11.** Write the following statements in words. Determine whether each statement is true or false.

a. $\exists m \in \mathbb{Z},\ m^2 = m.$
b. $\exists x \in \mathbb{R},\ x^2 = -1.$
c. $\exists x \in \mathbb{Z},\ \frac{1}{x} \notin \mathbb{Q}.$

Definition

Let $P(x)$ be a predicate and D the domain of x. In logic, the negation of quantifiers changes the statement's form:

- The negation of a **universal statement** of the form: $\forall x \in D,\ P(x)$ is logically equivalent to an existential statement where the predicate is negated:

$$\neg\left(\forall x \in D,\ P(x)\right) \iff \exists x \in D,\ \neg P(x)$$

- The negation of an **existential statement** of the form: $\exists x \in D,\ P(x)$ is logically equivalent to a universal statement where the predicate is negated:

$$\neg\left(\exists x \in D,\ P(x)\right) \iff \forall x \in D,\ \neg P(x)$$

These logical equivalences can be applied to various mathematical statements. Consider the following examples:

Example

$$\neg\left(\forall x \in \mathbb{R},\ x^2+1 \geq 2x\right) \iff \exists x \in \mathbb{R},\ x^2+1 < 2x.$$

This equivalence states that if it is not true that $x^2+1 \geq 2x$ for all $x \in \mathbb{R}$, then there exists some $x \in \mathbb{R}$ for which $x^2+1 < 2x$.

$$\neg\left(\exists x \in \mathbb{Q},\ x^2 = 2\right) \iff \forall x \in \mathbb{Q},\ x^2 \neq 2.$$

This equivalence tells us that if there does not exist an $x \in \mathbb{Q}$ such that $x^2 = 2$, then for all $x \in \mathbb{Q}$, $x^2 \neq 2$.

✎ **Wooclap 12.** Write down the negation of the following statement.

a. $\forall x \in \mathbb{R},\ x^2 \geq 0$.
b. $\forall y \in \mathbb{R},\ \left(y \neq 0 \Longrightarrow \frac{y+1}{y} < 1\right)$.
c. Every real number is either positive or negative.
d. The square of any integer is positive.
e. $\exists z \in \mathbb{Z}$, $(z$ is odd$) \vee (z$ is even$)$
f. $\exists n \in \mathbb{N}$, $(n$ is even$) \wedge (\sqrt{n}$ is prime$)$

✎ **Wooclap 13.** Let $P(x)$ be the statement "x *spends more than five hours every weekday in class*" where the domain for x consists of all students. Express each of these quantifications in English.

$a)\ \exists x\, P(x) \qquad b)\ \forall x\, P(x) \qquad c)\ \exists x \neg P(x) \qquad d)\ \forall x \neg P(x)$

✎ **Wooclap 14.** Let $P(x)$ be the statement "$x = x^2$". If the domain consists of the integers, what are these truth values?

$a)\ P(0) \quad b)\ P(1) \quad c)\ P(2) \quad d)\ P(-1) \quad e)\ \exists x\, P(x) \quad f)\ \forall x\, P(x)$.

When a statement contains multiple quantifiers, the order in which they appear is crucial and can significantly affect the truth set of the statement.

Example

- Everybody loves somebody.

  Let H be the set of people.

  Statement: $\forall x \in H, \exists y \in H,$ x loves y.
- Somebody loves everyone.

  Let H be the set of people.

  Statement: $\exists x \in H, \forall y \in H,$ x loves y.

✎ **Wooclap 15.** Write the following statements using quantifiers:

a. Everybody loves everybody.

b. The Commutative Law of Addition for $\mathbb{Z}$.

c. Everyone had a mother.

d. There is an oldest person.

✎ **Wooclap 16.** Write the following statements without using quantifiers:

a. $\forall x \in \mathbb{R}\ \exists y \in \mathbb{R},\ x + y = 0$.

b. $\exists x \in \mathbb{R}\ \forall y \in \mathbb{R},\ x + y = y$.

c. $\forall c \in$ colors, $\exists a \in$ animals, a is colored c.

d. $\exists \in$ books, $\forall \in$ people, p has read b.

Interpreting Statements with Multiple Quantifiers.

Statement	When True?	When False?
$\forall x,\ \forall y,\ P(x, y)$	The predicate $P(x, y)$ is true for all possible pairs (x, y).	There exists at least one pair (x, y) for which $P(x, y)$ is false.
$\forall x,\ \exists y,\ P(x, y)$	For every x, there is at least one y such that $P(x, y)$ is true.	There exists at least one x such that for every y, $P(x, y)$ is false.
$\exists x,\ \forall y,\ P(x, y)$	There exists at least one x such that $P(x, y)$ is true for all y.	For every x, there exists at least one y for which $P(x, y)$ is false.
$\exists x,\ \exists y,\ P(x, y)$	There exists at least one pair (x, y) for which $P(x, y)$ is true.	The predicate $P(x, y)$ is false for all possible pairs (x, y).

- Statement: $\forall x, \forall y, P(x, y)$
  Meaning: For all possible pairs (x, y), the predicate $P(x, y)$ is true.

Example

Let $P(x, y)$ be "x + y > 0," where $x, y \in \mathbb{R}^+$ (the set of all positive real numbers).

True When: For all positive real numbers x and y, $x + y > 0$. This is always true.

False When: If we change the domain to $\mathbb{R}$ (all real numbers), there exists a pair (e.g., $x = -1, y = -2$) for which $x + y > 0$ is false, since $-1 + (-2) = -3 \leq 0$.

Interpreting Statements with Multiple Quantifiers.

Statement	When True?	When False?
$\forall x,\ \forall y,\ P(x,y)$	The predicate $P(x,y)$ is true for all possible pairs (x,y).	There exists at least one pair (x,y) for which $P(x,y)$ is false.
$\forall x,\ \exists y,\ P(x,y)$	For every x, there is at least one y such that $P(x,y)$ is true.	There exists at least one x such that for every y, $P(x,y)$ is false.
$\exists x,\ \forall y,\ P(x,y)$	There exists at least one x such that $P(x,y)$ is true for all y.	For every x, there exists at least one y for which $P(x,y)$ is false.
$\exists x,\ \exists y,\ P(x,y)$	There exists at least one pair (x,y) for which $P(x,y)$ is true.	The predicate $P(x,y)$ is false for all possible pairs (x,y).

- Statement: $\forall x, \exists y, P(x,y)$
  Meaning: For every x, there exists at least one y such that $P(x,y)$ is true.

Example

$\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, P(x,y)$ is "x + y = 5".

True When: For every $x \in \mathbb{R}$, there exists a $y = 5 - x$ such that $x + y = 5$. This is always true because for any x, we can find a corresponding y.

False When: If $P(x,y)$ is "x + y = 5" and the domain of y is restricted to only positive numbers greater than 5, then for some x, there is no y that satisfies the condition (e.g., $x = 1, y > 5$, no y makes $1 + y = 5$).

Interpreting Statements with Multiple Quantifiers.

Statement	When True?	When False?
$\forall x,\ \forall y,\ P(x,y)$	The predicate $P(x,y)$ is true for all possible pairs (x,y).	There exists at least one pair (x,y) for which $P(x,y)$ is false.
$\forall x,\ \exists y,\ P(x,y)$	For every x, there is at least one y such that $P(x,y)$ is true.	There exists at least one x such that for every y, $P(x,y)$ is false.
$\exists x,\ \forall y,\ P(x,y)$	There exists at least one x such that $P(x,y)$ is true for all y.	For every x, there exists at least one y for which $P(x,y)$ is false.
$\exists x,\ \exists y,\ P(x,y)$	There exists at least one pair (x,y) for which $P(x,y)$ is true.	The predicate $P(x,y)$ is false for all possible pairs (x,y).

- Statement: $\exists x, \forall y, P(x,y)$
  Meaning: There exists at least one x such that for all y, the predicate $P(x,y)$ is true.

Example

$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, P(x,y)$ is "x is less than or equal to y" .

True When: If $x = 0$, for all $y \geq 0$, $0 \leq y$ is always true.

False When: If $P(x,y)$ is "x > y," there does not exist any x such that $x > y$ for all $y \in \mathbb{R}$, since for any chosen x, there is always a larger y.

Interpreting Statements with Multiple Quantifiers.

Statement	When True?	When False?
$\forall x,\ \forall y,\ P(x,y)$	The predicate $P(x,y)$ is true for all possible pairs (x,y).	There exists at least one pair (x,y) for which $P(x,y)$ is false.
$\forall x,\ \exists y,\ P(x,y)$	For every x, there is at least one y such that $P(x,y)$ is true.	There exists at least one x such that for every y, $P(x,y)$ is false.
$\exists x,\ \forall y,\ P(x,y)$	There exists at least one x such that $P(x,y)$ is true for all y.	For every x, there exists at least one y for which $P(x,y)$ is false.
$\exists x,\ \exists y,\ P(x,y)$	There exists at least one pair (x,y) for which $P(x,y)$ is true.	The predicate $P(x,y)$ is false for all possible pairs (x,y).

- Statement: $\exists x, \exists y, P(x,y)$
  Meaning: There exists at least one pair (x,y) such that $P(x,y)$ is true.

Example

$\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, P(x,y)$ is "$x \cdot y = 4$".

True When: There exist pairs such as $(2,2)$ or $(-2,-2)$ for which $x \cdot y = 4$.

False When: If $P(x,y)$ is "$x \cdot y = -1$," and the domain is restricted to non-negative real numbers $x, y \geq 0$, then no such pair exists because the product of two non-negative numbers cannot be negative.

Interpreting Statements with Multiple Quantifiers.

Statement	When True?	When False?
$\forall x,\ \forall y,\ P(x,y)$	The predicate $P(x,y)$ is true for all possible pairs (x,y).	There exists at least one pair (x,y) for which $P(x,y)$ is false.
$\forall x,\ \exists y,\ P(x,y)$	For every x, there is at least one y such that $P(x,y)$ is true.	There exists at least one x such that for every y, $P(x,y)$ is false.
$\exists x,\ \forall y,\ P(x,y)$	There exists at least one x such that $P(x,y)$ is true for all y.	For every x, there exists at least one y for which $P(x,y)$ is false.
$\exists x,\ \exists y,\ P(x,y)$	There exists at least one pair (x,y) for which $P(x,y)$ is true.	The predicate $P(x,y)$ is false for all possible pairs (x,y).

✎ **Wooclap 17.** Express the statement "*there is a number x such that when it is added to any number, the result is that number, and if it is multiplied by any number, the result is x*" as a logical expression.

Negation of Statements with Multiple Quantifiers.

To negate statements involving multiple quantifiers, negate each quantifier and negate the predicate.

- **Negation of**: $\forall x \in A,\ \forall y \in B,\ P(x, y)$
$$\neg\left(\forall x \in A,\ \forall y \in B,\ P(x, y)\right) \iff \exists x \in A,\ \exists y \in B,\ \neg P(x, y).$$
This means that if it is not true that $P(x, y)$ holds for all pairs (x, y), there must exist at least one pair (x, y) such that $P(x, y)$ is false.
- **Negation of**: $\forall x \in A,\ \exists y \in B,\ P(x, y)$
$$\neg\left(\forall x \in A,\ \exists y \in B,\ P(x, y)\right) \iff \exists x \in A,\ \forall y \in B,\ \neg P(x, y).$$
This means that if for every x there does not exist a y such that $P(x, y)$ is true, then there exists at least one x such that for all y, $P(x, y)$ is false.
- **Negation of**: $\exists x \in A,\ \forall y \in B,\ P(x, y)$
$$\neg\left(\exists x \in A,\ \forall y \in B,\ P(x, y)\right) \iff \forall x \in A,\ \exists y \in B,\ \neg P(x, y).$$
This means that if it is not true that there exists an x such that $P(x, y)$ is true for all y, then for every x, there exists a y such that $P(x, y)$ is false.
- **Negation of**: $\exists x \in A,\ \exists y \in B,\ P(x, y)$
$$\neg\left(\exists x \in A,\ \exists y \in B,\ P(x, y)\right) \iff \forall x \in A,\ \forall y \in B,\ \neg P(x, y).$$
This means that if there does not exist a pair (x, y) such that $P(x, y)$ is true, then $P(x, y)$ must be false for all pairs (x, y).

✎ **Wooclap 18.** Write the negation of the following statement:

a. Statement: $\forall x \in \mathbb{R},,\ \exists y \in \mathbb{R},\ x + y = 0.$

b. Statement: $\exists x \in \mathbb{R}, \forall y \in \mathbb{R},\ xy = 1.$

c. Statement: $\forall c \in$ colors,$\exists a \in$ animals, a is colored c.

d. Statement: $\exists b \in$ books,$\forall p \in$ people, p has read b

Thank you! Questions?