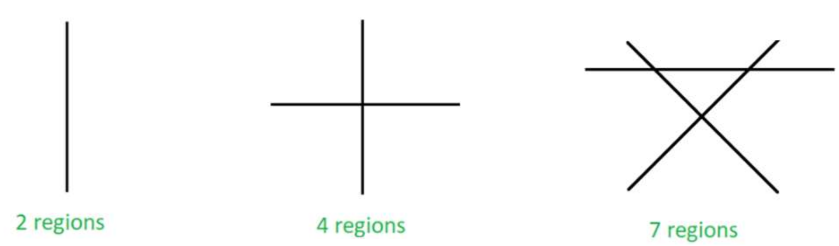
Module 3: Using RR for Problem Solving

1

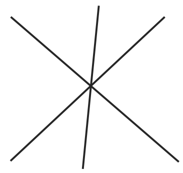
Lines in a Plane and Regions



- The above images show the maximum number of regions a line can divide a plane.
- One line can divide a plane into two regions…
- Two non-parallel lines can divide a plane into 4 regions and,
- Three non-parallel lines can divide into 7 regions, and so on.

2

Lines in General Position

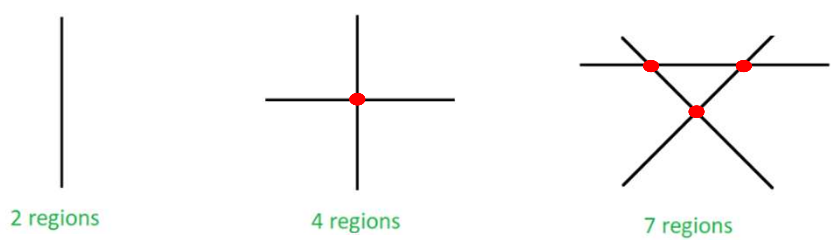


Concurrent lines

- A collection of n lines in the plane is said to be in general position if no two lines are parallel and no three are concurrent.
- When two or more lines pass through a single point, in a plane, they are concurrent with each other and are called concurrent lines.
- A point that is common to all those lines is called the point of concurrency.

3

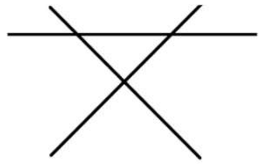
Lines in a Plane and Regions



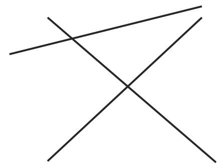
- New lines interact with the existing lines to create new regions.
- These interactions can be visualized by noticing that every time a line crosses another, a new 'intersection point' is formed, leading to new regions.
- For instance, when a third line crosses the first two, it not only creates its own separate spaces but also splits the existing regions in half at the points of intersection.

4

Lines in a Plane and Regions



7 regions

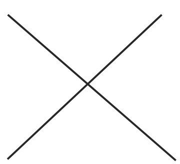


6 regions

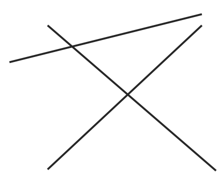
- Each time a line is added, it adds a region for each region it passes through, i.e., it divides the region in two.
- When the n^{th} line is added to a cluster of (n-1) lines, then the maximum number of extra regions formed is equal to n.
- The number of regions through which a line passes is equal to the number of lines crossed plus one.

5

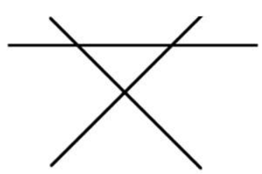
Lines in a Plane and Regions



4 regions



6 regions



7 regions

- Each time the line crosses another line, it creates a point of intersection, so we could count the number of lines crossed plus the number of lines added.

6

Lines in a Plane and Regions

- Let a_n be the number of regions into which n lines in general position divide the plane. How big can be a_n?
- We find that when the n^{th} line is added to a cluster of (n-1) lines, the maximum number of extra regions formed is equal to n.
- Therefore, the recurrence relation would be

  $$a_n = a_{n-1} + n \text{ for } n \geq 2, a_1 = 2$$
- Subsequently, we solve this simple RR to find an explicit formula

  $$a_n = n(n+1)/2 + 1.$$
- This formula gives the maximum number of regions formed by n lines in a plane.

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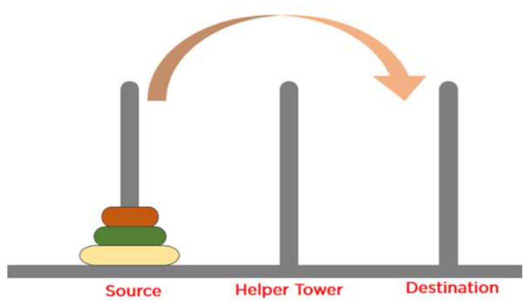
Tower of Hanoi



The objective is to transfer the entire disks to the Destination Peg with the following conditions (Lucas's rules):

1. move only one disk at a time, and
2. never move a larger one onto a smaller one.

Now the question is: How many moves are necessary and sufficient to perform the task?

 Figure Courtesy: Simplilearn Solutions

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Tower of Hanoi

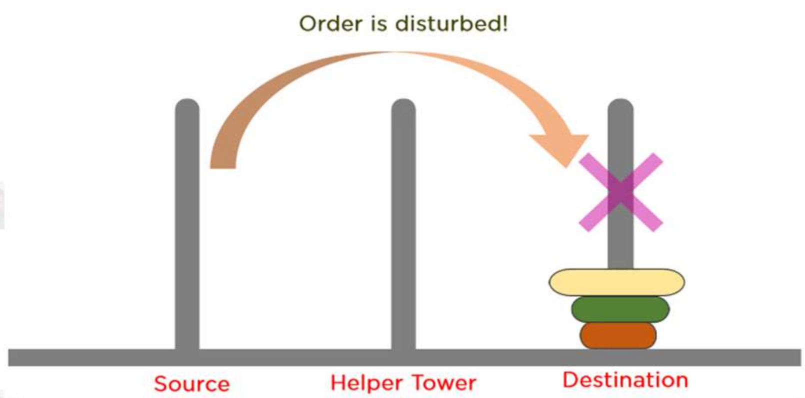

Figure Courtesy: Simplilearn Solutions

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Tower of Hanoi

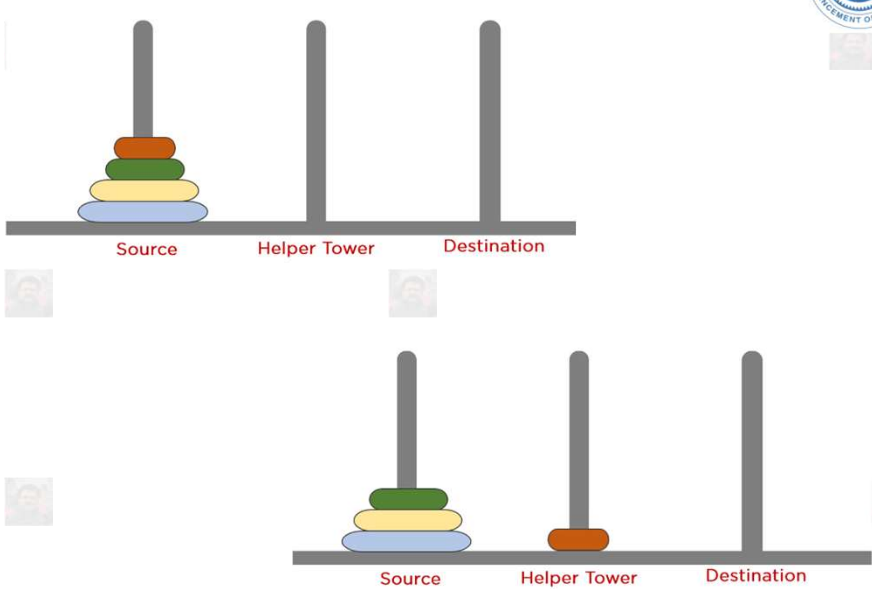

Figure Courtesy: Simplilearn Solutions

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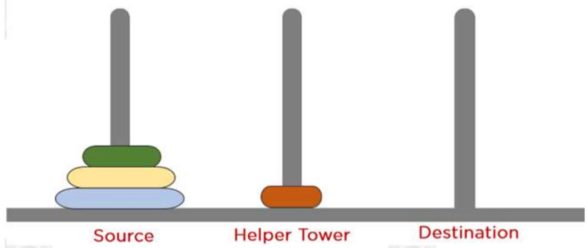
Tower of Hanoi



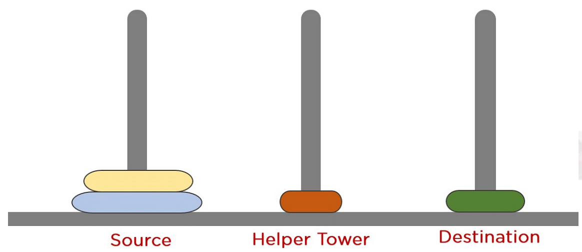


Figure Courtesy: Simplilearn Solutions

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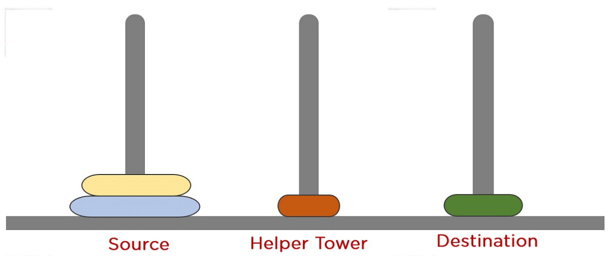
Tower of Hanoi



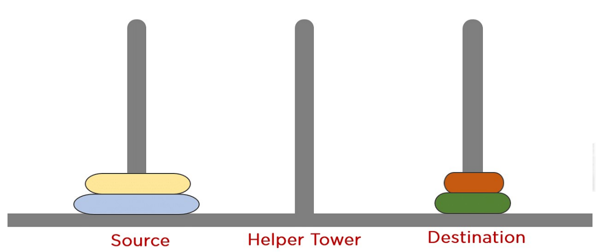


Figure Courtesy: Simplilearn Solutions

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Tower of Hanoi

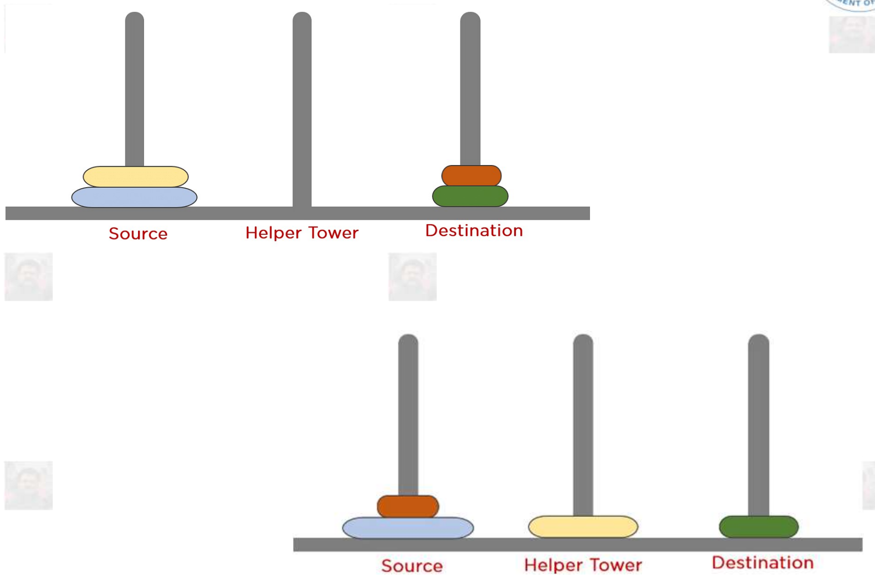

 Figure Courtesy: Simplilearn Solutions

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Tower of Hanoi

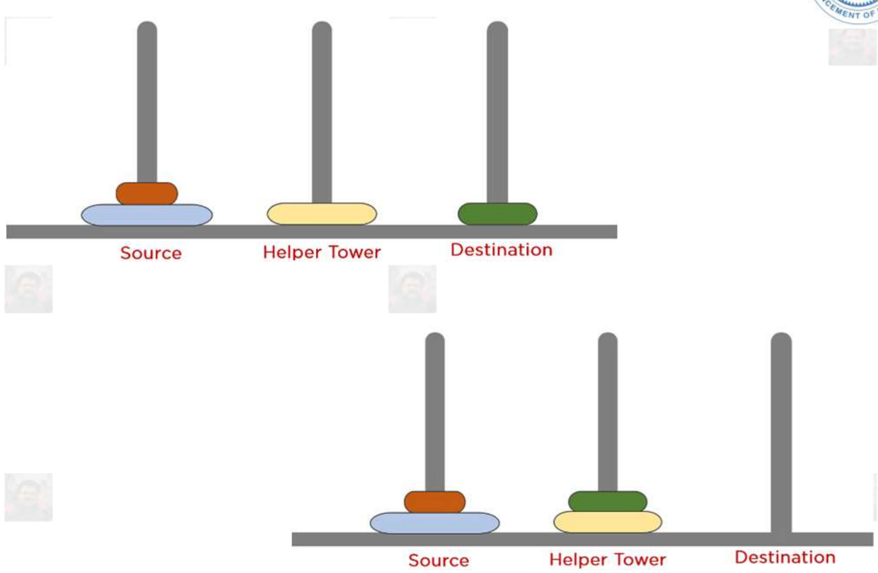

 Figure Courtesy: Simplilearn Solutions

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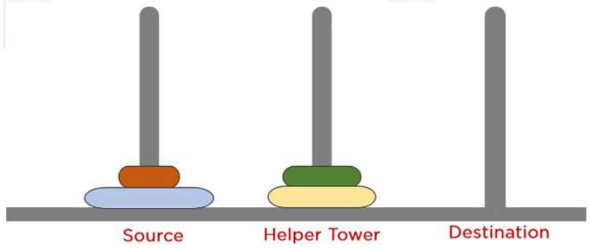
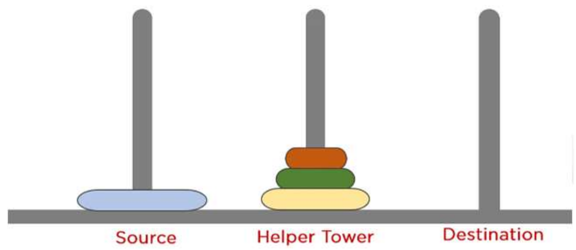
Tower of Hanoi





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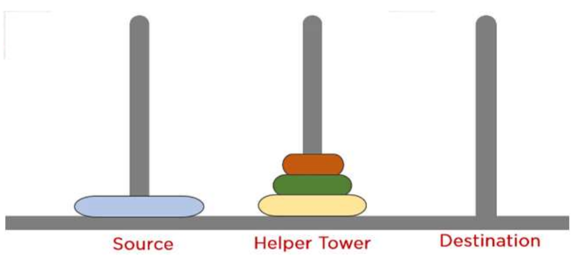
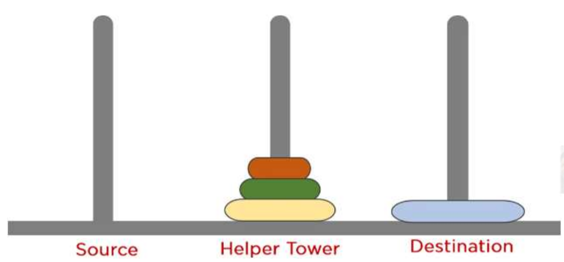
Tower of Hanoi





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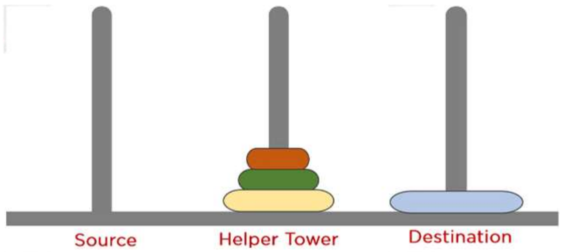
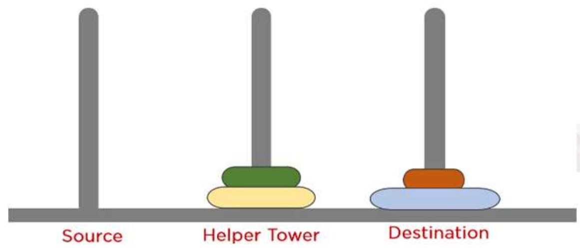
Tower of Hanoi





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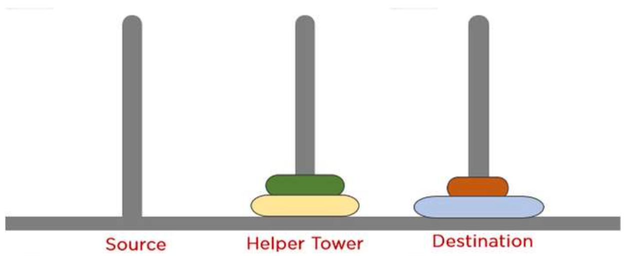
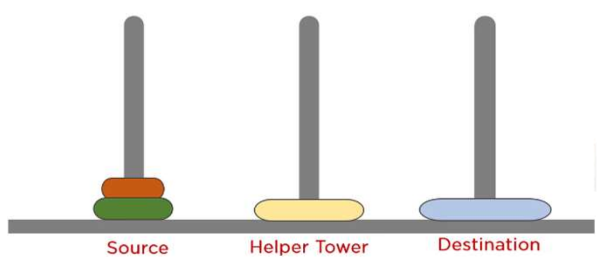
Tower of Hanoi





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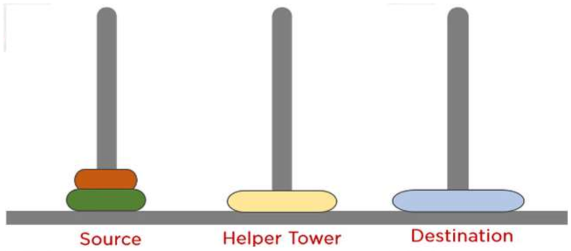
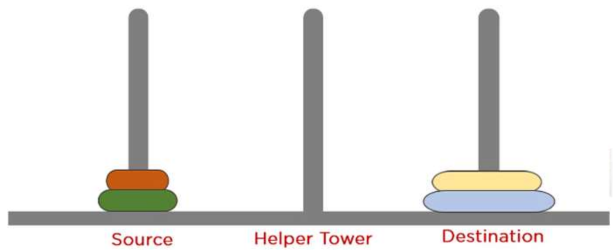
Tower of Hanoi





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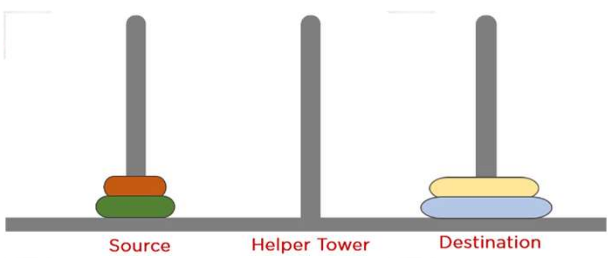
Tower of Hanoi



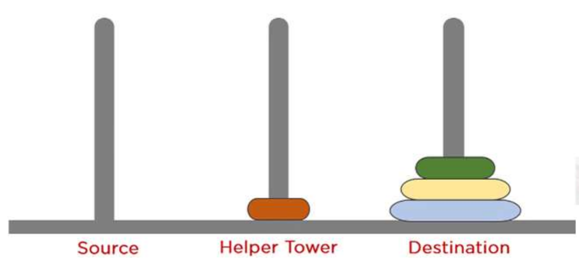


Figure Courtesy: Simplilearn Solutions

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Tower of Hanoi

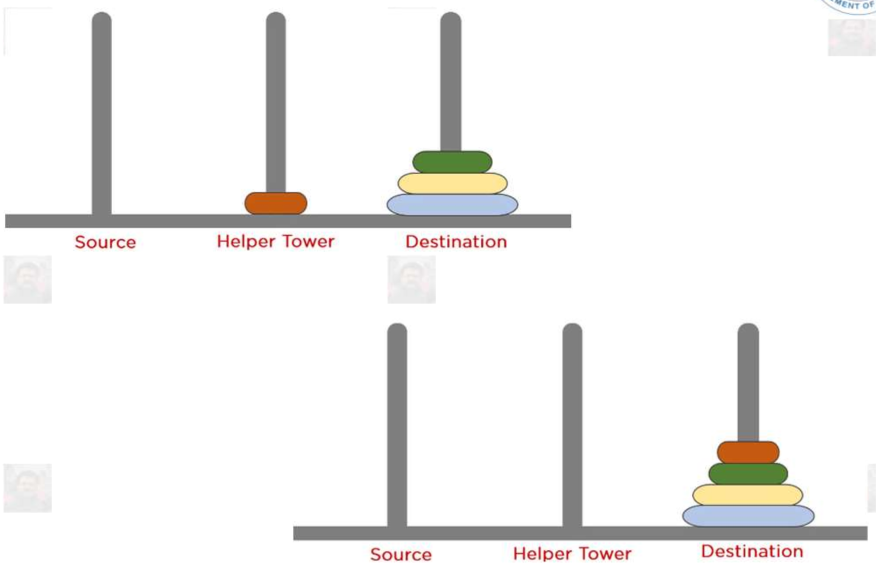

 Figure Courtesy: Simplilearn Solutions

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How can we transfer a tower of n disks?

- We discuss the case of transferring a tower of 3 disks:
  1. transfer the top 2 disks to the middle peg (3 moves)
  2. move the largest disk to the third peg (1 move)
  3. transfer other two disks to the largest disk (3 moves).
- This gives us a clue for transferring n disks in general:
  1. transfer the top *n−1* smaller disks to the middle peg (T_{n-1} moves)
  2. move the largest disk to the third peg (1 move)
  3. transfer the $n - 1$ smallest onto the largest disk (T_{n-1} moves).

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How many moves do we need?

- We can transfer n disks ($n > 0$) in at most $2T_{n-1} + 1$ moves
  - Hence, $T_n \leq 2T_{n-1} + 1$, for $n > 0$ ...i
- That is, $2T_{n-1} + 1$ moves suffice. Are $2T_{n-1} + 1$ moves necessary?
- At some point, we must move the largest disk to the third peg. When we do so, the $n - 1$ smallest must be on the middle peg, and it has taken at least T_{n-1} moves.
- At least 1 move is required to move the largest disk.
- After moving the largest disk, we again require at least T_{n-1} moves.
  - Hence, $T_n \geq 2T_{n-1} + 1$, for $n > 0$ ...ii

Recurrence Relation for Tower of Hanoi Problem

- From the equations i) and ii), together with the trivial solution for $n = 0$, yield
  - $T_0 = 0$ (boundary value)
  - $T_n = 2T_{n-1} + 1$ for $n > 0$.
- The above set of equalities is called a recurrence (recursion) relation.
- Note that the general value T_n is in terms of earlier ones T_{n-1}.

Tower of Brahma

- Legend has it that there is a temple near Hanoi, in Vietnam.
- In this temple, some monks are in the process of solving a giant puzzle consisting of 64 golden rings of different sizes on three diamond needles.
- God placed these golden rings on the first needle and ordered that the monks should transfer them to the third, according to Lucas's rule.

Tower of Brahma

- The monks reportedly work for day and night at their task. When they finish, the world will end!
- If the legend were true, and if the monks could move rings at a rate of 1 per second, it would take them $2^{64} - 1$ seconds or roughly 585 billion years.
- The universe is currently about 13.7 billion years old. Lucas's original puzzle is a bit more practical.
- With 8 rings, it requires $2^8 - 1 = 255$ moves, which takes about four minutes for the quick of hand.

How to solve non-homogeneous RRs?

- Thus, we see that the key to solving any non-homogeneous recurrence relations with constant coefficients is finding a particular solution.
- Then every solution is a sum of this solution and a solution of the associated homogeneous recurrence relation.
- Although there is no general method for finding such a solution that works for every function F(n), there are techniques that work for certain types of functions F(n), such as polynomials and powers of constant.

That's Interesting! ☺

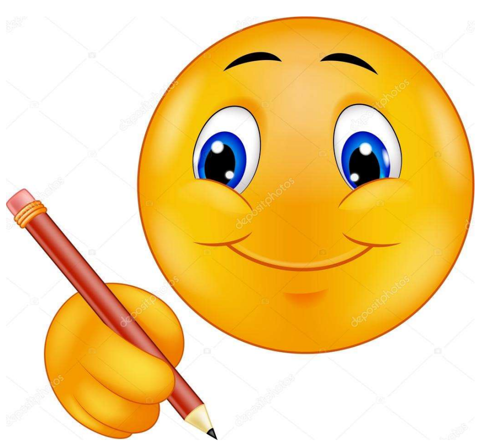

Image Courtesy: Depositphotos

Questions?

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Practice Problem

Q01: Find a recurrence relation for the number of regions created by n mutually intersecting circles on a piece of paper (no three circles have a common intersection point).

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