School of Engineering and Computer Science

Mathematical tools applied to Computer Science

Ch4: Graphs

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Week #**09** ✦ 07/NOV/2024 ✦

Week #**10** ✦ 14/NOV/2024 ✦

Definition (Graph)

A graph G is a pair $G = (V; E)$ where V is a set of vertices and E is a (multi)set of unordered pairs of vertices.

- The elements of E are called edges.
- We write $V(G)$ for the set of vertices and $E(G)$ for the set of edges of a graph G.
- Also, $|V(G)|$ denotes the number of vertices and $|E(G)|$ denotes the number of edges.

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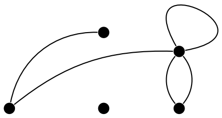
Graph



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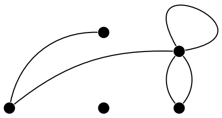
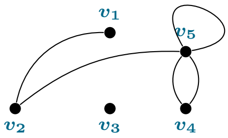


We label the vertices



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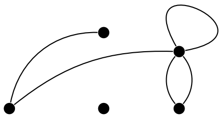

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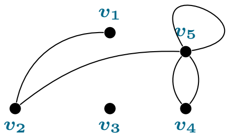

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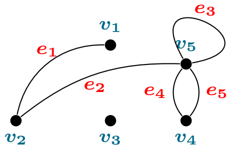

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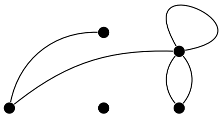

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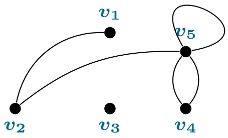

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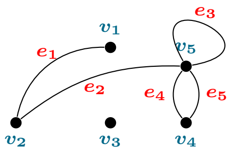

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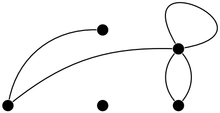

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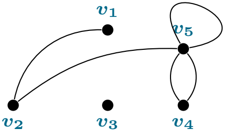

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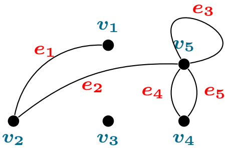

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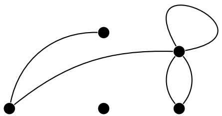

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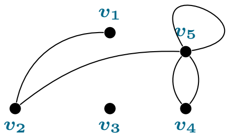

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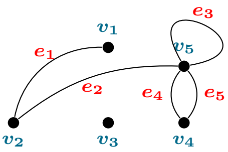

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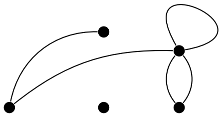
5. we say e_1 is *incident* on v_1 and v_2, and also that v_1 and v_2 are incident on e_1.
6. We have $V = \{v_1, \cdots, v_5\}$ for the vertices and
$E = \{(v_1, v_2), (v_2, v_5), (v_5, v_5), (v_5, v_4), (v_5, v_4)\} = \{e_1, \cdots, e_5\}$ for the edges.

Graph



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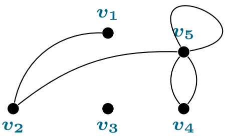

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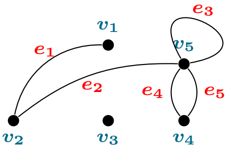
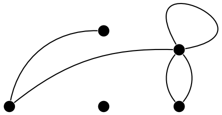


- The two edges (v_i, v_j) and (v_j, v_i) are the same. In other words, the pair is not ordered. Moreover v_i and v_j are **end vertices** of the edge (v_i, v_j).
- If a point is related to itself, a **loop** is drawn that extends out from the point and goes back to it. *The edge $e_3 = (v_5, v_5)$ is a **loop**.*
- If two distinct edges have the same pair of endpoints, then the edges are said to be **parallel**. *In the previous graph, e_2 and e_5 are parallel edges*
- A graph is **simple** if it has no parallel edges or loops.
- Edges are **adjacent** if they share a common end vertex. *We can say that $e_1 = (v_1, v_2)$ and $e_2 = (v_5, v_2)$ are **adjacent** edges.*

Graph



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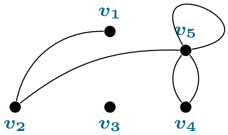

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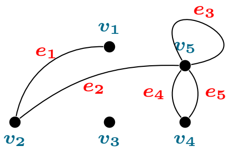
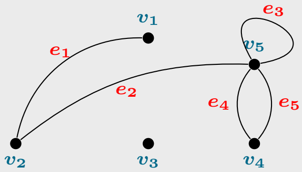


- Two vertices v_i and v_j are **adjacent** if they are connected by an edge.
  v_1 and v_2 are adjacent vertices.
- The **degree** of the vertex v_j, written as $d(v_j)$, is the number of edges with v_j as an end vertex. By convention, we count a loop twice and parallel edges contribute separately.
- A **pendant** vertex is a vertex whose degree is 1.
- An edge that has a pendant vertex as an end vertex is a **pendant edge**.
- An **isolated vertex** is a vertex whose degree is 0.

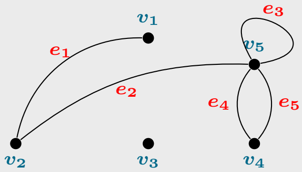
Example

- $v\ldots$ and $v\ldots$ are end vertices of e_5.
- $e\ldots$ and $e\ldots$ are parallel.
- $e\ldots$ is a loop.
- The graph is $\cdots$.
- e_1 and e_2 are $\cdots$.
- v_1 and v_2 are $\cdots$.
- The degree of v_1 is $\cdots$ so it is a pendant vertex.
- $e\ldots$ is a pendant edge.
- The degree of $v\ldots$ is 5.
- The degree of $v\ldots$ is 2.
- The degree of $v\ldots$ is 0 so it is an $\cdots$ vertex.



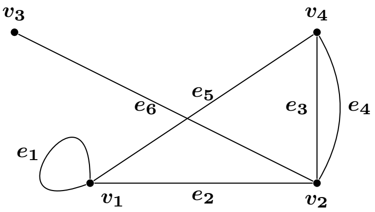
Example

- v_4 and v_5 are end vertices of e_5.
- e_4 and e_5 are parallel.
- e_3 is a loop.
- The graph is not simple.
- e_1 and e_2 are adjacent.
- v_1 and v_2 are adjacent.
- The degree of v_1 is 1 so it is a pendant vertex.
- e_1 is a pendant edge.
- The degree of v_5 is 5.
- The degree of v_4 is 2.
- The degree of v_3 is 0 so it is an isolated vertex.



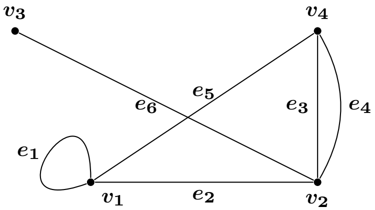
✎ **Exercise 1.** Draw a graph which has 4 vertices, 1 loop and one pair of parallel edges. Write down one pair of adjacent vertices and one pair of adjacent edges.

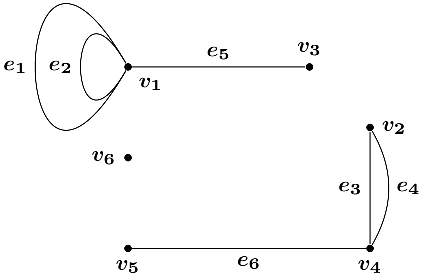
✎ **Exercise 2.** Write down the set V of vertices and the set E of edges for the following graph, G_1.



✎ **Exercise 3.** Draw a graph which has 6 vertices, 1 pair of parallel loops, 1 pair of parallel edges, one isolated vertex and which is made up of 3 'pieces' or 'components'

✎ **Exercise 4.** Write down the degree of each of the vertices in the following graphs:





For each of the above graphs calculate $\sum_{i=1}^{k} d(v_i)$, $k = 4$ in the first graph and $k = 6$ in the second.

Theorem (Euler's First Law)

In a graph G, the sum of the degrees of the vertices is equal to twice the number of edges. The graph $G = (V, E)$, where $V = \{v_1, \cdots, v_n\}$ and $E = \{e_1, \cdots, e_m\}$, satisfies

$$\sum_{i=1}^{n} d(v_i) = 2m$$

✎ **Exercise 5.** Draw a graph with 4 vertices having degrees $1,\ 2,\ 0$ and 2.

Theorem (Euler's Second Law)

In a graph G, the number of vertices with odd degree is even.

✎ **Exercise 6.** Draw two different graphs each with 3 vertices having degrees 1, 2 and 3.

Definition (Complete Graph)

A simple graph that contains every possible edge between all the vertices is called a complete graph. A complete graph with n vertices is denoted as K_n.

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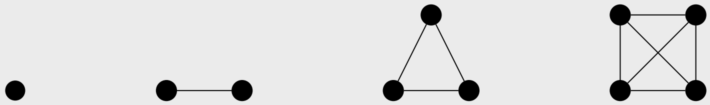
The following graphs are complete



K_1 K_2 K_3 K_4

Definition (Complete Graph)

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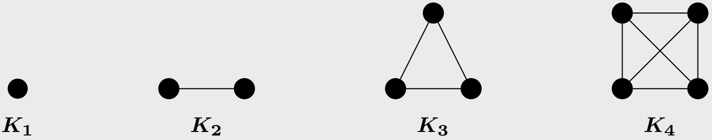


Proposition

A complete graph with n vertices K_n has $n(n-1)/2$ edges

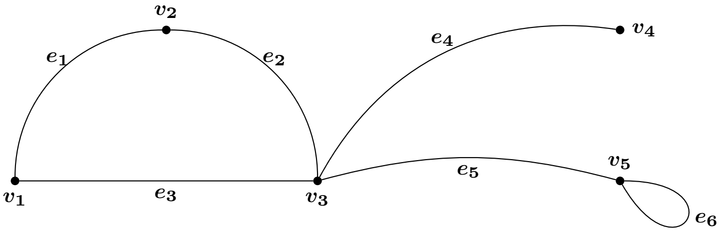
Definition (Subgraph)

The graph $G_1 = (V_1, E_1)$ is a subgraph of $G_2 = (V_2, E_2)$ if

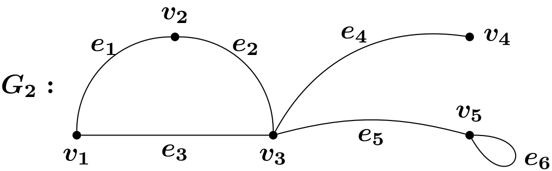
1. $V_1 \subset V_2$ and
2. Every edge of G_1 is also an edge of G_2.

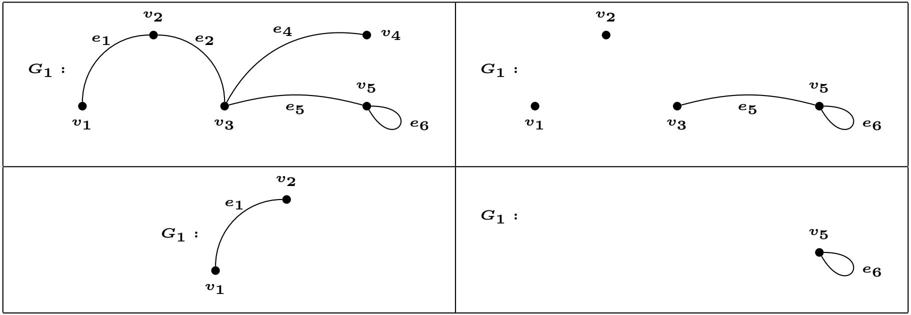
We have the graph

G_2 :



and some of its subgraphs are


G_2 :
v_1
v_2
v_3
v_4
v_5
e_1
e_2
e_3
e_4
e_5
e_6


G_1 :
v_1
v_2
v_3
v_4
v_5
e_1
e_2
e_4
e_5
e_6
G_1 :
v_1
v_2
v_3
v_5
e_5
e_6
G_1 :
v_1
v_2
e_1
G_1 :
v_5
e_6

✎ **Exercise 7.** Given the graph $G = (V, E)$ where:

$$V = \{1, 2, 3, 4\}$$

$$E = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{1, 3\}\}$$

List all subgraphs of G that include exactly 3 edges

Definition (Wlak)

A **walk** in the graph $G = (V, E)$ is a finite sequence of the form

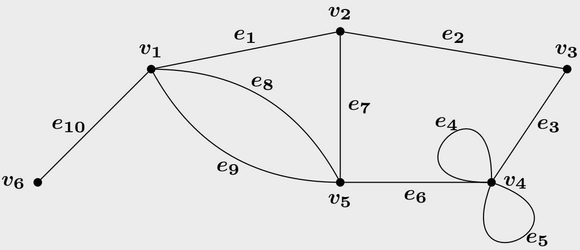
$$v_{i0},\ e_{j1},\ v_{i1},\ e_{j2},\ \cdots\ ,\ e_{jk},\ v_{ik}\ ,$$

which consists of alternating vertices and edges of G.

- The walk starts at a vertex.
- Vertices v_{it-1} and v_{it} are end vertices of e_{jt} $(t = 1, \cdots, k)$.
- v_{i0} is the initial vertex and v_{ik} is the terminal vertex. k is the length of the walk.
- A zero length walk is just a single vertex v_{i0} .
- It is allowed to visit a vertex or go through an edge more than once.
- A walk is *open* if $v_{i0} \neq v_{ik}$. Otherwise it is *closed*.

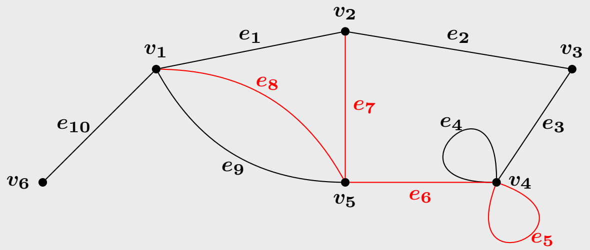
Example

G :



Example

G :

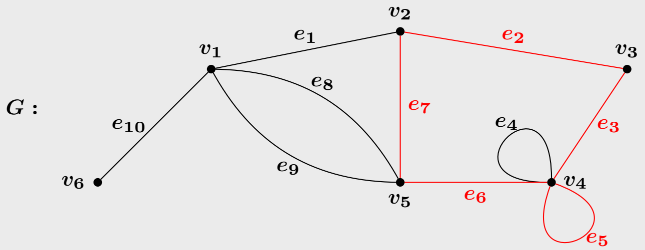


- In the above graph, the walk

$$v_2,\ e_7,\ v_5,\ e_8,\ v_1,\ e_8,\ v_5,\ e_6,\ v_4,\ e_5,\ v_4,\ e_5,\ v_4$$

is open.

Example



- In the above graph, the walk

$$v_2,\ e_7,\ v_5,\ e_8,\ v_1,\ e_8,\ v_5,\ e_6,\ v_4,\ e_5,\ v_4,\ e_5,\ v_4$$

  is open.
- On the other hand, the walk

$$v_4,\ e_5,\ v_4,\ e_3,\ v_3,\ e_2,\ v_2,\ e_7,\ v_5,\ e_6,\ v_4$$

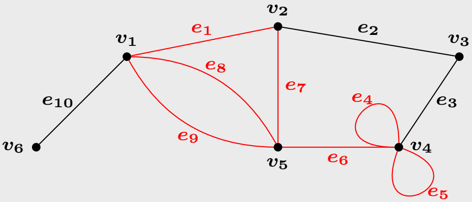
  is closed.

Definition (Trail)

A walk is a **trail** if any edge is traversed at most once. Then, the number of times that the vertex pair u, v can appear as consecutive vertices in a trail is at most the number of parallel edges connecting u and v.
(i.e. If the edges in a walk are distinct, then the walk is called a **trail**.)

Example

G :



The walk in the graph v_1, e_8, v_5, e_9, v_1, e_1, v_2, e_7, v_5, e_6, v_4, e_5, v_4, e_4, v_4 is a trail.

Definition (Circuit)

Similarly, a **trail** that begins and ends at the same vertex is called a closed trail, or **circuit**.

Definition (Path)

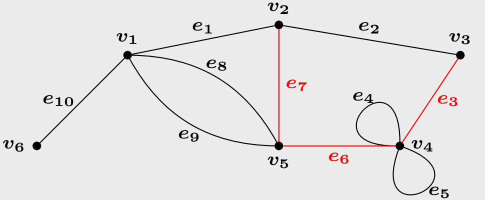
A walk is a **path** if any vertex is visited at most once except possibly the initial and terminal vertices when they are the same.
(i.e. If the vertices in a walk are distinct, then the walk is called a **path**.)

Remark: In this way, every path is a trail, but **not** every trail is a path.

Definition (Cycle)

A closed path is a **cycle**.

Example



The walk

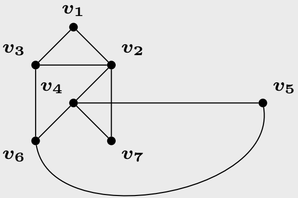
$$v_2,\ e_7,\ v_5,\ e_6,\ v_4,\ e_3,\ v_3$$

is a path and the walk

$$v_2,\ e_7,\ v_5,\ e_6,\ v_4,\ e_3,\ v_3,\ e_2,\ v_2$$

is a circuit.

Example

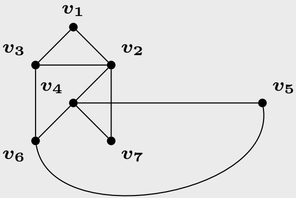


In the following graph

- $v_1,\ v_3,\ v_6,\ v_3,\ v_2,\ v_4$ is a $\cdots$ of length 5.
- The sequence $v_2,\ v_1,\ v_3,\ v_2, v_4$ represents a $\cdots$ of length 4.
- The sequence $v_4,\ v_7,\ v_2,\ v_1,\ v_3,\ v_6,\ v_5$ represents a $\cdots$ of length 6.
- Also, $v_7,\ v_4,\ v_2,\ v_3,\ v_1,\ v_2,\ v_7$ is a $\cdots$.
- While $v_5,\ v_4,\ v_2,\ v_1,\ v_3,\ v_6,\ v_5$ is a $\cdots$.

In general, it is possible for a walk, trail, or path to have length 0, but the least possible length of a circuit or cycle is 3.

Example

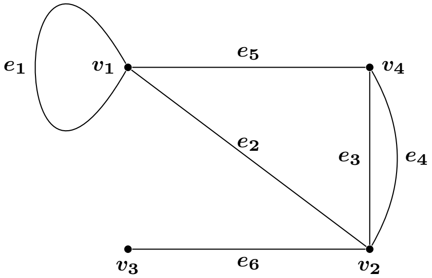


In the following graph

- $v_1, v_3, v_6, v_3, v_2, v_4$ is a **walk** of length 5.
- The sequence v_2, v_1, v_3, v_2, v_4 represents a **trail** of length 4.
- The sequence $v_4, v_7, v_2, v_1, v_3, v_6, v_5$ represents a **path** of length 6.
- Also, $v_7, v_4, v_2, v_3, v_1, v_2, v_7$ is a **circuit**.
- While $v_5, v_4, v_2, v_1, v_3, v_6, v_5$ is a **cycle**.

In general, it is possible for a walk, trail, or path to have length 0, but the least possible length of a circuit or cycle is 3.

✎ **Exercise 8.** Consider the following graph:



Write down a path from:

(a) v_1 to v_3

(b) v_4 to v_1

(c) v_4 to v_2

Definition (Connected Walk)

The walk starting at u and ending at v is called an $u - v$ walk. u and v are connected if there is a $u - v$ walk in the graph.

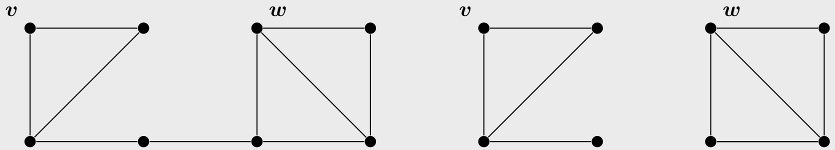
Proposition

If u and v are connected and v and w are connected, then u and w are also connected, i.e. if there is a $u - v$ walk and a $v - w$ walk, then there is also a $u - w$ walk.

Definition (Connected Graph)

A graph is **connected** if all the vertices are connected to each other. (A trivial graph is connected by convention.) Thus a graph is **connected** if and only if there is a path between each pair of vertices.

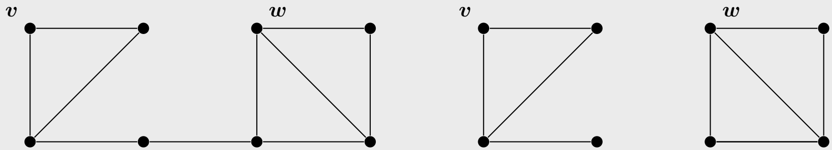
Example



connected

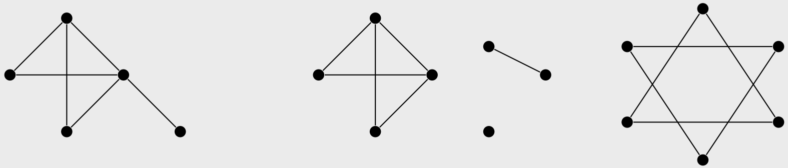
disconnected

Example



connected   disconnected

In the following figures G_1 is connected, and both G_2 and G_3 are not connected (or disconnected).

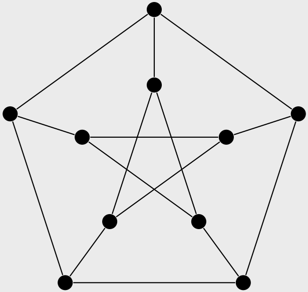


G_1 G_2 G_3

Definition (Regular graphs)

A graph in which each vertex has the same degree is a regular graph. If each vertex has degree r, the graph is regular of degree r or $r-$regular.

Example (Cubic graph)



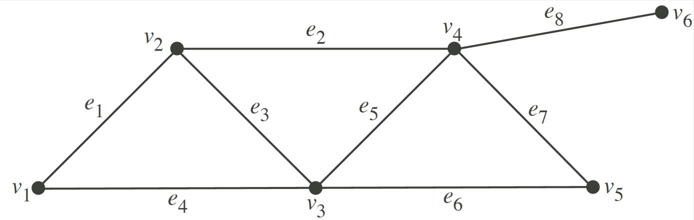
Definition (Cut set)

A **cut set** of the connected graph $G = (V, E)$ is an edge set $F \subset E$ such that

1. $G - F$ (remove the edges of F one by one) is not connected, and
2. $G - H$ is connected whenever $H \subset F$.

Example

In the graph



$\{e_1, e_4\}$, $\{e_6, e_7\}$, $\{e_1, e_2, e_3\}$, $\{e_8\}$, $\{e_3, e_4, e_5, e_6\}$, $\{e_2, e_5, e_7\}$, $\{e_2, e_5, e_6\}$ and $\{e_2, e_3, e_4\}$ are cut sets. Are there other cut sets?

Theorem

If F is a cut set of the connected graph G, then $G - F$ has two components.

Definition

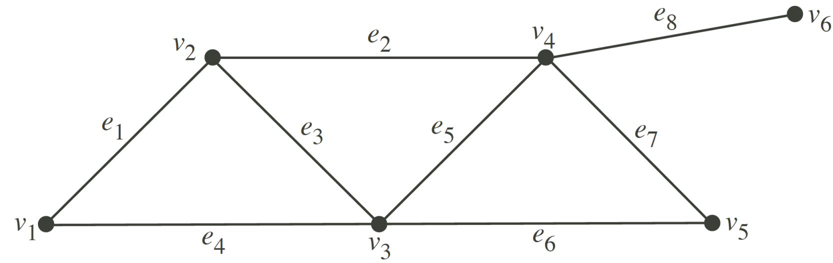
In a graph $G = (V, E)$, a pair of subsets V_1 and V_2 of V satisfying

$$V = V_1 \cup V_2, \quad V_1 \cap V_2 = \varnothing, \quad V_1 \neq \varnothing, \quad V_2 \neq \varnothing \, ,$$

is called a cut (or a *partition*) of G, denoted $\langle V_1, V_2 \rangle$. Usually, the cuts $\langle V_1, V_2 \rangle$ and $\langle V_2, V_1 \rangle$ are considered to be the same.

Example

In the graph



$\langle\{v_1, v_2, v_3\}, \{v_4, v_5, v_6\}\rangle$ is a cut.

✎ Exercise 9.

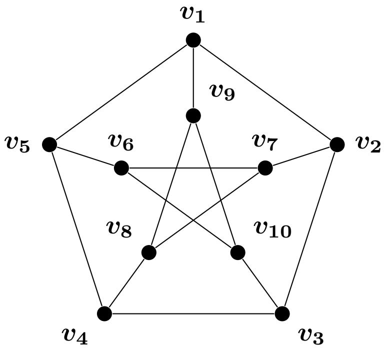
In the Petersen graph, find

(i) a trail of length $\mathbf{7}$;

(ii) a path of length $\mathbf{9}$;

(iii) cycles of lengths $\mathbf{5},\ \mathbf{6},\ \mathbf{8}$ and $\mathbf{9}$;

(iv) cutsets with $\mathbf{3},\ \mathbf{4}$ and $\mathbf{5}$ edges.



Definition (Bipartie Graph)

If there exists a cut $\langle V_1, V_2 \rangle$ for the graph $G = (V, E)$ so that $E = \langle V_1, V_2 \rangle$, i.e. the cut (considered as an edge set) includes every edge, then the graph G is **bipartite**.
In other words, if there exists a cut $\langle V_1, V_2 \rangle$ such that:

- $V_1 \cup V_2 = E$ and $V_1 \cap V_2 = \varnothing$.
- every edge of G connects a vertex in V_1 with a vertex in V_2.

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A bipartite graph $G = (V, E)$ is one whose vertices can be separated into two disjoint sets, where every edge joins a vertex in one set to a vertex in the other.

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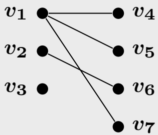
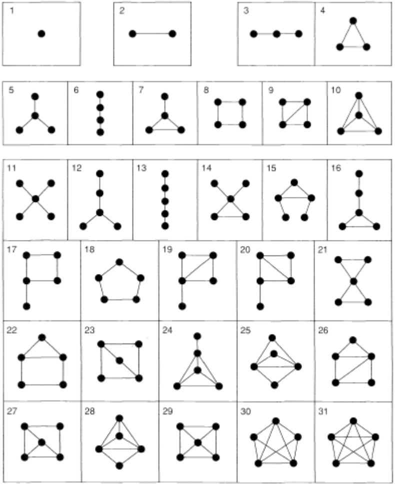
A bipartite graph $G = (V, E)$ is one whose vertices can be separated into two disjoint sets, where every edge joins a vertex in one set to a vertex in the other.

Example

The following graph is bipartite.
$V_1 = \{v_1, v_2, v_3\}$ and $V_2 = \{v_4, v_5, v_6, v_7\}$.



✎ **Exercise 10.** In the following table, locate all the regular graphs and the bipartite graphs



Definition (Digraph)

A **directed graph** or **digraph** is a pair (V, E) that formed by vertices connected by directed edges or arc.

Definition

A digraph $G = (V(G), E(G))$ consists of two sets, the nonempty set $V(G)$ of vertices of G and the set $E(G)$ of edges (or arcs) of G, together with a function γ from $E(G)$ to $V(G) \times V(G)$ that tells where the edges go.

- If e is an edge of G and $\gamma(e) = (v_i, v_j)$, then we say e goes from v_i to v_j, and we call v_i the initial vertex of e and v_j the terminal vertex of e.

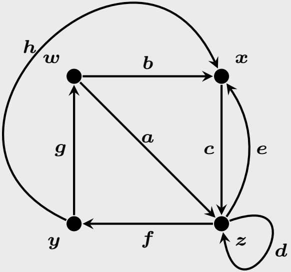
$$\underset{\text{initial vertex}}{v_i} \longrightarrow \underset{\text{terminal vertex}}{v_j} \qquad \begin{cases} v_j \text{ is a successor of } v_i \\ v_i \text{ is a predecessor of } v_j. \end{cases}$$

- The arc $e = (v_i, v_j)$ is said to be **outgoing** from v_i and **incident** to v_j.
- An arc $v_i \to v_i$ is called a **loop**.

Example

Consider the digraph G with vertex set $V(G) = \{w, x, y, z\}$, edge set $E(G) = \{a, b, c, d, e, f, g, h\}$, and γ given by the table below. We labeled the arrows to make the correspondence to $E(G)$ plain.

e	$\gamma(e)$
a	(w, z)
b	(w, x)
c	(x, z)
d	(z, z)
e	(z, x)
f	(z, y)
g	(y, w)
h	(y, x)



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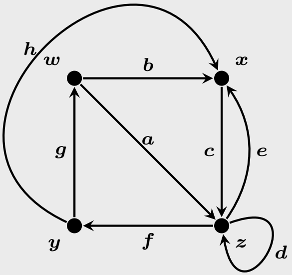


The sequence $e\ c\ f\ g\ a$ is a walk.

The sequence $f\ g\ b$ is a path of length 3 from z to x.

The paths $f\ g\ b\ f\ c$ is closed;

✎ Wooclap 1.

1. Let $A = \{0, 1, 2, 3\}$ and let R_1 be the relation on A given by $R_1 = \{(0, 0), (0, 1), (0, 2), (3, 0)\}$. Draw the directed graph of R_1 .
2. Let $A = \{0, 1, 2, 3\}$ and let R_2 be the relation on A given by $R_2 = \{(0, 0), (1, 2), (2, 2)\}$. Draw the directed graph of R_2.

The directed graph of an equivalence relation on A has the following properties:

- Each point of the graph has an arrow looping around from it back to itself. (Reflexivity)
- In each case where there is an arrow going from one point to a second, there is an arrow going from the second point back to the first. (Symmetry)
- In each case where there is an arrow going from one point to a second and from a second point to a third, there is an arrow going from the first point to the third. (Transitivity)

Example

1. Let $A = \{0, 1, 2\}$ and let R be the relation on A given by

$$R = \{(0,0), (1,1), (2,2), (0,1), (1,0)\}.$$

Draw the directed graph for R.

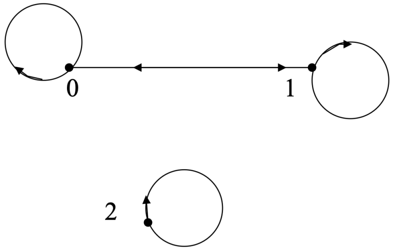
2. Let $A = \{2, 3, 4, 6, 7, 9\}$, and define a relation R on A by

$$\mathbb{R} = \{(a, b) : a \cong b(\mod 3)\}.$$

Draw the directed graph for R.

Solution

1. Previously R was shown to be an equivalence relation on A. The directed graph is then :



Solution

❷

$R =$
$\{(2,2),(3,3),(4,4),(6,6),(7,7),(9,9),(3,6),(6,3),(3,9),(9,3),(6,9),(9,6),(4,7),(7,4)$

It can be shown that R is an equivalence relation, and thus the directed graph is: