Fuzzy logic

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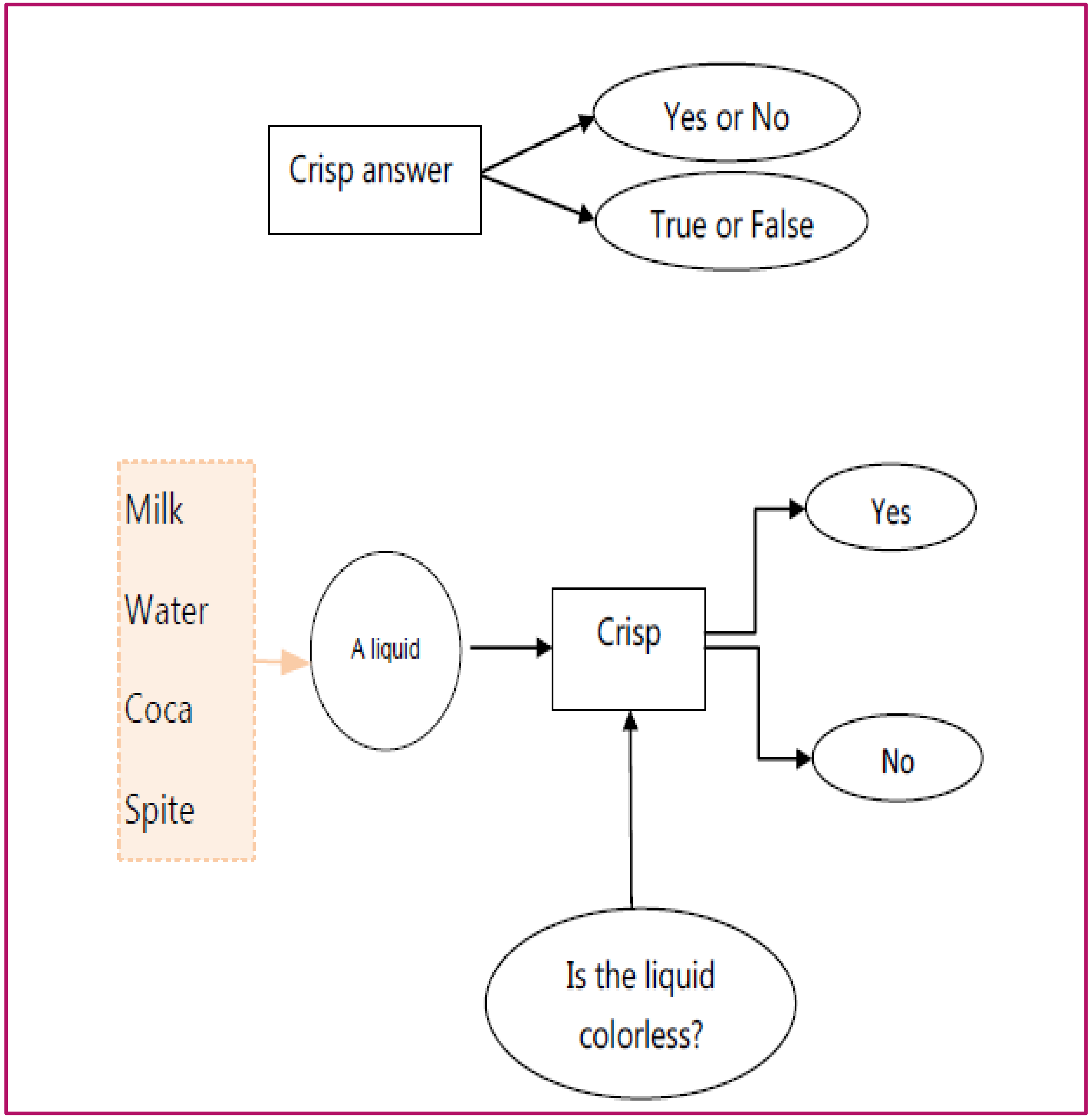
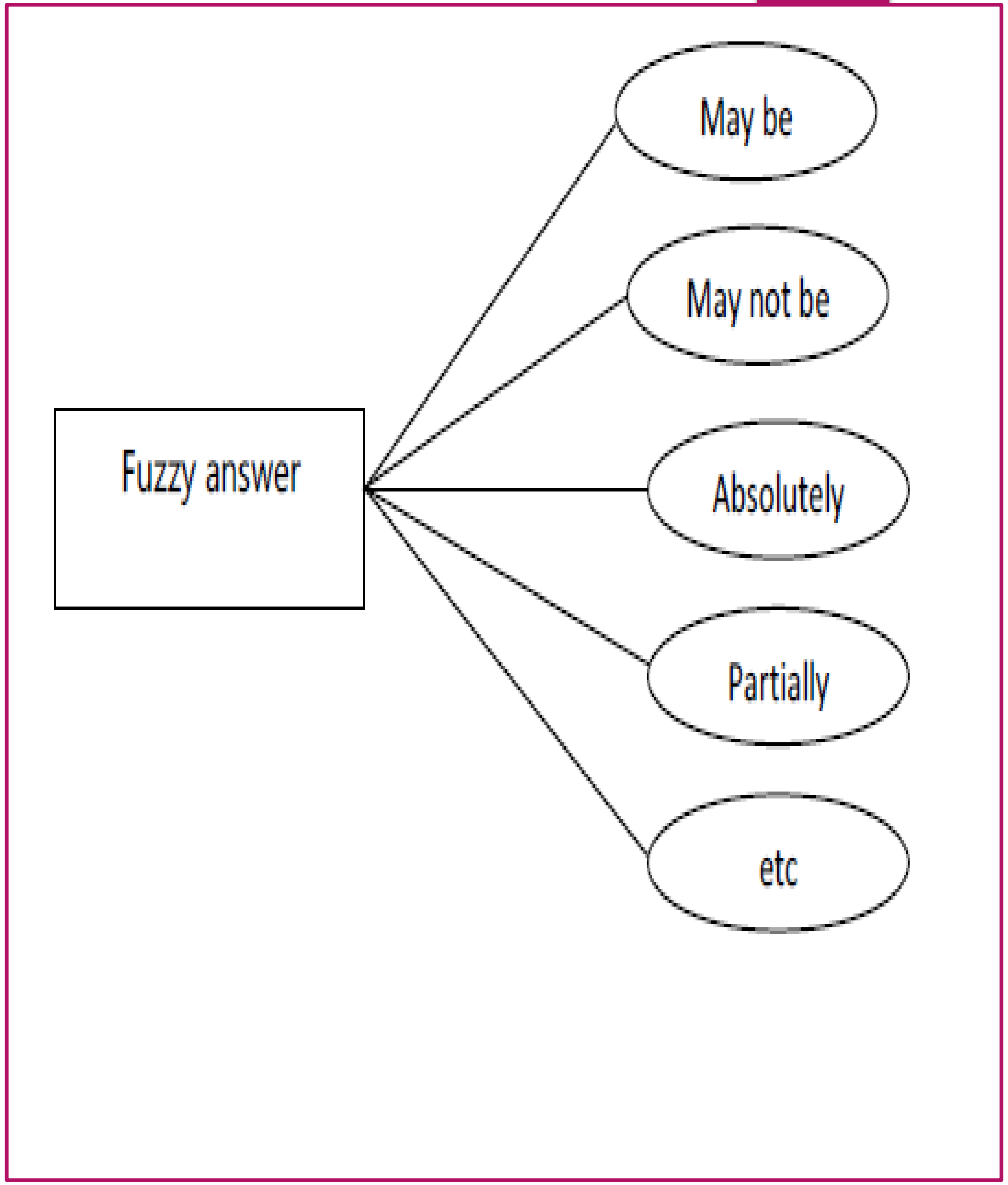
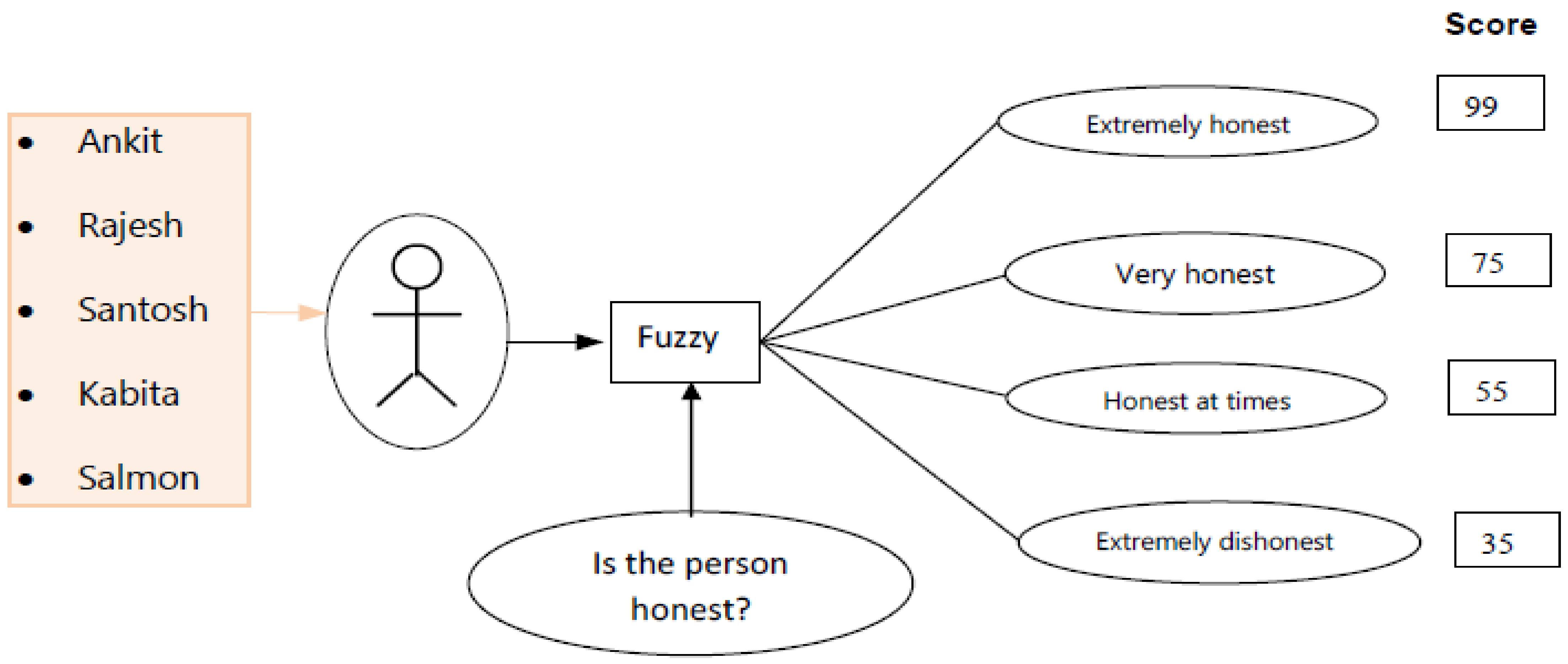
History of fuzzy logic

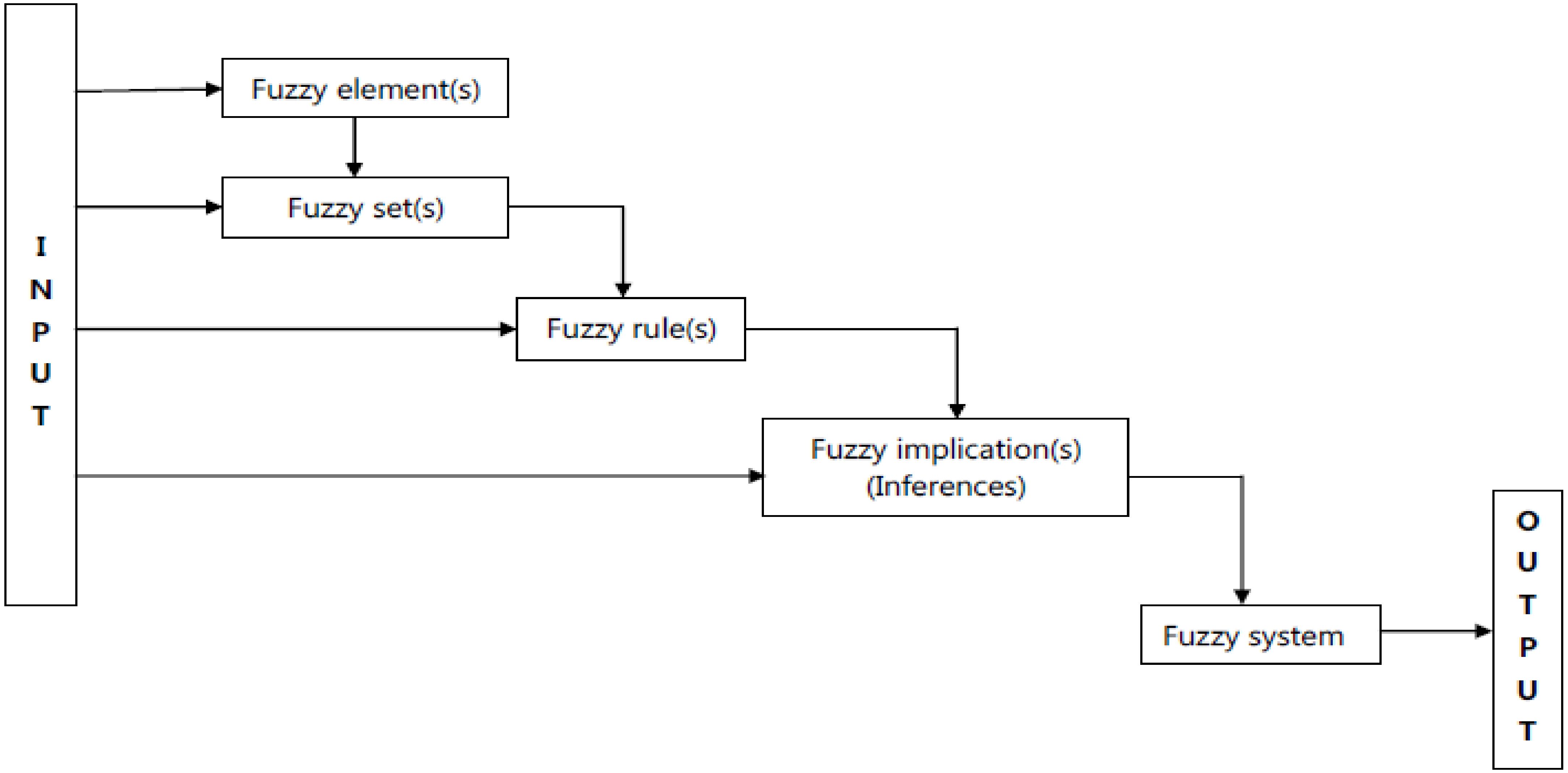
- The opposite word (antonym) of fuzzy is crisp. **Fuzzy** means un-clear or ambiguous, and **crisp** means clear, clean, and sharp.
- Fuzzy logic was proposed by Zadeh in 1965, and was applied in steam engine control by Mamdani in 1974.
- Fuzzy logic was made practically useful in Japan in the 1990s.

- ❑ Uncertainty
- ❑ Vagueness
- ❑ Impreciseness
- ❑ Inexactness


Crisp answer
Yes or No
True or False
Milk
Water
Coca
Spite
A liquid
Crisp
Yes
No
Is the liquid colorless?


Fuzzy answer
May be
May not be
Absolutely
Partially
etc


Score
Ankit
Rajesh
Santosh
Kabita
Salmon
Fuzzy
Is the person honest?
Extremely honest
99
Very honest
75
Honest at times
55
Extremely dishonest
35


INPUT
Fuzzy element(s)
Fuzzy set(s)
Fuzzy rule(s)
Fuzzy implication(s)
(Inferences)
Fuzzy system
OUTPUT

Conventional set vs. fuzzy set

- X: universe of discourse. Set A is defined by

$$\mathbf{A=\{ x \mid \mu(x)=T \wedge x \text{ in } X\}}$$

where $\mu(x)$ is a **membership function**.
  - For conventional set, the range of $\mu(x)$ is {T, F}
  - For fuzzy set, the range of $\mu(x)$ is [0,1]. So, the above definition cannot be used for fuzzy set. We cannot say clearly if x is in A or not when $0<\mu(x) < 1$.

Examples of fuzzy set

- ✓ young people, old people, kind people
- ✓ temperature is hot, just good, a little bit cold

Crisp vs Fuzzy

Crisp Set	Fuzzy Set
1. $S = \{ s \mid s \in X \}$	1. $F = (s, \mu) \mid s \in X$ and $\mu(s)$ is the degree of s.
2. It is a collection of elements.	2. It is collection of ordered pairs.
3. Inclusion of an element $s \in X$ into S is crisp, that is, has strict boundary **yes** or **no**.	3. Inclusion of an element $s \in X$ into F is fuzzy, that is, if present, then with a degree of **membership**.

Definition 1: Membership function (and Fuzzy set)

If X is a universe of discourse and $x \in X$, then a fuzzy set A in X is defined as a set of ordered pairs, that is

$A = \{(x, \mu_A(x)) | x \in X\}$ where $\mu_A(x)$ is called the membership function for the fuzzy set A.

Note:
$\mu_A(x)$ map each element of X onto a membership grade (or membership value) between 0 and 1 (both inclusive).

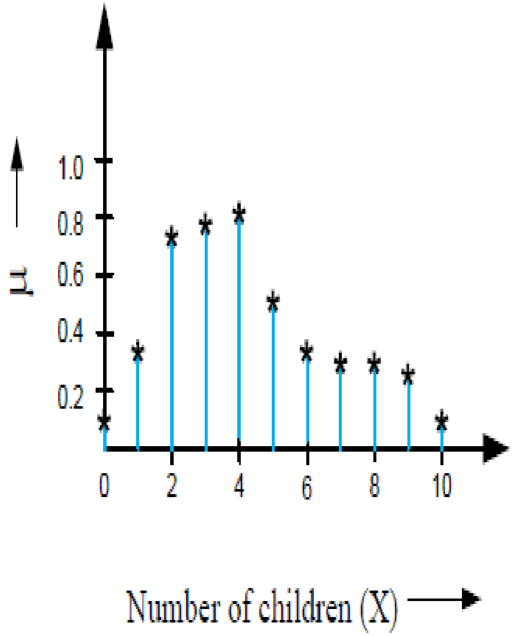
Example:

X = All cities in India

A = City of comfort

A={(New Delhi, 0.7), (Bangalore, 0.9), (Chennai, 0.8), (Hyderabad, 0.6), (Kolkata, 0.3), (Kharagpur, 0)}

Either elements or their membership values (or both) also may be of discrete values.

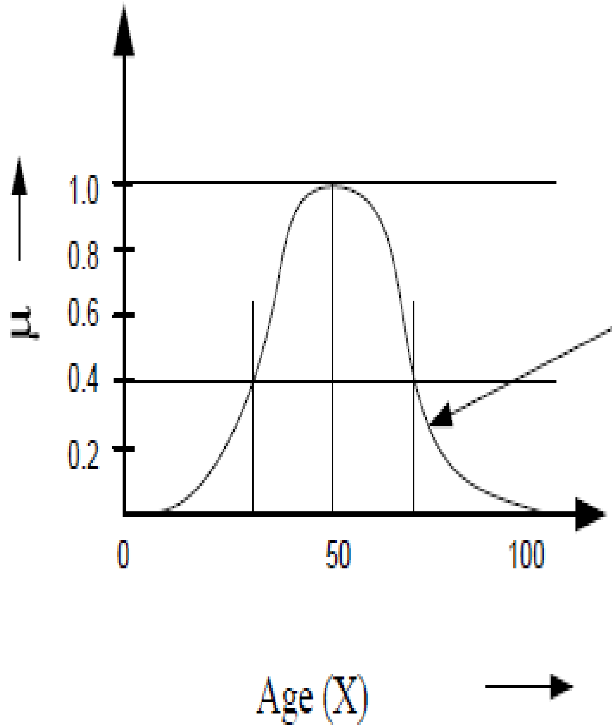


$A = \{(0,0.1),(1,0.30),(2,0.78)\ldots\ldots(10,0.1)\}$

Note : X = discrete value

How you measure happiness ??

A = “Happy family”



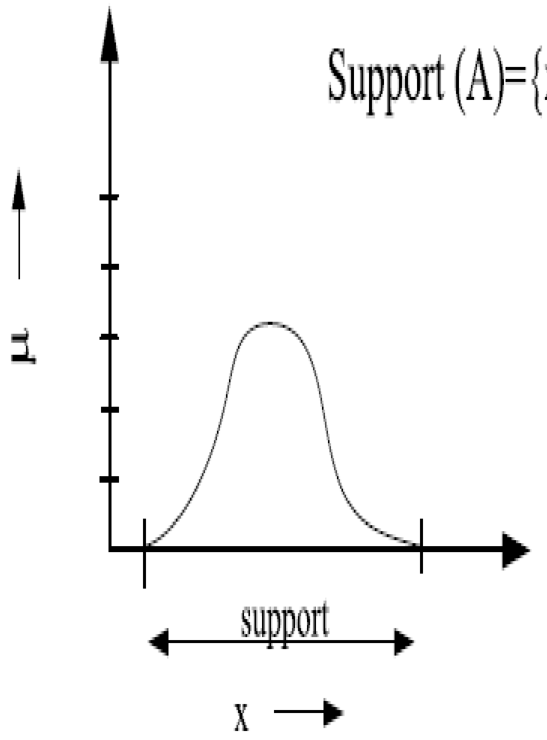
$$\mu_B(x) = \frac{1}{1+\left(\frac{x-50}{10}\right)^4}$$

$B=\{(x,\mu_B(x))\}$

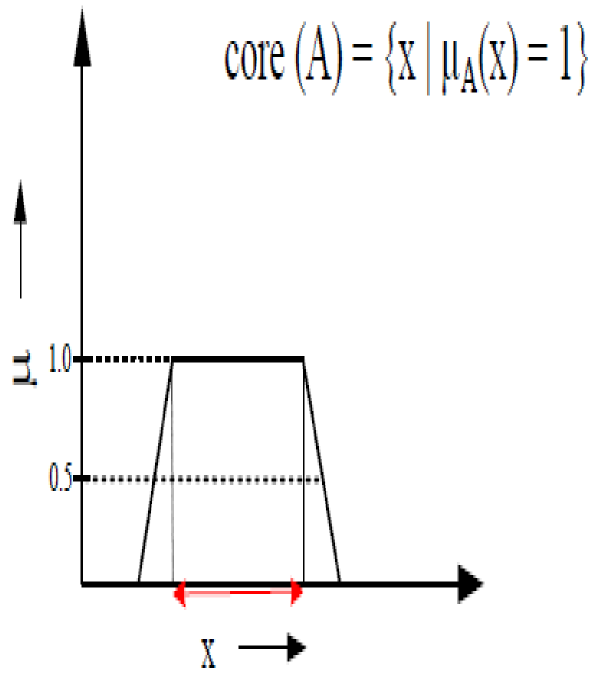
Note : x = real value
= R^+

B = “Middle aged”

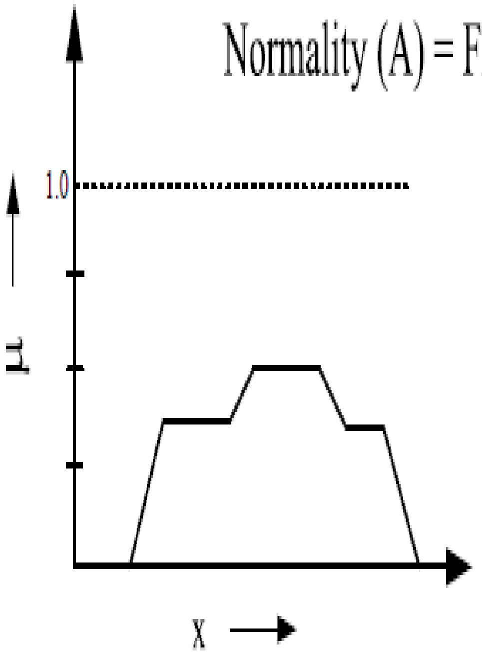
Support: The support of a fuzzy set A is the set of all points $x \in X$ such that $\mu_A(x) > 0$



Core: The core of a fuzzy set A is the set of all points x in X such that $\mu_A(x) = 1$

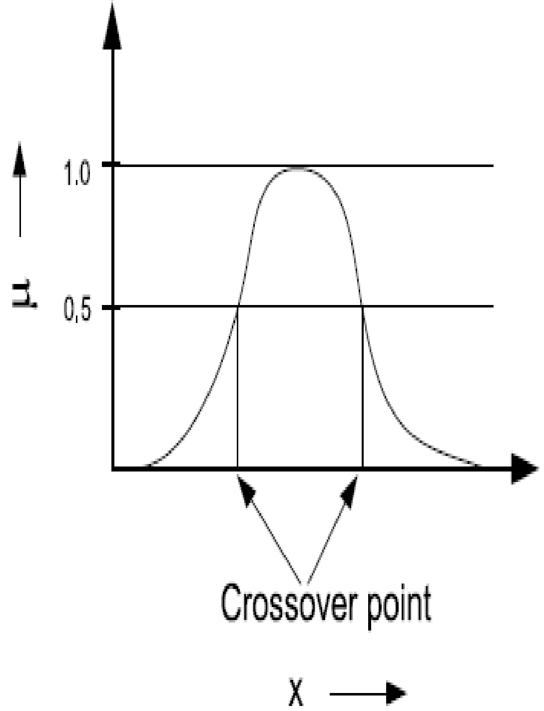


Normality : A fuzzy set A is a normal if its core is non-empty. In other words, we can always find a point $x \in X$ such that $\mu_A(x) = 1$.

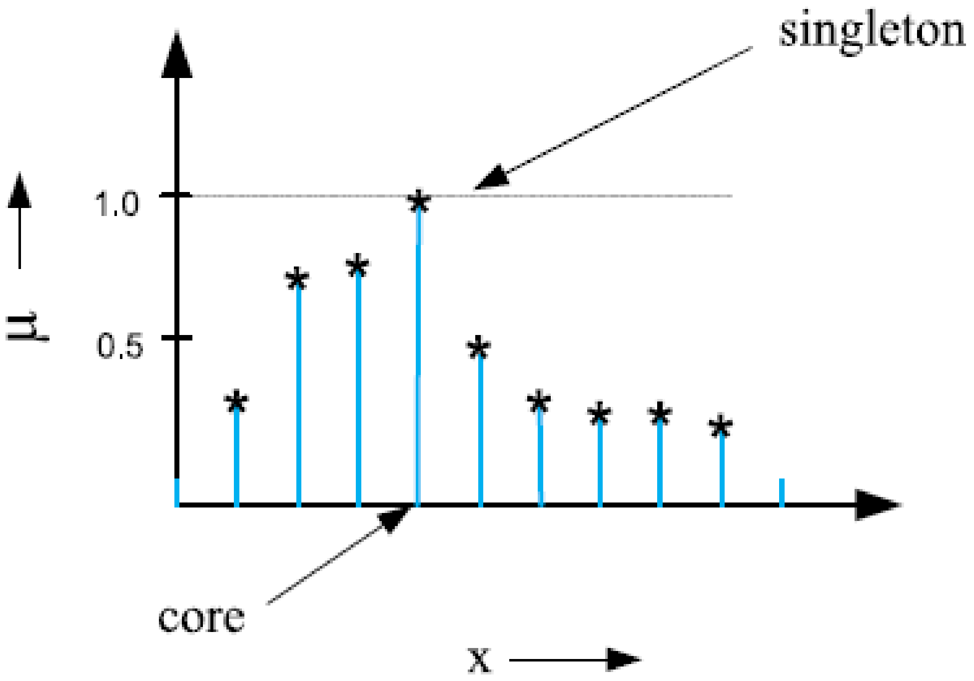


Crossover point : A crossover point of a fuzzy set A is a point $x \in X$ at which $\mu_A(x) = 0.5$. That is

Crossover $(A) = \{x | \mu_A(x) = 0.5\}$.

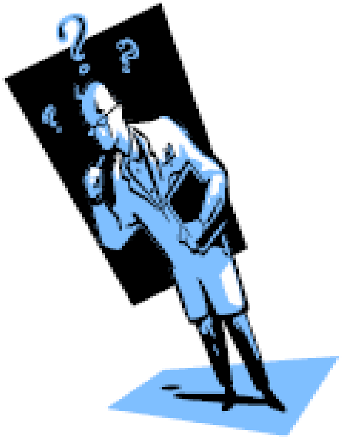


Fuzzy Singleton : A fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called a fuzzy singleton. That is $|A| = |\{ x \mid \mu_A(x) = 1\}| = 1$. Following fuzzy set is not a fuzzy singleton.



Description of fuzzy set

- Membership function of a fuzzy set A: $\mu_A: X \rightarrow [0,1]$
- If the universe of discourse $X=\{x_1,x_2,\ldots,x_N\}$, A is described as follows:



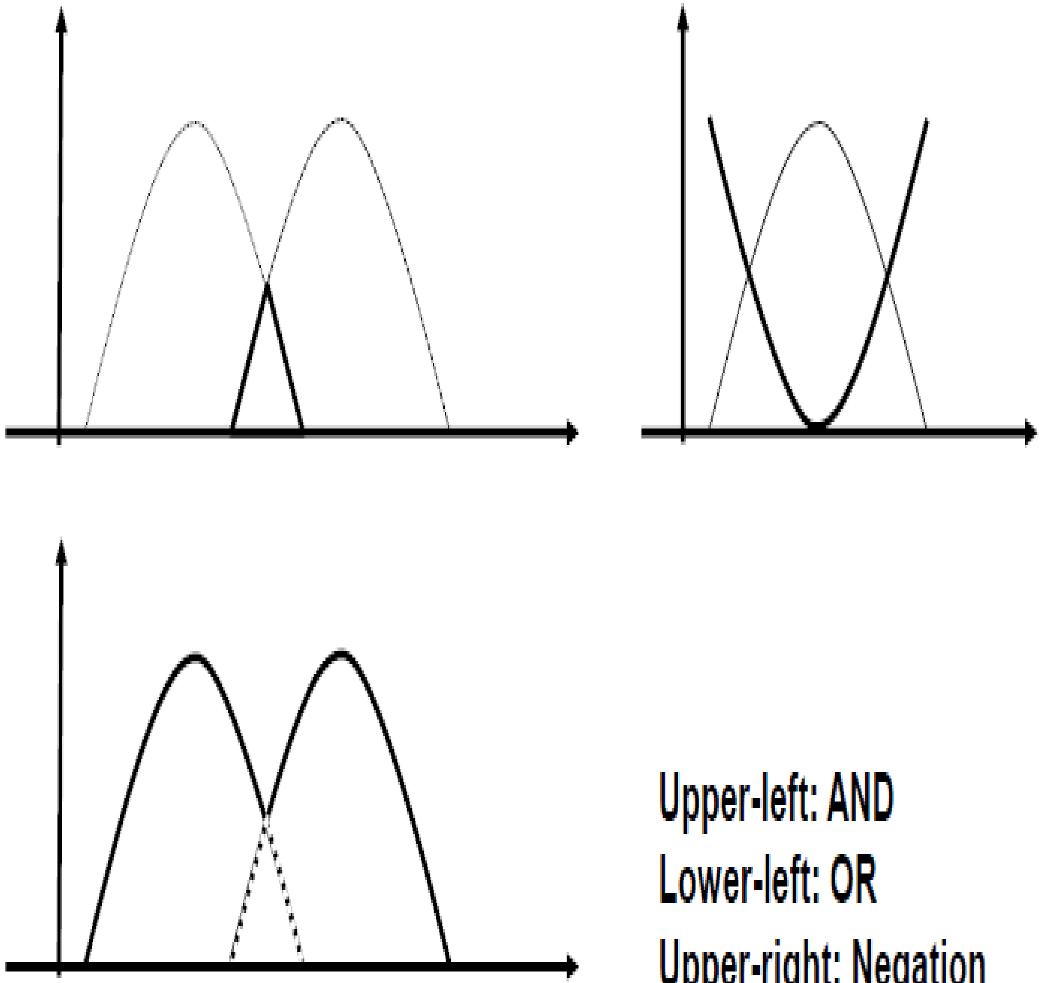
$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \ldots + \mu_A(x_N)/x_N$$
$$= \Sigma\mu_A(x_i)/x_i$$

where / is a separator, and + is the logic OR.

- If X is a continuous space, integral is used instead of the summation.

Operations of fuzzy sets

Meaning	Set notation	Logic notation	Membership function
Equivalence	$A = B$	$\mu_A(x) \Leftrightarrow \mu_B(x)$	$\mu_A(x) = \mu_B(x)$
Implication	$A \subseteq B$	$\mu_A(x) \Rightarrow \mu_B(x)$	$\mu_A(x) \leq \mu_B(x)$
Complement (negation)	$\overline{A}$	$\neg \mu_A(x)$	$\mu_{\overline{A}}(x) = 1 - \mu_A(x)$
Union, OR (disjunction)	$A \cup B$	$\mu_A(x) \vee \mu_B(x)$	$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$
Intersection, AND (conjunction)	$A \cap B$	$\mu_A(x) \wedge \mu_B(x)$	$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$



Upper-left: AND
Lower-left: OR
Upper-right: Negation

The universe of discourse:

X={Masahiro, Tsuyoshi, Takuya, Saburo, Masami}

Fuzzy sets A="young persons" and B="Tall persons" are defined by

- A = 0.4/Masahiro+0.6/Tsuyoshi+0.8/Takuya+1.0/Saburo+0.9/Masami
- B = 0.3/Masahiro+0.5/Tsuyoshi+0.9/Takuya+0.6/Saburo+0.9/Masami

The OR and AND of A and B are as follows:

- $A \cup B$=0.4/Masahiro+0.6/Tsuyoshi+0.9/Takuya+1.0/Saburo+0.9 /Masami
- $A \cap B$ =0.3/Masahiro+0.5/Tsuyoshi+0.8/Takuya+0.6/Saburo+0.9/Masami

Union ($A \cup B$):

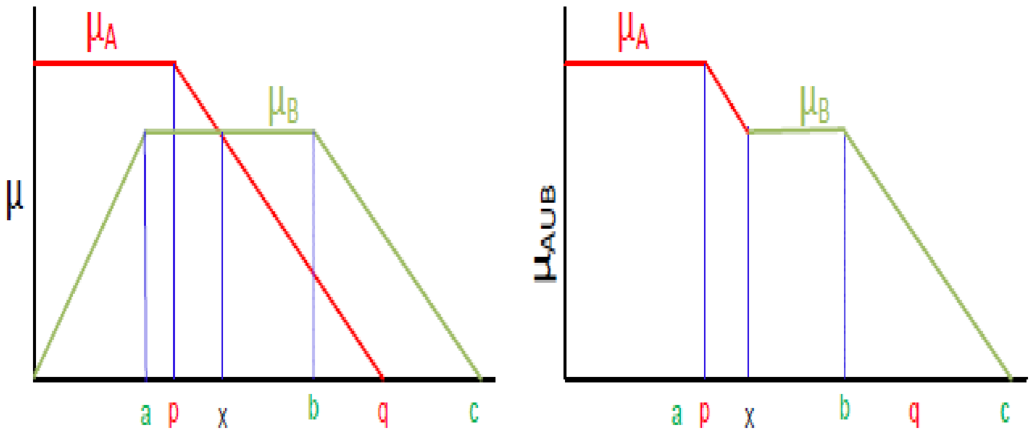
$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and

$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;

$C = A \cup B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.5)\}$



Intersection ($A \cap B$):

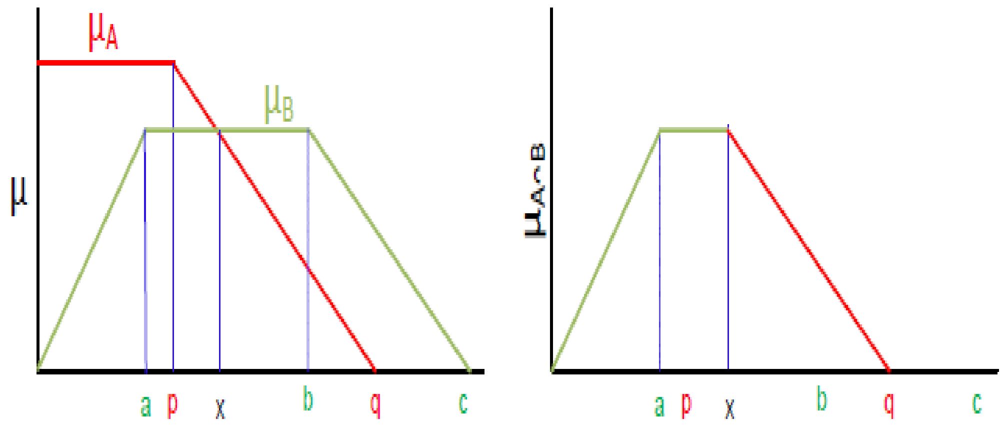
$$\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and

$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;

$C = A \cap B = \{(x_1, 0.2), (x_2, 0.1), (x_3, 0.4)\}$



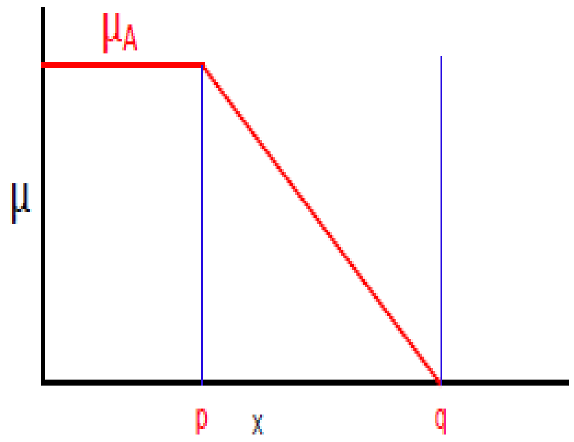
Complement (A^C):

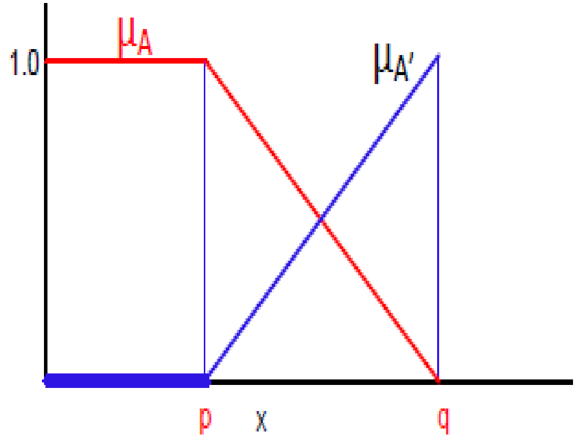
$$\mu_{A_{A^C}}(X) = 1 - \mu_A(X)$$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$

$C = A^C = \{(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)\}$





Algebric product or Vector product (A•B):

$$\mu_{A \bullet B}(X) = \mu_A(X) \bullet \mu_B(X)$$

Scalar product $(\alpha \times A)$:

$$\mu_{\alpha A}(X) = \alpha \cdot \mu_A(X)$$

Sum ($A + B$):

$$\mu_{A+B}(x) = \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$$

Difference ($A - B = A \cap B^C$):

$$\mu_{A-B}(x) = \mu_{A\cap B^C}(x)$$

Disjunctive sum: $A \oplus B = (A^C \cap B) \cup (A \cap B^C))$

Bounded Sum: $|\ \mathrm{A(x)} \oplus \mathrm{B(x)}\ |$

$$\mu_{|A(x)\oplus B(x)|} = \min\{1, \mu_A(x) + \mu_B(x)\}$$

Bounded Difference: $|\ \mathrm{A(x)} \ominus \mathrm{B(x)}\ |$

$$\mu_{|A(x)\ominus B(x)|} = \max\{0, \mu_A(x) + \mu_B(x) - 1\}$$

Equality ($A = B$):

$$\mu_A(x) = \mu_B(x)$$

Power of a fuzzy set A^α:

$$\mu_{A^\alpha}(x) = \{\mu_A(x)\}^\alpha$$

- If $\alpha < 1$, then it is called *dilation*
- If $\alpha > 1$, then it is called *concentration*

Basic fuzzy set operations: Cartesian product

Caretsian Product ($A \times B$):

$$\mu_{A \times B}(x, y) = min\{\mu_A(x), \mu_B(y)$$

Example 3:

A(x) = $\{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5), (x_4, 0.6)\}$
B(y) = $\{(y_1, 0.8), (y_2, 0.6), (y_3, 0.3)\}$

$$A \times B = \min\{\mu_A(x), \mu_B(y)\} = \begin{array}{c c} & \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} & \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.3 & 0.3 & 0.3 \\ 0.5 & 0.5 & 0.3 \\ 0.6 & 0.6 & 0.3 \end{bmatrix} \end{array}$$

Properties of fuzzy sets

Commutativity :

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associativity :

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributivity :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Idempotence :

$$A \cup A = A$$
$$A \cap A = \emptyset$$
$$A \cup \emptyset = A$$
$$A \cap \emptyset = \emptyset$$

Transitivity :

$$\text{If } A \subseteq B, B \subseteq C \text{ then } A \subseteq C$$

Involution :

$$(A^c)^c = A$$

De Morgan's law :

$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

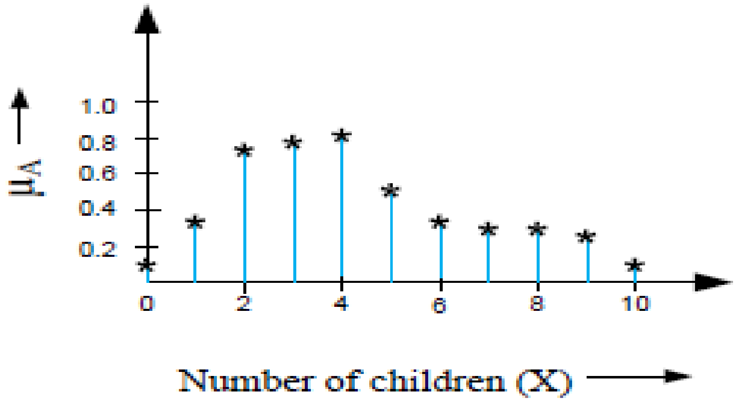
Membership Functions

A fuzzy set is completely characterized by its membership function (sometimes abbreviated as *MF* and denoted as μ). So, it would be important to learn how a membership function can be expressed (mathematically or otherwise).

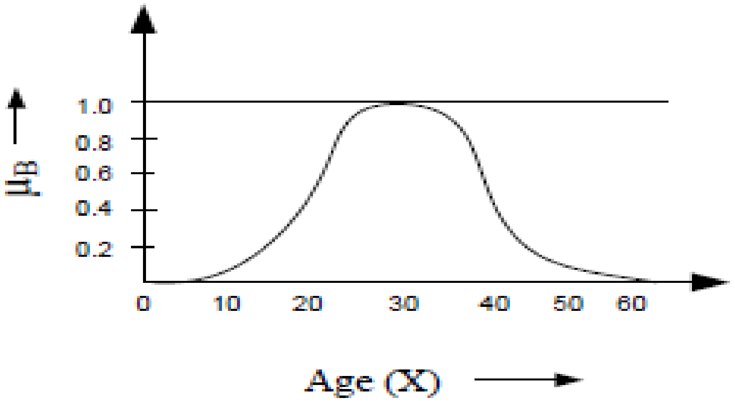
Note: A membership function can be on

(a) a discrete universe of discourse and
(b) a continuous universe of discourse.

Example:



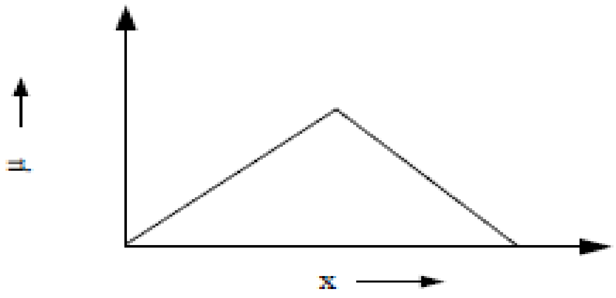
A = Fuzzy set of "Happy family"



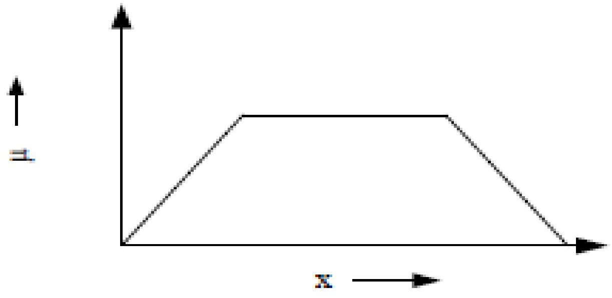
B = "Young age"

So, membership function on a discrete universe of course is trivial. However, a membership function on a continuous universe of discourse needs a special attention.

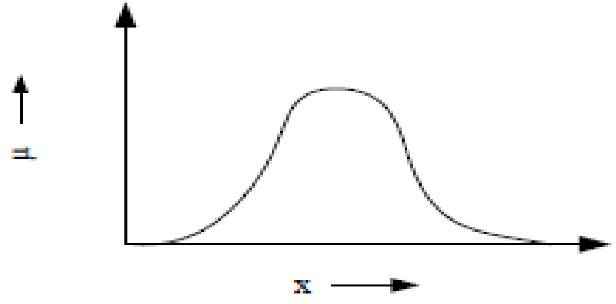
Following figures shows a typical examples of membership functions.



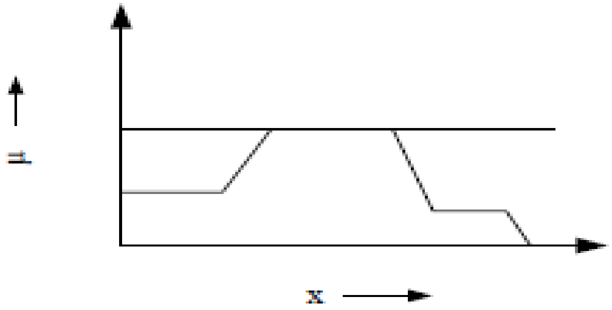
< triangular >



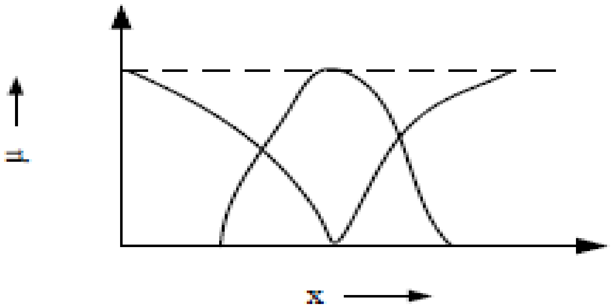
< trapezoidal >



< curve >



< non-uniform >



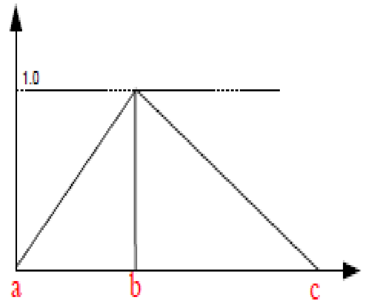
< non-uniform >

Fuzzy MFs : Formulation and parameterization

In the following, we try to parameterize the different MFs on a continuous universe of discourse.

Triangular MFs : A triangular MF is specified by three parameters $\{a, b, c\}$ and can be formulated as follows.

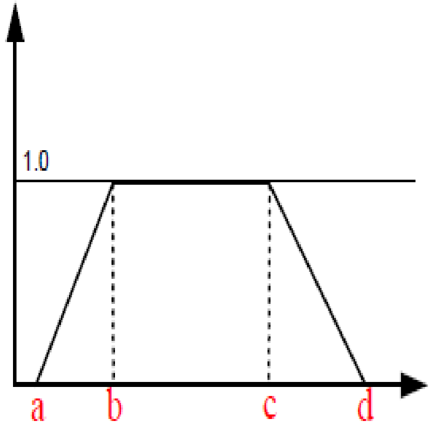
$$triangle(x; a, b, c) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ \frac{c-x}{c-b} & \text{if } b \leq x \leq c \\ 0 & \text{if } c \leq x \end{cases}$$



Fuzzy MFs: Trapezoidal

A trapezoidal MF is specified by four parameters $\{a, b, c, d\}$ and can be defined as follows:

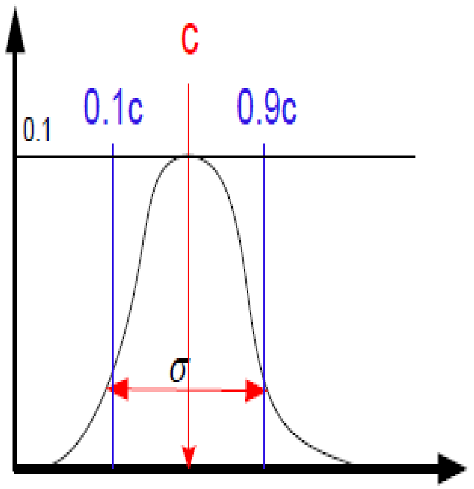
$$trapeziod(x; a, b, c, d) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c \leq x \leq d \\ 0 & \text{if } d \leq x \end{cases}$$



Fuzzy MFs: Gaussian

A Gaussian MF is specified by two parameters $\{c, \sigma\}$ and can be defined as below:

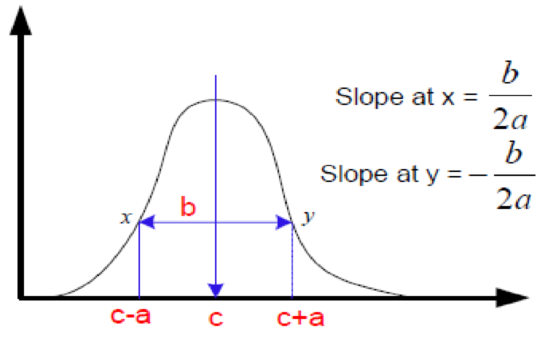
$$\text{gaussian}(x;c,\sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}.$$



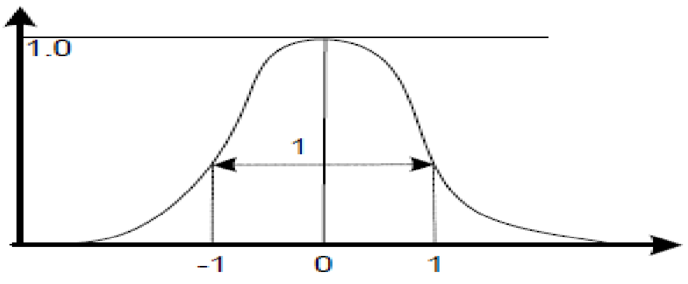
Fuzzy MFs: Generalized bell

It is also called Cauchy MF. A generalized bell MF is specified by three parameters $\{a, b, c\}$ and is defined as:

$$\text{bell}(x;\, a,\, b,\, c) = \frac{1}{1+\left|\frac{x-c}{a}\right|^{2b}}$$



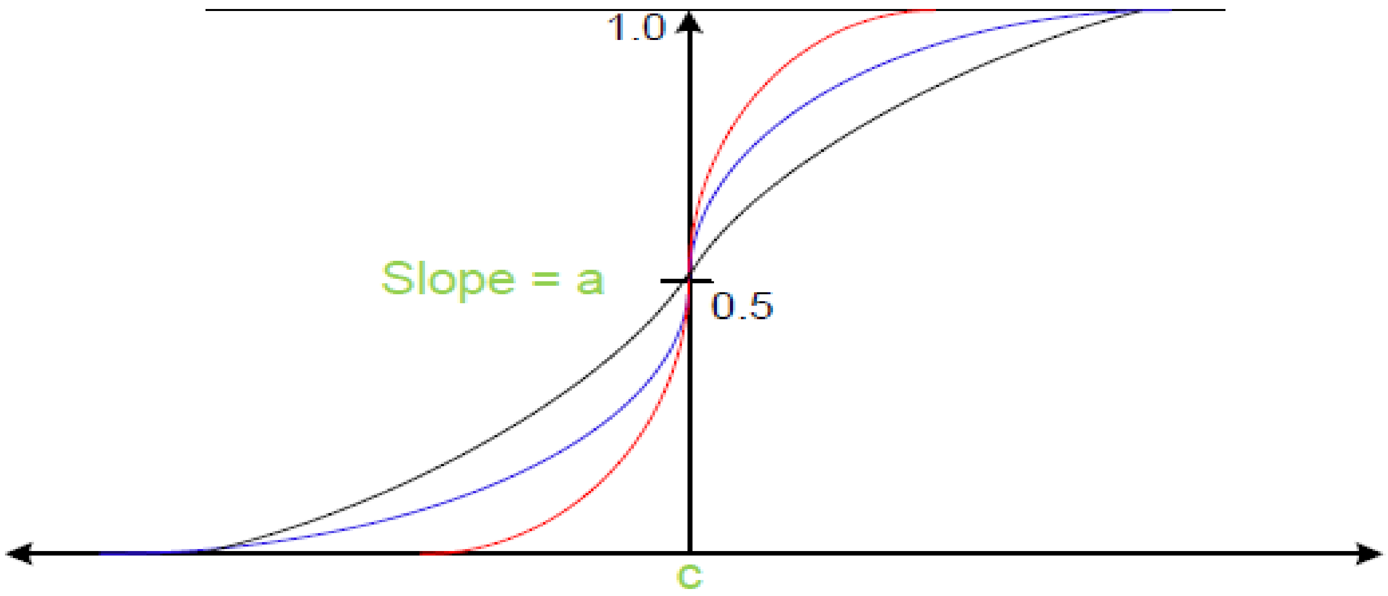
Example: $\mu(x) = \frac{1}{1+x^2}$;
$a = b = 1$ and $c = 0$;



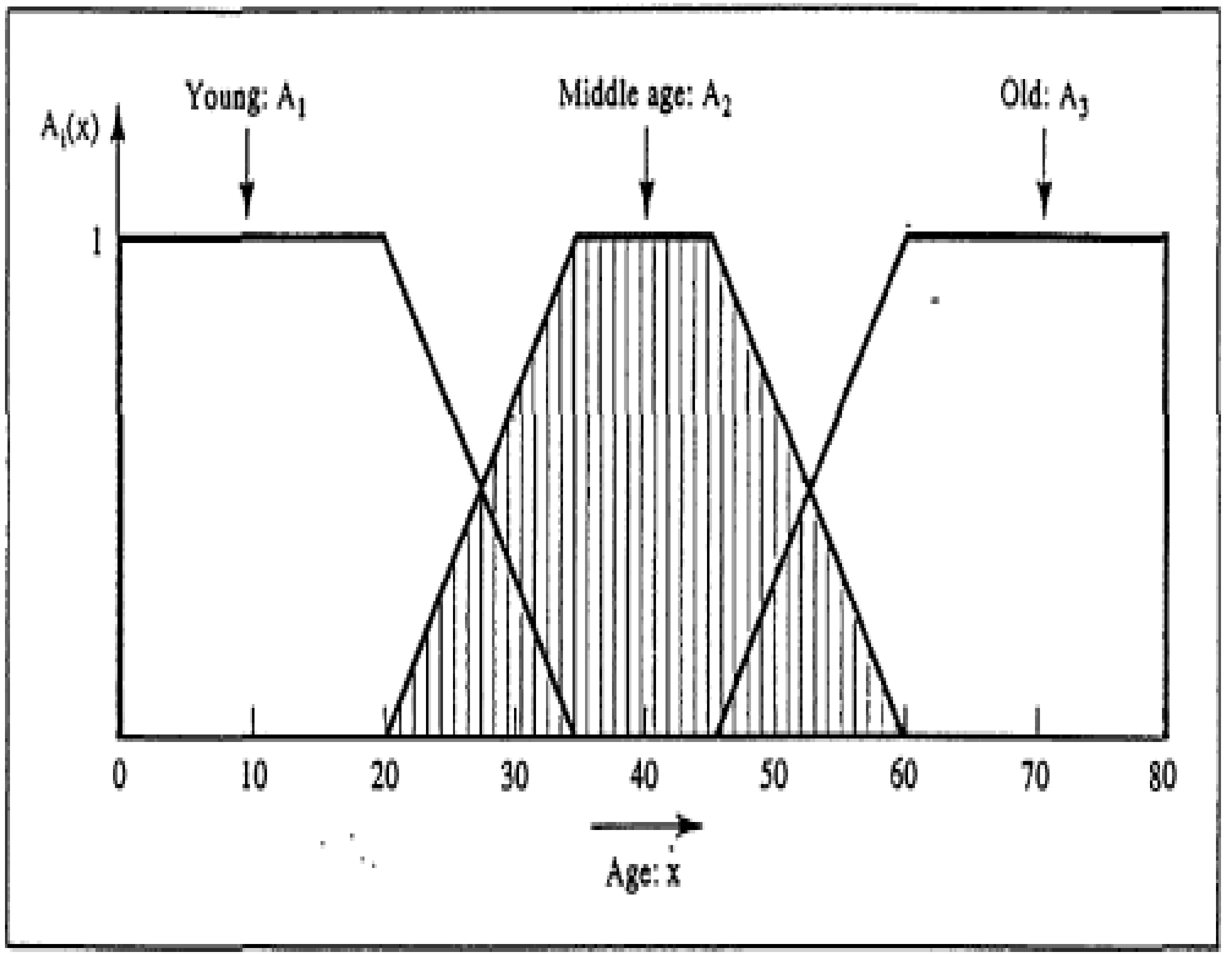
Fuzzy MFs: Sigmoidal MFs

Parameters: $\{a, c\}$; where c = crossover point and a = slope at c;

$$\text{Sigmoid}(x;a,c)=\frac{1}{1+e^{-\left[\frac{a}{x-c}\right]}}$$



Example: 0 to 35 young aged; 20 to 60 middle aged; and 45 to 80 old aged persons. Draw the membership curve and define the membership functions of “Age”

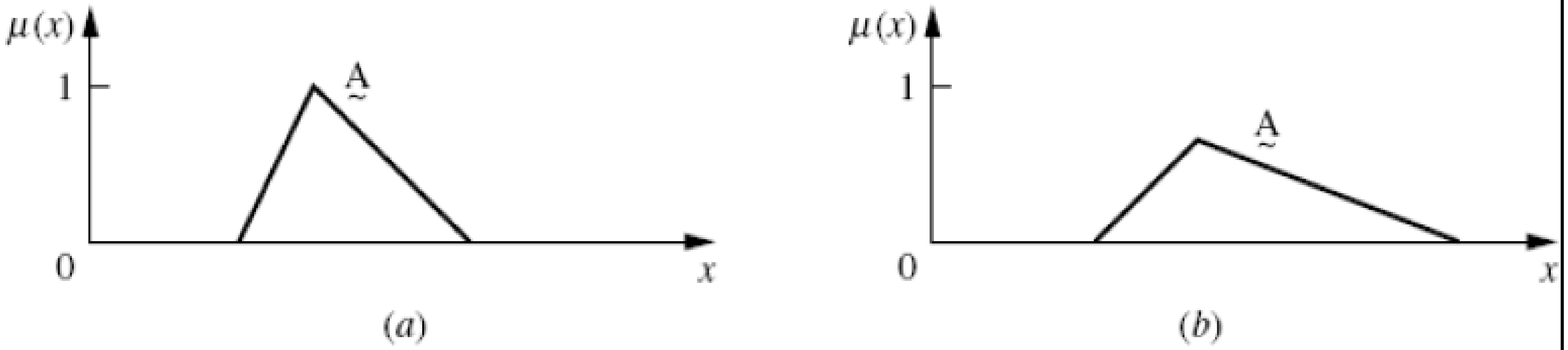


$$A_1(x) = \begin{cases} 1 & \text{when } x \leq 20 \\ (35 - x)/15 & \text{when } 20 < x < 35 \\ 0 & \text{when } x \geq 35 \end{cases}$$

$$A_2(x) = \begin{cases} 0 & \text{when either } x \leq 20 \text{ or } \geq 60 \\ (x - 20)/15 & \text{when } 20 < x < 35 \\ (60 - x)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } 35 \leq x \leq 45 \end{cases}$$

$$A_3(x) = \begin{cases} 0 & \text{when } x \leq 45 \\ (x - 45)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } x \geq 60 \end{cases}$$

- A *normal* fuzzy set is one whose membership function has at least one element *x* in the universe whose membership value is unity.



Convex fuzzy set

- A *convex* fuzzy set is described by a membership function whose membership values are strictly monotonically increasing, or strictly monotonically decreasing, or strictly monotonically increasing then strictly monotonically decreasing with increasing values for elements in the universe.

  (Said another way, if, for any elements *x*, *y*, and *z* in a fuzzy set A, the relation $x < y < z$ implies that

$$\mu_{\underset{\sim}{A}}(y) \geq \min[\mu_{\underset{\sim}{A}}(x), \mu_{\underset{\sim}{A}}(z)]$$

then A is said to be a convex fuzzy set.)

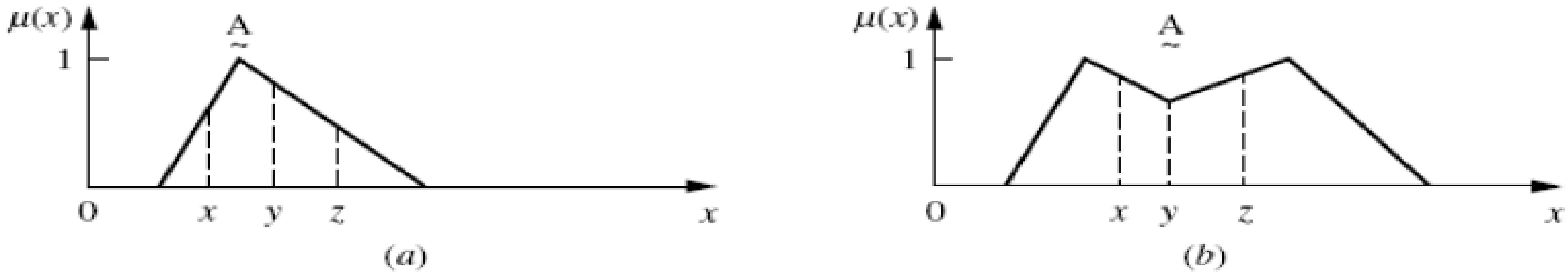


FIGURE
Convex, normal fuzzy set (*a*) and nonconvex, normal fuzzy set (*b*).

this definition of convexity is *different* from some definitions of the same term in mathematics. In some areas of mathematics, convexity of shape has to do with whether a straight line through any part of the shape goes outside the boundaries of that shape. This definition of convexity is *not* used here.

- The *crossover points* of a membership function are defined as the elements in the universe for which a particular fuzzy set $\underline{A}$ has values equal to 0.5, i.e., for which $\mu_A(x) = 0.5$.
- The *height* of a fuzzy set *A* is the maximum value of the membership function, i.e.,

$$\mathrm{hgt}(\underset{\sim}{\mathrm{A}}) = \max\{\mu_{\underset{\sim}{\mathrm{A}}}(x)\}$$

- If the hgt$(\underline{A}) < 1$, the fuzzy set is said to be subnormal.
- If $\underline{A}$ is a convex single-point normal fuzzy set defined on the real line, then $\underline{A}$ is often termed a *fuzzy number*.

A special property of two convex fuzzy sets, say $\underset{\sim}{A}$ and $\underset{\sim}{B}$, is that the intersection of these two convex fuzzy sets is also a convex fuzzy set, as shown in Fig. 4.4. That is, for $\underset{\sim}{A}$ and $\underset{\sim}{B}$, which are both convex, $\underset{\sim}{A} \cap \underset{\sim}{B}$ is also convex.

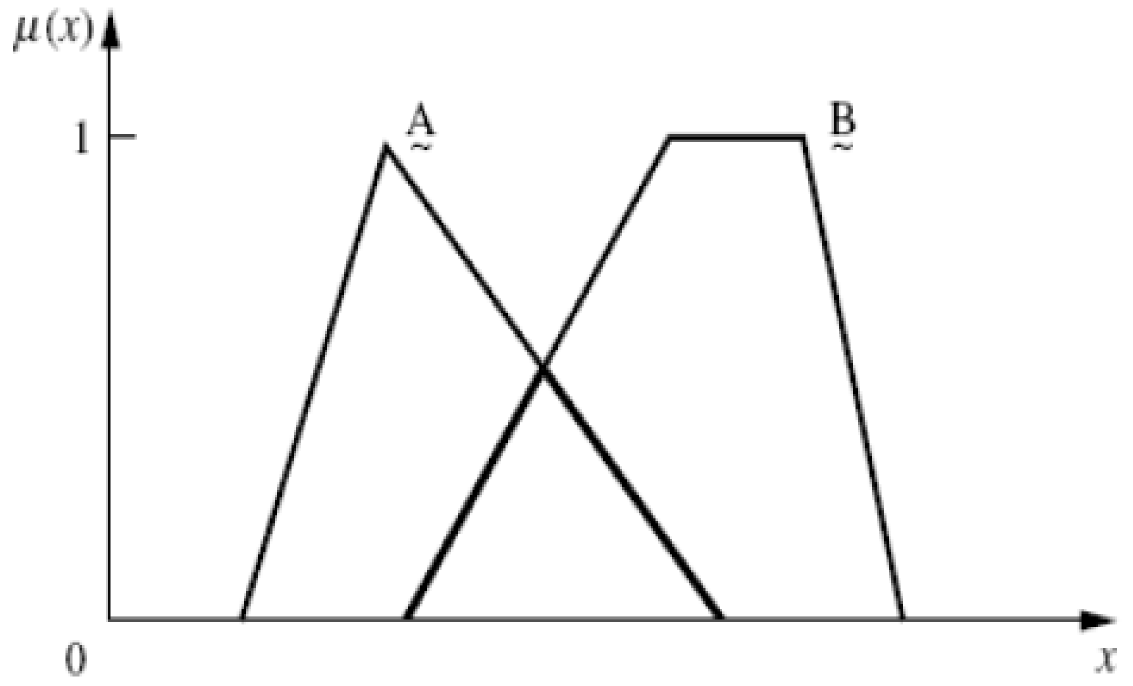


FIGURE
The intersection of two convex fuzzy sets produces a convex fuzzy set.

It will be Continued....