Matrix Theory: Assignment 6

Debolena Basak
Roll No.: AI20RESCH11003
PhD Artificial Intelligence

***Abstract*—This document is to find the QR decomposition of V and vertex of a parabola using SVD, then verifying it using Least Squares.**

Download all latex-tikz codes from

https://github.com/Debolena/EE5609/tree/master/Assignment_6

1 Problem

From Assignment 5,
1. Find the QR decomposition of V.
2. Obtain $\mathbf{c}$ using SVD by using $\eta/2$ instead of η and verify your solution using least squares.

2 Solution

2.1 **QR** *decomposition of* **V**

We have,

$$\mathbf{V} = \begin{pmatrix} 1 & -4 \\ -4 & 16 \end{pmatrix} \tag{2.1.1}$$

where, $\mathbf{V}$ can be written as,

$$\mathbf{V} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \end{pmatrix} \tag{2.1.2}$$

where $\mathbf{a}$ and $\mathbf{b}$ and are column vectors. Here,

$$\mathbf{a} = \begin{pmatrix} 1 \\ -4 \end{pmatrix} \tag{2.1.3}$$

$$\mathbf{b} = \begin{pmatrix} -4 \\ 16 \end{pmatrix} \tag{2.1.4}$$

The QR decomposition of a matrix $\mathbf{A}$ is given by,

$$\mathbf{A} = \mathbf{QR} \tag{2.1.5}$$

where $\mathbf{R}$ is a upper triangular matrix and $\mathbf{Q}$ is such that,

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \tag{2.1.6}$$

where,

$$\mathbf{Q} = \begin{pmatrix} \mathbf{q_1} & \mathbf{q_2} \end{pmatrix} \tag{2.1.7}$$

$$\mathbf{R} = \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \tag{2.1.8}$$

The values in $\mathbf{R}$ and $\mathbf{q_1}$, $\mathbf{q_2}$ are given by,

$$r_1 = \|\mathbf{a}\| \tag{2.1.9}$$

$$\mathbf{q_1} = \frac{\mathbf{a}}{r_1} \tag{2.1.10}$$

$$r_2 = \frac{\mathbf{q_1^T b}}{\|\mathbf{q_1}\|^2} \tag{2.1.11}$$

$$\mathbf{q_2} = \frac{\mathbf{b} - r_2\mathbf{q_1}}{\|\mathbf{b} - r_2\mathbf{q_1}\|} \tag{2.1.12}$$

$$r_3 = \mathbf{q_2^T b} \tag{2.1.13}$$

Using (2.1.3), (2.1.4) and the above formulas,

$$r_1 = \sqrt{17} \tag{2.1.14}$$

$$\mathbf{q_1} = \frac{1}{\sqrt{17}}\begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{17}} \\ -\frac{4}{\sqrt{17}} \end{pmatrix} \tag{2.1.15}$$

$$\|\mathbf{q_1}\| = 1 \tag{2.1.16}$$

$$r_2 = \begin{pmatrix} \frac{1}{\sqrt{17}} & -\frac{4}{\sqrt{17}} \end{pmatrix}\begin{pmatrix} -4 \\ 16 \end{pmatrix} = -\frac{68}{\sqrt{17}} \tag{2.1.17}$$

$$\mathbf{q_2} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.1.18}$$

$$r_3 = 0 \tag{2.1.19}$$

$$\tag{2.1.20}$$

Hence,

$$\mathbf{QR} = \begin{pmatrix} \mathbf{q}_1 & \mathbf{q}_2 \end{pmatrix} \begin{pmatrix} r_1 & r_2 \\ 0 & r_3 \end{pmatrix} \tag{2.1.21}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{17}} & 0 \\ -\frac{4}{\sqrt{17}} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{17} & -\frac{68}{\sqrt{17}} \\ 0 & 0 \end{pmatrix} \tag{2.1.22}$$

$$= \begin{pmatrix} 1 & -4 \\ -4 & 16 \end{pmatrix} \tag{2.1.23}$$

Clearly, (2.1.23) and (2.1.1) are equal. Hence, the **QR** decomposition holds.

2.2 *Finding vertex using SVD*

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 1 & -4 \\ -4 & 16 \end{pmatrix} \tag{2.2.1}$$

$$\mathbf{u} = \begin{pmatrix} 0 \\ -\frac{51}{2} \end{pmatrix} \tag{2.2.2}$$

$$f = 0 \tag{2.2.3}$$

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} = \frac{1}{\sqrt{17}} \begin{pmatrix} -4 & -1 \\ -1 & 4 \end{pmatrix} \tag{2.2.4}$$

$$\eta = \mathbf{u}^T \mathbf{p}_1 = \frac{51}{2\sqrt{17}} \tag{2.2.5}$$

The equation of perpendicular line passing through focus and intersecting parabola at vertex **c** is given as

$$\begin{pmatrix} \mathbf{u^T} + \frac{\eta}{2}\mathbf{p_1^T} \\ \mathbf{V} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -f \\ \frac{\eta}{2}\mathbf{p_1} - \mathbf{u} \end{pmatrix} \tag{2.2.6}$$

Using (2.2.2), (2.2.4), (2.2.5), (2.2.1) and (2.2.3),

$$\begin{pmatrix} -3 & -\frac{105}{4} \\ 1 & -4 \\ -4 & 16 \end{pmatrix} \mathbf{c} = \begin{pmatrix} 0 \\ -3 \\ \frac{99}{4} \end{pmatrix} \tag{2.2.7}$$

$$\implies \mathbf{Mc} = \mathbf{b} \tag{2.2.8}$$

where,

$$\mathbf{M} = \begin{pmatrix} -3 & -\frac{105}{4} \\ 1 & -4 \\ -4 & 16 \end{pmatrix} \tag{2.2.9}$$

$$\mathbf{b} = \begin{pmatrix} 0 \\ -3 \\ \frac{99}{4} \end{pmatrix} \tag{2.2.10}$$

To solve (2.2.8), we perform Singular Value Decomposition on **M** given as

$$\mathbf{M} = \mathbf{USV^T} \tag{2.2.11}$$

Putting this value of **M** in (2.2.8), we get

$$\mathbf{USV^Tc} = \mathbf{b} \tag{2.2.12}$$

$$\implies \mathbf{c} = \mathbf{VS_+U^Tb} \tag{2.2.13}$$

where, $\mathbf{S}_+$ is Moore-Penrose pseudo-inverse of **S**. Columns of **U** are eigen-vectors of $\mathbf{MM^T}$, columns of **V** are eigen-vectors of $\mathbf{M^TM}$ and **S** is diagonal matrix of singular value of eigenvalues of $\mathbf{M^TM}$.

$$\mathbf{MM^T} = \begin{pmatrix} -3 & \frac{-105}{4} \\ 1 & -4 \\ -4 & 16 \end{pmatrix} \begin{pmatrix} -3 & 1 & -4 \\ \frac{-105}{4} & -4 & 16 \end{pmatrix} \tag{2.2.14}$$

$$= \begin{pmatrix} 698.0625 & 102 & -408 \\ 102 & 17 & -68 \\ -408 & -68 & 272 \end{pmatrix} \tag{2.2.15}$$

$$\mathbf{M^TM} = \begin{pmatrix} -3 & 1 & -4 \\ -\frac{105}{4} & -4 & 16 \end{pmatrix} \begin{pmatrix} -3 & -\frac{105}{4} \\ 1 & -4 \\ -4 & 16 \end{pmatrix} \tag{2.2.16}$$

$$= \begin{pmatrix} 26 & 10.75 \\ 10.75 & 961.0625 \end{pmatrix} \tag{2.2.17}$$

Eigen values of $\mathbf{M^TM}$ can be found out as

$$\left| \mathbf{M^TM} - \lambda \mathbf{I} \right| = 0 \tag{2.2.18}$$

$$\implies \begin{vmatrix} 26 - \lambda & 10.75 \\ 10.75 & 961.0625 - \lambda \end{vmatrix} = 0 \tag{2.2.19}$$

Solving this we get the eigen values of $\mathbf{M^TM}$ as,

$$\lambda_1 = 961.1861 \tag{2.2.20}$$

$$\lambda_2 = 25.8764 \tag{2.2.21}$$

The corresponding eigen vectors are:

$$\mathbf{v}_1 = \begin{pmatrix} 0.0115 \\ 1 \end{pmatrix} \tag{2.2.22}$$

$$\mathbf{v}_2 = \begin{pmatrix} -86.994 \\ 1 \end{pmatrix} \tag{2.2.23}$$

Normalizing the values,

$$\mathbf{v}_1 = \begin{pmatrix} 0.0115 \\ 0.9999 \end{pmatrix} \tag{2.2.24}$$

$$\mathbf{v}_2 = \begin{pmatrix} -0.9999 \\ 0.0115 \end{pmatrix} \tag{2.2.25}$$

Hence,

$$\mathbf{V} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} \tag{2.2.26}$$

$$= \begin{pmatrix} 0.0115 & -0.9999 \\ 0.9999 & 0.0115 \end{pmatrix} \tag{2.2.27}$$

Eigen values of $\mathbf{M}\mathbf{M}^{\mathbf{T}}$ can be found by solving:

$$\left|\mathbf{M}\mathbf{M}^{\mathbf{T}} - \lambda \mathbf{I}\right| = 0 \tag{2.2.28}$$

$$\implies \begin{vmatrix} 698.0625 - \lambda & 102 & -408 \\ 102 & 17 - \lambda & -68 \\ -408 & -68 & 272 - \lambda \end{vmatrix} = 0 \tag{2.2.29}$$

Solving this, we get the eigen values of $\mathbf{M}\mathbf{M}^{\mathbf{T}}$ as:

$$\lambda_3 = 961.1861 \tag{2.2.30}$$

$$\lambda_4 = 25.8764 \tag{2.2.31}$$

$$\lambda_5 = 0 \tag{2.2.32}$$

The corresponding eigen vectors after normalizing are:

$$\mathbf{u}_1 = \begin{pmatrix} -0.8477 \\ -0.1286 \\ 0.5146 \end{pmatrix} \tag{2.2.33}$$

$$\mathbf{u}_2 = \begin{pmatrix} 0.5304 \\ -0.2056 \\ 0.8224 \end{pmatrix} \tag{2.2.34}$$

$$\mathbf{u}_3 = \begin{pmatrix} 0 \\ 0.9701 \\ 0.2425 \end{pmatrix} \tag{2.2.35}$$

$$\therefore \mathbf{U} = \begin{pmatrix} -0.8477 & 0.5304 & 0 \\ -0.1286 & -0.2056 & 0.9701 \\ 0.5146 & 0.8224 & 0.2425 \end{pmatrix} \tag{2.2.36}$$

After computing the singular values from the eigen values,

$$\mathbf{S} = \begin{pmatrix} \sqrt{\lambda_1} & 0 \\ 0 & \sqrt{\lambda_2} \\ 0 & 0 \end{pmatrix} \tag{2.2.37}$$

$$= \begin{pmatrix} 31.0030 & 0 \\ 0 & 5.0869 \\ 0 & 0 \end{pmatrix} \tag{2.2.38}$$

Therefore we get the SVD of $\mathbf{M}$ as:

$$\mathbf{M} = \begin{pmatrix} -0.8477 & 0.5304 & 0 \\ -0.1286 & -0.2056 & 0.9701 \\ 0.5146 & 0.8224 & 0.2425 \end{pmatrix} \begin{pmatrix} 31.0030 & 0 \\ 0 & 5.0869 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0.0115 & -0.9999 \\ 0.9999 & 0.0115 \end{pmatrix}^T \tag{2.2.39}$$

$$= \begin{pmatrix} -3 & -26.2500 \\ 1 & -4 \\ -4 & 16 \end{pmatrix} \tag{2.2.40}$$

Moore- penrose pseudo inverse of $\mathbf{S}$ is:

$$\mathbf{S}_+ = \begin{pmatrix} 0.0323 & 0 & 0 \\ 0 & 0.1966 & 0 \end{pmatrix} \tag{2.2.41}$$

Putting the values in (2.2.13),

$$\mathbf{U}^{\mathbf{T}}\mathbf{b} = \begin{pmatrix} 13.1213 \\ 20.9721 \\ 3.0923 \end{pmatrix} \tag{2.2.42}$$

$$\mathbf{S}_+\mathbf{U}^{\mathbf{T}}\mathbf{b} = \begin{pmatrix} 0.4238 \\ 4.1231 \end{pmatrix} \tag{2.2.43}$$

$$\mathbf{c} = \mathbf{S}_+\mathbf{U}^{\mathbf{T}}\mathbf{b} = \begin{pmatrix} -4.1180 \\ 0.4712 \end{pmatrix} \tag{2.2.44}$$

Verifying the above solution using least squares:

$$\mathbf{M}^{\mathbf{T}}\mathbf{M}\mathbf{c} = \mathbf{M}^{\mathbf{T}}\mathbf{b} \tag{2.2.45}$$

$$\implies \mathbf{M}^{\mathbf{T}}\mathbf{M}\mathbf{c} = \begin{pmatrix} -102 \\ 408 \end{pmatrix} \tag{2.2.46}$$

$$\implies \begin{pmatrix} 26 & \frac{43}{4} \\ \frac{43}{4} & \frac{15377}{16} \end{pmatrix} \mathbf{c} = \begin{pmatrix} -102 \\ 408 \end{pmatrix} \tag{2.2.47}$$

Forming the augmented matrix and row-reducing it:

$$\begin{pmatrix} 26 & \frac{43}{4} & -102 \\ \frac{43}{4} & \frac{15377}{16} & 408 \end{pmatrix} \tag{2.2.48}$$

$$\xleftrightarrow{R_2 \leftarrow R_1 - \frac{4}{43}26R_2} \begin{pmatrix} 26 & \frac{43}{4} & -102 \\ 0 & -\frac{397953}{172} & -\frac{46818}{43} \end{pmatrix} \tag{2.2.49}$$

$$\xleftrightarrow{R_2 \leftarrow -\frac{172}{397953}R_2} \begin{pmatrix} 26 & \frac{43}{4} & -102 \\ 0 & 1 & \frac{8}{17} \end{pmatrix} \tag{2.2.50}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{1}{26}R_1} \begin{pmatrix} 1 & \frac{43}{104} & -\frac{51}{13} \\ 0 & 1 & \frac{8}{17} \end{pmatrix} \tag{2.2.51}$$

$$\xleftrightarrow{R_1 \leftarrow R_1 - \frac{43}{104}R_2} \begin{pmatrix} 1 & 0 & -\frac{70}{17} \\ 0 & 1 & \frac{8}{17} \end{pmatrix} \tag{2.2.52}$$

From (2.2.52),

$$\mathbf{c} = \begin{pmatrix} -\frac{70}{17} \\ \frac{8}{17} \end{pmatrix} \tag{2.2.53}$$

$$= \begin{pmatrix} -4.1176 \\ 0.4706 \end{pmatrix} \tag{2.2.54}$$

Comparing (2.2.44) and (2.2.54), it can be said that the solution of $\mathbf{c}$ is verified.