School of Engineering and Computer Science

Mathematical tools applied to Computer Science

Ch1 : Elements of Logic

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Definition

In mathematics, an argument is a sequence of statements, known as the **premises)**, followed by another statement called the **conclusion**. An argument is used to establish the truth of the conclusion based on the premises.

Example

$\underbrace{\text{"If you have a current password, then you can log onto the network."}}_{\text{Premise 1}}$

$\underbrace{\text{"You have a current password."}}_{\text{Premise 2}}$

Therefore,

$\underbrace{\text{"You can log onto the network."}}_{\text{Conclusion}}$

An argument can be viewed as a condensed form of expressing a large conditional statement. In ess

Definition (Valid Argument)

A **valid** argument is one where, if all its premises are true, the conclusion **cannot** be false.

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Example

Premise 1: All humans are mortal.

Premise 2: John is a human.

Conclusion: Therefore, John is mortal.

This argument is valid because it follows the logical form: a universal statement (Premise 1), combined with a specific case (Premise 2). Since the conclusion "John is mortal" logically follows from the premises, the argument is considered valid.

Definition (Valid Argument)

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Definition (Invalid argument)

An **invalid** argument is one where, even if all its premises are true, the conclusion **CAN** be false

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Definition (Invalid argument)

An **invalid** argument is one where, even if all its premises are true, the conclusion **CAN** be false

Example

Premise 1: All humans are mortal.

Premise 2: John is mortal.

Conclusion: Therefore, John is human.

This argument is invalid because the conclusion does not logically follow from the given premises. While the premises are true, the conclusion is not necessarily true based on them, as being mortal does not imply that John is necessarily human (he could be any mortal being).

Example

1. Some football coaches are poets. But no poet knows how to play football. So, some football coaches do not know how to play football.
2. Some cars are purple, and some cars are Chevrolets. So, some cars are purple Chevrolets.
3. If Smith wins, Jones will be happy. However, Smith won't win. So, Jones won't be happy.
4. Only birds are blue. My pet is blue. So, my pet is a bird.
5. Ann and Bob both won't be home. So, Ann won't be home.
6. If Sue wins, then Ed will be happy. If Ed is happy, then George will be happy. So, if Sue wins, then George will be happy.
7. All dogs are wolves. No wolves are insects. So, no dogs are insects.
8. The United States comprises fifty states. Thus, the United States has an even number of states.
9. If John is a bachelor, then John is unmarried. However, John is not a bachelor. Therefore, John is unmarried.

Solution

1. **Validity**: Valid
   Reasoning: This is a valid argument based on the fact that if some football coaches are poets and no poet knows how to play football, then it logically follows that some football coaches do not know how to play football.

2. **Validity**: Invalid
   Reasoning: Just because some cars are purple and some are Chevrolets does not guarantee that there is an overlap between purple cars and Chevrolets. The argument lacks sufficient connection between the categories.

3. **Validity**: Invalid
   Reasoning: The argument commits the fallacy of denying the antecedent. Just because Smith won't win does not necessarily mean Jones won't be happy, as there could be other reasons Jones might be happy.

4. **Validity**: Valid
   Reasoning: This is a valid categorical syllogism. If only birds are blue and the pet is blue, then the pet must be a bird.

5. **Validity**: Valid
   Reasoning: If both Ann and Bob are not home, then it is certainly true that Ann is not home.

6. **Validity**: Invalid
   Reasoning: This argument is an example of the fallacy of the transitive property of conditional statements, which is not always valid unless the conditions are directly linked.

Solution

6. **Validity**: Invalid
Reasoning: This argument is an example of the fallacy of the transitive property of conditional statements, which is not always valid unless the conditions are directly linked.

7. **Validity**: Valid
Reasoning: This is a valid syllogism. If all dogs are wolves and no wolves are insects, then it logically follows that no dogs are insects.

8. **Validity**: Valid
Reasoning: This is valid because fifty is an even number, so the statement logically follows.

9. **Validity**: Valid
Reasoning: This is another example of the fallacy of denying the antecedent. John being unmarried is not necessarily tied to being a bachelor; there could be other reasons he is unmarried.

Definition (**Argument Form**)

An argument form is a logical structure that outlines the relationships between premises and conclusion in a generalizable way. It consists of a template where propositions are represented as variables, allowing one to analyze the validity of the argument based on its structure alone, independent of the specific content of the propositions.

An argument form is simply a different way of abbreviating and presenting a big conditional statement form. So an argument form is a statement form.

Example

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore\ q \end{array} \qquad \begin{array}{l} p \to q \\ \neg p \\ \hline \therefore\ \neg q \end{array} \qquad \begin{array}{l} p \to q \\ q \\ \hline \therefore\ p \end{array}$$

Definition (Valid Argument Form)

An argument form is considered valid if, for every possible substitution of truth values into its propositional variables, if the premises are all true, then the conclusion must also be true. In other words, a valid argument form guarantees that the conclusion logically follows from the premises.

Definition (Invalid Argument Form)

An argument form is considered invalid if there exists at least one scenario where the premises are all true, but the conclusion is false. This means that the conclusion does not necessarily follow from the premises, even if they are all true.

Example

Valid Argument Form: Consider the following argument form:

$$\begin{array}{l} p \to q \\ p \\ \hline \therefore\ q \end{array}$$

In this form, if the premises $p \to q$ (if p then q) and p are both true, then the conclusion q must be true as well. This is an example of a valid argument form because it preserves truth in every possible scenario where the premises are true.

Invalid Argument Form: Consider the following argument form:

$$\begin{array}{l} p \\ q \\ \hline \therefore\ p \wedge q \end{array}$$

In this form, even if both p and q are true, the conclusion $p \wedge q$ (both p and q are true) is not guaranteed. For instance, if p is true and q is false, the conclusion $p \wedge q$ would be false, demonstrating that this argument form is invalid.

Practically, how do we test an argument form to determine if it is valid?

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- List all possible combinations of truth values (true and false) for the propositional variables.
- Evaluate the truth value of each premise and the conclusion for each combination.

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- List all possible combinations of truth values (true and false) for the propositional variables.
- Evaluate the truth value of each premise and the conclusion for each combination.

Analyze the Results: Check whether there is any row in which all the premises are true, but the conclusion is false. If such a row exists, the argument form is invalid. If no such row exists, the argument form is valid.

Testing Validity with a Truth Table. Consider the argument form:

$$\begin{array}{l} p \to q \\ \underline{p \qquad\quad} \\ \therefore\ q \end{array}$$

Step 1: Identify the Propositions. Propositions: p and q

Step 2: Construct the Truth Table

	p	q	$p \to q$	Conclusion (q)
▶	T	T	T	T
	T	F	F	F
	F	T	T	F
	F	F	T	F

Step 3: Fill in the Truth Table

- For $p \to q$, the truth value is true unless p is true and q is false.
- The conclusion is just q.

Step 4: Analyze the Results

- In the second row, the premise $p \to q$ is false and p is true, and the conclusion q is false. However, this row does not matter for validity as not all premises are true in this case.
- In rows where all premises are true, the conclusion is also true:
- Row 1: Premises are $p \to q = T$ and $p = T$, and conclusion $q = T$ (Valid)
- Row 3: Premises are $p \to q = T$ and $p = F$, and conclusion $q = T$ (Valid)

Since there are no rows where all premises are true and the conclusion is false, the argument form is valid.

Summary To determine if an argument form is valid:

- Construct a truth table for the argument form.
- Evaluate the truth values for all possible scenarios.
- Verify if there is any instance where all premises are true and the conclusion is false. If there is none, the argument form is valid.

This method ensures a thorough check of the argument's validity based on logical consistency across all possible truth value combinations.

$p \to q$
p
___
$\therefore\ q$

	p	q	$p \to q$
▷	T	T	T
	T	F	F
	F	T	T
	F	F	T

In all of the rows where all premises are true, the conclusion is true.

$p \to q$
$\neg q$
___
$\therefore\ \neg p$

	p	q	$\neg p$	$\neg q$	$p \to q$
	T	T	F	F	T
	T	F	F	T	F
	F	T	T	F	T
▷	F	F	T	T	T

In every critical row, the conclusions is true, so we have valid argument form.

$p \to q$
q
___
$\therefore\ p$

	p	q	$p \to q$
▷	T	T	T
	T	F	F
▷	F	T	T
	F	F	T

There is a critical row with a false conclusion. So it is invalid argument form.

$p \to q$
$\neg p$
___
$\therefore\ \neg q$

	p	q	$\neg p$	$\neg q$	$p \to q$
	T	T	F	F	T
	T	F	F	T	F
▷	F	T	T	F	T
▷	F	F	T	T	T

There is a critical row with false conclusion. So the argument form is invalid.

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- Each has a specific name and serves as a key example in the study of logical reasoning.

Definition (Inference Rules)

Inference rules are fundamental argument forms used to construct more complex arguments. These rules are simple, established patterns of reasoning that can be applied to derive conclusions from premises.

⋆ Modus Ponens ⋆

(Modus Ponens=Mode that affirms)

$$\begin{array}{l} p \\ p \to q \\ \hline \therefore\ q \end{array}$$

Corresponding tautology

$$(p \land (p \to q)) \to q$$

Example

Let p be "It is snowing." Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"It is snowing."

"Therefore , I will study discrete math."

⋆ Modus Tollens ⋆

$$\begin{array}{l} \neg q \\ p \to q \\ \hline \therefore\ \neg p \end{array}$$

Corresponding tautology

$$(\neg q \wedge (p \to q)) \to \neg p$$

Example

Let p be "It is snowing." Let q be "I will study discrete math."

"If it is snowing, then I will study discrete math."
"I will not study discrete Math."

"Therefore , its is not snowing."

⋆ Chain Argument ⋆

Transitivity of Implication or Chain Argument

$$\begin{array}{l} p \to q \\ q \to r \\ \hline \therefore\ p \to r \end{array}$$

Corresponding tautology

$$((p \to q) \wedge (q \to r)) \to (p \to r)$$

Example

Let p be "It is snowing." Let q be "I will study discrete math." Let r be "I will get an A".

"If it is snows, then I will study discrete math."
"If i study discrete math, I will get A."

"Therefore , its is snows, I will get an A."

⋆ Disjunction Elimination ⋆

Disjunction Elimination or OR Elimination

$$\begin{array}{l} p \lor q \\ \neg p \\ \hline \therefore\ q \end{array}$$

Corresponding tautology

$$((p \lor q) \land \neg p) \to q$$

Example

Let p be "I will study discrete math." Let q be "I will study English literature."

"I will study discrete math or I will study English literature"
"I will not study discrete math."

"Therefore , I will study English literature.."

⋆ Disjunction Introduction ⋆

$$\begin{array}{l} p \\ \hline \therefore\ (p \vee q) \end{array}$$

Corresponding tautology

$$p \rightarrow (p \vee q)$$

Example

Let p be "I will study discrete math." Let q be "I will visit Las Vegas."

"I will study discrete math."

"Therefore , I will study discrete math or I will visit Las Vegas.."

⋆ Conjunction Elimination ⋆

$$\begin{array}{l} p \wedge q \\ \hline \therefore\ p \end{array}$$

Corresponding tautology

$$(p \wedge q) \rightarrow p$$

Example

Let p be "I will study discrete math." Let q be "I will study English literature."

"I will study discrete math and English literature."

"Therefore , I will study discrete math."

⋆ Conjunction Introduction ⋆

$$\begin{array}{l} p \\ q \\ \hline \therefore\ (p \wedge q) \end{array}$$

Corresponding tautology

$$((p) \wedge (q)) \rightarrow (p \wedge q)$$

Example

Let p be "I will study discrete math." Let q be "I will study English literature."

"I will study discrete math."
"I will study English literature."

"Therefore , I will study discrete math and I will study English literature."

✯ Resolution ✯

Resolution plays an important role in Artificial Intelligence and is used in the programming language Prolog.

$$\begin{array}{l} p \vee q \\ \neg p \vee r \\ \hline \therefore\ (q \vee r) \end{array}$$

Corresponding tautology

$$((p \vee q) \wedge (\neg p \vee r)) \rightarrow (q \vee r)$$

Example

Let p be "I will study discrete math."
Let q be "I will study databases."
Let r be "I will study English literature."

"I will study discrete math or I will study databases."
"I will not study discrete math or I will study English literature."
"Therefore , I will study databases or I will English literature."

Rule	Formula	Name
$p \to q$ p $\therefore q$ (Law of Detachment)	$((p \to q) \land p) \to q$	Modus Ponens
$p \to q$ $\neg q$ $\therefore \neg p$	$((p \to q) \land \neg q) \to \neg p$	Modus Tollens
$p \to q$ $q \to r$ $\therefore p \to r$ (Transitivity)	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical Syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive Syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$(p) \land (q) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

✎ **Wooclap 1.** Find the argument form for the following argument and determine whether it is valid. Can we conclude that the conclusion is true if the premises are true?

❶

If Socrates is human, then Socrates is mortal.
Socrates is human.

∴ Socrates is mortal.

❷

If George does not have eight legs, then he is not a spider.
George is a spider.

∴ George has eight legs.

✎ **Wooclap 2.** What rule of inference is used in each of these arguments?

a) Alice is a mathematics major. Therefore, Alice is either a mathematics major or a computer science major.

b) Jerry is a mathematics major and a computer science major. Therefore, Jerry is a mathematics major.

c) If it is rainy, then the pool will be closed. It is rainy. Therefore, the pool is closed.

d) If it snows today, the university will close. The university is not closed today. Therefore, it did not snow today.

e) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will sunburn.

⋆ Using Rules of Inference to Build Arguments ⋆

Example

Show that the premises

- P1 "*It is not sunny this afternoon and it is colder than yesterday*,"
- P2 "*We will go swimming only if it is sunny*,"
- P3 "*If we do not go swimming, then we will take a canoe trip*,"

and

- P4 "*If we take a canoe trip, then we will be home by sunset*"

lead to the conclusion

- C "*We will be home by sunset*."

Solution

1. Choose propositional variables:

   p "It is sunny this afternoon,"
   q "It is colder than yesterday,"
   r "We will go swimming,"
   s "We will take a canoe trip,"
   t "We will be home by sunset."

   Then the premises become

   $$\neg p \wedge q,\ r \rightarrow p,\ \neg r \rightarrow \qquad \text{and} \qquad s \rightarrow t\,.$$

   The conclusion is simply t .
   We need to give a valid argument with premises $\neg p \wedge q$, $r \rightarrow p$, $\neg r \rightarrow s$, and $s \rightarrow t$ and conclusion t.

Solution

2. Translation into propositional logic:

$$\begin{array}{l} \neg p \wedge q \\ r \rightarrow p \\ \neg r \rightarrow s \\ s \rightarrow t \\ \hline \therefore\ t \end{array}$$

3.

Step		Reason
1.	$\neg p \wedge q$	Premise
2.	$\neg p$	Simplification using (1)
3.	$r \rightarrow p$	Premise
4.	$\neg r$	Modus tollens using (2) and (3)
5.	$\neg r \rightarrow s$	Premise
6.	s	Modus ponens using (4) and (5)
7.	$s \rightarrow t$	Premise
8.	t	Modus ponens using (6) and (7)

✎ **Wooclap 3.** Show that the premises
"*If you send me an e-mail message, then I will finish writing the program,*"
"*If you do not send me an e-mail message, then I will go to sleep early,*"
and
"*If I go to sleep early, then I will wake up feeling refreshed*"
lead to the conclusion
"*If I do not finish writing the program, then I will wake up feeling refreshed.*"

⋆ Rules of Inference for Quantified Statements ⋆

Universal Instantiation(UI)

$$\begin{array}{l} \forall x\, P(x) \\ \hline \therefore\ P(c) \text{ for an arbitrary } c \end{array}$$

Definition (**Universal Instantiation (UI)**)

Universal Instantiation is a rule of inference stating that if a property $P(x)$ holds for all elements in a domain (i.e., $\forall x\, P(x)$), then it also holds for any specific element c in that domain. This allows us to infer $P(c)$ from $\forall x\, P(x)$.

Example

Let the domain consist of all students, and consider that KAMEL is a student.

Given: "All students are smart." ($\forall x$ Smart(x))

By Universal Instantiation, we can conclude: "Therefore, KAMEL is smart." (Smart(KAMEL))

Universal Generalization(UG)

$P(c)$ for an arbitrary c

$\therefore\ \forall x\, P(x)$

Used often implicitly in Mathematical Proofs.

Definition (**Universal Generalization (UG)**)

Universal Generalization is a rule of inference that allows one to conclude that a property $P(x)$ holds for all elements in a domain (i.e., $\forall x\, P(x)$) if it has been shown to hold for an arbitrary element c in that domain. This rule is commonly used in mathematical proofs to generalize from a specific instance to a universal statement.

Universal Generalization is often used implicitly in mathematical proofs. By proving that a statement is true for an arbitrary element c, we infer that it must be true for every element in the domain.

Existential Instantiation(EI)

$$\exists x\, P(x)$$

$\therefore\ P(c)$ for some elements c

Definition (**Existential Instantiation (EI)**)

Existential Instantiation is a rule of inference that allows one to infer that a particular instance exists when a property $P(x)$ is known to be true for at least one element in the domain (i.e., $\exists x\, P(x)$). This rule allows us to introduce a specific element c for which the property $P(c)$ holds, without specifying which element it is.

Example

Suppose we know that "There is someone who got an A in Mathematical Tools Applied to Computer Science."

By Existential Instantiation, we can say, "Let's call this person Amelie and assume that Amelie got an A."

Existential Generalization(EG)

$P(c)$ for some element c

$\therefore\ \exists x\, P(x)$

Definition (**Existential Generalization (EG)**)

Existential Generalization is a rule of inference that allows one to conclude that a property $P(x)$ is true for at least one element in a domain (i.e., $\exists x\, P(x)$), given that it is known to be true for some specific element c. This rule is used to move from a statement about a specific instance to a statement that asserts the existence of such an instance.

Example

Consider the statement: "Amelie got an A in the class."

By Existential Generalization, we can conclude: "Therefore, there exists someone who got an A in the class."

Name	Rule	Example
Universal Instantiation	$\forall x P(x)$ $\therefore\ P(c)$	"All women are brave." "Therefore, Lily is brave."
Universal Generalization	$P(c)$ for an arbitrary c $\therefore\ \forall x P(x)$	"Lily is brave." "Therefore, all women are brave."
Existential Instantiation	$\exists x P(x)$ $\therefore\ P(c)$ for some element c	"There is someone who ran a mile in 4 minutes." "Let's call him Sparky and say that Sparky ran a mile in 4
Existential Generalization	$P(c)$ for some element c $\therefore\ \exists x P(x)$	"Sparky ran a mile in 4 minutes." "Therefore, someone ran a mile in 4 minutes."

Example

Show that the premises
"*A student in this class has not read the book*,"
and
"*Everyone in this class passed the first exam*,"
lead to the conclusion
"*Someone who passed the first exam has not read the book*."

Example

Show that the premises
"*A student in this class has not read the book*,"
and
"*Everyone in this class passed the first exam*,"
lead to the conclusion
"*Someone who passed the first exam has not read the book*."

Solution:

1. Choose propositional function:

$C(x) :=$ "x is in this class,"
$B(x) :=$ "x has read the book,"
$P(x) :=$ "x passed the first exam."

2. premises $\exists x\ (C(x) \wedge \neg B(x))$
and $\forall x(C(x) \to P(x))$
Conclusion $\exists x(P(x) \wedge \neg B(x))$

Solution:

3.

Step		Reason
1.	$\exists x(C(x) \land \neg B(x))$	Premise
2.	$C(a) \land \neg B(a)$	(For some student a) "EI from (1)"
3.	$C(a)$	"Conjunction elimination from (2)"
4.	$\forall x(C(x) \to P(x))$	Premise
5.	$C(a) \to P(a)$	(let $x = a$) UI from (4)
6.	$P(a)$	Modus ponens from (3) and (5)
7.	$\neg B(a)$	Conjunction elimination from (2)
8.	$P(a) \land \neg B(a)$	Conjunction from (6) and (7)
9.	$\exists x(P(x) \land \neg B(x))$	Existential generalization from (8)

✎ Wooclap 4.

Show that the premises

> "*Students who pass the course either do the homework or attend lecture;*,"
> "*Bob did not attend every lecture*,"

and
"*Bob passed the course*,"
lead to the conclusion
"*Bob must have done the homework.*"

$$\forall x\,(P(x) \rightarrow Q(x))$$
$P(a),$ where a is a particular element in the domain

$\therefore\ Q(a)$

$$\forall x\,(P(x) \rightarrow Q(x))$$
$\exists Q(a),$ where a is a particular element in the domain

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$\therefore\ \exists P(a)$