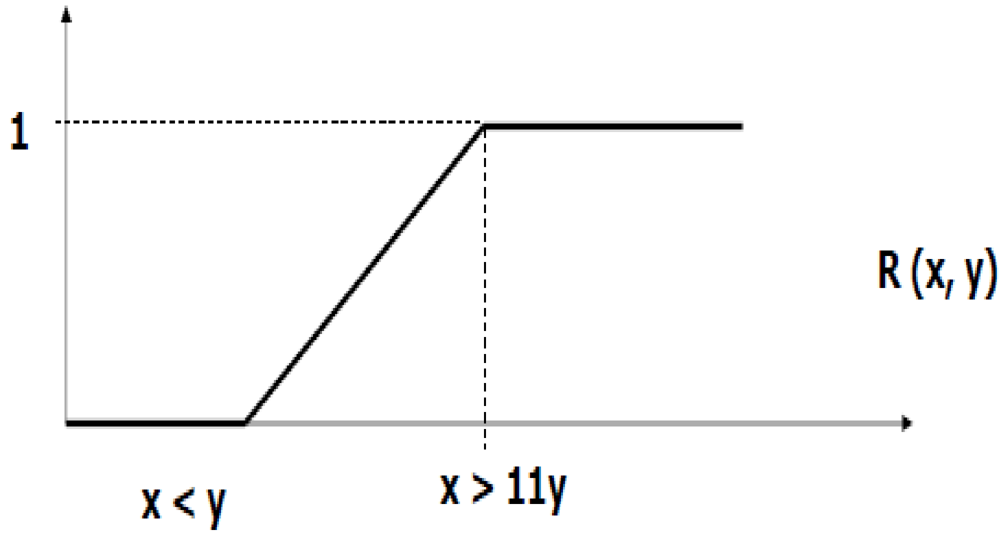
Fuzzy Relation

Definition

- *Fuzzy relations are mapping elements of one universe, to those of another universe, Y, through the Cartesian product of two universes. X, Universe X = {1, 2, 3}*
- *R (X, Y)= {[(x, y), μ_R (x, y)] | (x, y) ∈ (X × Y)}*
- *Where the fuzzy relation R has membership function*
- *μ_R (x, y) = $\mu_{A \times B}$ (x, y) = min (μ_A (x), μ_B (y))*
- *It represents the strength of association between elements of the two sets*
- *Ex: R = "x is considerably larger than y"*
- *R (X, Y)= Relation between sets X and Y*
- *R (x, y) = memebership function for the relation R (X, Y)*
- *R (X, Y) = {R (x, y) / (x, y) | (x, y) ∈ (X × Y)}*

- $$R(x, y) = \begin{cases} 0 & \text{for } x \le y \\ \{x - y)/(10-y), & \text{for } y < x \le 11y \\ 1 & \text{fro } x > 11y \end{cases}$$



Cartesian Product

- *Let $A_1, A_2, \ldots, A_n$ be fuzzy sets in $U_1, U_2, \ldots U_n$, respectively.*

  The Cartesian product of $A_1, A_2, \ldots, A_n$ is a fuzzy set in the space $U_1 \times U_2 \times \ldots \times U_n$ with the membership function as:

  $\mu_{A1 \times A2 \times \ldots \times An}(x_1, x_2, \ldots, x_n) = \min[\mu_{A1}(x_1), \mu_{A2}(x_2), \ldots \mu_{An}(x_n)]$
- *So, the Cartesian product of $A_1, A_2, \ldots, A_n$ are donated by $A_1 \times A_2 \times \ldots \times A_n$*

Crisp Relations

- *The relation between any two sets is the Cartesian product of the elements of $A_1 \times A_2 \times \ldots \times A_n$*
- *For X and Y universes* $X \times Y = \{(x, y) \mid x \in X, y \in Y\}$
- $$\mu_{x \times y}(x, y) = \begin{cases} 1, & (x, y) \in X \times Y \\ 0, & (x, y) \notin X \times Y \end{cases}$$
- *This relation can be represented in a matrix format*

Cartesian Product: Example

- *Let A =* $\{(3, 0.5), (5, 1), (7, 0.6)\}$
- *Let B =* $\{(3, 1), (5, 0.6)\}$
- Find the product
- *The product is all set of pairs from A and B with the minimum associated memberships*
- *Ax B =* $\{[(3, 3), \min(0.5, 1)], [(5, 3), \min(1, 1)], [(7, 3), \min(0.6, 1)], [(3, 5), \min(0.5, 0.6)], [(5, 5), \min(1, 0.6)], [(7, 5), \min(0.6, 0.6)]\}$

  $= \{[(3, 3), 0.5], [(5, 3), 1], [(7, 3), 0.6], [(3, 5), 0.5], [(5, 5), 0.6], [(7, 5), 0.6]\}$

Operations on Fuzzy Relations

- *Since the fuzzy relation from X to Y is a fuzzy set in X × Y, then the operations on fuzzy sets can be extended to fuzzy relations. Let R and S be fuzzy relations on the Cartesian space X × Y then:*
- *Union:* $\mu_{R \cup S}(x, y) = \max[\mu_R(x, y), \mu_S(x, y)]$
- *Intersection:* $\mu_{R \cap S}(x, y) = \min[\mu_R(x, y), \mu_S(x, y)]$
- *Complement:* $\mu_{\bar{R}}(x, y) = 1 - \mu_R(x, y)$

- *Assume two Universes:* A = {3, 4, 5} and B = {3, 4, 5, 6, 7}
- $\mu_R(x, y) = \begin{cases} (y-x)/(y+x+2) & \text{if } y > x \\ 0, & \text{if } y \le x \end{cases}$
- *This can be expressed as follow:*

$$R = \begin{array}{c} 3 \\ 4 \\ 5 \\ \\ \end{array} \begin{array}{c} \begin{bmatrix} 0 & 0.11 & 0.2 & 0.27 & 0.33 \\ 0 & 0 & 0.09 & 0.17 & 0.23 \\ 0 & 0 & 0 & 0.08 & 0.14 \end{bmatrix} \\ \begin{array}{ccccc} 3 & 4 & 5 & 6 & 7 \end{array} \end{array}$$

- *This matrix represents the membership grades between elements in X and Y*
- $\mu_R(x, y) = \{[0/(3, 3)], [0.11/(3, 4)], [0.2/(3, 5)], \ldots\ldots\ldots\ldots\ldots\ldots, [0.14/(5, 7)]\}$

Fuzzy Relations: Example

- Assume two fuzzy sets: $A = \{0.2/x_1 + 0.5/x_2 + 1/x_3\}$

  $B = \{0.3/y_1 + 0.9/y_2\}$

- Find the fuzzy relation (the Cartesian product)

$$A \times B = R = \begin{array}{c} x_1 \\ x_2 \\ x_3 \\ \ \end{array} \begin{array}{c} \begin{pmatrix} 0.2 & 0.2 \\ 0.3 & 0.5 \\ 0.3 & 0.9 \end{pmatrix} \\ \begin{matrix} y_1 & y_2 \end{matrix} \end{array}$$

- Consider the fuzzy relation. Express R using the resolution principle

$$R = \begin{pmatrix} 0.4 & 0.5 & 0 \\ 0.9 & 0.5 & 0 \\ 0 & 0 & 0.3 \\ 0.3 & 0.9 & 0.4 \end{pmatrix}$$

- $R = R_{0.3} + R_{0.4} + R_{0.5} + R_{0.9}$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Composition of Fuzzy Relations

- *Composition of fuzzy relations used to combine fuzzy relations on different product spaces*
- *Having a fuzzy relation; $R (X \times Y)$ and $S (Y \times Z)$, then Composition is used to determine a relation $T (X \times Z)$,*

- *Consider two fuzzy relation; $R (X \times Y)$ and $S (Y \times Z)$, then a relation $T (X \times Z)$, can be expressed as (max-min composition)*

$$T = R \circ S$$

$$\mu_T(x, z) = \text{max-min}\,[\mu_R(x, y), \mu_S(y, z)]$$

$$= \vee\,[\mu_R(x, y) \wedge \mu_S(y, z)]$$

- *If algebraic product is adopted, then max-product composition is adopted:*

$$T = R \circ S$$

$$\mu_T(x, z) = \max\,[\mu_R(x, y) \,.\, \mu_S(y, z)]$$

$$= \vee\,[\mu_R(x, y) \,.\, \mu_S(y, z)]$$

- *The max-min composition can be interpreted as indicating the strength of the existence of relation between the elements of X and Z*
- *Calculations of (R o S) is almost similar to matrix multiplication*
- *Fuzzy relations composition have the same properties of:*

Distributivity: R o (S U T) = (R o S) U (R o T)

Associativity: R o (S o T) = (R o S) o T

- *Assume the following universes: $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$, and $Z = \{z_1, z_2, z_3\}$, with the following fuzzy relations.*

$$R = \begin{array}{c|cc} & y_1 & y_2 \\ \hline x_1 & 0.7 & 0.5 \\ x_2 & 0.8 & 0.4 \end{array} \quad \text{and} \quad S = \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline y_1 & 0.9 & 0.6 & 0.2 \\ y_2 & 0.1 & 0.7 & 0.5 \end{array}$$

- *Find the fuzzy relation between X and Z using the max-min and max-product composition*

- ***By max-min composition***

$$\mu_T(x_1, z_1) = \max[\min(0.7, 0.9), \min(0.5, 0.1)] = 0.7$$

$$T = \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline x_1 & 0.7 & 0.6 & 0.5 \\ x_2 & 0.8 & 0.6 & 0.4 \end{array}$$

- ***By max-product composition***

$$\mu_T(x_2, z_2) = \max[(0.8, 0.6), (0.4, 0.7)] = 0.48$$

$$T = \begin{array}{c|ccc} & z_1 & z_2 & z_3 \\ \hline x_1 & 0.63 & 0.42 & 0.25 \\ x_2 & 0.72 & 0.48 & 0.20 \end{array}$$

Let us consider two kinds of troubles a PC may suffer from, viz., the *system hangs while running*, and *the system does not boot*. We symbolize the former by h and the later by b and define the set $A = \{h, b\}$ of PC troubles. Two possible causes of these troubles are *computer virus* (v) and *disc crash*(c) and they form the set $B = \{c, v\}$ of PC trouble makers. And finally, let the sources of the causes mentioned above are *internet* (i) and *obsolescence* (o) and $C = \{i, o\}$ is the set of PC trouble causes. The relation between PC troubles and their causes is expressed by R, a fuzzy relation over $A \times B$. Similarly, S is the fuzzy relation over $B \times C$, i.e., the relation between the causes of troubles and the sources of those causes. The relations R and S in terms of their relation matrices are shown below.

$$R = \begin{array}{c} \\ h \\ b \end{array} \begin{array}{c} \begin{array}{cc} v & c \end{array} \\ \begin{bmatrix} 0.7 & 0.2 \\ 0.5 & 0.8 \end{bmatrix} \end{array}, \qquad S = \begin{array}{c} \\ v \\ c \end{array} \begin{array}{c} \begin{array}{cc} i & o \end{array} \\ \begin{bmatrix} 0.9 & 0.7 \\ 0.1 & 0.2 \end{bmatrix} \end{array}$$

The relation between PC troubles and their ultimate sources, *i.e.*, between A and C, can be computed on the basis of R and S above as the max–min composition $R \circ S$. The first element of $R \circ S$, expressed as $(R \circ S)(h, i)$ is computed as follows.

$$(R \circ S)(h, i) = max\ \{min\ (R\ (h, v), S\ (v, i)), min\ (R\ (h, c), S\ (c, i))\}$$
$$= max\ \{min\ (0.7, 0.9), min\ (0.2, 0.1)\}$$
$$= max\ \{0.7, 0.1\}$$
$$= 0.7$$

The rest of the elements of $R \circ S$ can be found in a similar fashion.

$(R \circ S)(h, o) = 0.7$
$(R \circ S)(b, i) = 0.5$
$(R \circ S)(b, o) = 0.5$

And finally we get,

$$R \bullet S = \begin{array}{c c} & \begin{array}{cc} i & o \end{array} \\ \begin{array}{c} h \\ b \end{array} & \begin{bmatrix} 0.7 & 0.7 \\ 0.5 & 0.5 \end{bmatrix} \end{array}$$

Let A = {Mimi, Bob, Kitty, Jina} be a set of four children, B = {Tintin, Asterix, Phantom, Mickey} be a set of four comic characters, and C = {funny, cute, dreamy} be a set of three attributes. The fuzzy relations $R = x$ Likes y is defined on $A \times B$ and $S = x$ IS y is defined on $B \times C$ as shown in Table 2.20 and Table 2.21. Find R ° S.

Table 2.20. Relation matrix for $R = x$ Likes y

	$R \equiv$ *Likes*			
	Tintin	Asterix	Phantom	Mickey
Mimi	0.8	0.5	0.7	0.8
Bob	0.4	0.9	0.3	0.3
Kitty	0.6	0.7	0.4	0.9
Jina	0.3	0.8	0.2	0.5

Table 2.21. Relation matrix for $S = x$ IS y

	$S \equiv$ *IS*		
	funny	cute	dreamy
Tintin	0.6	0.7	0.3
Asterix	0.8	0.4	0.2
Phantom	0.1	0.2	0.1
Mickey	0.9	0.8	0.3

Table 2.22. Relation matrix for $R \circ S$

	ROS		
	funny	cute	dreamy
Mimi	0.8	0.8	0.3
Bob	0.8	0.4	0.3
Kitty	0.9	0.8	0.3
Jina	0.8	0.5	0.3

It will be Continued....