Matrix Theory: Assignment 9

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Abstract—**This document is based on finding eigen values of a symmetric matrix.**

Download all latex-tikz codes from

https://github.com/Debolena/EE5609/tree/master/Assignment_9

1 Problem

Which of the following are eigen values of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \quad ? \tag{1.0.1}$$

1) $+1$
2) -1
3) $+i$
4) $-i$

2 solution

2.1 Eigen values of a real symmetric matrix are real.

Proof:
Here $\mathbf{A}^T = \mathbf{A}$. Therefore matrix $\mathbf{A}$ is a symmetric matrix. Also $\mathbf{A}$ is a real matrix.
Let λ be a complex eigen value. Then the eigen vector $\mathbf{x}$ will have one or more complex elements. We have,

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x} \tag{2.1.1}$$

$\implies$ $\mathbf{Ax}$ and $\lambda\mathbf{x}$ are complex respectively.
$\implies$ their complex conjugates are also equal.
Let the conjugates of λ and $\mathbf{x}$ be $\bar{\lambda}$ and $\bar{\mathbf{x}}$ respectively.

$$\therefore \mathbf{A}\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}} \tag{2.1.2}$$

$$\left[\because \bar{\mathbf{Ax}} = \bar{\lambda\mathbf{x}} \implies \bar{\mathbf{A}}\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}} \implies \mathbf{A}\bar{\mathbf{x}} = \bar{\lambda}\bar{\mathbf{x}}\right] \tag{2.1.3}$$

Multiplying (2.1.1) by $\bar{\mathbf{x}}^T$ and (2.1.2) by $\mathbf{x}^T$ and subtracting,

$$\bar{\mathbf{x}}^{\mathbf{T}}\mathbf{A}\mathbf{x} - \mathbf{x}^{\mathbf{T}}\mathbf{A}\bar{\mathbf{x}} = \left(\lambda - \bar{\lambda}\right)\bar{\mathbf{x}}^{\mathbf{T}}\mathbf{x} \tag{2.1.4}$$

Each term on the LHS of (2.1.4) is scalar and $\mathbf{A}$ is symmetric

$$\therefore \bar{\mathbf{x}}^{\mathbf{T}}\mathbf{A}\mathbf{x} - \mathbf{x}^{\mathbf{T}}\mathbf{A}\bar{\mathbf{x}} = 0 \tag{2.1.5}$$

From (2.1.4) and (2.1.5),

$$\left(\lambda - \bar{\lambda}\right)\bar{\mathbf{x}}^{\mathbf{T}}\mathbf{x} = 0 \tag{2.1.6}$$

where $\bar{\mathbf{x}}^{\mathbf{T}}\mathbf{x}$ = sum of products of complex numbers times their conjugates.

$$\because \bar{\mathbf{x}}^{\mathbf{T}}\mathbf{x} \neq 0 \tag{2.1.7}$$

$$\therefore \left(\lambda - \bar{\lambda}\right) = 0 \tag{2.1.8}$$

$$\implies \lambda = \bar{\lambda} \tag{2.1.9}$$

This implies λ is real.
$\therefore$ The eigen values are real. (*proved*).
Thus, we can eliminate option 3 and 4.
The sum of eigen values of a matrix is equal to the trace of the matrix.
From (1.0.1), trace of $\mathbf{A}$ = 0, which is only possible if the eigen values are +1 and -1.
Therefore, option 1 and 2 are the correct choices.