Matrix Theory: Assignment 10

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***Abstract*—This document is based on checking some properties of orthogonal matrix.**

Download all latex-tikz codes from

https://github.com/Debolena/EE5609/tree/master/Assignment_10

1 Problem

Let $\mathbf{A}$ be a real n x n orthogonal matrix, that is, $\mathbf{A^T A} = \mathbf{AA^T} = \mathbf{I_n}$, the n x n identity matrix. which of the following statements are necessarily true?

1) $< \mathbf{Ax}, \mathbf{Ay} >=< \mathbf{x}, \mathbf{y} > \quad \forall \mathbf{x}, \mathbf{y} \in \mathbf{R}^n$
2) All eigen values of $\mathbf{A}$ are either +1 or -1.
3) The rows of $\mathbf{A}$ form an orthonormal basis of $\mathbf{R}^n$.
4) $\mathbf{A}$ is diagonalizable over $\mathbf{R}$.

2 solution

2.1 Option 1

$$< \mathbf{Ax}, \mathbf{Ay} > = (\mathbf{Ax})^\mathbf{T} \mathbf{Ay} \tag{2.1.1}$$
$$= \mathbf{x^T A^T A y} \tag{2.1.2}$$
$$= \mathbf{x^T y} \quad \because \mathbf{A^T A} = \mathbf{I} \tag{2.1.3}$$
$$= < \mathbf{x}, \mathbf{y} > \tag{2.1.4}$$

Hence, option 1 is correct.

2.2 Option 2

Let λ be the eigen value and $\mathbf{v}$ be the eigen vector corresponding to it.
Then,

$$\mathbf{Av} = \lambda \mathbf{v} \tag{2.2.1}$$
$$\implies \|\mathbf{Av}\|^2 = \|\lambda \mathbf{v}\|^2 \tag{2.2.2}$$
$$\implies \|\mathbf{Av}\|^2 = |\lambda|^2 \|\mathbf{v}\|^2 \tag{2.2.3}$$

Now,

$$\|\mathbf{Av}\|^2 = (\mathbf{Av})^T \mathbf{Av} \tag{2.2.4}$$
$$= \mathbf{v^T A^T A v} \tag{2.2.5}$$
$$= \mathbf{v^T I v} \tag{2.2.6}$$
$$= \mathbf{v^T v} \tag{2.2.7}$$
$$= \|\mathbf{v}\|^2 \tag{2.2.8}$$

Comparing (2.2.3) and (2.2.8), we get,

$$|\lambda|^2 = 1 \tag{2.2.9}$$
$$\implies |\lambda| = \pm 1 \tag{2.2.10}$$

But $|\lambda|$ cannot be -1.

$$\therefore |\lambda| = 1 \tag{2.2.11}$$
$$\implies \lambda = \pm 1 \tag{2.2.12}$$

Thus, option 2 is correct.

2.3 Option 3

Let $\mathbf{r_1}, \mathbf{r_2}, ..., \mathbf{r_n}$ denote the row vectors of $\mathbf{A}$.
Then,

$$\mathbf{AA^T} = \begin{pmatrix} \mathbf{r_1^T r_1} & \mathbf{r_1^T r_2} & ... & \mathbf{r_1^T r_n} \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \\ \mathbf{r_n^T r_1} & \mathbf{r_n^T r_2} & ... & \mathbf{r_n^T r_n} \end{pmatrix} \tag{2.3.1}$$

But, $\mathbf{A}$ is orthogonal. So, $\mathbf{AA^T} = \mathbf{I}$. It therefore follows that

1) All diagonal elements of (2.3.1) are 1.
2) All off- diagonal elements of (2.3.1) are 0.

That is, for all $i, j = 1, 2, ..., n$,

$$\mathbf{r_i^T r_j} = 1, \quad i = j \tag{2.3.2}$$
$$= 0, \quad i \neq j \tag{2.3.3}$$

Therefore, $\mathbf{r_1}, \mathbf{r_2}, ...\mathbf{r_n}$ are orthonormal and form a basis of $\mathbf{R}^n$.
Hence, option 3 is correct.

2.4 Option 4

Counter Example:
Let us consider a matrix in $\mathbf{R}^2$

$$\mathbf{Q} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.4.1}$$

$$\therefore \mathbf{Q}^T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{2.4.2}$$

Check that $\mathbf{AA^T} = \mathbf{I}, \therefore \mathbf{Q}$ is orthogonal.
The characteristic equation is:

$$\left|\mathbf{Q} - \lambda\mathbf{I}\right| = 0 \tag{2.4.3}$$

$$\implies \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \tag{2.4.4}$$

$$\implies \lambda^2 + 1 = 0 \tag{2.4.5}$$

$$\implies \lambda = \pm i \notin \mathbf{R} \tag{2.4.6}$$

which implies $\mathbf{Q}$ is not diagonalizable over $\mathbf{R}$.

Hence, we can conclude that option 1, 2 and 3 are correct.