Matrix Theory: Assignment 8

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***Abstract*—This document is based on orthonormal basis and orthonormal matrix.**

Download all latex-tikz codes from

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https://github.com/Debolena/EE5609/tree/master/
    Assignment_8
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1 Problem

Let $\mathbf{R}^n, n \geq 2$ be equipped with standard inner product. Let $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ be n column vectors forming an orthornormal basis of $\mathbf{R}^n$. Let $\mathbf{A}$ be a n x n matrix formed by the column vectors, $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$. Then,

1) $\mathbf{A} = \mathbf{A}^{-1}$
2) $\mathbf{A} = \mathbf{A}^T$
3) $\mathbf{A}^{-1} = \mathbf{A}^T$
4) $Det(\mathbf{A}) = 1$

2 solution

Given, $\mathbf{v_1}, \mathbf{v_2}, ..., \mathbf{v_n}$ are orthonormal and form basis.
So, when they form column vectors of matrix $\mathbf{A}$, we can say that $\mathbf{A}$ is also orthonormal.

$$\therefore \quad \mathbf{A^TA} = \mathbf{I} \tag{2.0.1}$$

$$\implies \mathbf{A^TAA^{-1}} = \mathbf{IA^{-1}} \tag{2.0.2}$$

$$\implies \mathbf{A^T} = \mathbf{A^{-1}} \tag{2.0.3}$$

Clearly, option 3 is the correct answer.

2.1 Example:

Let us consider an orthonormal basis for $\mathbf{R}^2$.

We can check that $\mathbf{S}$= $\{ \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}, \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \}$ forms an orthonormal basis.
Thus the matrix

$$\mathbf{Q} = \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{2.1.1}$$

is the orthonormal matrix whose column vectors are the basis of $\mathbf{R}^2$.

2.2 Option 4 cannot be true

For an orthonormal matrix $\mathbf{A}$,

$$\mathbf{A^TA} = \mathbf{I} \tag{2.2.1}$$

$$\implies \det\left(\mathbf{A^TA}\right) = \det(\mathbf{I}) \tag{2.2.2}$$

$$\implies \det\left(\mathbf{A}^T\right)\det(\mathbf{A}) = 1 \tag{2.2.3}$$

$$\implies \det(\mathbf{A})^2 = 1 \quad \because \det(\mathbf{A}) = \det\left(\mathbf{A^T}\right) \tag{2.2.4}$$

$$\implies \det(\mathbf{A}) = \pm 1 \tag{2.2.5}$$

Also, here a contradictory example:
Let,

$$\mathbf{R} = \begin{pmatrix} -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix} \tag{2.2.6}$$

Clearly, $\mathbf{R}$ is an orthonormal matrix and the column vectors of it form an orthonormal basis of $\mathbf{R}^2$. But,

$$\det \mathbf{R} = \begin{vmatrix} -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{vmatrix} \tag{2.2.7}$$

$$= -1 \tag{2.2.8}$$

From the above two arguments it is clear that option 4 cannot be true.