Matrix Theory: Assignment 3

Debolena Basak
Roll No.: AI20RESCH11003
PhD Artificial Intelligence

***Abstract*—This a problem on congruency of triangles in a quadrilateral.**

Download all latex-tikz codes from

https://github.com/Debolena/EE5609/blob/master/Assignment_3/latex_code.tex

1 Problem

ABCD is a quadrilateral in which AD = BC and $\angle DAB = \angle CBA$. Prove that

$$a)\quad \triangle ABD \cong \triangle BAC \tag{1.0.1}$$

$$b)\quad BD = AC \tag{1.0.2}$$

$$c)\quad \angle ABD = \angle BAC \tag{1.0.3}$$

2 Figure

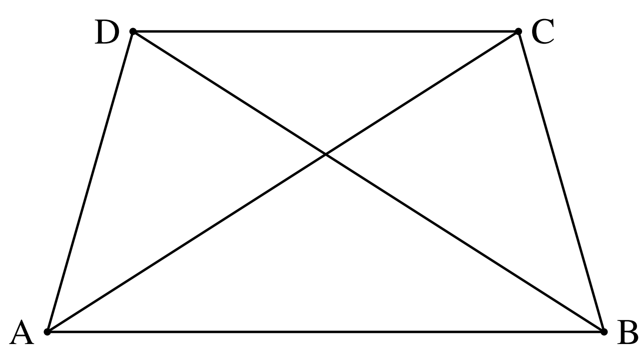

Fig. 0: Quadrilateral ABCD

3 Solution

ABCD is a quadrilateral. We are given that AD=BC and $\angle DAB = \angle CBA$.

We have to show that $\triangle ABD \cong \triangle BAC$.

$$\angle DAB = \angle CBA \qquad (Given) \tag{3.0.1}$$

$$AD = BC \qquad (Given) \tag{3.0.2}$$

$$AB = BA \qquad (Common\quad base) \tag{3.0.3}$$

Hence, by SAS congruency, $\triangle ABD \cong \triangle BAC$.

b) We have,

$$\angle DAB = \angle CBA \tag{3.0.4}$$

$$\implies \cos\angle DAB = \cos\angle CBA \tag{3.0.5}$$

$$\frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{D})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{D}\|} = \frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{C})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{3.0.6}$$

,using the formula of dot product, i.e.,

$$\mathbf{a}.\mathbf{b} = \|\mathbf{a}\|\,.\,\|\mathbf{b}\|\,.\cos\theta \tag{3.0.7}$$

$$\implies \cos\theta = \frac{\mathbf{a}.\mathbf{b}}{\|\mathbf{a}\|\,.\,\|\mathbf{b}\|} \tag{3.0.8}$$

We are given AD=BC and we know AB=BA always. Thus,

$$\|\mathbf{A}-\mathbf{D}\| = \|\mathbf{B}-\mathbf{C}\| \tag{3.0.9}$$

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{B}-\mathbf{A}\| \tag{3.0.10}$$

Then, from (3.0.6), we have,

$$(\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{D}) = (\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{C}) \tag{3.0.11}$$

$$\implies \|\mathbf{A}-\mathbf{B}\|^2 - (\mathbf{B}-\mathbf{D})^T(\mathbf{B}-\mathbf{A}) = \|\mathbf{A}-\mathbf{B}\|^2 - (\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B}) \tag{3.0.12}$$

$$\implies (\mathbf{B}-\mathbf{D})^T(\mathbf{B}-\mathbf{A}) = (\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B}) \tag{3.0.13}$$

$$\implies \|\mathbf{B}-\mathbf{D}\|\,\|\mathbf{B}-\mathbf{A}\|\cos\angle ABD = \|\mathbf{A}-\mathbf{C}\|\,\|\mathbf{A}-\mathbf{B}\|\cos\angle BAC \tag{3.0.14}$$

$$\implies \|\mathbf{B}-\mathbf{D}\|\cos\angle ABD = \|\mathbf{A}-\mathbf{C}\|\cos\angle BAC \tag{3.0.15}$$

We have to prove: $\|\mathbf{B}-\mathbf{D}\| = \|\mathbf{A}-\mathbf{C}\|$.
From (3.0.13),

$$(\mathbf{B}-\mathbf{D})^T(\mathbf{B}-\mathbf{A}) = (\mathbf{A}-\mathbf{C})^T(\mathbf{A}-\mathbf{B}) \tag{3.0.16}$$

$$\implies \|\mathbf{B}-\mathbf{D}\|^2 - (\mathbf{D}-\mathbf{A})^T(\mathbf{D}-\mathbf{B}) = \|\mathbf{A}-\mathbf{C}\|^2 - (\mathbf{C}-\mathbf{B})^T(\mathbf{C}-\mathbf{A}) \tag{3.0.17}$$

$$\implies \|\mathbf{B}-\mathbf{D}\|^2 - (\|\mathbf{A}-\mathbf{D}\|^2 - (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{D})) = \|\mathbf{A}-\mathbf{C}\|^2 - (\|\mathbf{B}-\mathbf{C}\|^2 - (\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{C})) \tag{3.0.18}$$

We are given that,

$$\|\mathbf{A}-\mathbf{D}\| = \|\mathbf{B}-\mathbf{C}\| \tag{3.0.19}$$

$$\therefore \|\mathbf{B}-\mathbf{D}\|^2 + (\mathbf{A}-\mathbf{B})^T(\mathbf{A}-\mathbf{D}) = \|\mathbf{A}-\mathbf{C}\|^2 + (\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{C}) \tag{3.0.20}$$

$$\implies \|\mathbf{B}-\mathbf{D}\|^2 + \|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{A}-\mathbf{D}\|\cos\angle DAB = \|\mathbf{A}-\mathbf{C}\|^2 + \|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{B}-\mathbf{C}\|\cos\angle CBA \tag{3.0.21}$$

From the question, $\angle DAB = \angle CBA$ and $\|\mathbf{A}-\mathbf{D}\| = \|\mathbf{B}-\mathbf{C}\|$. We also know $\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{B}-\mathbf{A}\|$. Thus, from (3.0.21), we get,

$$\|\mathbf{B}-\mathbf{D}\|^2 = \|\mathbf{A}-\mathbf{C}\|^2 \tag{3.0.22}$$

$$\implies \|\mathbf{B}-\mathbf{D}\| = \|\mathbf{A}-\mathbf{C}\| \tag{3.0.23}$$

$$\therefore BD = AC \tag{3.0.24}$$

c) From (3.0.15) and (3.0.23), we have

$$\cos\angle ABD = \cos\angle BAC \tag{3.0.25}$$

$$\implies \angle ABD = \angle BAC \tag{3.0.26}$$