Assignment 4: Matrix Theory

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***Abstract*—This is a problem of a line, circle and tangent.**

Download all python codes from

https://github.com/Debolena/EE5609/blob/master/Assignment_4/figure.py

and all the latex-tikz codes from

https://github.com/Debolena/EE5609/tree/master/Assignment_4

1 Problem

Find the points of intersection of the line

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = 12 \tag{1.0.1}$$

and the circle

$$\|\mathbf{x}\|^2 = 13 \tag{1.0.2}$$

and for what values of c the line

$$\begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} = c \tag{1.0.3}$$

touches the circle.

2 Solution

If $\mathbf{P}$ be a point on the line and $\mathbf{n}$ is the normal vector, the equation of the line can be expressed as

$$\mathbf{n}^T(\mathbf{x} - \mathbf{P}) = 0 \tag{2.0.1}$$

$$\implies \mathbf{n}^T\mathbf{x} = c \tag{2.0.2}$$

where

$$c = \mathbf{n}^T\mathbf{P} \tag{2.0.3}$$

From (1.0.1) and (2.0.2),

$$\mathbf{n}^T = \begin{pmatrix} 3 & 2 \end{pmatrix} \tag{2.0.4}$$

We know,

$$\mathbf{m}^T\mathbf{n} = 0 \tag{2.0.5}$$

$$\implies \mathbf{m}^T \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 0 \tag{2.0.6}$$

$$\implies \mathbf{m}^T = \begin{pmatrix} -2 & 3 \end{pmatrix} \tag{2.0.7}$$

Now,

$$\mathbf{n}^T\mathbf{P} = c \tag{2.0.8}$$

$$\implies \begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{P} = 12 \tag{2.0.9}$$

$\mathbf{P}$ can be

$$\begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 6 \end{pmatrix} \tag{2.0.10}$$

Let us take

$$\mathbf{P} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \mathbf{q} \tag{2.0.11}$$

The circle equation:

$$\mathbf{x}^T\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.12}$$

From (1.0.2),

$$\mathbf{u} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.13}$$

$$f = -13 \tag{2.0.14}$$

The points of intersection of the line

$$L : \mathbf{x} = \mathbf{q} + \mu\mathbf{m} \quad , \mu \in \mathbb{R} \tag{2.0.15}$$

with the conic section

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.16}$$

are given by

$$\mathbf{x}_i = \mathbf{q} + \mu_i\mathbf{m} \tag{2.0.17}$$

where,

$$\mu_i = \frac{1}{\mathbf{m}^T\mathbf{V}\mathbf{m}}\Big(-\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u}) \pm \sqrt{[\mathbf{m}^T(\mathbf{V}\mathbf{q}+\mathbf{u})]^2 - (\mathbf{q}^T\mathbf{V}\mathbf{q}+2\mathbf{u}^T\mathbf{q}+f)(\mathbf{m}^T\mathbf{V}\mathbf{m})}\Big) \tag{2.0.18}$$

For circle,

$$\mathbf{V} = \mathbf{I} \tag{2.0.19}$$

$$\therefore \mu_i = \frac{1}{13}\left(-5 \pm \sqrt{25-(13-13)\,13}\right) \tag{2.0.20}$$

$$= \frac{1}{13}(-5 \pm 5) \tag{2.0.21}$$

$$= 0, -\frac{10}{13} \tag{2.0.22}$$

Using (2.0.17), the points of intersection are given by

$$\mathbf{x} = \begin{pmatrix}2\\3\end{pmatrix}, \begin{pmatrix}\frac{46}{13}\\\frac{9}{13}\end{pmatrix} \tag{2.0.23}$$

Points of contact are given by

$$\mathbf{q} = \mathbf{V}^{-1}(\kappa\mathbf{n}-\mathbf{u}) \tag{2.0.24}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{2.0.25}$$

Since for circle,

$$\mathbf{V} = \mathbf{I} \tag{2.0.26}$$

$$\therefore \mathbf{V}^{-1} = \mathbf{I} \quad \because \mathbf{I}^{-1} = \mathbf{I} \tag{2.0.27}$$

$$\therefore \kappa = \pm\sqrt{\frac{-f}{\mathbf{n}^T\mathbf{n}}} \quad \because \mathbf{u}^T\mathbf{u} = 0 \tag{2.0.28}$$

$$= \pm\sqrt{\frac{13}{\begin{pmatrix}3 & 2\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix}}} \tag{2.0.29}$$

$$= \pm\sqrt{\frac{13}{13}} \tag{2.0.30}$$

$$= \pm 1 \tag{2.0.31}$$

$$\therefore \mathbf{q} = \pm 1\begin{pmatrix}3\\2\end{pmatrix} \tag{2.0.32}$$

$$= \begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}-3\\-2\end{pmatrix} \tag{2.0.33}$$

From (1.0.3),

$$c = \begin{pmatrix}3 & 2\end{pmatrix}\begin{pmatrix}3\\2\end{pmatrix} = 13, \tag{2.0.34}$$

$$\begin{pmatrix}3 & 2\end{pmatrix}\begin{pmatrix}-3\\-2\end{pmatrix} = -13 \tag{2.0.35}$$

The line (1.0.3) touches the circle for $c = 13, -13$.