Structural Econometrics of Auctions: A Review

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Contents

Abstract

We review the literature concerned with the structural econometrics of observational data from auctions, discussing the problems that have been solved and highlighting those that remain unsolved as well as suggesting areas for future research. Where appropriate, we discuss different modeling choices as well as the fragility or robustness of different methods.

Keywords: auctions; structural econometrics.

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Introduction and Motivation

One of the great success stories in economics during the latter half of the twentieth century was the systematic theoretical investigation of incomplete information in various economic environments—for example, moral hazard in insurance markets and adverse selection in such market institutions as auctions, to name just two. Developed in tandem with the incredible advances in game theory since the Second World War, an important by-product of this research program was the formulation of many incomplete information problems as the design of optimal mechanisms.[1]

From an econometrician's perspective, theoretical models of incomplete information are particularly compelling because they provide richer explanations of the observed heterogeneity in data than ones relying on measurement error or productivity shocks. For example, when individual agents use private information strategically to make decisions, the equilibrium interactions of those decisions with the decisions of others can generate new explanations of observed phenomena, particularly in the field of industrial organization.

[1]The book by Tilman Börgers [2015], with chapter by Daniel Krähmer and Roland Strausz, provides a useful introduction to the topic.

As is common in economics, however, econometric and empirical developments involving models of incomplete information lagged behind those in theory. For even though using theoretical models to put structure on economic data was advocated by the late Nobel laureates Trygve Haavelmo [1944] and Tjalling C. Koopmans [1947, 1949, 1953] as well as associates of Koopmans at the Cowles Foundation, Yale University, in the 1950s (for example, the late Nobel laureate Leonid Hurwicz [1950]), the approach did not become generally accepted, let alone widespread, until the last quarter of the twentieth century, particularly in response to the devastating evaluation of reduced-form methods by the Nobel laureate Robert E. Lucas, Jr. [1976], which is now commonly referred to as the Lucas critique.

In this review, we summarize a subset of this literature—that devoted to the structural econometric analysis of observational data from auctions, as opposed to data from either laboratory or field experiments, although we mention some studies involving the latter two in passing.

A classic, canonical problem in economic theory involves a seller of an object who faces several potential buyers. Although the seller probably knows the object's value to himself, he typically has little or no information concerning what value any one of the potential buyers places on the object. Similarly, each potential buyer may have a good notion of the object's value to himself, but probably knows little about the seller's valuation or the valuations of the other potential buyers. In short, the seller as well as the potential buyers face incomplete, asymmetric information.

How should the seller dispose of the object? One commonly used method of sale involves announcing a take-it-or-leave-it price and then selling the object to the first who accepts that price. Another involves the seller's engaging in pair-wise negotiations with individual potential buyers, either sequentially or simultaneously. Yet a third involves selling the object at auction. In short, a set of different selling mechanisms exists, from which the seller must choose, guided by some objective—an auction being one of those potential choices. The choice of mechanism by the seller typically depends on many factors, for example, the objective of the seller and transaction costs to name just two.

Auctions are ubiquitous in market economies. They are also ancient, their durability suggesting that the institution serves an important allocational role. Perhaps the most important feature defining environments within which auctions are used is the existence of incomplete information of some sort—the asymmetry of information between the seller and the potential buyers. Because the potential buyers have no incentive to tell the seller or any of their competitors anything about those valuations, the auction plays an important allocative role by inducing the potential buyers to reveal to the seller information concerning their valuations of the object.

The way potential buyers form their valuations remains an open question in economics. In fact, in auction theory, researchers are unusually vague about what generates the demand structure, unlike in standard demand theory where considerable care is taken to specify the structure of preferences. Suffice it to say, however, when economic theorists come to modeling the asymmetry in information as well as the heterogeneity in valuations across agents, they employ random variables. Typically, each potential buyer is assumed to demand at most one unit of the object in question. In the simplest model, for each potential buyer, the marginal utility of the one unit is assumed an independent realization of a continuous random variable. By and large, the budget constraint and issues of substitution are ignored.

With such an austere model, it is indeed somewhat surprising that economists can provide any practical insights concerning how to bid at auctions, let alone how to structure the institution with any purpose. In the workhorse model of auction theory, which was first developed by the late Nobel laureate William S. Vickrey [1961], each of a known number N of potential buyers draws an individual-specific random valuation independently from the same differentiable cumulative distribution function (cdf) $F_V(v)$ that has corresponding probability density function (pdf) $f_V(v) = \mathrm{d}F_V(v)/\mathrm{d}v$. In Vickrey's model, he assumed that V is distributed uniformly on the unit interval: the specific value of his draw is that potential buyer's private information; it represents the monetary value of the object to him. Economic theorists refer to this as the symmetric independent private-values (IPV) model because the draws are independent and the valuations are bidder specific. Also, because the valuations are drawn from the same distribution (urn so to speak), the bidders are *ex ante* symmetric.

Different auction formats (open-outcry versus sealed-bid) and different pricing rules (pay-your-bid versus second-price) provide potential buyers with different incentives concerning how to bid. For example, under the pay-your-bid rule, a bidder's action (his bid) determines what he pays should he win, whereas under the second-price rule, the action (bid) of his nearest rival determines what the winner pays.

In equilibrium, different functions map the private information of potential buyers (their values) into their actions (their bids). For example, open-outcry (sometimes referred to as oral) auctions can be conducted in two different ways. In the first, the price is set very low, perhaps at zero, and then allowed to rise more or less continuously until only one participant remains active in the auction. That remaining active bidder is the winner, and he pays what the last other active bidder was willing to pay—sometimes plus a small increment. (Later, in the context of electronic auctions held on the Internet, the relative magnitude of that increment is shown to be important.)

Economic theorists have typically chosen to model oral auctions as clocks, where the price rises continuously with the movement of a clock hand. In this case, the winner of the auction is the participant with the highest valuation and he pays what his nearest rival (that participant with the second highest value) was willing to pay.

Thus, the oral, ascending-price auction guarantees the efficient allocation of the object: the participant with highest valuation wins the auction. Such an auction is sometimes referred to as a second-price auction because in the absence of bid increments the winning bid is the second-highest bid, which happens to be the second-highest valuation as well.

In economics, this outcome has special meaning: the second-highest valuation represents the opportunity cost of the object for sale—its value in the next-best alternative. Thus, one can see why economists are naturally attracted to mechanisms that have this property.

As a technical aside, within the IPV model, the equilibrium at an oral, ascending-price auction (sometimes referred to as an English auction) has a special structure. It is a weakly dominant-strategy equilibrium: each participant has an incentive to reveal his private information, that is, to tell the truth concerning his value by continuing to bid up to his value, regardless of what his rivals do.

The second way to conduct an oral auction involves initially setting the price very high, and then allowing it to fall continuously; the winner is the first participant to cry out a bid, and he pays his bid. In practice, these oral auctions are typically implemented using a clock, where the hand (or a digital panel) lists the current price. Participants affirm their willingness to pay the current price by pushing a button which stops the clock at that price. Such auctions are often referred to as Dutch auctions, perhaps because they are frequently used in the Netherlands to sell fish and flowers.

Because the price that the winner pays is related to his action (crying out or pushing the button), he has an incentive to shave his bid—to wait longer before pushing the button to stop the clock. In theoretical models of Dutch auctions within the symmetric IPV model, the equilibrium is not a dominant-strategy equilibrium, but rather a Bayes-Nash equilibrium, which is a much stronger notion of equilibrium. Although the Bayes-Nash equilibrium bid function is an increasing function of a bidder's value, it has a slope that is less than one: each bidder is deceptive when bidding; he does not tell the truth, but rather bids less than his value. Again, however, the winner is the participant with the highest valuation, so objects are allocated efficiently at Dutch auctions.

In general, economic theorists have found that the seller's expected revenue depends first and foremost on the information available to potential buyers and then on the auction format and pricing rules employed, the amount of competition, and the attitudes of potential buyers toward risk. Within the symmetric IPV model, however, under the assumption that potential buyers are risk neutral with respect to winning the object for sale, a remarkable result obtains: revenue equivalence (in expectation). That is, if the same object were sold under the two different institutions, then the average winning bid at the English auction would equal the average winning bid at the Dutch auction.

To most people, this revenue equivalence result is at first somewhat surprising because at English auctions considerable information is revealed during the course of bidding, whereas at Dutch auctions no information is revealed until the winner has been determined. Within the symmetric IPV model, however, information plays no extra role in determining the average winning price since each bidder's private information (his value) is, by assumption, statistically independent of the private information of his rivals (their values): knowing something about the values of his rivals provides no extra information to a bidder concerning his own valuation, or his likelihood of winning the auction. Therefore, no bidder at an English auction can learn anything more about his valuation from the actions (bids) of his rivals. Once one realizes this fact, the equivalence of average winning bids is clear: at a Dutch auction, assuming he wins because he has the highest value, a representative participant forms his bid so that he will, on average, just beat his nearest rival, the bidder with the second-highest valuation.

Similar analyses have been performed for the sealed format under different pricing rules. In fact, theorists have shown that sealed auctions at which the highest bidder wins the auction and pays what he bid are strategically equivalent to Dutch auctions. Consequently, the Bayes-Nash equilibrium bid function at a sealed, pay-your-bid auction is identical to that at a Dutch auction. It has also been shown that sealed auctions at which the highest bidder wins the auction, but pays the bid of his closest rival, are strategically equivalent to English auctions, so it is a dominant strategy at these auctions for bids to tell the truth, too.

Under the assumption of risk-neutral potential buyers, expected revenue equivalence follows. That is, if the same object were sold under the two different institutions, then the average selling price at a sealed, pay-your-bid auction would equal the average selling price at a sealed, second-price (also known as Vickrey) auction. This result, which was first outlined by Vickrey [1961] and then proved in general by the Nobel laureate Roger B. Myerson [1981] as well as John G. Riley and William F. Samuelson [1981], is the celebrated Revenue Equivalence Theorem

(RET), perhaps the best known result in auction theory.

In its full generality, the RET states that any combination of auction format and pricing rule that has the same probability of assigning a winning bidder generates the same expected revenue to the seller. In particular, the RET predicts that the expected revenues garnered by the seller at sealed auction formats will be the same as those earned at oral auction formats under either pay-your-bid or second-price rules, at least for one-shot, single-object auctions when the distribution from which the values are drawn is the same for all potential buyers, who are also risk neutral.

From a policymaker's perspective, an important issue involves choosing the selling mechanism that garners the most revenue for the seller, on average. To a large extent, the structure of the optimal selling mechanism depends on the informational environment. Within the symmetric IPV model, given the RET, one question arises naturally: Can one still improve on the structure of the four combinations of auction format and pricing rule? Myerson as well as Riley and Samuelson showed that devising a selling mechanism that maximizes the seller's expected gain involves choosing the reserve price r, the minimum price that must be bid, optimally. In the previous notation, the optimal reserve price r^* solves the following equation:

$$r^* = v_0 + \frac{[1 - F_V(r^*)]}{f_V(r^*)}, \tag{1.1}$$

where v_0 denotes the seller's valuation of the object at auction.

The presence of a binding reserve price means that some fraction of the time the object at auction goes unsold. In other words, an inefficiency is introduced because $r^* > v_0$, meaning some bidders might value the object more than the seller, but the object goes unsold. This inefficiency highlights the tension between expected revenue maximization on the part of the seller in particular and allocative efficiency in the economy in general.

Historically, the literature concerned with mechanism design was sometimes criticized as lacking practical value because the optimal selling mechanism (in this case, the optimal reserve price r^*) typically depends on a primitive like $F_V(\cdot)$, the distribution of the valuations, which is often unknown to the designer. In the past, because the distribution of valuations has been unknown, calculating the optimal reserve price, the optimal selling mechanism, for a real-world auction seemed impossible.

From an econometrician's perspective, auctions are particularly attractive because the rules of an auction govern how the potential buyers must behave during the selling process—specifically, how bids must be tendered, who wins the auction, what the winner pays, and so forth. These rules place incredible structure on the data generating process, unlike in some other economic applications. In particular, under certain conditions, the twin hypotheses of optimization and equilibrium allow the econometrician to identify the unobserved distribution of valuations from the observed distribution of bids. In other words, part of the structural econometric approach to auctions is an identification strategy. Another part involves reverse-engineering an estimate of the distribution of latent types (for example, valuations) from the observed distribution of actions (the bids). Yet a third part is referred to as comparative institutional design—using the estimate of the distribution of latent valuations to improve on auction design.

For example, at auctions within the IPV model, the equilibrium bidding strategies of potential buyers are increasing functions of their valuations. At English auctions, under the clock model, for instance, the dominant strategy of bidders who lose the auction is to bid their valuations. Thus, in principle, it is possible to estimate the underlying probability law of valuations using the empirical distribution of bids from a cross-section of auctions. Because a researcher can recover the primitives of the economic model, the Lucas critique is circumvented; the researcher can then also entertain comparative institutional design, for instance, comparing outcomes under alternative market institutions not observed in the data.

Despite this success, several potential problems remain. For example, at English auctions, as shown later, the winning bid does not reveal complete information concerning the winner's actual valuation of the object for sale. Next, in the presence of a binding reserve price, the empirical distribution of observed bids represents a truncated sample of data: only those potential buyers whose valuations exceeded the reserve price chose to bid. Finally, in the presence of a binding reserve price, the joint distribution of bidding and nonparticipation depends on the number of potential buyers, but finding a measure of potential competition is often impossible; when it can be done, the specific proxy is often inaccurate. In short, for the past three decades or so, econometricians have had much to occupy themselves, at least when it comes to analyzing observational data from auctions.

At the polar extreme of the IPV model is the pure common-value (CV) model. Within the pure CV model, an alternative device is used to describe the motivation of potential buyers—in particular, a continuous random variable that represents individual-specific signals concerning the object's true, but unknown, value. This true, but unknown, value will be revealed only after the auction has ended, when the winner has been determined and the transaction price paid. Regardless of the winner, however, the value of the object is the same to all.

In the baseline model, the conceptual experiment involves each potential buyer's receiving a draw from a signal distribution. Conditional on his draw, a bidder is then assumed to act purposefully, using the information in his signal along with Bayes' rule to maximize either the expected profit or the expected utility of profit from winning the auction. Another frequently-made assumption is that the signal draws of potential buyers are independent and that those potential buyers are *ex ante* symmetric—their draws coming from the same distribution of signals.

Under these assumptions, a researcher can then focus on a representative agent's decision rule (policy function) when characterizing equilibrium behavior. Robert B. Wilson [1977] invented this model to demonstrate that the winner's curse could not obtain in equilibrium among rational bidders.

The winner's curse is perhaps the second-best known result in auction theory. First conjectured to be a problem in oil exploration by Edward C. Capen, Robert V. Clapp, and William M. Campbell [1971], the prospect of the phenomenon has captured the imaginations of many. For a readable history of the topic, see the book by the Nobel laureate

Richard H. Thaler [1992]; the book by John H. Kagel and Dan Levin [2002] provides additional information—both theoretical and experimental.

In oil exploration, for instance, the idea is that within the CV model, even if signals are unbiased estimates of the true value of the as yet undiscovered mineral resource, the maximum signal is an overestimate of the oil's true value. Thus, potential buyers who do not take this fact into account and who then bid their signals will systematically overbid and lose money: the winner is cursed.

One of Wilson's contributions was to point out that rational bidders in equilibrium will anticipate this overestimation problem, and adjust accordingly. That is, among rational bidders the winner's curse cannot obtain as an equilibrium.

More importantly, however, Wilson demonstrated that when the number of potential buyers is large (tends to infinity) the winning bid at sealed, pay-your-bid auctions converges almost surely to the true, but unknown, value of the object. In other words, the auction format and pricing rule play an important role in aggregating the disparate, individual pieces of information held by the bidders. In short, auctions play a critical role in the process referred to as price discovery.

Paul R. Milgrom [1979] subsequently provided a precise characterization of the structure the signal distribution must possess in order for this convergence property to hold; Wolfgang Pesendorfer and Jeroen M. Swinkels [1997] have referred to this as full information aggregation.

In summary, within both the IPV and the CV models, results depend on five important assumptions: (1) noncoöperative behavior among participants; (2) only one object is for sale, so these are one-shot, single-object auction games; (3) the valuation or signal draws are independent; (4) potential buyers draw their values from the same distribution; (5) potential buyers are risk neutral. To understand the effects of relaxing each of these five assumptions individually, we examine here only the IPV model, the most commonly used informational paradigm in auction theory.

When potential buyers are symmetrically risk-averse (that is, each has the same von Neumann-Morgenstern utility function), participants at sealed, pay-your-bid and oral, descending-price auctions bid more aggressively than they would had the same object been sold at oral, ascending-price or sealed, second-price auctions. Under each pricing rule, both auction formats yield efficient allocations: the highest-valuation bidder wins the auctions. Expected winning bids are, however, higher at pay-your-bid auctions (either Dutch or sealed) than at second-price auctions (either English or Vickrey); see Charles A. Holt, Jr. [1980], Riley and Samuelson [1981] as well as Steven A. Matthews [1983]. In other words, the RET breaks down.

When potential buyers are risk neutral with respect to winning the object, but their valuations are drawn independently from different distributions (different urns so to speak), some important economic issues arise. If potential buyers are *ex ante* asymmetric, often referred to as the asymmetric IPV model, then at English and Vickrey auctions it remains a weakly dominant strategy for bidders to reveal their private information—that is, to tell the truth and to bid up to their private values. In short, under the clock model, the highest-valuation bidder wins the auction and pays what his nearest rival was willing to pay, so the outcome is efficient. The winning bid is the second-highest valuation, which represents the opportunity cost of the object for sale.

On the other hand, within the asymmetric IPV model, at pay-your-bid auctions, Bayes-Nash equilibrium behavior is much more complicated. Characterizing this equilibrium behavior requires solving systems of nonlinear differential equations for which a mathematical property known as the Lipschitz condition does not hold. Consequently, only numerical methods are in general feasible and few clean results exist in particular.

Here is what is known: Consider the case of just two bidders, one of whom is strong and one who is weak. In this case, weak means that the cdf of valuations of the weak bidder is everywhere to the left of that of the strong bidder; see the papers by the Nobel laureate Eric S. Maskin and Riley [2000a, 2000b]. Under these assumptions, for the same valuation, the strong bidder will bid less than the weak bidder: Vijay Krishna [2010] has described this as "weakness breeds aggression." Suffice it to say that the winning bid need not bear any relation to the opportunity cost, either in realization or in expectation. Most important, disturbingly, it is possible for the highest-valuation bidder to lose the auction. In other words, the outcomes at sealed, pay-your-bid and oral, descending-price auctions can be inefficient. In general, however, except by using the numerical methods described in Timothy P. Hubbard and Paarsch [2014], few predictions concerning the expected-revenue ranking of the different auction formats and pricing rules exist, which makes structural econometric empirical work especially important.

When bidders' valuations are symmetric, but dependent, the revelation of private information through bidding can be important to the equilibrium outcome. Thus, the winning bids at English auctions are more informative than those at sealed or Dutch auctions because considerably more information is revealed during the course of bidding at English auctions; see, for example, the work of Pesendorfer and Swinkels [2000]; Han Hong and Matthew Shum [2004] as well as Hong, Paarsch, and Pai Xu [2013].

In order to construct equilibria to auction games under dependence, economic theorists have been forced to impose a specific structure on the form of the dependence. Mathematicians, following the path blazed by Samuel Karlin [1968], refer to this structure as multivariate total positivity of order two, or MTP_2 for short, whereas in an influential and classic paper, Milgrom and Robert J. Weber [1982] coined the term affiliation to describe this form of dependence.

Under symmetric affiliation, Milgrom and Weber derived a powerful result and coined the term linkage principle to describe it. In single-object auction models, where the signals of the risk-neutral potential buyers are symmetrically affiliated, the linkage principle states that a seller can expect to increase revenues by providing more information to bidders, both before and during the auction. An implication of the linkage principle is that English auctions will, on average, earn more revenue for the seller than sealed auctions, under which no information is released, or similar auction formats that reveal less information to potential buyers. Again, the RET breaks down. According to Motty

Perry and Philip J. Reny [1999], "the linkage principle has come to be considered one of *the* fundamental lessons provided by auction theory."

Thus, the presence of some degree of dependence (a common-value component so to speak) in the signals of potential buyers is critical to the validity of the linkage principle. The affiliated-values (AV) model is a generalization of the pure CV model, and the symmetric IPV model is nested within the AV model. Within the symmetric AV model, the conditional expectation of any monotonic function of the signals of all bidders is an increasing function of any individual bidder's own signal. When the signals of bidders are dependent in this manner, information released by the seller or information the seller provides concerning the bids made by other participants (by virtue of the seller's choice of auction format) helps bidders refine their beliefs concerning the true value of the object for sale, which in turn induces them to bid more aggressively than they would in the absence of such information.

Of course, when bidders coöperate with one another (for example, collude), knowing the exact nature of such behavior is critical to any analysis. As a case in point, collusion is easier to sustain in environments that are rich in information: more information is released at English auctions than at sealed ones, or other less open auction formats. At the risk of belaboring the obvious, investigating collusion is difficult because collusive arrangements often differ substantially across economic environments; a deep understanding of economic institutions is required.

John Asker [2010] produced an excellent example of what the structural approach can deliver when the workings of a cartel are known. Asker focused on a cartel of stamp dealers in the 1990s. His data contained records of an entire year's worth of activities involving the bidding ring members—including side payments, and behavior in both "knockout" and "target" auctions. The structural approach allowed him to assess damages imposed by the ring, to consider the extent to which market efficiency was compromised, and to evaluate how much the ring benefited from collusion.

Like the work of Robert H. Porter and J. Douglas Zona [1993, 1999] as well as Martin Pesendorfer [2000], Asker's research documented and investigated a collusive scheme *ex post*, although economists are making progress on testing for collusion too; see, for example, the research of Vadim Marmer, Artyom Shneyerov, and Uma Kaplan [2017].

Admitting several objects complicates matters considerably as the research of Weber [1983], for example, has shown. In fact, economic theorists distinguish between multi-object and multi-unit auctions. At multi-unit auctions, it matters not which unit a bidder wins, but rather the aggregate number of units he wins, while at multi-object auctions it matters which specific object(s) a bidder wins. An example of a multi-object auction would involve the sale of an apple and an orange, whereas one of a multi-unit auction would involve the sale of two identical apples.

When several units of the same object are sold sequentially, the analysis depends on whether potential buyers demand just one unit (referred to by Milgrom [2004] as singleton demand) or several units. Within the symmetric IPV model, when potential buyers have singleton demand, Weber [1983] demonstrated that the equilibrium price path under the four combinations of auction formats and pricing rules follows a martingale: the expectation of the price of the next unit at auction equals the price of the last unit sold.

Characterizing equilibrium in a private-values model of a sequential, multi-unit auction when multi-unit demand is admitted involves solving an asymmetric-auction game for each unit. In a special case, when potential buyers have multi-unit demand that follows a Poisson process, Stephen G. Donald, Paarsch, and Jacques Robert [2006] demonstrated that the equilibrium price path follows a submartingale: on average, the equilibrium price rises over consecutive auctions. As one can see, small changes can have large effects on the equilibrium predictions, even within the simplest of informational paradigms, the IPV model.

To our knowledge, only Perry and Reny [1999] have investigated the effect of affiliation in multi-unit auctions. In fact, they provided a counterexample that demonstrates the Milgrom-Weber ranking breaks down in multi-unit auctions with affiliation.

When several, say K, units of a good are simultaneously for sale, at least two important questions arise: Who will be the winning bidders? What price(s) will those winners pay? For example, Milgrom [1981] developed a natural generalization of the Wilson [1977] model. In Milgrom's model, each bidder submits a price and the auctioneer then aggregates these demands, allocating the units to those bidders with the highest K submitted bids. The winners then pay a uniform price—specifically, the highest rejected bid.

Pesendorfer and Swinkels [1997] built on this research by investigating a sequence of auctions at which both the number of potential buyers N_t and the number of units K_t increase. They demonstrated that a necessary and sufficient condition for full information aggregation is that $K_t \to \infty$ and $(N_t - K_t) \to \infty$, a condition they referred to as double largeness. Under this condition, non-negligible supply can be a substitute for the strong signal structure required in Wilson [1977] as well as Milgrom [1979, 1981]. Ilan Kremer [2002] has investigated this further.

Even though it is heartening to know there are conditions under which transaction prices will converge in probability to the true, but unknown, values of objects for sale, the rate at which these prices converge is probably of more practical relevance and value. In particular, Hong and Shum [2004] asked the following question: How large must N be to be large enough? They then investigated the rates of information aggregation in common-value environments. Knowing the conditions under which transaction prices provide potentially useful estimates of the unknown values of objects is important to understanding the price discovery process because in practice neither the number of bidders nor the number of units for sale at an auction ever really gets to infinity.

Of course, the pricing rule investigated in Wilson [1977] and Milgrom [1979, 1981] as well as Pesendorfer and Swinkels [1997, 2000] is not the only pricing rule that could be used under a sealed format. For example, another pricing rule would involve allocating the K units to those bidders who tendered the highest K bids, but each winner would then pay what he bid for the unit(s) he won. In general, at multi-unit auctions, different auction formats and different pricing rules induce different equilibrium behavior and, thus, translate into different transaction prices as well as potentially different expected revenues for sellers. Hence, as Matthew O. Jackson and Kremer [2004, 2006] have emphasized, understanding the effects of auction formats and pricing rules has important practical relevance. Even small changes can have effects, as has been illustrated by Claudio Mezzetti and Ilia Tsetlin [2008, 2009].

At the risk of sounding banal, when potential buyers demand more than one object, investigating multi-object auctions is even more complicated than multi-unit auctions. One mechanism that has been investigated extensively is the multi-object extension of the Vickrey auction, referred to in the economics literature as the Vickrey-Clarke-Groves (VCG) mechanism—in honor of the contributions of the late Edward H. Clarke [1971] and Theodore Groves [1973] as well as Vickrey. Although the VCG mechanism has many attractive features, if say K objects exist, then in order to know how to bid a potential buyer must evaluate all of the $K!$ combinations of the objects. For large numbers of objects, searching through $K!$ different combinations is a computationally intractable problem, belonging to the complexity class known as NP-hard. In short, for this reason (and others) the VCG mechanism is impractical for many real-world applications; Michael H. Rothkopf, Thomas J. Teisberg, and Edward P. Kahn [1990]; Lawrence M. Ausubel and Milgrom [2006] as well as Rothkopf [2007] have all provided helpful discussions.

An alternative to the VCG mechanism is the so-called generalized second-price (GSP) auction, which scales well for large numbers of objects. The GSP auction has generated literally hundreds of billions in advertising revenues for the Internet search company Google; for more details on this auction, see the paper by Benjamin Edelman, Michael Ostrovsky, and Michael Schwarz [2007].

That noted, for relatively small K, the VCG mechanism has been implemented successfully on the Internet, too. For instance, Google actually switched from the GSP auction to the VCG mechanism for its auctions of contextual advertisements in 2012; Google's chief economist, Hal R. Varian, and Christopher Harris [2014] have documented some of the reasons behind this change. In addition, we have learned from industry experts that other Internet firms have used the

VCG mechanism at online advertising auctions.

In summary, the two most important considerations in auction theory are allocative efficiency and price discovery. Within the symmetric IPV model, allocative efficiency is guaranteed; under risk neutrality, revenue equivalence obtains. Despite these positive results, the seller can increase his expected revenues by introducing a binding reserve price. An optimally-chosen, binding reserve price increases expected revenues, but introduces inefficiencies, too: some fraction of the time the object goes unsold. Within the pure CV model, price discovery is guaranteed under a variety of assumptions. Symmetric affiliation permits a ranking of auction formats and pricing rules: those formats and rules that release more information will yield higher average revenues. In the presence of asymmetries, allocative efficiency and price discovery can breakdown.

Having whetted the reader's appetite with an *amuse bouche* of economic theory as it pertains to auctions, in the remaining sections of this review we employ these results to put structure on the data generating processes of observational data from auctions. To this end, in the next section, we develop several different theoretical models of equilibrium bidding under different auction formats and pricing rules within the three commonly-used informational environments; these models provide the foundations on which the econometric models developed later in the review rest. Following that, in section 3 we illustrate how to construct the mapping from the theoretical models to the observable data that can be used to estimate empirical specifications. In section 4, we examine the thorny issue of identification, while in section 5 we outline two basic approaches to doing empirical work using auction data—the reduced-form and the structural. We devote section 6 to describing several successful strategies used when estimating empirical specifications. In the next three sections, we then document the nitty gritty of dealing with asymmetries and dependence as well as multi-unit and multi-object auctions—computationally intensive work. We summarize and conclude as well as provide suggestions for future research in section 10, and collect in appendices any results too cumbersome for inclusion in the text.

Before proceeding any further, however, we should also note that this review is intended to augment those reviews and surveys of the econometrics of auctions that came before it—specifically, those by Kenneth Hendricks and Paarsch [1995]; Isabelle Perrigne and Quang H. Vuong [1999]; Paarsch and Hong [2006]; Susan C. Athey and Philip A. Haile [2007]; Hendricks and Porter [2007] as well as Brent R. Hickman, Hubbard, and Yiğit Sağlam [2012]. It is impossible for us to acknowledge fully the influence that these papers have had on our work. Suffice it to say, we owe a debt to those who came before us.

1.1 Recommended, Additional Reading concerning Auctions

The classic reference concerning auctions is the book by Ralph Cassady, Jr. [1967]. That said, even though this book has many interesting anecdotes and facts, it is somewhat out of date, and provides no guidance in theoretical modeling. At the risk of shameless self-promotion, for those with little experience in economics, we recommend the elementary book by Hubbard and Paarsch [2015], which is concerned only with auctions. For those with a bit more training in economics, we recommend the beautifully written book by the late John McMillan [2002], which is concerned with markets in general, but deals with auctions as well. At the next level, even though each is over thirty years old, we recommend the surveys by Milgrom [1987] as well as R. Preston McAfee and McMillan [1987b]—definitely worth reading. For a light technical treatment of market design, the book by Guillaume Haeringer [2017] is a wonderful read. For graduate students and professional economists, the books by Paul D. Klemperer [2004], Milgrom [2004], and Krishna [2010] are essential reading. In fact, in our opinion, no one interested in conducting research in the structural econometrics of auctions should begin without having read Krishna's textbook first.

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