Matrix Theory: Assignment 5

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***Abstract*—This a problem on tracing a parabola using vector algebra.**

Download Python code from

https://github.com/Debolena/EE5609/blob/master/Assignment_5/parabola%20plot.py

Download all latex-tikz codes from

https://github.com/Debolena/EE5609/tree/master/Assignment_5

1 Problem

Trace the parabola:

$$(x-4y)^2 = 51y \tag{1.0.1}$$

2 Solution

Expanding the given equation, we have,

$$x^2 - 8xy + 16y^2 - 51y = 0 \tag{2.0.1}$$

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.2}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.3}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.4}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.5}$$

From equation (2.0.1) , we get

$$\mathbf{V} = \begin{pmatrix} 1 & -4 \\ -4 & 16 \end{pmatrix} \tag{2.0.6}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -\frac{51}{2} \end{pmatrix} \tag{2.0.7}$$

$$f = 0 \tag{2.0.8}$$

Expanding the determinant of $\mathbf{V}$ we observe,

$$\begin{vmatrix} 1 & -4 \\ -4 & 16 \end{vmatrix} = 0 \tag{2.0.9}$$

Therefore, 2.0.1 is a parabola.

The characteristic equation of $\mathbf{V}$ is given as follows,

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 1 & 4 \\ 4 & \lambda - 16 \end{vmatrix} = 0 \tag{2.0.10}$$

$$\implies \lambda^2 - 17\lambda = 0 \tag{2.0.11}$$

The eigenvalues are given by

$$\lambda_1 = 0, \lambda_2 = 17 \tag{2.0.12}$$

For $\lambda_1 = 0$, the eigen vector $\mathbf{p}$ is given by

$$\mathbf{V}\mathbf{p} = 0 \tag{2.0.13}$$

Row reducing $\mathbf{V}$

$$\begin{pmatrix} 1 & -4 \\ -4 & 16 \end{pmatrix} \xleftrightarrow[R_2 = R_2 + R_1]{R_2 = R_2/4} \begin{pmatrix} 1 & -4 \\ 0 & 0 \end{pmatrix} \tag{2.0.14}$$

$$\implies \mathbf{p}_1 = \frac{1}{\sqrt{17}}\begin{pmatrix} -4 \\ -1 \end{pmatrix} \tag{2.0.15}$$

Similarly,

$$\mathbf{p}_2 = \frac{1}{\sqrt{17}}\begin{pmatrix} -1 \\ 4 \end{pmatrix} \tag{2.0.16}$$

Thus,

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{\sqrt{17}}\begin{pmatrix} -4 & -1 \\ -1 & 4 \end{pmatrix} \tag{2.0.17}$$

The equation of the parabola is:

$$\mathbf{y^T D y} = -2\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \tag{2.0.18}$$

where

$$\eta = \mathbf{u}^T\mathbf{p_1} = \frac{51}{2\sqrt{17}} \tag{2.0.19}$$

and focal length of the parabola is given by

$$\frac{\left|2\mathbf{u}^T\mathbf{p_1}\right|}{\lambda_2} = \frac{3}{\sqrt{17}} \tag{2.0.20}$$

Now,

$$\begin{pmatrix}\mathbf{u^T}+\eta\mathbf{p_1^T}\\ \mathbf{V}\end{pmatrix}\mathbf{c} = \begin{pmatrix}-f\\ \eta\mathbf{p_1}-\mathbf{u}\end{pmatrix} \tag{2.0.21}$$

using equations (2.0.6), (2.0.7) and (2.0.21)

$$\begin{pmatrix}-6 & -27\\ 1 & -4\\ -4 & 16\end{pmatrix}\mathbf{c} = \begin{pmatrix}0\\ -6\\ 24\end{pmatrix} \tag{2.0.22}$$

Forming the augmented matrix and row reducing it:

$$\begin{pmatrix}-6 & -27 & 0\\ 1 & -4 & -6\\ -4 & 16 & 24\end{pmatrix} \tag{2.0.23}$$

$$\xleftrightarrow{R_3\leftarrow R_3+4R_2}\begin{pmatrix}-6 & -27 & 0\\ 1 & -4 & -6\\ 0 & 0 & 0\end{pmatrix} \tag{2.0.24}$$

$$\xleftrightarrow{R_1\leftarrow R_1/(-6)}\begin{pmatrix}1 & 9/2 & 0\\ 1 & -4 & -6\\ 0 & 0 & 0\end{pmatrix} \tag{2.0.25}$$

$$\xleftrightarrow{R_2\leftarrow R_2-R_1}\begin{pmatrix}1 & 9/2 & 0\\ 0 & -17/2 & -6\\ 0 & 0 & 0\end{pmatrix} \tag{2.0.26}$$

$$\xleftrightarrow{R_2\leftarrow(-\frac{2}{17})R_2}\begin{pmatrix}1 & 9/2 & 0\\ 0 & 1 & 12/17\\ 0 & 0 & 0\end{pmatrix} \tag{2.0.27}$$

$$\xleftrightarrow{R_1\leftarrow R_1-(\frac{9}{2})R_2}\begin{pmatrix}1 & 0 & -54/17\\ 0 & 1 & 12/17\\ 0 & 0 & 0\end{pmatrix} \tag{2.0.28}$$

Thus the vertex is:

$$\mathbf{c} = \begin{pmatrix}-\frac{54}{17}\\ \frac{12}{17}\end{pmatrix} \tag{2.0.29}$$

$$\approx \begin{pmatrix}-3.18\\ 0.71\end{pmatrix} \tag{2.0.30}$$

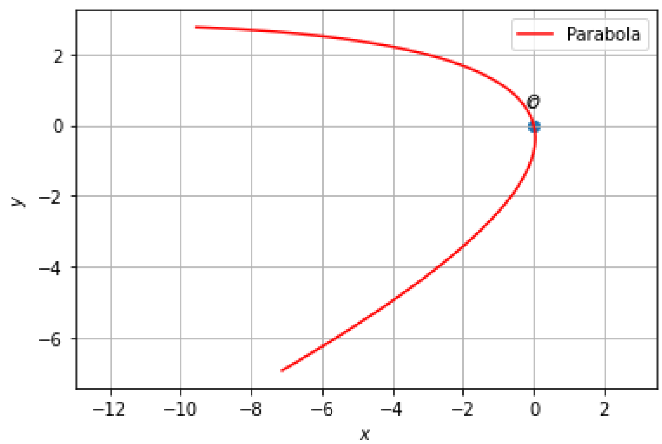

Fig. 0: Parabola