School of Engineering and Computer Science

Mathematical tools applied to Computer Science

Ch3: Relations and Functions

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Week $\#$**5** ✦ 10/OCT/2024 ✦
Week $\#$**6** ✦ 17/OCT/2024 ✦
Week $\#$**7** ✦ 24/OCT/2024 ✦

Totally Orderings

Definition (Ordered Pair)

Let A and B be sets and let $a \in A$ and $b \in B$. An ordered pair (a, b) is a pair of elements with the property that:

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- A pair set $\{a, b\}$ is NOT an ordered pair, since $\{a, b\} = \{b, a\}$
- It should be clear from the context when (a, b) is an ordered pair, and when $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ is an open interval of real numbers.

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Example

- Points in the plane $\mathbb{R}^2$ are represented as ordered pairs. From the graph it can be seen $(1, 2) \neq (2, 1)$ and $(-1, -2) \neq (-2, -1)$.
- Complex numbers $a + ib$ where $i^2 = -1$ and $a,\ b \in \mathbb{R}$, are ordered pairs in the sense that, $a + ib = c + id \iff (a = c) \wedge (b = d)$.

Definition (Cartesian Products)

Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

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✎ **Wooclap 1.**

1. What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?
2. Show that the Cartesian product $B \times A$ is not equal to the Cartesian product $A \times B$.
3. What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?
4. How many different elements does $A \times B$ have if A has m elements and B has n elements?

Notation

We use the notation A^2 to denote $A \times A$, the Cartesian product of the set A with itself. Similarly, $A^3 = A \times A \times A$, $A^4 = A \times A \times A \times A$, and so on.

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Example

If $A = \{1, 2\}$, then $A^2 = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$ and $A^3 = \{(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)\}$.

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✎ **Wooclap 2.**

1. Find A^2 if $A = \{0, 1, a\}$.
2. Find A^3 if $A = \{0, a\}$.
3. How many different elements does A^n have when A has m elements and n is a positive integer?

Notation

We use the notation A^2 to denote $A \times A$, the Cartesian product of the set A with itself. Similarly, $A^3 = A \times A \times A$, $A^4 = A \times A \times A \times A$, and so on.

Example

If $A = \{1, 2\}$, then $A^2 = \{(1,1), (1,2), (2,1), (2,2)\}$ and $A^3 = \{(1,1,1), (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)\}$.

✎ **Wooclap 3.**

1. Find A^2 if $A = \{0, 1, a\}$.
2. Find A^3 if $A = \{0, a\}$.
3. How many different elements does A^n have when A has m elements and n is a positive integer?

Theorem :

If A and B are sets, then $(x, y) \notin A \times B \Longleftrightarrow x \notin A \quad \text{or} \quad y \notin B$.

Definition (Binary Relation)

Let A and B be two sets. A **binary relation** from A to B is a subset $\mathcal{R}$ of the Cartesian product $A \times B$.

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Notation:

- We write $a \, \mathcal{R} \, b$ to denote that $(a, b) \in \mathcal{R}$, meaning a is related to b through the relation $\mathcal{R}$. Conversely, $a \neg\mathcal{R} \, b$ indicates that $(a, b) \notin \mathcal{R}$.
- When (a, b) is an element of $\mathcal{R}$, we say that a **is related to** b **by** $\mathcal{R}$.
- If $\mathcal{R} \subseteq A \times A$, we say that $\mathcal{R}$ is a **relation on** A.

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- If $\mathcal{R} \subseteq A \times A$, we say that $\mathcal{R}$ is a **relation on** A.

Example

- Let A represent the set of all male human beings and B represent the set of all human beings. The relation T from A to B can be described as:

$$T = \{(x, y) : x \text{ is the father of } y\}.$$

- Consider the set $W = \Big\{(1, 2), (2, 1), (5, \pi)\Big\}$. This is an example of a relation where no specific rule defines it—sometimes, relations are simply defined by listing their elements.

Definition (Functions as Relations)

Recall that a function f from a set A to a set B assigns exactly one element of B to each element of A. The notation for a function is:

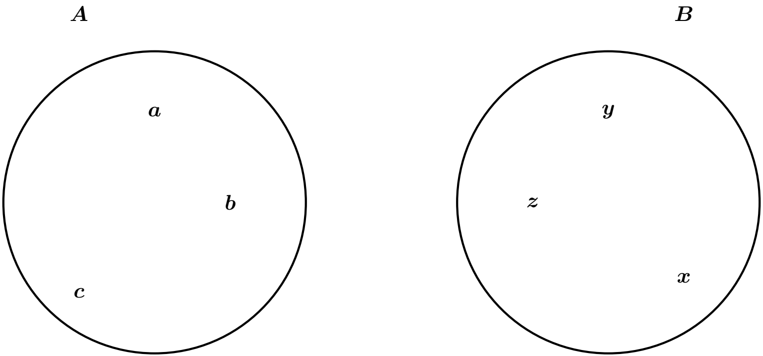
$$\begin{array}{cccccc} f & : & A & \longmapsto & B \\ & & a & \longmapsto & b = f(a) \end{array}$$

The **graph of** f is the set of ordered pairs (a, b) such that $b = f(a)$. This can be expressed as:

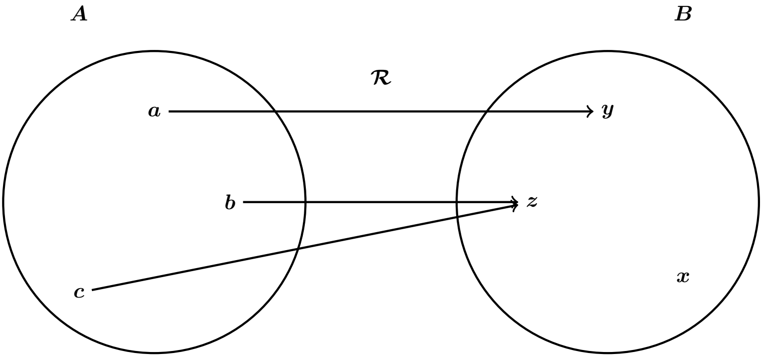
$$\text{graph}(f) := \{(a, b) \in A \times B : b = f(a)\}$$

Since the graph of f is a subset of $A \times B$, it is also a relation from A to B. Therefore, the relation defined by f is:

$$\mathcal{R}_f = \text{graph}(f) := \{(a, b) \in A \times B : b = f(a)\}$$



$$A \times B = \Big\{(a,x),\ (b,x),\ (c,x),\ (a,y),\ (b,y),\ (c,y),\ (a,z),\ (b,z),\ (c,z)\Big\}$$



$$\mathcal{R} = \Big\{ (a, y),\ (b, z),\ (c, z) \Big\}$$

Example

Let A be the set $\{1, 2, 3, 4\}$. Which ordered pairs are in the relation $R = \{(a, b) : a \text{ divides } b\}$?

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Solution

For each pair (a, b), a divides b if and only if b is divisible by a, i.e., $b = 0\ [\mod a]$.
We check for each element a and b:

- For $a = 1$: Since 1 divides every number, the pairs are:$(1, 1), (1, 2), (1, 3), (1, 4)$
- For $a = 2$: 2 divides 2 and 4, so the pairs are: $(2, 2), (2, 4)$
- For $a = 3$: 3 divides only 3, so the pair is: $(3, 3)$
- For $a = 4$: 4 divides only 4, so the pair is: $(4, 4)$

Thus, the relation R consists of the following ordered pairs:

$$R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)\}$$

Example

Consider these relations on the set of integers:

$$\mathcal{R}_1 = \{(a, b) : a \leq b\},$$
$$\mathcal{R}_2 = \{(a, b) : a > b\},$$
$$\mathcal{R}_3 = \{(a, b) : a = b \text{ or } a = -b\},$$
$$\mathcal{R}_4 = \{(a, b) : a = b\},$$
$$\mathcal{R}_5 = \{(a, b) : a = b + 1\},$$
$$\mathcal{R}_6 = \{(a, b) : a + b \leq 3\}.$$

Which of these relations contain each of the pairs $(1, 1)$, $(1, 2)$, $(2, 1)$, $(1, -1)$, and $(2, 2)$?

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$$\mathcal{R}_6 = \{(a, b) : a + b \leq 3\}.$$

Which of these relations contain each of the pairs $(1, 1)$, $(1, 2)$, $(2, 1)$, $(1, -1)$, and $(2, 2)$?

Solution

- $(1, 1)$ is in $\mathcal{R}_1, \mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_6$.
- $(1, 2)$ is in $\mathcal{R}_1, \mathcal{R}_6$.
- $(2, 1)$ is in $\mathcal{R}_2, \mathcal{R}_5, \mathcal{R}_6$.
- $(1, -1)$ is in $\mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_6$.
- $(2, 2)$ is in $\mathcal{R}_1, \mathcal{R}_3, \mathcal{R}_4$.

✎ **Wooclap 4.** Consider the relation $\mathcal{R}$ on $\mathbb{R}$ given by $\mathcal{R} = \Big\{(x, y) : x = y\Big\}$.

1. Sketch the graph of $\mathcal{R} \in \mathbb{R}^2$.
2. Are the following true or false?
$$1 \,\mathcal{R}\, 1\,, \quad 1 \,\mathcal{R}\, 2\,, \quad (-3, 3) \in \mathcal{R}\,.$$
3. If $a \,\mathcal{R}\, 100$, what is the value of a?

✎ **Wooclap 5.** Consider the relation $\mathcal{R}$ on $\mathbb{R}$ given by $\mathcal{R} = \left\{(x, y) : x = y\right\}$.

1. Sketch the graph of $\mathcal{R} \in \mathbb{R}^2$.
2. Are the following true or false?
$$1\,\mathcal{R}\,1\,,\quad 1\,\mathcal{R}\,2\,,\quad (-3, 3) \in \mathcal{R}\,.$$
3. If $a\,\mathcal{R}\,100$, what is the value of a?

✎ **Wooclap 6.** Let $X = \{0, 1, 2, 3\}$, and let the relation $\mathcal{R}$ on X be given by

$$\mathcal{R} = \left\{(x, y) : \exists z \in \mathbb{N}^*, x + z = y\right\}.$$

1. What is an easier way of expressing the relation $\mathcal{R}$?
2. List all the elements of $\mathcal{R}$.
3. Sketch $X \times X$, and circle the elements of $\mathcal{R}$.

✎ **Wooclap 7.** Consider the relation $\mathcal{R}$ on $\mathbb{R}$ given by $\mathcal{R} = \left\{(x, y) : x = y\right\}$.

1. Sketch the graph of $\mathcal{R} \in \mathbb{R}^2$.
2. Are the following true or false?

$$1\,\mathcal{R}\,1\,,\quad 1\,\mathcal{R}\,2\,,\quad (-3, 3) \in \mathcal{R}\,.$$

3. If $a\,\mathcal{R}\,100$, what is the value of a?

✎ **Wooclap 8.** Let $X = \{0, 1, 2, 3\}$, and let the relation $\mathcal{R}$ on X be given by

$$\mathcal{R} = \left\{(x, y) : \exists z \in \mathbb{N}^*, x + z = y\right\}.$$

1. What is an easier way of expressing the relation $\mathcal{R}$?
2. List all the elements of $\mathcal{R}$.
3. Sketch $X \times X$, and circle the elements of $\mathcal{R}$.

✎ **Wooclap 9.** Let S be the relation on $\mathbb{Z} - \{0\}$ given by $S = \left\{(x, y) : \exists z \in \mathbb{Z}, xz = y\right\}$.

1. Describe the relation S.
2. Are the following true or false?

$$(2, -4) \in S\,,\quad -3\,S\,0\,,\quad (3, 5) \in S\,.$$

Definition (Domain)

Let $\mathcal{R}$ be a relation from A to B. The domain of $\mathcal{R}$, denoted as Dom$\mathcal{R}$, is defined as:

$$\text{Dom}\mathcal{R} = \{x \in A : \exists y \in B,\ (x, y) \in \mathcal{R}\}$$

Notes:

- The domain Dom$\mathcal{R}$ is a subset of A, meaning Dom$\mathcal{R} \subseteq A$.
- Dom$\mathcal{R}$ consists of all **first** elements of the ordered pairs in the relation $\mathcal{R}$.

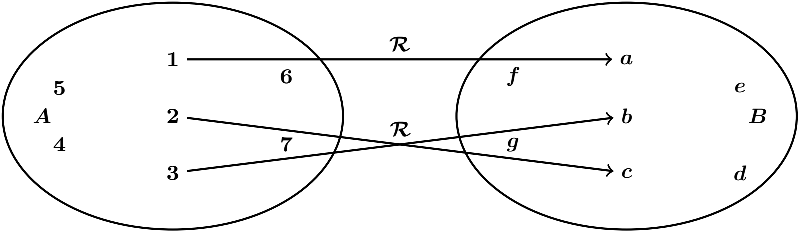
Definition (Range)

Let $\mathcal{R}$ be a relation from A to B. The range of $\mathcal{R}$, denoted as Range$\mathcal{R}$, is defined as:

$$\text{Range}\mathcal{R} = \{y \in B : \exists x \in A,\ (x, y) \in \mathcal{R}\}$$

Notes:

- The range Range$\mathcal{R}$ is a subset of B, meaning Range$\mathcal{R} \subseteq B$.
- Range$\mathcal{R}$ consists of all **second** elements of the ordered pairs in the relation $\mathcal{R}$.


1
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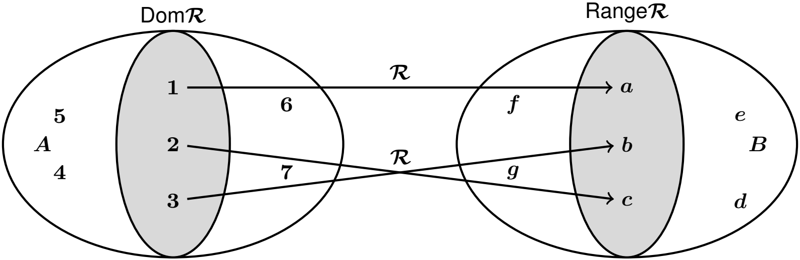
B



- Dom$\mathcal{R} = \left\{\mathbf{1},\ \mathbf{2},\ \mathbf{3}\right\}$
- Range$\mathcal{R} = \left\{\boldsymbol{a},\ \boldsymbol{b},\ \boldsymbol{c}\right\}$
- $\mathcal{R} = \left\{(\mathbf{1}, \boldsymbol{a}),\ (\mathbf{2}, \boldsymbol{c}),\ (\mathbf{3}, \boldsymbol{b})\right\}$

✎ Wooclap 10.

❶ Let $A = \{0, 1, 2, 3\}$ and let $\mathcal{R}_1$ be the relation on A given by $\mathcal{R}_1 = \Big\{(0, 0), (0, 1), (0, 2), (3, 0)\Big\}$.

Determine: Dom$\mathcal{R}_1$ and Range$\mathcal{R}_1$.

❷ Let $\mathcal{R}_2$ be the relation on $\mathbb{Z}$ given by $\mathcal{R}_2 = \Big\{(x, y) : xy \neq 0\Big\}$.

Determine: Dom$\mathcal{R}_2$ and Range$\mathcal{R}_2$.

❸ Let $\mathcal{R}_3$ be the relation from $\mathbb{Z}$ to $\mathbb{Q}$ given by $\mathcal{R}_3 = \Big\{(x, y) : x \neq 0 \wedge y = \frac{1}{x}\Big\}$.

Determine: Dom$\mathcal{R}_3$ and Range$\mathcal{R}_3$.

Definition

Let $\mathcal{R}$ be a relation from set A to set B. The **inverse relation**, denoted $\mathcal{R}^{-1}$, is defined as:

$$\mathcal{R}^{-1} = \left\{(y, x) : (x, y) \in \mathcal{R}\right\}.$$

- For a relation $\mathcal{R}$ from A to B, the inverse relation $\mathcal{R}^{-1}$ is formed by interchanging the elements of all ordered pairs in $\mathcal{R}$. This is generally more straightforward for finite relations (defined by a list of pairs) than for infinite relations (defined by a formula).
- The **domain** of $\mathcal{R}^{-1}$ is equal to the **range** of $\mathcal{R}$, denoted as Dom$\mathcal{R}^{-1}$ $=$ Range$\mathcal{R} \subseteq B$, and the **range** of $\mathcal{R}^{-1}$ is equal to the **domain** of $\mathcal{R}$, denoted as Range$\mathcal{R}^{-1}$ $=$ Dom$\mathcal{R} \subseteq A$.

Example

Define a relation $\mathcal{R}$ on $\mathbb{R}$ as: $\mathcal{R} = \left\{(x, y) : y = 2x\right\}$

1. List three elements of $\mathcal{R}$:

2. List three elements of $\mathcal{R}^{-1}$:

3. Provide a simple definition for $\mathcal{R}^{-1}$:

4. Sketch a graph of $\mathcal{R}$ and $\mathcal{R}^{-1}$ on coordinate axes, and highlight the elements of $\mathcal{R}^{-1}$ with circles.

Example

Define a relation $\mathcal{R}$ on $\mathbb{R}$ as: $\mathcal{R} = \left\{ (x, y) : y = 2x \right\}$

1. List three elements of $\mathcal{R}$:
   - $(1, 2), \quad (2, 4), \quad (3, 6)$
2. List three elements of $\mathcal{R}^{-1}$:
3. Provide a simple definition for $\mathcal{R}^{-1}$:
4. Sketch a graph of $\mathcal{R}$ and $\mathcal{R}^{-1}$ on coordinate axes, and highlight the elements of $\mathcal{R}^{-1}$ with circles.

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1. List three elements of $\mathcal{R}$:
   - $(1, 2), \quad (2, 4), \quad (3, 6)$
2. List three elements of $\mathcal{R}^{-1}$:
   - $(2, 1), \quad (4, 2), \quad (6, 3)$
3. Provide a simple definition for $\mathcal{R}^{-1}$:
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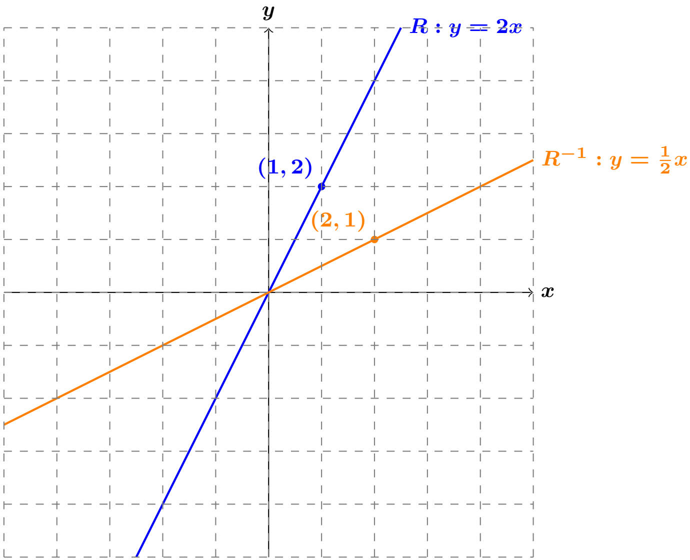
2. List three elements of $\mathcal{R}^{-1}$:
   - $(2, 1), \quad (4, 2), \quad (6, 3)$
3. Provide a simple definition for $\mathcal{R}^{-1}$:
   - $\mathcal{R}^{-1} = \left\{(y, x) : y = 2x\right\} = \left\{(x, y) : x = 2y\right\} = \left\{(x, y) : y = \frac{1}{2}x\right\}$
4. Sketch a graph of $\mathcal{R}$ and $\mathcal{R}^{-1}$ on coordinate axes, and highlight the elements of $\mathcal{R}^{-1}$ with circles.


y
$R : y = 2x$
$R^{-1} : y = \frac{1}{2}x$
$(1, 2)$
$(2, 1)$
x

Definition (Reflexive Relation)

A relation $\mathcal{R}$ on a set A is said to be **reflexive** if for every element $a \in A$, the pair (a, a) belongs to $\mathcal{R}$. In other words, $\mathcal{R}$ is reflexive if every element of the set is related to itself.

$$\forall a \in A, \ (a, a) \in \mathcal{R}.$$

Example

Consider the following relations on the set $A = \{1, 2, 3, 4\}$:

- $\mathcal{R}_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$
- $\mathcal{R}_2 = \{(1,1),(1,2),(2,1)\}$
- $\mathcal{R}_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$
- $\mathcal{R}_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$
- $\mathcal{R}_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$
- $\mathcal{R}_6 = \{(3,4)\}$

Determine which of these relations are reflexive.

Solution

The relations $\mathcal{R}_3$ and $\mathcal{R}_5$ are reflexive because they contain all the pairs (a, a) for $a \in \{1, 2, 3, 4\}$, specifically the pairs $(1, 1)$, $(2, 2)$, $(3, 3)$, and $(4, 4)$.
The other relations are not reflexive because they are missing at least one of these self-pairs. Specifically:

- $\mathcal{R}_1$ is missing the pair $(3, 3)$,
- $\mathcal{R}_2$ is missing $(2, 2)$, $(3, 3)$, and $(4, 4)$,
- $\mathcal{R}_4$ lacks all the self-pairs, and
- $\mathcal{R}_6$ only contains the pair $(3, 4)$ and no self-pairs.

Example

Which of the following relations are reflexive?

$$\mathcal{R}_1 = \{(a,b) : a \leq b\},$$

$$\mathcal{R}_2 = \{(a,b) : a > b\},$$

$$\mathcal{R}_3 = \{(a,b) : a = b \text{ or } a = -b\},$$

$$\mathcal{R}_4 = \{(a,b) : a = b\},$$

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Example

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solution

The reflexive relations are R_1 (because $a \leq a$ for every integer a), R_3, and R_4. For each of the other relations in this example it is easy to find a pair of the form (a, a) that is not in the relation.

Solution

$\mathcal{R}_1$: $\mathcal{R}_1 = \{(a, b) : a \leq b\}$
For any a, $a \leq a$ is always true. Therefore, $\mathcal{R}_1$ is **reflexive**.

$\mathcal{R}_2$: $\mathcal{R}_2 = \{(a, b) : a > b\}$
For a, $a > a$ is never true. Therefore, $\mathcal{R}_2$ is **not reflexive**.

$\mathcal{R}_3$: $\mathcal{R}_3 = \{(a, b) : a = b$ or $a = -b\}$
For reflexivity, we require $a = a$, which holds for any a. Therefore, $\mathcal{R}_3$ is **reflexive**.

$\mathcal{R}_4$: $\mathcal{R}_4 = \{(a, b) : a = b\}$
Since $a = a$ is true for any a, $\mathcal{R}_4$ is **reflexive**.

$\mathcal{R}_5$: $\mathcal{R}_5 = \{(a, b) : a = b + 1\}$
Reflexivity would require $a = a + 1$, which is never true for any a. Therefore, $\mathcal{R}_5$ is **not reflexive**.

$\mathcal{R}_6$: $\mathcal{R}_6 = \{(a, b) : a + b \leq 3\}$
For reflexivity, we require $a + a \leq 3$, or equivalently $2a \leq 3$. This holds only for $a \leq 1.5$, so $\mathcal{R}_6$ is **not reflexive** for all values of a.

Definition (Symmetric Relation)

A relation $\mathcal{R}$ on a set A is called **symmetric** if for all $a, b \in A$, whenever $(a, b) \in \mathcal{R}$, it follows that $(b, a) \in \mathcal{R}$. In other words, if one element is related to another, the reverse must also hold.

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Definition (Antisymmetric Relation)

A relation $\mathcal{R}$ on a set A is called **antisymmetric** if for all $a, b \in A$, whenever both $(a, b) \in \mathcal{R}$ and $(b, a) \in \mathcal{R}$, it must be the case that $a = b$. This means that no two distinct elements can be mutually related unless they are the same element.

Definition (Symmetric Relation)

A relation $\mathcal{R}$ on a set A is called **symmetric** if for all $a, b \in A$, whenever $(a, b) \in \mathcal{R}$, it follows that $(b, a) \in \mathcal{R}$. In other words, if one element is related to another, the reverse must also hold.

Definition (Antisymmetric Relation)

A relation $\mathcal{R}$ on a set A is called **antisymmetric** if for all $a, b \in A$, whenever both $(a, b) \in \mathcal{R}$ and $(b, a) \in \mathcal{R}$, it must be the case that $a = b$. This means that no two distinct elements can be mutually related unless they are the same element.

Note

A relation can be both symmetric and antisymmetric, only symmetric, only antisymmetric, or neither. The key difference is that symmetric relations require mutual connections, while antisymmetric relations forbid mutual connections unless the elements are identical.

Example

Determine which of the following relations are symmetric and which are antisymmetric:

$$\mathcal{R}_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\},$$
$$\mathcal{R}_2 = \{(1,1),(1,2),(2,1)\},$$
$$\mathcal{R}_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\},$$
$$\mathcal{R}_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\},$$
$$\mathcal{R}_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\},$$
$$\mathcal{R}_6 = \{(3,4)\}.$$

Solution

- $\mathcal{R}_1$:
  Symmetric? No, because $(3, 4) \in \mathcal{R}_1$ but $(4, 3) \notin \mathcal{R}_1$.
  Antisymmetric? No, because $(1, 2) \in \mathcal{R}_1$ and $(2, 1) \in \mathcal{R}_1$ but $1 \neq 2$.

Solution

- $\mathcal{R}_1$:
  Symmetric? No, because $(3, 4) \in \mathcal{R}_1$ but $(4, 3) \notin \mathcal{R}_1$.
  Antisymmetric? No, because $(1, 2) \in \mathcal{R}_1$ and $(2, 1) \in \mathcal{R}_1$ but $1 \neq 2$.
- $\mathcal{R}_2$:
  Symmetric? Yes, because for all pairs $(a, b) \in \mathcal{R}_2$, we also have $(b, a) \in \mathcal{R}_2$.
  Antisymmetric? No, because $(1, 2) \in \mathcal{R}_2$ and $(2, 1) \in \mathcal{R}_2$ but $1 \neq 2$.

Solution

- $\mathcal{R}_1$:
  Symmetric? No, because $(3, 4) \in \mathcal{R}_1$ but $(4, 3) \notin \mathcal{R}_1$.
  Antisymmetric? No, because $(1, 2) \in \mathcal{R}_1$ and $(2, 1) \in \mathcal{R}_1$ but $1 \neq 2$.
- $\mathcal{R}_2$:
  Symmetric? Yes, because for all pairs $(a, b) \in \mathcal{R}_2$, we also have $(b, a) \in \mathcal{R}_2$.
  Antisymmetric? No, because $(1, 2) \in \mathcal{R}_2$ and $(2, 1) \in \mathcal{R}_2$ but $1 \neq 2$.
- $\mathcal{R}_3$:
  Symmetric? Yes, all reverse pairs are present, such as $(1, 2)$ and $(2, 1)$, $(1, 4)$ and $(4, 1)$.
  Antisymmetric? No, because $(1, 2) \in \mathcal{R}_3$ and $(2, 1) \in \mathcal{R}_3$, but $1 \neq 2$.

Solution

- $\mathcal{R}_1$:

  Symmetric? No, because $(3, 4) \in \mathcal{R}_1$ but $(4, 3) \notin \mathcal{R}_1$.

  Antisymmetric? No, because $(1, 2) \in \mathcal{R}_1$ and $(2, 1) \in \mathcal{R}_1$ but $1 \neq 2$.
- $\mathcal{R}_2$:

  Symmetric? Yes, because for all pairs $(a, b) \in \mathcal{R}_2$, we also have $(b, a) \in \mathcal{R}_2$.

  Antisymmetric? No, because $(1, 2) \in \mathcal{R}_2$ and $(2, 1) \in \mathcal{R}_2$ but $1 \neq 2$.
- $\mathcal{R}_3$:

  Symmetric? Yes, all reverse pairs are present, such as $(1, 2)$ and $(2, 1)$, $(1, 4)$ and $(4, 1)$.

  Antisymmetric? No, because $(1, 2) \in \mathcal{R}_3$ and $(2, 1) \in \mathcal{R}_3$, but $1 \neq 2$.
- $\mathcal{R}_4$:

  Symmetric? No, because no reverse pairs are present.

  Antisymmetric? Yes, there are no pairs (a, b) and (b, a) with $a \neq b$.

Solution

- $\mathcal{R}_1$:
  Symmetric? No, because $(3, 4) \in \mathcal{R}_1$ but $(4, 3) \notin \mathcal{R}_1$.
  Antisymmetric? No, because $(1, 2) \in \mathcal{R}_1$ and $(2, 1) \in \mathcal{R}_1$ but $1 \neq 2$.
- $\mathcal{R}_2$:
  Symmetric? Yes, because for all pairs $(a, b) \in \mathcal{R}_2$, we also have $(b, a) \in \mathcal{R}_2$.
  Antisymmetric? No, because $(1, 2) \in \mathcal{R}_2$ and $(2, 1) \in \mathcal{R}_2$ but $1 \neq 2$.
- $\mathcal{R}_3$:
  Symmetric? Yes, all reverse pairs are present, such as $(1, 2)$ and $(2, 1)$, $(1, 4)$ and $(4, 1)$.
  Antisymmetric? No, because $(1, 2) \in \mathcal{R}_3$ and $(2, 1) \in \mathcal{R}_3$, but $1 \neq 2$.
- $\mathcal{R}_4$:
  Symmetric? No, because no reverse pairs are present.
  Antisymmetric? Yes, there are no pairs (a, b) and (b, a) with $a \neq b$.
- $\mathcal{R}_5$:
  Symmetric? No, because $(1, 2) \in \mathcal{R}_5$ but $(2, 1) \notin \mathcal{R}_5$.
  Antisymmetric? Yes, there are no reverse pairs with $a \neq b$.

Solution

- $\mathcal{R}_1$:
  Symmetric? No, because $(3,4) \in \mathcal{R}_1$ but $(4,3) \notin \mathcal{R}_1$.
  Antisymmetric? No, because $(1,2) \in \mathcal{R}_1$ and $(2,1) \in \mathcal{R}_1$ but $1 \neq 2$.
- $\mathcal{R}_2$:
  Symmetric? Yes, because for all pairs $(a,b) \in \mathcal{R}_2$, we also have $(b,a) \in \mathcal{R}_2$.
  Antisymmetric? No, because $(1,2) \in \mathcal{R}_2$ and $(2,1) \in \mathcal{R}_2$ but $1 \neq 2$.
- $\mathcal{R}_3$:
  Symmetric? Yes, all reverse pairs are present, such as $(1,2)$ and $(2,1)$, $(1,4)$ and $(4,1)$.
  Antisymmetric? No, because $(1,2) \in \mathcal{R}_3$ and $(2,1) \in \mathcal{R}_3$, but $1 \neq 2$.
- $\mathcal{R}_4$:
  Symmetric? No, because no reverse pairs are present.
  Antisymmetric? Yes, there are no pairs (a,b) and (b,a) with $a \neq b$.
- $\mathcal{R}_5$:
  Symmetric? No, because $(1,2) \in \mathcal{R}_5$ but $(2,1) \notin \mathcal{R}_5$.
  Antisymmetric? Yes, there are no reverse pairs with $a \neq b$.
- $\mathcal{R}_6$:
  Symmetric? No, because $(3,4) \in \mathcal{R}_6$ but $(4,3) \notin \mathcal{R}_6$.
  Antisymmetric? Yes, because there are no reverse pairs.

Example

Which of the following relations are symmetric and which are antisymmetric?

$$\mathcal{R}_1 = \{(a,b) : a \leq b\},$$
$$\mathcal{R}_2 = \{(a,b) : a > b\},$$
$$\mathcal{R}_3 = \{(a,b) : a = b \text{ or } a = -b\},$$
$$\mathcal{R}_4 = \{(a,b) : a = b\},$$
$$\mathcal{R}_5 = \{(a,b) : a = b + 1\},$$
$$\mathcal{R}_6 = \{(a,b) : a + b \leq 3\}.$$

Solution

- $\mathcal{R}_1 = \{(a, b) : a \leq b\}$:
  Symmetric? No, because if $a \leq b$, it does not imply $b \leq a$ unless $a = b$.
  Antisymmetric? Yes, since if $a \leq b$ and $b \leq a$, then $a = b$.

Solution

- $\mathcal{R}_1 = \{(a, b) : a \leq b\}$:
  Symmetric? No, because if $a \leq b$, it does not imply $b \leq a$ unless $a = b$.
  Antisymmetric? Yes, since if $a \leq b$ and $b \leq a$, then $a = b$.

- $\mathcal{R}_2 = \{(a, b) : a > b\}$:
  Symmetric? No, because if $a > b$, it does not imply $b > a$.
  Antisymmetric? Yes, because no two distinct elements a and b can satisfy both $a > b$ and $b > a$ simultaneously.

Solution

- $\mathcal{R}_1 = \{(a, b) : a \leq b\}$:
  Symmetric? No, because if $a \leq b$, it does not imply $b \leq a$ unless $a = b$.
  Antisymmetric? Yes, since if $a \leq b$ and $b \leq a$, then $a = b$.

- $\mathcal{R}_2 = \{(a, b) : a > b\}$:
  Symmetric? No, because if $a > b$, it does not imply $b > a$.
  Antisymmetric? Yes, because no two distinct elements a and b can satisfy both $a > b$ and $b > a$ simultaneously.

- $\mathcal{R}_3 = \{(a, b) : a = b \text{ or } a = -b\}$:
  Symmetric? Yes, if $a = b$ or $a = -b$, then $b = a$ or $b = -a$ holds, making it symmetric.
  Antisymmetric? No, because $a = -b$ allows both (a, b) and (b, a) with $a \neq b$.

Solution

- $\mathcal{R}_1 = \{(a, b) : a \leq b\}$:
  Symmetric? No, because if $a \leq b$, it does not imply $b \leq a$ unless $a = b$.
  Antisymmetric? Yes, since if $a \leq b$ and $b \leq a$, then $a = b$.

- $\mathcal{R}_2 = \{(a, b) : a > b\}$:
  Symmetric? No, because if $a > b$, it does not imply $b > a$.
  Antisymmetric? Yes, because no two distinct elements a and b can satisfy both $a > b$ and $b > a$ simultaneously.

- $\mathcal{R}_3 = \{(a, b) : a = b \text{ or } a = -b\}$:
  Symmetric? Yes, if $a = b$ or $a = -b$, then $b = a$ or $b = -a$ holds, making it symmetric.
  Antisymmetric? No, because $a = -b$ allows both (a, b) and (b, a) with $a \neq b$.

- $\mathcal{R}_4 = \{(a, b) : a = b\}$:
  Symmetric? Yes, since $a = b$ implies $b = a$.
  Antisymmetric? Yes, because if $(a, b) \in \mathcal{R}_4$ and $(b, a) \in \mathcal{R}_4$, then $a = b$.

Solution

- $\mathcal{R}_5 = \{(a, b) : a = b + 1\}$:
  Symmetric? No, because $a = b + 1$ does not imply $b = a + 1$.
  Antisymmetric? Yes, because no distinct elements can satisfy both $a = b + 1$ and $b = a + 1$.

Solution

- $\mathcal{R}_5 = \{(a, b) : a = b + 1\}$:
  Symmetric? No, because $a = b + 1$ does not imply $b = a + 1$.
  Antisymmetric? Yes, because no distinct elements can satisfy both $a = b + 1$ and $b = a + 1$.

- $\mathcal{R}_6 = \{(a, b) : a + b \leq 3\}$:
  Symmetric? Yes, since $a + b \leq 3$ implies $b + a \leq 3$.
  Antisymmetric? No, because there can be distinct pairs (a, b) and (b, a) where $a + b \leq 3$.

Solution

- $\mathcal{R}_5 = \{(a, b) : a = b + 1\}$:
  Symmetric? No, because $a = b + 1$ does not imply $b = a + 1$.
  Antisymmetric? Yes, because no distinct elements can satisfy both $a = b + 1$ and $b = a + 1$.

- $\mathcal{R}_6 = \{(a, b) : a + b \leq 3\}$:
  Symmetric? Yes, since $a + b \leq 3$ implies $b + a \leq 3$.
  Antisymmetric? No, because there can be distinct pairs (a, b) and (b, a) where $a + b \leq 3$.

Summary:

Symmetric relations: $\mathcal{R}_3, \mathcal{R}_4, \mathcal{R}_6$

Antisymmetric relations: $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_4, \mathcal{R}_5$

Example

Is the divides relation on the set of positive integers symmetric? Is it antisymmetric?

$$\mathcal{R} = \{(a, b) : a \text{ divides } b\}$$

Solution

Symmetric Relation:

Counterexample: Let $a = 2$ and $b = 4$.

Here, 2 divides 4 (i.e., $2 \mid 4$).

However, 4 does not divide 2 (i.e., $4 \nmid 2$).

Since we found a pair $(2, 4) \in \mathcal{R}$ for which $(4, 2) \notin \mathcal{R}$, we conclude that:

The divides relation is **not symmetric**.

Antisymmetric Relation:

Analysis: Assume $a \mid b$ and $b \mid a$. This means there exist integers k and m such that:

$$b = a \cdot k \quad \text{and} \quad a = b \cdot m$$

Substituting $b = a \cdot k$ into $a = b \cdot m$ gives us:

$$a = (a \cdot k) \cdot m = a \cdot (k \cdot m)$$

For this to hold true for all positive integers a, we need $k \cdot m = 1$.

The only solution for k and m in the positive integers is $k = 1$ and $m = 1$, which implies: $a = b$

Thus, if $a \mid b$ and $b \mid a$, we conclude that $a = b$.

Definition (Transitive Relation)

A relation $\mathcal{R}$ on a set A is called **transitive** if, for all $a, b, c \in A$, whenever $(a, b) \in \mathcal{R}$ and $(b, c) \in \mathcal{R}$, it follows that $(a, c) \in \mathcal{R}$.

This property implies that if one element is related to a second element, and that second element is related to a third element, then the first element must also be related to the third element.

Definition (Transitive Relation)

A relation $\mathcal{R}$ on a set A is called **transitive** if, for all $a, b, c \in A$, whenever $(a, b) \in \mathcal{R}$ and $(b, c) \in \mathcal{R}$, it follows that $(a, c) \in \mathcal{R}$.

This property implies that if one element is related to a second element, and that second element is related to a third element, then the first element must also be related to the third element.

Example

Consider the relation $\mathcal{R}$ defined on the set of integers $\mathbb{Z}$ as follows:

$$\mathcal{R} = \{(a, b) : a \leq b\}$$

This relation is transitive because if $a \leq b$ and $b \leq c$, then it necessarily follows that $a \leq c$.
Conversely, consider the relation S defined as:

$$S = \{(a, b) : a \text{ is a sibling of } b\}$$

This relation is **not transitive** because being a sibling does not imply that if a is a sibling of b and b is a sibling of c, then a must be a sibling of c.

Example

Which of the following relations are transitive?

$$\mathcal{R}_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\},$$
$$\mathcal{R}_2 = \{(1,1),(1,2),(2,1)\},$$
$$\mathcal{R}_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\},$$
$$\mathcal{R}_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\},$$
$$\mathcal{R}_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\},$$
$$\mathcal{R}_6 = \{(3,4)\}.$$

Solution

$\mathcal{R}_1$: Not Transitive, since $(1, 2)$ and $(2, 1)$ do not imply $(1, 1)$.

$\mathcal{R}_2$: Not Transitive, since $(2, 1)$ and $(1, 2)$ do not imply $(2, 2)$.

$\mathcal{R}_3$: Not Transitive, since $(4, 1)$ and $(1, 2)$ do not imply $(4, 2)$.

$\mathcal{R}_4$: Transitive, because $(3, 2)$ and $(2, 1)$, $(4, 2)$ and $(2, 1)$, $(4, 3)$ and $(3, 1)$, and $(4, 3)$ and $(3, 2)$ are the only such sets of pairs, and $(3, 1)$, $(4, 1)$, and $(4, 2)$ belong to $\mathcal{R}_4$.

$\mathcal{R}_5$: Transitive, all transitive pairs hold.

$\mathcal{R}_6$: Transitive (vacuously), since there are no pairs to violate transitivity.

✎ **Wooclap 11.**

Let E be a set and let $\mathcal{R}$ be a binary relation in E. Check in each of the following cases if $\mathcal{R}$ is reflexive, symmetric, antisymmetric or transitive:

(a) $E = P(A)$, where A s a non-empty set and $X\mathcal{R}Y \iff X \cap Y = \varnothing$.

(b) $E = \mathbb{N} \times \mathbb{N}$ and $(a, b)\mathcal{R}(c, d) \iff c \leq d$.

(c) $E = \{0, 1\}$ and $G_{\mathcal{R}} = \{(1, 0)\}$.

(d) $E = \{0, 1\}$ and $G_{\mathcal{R}} = \{(0, 0),\ (1, 0),\ (0, 1)\}$.

Definition

A relation $\mathcal{R}$ on a set A is called an **equivalence relation** if it satisfies three properties:

- **Reflexivity:** For every element $a \in A$, $(a, a) \in \mathcal{R}$.
- **Symmetry:** For all $a, b \in A$, if $(a, b) \in \mathcal{R}$, then $(b, a) \in \mathcal{R}$.
- **Transitivity:** For all $a, b, c \in A$, if $(a, b) \in \mathcal{R}$ and $(b, c) \in \mathcal{R}$, then $(a, c) \in \mathcal{R}$.

- Equivalence relations are fundamental in mathematics and computer science, as they enable the classification of elements into distinct equivalence classes.
- When two elements are related through an equivalence relation, we can meaningfully say they are equivalent or belong to the same class.
- To establish that a relation R is an equivalence relation, it is necessary to demonstrate that all three properties hold.
- Conversely, to disprove that a relation R is an equivalence relation, it suffices to provide a counterexample for any one of the three properties.

Example

Consider the relation $\mathcal{R}$ on the set of integers defined by $a\mathcal{R}b$ if and only if $a = b$ or $a = -b$. We can verify that:

- $\mathcal{R}$ is reflexive, as $a = a$ for any integer a.
- $\mathcal{R}$ is symmetric, since if $a = b$ or $a = -b$, then it follows that $b = a$ or $b = -a$.
- $\mathcal{R}$ is transitive; if $a = b$ or $a = -b$ and $b = c$ or $b = -c$, then it can be shown that a relates to c.

Hence, $\mathcal{R}$ is indeed an equivalence relation.

Reflexive Relation

	A	B	C
A	1	0	0
B	0	1	1
C	0	0	1

(A, A), (B, B), and (C, C) are 1.

Antisymmetric Relation

	A	B	C
A	1	1	0
B	0	1	0
C	0	0	1

Only (A, B) and (B, A) are both 1 if A = B.

Symmetric Relation

	A	B	C
A	1	1	0
B	1	1	1
C	0	1	1

If (A, B) is 1, then (B, A) is also 1.

Transitive Relation

	A	B	C
A	1	1	1
B	0	1	1
C	0	0	1

If (A, B) and (B, C) are both 1, then (A, C) must also be 1.

Equivalence Relation

	A	B	C
A	1	1	0
B	1	1	0
C	0	0	1

Reflexive, symmetric, and transitive properties hold.

✎ **Wooclap 12.** Let $\mathcal{R}$ be the relation on the set of real numbers such that $a\mathcal{R}b$ if and only if $a - b$ is an integer.

Is $\mathcal{R}$ an equivalence relation?

✎ **Wooclap 13.** Let $\mathcal{R}$ be the relation on the set of real numbers such that $a\mathcal{R}b$ if and only if $a-b$ is an integer.

Is $\mathcal{R}$ an equivalence relation?

Solution

- **Reflexive** Because $a-a=0$ is an integer for all real numbers a, $a\mathcal{R}a$ for all real numbers a. Hence, $\mathcal{R}$ is reflexive.
- **Symmetric** Now suppose that $a\mathcal{R}b$. Then $a-b$ is an integer, so $b-a$ is also an integer. Hence, $b\mathcal{R}a$. It follows that $\mathcal{R}$ is symmetric.
- **Transitive** If $a\mathcal{R}b$ and $b\mathcal{R}c$, then $a-b$ and $b-c$ are integers. Therefore, $a-c=(a-b)+(b-c)$ is also an integer. Hence, $a\mathcal{R}c$. Thus, $\mathcal{R}$ is transitive.

Consequently, $\mathcal{R}$ is an equivalence relation.

Definition

Two elements a and b that are related by an equivalence relation are said to be **equivalent**. The notation $a \sim b$ is commonly used to indicate that a and b are equivalent under a specific equivalence relation.

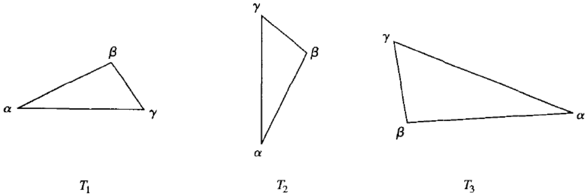
- If $s \sim t$, we say that s and t are **equivalent**. Depending on the context, we may also refer to them as **similar**, **congruent**, or use more specialized terms like **isomorphic**.
- Other notations that may be used to express equivalence relations include $s \approx t$, $s \cong t$, $s \equiv t$, and $s \leftrightarrow t$.
- All these notations signify that s and t share an equal (or equivalent) status, which is a reasonable perspective based on the symmetry property denoted by **(S)**.
- It's important to remember that the specific choice of notation may vary across different mathematical contexts, but the underlying concept of equivalence remains consistent.

Example

Triangles T_1 and T_2 in the plane are said to be:

- **Similar** We write $T_1 \approx T_2$, if their angles can be matched up so that corresponding angles are equal.
- **Congruent** If the corresponding sides are also equal, we say that the triangles are congruent, and we write $T_1 \cong T_2$.

In the figure below we have $T_1 \cong T_2$, $T_1 \approx T_3$, and $T_2 \approx T_3$, but T_3 is not congruent to T_1 or to T_2. Both $\approx$ and $\cong$ are equivalence relations on the set of all triangles in the plane. All the laws **(R)**, **(S)**, and **(T)** are evident for these relations.



✎ **Wooclap 14.** Let R_3 be the relation on $\mathbb{Z}$ given by $R_3 = \{(a, b) : ab \neq 0\}$.

1. Prove or disprove R_3 is an equivalence relation.
2. Is R_3 symmetric or transitive?
3. How can we adjust the relation so it becomes an equivalence relation?

✎ **Wooclap 15.** Let $A = \{0, 1, 2\}$ and let R be the relation on A given by

$$R = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}$$

Prove or disprove R is an equivalence relation on A.

Example

Let m be an integer with $m > 1$. Show that the relation

$$R_{\text{mod}} = \left\{(a, b) \in \mathbb{Z} \times \mathbb{Z} \ : \ a \equiv b \ (\text{mod } m)\right\}$$

is an equivalence relation on the set of integers.

Example

Let m be an integer with $m > 1$. Show that the relation

$$R_{\text{mod}} = \left\{ (a, b) \in \mathbb{Z} \times \mathbb{Z} \ : \ a \equiv b \pmod{m} \right\}$$

is an equivalence relation on the set of integers.

To do this, we need to verify that R_{mod} satisfies the three properties of an equivalence relation: reflexivity, symmetry, and transitivity.

1. **Reflexivity:** We need to show that for all $a \in \mathbb{Z}$, $a \equiv a \pmod{m}$.
   By the definition of congruence modulo m,

   $$a \equiv b \pmod{m} \text{ means that } m \text{ divides } a - b,$$

   i.e., there exists an integer k such that $a - b = km$. If we take $a = b$, then $a - a = 0$, which is clearly divisible by m because $0 = m \times 0$.
   Thus, $a \equiv a \pmod{m}$ for all $a \in \mathbb{Z}$, proving that R_{mod} is **reflexive**.

❷ **Symmetry:** We need to show that if $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$ for all $a, b \in \mathbb{Z}$.
If $a \equiv b \pmod{m}$, then by definition, m divides $a - b$. This means there exists some integer k such that $a - b = km$. Rewriting this equation as $b - a = -km$, we see that m also divides $b - a$.
Therefore, $b \equiv a \pmod{m}$, proving that R_{mod} is **symmetric**.

2. **Symmetry:** We need to show that if $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$ for all $a, b \in \mathbb{Z}$.
If $a \equiv b \pmod{m}$, then by definition, m divides $a - b$. This means there exists some integer k such that $a - b = km$. Rewriting this equation as $b - a = -km$, we see that m also divides $b - a$.
Therefore, $b \equiv a \pmod{m}$, proving that R_{mod} is **symmetric**.

3. **Transitivity:** We need to show that if $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$ for all $a, b, c \in \mathbb{Z}$.
Suppose $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. By the definition of congruence, we have:

$$m \mid (a - b) \quad \text{and} \quad m \mid (b - c),$$

which means there exist integers k_1 and k_2 such that:

$$a - b = k_1 m \quad \text{and} \quad b - c = k_2 m.$$

Now, adding these two equations gives:

$$(a - b) + (b - c) = k_1 m + k_2 m,$$

which simplifies to:

$$a - c = (k_1 + k_2)m.$$

Since $a - c$ is a multiple of m, we conclude that $m \mid (a - c)$, which means $a \equiv c \pmod{m}$.
Thus, R_{mod} is **transitive**.

2. **Symmetry:** We need to show that if $a \equiv b \pmod{m}$, then $b \equiv a \pmod{m}$ for all $a, b \in \mathbb{Z}$.
If $a \equiv b \pmod{m}$, then by definition, m divides $a - b$. This means there exists some integer k such that $a - b = km$. Rewriting this equation as $b - a = -km$, we see that m also divides $b - a$.
Therefore, $b \equiv a \pmod{m}$, proving that R_{mod} is **symmetric**.

3. **Transitivity:** We need to show that if $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$, then $a \equiv c \pmod{m}$ for all $a, b, c \in \mathbb{Z}$.
Suppose $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. By the definition of congruence, we have:

$$m \mid (a - b) \quad \text{and} \quad m \mid (b - c),$$

which means there exist integers k_1 and k_2 such that:

$$a - b = k_1 m \quad \text{and} \quad b - c = k_2 m.$$

Now, adding these two equations gives:

$$(a - b) + (b - c) = k_1 m + k_2 m,$$

which simplifies to:

$$a - c = (k_1 + k_2)m.$$

Since $a - c$ is a multiple of m, we conclude that $m \mid (a - c)$, which means $a \equiv c \pmod{m}$.
Thus, R_{mod} is **transitive**.

- **Conclusion:**
Since R_{mod} satisfies reflexivity, symmetry, and transitivity, it is an equivalence relation on $\mathbb{Z}$.

The fundamental property of equivalence relations which makes them important is that each one determines a partition of the set A into a family of disjoint sets.

Definition

Let $\mathcal{R}$ be an equivalence relation on the set A. Then for each $a \in A$, we define the equivalence class of a as class$(a) = \{b \in A : (a, b) \in \mathcal{R}\}$.

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Example

Let $A = \{0, 1, 2\}$ and let $\mathcal{R}$ be the relation on A given by $\mathcal{R} = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 0)\}$. For each element in A, we define equivalence classes as follows:

- class$(0) = \{b \in A : (0, b) \in \mathcal{R}\} = \{0, 1\}$
- class$(1) = \{b \in A : (1, b) \in \mathcal{R}\} = \{1, 0\} = class(0)$
- class$(2) = \{b \in A : (2, b) \in \mathcal{R}\} = \{2\}$

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✎ **Wooclap 16.** Consider the relation $\mathcal{R}$ on $\mathbb{Z}$ given by $\mathcal{R} = \left\{(a, b) : a \cong b(\mod 3)\right\}$. What kind of numbers are in class(2) (otherwise written as $[2]$)?

Definition

A relation $\mathcal{R}$ on a set S is called a **partial ordering** or **partial order** if it satisfies the following three properties:

- **Reflexive**: For all $a \in S$, $(a, a) \in \mathcal{R}$ (i.e., every element is related to itself).
- **Antisymmetric**: For all $a, b \in S$, if $(a, b) \in \mathcal{R}$ and $(b, a) \in \mathcal{R}$, then $a = b$.
- **Transitive**: For all $a, b, c \in S$, if $(a, b) \in \mathcal{R}$ and $(b, c) \in \mathcal{R}$, then $(a, c) \in \mathcal{R}$.

A set S together with a partial ordering $\mathcal{R}$ is called a **partially ordered set**, or **poset**, and is denoted by $(S, \mathcal{R})$. The elements of S are referred to as **elements of the poset**.

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A set S together with a partial ordering $\mathcal{R}$ is called a **partially ordered set**, or **poset**, and is denoted by $(S, \mathcal{R})$. The elements of S are referred to as **elements of the poset**.

In a poset, not all elements need to be comparable under the relation $\mathcal{R}$. That is, there may exist distinct elements $a, b \in S$ such that neither $(a, b) \in \mathcal{R}$ nor $(b, a) \in \mathcal{R}$. This distinguishes a partial order from a total order, where all pairs of elements are comparable.

Partial Order			
	A	B	C
A	1	1	0
B	0	1	0
C	1	1	1

Example

Show that the **greater than or equal** relation ($\geq$) is a partial ordering on the set of integers.

Example

Show that the **greater than or equal** relation $(\geq)$ is a partial ordering on the set of integers.

Solution

(R) For any integer a, it is clear that $a \geq a$ because any number is greater than or equal to itself. Thus, the reflexivity property is satisfied.

(A) Suppose $a \geq b$ and $b \geq a$ for some integers a and b. This implies both $a \geq b$ and $b \geq a$, which can only happen if $a = b$. Therefore, $\mathcal{R}$ satisfies the antisymmetry property.

(T) Suppose $a \geq b$ and $b \geq c$ for some integers a, b, and c. By the definition of the relation $\geq$, this means that:

$$a \geq b \quad \text{and} \quad b \geq c.$$

Since $a \geq b$ and $b \geq c$, we can conclude that $a \geq c$, because the greater-than-or-equal relation is transitive on the set of integers.

Thus, the transitivity property is satisfied.

- **Conclusion:** Since the relation $(\geq)$ on $\mathbb{Z}$ satisfies **reflexivity**, **antisymmetry**, and **transitivity**, it is a **partial ordering** on the set of integers. Therefore, $(\mathbb{Z}, \geq)$ is a **poset** (partially ordered set).

Example

Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Example

Show that the inclusion relation $\subseteq$ is a partial ordering on the power set of a set S.

Solution

(R) For any subset A, it is trivially true that $A \subseteq A$ since every element of A is included in A. Thus, the reflexivity property is satisfied.

(A) Assume $A \subseteq B$ and $B \subseteq A$ for subsets A and B. By the definition of subset inclusion:

Since $A \subseteq B$, every element of A is in B.
Since $B \subseteq A$, every element of B is in A.

Therefore, A and B must contain the same elements, which implies $A = B$. Thus, the antisymmetry property is satisfied.

(T) Assume $A \subseteq B$ and $B \subseteq C$ for subsets $A, B,$ and C. By the definition of subset inclusion:

Since $A \subseteq B$, every element of A is in B.
Since $B \subseteq C$, every element of B is in C.

Thus, every element of A must also be in C. Therefore, $A \subseteq C$. This shows that the transitivity property is satisfied.

- **Conclusion:** Since the inclusion relation $\subseteq$ on $\mathcal{P}(S)$ satisfies **reflexivity**, **antisymmetry**, and **transitivity**, it is a **partial ordering** on the power set of the set S. Therefore, $(\mathcal{P}(S), \subseteq)$ is a **poset** (partially ordered set).

Example

Let $\mathcal{R}$ be the relation on the set of people such that $x\mathcal{R}y$ if x and y are people and x is older than y. Show that $\mathcal{R}$ is not a partial ordering.

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Solution

(A) Note that $\mathcal{R}$ is **antisymmetric** because if a person x is older than a person y, then y is not older than x. That is, if $x\mathcal{R}y$, then $y\neg\mathcal{R}x$.

(T) The relation $\mathcal{R}$ is **transitive** because if person x is older than person y and y is older than person z, then x is older than z. That is, if $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$.

R However, $\mathcal{R}$ is not **reflexive**, because no person is older than himself or herself. That is, $x\neg\mathcal{R}x$ for all people x.

It follows that $\mathcal{R}$ is not a partial ordering.

✎ Wooclap 17.

Let E be a non-empty set and let F be an ordered set. Let Ω be the set of all mappings of E to F. Define on Ω the relation $\mathcal{R}$ by

$$u\mathcal{R}v \Longleftrightarrow u(x) \leq v(x)\,, \quad \forall x \in E\,.$$

Show that $\mathcal{R}$ is an ordering relation on Ω.

Notation: The notation $a \preccurlyeq b$ is used to indicate that $(a, b) \in \mathcal{R}$ in an arbitrary poset $(S, \mathcal{R})$.

In a poset $(S, \preccurlyeq)$, it is not required that for any two elements a and b either $a \preccurlyeq b$ or $b \preccurlyeq a$. That is, elements of a poset are not always comparable.

Definition (Comparable relation)

The elements a and b of a poset $(S, \preccurlyeq)$ are said to be **comparable** if either $a \preccurlyeq b$ or $b \preccurlyeq a$. Conversely, if neither $a \preccurlyeq b$ nor $b \preccurlyeq a$, then a and b are called **incomparable**.

Example

Consider the poset $(\mathbb{Z}^+, |)$, where the relation | denotes divisibility among the positive integers. Are the integers 3 and 9 comparable? What about 5 and 7?

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Example

Consider the poset $(\mathbb{Z}^+, |)$, where the relation | denotes divisibility among the positive integers. Are the integers 3 and 9 comparable? What about 5 and 7?

Solution

The integers 3 and 9 are **comparable**, because $3 \mid 9$ (i.e., 3 divides 9). However, the integers 5 and 7 are **incomparable**, because neither $5 \mid 7$ nor $7 \mid 5$ (i.e., neither divides the other).

✎ **Wooclap 18.** Consider the poset $(\mathcal{P}(\{a,b,c,d\}), \subseteq)$, where $\mathcal{P}(\{a,b,c,d\})$ is the power set of $\{a,b,c,d\}$ and the relation $\subseteq$ denotes set inclusion.

1. Are the subsets $X = \{a,b\}$ and $Y = \{a,c\}$ comparable?
2. Are the subsets $X = \{a,b\}$ and $Z = \{a,b,d\}$ comparable?
3. What is the maximum subset of the power set $\mathcal{P}(\{a,b,c,d\})$ with respect to the inclusion relation $\subseteq$?

Solution

1. **Incomparable:** The subsets $X = \{a, b\}$ and $Y = \{a, c\}$ are incomparable, because neither $X \subseteq Y$ nor $Y \subseteq X$. Each set contains different elements besides a, so one is not contained within the other.

2. **Comparable:** The subsets $X = \{a, b\}$ and $Z = \{a, b, d\}$ are comparable, because $X \subseteq Z$. Every element of X is also in Z, which makes them comparable.

3. **Maximum Subset:** The maximum subset of the poset $(\mathcal{P}(\{a, b, c, d\}), \subseteq)$ is $\{a, b, c, d\}$, because it contains all the elements of the power set and no other subset can contain more elements with respect to set inclusion.

Definition (covering relation)

Let $(S, \preccurlyeq)$ be a poset.

- We say that an element $y \in S$ covers an element $x \in S$ if $x \prec y$ and there is no element $z \in S$ such that $x \prec z \prec y$.
- The set of pairs (x, y) such that y covers x is called the **covering relation** of $(S, \preccurlyeq)$.

- A partial order is a type of relation, and it can be represented as a directed graph (digraph).

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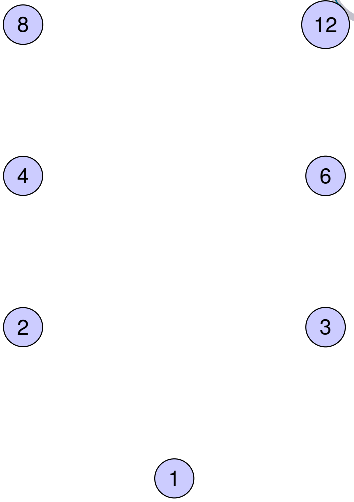
- Each element is represented as a node.
- We usually draw the diagram of a finite poset in the plane in such a way that, if y covers x, then the point representing y is higher than the point representing x.



$$(\{1,2,3,4,6,8,12\},|)$$

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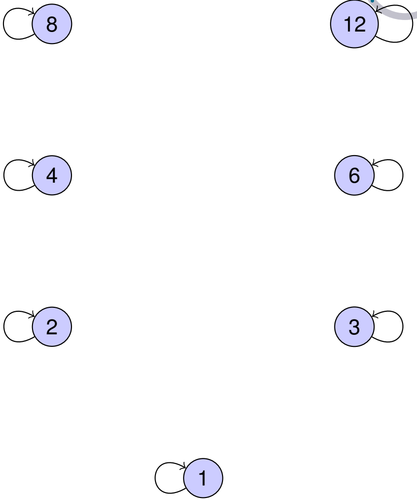

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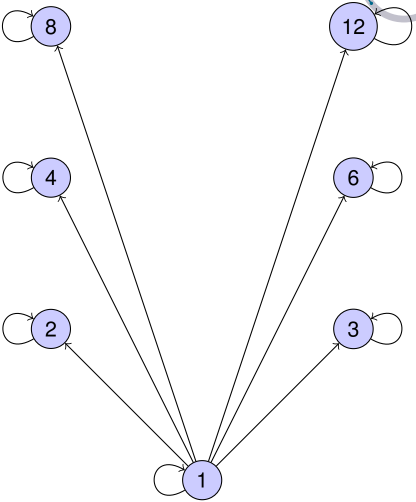

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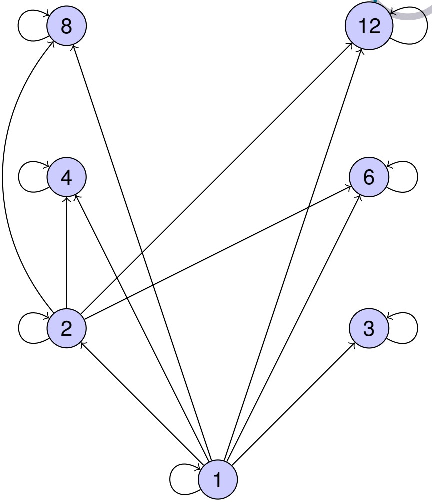

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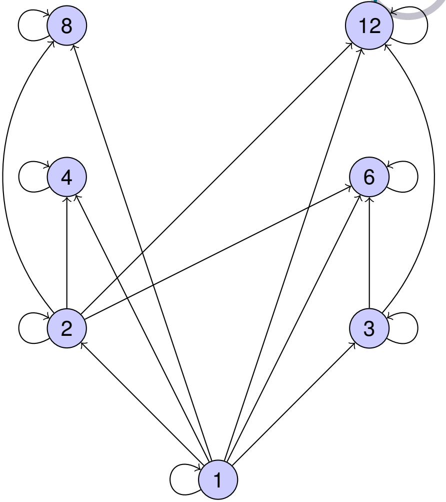

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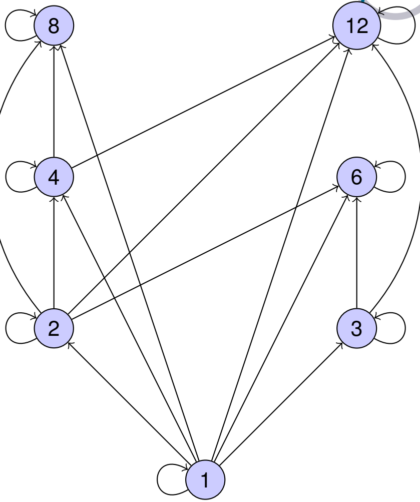

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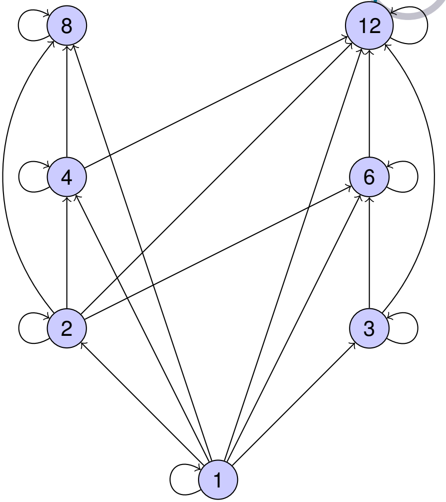

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- This allows to simplify the graphical representation of a partially ordered set by taking the following steps:
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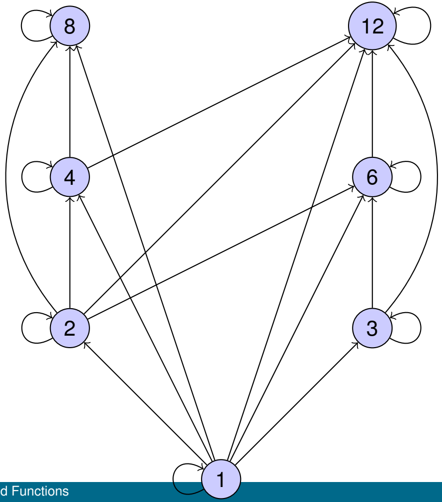
Definition (Hasse Diagram)

The resulting graph looks far simpler and is called a **Hasse diagram**, named after the German mathematician Helmut Hasse (1898-1979).

☆ Constructing the Hasse Diagram ☆

In general, we can represent a finite poset $(S, \preccurlyeq)$ using this procedure:

1. Start with the directed graph for this relation. Because a partial ordering is **reflexive**, a loop (a, a) is present at every vertex a. Remove these loops.



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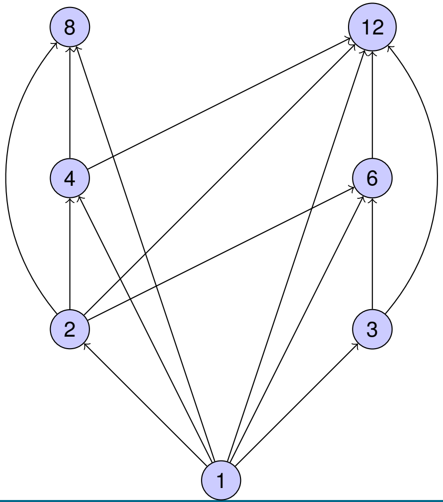

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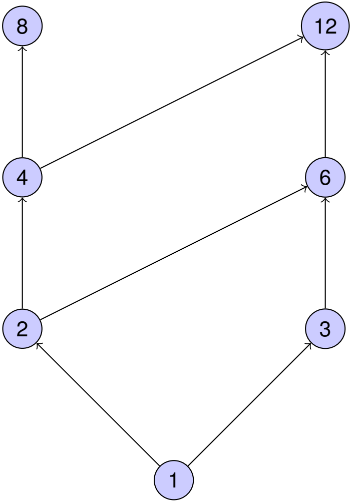
3. Finally, arrange each edge so that its initial vertex is below its terminal vertex. Remove all the arrows on the directed edges, because all edges point *upward* toward their terminal vertex.



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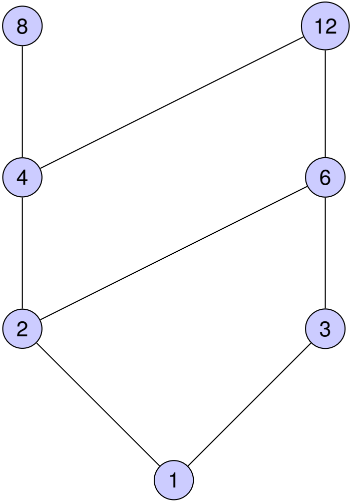


Example (The Hasse Diagram of $(P(\{a,b,c\}),\subseteq)$)

Draw the Hasse diagram for the partial ordering $\{(A,B) : A \subseteq B\}$ on the power set $P(S)$ where $S = \{a,b,c\}$.

Example (The Hasse Diagram of $(P(\{a,b,c\}), \subseteq)$)

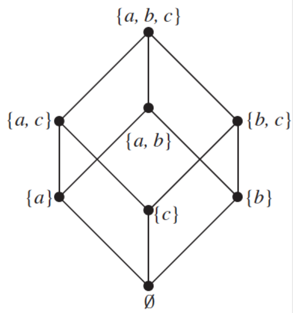
Draw the Hasse diagram for the partial ordering $\{(A,B) : A \subseteq B\}$ on the power set $P(S)$ where $S = \{a,b,c\}$.

Solution

We have

$$P(S) = \Big\{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\Big\}.$$

The Hasse diagram for this partial ordering is obtained from the associated digraph by deleting all the loops and all the edges that occur from transitivity, namely, $(\emptyset, \{a,b\}), (\emptyset, \{a,c\})$, $(\emptyset, \{b,c\}), (\emptyset, \{a,b,c\}), (\{a\}, \{a,b,c\}), (\{b\}, \{a,b,c\})$, and $(\{c\}, \{a,b,c\})$. Finally all edges point upward, and arrows are deleted. The resulting Hasse diagram is illustrated in this figure.



Definition (maximal)

- An element of a poset is called **maximal** if it is not less than any element of the poset.
- That is, a is **maximal** in the poset $(S, \preccurlyeq)$ if there is no $b \in S$ such that $a \preccurlyeq b$.
- An element with no other element greater than it.

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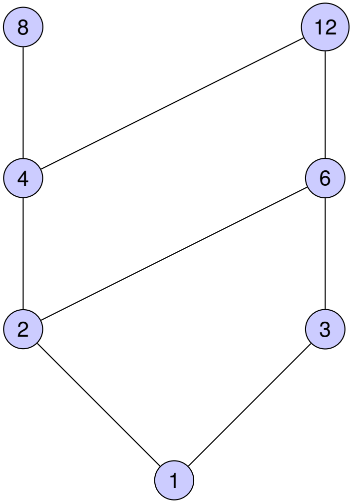
Maximal and **minimal** elements are easy to spot using a **Hasse diagram**. They are the *top* and *bottom* elements in the diagram.


8
12
4
6
2
3
1

Example

Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are **maximal**, and which are **minimal**?

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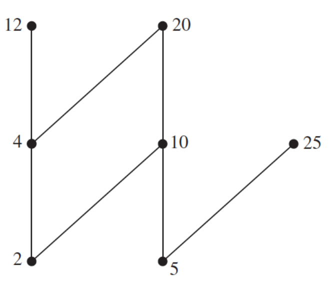
Which elements of the poset $(\{2, 4, 5, 10, 12, 20, 25\}, |)$ are **maximal**, and which are **minimal**?

Solution

The Hasse diagram in the figure below for this poset shows that the maximal elements are 12, 20, and 25, and the minimal elements are 2 and 5.



Remark:

- As this example shows, a poset can have more than one maximal element and more than one minimal element.
- Sometimes there is an element in a poset that is greater than every other element. Such an element is called the **greatest** element.

Definition (Greatest and Least element)

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- a is the **least element** of $(S, \preccurlyeq)$ if $a \preccurlyeq b$ for all $b \in S$. The least element is unique when it exists.

Definition (Greatest and Least element)

- a is the **greatest element** of the poset $(S, \preccurlyeq)$ if $b \preccurlyeq a$ for all $b \in S$. The greatest element is unique when it exists.
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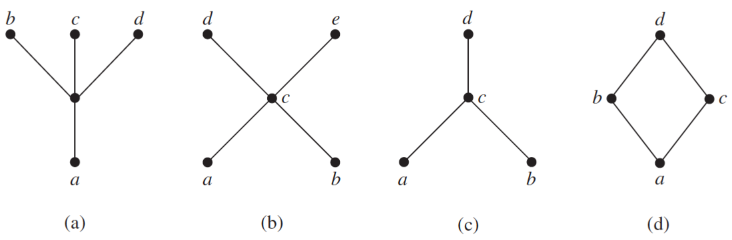
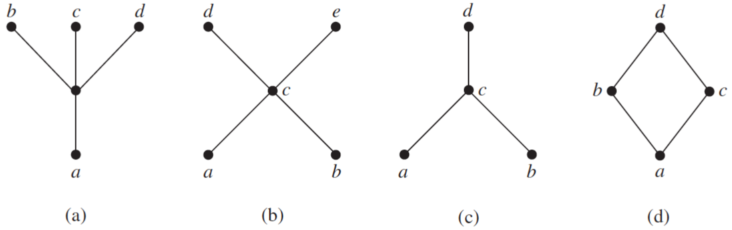
Example

Determine whether the posets represented by each of the Hasse diagrams in the figure below have a greatest element and a least element.





Solution

- F.a The least element of the poset with Hasse diagram (a) is a. This poset has no greatest element.
- F.b The poset with Hasse diagram (b) has neither a least nor a greatest element.
- F.c The poset with Hasse diagram (c) has no least element. Its greatest element is d.
- F.d The poset with Hasse diagram (d) has least element a and greatest element d.

Example

Let S be a set. Determine whether there is a greatest element and a least element in the poset $(P(S), \subseteq)$.

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Solution

The least element is the empty set, because $\emptyset \subseteq T$ for any subset T of S. The set S is the greatest element in this poset, because $T \subseteq S$ whenever T is a subset of S.

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Is there a greatest element and a least element in the poset $(Z^+, |)$?

Solution

The integer 1 is the least element because $1 \mid n$ whenever n is a positive integer. Because there is no integer that is divisible by all positive integers, there is no greatest element.

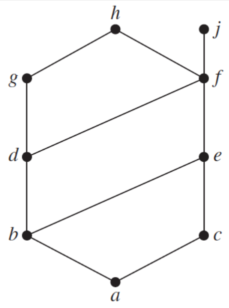
Sometimes it is possible to find an element that is greater than or equal to all the elements in a subset A of a poset $(S, \preccurlyeq)$.

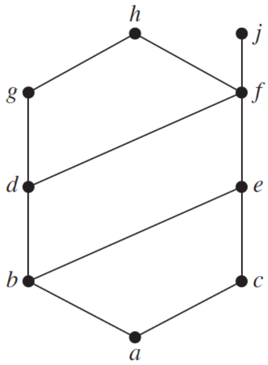
Definition (Upper and Lower bound)

- If u is an element of S such that $a \preccurlyeq u$ for all elements $a \in A$, then u is called an **upper bound** of A
- If l is an element of S such that $l \preccurlyeq a$ for all elements $a \in A$, then l is called a **lower bound** of A.

Example

Find the lower and upper bounds of the subsets $\{a, b, c\}$, $\{j, h\}$, and $\{a, c, d, f\}$ in the poset with the Hasse diagram shown in the figure below.





Solution

- The upper bounds of $\{a, b, c\}$ are e, f, j, and h, and its only lower bound is a.
- There are no upper bounds of $\{j, h\}$, and its lower bounds are a, b, c, d, e, and f.
- The upper bounds of $\{a, c, d, f\}$ are f, h, and j , and its lower bound is a.

Definition (least upper bound)

The element x is called the **least upper bound** of the subset A if x is an upper bound that is less than every other upper bound of A. The **least upper bound** of A is unique if it exists

That is, x is the least upper bound of A if $a \preccurlyeq x$ whenever $a \in A$, and $x \preccurlyeq z$ whenever z is an upper bound of A.

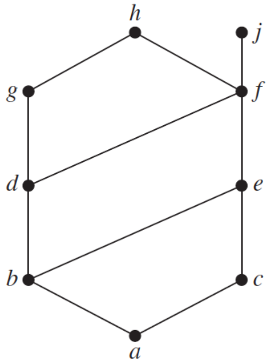
Definition (greatest lower bound)

the element y is called the **greatest lower bound** of A if y is a lower bound of A and $z \preccurlyeq y$ whenever z is a lower bound of A. The **greatest lower bound** of A is unique if it exists.

Notations: The **greatest lower bound** and **least upper bound** of a subset A are denoted by glb(A) and lub(A), respectively.

Example

Find the greatest lower bound and the least upper bound of $\{b, d, g\}$, if they exist, in the poset shown in the previous figure.



Solution

The upper bounds of $\{b, d, g\}$ are g and h. Because $g \prec h$, g is the least upper bound.
The lower bounds of $\{b, d, g\}$ are a and b. Because $a \prec b$, b is the greatest lower bound.

Example

Find the **greatest lower bound** and **the least upper bound** of the sets $\{3, 9, 12\}$ and $\{1, 2, 4, 5, 10\}$, if they exist, in the poset $(Z^+, |)$.

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Solution

(GLB) An integer is a lower bound of $\{3, 9, 12\}$ if $3, 9$, and 12 are divisible by this integer. The only such integers are 1 and 3. Because $1 \mid 3$, 3 is the greatest lower bound of $\{3, 9, 12\}$.
The only lower bound for the set $\{1, 2, 4, 5, 10\}$ with respect to | is the element 1. Hence, 1 is the greatest lower bound for $\{1, 2, 4, 5, 10\}$.

(LUB) An integer is an upper bound for $\{3, 9, 12\}$ if and only if it is divisible by $3, 9$, and 12. The integers with this property are those divisible by the least common multiple of $3, 9$, and 12, which is 36. Hence, 36 is the least upper bound of $\{3, 9, 12\}$.
A positive integer is an upper bound for the set $\{1, 2, 4, 5, 10\}$ if and only if it is divisible by $1, 2, 4, 5$, and 10. The integers with this property are those integers divisible by the least common multiple of these integers, which is 20. Hence, 20 is the least upper bound of $\{1, 2, 4, 5, 10\}$.

The adjective **partial** is used to describe partial orderings because not all pairs of elements are necessarily comparable. When every two elements in the set are comparable, the relation is called a **total ordering**.

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Definition

If $(S, \preccurlyeq)$ is a poset and every two elements of S are **comparable**, meaning that for all $a, b \in S$, either $a \preccurlyeq b$ or $b \preccurlyeq a$, then S is called a **totally ordered** or **linearly ordered** set.

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Example

1. The poset $(\mathbb{Z}, \leq)$ (the set of integers with the usual "less than or equal to" relation) is a totally ordered set, because for any two integers a and b, either $a \leq b$ or $b \leq a$.
2. The poset $(\mathbb{Z}^+, |)$ (the set of positive integers with the divisibility relation) is **not totally ordered**, because it contains elements that are incomparable, such as 5 and 7 (since $5 \nmid 7$ and $7 \nmid 5$).

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- A totally ordered set is also known as a **chain**.
- In a **totally ordered set**, all pairs of elements must be comparable. This is not the case for partial orders, where some elements can be incomparable.

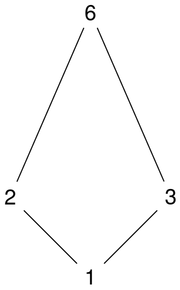
✎ Wooclap 19.

1. Consider the poset $(\mathbb{Z}^+, |)$, where $|$ denotes the divisibility relation among positive integers. Determine whether the following pairs of integers are comparable:
   a) $\mathbf{4}$ and $\mathbf{8}$
   b) $\mathbf{6}$ and $\mathbf{9}$
   c) $\mathbf{3}$ and $\mathbf{12}$

2. Let $\boldsymbol{S = \{\{1\}, \{1,2\}, \{1,2,3\}\}}$ and consider the poset $\boldsymbol{(S, \subseteq)}$, where $\subseteq$ is the subset relation.
   a) Is $\boldsymbol{(S, \subseteq)}$ totally ordered? Justify your answer.
   b) List all pairs of elements that are comparable in $\boldsymbol{S}$.

3. Prove or disprove: The set $\mathbb{N}$ (the set of natural numbers) with the relation $\leq$ is totally ordered.

4. **Challenge:** Provide an example of a poset that is not totally ordered, and identify a pair of incomparable elements within the set.



Partial Order (Divisibility on {1, 2, 3, 6}) **Total Order (Less than or equal to on {1, 2, 3, 4})**