Chapter 3

The z-transform and Analysis of LTI Systems

Contents

Primary points

- Convolution of discrete-time signals simply becomes multiplication of their z-transforms.
- Systematic method for finding the **impulse response** of LTI systems described by **difference equations**: **partial fraction expansion**.
- Characterize LTI discrete-time systems in the z-domain

Secondary points

- Characterize discrete-time signals
- Characterize LTI discrete-time systems and their response to various input signals

3.1

The z-transform

We focus on the bilateral z-transform.

3.1.1 The bilateral z-transform

The **direct z-transform** or **two-sided z-transform** or **bilateral z-transform** or just the **z-transform** of a discrete-time signal $x[n]$ is defined as follows.

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \quad \text{or} \quad X(\cdot) = Z\{x[\cdot]\} \quad \text{or shorthand:} \quad x[n] \overset{Z}{\leftrightarrow} X(z)\,.}$$

- Note capital letter for transform.
- In the math literature, this is called a **power series**.
- It is a mapping from the space of discrete-time signals to the space of functions defined over (some subset of) the complex plane.
- We will also call the complex plane the z-plane.

We will discuss the **inverse z-transform** later.

Convergence

Any time we consider a summation or integral with infinite limits, we must think about **convergence**.

We say an infinite series of the form $\sum_{n=-\infty}^{\infty} c_n$ **converges** [1, p. 141] if there is a $c \in \mathbb{C}$ such that $\lim_{N\to\infty} \left| c - \sum_{n=-N}^{N} c_n \right| = 0$.

- Some infinite series do converge to a finite value, *e.g.*, $1 + 1/2 + 1/4 + 1/8 + \cdots = \frac{1}{1-1/2} = 2$,
  since $\left| 2 - \sum_{n=0}^{N} (1/2)^n \right| = \left| 2 - \frac{1-(1/2)^{N+1}}{1-1/2} \right| = (1/2)^N \to 0$ as $N \to \infty$.
- One can also extend the notion of convergence to include "convergence to ∞" [2, p. 37].
  Example. The infamous **harmonic series** is an infinite series that converges to infinity: $1 + 1/2 + 1/3 + 1/4 + \cdots = \infty$.
- Some infinite series simply do not converge, *e.g.*, $1 - 1 + 1 - 1 + \cdots = \ ?$

The z-transform of a signal is an infinite series for each possible value of z in the complex plane. Typically only some of those infinite series will converge. We need terminology to distinguish the "good" subset of values of z that correspond to convergent infinite series from the "bad" values that do not.

Definition of ROC

On *p. 152*, the textbook, like many DSP books, defines the **region of convergence** or **ROC** to be:

"the set of all values of z for which $X(z)$ attains a finite value."

Writing each z in the polar form $z = r\, e^{\jmath\phi}$, on *p. 154*, the book says that: "finding the ROC for $X(z)$ is equivalent to determining the range of values of r for which the sequence $x[n]\, r^{-n}$ is **absolutely summable**."

Unfortunately, that claim of equivalence is *incorrect* if we use the book's definition of ROC on *p. 152*. There are examples of signals, such as $x[n] = \frac{1}{n} u[n-1]$, for which certain values of z lead to a convergent infinite series, but yet $x[n]\, r^{-n}$ is *not* absolutely summable.

So we have two possible *distinct* definitions for the ROC: "the z values where $X(z)$ is finite," or, "the z values where $x[n]\, z^{-n}$ is absolutely summable." Most DSP textbooks are not rigorous about this distinction, and in fact either definition is fine from a practical perspective. The definitions are compatible in the case of z-transforms that are rational, which are the most important type for practical DSP use. To keep the ROC properties (and Fourier relations) simple, we adopt the following definition.

The ROC is the set of values $z \in \mathbb{C}$ for which the sequence $x[n]\, z^{-n}$ is absolutely summable, *i.e.*, $\left\{ z \in \mathbb{C} : \sum_{n=-\infty}^{\infty} \left| x[n]\, z^{-n} \right| < \infty \right\}.$

All absolutely summable sequences have convergent infinite series [1, p. 144]. But there are some sequences, such as $(-1)^n/n$, that are not absolutely summable yet have convergent infinite series. These will not be included in our definition of ROC, but this will not limit the practical utility.

Skill: ***Finding a z-transform completely, including both $X(z)$ and the ROC.***

Example. $x[n] = \delta[n]$. $X(z) = 1$ and ROC = $\mathbb{C}$ = entire z-plane.

Example. $x[n] = \delta[n-k]$. $X(z) = z^{-k}$ and

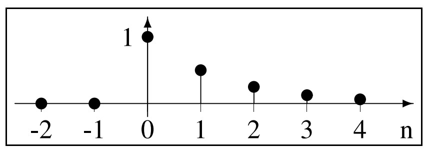
$$\text{ROC} = \begin{cases} \mathbb{C}, & k = 0 \\ \mathbb{C} - \{0\}, & k > 0 \\ \mathbb{C} - \{\infty\}, & k < 0. \end{cases} \qquad \boxed{\delta[n-k] \overset{Z}{\leftrightarrow} z^{-k}}$$

Example. $x[n] = \{4, \underline{3}, 0, \pi\}$. $X(z) = 4z + 3 + \pi z^{-2}$, ROC = $\mathbb{C} - \{0\} - \{\infty\}$

For a **finite-duration signal**, the ROC is the entire z-plane, possibly excepting $z = 0$ and $z = \infty$.

Why? Because for $k > 0$: z^k is infinite for $z = \infty$ and z^{-k} is infinite for $z = 0$; elsewhere, polynomials in z and z^{-1} are finite.

Example. $x[n] = p^n\, u[n]$. **Skill:** ***Combining terms to express as geometric series.***



$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = \sum_{n=0}^{\infty} p^n z^{-n} = \sum_{n=0}^{\infty} (pz^{-1})^n = 1 + \left(\frac{p}{z}\right) + \left(\frac{p}{z}\right)^2 + \left(\frac{p}{z}\right)^3 + \cdots = \frac{1}{1 - pz^{-1}}.$$

The series converges iff $\left|pz^{-1}\right| < 1$, *i.e.*, if $\{|z| > |p|\}$.

$$\boxed{p^n\, u[n] \overset{Z}{\leftrightarrow} \frac{1}{1 - pz^{-1}}, \text{ for } |z| > |p|}$$ ***Picture 3.2 shading outside circle radius*** $|p|$

Smaller $|p|$ means faster decay means larger ROC.

Example. Important special case: $p = 1$ leaves just the unit step function. $\boxed{u[n] \overset{Z}{\leftrightarrow} U(z) = \frac{1}{1 - z^{-1}},\ |z| > 1}$

Example. $x[n] = -p^n\, u[-n-1]$ for $p \neq 0$. ***Picture*** . An **anti-causal** signal.

$$X(z) = \sum_{n=-\infty}^{-1} -p^n z^{-n} = -\sum_{k=1}^{\infty} (p^{-1}z)^k = -(p^{-1}z) \sum_{k=0}^{\infty} (p^{-1}z)^k = -p^{-1}z \frac{1}{1 - p^{-1}z} = \frac{1}{1 - pz^{-1}}.$$

The series converges iff $\left|p^{-1}z\right| < 1$, *i.e.*, if $|z| < |p|$. ***Picture 3.3 shading inside circle radius*** $|p|$

Note that the last two examples *have the same formula* for $X(z)$. The ROC is essential for resolving this ambiguity!

Laplace analogy

$$\begin{aligned} e^{\lambda t}\, u(t) &\overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s - \lambda}, &\quad \text{real}(s) > \text{real}(\lambda) \\ -e^{\lambda t}\, u(-t) &\overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s - \lambda}, &\quad \text{real}(s) < \text{real}(\lambda) \end{aligned}$$

General shape of ROC

In the preceding two examples, we have seen ROC's that are the interior and exterior of circles. What is the general shape?

$$\boxed{\text{The ROC is always an annulus, } i.e., \{r_2 < |z| < r_1\}.}$$

Note that r_2 can be zero (possibly with $\leq$) and r_1 can be ∞ (possibly with $\leq$).

Explanation. Let $z = r\,e^{\jmath\theta}$ be polar form.

$$\begin{aligned}
|X(z)| &= \left|\sum_{n=-\infty}^{\infty} x[n]\,z^{-n}\right| \leq \sum_{n=-\infty}^{\infty} |x[n]|\,r^{-n} \text{ by triangle inequality} \\
&= \sum_{n=-\infty}^{-1} |x[n]|\;r^{-n} + \sum_{n=0}^{\infty} |x[n]|\;r^{-n} \\
&= \sum_{n=1}^{\infty} |x[-n]|\,r^{n} + \sum_{n=0}^{\infty} \frac{|x[n]|}{r^n}.
\end{aligned}$$

The ROC is the subset of $\mathbb{C}$ where *both* of the above sums are finite.

If the right sum (the "causal part") is finite for some z_2 with magnitude $r_2 = |z_2|$,
then that sum will also be finite for any z with magnitude $r \geq r_2$, since for such an r each term in the sum is smaller.
So the ROC for the right sum is the subset of $\mathbb{C}$ for which $|z| > r_2$, which is the exterior of some circle.

Likewise if the left sum (the "anti-causal part") is finite for some z_1 with magnitude $r_1 = |z_1|$,
then that sum will also be finite for any z with magnitude $r \leq r_1$, since for such an r each term in the sum is smaller.
So the ROC for the left sum is the subset of $\mathbb{C}$ for which $|z| < r_1$, for some r_1, which is the interior of some circle.

$$\boxed{\text{The ROC of a causal signal is the exterior of a circle of some radius } r_2.}$$

$$\boxed{\text{The ROC of an anti-causal signal is the interior of a circle of some radius } r_1.}$$

For a general signal $x[n]$, the ROC will be the *intersection* of the ROC of its causal and noncausal parts, which is an annulus.
If $r_2 < r_1$, then that intersection is a (nonempty) annulus. Otherwise the z-transform is undefined (does not exist).

Simple example of a signal which has empty ROC?
$x[n] = 1 = u[n] + u[-n-1]$.
Recall $u[n] \overset{Z}{\leftrightarrow} X(z) = \frac{1}{1-z^{-1}}$ for $\{|z| > 1\}$.
ROC for the causal part is $\{|z| > 1\}$,
ROC for the anti-causal part is $\{|z| < 1\}$.

TABLE 3.1 - discuss here

Table shows signals decreasing away from zero, since for non-decreasing signals the z-transform is usual undefined (empty ROC).
Energy signals must eventually diminish to zero.

Subtleties in defining the ROC (optional reading!)

We elaborate here on why the two possible definitions of the ROC are *not* equivalent, contrary to to the book's claim on *p. 154*.

Consider the **harmonic series** signal $x[n] = \frac{1}{n} u[n-1]$. (A signal with no practical importance.)
The z-transform of this signal is

$$X(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^{-n}.$$

Consider first the exterior of the unit circle. If $r = |z| > 1$ then

$$\sum_{n=1}^{\infty} \left| \frac{1}{n} z^{-n} \right| = \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{1}{r}\right)^n < \sum_{n=1}^{\infty} \left(\frac{1}{r}\right)^n < \infty.$$

So $\{|z| > 1\}$ will be included in the ROC, by either definition.

Now consider the interior of the unit circle. If $r = |z| < 1$ then

$$\sum_{n=1}^{N} \left| \frac{1}{n} z^{-n} \right| = \sum_{n=1}^{N} \frac{1}{n} \left(\frac{1}{r}\right)^n > \sum_{n=1}^{N} \frac{1}{n} \to \infty.$$

So $\{|z| < 1\}$ will not be in the ROC, by the "absolutely summable" definition.

Now consider the point $z = 1$. At this point $X(1) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$. So there is a pole at $z = 1$ and $z = 1$ is not in the ROC by either definition.

Now consider the point $z = -1$. At this point $X(-1) = \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n = -\log 2$, which is a well-defined finite value!
See `http://mathworld.wolfram.com/HarmonicSeries.html` for more information.
It is easy to verify that sum using the Taylor expansion of $\log$ around 1, evaluated at 2.

- So the point $z = -1$ would be included in the ROC defined by the "attains a finite value" definition.
- However, at $z = -1$ the series $\frac{1}{n}(-1)^{-n}$ is not absolutely summable, since $\sum_{n=1}^{\infty} \left|\frac{1}{n}(-1)^{-n}\right| = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$. So the point $z = -1$ is not included in the "absolutely summable" definition of the ROC.

Furthermore, there are other points around the unit circle where the z-transform series is convergent but not absolutely summable.

Consider $z = e^{\jmath 2\pi M/N}$, with N even and M odd.

$$\sum_{n=1}^{\infty} \frac{1}{n} e^{\jmath 2\pi (M/N) n} = \sum_{k=0}^{\infty} \sum_{l=1}^{N} \frac{1}{Nk + l} e^{\jmath 2\pi (M/N)(Nk+l)} = \sum_{k=0}^{\infty} \sum_{n=1}^{N} \frac{1}{Nk + n} e^{\jmath 2\pi (M/N) n}.$$

$$\begin{aligned}
\sum_{n=1}^{N} \frac{1}{Nk+n} e^{\jmath 2\pi (M/N) n} &= \sum_{n=1}^{N/2} \left[\frac{1}{Nk+n} e^{\jmath 2\pi (M/N) n} + \frac{1}{Nk+n+N/2} e^{\jmath 2\pi (M/N)(n+N/2)} \right] \\
&= \sum_{n=1}^{N/2} \left[\frac{1}{Nk+n} - \frac{1}{Nk+n+N/2} \right] e^{\jmath 2\pi (M/N) n} = \sum_{n=1}^{N/2} \frac{N/2}{(Nk+n)(Nk+n+N/2)} e^{\jmath 2\pi (M/N) n}.
\end{aligned}$$

This is like $1/k^2$, so it will be convergent.

3.1.2

The inverse z-transform

One method for determining the inverse is contour integration using the **Cauchy integral theorem**. ***See 3.4.***

Key point: we want to avoid this! By learning z-transform properties, can expand small table of z-transforms into a large set.

3.2

Properties of the z-transform

For each property must consider both "what happens to formula $X(z)$" *and* what happens to ROC.

Linearity

If $x_1[n] \overset{Z}{\leftrightarrow} X_1(z)$ and $x_2[n] \overset{Z}{\leftrightarrow} X_2(z)$ then

$$x[n] = a_1\, x_1[n] + a_2\, x_2[n] \overset{Z}{\leftrightarrow} a_1\, X_1(z) + a_2\, X_2(z)$$

Follows directly from definition.

Very useful for finding z-transforms and inverse z-transforms!

The ROC of the sum contains at least as much of the z-plane as the intersection of the two ROC's.

Example: $x[n] = \cos(\omega_0 n + \phi)\, u[n]$ (causal sinusoid).
By Euler's identity, $x[n] = \frac{1}{2}\left(e^{\jmath(\omega_0 n+\phi)} + e^{-\jmath(\omega_0 n+\phi)}\right) u[n] = \frac{1}{2}\, e^{\jmath\phi} \left(e^{\jmath\omega_0}\right)^n u[n] + \frac{1}{2}\, e^{-\jmath\phi} \left(e^{-\jmath\omega_0}\right)^n u[n]$.
Applying previous example with "$p = e^{\pm\jmath\omega_0}$" and linearity:

$$X(z) = \frac{\frac{1}{2}\, e^{\jmath\phi}}{1 - e^{\jmath\omega_0}\, z^{-1}} + \frac{\frac{1}{2}\, e^{-\jmath\phi}}{1 - e^{-\jmath\omega_0}\, z^{-1}} = \frac{\frac{1}{2}\, e^{\jmath\phi}\left(1 - e^{-\jmath\omega_0}\, z^{-1}\right) + \frac{1}{2}\, e^{-\jmath\phi}\left(1 - e^{\jmath\omega_0}\, z^{-1}\right)}{(1 - e^{\jmath\omega_0}\, z^{-1})(1 - e^{-\jmath\omega_0}\, z^{-1})} = \frac{\cos\phi - z^{-1}\cos(\omega_0 - \phi)}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}.$$

What is the ROC? $\{|z| > |p| = 1\}$, as one expects since $|\cos(\omega n)| \leq 1$.

Time shifting

If $x[n] \overset{Z}{\leftrightarrow} X(z)$, then $x[n-k] \overset{Z}{\leftrightarrow} z^{-k}\, X(z)$.
Simple proof by change of index variable.
ROC is unchanged, except for adding or deleting $z = 0$ or $z = \infty$.

Now clear why unit delay was labeled z^{-1}.

Scaling the z-domain, aka modulation

If $x[n] \overset{Z}{\leftrightarrow} X(z)$ with ROC = $\{r_1 < |z| < r_2\}$, then $a^n\, x[n] \overset{Z}{\leftrightarrow} X\left(a^{-1}z\right)$ with ROC = $\{|a|r_1 < |z| < |a|r_2\}$.

Example. Decaying sinusoid: $x[n] = \frac{1}{2^n}\cos(\omega_0 n)\, u[n]$.

$$X(z) = \frac{1 - \frac{1}{2}z^{-1}\cos\omega_0}{1 - z^{-1}\cos\omega_0 + \frac{1}{4}z^{-2}}$$

with ROC = $\left\{|z| > \frac{1}{2}\right\}$.

Time reversal

If $x[n] \overset{Z}{\leftrightarrow} X(z)$ with ROC = $\{r_1 < |z| < r_2\}$, then $x[-n] \overset{Z}{\leftrightarrow} X\left(z^{-1}\right)$ with ROC = $\{1/r_2 < |z| < 1/r_1\}$.

Simple proof by change of summation index, since positive powers of z become negative and vice versa.

Differentiation in z-domain

If $x[n] \overset{Z}{\leftrightarrow} X(z)$ then $n\, x[n] \overset{Z}{\leftrightarrow} -z\frac{d}{dz}\, X(z)$. The ROC is unchanged.

Proof:

$$-z\frac{d}{dz}\, X(z) = -z\frac{d}{dz}\sum_{n=-\infty}^{\infty} x[n]\, z^{-n} = -z\sum_{n=-\infty}^{\infty} x[n]\,(-n)\, z^{-n-1} = \sum_{n=-\infty}^{\infty} (n\, x[n])\, z^{-n} = Z\left\{n\, x[n]\right\}.$$

Caution for derivative when $n = 0$.

Example: $x[n] = n\, u[n]$ (unit ramp signal). We know $U(z) = 1/(1 - z^{-1})$ for $\{|z| > 1\}$. So

$$X(z) = -z\frac{d}{dz}U(z) = -z\frac{-z^{-2}}{(1 - z^{-1})^2} = \frac{z^{-1}}{(1 - z^{-1})^2}, \qquad \{|z| > 1\}.$$

Convolution

$$\text{If } x_1[n] \overset{Z}{\leftrightarrow} X_1(z) \text{ and } x_2[n] \overset{Z}{\leftrightarrow} X_2(z) \text{ then } x[n] = x_1[n] * x_2[n] \overset{Z}{\leftrightarrow} X(z) = X_1(z)\, X_2(z)$$

Proof:

$$\begin{aligned}
X(z) &= \sum_{n=-\infty}^{\infty} x[n]\, z^{-n} \\
&= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k] \right] z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x_1[k] \left[\sum_{n=-\infty}^{\infty} x_2[n-k]\, z^{-n} \right] \\
&= \sum_{k=-\infty}^{\infty} x_1[k]\, z^{-k}\, X_2(z) \\
&= X_1(z)\, X_2(z)
\end{aligned}$$

The ROC of the convolution contains *at least as much* of the z-plane as the intersection of the ROC of $X_1(z)$ and the ROC of $X_2(z)$.

Recipe for convolution without tears:
- Compute both z-transforms
- Multiply
- Find inverse z-transform. (Hopefully already in table...)

Example. $x[n] = u[n] * u[n-1]$

$$X(z) = \frac{1}{1-z^{-1}} \cdot z^{-1} \frac{1}{1-z^{-1}} = \frac{z^{-1}}{(1-z^{-1})^2}$$

using the time-shift property. So $x[n] = n\, u[n]$ from previous example.
Contrast with continuous-time: $u(t) * u(t) = tu(t)$.

ROC for both $u[n]$ and $u[n-1]$ is $\{|z| > 1\}$. Same ROC for their convolution.

Convolution and LTI systems

If $x[n] \to \boxed{\text{LTI } h[n]} \to y[n]$, then since $y[n] = x[n] * h[n]$, $Y(z) = H(z)\, X(z)$.

Example: where ROC after convolution is larger than intersection.
$h[n] = \delta[n] - \delta[n-1]$ (discrete-time differentiator).
$x[n] = u[n-2]$ (delayed step function).
$H(z) = 1 - z^{-1}$ for $z \neq 0$.
$X(z) = \frac{z^{-2}}{1-z^{-1}}$ for $\{|z| > 1\}$. (Why?)

$y[n] = x[n] * h[n]$, so

$$Y(z) = H(z)\, X(z) = (1 - z^{-1}) \frac{z^{-2}}{1-z^{-1}} = z^{-2}$$

which has ROC $= \mathbb{C} - \{0\}$, which is "bigger" than intersection of ROC$_X$ and ROC$_H$.
What is $y[n]$? $y[n] = \delta[n-2]$.

Correlation of two sequences

If $x[n] \overset{Z}{\leftrightarrow} X(z)$ and $y[n] \overset{Z}{\leftrightarrow} Y(z)$ ***are both real*** then

$$r_{xy}[l] = \sum_{n=-\infty}^{\infty} x[n]\, y[n-l] \overset{Z}{\leftrightarrow} R_{xy}(z) = X(z)\, Y(z^{-1})$$

since $r_{xy}[l] = x[l] * y[-l]$ and by convolution and time-reversal properties.

The ROC is at least as large as the intersection of the ROC of $X(z)$ with the ROC of $Y(z^{-1})$.

Multiplication of two sequences ***(mention only)***

If $x_1[n] \overset{Z}{\leftrightarrow} X_1(z)$ and $x_2[n] \overset{Z}{\leftrightarrow} X_2(z)$ then

$$x[n] = x_1[n]\, x_2[n] \overset{Z}{\leftrightarrow} X(z) = \frac{1}{2\pi \jmath} \oint X_1(v)\, X_2^*\!\left(\frac{z}{v}\right) v^{-1}\, \mathrm{d}v$$

Read about ROC

Parseval's relation ***(mention only)***

$$\sum_{n=-\infty}^{\infty} x_1[n]\, x_2^*[n] = \frac{1}{2\pi \jmath} \oint X_1(z)\, X_2^*\!\left(\frac{1}{z^*}\right) z^{-1}\, \mathrm{d}z$$

provided that $r_{1l} r_{2l} < 1 < r_{1u} r_{2u}$

Initial value theorem

If $x[n]$ is **causal**, then

$$x[0] = \lim_{z\to\infty} X(z)$$

Proof: simple from definition: $X(z) = x[0] + x[1]\, z^{-1} + x[2]\, z^{-2} + \cdots$

Final value theorem

If $x[n]$ is causal then

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1} (z-1)\, X(z)\,.$$

The limit exists provided the ROC of $(z-1)\, X(z)$ includes the unit circle.

Comparison to Laplace properties

Compared to corresponding properties for Laplace transform, there are some missing.
Which ones?

Conjugation

$x^*[n] \overset{Z}{\leftrightarrow} X^*(z^*)$

So if $x[n]$ is real, then $X(z) = X^*(z^*)$.
(For later: If in addition, $X(z)$ is rational, then the polynomial coefficients are real.)

Laplace properties for which z-transform analogs are less obvious because time index n is an integer in DT.

Property	Continuous-Time Laplace transform	Discrete-Time z-transform
Time scaling	$f(at) \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{\lvert a \rvert} F\left(\frac{s}{a}\right),\ a \neq 0.$	?
Differentiation/difference in the time domain	$\frac{\mathrm{d}}{\mathrm{d}t} x_{\mathrm{a}}(t) \overset{\mathcal{L}}{\leftrightarrow} s\, X_{\mathrm{a}}(s)$	$x[n] - x[n-1] \overset{Z}{\leftrightarrow} (1 - z^{-1})\, X(z)$
Integration/summation in the time domain	$\int_{-\infty}^{t} x_{\mathrm{a}}(\tau)\, \mathrm{d}\tau \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{s} X_{\mathrm{a}}(s)$	$\sum_{k=-\infty}^{n} x[k] \overset{Z}{\leftrightarrow} \frac{1}{1-z^{-1}}\, X(z)$

In discrete time, the analog of **time scaling** is **up-sampling** and **down-sampling**.

Time expansion (up-sampling)

Define the M-times **upsampled** version of $x[n]$ as follows:

$$\begin{aligned} y[n] &= \begin{cases} x[n/M], & \text{if } n \text{ is a multiple of } M \\ 0, & \text{otherwise} \end{cases} \\ \text{for } M = 2: &= \{\ldots, 0, x[-2], 0, x[-1], 0, x[0], 0, x[1], 0, x[2], 0, \ldots\}. \end{aligned}$$

Then $Y(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-nM} = X\left(z^M\right)$, with $\mathrm{ROC}_Y = \left\{z \in \mathbb{C} : z^M \in \mathrm{ROC}_X\right\}$.

$$\boxed{x[n] \uparrow M \overset{Z}{\leftrightarrow} X\left(z^M\right)}$$

Example. Find z-transform of $y[n] = \left\{\underline{1}, 0, 0, 1/8, 0, 0, 1/8^2, \ldots\right\}$. The brute-force way to solve this problem is as follows:

$$Y(z) = 1 + (1/8)z^{-3} + (1/8)^2 z^{-6} + \cdots = \sum_{k=0}^{\infty} (1/8)^k z^{-3k} = \sum_{k=0}^{\infty} \left(\frac{1}{8z^3}\right)^k = \frac{1}{1 - (1/8)z^{-3}},$$

if $|(1/8)z^{-3}| < 1$ *i.e.*, $|z| > 1/2$ = ROC.

The alternative approach is to use upsampling properties. $y[n]$ is formed by upsampling by a factor of $m = 3$ the signal $x[n] = (1/8)^n\, u[n] \overset{Z}{\leftrightarrow} X(z) = \frac{1}{1-(1/8)z^{-1}}$ for ROC = $\{|z| > 1/8\}$. Thus $Y(z) = X\left(z^3\right) = \frac{1}{1-(1/8)z^{-3}}$ for ROC = $\left\{|z^3| > 1/8\right\}$.

Down-sampling

One way to "down sample" is to zero out all samples except those that are multiples of m: Define

$$\begin{aligned} y[n] &= \begin{cases} x[n], & n \text{ not a multiple of } m \\ 0, & \text{otherwise} \end{cases} \\ \text{for } m = 2: &= \{\ldots, 0, x[-4], 0, x[-2], 0, x[0], 0, x[2], 0, x[4], 0, \ldots\}. \end{aligned}$$

General case left as exercise.

Example: $m = 2$.

Trick: write $y[n] = \frac{1}{2}\left(1 + (-1)^n\right) x[n] = \frac{1}{2}\, x[n] + \frac{1}{2}(-1)^n\, x[n]\,.$
Using linearity and z-domain scaling property: $Y(z) = \frac{1}{2}\left[X(z) + X(-z)\right].$
ROC of $Y(z)$ is at least as large as ROC of $X(z)$.

Formula that is useful for such derivations:

$$\cdots + g[-2] + g[0] + g[2] + g[4] + \cdots = \sum_{n=-\infty}^{\infty} g[2n] = \sum_{m=-\infty}^{\infty} \frac{1}{2}\left(1 + (-1)^m\right) g[m]\,.$$

3.3 Rational z-transforms

All of the above examples had z-transforms that were rational functions, *i.e.*, a ratio of two polynomials in z or z^{-1}.

$$X(z) = \frac{B(z)}{A(z)} = g \frac{\prod_k (z - z_k)}{\prod_k (z - p_k)}.$$

This is a very important class (*i.e.*, for LTI systems described by difference equations).

3.3.1 Poles and zeros

- The **zeros** of a z-transform $X(z)$ are the values of z where $X(z) = 0$.
- The **poles** of a z-transform $X(z)$ are the values of z where $X(z) = \infty$. (*cf.* mesh plot of $X(z)$)

If $X(z)$ is a **rational** function, *i.e.*, a ratio of two polynomials in z, then

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_M z^{-M}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Without loss of generality, we assume $a_0 \neq 0$ and $b_0 \neq 0$, so we can rewrite

$$X(z) = \frac{b_0}{a_0} \frac{z^{-M}}{z^{-N}} \frac{z^M + \frac{b_1}{b_0} z^{M-1} + \cdots + \frac{b_M}{b_0}}{z^N + \frac{a_1}{a_0} z^{N-1} + \cdots + \frac{a_N}{a_0}} \triangleq \frac{b_0}{a_0} z^{N-M} \frac{N'(z)}{D'(z)}$$

$N'(z)$ has M finite roots at $z_1, \ldots, z_M$, and $D'(z)$ has N finite roots at $p_1, \ldots, p_N$. So we can rewrite $X(z)$:

$$X(z) = \frac{b_0}{a_0} z^{N-M} \frac{(z - z_1)(z - z_2) \cdots (z - z_M)}{(z - p_1)(z - p_2) \cdots (z - p_N)}$$

or

$$\boxed{X(z) = G z^{N-M} \frac{\prod_{k=1}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)},}$$

where $G \triangleq \frac{b_0}{a_0}$. Thus

- $X(z)$ has M finite zeros at $z_1, \ldots, z_M$
- $X(z)$ has N finite poles at $p_1, \ldots, p_N$
- If $N > M$, $X(z)$ has $N - M$ zeros at $z = 0$
- If $N < M$, $X(z)$ has $M - N$ poles at $z = 0$
- There can also be poles or zeros at $z = \infty$, depending if $X(\infty) = \infty$ or $X(\infty) = 0$
- Counting all of the above, there will be the same number of poles and zeros.

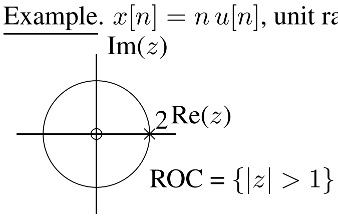
Because of the boxed form above, $X(z)$ is *completely determined by is pole-zero locations* up to the scale factor G. The scale factor only affects the *amplitude* (or units) of the signal or system, whereas the poles and zeros affect the *behavior*.

A **pole-zero plot** is a graphic description of rational $X(z)$, up to the scale factor. Use $\circ$ for zeros and $\times$ for poles. Multiple poles or zeros indicated with adjacent number.

$$\boxed{\text{By definition, the ROC will not contain any poles.}}$$

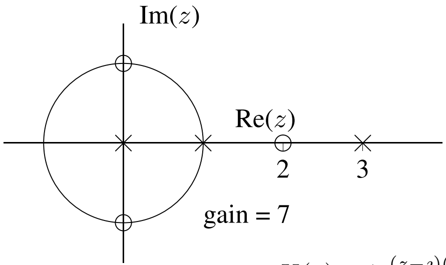
Skill: *Go from $x[n]$ to $X(z)$ to pole-zero plot.*

Example. $x[n] = n\, u[n]$, unit ramp signal. Previously showed that $X(z) = \frac{z^{-1}}{(1-z^{-1})^2} = \frac{z}{(z-1)^2}$, $\{|z| > 1\}$.



Skill: *Go from pole-zero plot to $X(z)$ to $x[n]$.*

Example. What are possible ROC's in following case? Answer: $\{|z| < 1\}$, $\{1 < |z| < 3\}$, or $\{3 < |z|\}$.



$X(z) = 7\frac{(z-\jmath)(z+\jmath)(z-2)}{(z-0)(z-1)(z-3)} = 7\frac{(1-\jmath z^{-1})(1+\jmath z^{-1})(1-2z^{-1})}{(1-z^{-1})(1-3z^{-1})}$. But what is $x[n]$? (PFE soon...)

3.3.2

Pole location and time-domain behavior for causal signals

The roots of a polynomial with real coefficients (the usual case) are either real or complex conjugate pairs. Thus we focus on these cases.

Single real pole

$$x[n] = p^n\, u[n] \overset{Z}{\leftrightarrow} X(z) = \frac{1}{1 - pz^{-1}} = \frac{z}{z - p}.$$

Fig. 3.11

- signal decays if pole is inside unit circle
- signal blows up if pole is outside unit circle
- signal alternates sign if pole is in left half plane, since $(-|p|)^n = (-1)^n |p|^n$

Double real pole

$$x[n] = np^n\, u[n] \overset{Z}{\leftrightarrow} X(z) = -z\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{1 - pz^{-1}} = \frac{pz^{-1}}{(1 - pz^{-1})^2} = \frac{pz}{(z - p)^2}$$

Fig. 3.12

Generalization to multiple real poles?

Pair of complex-conjugate poles

From *Table 3.3*:

$$a^n \sin(\omega_0 n)\, u[n] \overset{Z}{\leftrightarrow} \frac{az^{-1}\sin\omega_0}{1 - 2az^{-1}\cos\omega_0 + a^2 z^{-2}} = \frac{az\sin\omega_0}{z^2 - 2az\cos\omega_0 + a^2} = \frac{az\sin\omega_0}{(z - a\,\mathrm{e}^{\jmath\omega_0})(z - a\,\mathrm{e}^{-\jmath\omega_0})},$$

where a is assumed real. The roots of the denominator polynomial are

$$z = \frac{2a\cos\omega_0 \pm \sqrt{(2a\cos\omega_0)^2 - 4a^2}}{2} = a\cos\omega_0 \pm a\sqrt{\cos^2\omega_0 - 1} = a[\cos\omega_0 \pm \sqrt{-\sin^2\omega_0}] = a[\cos\omega_0 \pm j\sin\omega_0] = a\,\mathrm{e}^{\pm\jmath\omega_0}.$$

Thus the poles of the transform of the above signal are at $p = a\,\mathrm{e}^{\jmath\omega_0}$ and $p^* = \mathrm{e}^{-\jmath\omega_0}$.

Thus the following signal has a pair of complex-conjugate poles:

$$x[n] = a^n \sin(\omega_0 n)\, u[n] \overset{Z}{\leftrightarrow} X(z) = \frac{az\sin\omega_0}{(z - p)(z - p^*)}.$$

(Also see (3.6.43).)
Fig. 3.13

What determines the rate of oscillation? ω_0

Qualitative relationship with Laplace: $z \equiv e^{sT}$, in terms of pole-zero locations.

3.3.3

The system function of a LTI system

As noted previously: $x[n] \to \boxed{\text{LTI } h[n]} \to y[n] = x[n] * h[n] \overset{Z}{\leftrightarrow} \boxed{Y(z) = H(z)\, X(z)\,.}$

- Forward direction: transform $h[n]$ and $x[n]$, multiply, then inverse transform.
- Reverse engineering: put in known signal $x[n]$ with transform $X(z)$; observe output $y[n]$; compute transform $Y(z)$. Divide the two to get the **system function** or **transfer function** $\boxed{H(z) = Y(z) \,/\, X(z)\,.}$
  If you can choose any input $x[n]$, what would it be? Probably $x[n] = \delta[n]$ since $X(z) = 1$, so output is directly the impulse response.
- The third rearrangement $\boxed{X(z) = Y(z) \,/\, H(z)}$ is also useful sometimes.

Now apply these ideas to the analysis of LTI systems that are described by general linear constant-coefficient difference equations (LCCDE) (or just **diffeq** systems):

$$y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k\, x[n-k]\,.$$

Goal: find impulse response $h[n]$. Not simple with time-domain techniques. Systematic approach uses z-transforms.

Applying linearity and shift properties taking z-transform of both sides of the above:

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k}\, Y(z) + \sum_{k=0}^{M} b_k z^{-k}\, X(z)$$

so

$$\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] Y(z) = \left[\sum_{k=0}^{M} b_k z^{-k}\right] X(z)$$

so, defining $a_0 \triangleq 1$,

$$\boxed{H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}},}$$

What is the name for this type of system function? It is a **rational** system function. (Ratio of polynomials in z.)

Now we can see why "$-$" sign in difference equation.
We can also see why studying rational z-transforms is very important.

The system function for a LCCDE system is rational.

Skill: ***Convert between LCCDE and system function.***

What about irrational system functions? (optional reading)

Although all diffeq systems have rational z-transforms, diffeq systems are just a (particularly important) type of system within the broader family of LTI systems. There do exist (in principle at least) LTI systems that do not have rational system functions.

Example. Consider the LTI system having the impulse response $h[n] = \frac{1}{n}\, u[n]$.
The system function for this (IIR) system is $H(z) = \sum_{n=0}^{\infty} \frac{1}{n} z^{-n} = \log z^{-1} = -\log z$, which certainly is not rational.

However, this system does not have any known practical use, and would be entirely impractical to implement!

All zero system

If $N = 0$ or equivalently $a_1 = \cdots = a_N = 0$, then the system function simplifies to

$$H(z) = \sum_{k=0}^{M} b_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} b_k z^{M-k} = \frac{\prod_{k=1}^{M} (z - z_k)}{z^M}.$$

The M poles at $z = 0$ are called **trivial poles**.

Why are they called trivial poles? One reason is that they correspond only to a time shift. The other is that if a system has a pole outside the unit circle, then certain bounded inputs will produce an unbounded output (unstable). But a pole at zero does not cause this unstable behavior, so its effect is in some sense trivial.

Then there are M roots of the "numerator" polynomial that are nontrivial zeros. Thus this is called a **all-zero system**.
The impulse response is FIR:

$$h[n] = \sum_{k=0}^{N} b_k \, \delta[n - k] \, .$$

All pole system

If $M = 0$ or equivalently $b_1 = \cdots = b_M = 0$, then the system function reduces to

$$H(z) = \frac{b_0}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 z^N}{\sum_{k=0}^{N} a_k z^{N-k}} = b_0 \frac{z^N}{\prod_{k=1}^{N} (z - p_k)},$$

where $a_0 \triangleq 1$. This system function has N **trivial zeros** at $z = 0$ that are relatively unimportant, and the denominator polynomial has N roots that are the poles of $H(z)$. Thus this is called a **all-pole system**.
The impulse response is IIR.

Otherwise the impulse response is called a **pole-zero system**, and the impulse response is IIR.

Skill: ***Find impulse response*** $h[n]$ ***for rational system function*** $H(z)$**.**

Example. Find impulse response $h[n]$ for a system described by the following input-output relationship: $y[n] = -\, y[n-2] + x[n]$.

Recall that earlier we found the impulse response of $y[n] = y[n-1] + x[n]$ by a "trick."
Now we can approach such problems systematically.

Do not bother using above formulas, just use the *principle* of going to the transform domain.

Write z-transforms: $Y(z) = -z^{-2}\, Y(z) + X(z)$, so $(1 + z^{-2})\, Y(z) = X(z)$ and $H(z) = \dfrac{1}{1 + z^{-2}}$.

From *Table 3.3*: $\cos(n\pi/2)\, u[n] \overset{Z}{\leftrightarrow} \dfrac{1}{1 + z^{-2}}$, so $h[n] = \cos(n\pi/2)\, u[n] = \{\underline{1}, 0, -1, 0, \ldots\}$.

Note that there is more than one choice (causal and anti-causal) for the inverse z-transform since ROC never discussed.
Why did I choose the causal sequence? Because all LTI systems described by difference equations are **causal**.

In the above case, we could work from Table 3.3 to find $h[n]$ from $H(z)$. But what if the example were $y[n] = y[n-3] + x[n]$?
Looks simple, should be do-able. By same approach, $H(z) = \dfrac{1}{1 - z^{-3}}$, which is not in our table.
So what do we do? We need inverse z-transform method(s)!

Summary

The above concepts are very important!

3.4

Inversion of the z-transform

Skill: ***Choosing and performing simplest approach to inverting a z-transform.***

Methods for inverse z-transform
- Table lookup (already illustrated), using *properties*
- Contour integration
- Series expansion into powers of z and z^{-1}
- Partial-fraction expansion and table lookup

Practical problems requiring inverse z-transform?
- Given a system function $H(z)$, *e.g.*, described by a pole-zero plot, find $h[n]$.
  This is particularly important since we will *design* filters "in the z-domain."
- When performing convolution via z-transforms: $Y(z) = H(z)\, X(z)$, leading to $y[n]$.

3.4.1

The inverse z-transform by contour integration

$$\boxed{x[n] = \frac{1}{2\pi \jmath} \oint X(z)\, z^{n-1}\, \mathrm{d}z}$$

The integral is a **contour integral** over a **closed path** C that must
- enclose the origin,
- lie in the ROC of $X(z)$.

Typically C is just a circle centered at the origin and within the ROC.

Cauchy residue theorem. ***skip*** : ***see text***

$$\frac{1}{2\pi \jmath} \oint z^{n-1-k}\, \mathrm{d}z = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases} = \delta[n-k]\,.$$

The rest of this section might be called "how to avoid using this integral."

3.4.2

The inverse z-transform by power series expansion, aka "**coefficient matching**"

If we can expand the z-transform into a power series (considering its ROC), then "by the uniqueness of the z-transform:"

$$\boxed{\text{if } \; X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n} \text{ then } \; x[n] = c_n,}$$

i.e., the signal sample values in the time-domain are the corresponding coefficients of the power series expansion.

Example. Find impulse response $h[n]$ for system described by $y[n] = 2\,y[n-3] + x[n]$.

By the usual Y/X method, we find $H(z) = \frac{1}{1-2z^{-3}}$.
From the diffeq, we this is a causal system. Do we want an expansion in terms of powers of z or z^{-1}? We want z^{-1}.

Using geometric series: $H(z) = \frac{1}{1-2z^{-3}} = \sum_{k=0}^{\infty}(2z^{-3})^k = \sum_{k=0}^{\infty} 2^k z^{-3k} = 1 + 2z^{-3} + 2^2 z^{-6} + \ldots$.
Thus $h[n] = \{\underline{1}, 0, 0, 2, 0, 0, 4, \ldots\} = \sum_{k=0}^{\infty} 2^k\,\delta[n-3k]$.

This case was easy since the power series was just the familiar geometric series.
In general one must use tedious **long division** if the power series is not easy to find.

Very useful for checking the first few coefficients!

Example. Find the impulse response $h_2[n]$ for the system described by $y[n] = 2\,y[n-3] + x[n] + 5\,x[n-1]$.
We have

$$H_2(z) = \frac{1+5z^{-1}}{1-2z^{-3}} = \frac{1}{1-2z^{-3}} + \frac{5z^{-1}}{1-2z^{-3}} = H(z) + 5z^{-1}\,H(z) \Longrightarrow h_2[n] = h[n] + 5\,h[n-1] = \{\underline{1}, 5, 0, 2, 10, 0, 4, 20, \ldots\}.$$

Example. What if we knew we had an anti-causal system? (*e.g.*, $y[n] = 2\,y[n+3] + x[n+1]$).

Rewrite $H(z) = z/(1-2z^3) = z\sum_{k=0}^{\infty}(2z^3)^k = \sum_{k=0}^{\infty} 2^k z^{3k+1} \Longrightarrow$
$h[n] = \sum_{k=0}^{\infty} 2^k\,\delta[n+(3k+1)] = \{\ldots, 4, 0, 0, 2, 0, 0, 1, \underline{0}, 0, 0, \ldots\}$.

But we still need a systematic method for general cases.

To PFE or not to PFE?

Before delving into the PFE, it is worth noting that there are often multiple mathematically equivalent answers to discrete-time inverse z-transform problems.

Example. Find the impulse response $h[n]$ of the causal system having system function $H(z) = \dfrac{1+5z^{-1}}{1-2z^{-1}}$.

Approach 1: expand $H(z)$ into two terms and use linearity and shift properties:

$$H(z) = \frac{1}{1-2z^{-1}} + 5z^{-1}\frac{1}{1-2z^{-1}} \Longrightarrow h[n] = 2^n\,u[n] + 5 \cdot 2^{n-1}\,u[n-1].$$

Approach 2: perform "**long division**:"

$$H(z) = -\frac{5}{2} + \left[\frac{1+5z^{-1}}{1-2z^{-1}} + \frac{5}{2}\right] = \frac{5}{2} + \frac{\frac{7}{2}}{1+2z^{-1}} \Longrightarrow h[n] = -\frac{5}{2}\,\delta[n] \underbrace{+\frac{7}{2}2^n\,u[n]}_{\text{due to pole}}.$$

Which answer is correct for $h[n]$? Both!
(Equality is not immediately obvious, but one can show that they are equal using $\delta[n] = u[n] - u[n-1]$.)

However, the second form is preferable because this system has one pole, at $z = 2$, so it is preferable to use the form that has exactly one term for each pole. The asymptotic (large n) behavior is more apparent in the second form.

3.4.3

The inverse z-transform by partial-fraction expansion

General strategy: suppose we have a "complicated" z-transform $X(z)$ for which we would like to find the corresponding discrete-time signal $x[n]$. If we can express $X(z)$ as a linear combination of "simple" functions $\{X_k(z)\}$ whose inverse z-transform is known, then we can use linearity to find $x[n]$. In other words:

$$X(z) = \alpha_1 X_1(z) + \cdots + \alpha_K X_K(z) \Longrightarrow x[n] = \alpha_1 x_1[n] + \cdots + \alpha_K x_K[n] .$$

In principle one can apply this strategy to any $X(z)$. But whether "simple" $X_k(z)$'s can be found will depend on the particular form of $X(z)$.

Fortunately, for the class of **rational** z-transforms, a decomposition into simple terms is *always* possible, using the **partial-fraction expansion** (**PFE**) method.

What are the "simple forms" we will try to find? They are the "single real pole," "double real pole," and "complex conjugate pair" discussed previously, summarized below.

Type	$X(z)$	$x[n]$
polynomial in z	$\sum_k c_k z^{-k}$	$\sum_k c_k \,\delta[n-k]$
single real pole	$\dfrac{1}{1-pz^{-1}}$	$p^n\, u[n]$
double real pole	$\dfrac{pz^{-1}}{(1-pz^{-1})^2}$	$np^n\, u[n]$
double real pole	$\dfrac{1}{(1-pz^{-1})^2}$	$(n+1)p^n\, u[n]$
triple real pole	$\dfrac{1}{(1-pz^{-1})^3}$	$\dfrac{(n+2)(n+1)}{2} p^n\, u[n]$
complex conjugate pair	$\dfrac{az\sin\omega_0}{(z-a\,e^{\jmath\omega_0})(z-a\,e^{-\jmath\omega_0})}$	$a^n \sin(\omega_0 n)\, u[n]$
complex conjugate pair $p = \lvert p\rvert\, e^{\jmath\omega_0}$	$\dfrac{r}{1-pz^{-1}} + \dfrac{r^*}{1-p^* z^{-1}}$	$2\,\lvert r\rvert\, \lvert p\rvert^n \cos(\omega_0 n + \angle r)\, u[n]$

Step 1: Decompose $X(z)$ into proper form + polynomial

As usual, we assume $a_0 = 1$, without loss of generality, so we can write the rational z-transform as follows:

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}.$$

Such a rational function is called **proper** iff $a_N \neq 0$ and $M < N$. We want to work with proper rational functions.

We can always rewrite an improper rational function ($M \geq N$) as the sum of a polynomial and a proper rational function.

$$\boxed{\text{If } M \geq N, \text{ then } \frac{P_M(z^{-1})}{P_N(z^{-1})} = P_{M-N}(z^{-1}) + \frac{P_{N-1}(z^{-1})}{P_N(z^{-1})}.}$$

Example.

$$\begin{aligned}
X(z) &= \frac{1+z^{-2}}{1+2z^{-1}} = \frac{1}{2}z^{-1} + \left[\frac{1+z^{-2}}{1+2z^{-1}} - \frac{1}{2}z^{-1}\right] = \frac{1}{2}z^{-1} + \frac{1+z^{-2}-\frac{1}{2}z^{-1}\left(1+2z^{-1}\right)}{1+2z^{-1}} = \frac{1}{2}z^{-1} + \frac{1-\frac{1}{2}z^{-1}}{1+2z^{-1}} \\
&= \frac{1}{2}z^{-1} - \frac{1}{4} + \left[\frac{1-\frac{1}{2}z^{-1}}{1+2z^{-1}} + \frac{1}{4}\right] = -\frac{1}{4} + \frac{1}{2}z^{-1} + \frac{1-\frac{1}{2}z^{-1}+\frac{1}{4}\left(1+2z^{-1}\right)}{1+2z^{-1}} = -\frac{1}{4} + \frac{1}{2}z^{-1} + \frac{\frac{5}{4}}{1+2z^{-1}}.
\end{aligned}$$

In general this is always possible using long division.
The polynomial part is trivial to invert. Therefore, from now on we focus on **proper** rational functions.

Step 2: Find roots of denominator (poles)

The MATLAB `roots` command is useful here, or the quadratic formula when $N = 2$.

We call the roots $p_1, \ldots, p_N$, since these roots are the **poles** of $X(z)$.

Step 3a: PFE for distinct roots

One can use z or z^{-1} for PFE. The book chooses z. We choose z^{-1} to match MATLAB's `residuez`.

If the poles $p_1, \ldots, p_N$ are all different (distinct) then the expansion we seek has the form

$$X(z) = \frac{r_1}{1 - p_1 z^{-1}} + \cdots + \frac{r_N}{1 - p_N z^{-1}}, \tag{3-1}$$

where the r_k's are real or complex numbers called **residues**.

For distinct roots:

$$\boxed{r_k = (1 - p_k z^{-1})\, X(z) \Big|_{z=p_k}}$$

Proof:

$$(1 - p_k z^{-1})\, X(z) = (1 - p_k z^{-1}) \frac{r_1}{1 - p_1 z^{-1}} + \cdots + r_k + \cdots + (1 - p_k z^{-1}) \frac{r_N}{1 - p_N z^{-1}},$$

and evaluate the LHS and RHS at $z = p_k$.

Step 4a: inverse z-transform

Assuming $x[n]$ is causal (*i.e.*, ROC = $\{|z| > \max_k |p_k|\}$):

$$\boxed{x[n] = r_1 p_1^n\, u[n] + \cdots + r_N p_N^n\, u[n]\,.}$$

The discrete-time signal corresponding to a rational function in proper form *with distinct roots* is a weighted sum of geometric progression signals.

Complex conjugate pairs

In the usual case where the polynomial coefficients are real, any complex poles occur in conjugate pairs. Furthermore, the corresponding residues in the PFE also occur in complex-conjugate pairs.

PFE residues occur in complex-conjugate pairs for complex-conjugate roots.

skip Proof (for the distinct-root case with real coefficients):
Let p and p^* denote a complex-conjugate pair of roots. Suppose $X(z) = \frac{Y(z)}{(1-pz^{-1})(1-p^* z^{-1})}$ where $Y(z)$ is a ratio of polynomials in z with real coefficients. Then

$$\begin{aligned}
r_1 &= (1 - pz^{-1})\, X(z) \Big|_{z=p} = \frac{Y(z)}{1 - p^* z^{-1}} \Big|_{z=p} = \frac{Y(p)}{1 - p^*/p} \\
r_2 &= (1 - p^* z^{-1})\, X(z) \Big|_{z=p^*} = \frac{Y(z)}{(1 - pz^{-1})} \Big|_{z=p^*} = \frac{Y(p^*)}{1 - p/p^*} = \left[\frac{Y^*(p^*)}{1 - p^*/p}\right]^* = \left[\frac{Y(p)}{1 - p^*/p}\right]^* = r_1^*
\end{aligned}$$

since $Y^*(p) = Y(p^*)$ because $Y(z)$ has real coefficients.

Example.

$$X(z) = \frac{r}{1 - pz^{-1}} + \frac{r^*}{1 - p^* z^{-1}}$$

thus

$$x[n] = [r p^n + r^* (p^*)^n]\, u[n]\,.$$

Since this is of the form $a + a^*$, it must be real, so it is useful to express it using real quantities.

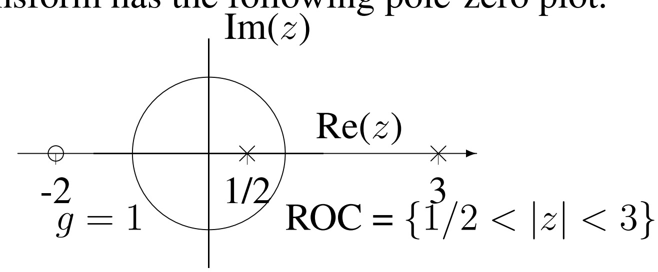
$$x[n] = 2\,\mathrm{real}(r p^n)\, u[n] = 2\,\mathrm{real}\big(|r|\, e^{\jmath\phi}\, |p|^n\, e^{\jmath\omega_0 n}\big)\, u[n] = 2\,|r|\,|p|^n \cos(\omega_0 n + \phi)\, u[n]$$

where $p = |p|\, e^{\jmath\omega_0}$ and $r = |r|\, e^{\jmath\phi}$. Note the different roles of $\angle p = \omega_0$ (frequency) and $\angle r = \phi$ (phase).

3.4.3

Example. Inverse z-transform by PFE

Find the signal $x[n]$ whose z-transform has the following pole-zero plot.



Find the formula for $X(z)$ and manipulate it (resorting to long division if necessary) to put in "proper form:"

$$
\begin{aligned}
X(z) &= \frac{z+2}{(z-3)(z-1/2)} \\
&= \frac{z^{-1}+2z^{-2}}{(1-3z^{-1})(1-\frac{1}{2}z^{-1})} \qquad \text{(negative powers of } z \text{ in denominator)} \\
&= \frac{4}{3} + \left[\frac{z^{-1}+2z^{-2}}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}} - \frac{2}{3/2}\right] \qquad \text{(avoiding long division)} \\
&= \frac{4}{3} + \frac{z^{-1}+2z^{-2}-\frac{4}{3}\left[1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}\right]}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}} \\
&= \frac{4}{3} + \frac{-\frac{4}{3}+\frac{17}{3}z^{-1}}{(1-3z^{-1})(1-\frac{1}{2}z^{-1})} \qquad \text{(proper form!)} \\
&= \frac{4}{3} + \frac{r_1}{1-3z^{-1}} + \frac{r_2}{1-\frac{1}{2}z^{-1}} \qquad \text{(PFE)}
\end{aligned}
$$

residue values:

$$
r_1 = \left.\frac{-\frac{4}{3}+\frac{17}{3}z^{-1}}{1-\frac{1}{2}z^{-1}}\right|_{z=3} = \frac{2}{3}, \qquad r_2 = \left.\frac{-\frac{4}{3}+\frac{17}{3}z^{-1}}{1-3z^{-1}}\right|_{z=1/2} = -2
$$

$$
X(z) = \frac{4}{3} + \frac{\frac{2}{3}}{1-3z^{-1}} + \frac{-2}{1-\frac{1}{2}z^{-1}} \qquad \text{(could multiply out to check).}
$$

Considering the ROC, we conclude

$$
x[n] = \frac{4}{3}\delta[n] \underbrace{-\frac{2}{3}3^n u[-n-1]}_{\text{anti-causal}} -2\left(\frac{1}{2}\right)^n u[n].
$$

MATLAB approach: `[r p k] = residuez([0 1 2], [1 -7/2 3/2])`
returns (in decimals): `r = [2/3 -2], p = [3 1/2], k = 4/3.`

General PFE formula for single poles, for proper form[1] with $M < N$:

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots b_M z^{-M}}{\prod_{k=1}^{N}(1 - p_k z^{-1})} = \frac{r_1}{1 - p_1 z^{-1}} + \cdots + \frac{r_N}{1 - p_N z^{-1}}$$

where the **residue** is given by:

$$\boxed{r_j = (1 - p_k z^{-1})\, X(z) \Big|_{z=p_k} = \left.\frac{B(z)}{\prod_{l \neq k}(1 - p_l z^{-1})}\right|_{z=p_k}.}$$

If p_k is a **repeated (2nd-order) pole**:

$$\boxed{\begin{aligned} X(z) &= \cdots + \frac{r_{k,1}}{1 - p_k z^{-1}} + \frac{r_{k,2}}{(1 - p_k z^{-1})^2} + \cdots \\ r_{k,1} &= \frac{1}{-p_k} \frac{\mathrm{d}}{\mathrm{d}z^{-1}} (1 - p_k z^{-1})^2 X(z) \Big|_{z=p_k} \\ r_{k,2} &= (1 - p_k z^{-1})^2 X(z) \Big|_{z=p_k}. \end{aligned}}$$

In general, if p_k is an L**th-order repeated pole**, then

$$X(z) = \cdots + \sum_{l=1}^{L} \frac{r_{k,l}}{(1 - p_k z^{-1})^l} + \cdots$$

where

$$r_{k,l} = \frac{1}{(L-l)!\,(-p_k)^{(L-l)}} \frac{d^{L-l}}{d(z^{-1})^{L-l}} (1 - p_k z^{-1})^L X(z) \Big|_{z=p_k}, \quad l = 1, \ldots, L.$$

Rarely would one do this by hand for $L > 2$. Use `residuez` instead!

Fact. For real signals, any complex poles appear in complex conjugate pairs,
and the corresponding residues come in complex conjugate pairs:

$$X(z) = \cdots + \frac{r}{1 - p z^{-1}} + \frac{r^*}{1 - p^* z^{-1}} + \cdots$$

Letting $p = |p|\, \mathrm{e}^{\jmath\omega_0}$ and $r = |r|\, \mathrm{e}^{\jmath\phi}$ (note the difference in meaning of the angles!):

$$\begin{aligned} x[n] &= r p^n\, u[n] + r^* (p^*)^n\, u[n] \\ &= |r|\, \mathrm{e}^{\jmath\phi} (|p|\, \mathrm{e}^{\jmath\omega_0})^n\, u[n] + |r|\, \mathrm{e}^{-\jmath\phi} (|p|\, \mathrm{e}^{-\jmath\omega_0})^n\, u[n] \\ &= |r| \left(|p|^n\, \mathrm{e}^{\jmath\phi}\, \mathrm{e}^{\jmath\omega_0 n} + \mathrm{e}^{-\jmath\phi}\, \mathrm{e}^{-\jmath\omega_0 n}\right) u[n] \\ &= 2\, |r|\, |p|^n \cos(\omega_0 n + \phi)\, u[n]\,. \end{aligned}$$

[1] If not in proper form, then first do long division.

Example. Finding the impulse response of a diffeq system.

Find the impulse response of the system described by the following diffeq:

$$y[n] = \frac{4}{3}\,y[n-1] - \frac{7}{12}\,y[n-2] + \frac{1}{12}\,y[n-3] + x[n] - x[n-3]\,.$$

Step 0: Find the system function. ________________________ (linearity, shift property)

$$Y(z) = \frac{4}{3}z^{-1}\,Y(z) - \frac{7}{12}z^{-2}\,Y(z) + \frac{1}{12}z^{-3}\,Y(z) + X(z) - z^{-3}\,X(z)$$

$$\left[1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}\right] Y(z) = \left[1 - z^{-3}\right] X(z)$$

so (by convolution property):

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-3}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}}.$$

Step 1: Decompose system function into proper form + polynomial. ________________

In this case we can see by comparing the coefficients of the z^{-3} terms that the coefficient for the 0th-order term will be 1/(1/12) = 12.

$$H(z) = 12 + \left[\frac{1 - z^{-3}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}} - 12\right] = 12 + P(z)$$

where

$$\begin{aligned} P(z) &= \frac{1 - z^{-3}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}} - 12 \\ &= \frac{1 - z^{-3} - 12\left[1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}\right]}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}} \\ &= \frac{-11 + 16z^{-1} - 7z^{-2}}{1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3}}. \end{aligned}$$

Note that $P(z)$ is a proper rational function.
Since $H(z) = 12 + P(z)$, we see that $h[n] = 12\,\delta[n] + p[n]$.
We now focus on finding $p[n]$ from $P(z)$ by PFE.

Step 2: Find poles (roots of denominator).

The MATLAB command `roots([1 -4/3 7/12 -1/12])` returns `0.5 0.5 0.33`, so we check and verify that the denominator can be factored:

$$1 - \frac{4}{3}z^{-1} + \frac{7}{12}z^{-2} - \frac{1}{12}z^{-3} = \left(1 - \frac{1}{2}z^{-1}\right)^2 \left(1 - \frac{1}{3}z^{-1}\right),$$

so in factored form:

$$P(z) = \frac{-11 + 16z^{-1} - 7z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2 \left(1 - \frac{1}{3}z^{-1}\right)}.$$

Step 3: Find PFE

Since there is one repeated root, the PFE form is

$$P(z) = \frac{r_{1,1}}{1 - \frac{1}{2}z^{-1}} + \frac{r_{1,2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2} + \frac{r_2}{1 - \frac{1}{3}z^{-1}}. \qquad \text{(3-2)}$$

For a single pole at $z = p_k$, we find the residue using this formula:

$$r_k = (1 - p_k z^{-1})\, P(z) \Big|_{z=p_k}.$$

Thus for the single pole at $z = 1/3$:

$$r_2 = (1 - \frac{1}{3}z^{-1})\, P(z)\Big|_{z=1/3} = \frac{-11 + 16z^{-1} - 7z^{-2}}{\left(1 - \frac{1}{2}z^{-1}\right)^2}\Bigg|_{z=1/3} = -104.$$

For a double pole at $z = p_k$, the residues are given by

$$r_{k,1} = \frac{1}{-p_k}\frac{\mathrm{d}}{\mathrm{d}z^{-1}}(1 - p_k z^{-1})^2\, P(z)\Big|_{z=p_k}, \quad \text{and} \quad r_{k,2} = (1 - p_k z^{-1})^2\, P(z)\Big|_{z=p_k}.$$

Thus for the double pole at $z = 1/2$:

$$\begin{aligned} r_{1,1} &= \frac{1}{-1/2}\,\frac{d}{dz^{-1}}(1 - \frac{1}{2}z^{-1})^2\, P(z)\Big|_{z=1/2} = -2\,\frac{d}{dz^{-1}}\frac{-11 + 16z^{-1} - 7z^{-2}}{1 - \frac{1}{3}z^{-1}}\Bigg|_{z=1/2} \\ &= -2\,\frac{(1 - \frac{1}{3}z^{-1})(16 - 14z^{-1}) - (-11 + 16z^{-1} - 7z^{-2})(-\frac{1}{3})}{(1 - \frac{1}{3}z^{-1})^2}\Bigg|_{z=1/2} = 114, \end{aligned}$$

and

$$r_{1,2} = (1 - \frac{1}{2}z^{-1})^2\, P(z)\Big|_{z=1/2} = \frac{-11 + 16z^{-1} - 7z^{-2}}{(1 - \frac{1}{3}z^{-1})}\Bigg|_{z=1/2} = -21.$$

Substituting in these residues into equation (3-2):

$$P(z) = \frac{114}{1 - \frac{1}{2}z^{-1}} + \frac{-21}{(1 - \frac{1}{2}z^{-1})^2} + \frac{-104}{1 - \frac{1}{3}z^{-1}}.$$

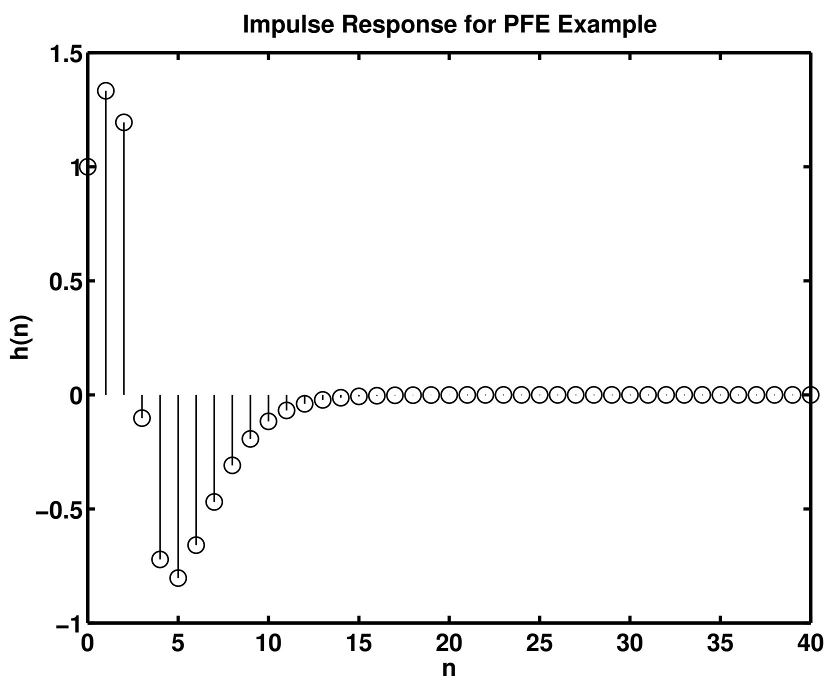
Step 4: Inverse z-transform (using table lookup)

$$p[n] = 114\left(\frac{1}{2}\right)^n u[n] - 21(n+1)\left(\frac{1}{2}\right)^n u[n] - 104\left(\frac{1}{3}\right)^n u[n] \,.$$

Substituting into proper form decomposition above yields our final answer:

$$\boxed{h[n] = 12\,\delta[n] + \left[(114 - 21(n+1))\left(\frac{1}{2}\right)^n - 104\left(\frac{1}{3}\right)^n\right] u[n] \,.}$$

The Resulting Impulse Response



Sanity check: $h[0] = 1$, as it should because for this system $y[0] = x[0]$ for a causal input.

Using MATLAB for PFE

Most of the above work is built into the following MATLAB command:

```
[r p k] = residuez([1 0 0 -1], [1 -4/3 7/12 -1/12])
```

which returns

- `r = [114 -21 -104]` (residues)
- `p = [0.5 0.5 0.3333]` (poles)
- `k = [12]` (direct terms)

Furthermore, using MATLAB's `impz` command, one can compute values of $h[n]$ directly from $\{b_k\}$ and $\{a_k\}$ (but it does not provide a *formula* for $h[n]$).

skim

3.4.4
Decomposition of rational z-transforms

If $a_0 = 1$ then

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = b_0 \frac{\prod_{k=1}^{M} (1 - z_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})}.$$

Product form.

Combine complex-conjugate pairs

$b_{1k} = -2\,\mathrm{real}(z_k)$

$b_{2k} = |z_k|^2$

$a_{1k} = -2\,\mathrm{real}(p_k)$

$a_{2k} = |p_k|^2$

$$X(z) = b_0 \frac{\prod_{k=?}^{?} (1 - z_k z^{-1})}{\prod_{k=?}^{?} (1 - p_k z^{-1})} \frac{\prod_{k=?}^{?} (1 + b_{1k} z^{-1} + b_{2k} z^{-k})}{\prod_{k=?}^{?} (1 + a_{1k} z^{-1} + a_{2k} z^{-k})}.$$

useful for implementing, see Ch 7, 8

just skim for now!

3.5
The One-Sided z-transform

skim

Useful for analyzing response of non-relaxed systems.

Definition:

$$X^+(z) \triangleq \sum_{n=0}^{\infty} x[n]\, z^{-n}$$

3.5.1

3.5.2 **Solution of difference equations with nonzero initial conditions**

3.6

Analysis of LTI Systems in the z-domain

Main goals:
- Characterize response to inputs.
- Characterize system properties (stability, causality, etc.) in z-domain.

3.6.1

Response of systems with rational system functions

$X(z) \to \boxed{H(z)} \to Y(z)$. Goal: characterize $y[n]$

Assume
- $H(z)$ is a pole-zero system, *i.e.*, $H(z) = B(z) \,/\, A(z)$.
- Input signal has a rational z-transform of the form $X(z) = N(z)/Q(z)$.

Then
$$Y(z) = H(z)\,X(z) = \frac{B(z)\,N(z)}{A(z)\,Q(z)}.$$

So the output signal also has a rational z-transform.

How do we find $y[n]$? Since $Y(z)$ is rational, we use PFE to find $y[n]$.

Assume
- Poles of system $p_1, \ldots, p_N$ are unique
- Poles of input signal $q_1, \ldots, q_L$ are unique
- Poles of system and input signal are all different
- Zeros of system and input signal differ from all poles (so no pole-zero cancellation)
- Proper form
- Causal input sequence and causal LTI system

Then
$$X(z) = \sum_{k=1}^{L} \frac{\alpha_k}{1 - q_k z^{-1}} \xrightarrow{\mathcal{T}} Y(z) = \sum_{k=1}^{N} \frac{r_k}{1 - p_k z^{-1}} + \sum_{k=1}^{L} \frac{s_k}{1 - q_k z^{-1}}$$

so (assuming a causal system) the response is:
$$y[n] = \underbrace{\sum_{k=1}^{N} r_k p_k^n \, u[n]}_{\text{natural}} + \underbrace{\sum_{k=1}^{L} s_k q_k^n \, u[n]}_{\text{forced}}.$$

The output signal for a causal pole-zero system with input signal having rational z-transform is a weighted combination of geometric progression signals.

If there are repeated poles, then of course the PFE has terms of the form $np^n \, u[n]$ etc.

The output signal has two parts
- The p_k terms are the **natural response** $y_{\mathrm{nr}}[n]$ of the system. (The input signal affects only the residues r_k).
  Each term of the form $p_k^n \, u[n]$ is called a **mode** of the system.
- The q_k terms are the **forced response** $y_{\mathrm{fr}}[n]$ of the system. (The system affects "only" the residues s_k.)

Transient response from pole-zero plot

What about systems that are not necessarily in proper form?
There may be additional $k_l \, \delta[n-l]$ terms in the impulse response.

From the pole-zero plot corresponding to $H(z)$, we can identify how many $k_l \, \delta[n-l]$ terms will occur in the impulse response.
For causal systems:
- If there are one or more zeros at $z = 0$, then there will be no $\delta[n]$ terms in $h[n]$.
- If there are no poles or zeros at $z = 0$, then there will be one term of the form $k_0 \, \delta[n]$ in the the impulse response.

- If there are $N_1 \geq 1$ poles at $z = 0$, then $h[n]$ will include $N_1 + 1$ terms of the form $k_l\, \delta[n - l]$.

For IIR filters, the δ terms are less important than the terms in the impulse response (and in the transient response) that correspond to nonzero poles.

3.6.2

Response of pole-zero systems with nonzero initial conditions

skim

3.6.3

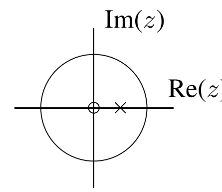
Transient and steady-state response

Define $y_{\mathrm{nr}}[n]$ to be the natural response of the system, *i.e.*, $y_{\mathrm{nr}}[n] = \sum_{k=1}^{N} r_k p_k^n \, u[n]$.

- If all the poles have magnitude less than unity, then this response decays to zero as $n \to \infty$.
- In such cases we also call the natural response the **transient response**.
- Smaller magnitude poles lead to faster signal decay. So *the closer the pole is to the unit circle, the longer the transient response*.

The forced response has the form $y_{\mathrm{fr}}[n] = \sum_{k=1}^{L} s_k q_k^n \, u[n]$.

- If all of the input signal poles are within the unit circle, then the forced response will decay towards zero as $n \to \infty$.
- If the input signal has a pole *on the unit circle* then there is a persistent sinusoidal component of the input signal. The forced response to such a sinusoid is also a persistent sinusoid.
- In this case, the forced response is also called the **steady-state** response.



Example. System (initially relaxed) described by diffeq: $y[n] = \frac{1}{2}\, y[n-1] + x[n]$.

What are the poles of the system? At $p = 0.5$. $H(z) = \dfrac{1}{1 - \frac{1}{2} z^{-1}}$.

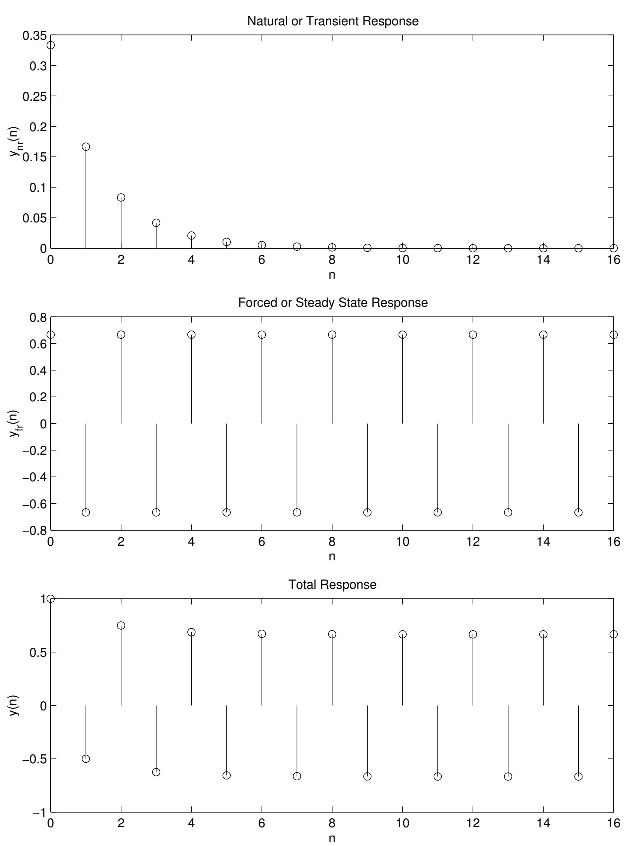
Signal: $x[n] = (-1)^n \, u[n]$. Pole at $q = -1$. $X(z) = \dfrac{1}{1 + z^{-1}}$.

$$Y(z) = H(z)\, X(z) = \frac{1}{(1 - \frac{1}{2} z^{-1})(1 + z^{-1})} = \frac{1}{1 + \frac{1}{2} z^{-1} - \frac{1}{2} z^{-2}} = \frac{1/3}{1 - \frac{1}{2} z^{-1}} + \frac{2/3}{1 + z^{-1}}$$

where I found the PFE using `[r p k] = residuez(1, [1 1/2 -1/2])`. So

$$y[n] = \underbrace{\frac{1}{3}\left(\frac{1}{2}\right)^n u[n]}_{\text{natural / transient}} + \underbrace{\frac{2}{3}(-1)^n \, u[n]}_{\text{forced / steady state}} .$$

Where did 2/3 come from? $H(-1) = 2/3$.



Geometric progression signals are "almost" eigenfunctions of LTI systems

Fact: the forced response of an LTI system with rational system function $H(z)$ that is driven by a geometric progression input signal $x[n] = q^n \, u[n]$ is that same geometric progression scaled by $H(q)$, *i.e.*,

$$x[n] = q^n \, u[n] \xrightarrow{\mathcal{T}} y[n] = y_{\mathrm{nr}}[n] + H(q) \, q^n \, u[n],$$

if no poles in system at $z = q$.

$$Y(z) = H(z) \, X(z) = H(z) \, \frac{1}{1 - qz^{-1}} = \frac{B(z)}{A(z)(1 - qz^{-1})} = \frac{P(z)}{A(z)} + \frac{r}{1 - qz^{-1}}$$

by PFE if no roots of $A(z)$ at $z = q$.

Residue:

$$r = (1 - qz^{-1}) \, Y(z)\big|_{z=q} = (1 - qz^{-1}) \, H(z) \, \frac{1}{1 - qz^{-1}} \bigg|_{z=q} = H(q)$$

so

$$y[n] = y_{\mathrm{nr}}[n] + H(q) \, q^n \, u[n] \, .$$

In particular, if $q = \mathrm{e}^{\jmath\omega_0}$, then the input signal is a **causal sinusoid**, and the forced response is a steady-state response. And if the LTI system is stable, then it has no poles on the unit circle, so the condition that $A(z)$ have no roots at $z = q$ is satisfied. So the steady-state response is

$$y_{\mathrm{fr}}[n] = H(\mathrm{e}^{\jmath\omega_0}) \, \mathrm{e}^{\jmath\omega_0 n} \, u[n] = |H(\mathrm{e}^{\jmath\omega_0})| \, \mathrm{e}^{\jmath(\omega_0 n + \angle H(\mathrm{e}^{\jmath\omega_0}))} \, u[n]$$

which is a causal sinusoidal signal.

Thus the interpretation of $H(\mathrm{e}^{\jmath\omega_0})$ as a **frequency response** is entirely appropriate, even in the case of non-eternal sinusoidal signals!

Note that if the system is stable, then the poles are inside the unit circle so the natural response will be a transient response in this case, so eventually the output just looks essentially like the steady-state sinusoidal response.

3.6.4

Causality and stability

We previously described six system properties: **linearity**, **invertibility**, **stability**, **causality**, **memory**, **time-invariance**.

- We first described these properties in general.
- We then characterized these properties in terms of the impulse response $h[n]$ of an LTI system,
  because any LTI system is described completely by its impulse response $h[n]$.
  - **causality**: $h[n] = 0 \; \forall \, n < 0$.
  - **stability**: $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.
- Now we characterize these properties in the z-domain.
  If it exists, the system function $H(z)$ (including its ROC) also describes completely an LTI system, since we can find $h[n]$ from $H(z)$, *i.e.*, we can determine the output $y[n]$ for any input signal $x[n]$ if we know $H(z)$ and its ROC.

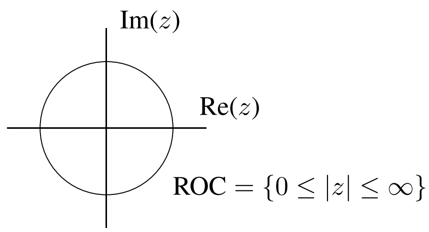
Skill: ***Examine conditions for causality, stability, invertibility, memory in the z-domain.***

Memory

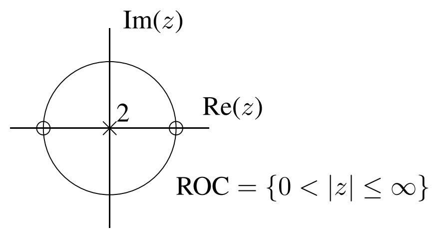
What is the system function and ROC of a memoryless system?

An LTI system is memoryless iff $h[n] = b_0 \, \delta[n]$. So $H(z) = b_0$. So $H(z)$ has no poles or zeros, and ROC $= \mathbb{C}$.

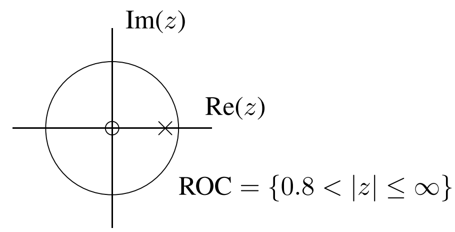
In terms of dynamic systems, recall that previously we noted that FIR systems are "all zero" systems (poles at origin only).



Memoryless



FIR



IIR

Causality

Previous time-domain condition for causality: LTI system is causal iff its impulse response $h[n]$ is 0 for $n < 0$.

How can we express this in the z-domain?
We showed earlier that the ROC of the z-transform of a right-sided signal is the exterior of a circle.
But is ROC = "exterior of a circle" enough? No!

Example. $h[n] = u[n+1] \overset{Z}{\leftrightarrow} \dfrac{z}{1-z^{-1}} = \dfrac{z^2}{z-1}$ for $\{1 < |z| < \infty\}$.
The ROC is a circle's exterior, and $h[n]$ is right-sided, but the system is *not* causal.

For a causal system, the system function (assuming it exists) has a series expansion that involves only *non-positive* powers of z:

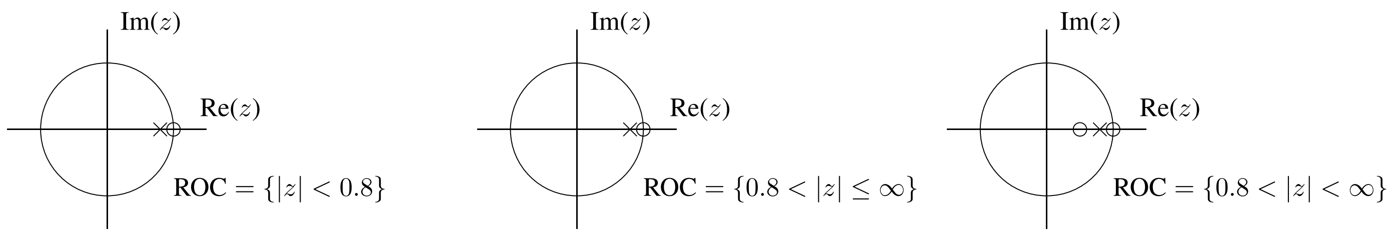
$$H(z) = \sum_{n=0}^{\infty} h[n]\, z^{-n} = h[0] + h[1]\, z^{-1} + h[2]\, z^{-2} + \cdots .$$

So the ROC of such an $H(z)$ will include $|z| = \infty$. (In fact, $\lim_{z\to\infty} H(z) = h[0]$, which must be finite.)

An LTI system with impulse response $h[n]$ is **causal** iff the ROC of the system function is the exterior of a circle of radius $r < \infty$ *including* $z = \infty$, *i.e.*, ROC = $\{r < |z| \leq \infty\}$, or, in the trivial case of a memoryless system, ROC = $\{0 \leq |z| \leq \infty\}$.

Example. (***skip***) Is the LTI system with system function $H(z) = z^2 - z^{-1}$ causal? The ROC is $\mathbb{C} - \{\infty\} - \{0\}$, which is the exterior of a circle of radius 0, excluding ∞. Thus noncausal, which we knew since $h[n] = \delta[n+2]$.

Example. Which of the following pole-zero plots correspond to causal systems?



Only the middle one. For the right one $H(z) = g\frac{(z-1)(z-1/2)}{z-0.8}$ which is infinite at $z = \infty$. It is noncausal.

A given pole-zero plot for a rational system function corresponds to a causal LTI system iff there are at least as many (finite) poles as (finite) zeros and the ROC is the exterior of the circle intersecting the outermost pole.

Stability

Recall time-domain condition for stability: an LTI system is BIBO stable iff $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$.
How to express in the z-domain?
Recall definition of the ROC of a system function:

$$z \in \text{ROC iff } \{h[n]\, z^{-n}\} \text{ is absolutely summable, } \textit{i.e.}, S(z) = \sum_{n=-\infty}^{\infty} |h[n]|\, |z|^{-n} < \infty.$$

- Suppose system is stable. What can we say about ROC?
  If the system is stable, then on the unit circle, where $|z| = 1$, we see $S(z) < \infty$.
  Thus BIBO stable system $\Longrightarrow$ ROC includes unit circle.
- Conversely, if the ROC includes the unit circle, then it includes the point $z = 1$, so $S(1) < \infty$, which implies $\sum_{n=-\infty}^{\infty} |h[n]| < \infty$ so the system is BIBO stable.

$$\boxed{\text{An LTI system is BIBO stable iff the ROC of its system function includes the unit circle.}}$$

Example. Suppose an LTI system has a pole on the unit circle at $z = e^{\jmath\omega_0}$. If we apply the bounded input $e^{\jmath\omega_0 n}\, u[n]$, then the steady state response (see 3.6.6 below) will include a term like $n\, e^{\jmath\omega_0 n}\, u[n]$, which is unbounded.

So poles on the unit circle preclude stability.

Example. $y[n] = -\,y[n-1] + x[n] \Longrightarrow H(z) = \frac{1}{1+z^{-1}} = \frac{z}{z+1}$ which has a pole at $z = -1$ so this system is unstable.

In general causality and stability are unrelated properties.

However, for a causal system we can narrow the condition for stability.

For a causal system, the ROC is the exterior of a circle. For it to be stable as well, the ROC must include the unit circle, so the radius r for the ROC must be less than 1. There cannot be any poles in the ROC, so all the poles must be inside (or on the boundary of) the circle of radius $r < 1$, which are thus inside the unit circle.

$$\boxed{\text{A causal LTI system is BIBO stable iff all of the poles of its system function are \textit{inside} the unit circle.}}$$

Example. Accumulator: $y[n] = y[n-1] + x[n]$ has $H(z) = \frac{1}{1-z^{-1}}$. Stable? No: causal but pole at $z = 1$ so unstable.

Recall earlier pictures showing that causal signals with poles outside unit circle are blowing up.

Intuition: signals with poles on the unit circle are the most "persistent" of the bounded signals, since they are oscillatory with no decay. So for the system to have bounded output for such bounded input signals, its ROC must include the unit circle.

skip *3.6.6* Multiple-order poles and stability

Can poles of system function lie on the unit circle and still have the system be stable? No.

Example. Consider $h[n] = u[n]$, so $H(z) = 1/(1 - z^{-1})$, which has a pole at $z = 1$. Now consider the input $x[n] = u[n]$, which is certainly a bounded input. The output $y[n] = (n+1)\, u[n]$, as we derived long ago. So the output is not bounded.

This can happen anywhere on the unit circle.
Therefore for a causal system to be stable, all the poles of its system function must lie strictly inside the unit circle.

3.6.5 Pole-zero cancellations

When a system has a pole and a zero at *exactly* the same location, they cancel each other out.

Example. Is the system $y[n] = 3\,y[n-1] + x[n]$ stable? No, since it hsa a pole at $z = 3$.

Example. Find system function and pole-zero plot and assess stability for diffeq system $y[n] = 3\,y[n-1] + \frac{1}{3}\,x[n] - x[n-1]$.
Since $[1 - 3z^{-1}]\,Y(z) = [\frac{1}{3} - z^{-1}]\,X(z)$, the system function is $H(z) = Y(z)\,/\,X(z) = \frac{\frac{1}{3}-z^{-1}}{1-3z^{-1}} = \frac{1}{3}$ and $h[n] = \frac{1}{3}\,\delta[n]$.

The pole and zero at $z = 3$ cancel, so yes, theoretically this is a stable LTI system.

picture of direct form I implementation $H_1(z) = \frac{1}{3} - z^{-1}$, $H_2(z) = \frac{1}{1-3z^{-1}}$.

In practice there may be imperfect pole-zero cancellation. For example, in binary representation,

$$1/3 = .010101\ldots = \sum_{k=0}^{\infty} 2^{-(2k+1)} = 1/4 + 1/16 + 1/64 + \cdots$$

which cannot be represented exactly with a finite number of bits. With 8 bits (.01010101), we get 0.333251953125 not $1/3$.

Invertibility

In time domain, an LTI system with impulse response $h[n]$ is invertible iff there exists an LTI system having some impulse response $h_I[n]$ that satisfies: $h[n] * h_I[n] = \delta[n]$.

In z-domain: $H(z)\,H_I(z) = 1$, so $H_I(z) = \frac{1}{H(z)}$.

Example. $H(z) = \frac{7}{5}\frac{z-2}{z-1/2} \Longrightarrow H_I(z) = \frac{5}{7}\frac{z-1/2}{z-2}$.

So the poles becomes zeros and the zeros become poles.

Thus, in principle, any LTI system with rational system function is invertible.

However, in practice usually we want a *stable* inverse.

A causal, stable LTI system has a causal stable inverse iff all of its poles and zeros are within the unit circle.

3.6.7
The Schur-Cohn stability test ***skip***

We now have two valid procedures for checking stability of *causal* LTI systems:
- Check if $\sum_{n=0}^{\infty} |h[n]| < \infty$.
- Check if poles of system lie inside unit circle.

To perform either one of these checks, generally one needs a concrete expression for $h[n]$ or for $H(z)$.

For a rational system function $H(z) = B(z)\,/\,A(z)$, the poles are the roots of the denominator polynomial: $A(z) = 1 + a_1 z^{-1} + \cdots\, a_N z^{-N}$. Given concrete numerical values for the a_k coefficients, the usual approach to testing stability would be to just use the MATLAB `roots` function and check the magnitudes of the roots.

But in the design process, often we have ranges of possible values for the coefficients, and we cannot check all of them using a numerical root-finding routine. And for degrees greater than 2, there is no simple method for analytically finding the roots.

The Schur-Cohn test provides a method for verifying stability of discrete-time LTI systems having rational system functions *without explicitly finding the roots* of the denominator polynomial. This is important practically since generally we want stable systems.

This test is the analog of the **Routh-Hurwitz criterion** used for testing stability of continuous-time systems.

The Schur-Cohn Stability Test

The Schur-Cohn test provides a method for verifying stability of LTI systems with rational system functions *without explicitly finding the roots* of the denominator polynomial. This is very important practically since generally we want stable systems.

Procedure.

- Initialization: $A_N(z) = A(z) = \sum_{k=0}^{N} a_k z^{-k}$, $a_N(k) = a_k$
- Define: $A_m(z) = \sum_{k=0}^{m} a_m(k) z^{-k}$, where $a_m(0) = 1$
- Define: $B_m(z) = z^{-m} A_m(z^{-1}) = \sum_{k=0}^{m} a_m(m-k) z^{-k}$.
  This is called the **reverse polynomial**, since order of coefficients are reversed.
- Define: $K_m = a_m(m),\ m = 1, \ldots, N$
- Recursion: $\boxed{A_{m-1}(z) = \dfrac{A_m(z) - K_m\, B_m(z)}{1 - K_m^2}}$ for $m = N, N-1, \ldots, 1$
- Test: The roots of $A(z)$ are all inside the unit circle iff $\boxed{|K_m| < 1}$ for $m = 1, 2, \ldots, N$.

The following second-order analysis serves as an "example."

3.6.8 Stability of second-order systems

For first-order systems $y[n] = a\, y[n-1] + x[n]$, stability is trivial: check if $|a| < 1$.

Next interesting case is second-order systems:

$$y[n] = -a_1\, y[n-1] - a_2\, y[n-2] + b_0\, x[n] \Longrightarrow H(z) = \frac{b_0}{1 + a_1 z^{-1} + a_2 z^{-2}}\ .$$

Question. What values of a_1 and a_2 lead to a stable system?

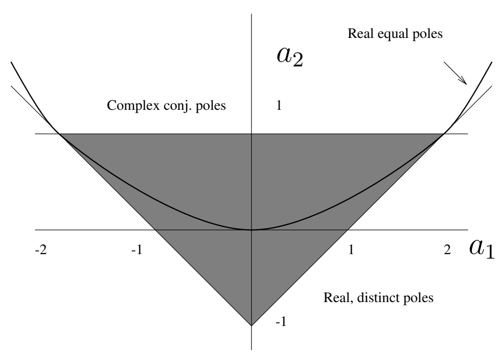
In this 2nd order case we could determine the roots using the quadratic formula.
That is not feasible for $N > 2$, so we use the Schur-Cohn method as an example.

$A_2(z) = 1 + a_1 z^{-1} + a_2 z^{-2}$ so $K_2 = a_2(2) = a_2$

$$A_1(z) = \frac{A_2(z) - K_2\, B_2(z)}{1 - K_2^2} = \frac{1 + a_1 z^{-1} + a_2 z^{-2} - a_2(a_2 + a_1 z^{-1} + z^{-2})}{1 - a_2^2} = \frac{1 - a_2^2 + a_1(1 - a_2) z^{-1}}{1 - a_2^2} = 1 + \frac{a_1}{1 + a_2} z^{-1},$$

so $K_1 = \dfrac{a_1}{1 + a_2}$. Thus $H(z)$ is stable iff $\boxed{|a_2| < 1}$ and $\boxed{\left|\dfrac{a_1}{1 + a_2}\right| < 1}$ or $|1 + a_2| > |a_1|$.

When $|a_2| < 1$, $|1 + a_2| = 1 + a_2$, so we need $1 + a_2 > |a_1|$, *i.e.*, $-(1 + a_2) < a_1 < 1 + a_2$.



Restricting our designs to coefficients in that triangle will ensure stability, without explicitly finding the roots.

In this 2nd order case the roots are given by the quadratic formula: $p = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2 - 4a_2}{4}}$

- Real and equal poles when $a_1^2 = 4a_2$, *i.e.*, on the parabola $a_2 = a_1^2/4$ that touches corners of triangle.
- Real and distinct poles when $a_1^2 > 4a_2$, which is below parabola.
- Complex poles otherwise, above parabola.

The book derives the corresponding impulse response for each case.

3.7

Summary

- z-transform and its properties
- convolution property for z-domain convolution
- system function of LTI systems
- finding impulse response of diffeq system having rational system function
- characterizing properties of output signals (forced, natural, transient, steady-state response)
- characterizing system properties (causality and stability) in z-domain

We now have many representations of systems:

- time domain:
  - block diagram
  - impulse response
  - difference equation
- transform domain:
  - system function
  - pole-zero plot
  - frequency response (soon)

Skill: ***Convert between these six system representations.*** (See diagram.)

- Use z for going between $H(z)$ and pole-zero plot.
- Use z^{-1} for PFE and for finding diffeq coefficients.

Where is 2D and image processing examples? Although 2D z-transform's have been studied, *e.g.*, [3], they are not particularly useful in image processing, especially compared to the Fourier transform. In contrast, the 1D z-transform is the foundation for 1D filter design.

Bibliography

[1] M. Rosenlicht. *Introduction to analysis*. Dover, New York, 1985.

[2] H. L. Royden. *Real analysis*. Macmillan, New York, 3 edition, 1988.

[3] J. S. Lim. *Two-dimensional signal and image processing*. Prentice-Hall, New York, 1990.

[4] A. K. Jain. *Fundamentals of digital image processing*. Prentice-Hall, New Jersey, 1989.

Discrete-time systems described by difference equations (FIR and IIR)

Difference equation:

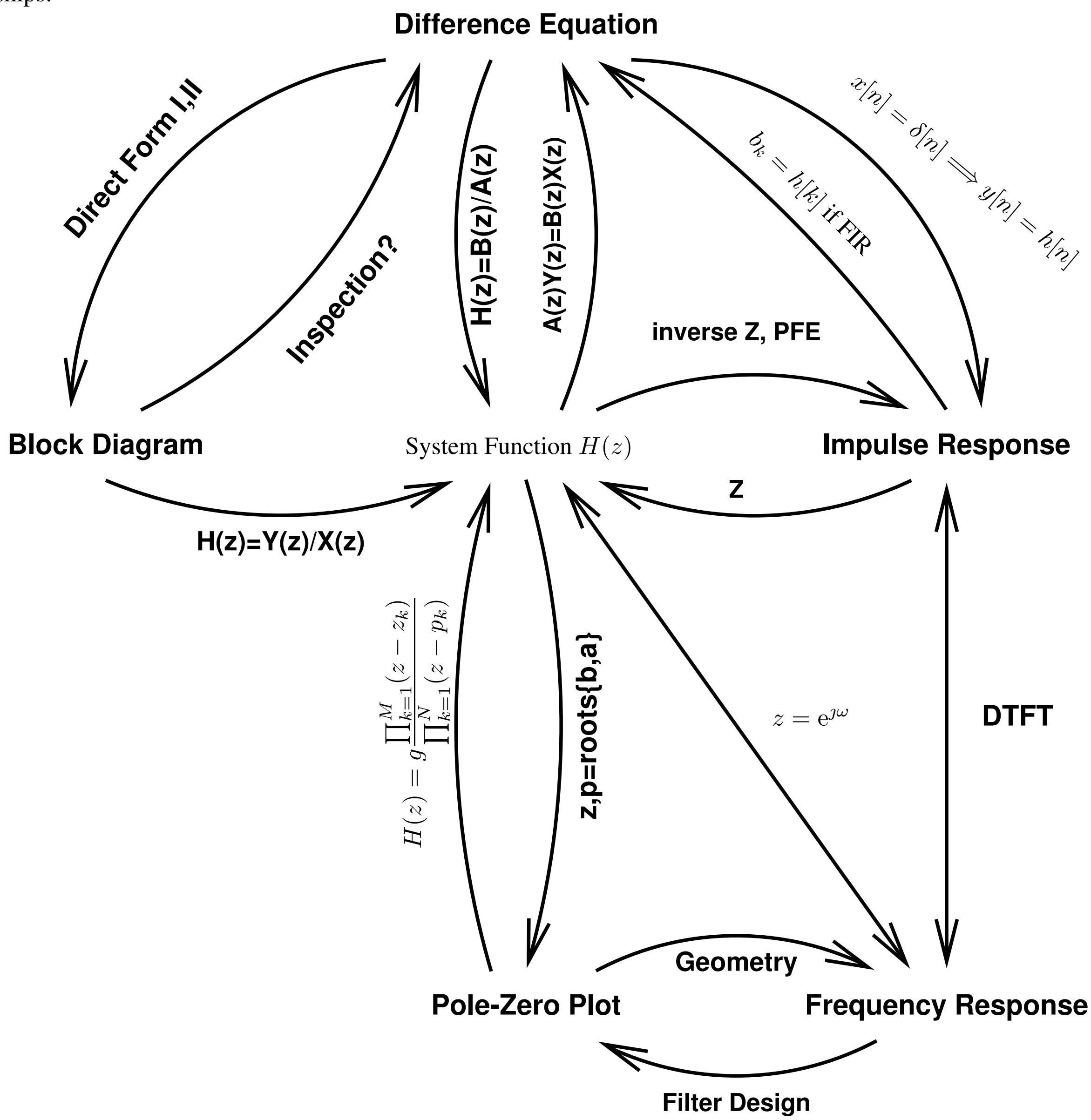
$$y[n] = -\sum_{k=1}^{N} a_k\, y[n-k] + \sum_{k=0}^{M} b_k\, x[n-k]$$

System function (in expanded polynomial and in factored polynomial forms):

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=1}^{N} a_k z^{-k}} = b_0 z^{N-M} \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

Frequency magnitude response: $|\mathcal{H}(\omega)| = b_0 \frac{\prod_k |e^{\jmath\omega} - z_k|}{\prod_k |e^{\jmath\omega} - p_k|}$

Relationships:



Each representation corresponds to a type of input/output relationship, *e.g.*, convolution.