07. Simple regression

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Data Analysis 2: Regression analysis

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Motivation

- What's data analysis?
- We build some model to get answers to our questions.
- Define a problem
  - Collect data (manage, wrangle, clean, etc) <— DA1
- Learn about patterns
- Use information to help decision in business, politics, economic policy
- Regression analysis is basic tool to do that
- In the end: *"All models are wrong, but some are useful."* - George Box

Case study motivation

- Spend a night in Vienna and you want to find a good deal for your stay.
- Travel time to the city center is rather important.
- Looking for a good deal: as low a price as possible and as close to the city center as possible.
- Collect data on suitable hotels, compare average prices for various distances from center.
- Look for hotels where price is cheap relative to what being close to the center would normally cost.

Introduction

- Regression is the most widely used method of comparison in data analysis.
- Simple regression analysis amounts to comparing average values of a dependent variable (y) for observations that are different in the explanatory variable (x).
- Simple regression: *comparing conditional means*.
- Doing so uncovers the pattern of association between y and x. What you use for y and for x is important and not inter-changeable!

Regression

- Simple regression analysis uncovers mean-dependence between two variables.
    - It amounts to comparing average values of one variable, called the dependent variable (y) for observations that are different in the other variable, the explanatory variable (x).
- Multiple regression analysis involves more variables -> later.

Regression - uses

- Discovering patterns of association between variables is often a good starting point even if our question is more ambitious.
- Causal analysis: uncovering the *effect* of one variable on another variable. Concerned with a parameter.
- Predictive analysis: what to expect of a y variable (long-run polls, hotel prices) for various values of another x variable (immediate polls, distance to the city center). Concerned with predicted value of y using x.

Regression - names and notation

- Regression analysis is a method that uncovers the average value of a variable y for different values of another variable x.

$$E[y|x] = f(x) \tag{1}$$

We use a simpler shorthand notation

$$y^E = f(x) \tag{2}$$

- dependent variable or left-hand-side variable, or simply the y variable,
- explanatory variable, right-hand-side variable, or simply the x variable
- "regress y on x," or "run a regression of y on x"= do simple regression analysis with y as the dependent variable and x as the explanatory variable.

Regression - type of patterns

Regression may find

- Linear patterns: positive (negative) association - average y tends to be higher (lower) at higher values of x.
- Non-linear patterns: association may be non-monotonic - y tends to be higher for higher values of x in a certain range of the x variable and lower for higher values of x in another range of the x variable
- No association or relationship

Non-parametric and parametric regression

- Non-parametric regressions describe the $y^E = f(x)$ pattern without imposing a specific functional form on f.
  - Let the data dictate what that function looks like, at least approximately.
  - Can spot (any) patterns well
- Parametric regressions impose a functional form on f. Parametric examples include:
  - linear functions: $f(x) = a + bx$;
  - exponential functions: $f(x) = ax^b$;
  - quadratic functions: $f(x) = a + bx + cx^2$,
  - or any functions which have parameters of a, b, c, etc.
  - Restrictive, but they produce readily interpretable numbers.

Non-parametric regression

- Non-parametric regressions come (also) in various forms.
- When x has few values and there are many observations in the data, the best and most intuitive non-parametric regression for $y^E = f(x)$ shows average y for each and every value of x. use bins
- There is no functional form imposed on f here.
  - The most straightforward example if you have ordered variables.
  - For example, Hotels: average price of hotels with the same numbers of stars and compare these averages = non-parametric regression analysis.

Non-parametric regression: bins

- With many x values - two ways to do non-parametric regression analysis: bins and smoothing.
- Bins - based on grouped values of x
  - Bins are disjoint categories (no overlap) that span the entire range of x (no gaps).
  - Many ways to create bins - equal size, equal number of observations per bin, or bins defined by analyst.

Non-parametric regression: lowess (loess)

- Produce "smooth" graph - both continuous and has no kink at any point.
- also called smoothed conditional means plots = non-parametric regression shows conditional means, smoothed to get a better image.
- Lowess = most widely used non-parametric regression methods that produce a smooth graph.
  - *locally weighted scatterplot smoothing* (sometimes abbreviated as "loess").
- A smooth curve fit around a bin scatter.
  - Related to density plots, set the bandwidth for smoothing
    - 'Bias-variance trade-off': wider bandwidth results in a smoother graph but may miss important details of the pattern (higher bias, smaller variance); narrower bandwidth produces a more rugged-looking graph (small bias, higher variance)

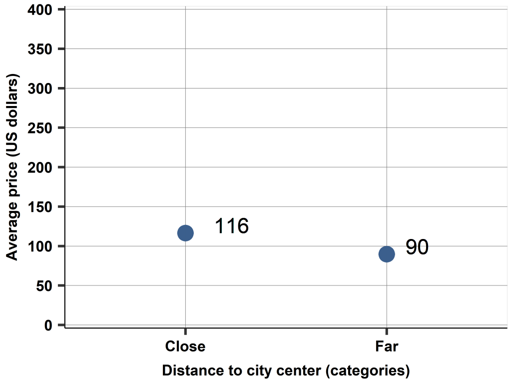
Non-parametric regression: lowess (loess)

- Smooth non-parametric regression methods, including lowess, do not produce numbers that would summarize the $y^E = f(x)$ pattern.
- Provide a value y^E for each of the particular x values that occur in the data, as well as for all x values in-between.
- Graph – we interpret these graphs in qualitative, not quantitative ways.
- They can show interesting shapes in the pattern, such as non-monotonic parts, steeper and flatter parts, etc.
- Great way to find relationship patterns

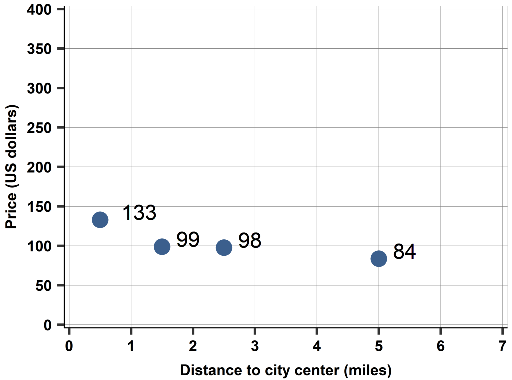
Case Study: Finding a good deal among hotels

- We look at Vienna hotels for a 2017 November weekday.
- we focus on hotels that are (i) in Vienna actual,(ii) not too far from the center, (iii) classified as hotels, (iv) 3-4 stars, and (v) have no extremely high price classified as error.
- There are 428 hotel prices for that weekday in Vienna, our focused sample has $N = 207$ observations.

Case Study: Finding a good deal among hotels

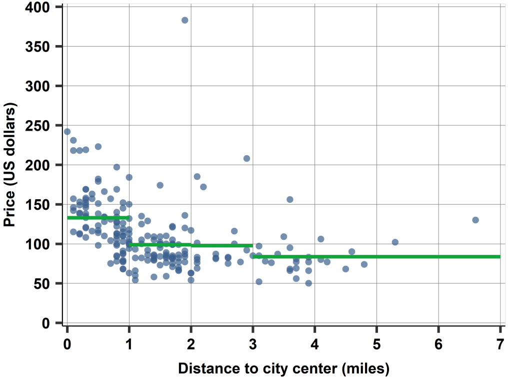


Bin scatter non-parametric regression, 2 bins

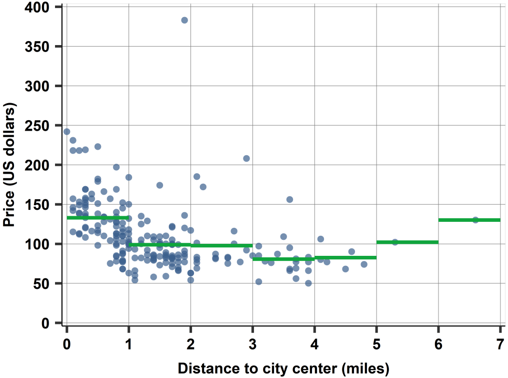


Bin scatter non-parametric regression, 4 bins

Case Study: Finding a good deal among hotels



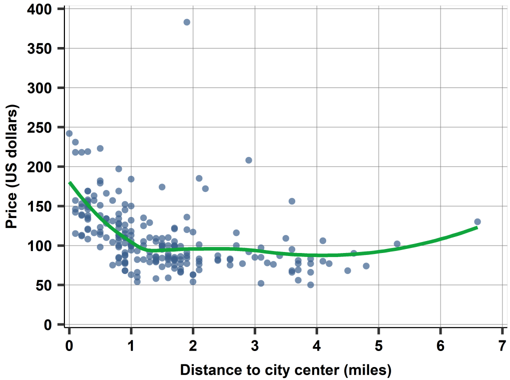
Scatter and bin scatter non-parametric regression, 4 bins



Scatter and bin scatter non-parametric regression, 7 bins

Case Study: Finding a good deal among hotels

- lowess non-parametric regression, together with the scatterplot.
- bandwidth selected by software is 0.8 miles.
- The smooth non-parametric regression retains some aspects of previous bin scatter – a smoother version of the corresponding non-parametric regression with disjoint bins of similar width.



Linear regression

Linear regression is the most widely used method in data analysis.

- imposes linearity of the function f in $y^E = f(x)$.
- Linear functions have two parameters, also called coefficients: the intercept and the slope.

$$y^E = \alpha + \beta x \tag{3}$$

- Linearity in terms of its coefficients.
  - can have any function, including any nonlinear function, of the original variables themselves (think of logarithms, squares, etc.).
- linear regression is a line through the $x - y$ scatterplot.
  - This line is the best-fitting line one can draw through the scatterplot.
  - It is the best fit in the sense that it is the line that is closest to all points of the scatterplot.

Linear regression - assumption vs approximation

- *Linearity as an assumption*:
  - by doing linear regression analysis we assume that the regression function is linear in its coefficients.
  - this may be true or not.
- *Linearity as an approximation*.
  - Whatever the form of the $y^E = f(x)$ relationship, the $y^E = \alpha + \beta x$ regression fits a line through it.
  - This may or may not be a good approximation.
  - By fitting a line we approximate the average slope of the $y^E = f(x)$ curve.

近似

Linear regression coefficients

Coefficients have a clear interpretation – based on comparing conditional means.

$$E[y|x] = \alpha + \beta x$$

Two coefficients:

- intercept: α = average value of y when x is zero:
- $E[y|x = 0] = \alpha + \beta \times 0 = \alpha$.

- slope: β. = expected difference in y corresponding to a one unit difference in x.
- $E[y|x = x_0 + 1] - E[y|x_0] = (\alpha + \beta \times (x_0 + 1)) - (\alpha + \beta \times x_0) = \beta$.

increment

Regression - slope coefficient

- slope: β = expected difference in y corresponding to a one unit difference in x.
- *y is higher, on average, by β for observations with a one-unit higher value of x.*
- Comparing two observations that differ in x by one unit, we expect y to be β higher for the observation with one unit higher x.

- Be careful...
  - "decrease/increase" – not right, unless time series or causal relationship only
  - "effect" – not right, unless causal relationship
  - comparing conditional means – always true whether or not the more ambiguous interpretations are true
  - See more on the uploaded example! (ceulearning)

Regression: binary explanatory

Simplest case:

- x is a binary variable, zero or one.
- α is the average value of y when x is zero ($E[y|x = 0] = \alpha$).
- β is the difference in average y between observations with $x = 1$ and observations with $x = 0$
    - $E[y|x = 1] - E[y|x = 0] = \alpha + \beta \times 1 - \alpha + \beta \times 0 = \beta$.
    - The average value of y when x is one is $E[y|x = 1] = \alpha + \beta$.
- Graphically, the regression line of linear regression goes through two points: average y when x is zero (α) and average y when x is one ($\alpha + \beta$).

Regression coefficient formula

Notation:

- General coefficients are α and β.
- Calculated *estimates* - $\hat{\alpha}$ and $\hat{\beta}$ (use data and calculate the statistic)
- The slope coefficient formula is

$$\hat{\beta} = \frac{Cov[x,y]}{Var[x]} = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

- Slope coefficient formula is normalized version of the covariance between x and y.
  - The slope measures the covariance relative to the variation in x.
  - That is why the slope can be interpreted as differences in average y corresponding to differences in x.

Regression coefficient formula

- The intercept – average y minus average x multiplied by the estimated slope $\hat{\beta}$.

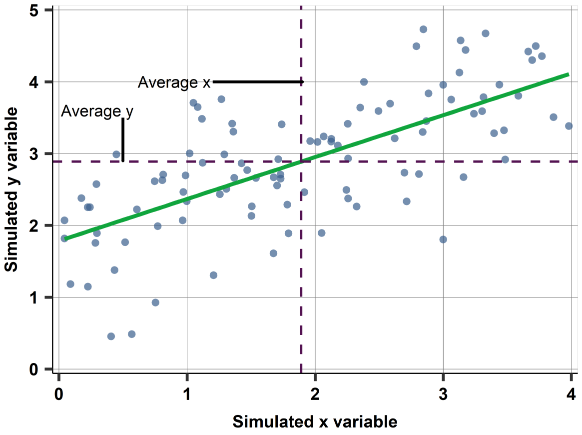
$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$$

- The formula of the intercept reveals that the regression line always goes through the point of average x and average y.
- Note, you can manipulate and get: $\bar{y} = \hat{\alpha} + \hat{\beta}\bar{x}$.

Ordinary Least Squares (OLS)

最小二乘法

- OLS gives the best-fitting linear regression line.
- A vertical line at the average value of x and a horizontal line at the average value of y. The regression line goes through the point of average x and average y.



More on OLS

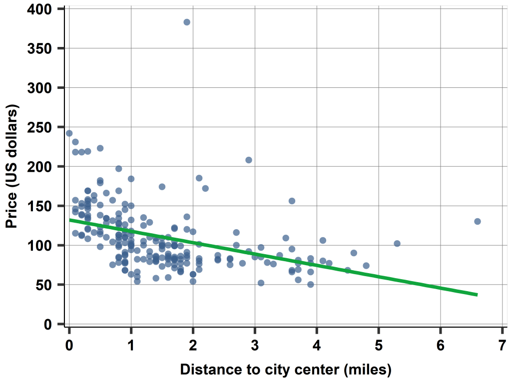
- The idea underlying OLS is to find the values of the intercept and slope parameters that make the regression line fit the scatterplot 'best'.
- OLS method finds the values of the coefficients of the linear regression that minimize the sum of squares of the difference between actual y values and their values implied by the regression, $\hat{\alpha} + \hat{\beta}x$.

$$min_{\alpha,\beta} \sum_{i=1}^{n} (y_i - \alpha - \beta x_i)^2$$

- For this minimization problem, we can use calculus to give $\hat{\alpha}$ and $\hat{\beta}$, the values for α and β that give the minimum.
- HW: show the formula which minimize α, β and prove that this is indeed a minimum!

Case Study: Finding a good deal among hotels

- The linear regression of hotel prices (in \$) on distance (in miles) produces an intercept of 133 and a slope -14.
- The intercept is 133, suggesting that the average price of hotels right in the city center is \$ 133.
- The slope of the linear regression is -14. Hotels that are 1 mile further away from the city center are, on average, \$ 14 cheaper in our data.



Case Study: Finding a good deal among hotels

- Compare linear model and non-parametric ones
- Linear is an average that fails to capture steep decline close to center
- Not bad approximation overall, but can be improved...

Predicted values

- The predicted value of the dependent variable = best guess for its average value if we know the value of the explanatory variable, using our model.
- The predicted value can be calculated from the regression for any x.
- The predicted values of the dependent variable are the points of the regression line itself.
- The predicted value of dependent variable y is denoted as $\hat{y}$.

$$\hat{y} = \hat{\alpha} + \hat{\beta}x$$

- Predicted value can be calculated for any model of y.
  - Interpolation: predict *within* observed x values - feasible if good model.
  - Extrapolation: predict *outside* observed x values - adventurous, only if meaningful and have high external validity

Residuals

- The residual is the difference between the actual value of the dependent variable for an observation and its predicted value :

$$e_i = y_i - \hat{y}_i, \qquad \text{where} \qquad \hat{y}_i = \hat{\alpha} + \hat{\beta} x_i$$

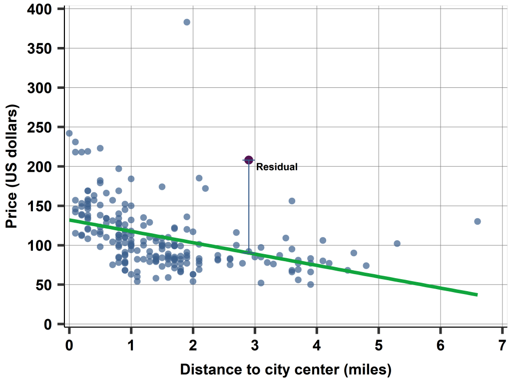
- The residual is meaningful only for actual observation. It compares observation i's difference for actual and predicted value.
- The residual is the vertical distance between the scatterplot point and the regression line.
  - For points above the regression line the residual is positive.
  - For points below the regression line the residual is negative.

Some further comments on residuals

- The residual may be important on its own right.
  - If we certain about our model: identifies observations that are special in that they have a dependent variable that is much higher or much lower than “it should be" as predicted by the regression.
  - If we are not certain in our model: how our predicted errors look like - can use it as a measure of model fit.
- Residuals sum up to zero if a linear regression is fitted by OLS.
  - It is a property of OLS: $E[e_i] = 0$
  - Remember: we minimized the *sum* of squared errors...

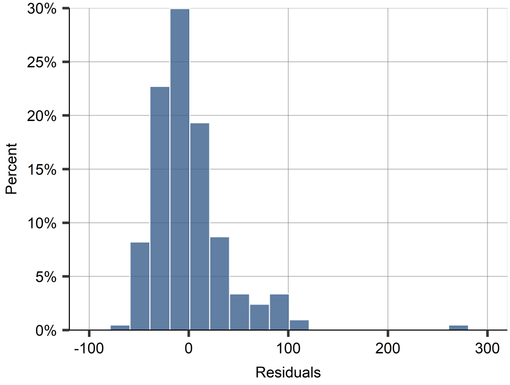
Case Study: Finding a good deal among hotels

- Residual is vertical distance
- Positive residual shown here - price is above what predicted by regression line



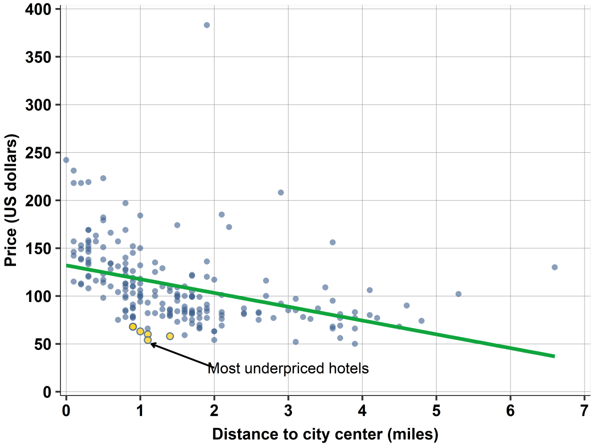
Case Study: Finding a good deal among hotels

- Can look at residuals from linear regressions
- Centered around zero
- Both positive and negative



Case Study: Finding a good deal among hotels

- If linear regression is accepted model for prices
- Draw a scatterplot with regression line
- With the model you can capture the over and underpriced hotels



Case Study: Finding a good deal among hotels

A list of the hotels with the five lowest value of the residual.

No.	hotel_id	distance	price	predicted price	residual
1	22080	1.1	54	116.17	-62.17
2	21912	1.1	60	116.17	-56.17
3	22152	1	63	117.61	-54.61
4	22408	1.4	58	111.85	-53.85
5	22090	0.9	68	119.05	-51.05

- Bear in mind, we can (and will) do better - this is not the best model for price prediction.
  - Non-linear pattern
  - Functional form
  - Taking into account differences beyond distance

Model fit - R^2

- *Fit of a regression* captures how predicted values compare to the actual values.
- *R-squared* (R^2) – how much of the variation in y is captured by the regression, and how much is left for residual variation

$$R^2 = \frac{Var[\hat{y}]}{Var[y]} = 1 - \frac{Var[e]}{Var[y]} \tag{4}$$

where, $Var[\hat{y}] = \frac{1}{n}\sum_{i=1}^{n}(\hat{y}_i - \bar{y})^2$, and $Var[e] = \frac{1}{n}\sum_{i=1}^{n}(e_i)^2$.

- Decomposition of the overall variation in y into variation in predicted values "explained by the regression") and residual variation ("not explained by the regression"):

$$Var[y] = Var[\hat{y}] + Var[e] \tag{5}$$

Model fit - R^2

- R-squared (or R^2) can be defined for both parametric and non-parametric regressions.
- Any kind of regression produces predicted $\hat{y}$ values, and all we need to compute R^2 is its variance compared to the variance of y.
- The value of R-squared is always between zero and one.
- R-squared is zero, if the predicted values are just the average of the observed outcome $\hat{y}_i = \bar{y}_i, \forall i$.
  - In linear regression, this corresponds to a slope of zero: the regression line is completely flat. $\beta = 0$ thus y and x are mean-independent.

Model fit - truth vs model

The 'true model', that we do not know:

$$y_i = f(x) + \varepsilon_i$$

Fit depends:

1. How well the particular version of the regression captures the actual function of $f(x)$
   - Can be helped by choice of model (parametric vs non-parametric, use of variables, functional form, ect.)
2. How far the realizations of y_i are spread around the true functional form of f due to ε_i

Model fit - how to use R^2

- R-squared may help in choosing between different versions of regression for the *same data*.
  - Choose between regressions with different functional forms
  - Predictions are *likely* to be better with high R^2
- R-squared matters less when the goal is to characterize the association between y and x
  - We would like to understand how x and y are related and we want to describe this pattern with interpretable coefficients.
  - The regression that best approximates that pattern may have a high R-squared or a low R-squared. (Remember: the role of ε_i.)

Correlation and linear regression

- Linear regression is closely related to correlation.
- Remember, the OLS formula for the slope

$$\hat{\beta} = \frac{Cov[y, x]}{Var[x]}$$

- In contrast with the correlation coefficient, its values can be anything. Furthermore y and x are *not interchangeable*.
- Covariance and correlation coefficient can be substituted to get $\hat{\beta}$:

$$\hat{\beta} = Corr[x, y]\frac{Std[y]}{Std[x]}$$

- Covariance, the correlation coefficient, and the slope of a linear regression capture similar information: the degree of association between the two variables.

Correlation and R^2 in linear regression

- R-squared of the simple linear regression is the square of the correlation coefficient.

$$R^2 = (Corr[y,x])^2$$

- So the R-squared is yet another measure of the association between the two variables.
- To show this equality holds, the trick is to substitute the numerator of R-squared and manipulate:

$$R^2 = \frac{Var[\hat{y}]}{Var[y]} = \frac{Var[\hat{\alpha} + \hat{\beta}x]}{Var[y]} = \frac{\hat{\beta}^2 Var[x]}{Var[y]} = \left(\hat{\beta}\frac{Std[x]}{Std[y]}\right)^2 = (Corr[y,x])^2$$

Reverse regression

- One can change the variables, but the interpretation is going to change as well!

$$x^E = \gamma + \delta y$$

- The OLS estimator for the slope coefficient here is $\hat{\delta} = \frac{Cov[y,x]}{Var[y]}$.
- The OLS slopes of the original regression and the reverse regression are related:

$$\hat{\beta} = \hat{\delta}\frac{Var[y]}{Var[x]}$$

  - Different, unless $Var[x] = Var[y]$,
  - but always have have the same sign.
  - both are larger in magnitude the larger the covariance.
- R^2 for the simple linear regression and the reverse regression is the same.

Regression and causation

- Be very careful to use neutral language, not talk about causation, when doing simple linear regression!
- Think back to sources of variation in x
  - Do you control for variation in x? Or do you only observe them?
- Regression is a method of comparison: it compares observations that are different in variable x and shows corresponding average differences in variable y.
  - Regardless of the relation of the two variable.

Regression and causation - variation in x

- The key is the source of variation in x - the method will never do the causal claim.
- It is always the data that makes it possible to claim causal relationship. More precisely, how the data was collected, how variation in x was provided.

Example: advertising (x) and sales (y)

- Observational data, collected from a firm and using regression –> no causal claim.
  - In holidays more people go shopping and firms are increasing their advertisements also in these days –> Sales are not increased by advertisement but because of holiday.
- If firm consciously experiments by allocating varying resources to advertising, in a random fashion, and keep track of sales. A regression of sales on the amount of advertising can uncover the effect of advertising here. (More in DA4)
  - Same method, but can do causal claim because of variation in advertisement.

Regression and causation - possible relations

- Slope of the $y^E = \alpha + \beta x$ regression is not zero in our data ($\beta \neq 0$) and the linear regression captures the y–x association reasonably well, one of three things – which are not mutually exclusive – may be true:
  - x causes y:
    - If this is the single one thing behind the slope, it means that we can expect y to increase by β units if we were to increase x by one unit.
  - y causes x.:
    - If this is the single one thing behind the slope, it means that we can expect x to increase if we were to increase y:
  - A third variable causes both x and y (or many such variables do):
    - If this is the single one thing behind the slope it means that we cannot expect y to increase if we were to increase x (or the other way around).
- In reality if we have observational data, there is a mix of these relations. E.g. if y has an effect on x and x has an effect on y we call it 'endogeneity'.
  - E.g. hotel ratings (x) and price (y)

Regression and causation

- The proper interpretation of the slope is necessary regardless the data is observational or comes from a controlled experiment.
  - Safe way: A positive slope in a regression of sales on advertising, means that sales tend to be higher on average when advertising is higher.
- Instead of "correlation (regression) does not imply causation"–> we should not infer cause and effect from comparisons in observational data.
- Suggested approach is two steps:
  - First interpret precisely the object (correlation ot slope coefficient)
  - Conclude and discuss causal claims if any

Case Study: Finding a good deal among hotels

- Fit and causation
- The R-squared of the regression is $0.16 = 16\%$.
  - This means that of the overall variation in hotel prices, 16% is explained by the linear regression with distance to the city center; the remaining 84% is left unexplained.
- 16% - good for cross-sectional regression with a single explanatory variable.
  - In any case it is the fit of the best-fitting line.

Case Study: Finding a good deal among hotels

- Slope is -14
- Does that mean that a longer distance causes hotels to be cheaper by that amount?

Summary take-away

- Regression – method to compare average y across observations with different values of x.
- Non-parametric regressions (bin scatter, lowess) visualize complicated patterns of association between y and x, but no interpretable number.
- Linear regression – linear approximation of the average pattern of association y and x
- In $y^E = \alpha + \beta x$, β shows how much larger y is, on average, for observations with a one-unit larger x
- When β is not zero, one of three things (+ any combination) may be true:
  - x causes y
  - y causes x
  - a third variable causes both x and y.

- If you are to study more econometrics, advanced statistics - Go through textbook under the hood derivations sections!